Lecture 9&10

Continuous Probability Distributions

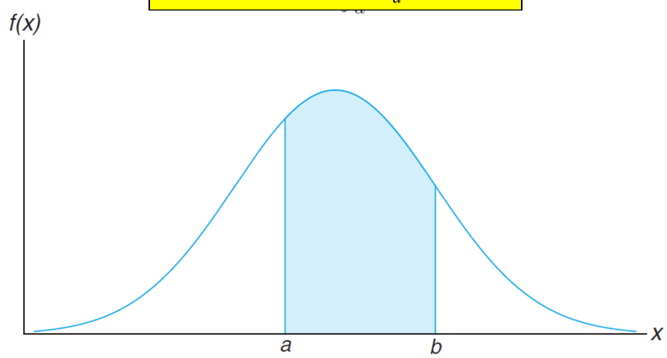
A continuous random variable has a probability of 0 of assuming *exactly* any of its values.

$$P(a < X \leq b) = P(a < X < b) + P(X = b) = P(a < X < b).$$

Consequently, its probability distribution cannot be given in tabular form but it can be stated as a formula. In dealing with continuous variables, $f(x)$ is usually called the **probability density function**, or simply the **density function**, of X.

A probability density function is constructed so that the area under its curve bounded by the x axis is equal to 1 when computed over the range of X for which $f(x)$ is defined.

$$\boxed{P(a < X < b) = \int_a^b f(x)dx}$$



Definition: The function $f(x)$ is a **probability density function** (pdf) for the continuous random variable X, defined over the set of real numbers, if

1. $f(x) \geq 0$, for all $x \in R$.
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.
3. $P(a < X < b) = \int_a^b f(x)dx$.

Mathematical Expectation (Mean of a Random Variable)

Definition: Let X be a random variable with probability distribution $f(x)$. The **mean**, or **expected value**, of X is

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx \qquad \text{if } X \text{ is continuous.}$$

Theorem: Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx \qquad \text{if } X \text{ is continuous.}$$

Also, if a and b are constants, then

$$E(aX \pm b) = aE(X) \pm b.$$

Variance of Random Variables

Definition: Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is

$$\boxed{\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx, \quad \text{if } X \text{ is continuous.}}$$

Theorem: The variance of a random variable X is $\boxed{\sigma^2 = E(X^2) - \mu^2.}$

Theorem: Let X be a random variable with probability distribution $f(x)$. The variance of the random variable $g(X)$

is $$\boxed{\sigma^2_{g(X)} = E\{[g(X)-\mu_{g(X)}]^2\} = \int_{-\infty}^{\infty} [g(X)-\mu_{g(X)}]^2 f(x)dx \qquad \text{if } X \text{ is continuous.}}$$

Example 10.1: Suppose that the error in the reaction temperature, in ◦C, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify that $f(x)$ is a density function.
(b) Find $P(0 < X \leq 1)$.
(c) The expected value of $g(X) = 4X + 3$
(d) The variance of the random variable $g(X)$.

***Solution*:**

(a) Obviously, $f(x) \geq 0$. To verify condition 2 in the previous definition, we have

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{2} \frac{x^2}{3} dx = \frac{x^3}{9}\Big|_{-1}^{2} = \frac{8}{9} + \frac{1}{9} = 1.$$

(b) $P(0 < X \leq 1) = \displaystyle\int_0^1 \frac{x^2}{3} dx = \frac{x^3}{9}\Big|_0^1 = \frac{1}{9}.$

(c) $E(4X + 3) = \displaystyle\int_{-1}^{2} \frac{(4x+3)x^2}{3} dx = \frac{1}{3}\int_{-1}^{2} (4x^3 + 3x^2)dx = 8.$

Or simply,

$$E(4X+3) = 4E(X) + 3.$$

Now

$$E(X) = \int_{-1}^{2} x(\frac{x^2}{3})dx = \int_{-1}^{2} \frac{x^3}{3} dx = \frac{5}{4}.$$

Therefore,

$$E(4X+3) = (4)\left(\frac{5}{4}\right) + 3 = 8, \qquad \text{as before.}$$

(d) $\sigma^2_{4X+3} = \text{Var}(4X+3) = E\{[(4X+3)-8]^2\} = E[(4X-5)^2]$

$$= \int_{-1}^{2} (4x-5)^2 \frac{x^2}{3} dx = \frac{1}{3}\int_{-1}^{2} (16x^4 - 40x^3 + 25x^2)dx = \frac{51}{5}. \Rightarrow \boxed{\sigma = \sqrt{\frac{51}{5}}.}$$

Or simply,

$$Var(4X+3) = 16\text{Var}(X)$$

$$E(X) = \frac{5}{4}$$

$$E(X^2) = \int_{-1}^{2} x^2(\frac{x^2}{3})dx = \int_{-1}^{2} \frac{x^4}{3} dx = \frac{11}{5}.$$

$$\sigma^2 = \frac{11}{5} - \left(\frac{5}{4}\right)^2 = \frac{51}{80}.$$

$$\Rightarrow Var(4X+3) = 16\text{Var}(X) = 16(\frac{51}{80}) = \frac{51}{5} \Rightarrow \boxed{\sigma = \sqrt{\frac{51}{5}}.}$$

Example 10.2: Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} C(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of C?
(b) Find $P(X > 1)$.

Solution:

(a) Since f is a probability density function, we must have $\int_{-\infty}^{\infty} f(x)dx = 1,$ implying that

$$C\int_{0}^{2} (4x - 2x^2)dx = 1$$

or

$$C[2x^2 - \frac{2x^3}{3}]\Big|_{x=0}^{x=2} = 1$$

or

$$C = \frac{3}{8}$$

Hence,

(b) $P(X > 1) = \int_1^\infty f(x)dx = \frac{3}{8}\int_1^2 (4x - 2x^2)dx = \frac{1}{2}$

Example 10.3: The amount of time in hours that a computer functions before breaking down is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \lambda e^{-x/100} & x \geq 0 \\ 0 & x<0 \end{cases}$$

(a) Find λ.

(b) What is the probability that a computer will function between 50 and 150 hours before breaking down?

(c) What is the probability that it will function for fewer than 100 hours?

***Solution*:**

(a) Since $1 = \int_{-\infty}^{\infty} f(x)dx = \lambda \int_0^\infty e^{-x/100}dx$

we obtain $1 = -\lambda(100)e^{-x/100} \Big|_0^\infty = 100\lambda$ or $\lambda = \frac{1}{100}$

(b) $P(50 < X < 150) = \int_{50}^{150} \frac{1}{100}e^{-x/100}dx = -e^{-x/100} \Big|_{50}^{150} = e^{-1/2} - e^{-3/2} \approx 0.384$

(c) Similarly, $P(X < 100) = \int_0^{100} \frac{1}{100}e^{-x/100}dx = -e^{-x/100} \Big|_0^{100} = 1 - e^{-1} \approx 0.633$

In other words, approximately 63.3 % of the time, a computer will fail before registering 100 hours of use.

Example 10.4: The lifetime in hours of a certain kind of radio tube is a random variable having a probability density function given by

$$f(x) = \begin{cases} 0 & x < 100 \\ \dfrac{100}{x^2} & x \geq 100 \end{cases}$$

What is the probability that exactly 2 of 5 such tubes in a radio set will have to be replaced within the first 150 hours of operation? Assume that the events E_i, $i = 1, 2, 3, 4, 5$, that the ith such tube will have to be replaced within this time are independent.

***Solution*:**

From the statement of the problem, we have

$$P(E_i) = \int_0^{150} f(x)dx$$

$$= 100\int_{100}^{150} x^{-2}dx = \frac{1}{3}$$

Hence, from the independence of the events E_i, it follows that the desired probability is

$$\binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = \frac{80}{243}$$

Example 10.7: Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \dfrac{20,000}{x^3}, & x > 100, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected life of this type of device.

***Solution*:**

$$\mu = E(X) = \int_{100}^{\infty} x\frac{20,000}{x^3}dx = \int_{100}^{\infty}\frac{20,000}{x^2}dx = 200.$$

Definition: The **cumulative distribution function** $F(x)$ of a continuous random variable X with density function $f(x)$ is

$$\boxed{F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt, \quad \text{for} \quad -\infty < x < \infty.}$$

1. $0 \le F(x) \le 1$.

2. $F(-\infty) = 0$ and $F(\infty) = 1$

3. $P(X<a) = F(a)$ and 4. $P(X>a) = 1 - F(a)$.

As an immediate consequence of this definition, one can write the two results

5. $P(a < X < b) = F(b) - F(a)$ and 6. $f(x) = \dfrac{dF(x)}{dx}$

if the derivative exists.

Example 10.5: For the density function $f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere} \end{cases}$, find $F(x)$, and use it to evaluate $P(0 < X \le 1)$.

***Solution*:**

For $x < -1$, $\quad F(x) = \int_{-\infty}^{x} f(t)dt = 0.$

For $-1 \le x < 2$, $\quad F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-1}^{x}\frac{t^2}{3}dt = \frac{t^3}{9}\Big|_{-1}^{x} = \frac{x^3+1}{9}.$

For $x \ge 2$, $\quad F(x) = \int_{-\infty}^{x} f(t)dt = 0 + \int_{-1}^{2}\frac{t^2}{3}dt + 0 = 0 + 1 + 0 = 1.$

Therefore,

$$F(x) = \begin{cases} 0, & x < -1, \\ \dfrac{x^3+1}{9}, & -1 \le x < 2, \\ 1, & x \ge 2. \end{cases}$$

The cumulative distribution function $F(x)$ is expressed in the following figure. Now

$$P(0 < X \le 1) = F(1) - F(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9},$$

$$\boxed{\text{note, } f(x) = \frac{dF(x)}{dx} = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}}$$