# CSE 6117 Distributed Computing Assignment 5

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03-02-2016

## The Problem

Given a k-set consensus (algorithm 1), we have a synchronous message passing model with the following two conditions which must be satisfied.

- **Validity:** Every output value is the input value of some process.
- **$k$-Agreement:** The number of different output values is at most $k$.

SET-AGREE(input: N);
$pref < -input$;
**for** $i < -1...f/k+1$ **do**
  send a message containing *pref* to all processes including self;
  pref $<-$ minimum value received in this round;
**end**
return *pref*;

**Algorithm 1**: k-set consensus algorithm

## Show that this algorithm solves k-set consensus in a synchronous message passing system where up to $f$ halting failuers can occur ($f < n$)

**Hint:** Let $P_i$ be the set of values in the *pref* variables of all the live processes at the end of round $i$. How is $P_i$ related to $P_{i-1}$? How big can $P_i$ be if $f_i$ processes fail during round $i$?

**Assumptions** We consider the worst case when we have $n$ processes each of which initially have all distinct values.

**Case 1** Consider what happens, if at any given round during this algorithm, no failures occur. Then there is a minimum value $v$ by processor $i$ $v_i$ such that every other process receives this value. And hence all remaining processes will have this minimum value and our problem reduces down to a 1-set consensus.

**Case 2** There is some number of processes $f_i$ which die at the end of round $i$. We assume that these processes which are halting had the minimum preference values, otherwise there would be a convergence in the consensus problem and we would again arrive at some trivial 1-set consensus.

**Setting Up The Problem** If we assume that during any round $i$ there is some $f_i$ halting failures occurring, then we note that $|P_i| \leq |P_{i-1}|$. In other words, there is some subset of $P_{i-1}$ which is remaining in the $i^{th}$ round. Formally, $P_i \subseteq P_{i-1}$. The set of distinct values can never increase from one round to the next since we have some $f_i$ failures occurring, and since each of these processes had distinct values, we are effectively reducing the size of the set. If there was some $v_i$ floating that was the minimum, then this must have failed before sending to at least $\leq k$ processes at that round.

**Size of $P_i$** has to be $|P_i| = f_i + 1$ since at the very least there should be one process remaining at the end of this round. We also note that since the size of $|P_i|$ is decreasing at each round and since we have distinct values in this round, then that implies that there should have been failures in previous rounds as well, hence we note that at least $f_i * i$ failures in total have occurred thus far.

**How and when Halting Failures Occur** Given a set of processes $p_0...p_r$, and if $a_0...a_r$ are the values in $P_i$ in ascending order where each of the values are distinct, then we know that in the previous round there's some processes that failed otherwise they would have taken a smaller value. Hence, if we follow this same logic, before the end of this round, some $f_i$ processes from $a_0....a_{f_i}$ each send a value to 1 other distinct process. If all the other processes have values greater than the ones received just before the process fails, that will be their new value. For the remaining $a_{f_{i+1}}....a_r$ processes, they will receive values from every other process and hence will reach a consensus. Hence why we have the constraint of $|P_i| = f_i + 1$ (note we are considering worst case analysis, if less than this number of processes die, then we can achieve a consensus faster, but we want to show the upper bound of at most k).

**Induction Hypothesis** We know that $|P_i| = f_i + 1$ processes remain at round $i$. If we suppose that the induction hypothesis also holds for $i - 1$, then $|P_{i-1}| \geq |P_i| \geq f_i + 1$ and hence at least $f_{i-1}$ processes died in the previous round. Total of $\sum_{a=0}^{a=i-1} f_a$ processes have died in the previous round.

**Show Consensus is Possible** Given that the total number of processes that can die is at most $\sum_{a=0}^{a=f/k+1} f_a = f$ and that we have $f < n$, we have shown that there is at least one process remaining at the end of each iteration. Hence at the very minimum $k = 1$ consensus is possible. Now let's assume that we have some final round $r$ such that $|P_r| = f_r + 1$, then it follows that $r = \lfloor \frac{f}{k} \rfloor + 1$ gives $f < kr$ where r is the final round. If we set the number of distinct values in the final round to $|P_r| = k + 1$, then we arrive at a contradiction since we had stated earlier that if at any given round $|P_r| = f_i + 1$ then the total number of processes that have died is at least $f_i * i$. However, note that $f > k * (i = r) \Leftrightarrow f > kr$ but, given the constraint on the number of rounds $r$, $f < kr$, hence it follows that using this algorithm, we will not get more than $k$ distinct values, and at least 1-set consensus. Given the number of failures, there is a constraint of $1 \leq j \leq k$ on the maximum number of distinct values we can have.

**Validity is Satisfied:** For every process that had failed, in a previous round, if was still alive, it's output was the input to some other process. The remaining processes output value to some other process as well, and the minimum value amongst these was chosen by all the other processes which did not receive a message from one of the processes with a smaller minimum value before it halted.

**k-Agreement is Satisfied:** By the induction hypothesis, there is at least $|P_i| = f_i + 1$ distinct values floating in the network, by the constraint on the number of rounds, we have at most $k$ different output values since $f < kr$.

# References

Chaudhuri, Soma. Tight Bounds for K-set Agreement. Cambridge, MA: Digital, Cambridge Research Laboratory, 1998. Print.