Lecture 6

Additive Rules for Probability:

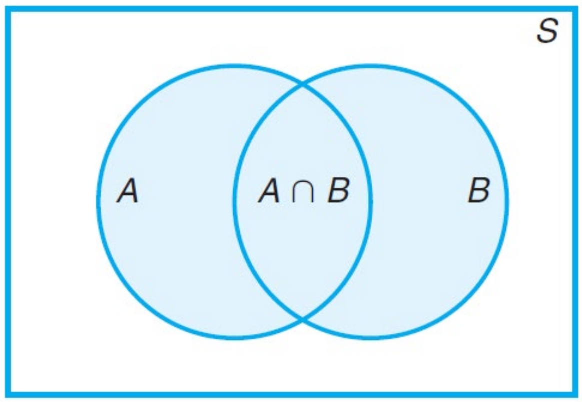
Happens in a single trial and requires elimination of any "double count".

Occurrence at the same time (in a single trial)

Theorem: If A and B are two events, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



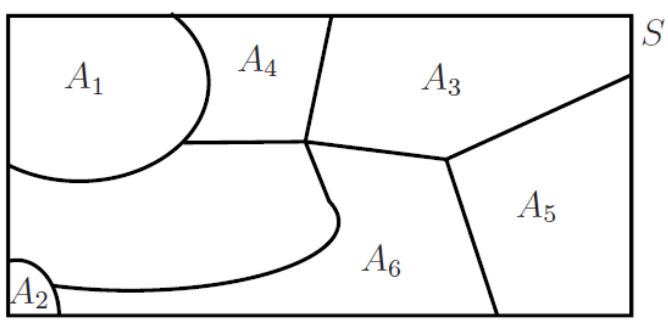
Corollary: If A and B are mutually exclusive (can't happen at the same time), then

$$P(A \cup B) = P(A) + P(B).$$

Corollary: If $A_1, A_2, \ldots, An$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

A collection of events $\{A_1, A_2, \ldots, A_n\}$ of a sample space S is called a **partition** of S if $A_1, A_2, \ldots, A_n$ are mutually exclusive and $A_1 \cup A_2 \cup \cdots \cup A_n = S$.



a partition is a collection of non-empty, non-overlapping subsets of a sample space whose union is the sample space itself

Thus, we have

Corollary: If $A_1, A_2, \ldots, An$ is a partition of sample space S, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) = P(S) = 1.$$

As one might expect, the previous theorem extends in an analogous fashion.

Example 6.7: What is the probability of getting a total of 7 or 11 when a pair of fair dice is tossed?

***Solution*:** Let A be the event that 7 occurs and B the event that 11 comes up. Now, a total of 7 occurs for 6 of the 36 sample points, and a total of 11 occurs for only 2 of the sample points. Since all sample points are equally likely, we have $P(A) = 1/6$ and $P(B) = 1/18$. The events A and B are mutually exclusive, since a total of 7 and 11 cannot both occur on the same toss. Therefore,

$$P(A \cup B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}.$$

This result could also have been obtained by counting the total number of points for the event $A \cup B$, namely 8, and writing

$$P(A \cup B) = \frac{n}{N} = \frac{8}{36} = \frac{2}{9}$$

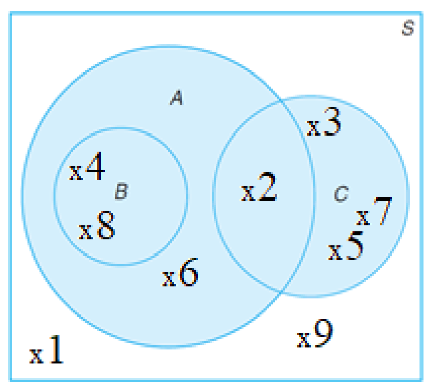
Example: $P(\heartsuit \text{ or } \spadesuit) = P(\heartsuit) + P(\spadesuit) - P(\heartsuit \text{ and } \spadesuit) = \frac{13}{52} + \frac{13}{52} = \frac{1}{2}$

Example: $P(\clubsuit \text{ or K}) = P(\clubsuit) + P(\text{K}) - P(\clubsuit \text{ and K}) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{4}{13}$

Example 6.11: 9 identical cards numbered from 1 to 9. card was drawn randomly.

Let A be the event that the card has an even number.
Let B be the event that the card has a number divisible by 4.
Let C be the event that the card has a prime number.
Calculate the probability of:

i. The occurrence of A **<u>and</u>** B (together).
ii. The occurrence of A **<u>or</u>** B.
iii. The occurrence of A **<u>only but not</u>** B.
iv. The **<u>nonoccurrence</u>** of A.
v. The occurrence of B **<u>and</u>** C together.



***Solution*:**

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \Rightarrow n(S) = 9.$

$A = \{2, 4, 6, 8\}.$

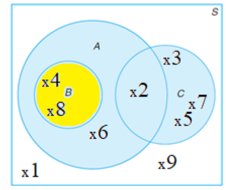
$B = \{4, 8\}.$

$C = \{2, 3, 5, 7\}.$

i. P(A **and** B together)

$A \cap B = \{4, 8\} \Rightarrow n(A \cap B) = 2$

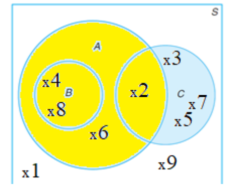
$$\Rightarrow P(A \cap B) = \frac{2}{9}$$



P(A **or** B)

$A \cup B = \{2, 4, 6, 8\} \Rightarrow n(A \cup B) = 4$

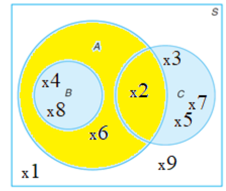
$$\Rightarrow P(A \cup B) = \frac{4}{9}$$



ii. P(A **only but not** B)

$A - B = \{2, 6\} \Rightarrow n(A - B) = 2$

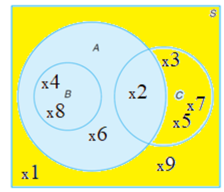
$$\Rightarrow P(A - B) = \frac{2}{9}$$



iii. P(**nonoccurrence** of A)

$A' = \{1, 3, 5, 7, 9\} \Rightarrow n(A') = 5$

$$\Rightarrow P(A') = \frac{5}{9}$$

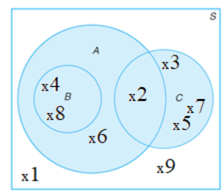


iv. P(B **and** C together)

$B \cap C = \phi \Rightarrow n(B \cap C) = 0$

$$\Rightarrow P(B \cap C) = 0$$

B and C are mutually exclusive.



Conditional Probability

The probability of an event A occurring when it is known that some event B has occurred is called a **conditional probability** and is denoted by $P(A|B)$. The symbol $P(A|B)$ is usually read "the probability that A occurs given that B occurs" or simply "the probability of A, given B."

- Consider a fair die being tossed, the event A: "6 appeared" $\Rightarrow$

  Relative to the sample space $S = \{1, 2, 3, 4, 5, 6\}$, $P(A) = \frac{1}{6}$.

- Suppose that the die has already been tossed, and event B occurred: "Even number appeared" What's the chance of A now?

  Relative to the sample space $S' = \{2, 4, 6\}$, $P(A) = \frac{1}{3}$.

$P(A, \text{Relative to } S) = \frac{1}{6} \Rightarrow P(A) = \frac{1}{6}$.

$P(A, \text{Relative to } S') = \frac{1}{3} \Rightarrow P(A|B) = \frac{1}{3}$.

$$P(A|B) = \frac{1}{3} = \frac{1/6}{3/6} = \frac{P(A \cap B)}{P(B)},$$

Definition: The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) > 0.$$

Example 6.12: The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane (a) arrives on time, given that it departed on time, and (b) departed on time, given that it has arrived on time.

Solution: (a) The probability that a plane arrives on time, given that it departed on time, is

$$P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{0.78}{0.83} = 0.94.$$

(b) The probability that a plane departed on time, given that it has arrived on time, is

$$P(D|A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82} = 0.95.$$

Also, The probability that the flight arrives on time, given that it did not depart on time will be $P(A \mid D')$:

$$P(A \mid D') = \frac{P(A \cap D')}{P(D')} = \frac{0.82 - 0.78}{0.17} = 0.24$$

As a result, the probability of an on-time arrival is diminished severely in the presence of the additional information.

Independent Events

Example: In the die-tossing experiment, we note that $P(A|B) = \frac{1}{3}$ whereas $P(A) = \frac{1}{6}$. That is, $P(A|B) \neq P(A)$, indicating that A depends on B.

Now consider an experiment in which 2 cards are drawn in succession from an ordinary deck, with replacement ($\Rightarrow A$ and B are **independent**). The events are defined as

A: the first card is an ace,

B: the second card is a spade.

$$P(B|A) = \frac{13}{52} = \frac{1}{4} \quad \text{and} \quad P(B) = \frac{13}{52} = \frac{1}{4}.$$

That is, $P(B|A) = P(B)$. When this is true, the events A and B are said to be **independent**.

Example: Suppose we have 2 questions; the first one is T/F, the second is MCQ with five choices.

$P(\text{Guess Answer in T/F}) = \frac{1}{2}$, $P(\text{Guess Answer in MCQ}) = \frac{1}{5}$

$P(\text{T/F}) * P(\text{MCQ}) = \frac{1}{2} * \frac{1}{5} = \frac{1}{10} = P(\text{T/F and MCQ})$

Example:

$$\text{w/o replacement } P(Q|9) = \frac{4}{51} \Rightarrow \textbf{dependent},$$

$$\text{w/ replacement } P(Q|9) = P(Q) = \frac{4}{52} = \frac{1}{13} \Rightarrow \textbf{independent},$$

$$\text{w/o replacement } P(Q|Q) = \frac{3}{51},$$

$$\text{w/ replacement } P(Q|Q) = \frac{4}{52} = \frac{1}{13},$$

$$\text{w/o replacement } P(\heartsuit|\text{J}\diamondsuit) = \frac{13}{51},$$

$$\text{w/ replacement } P(\heartsuit|\text{J}\diamondsuit) = \frac{13}{52},$$

Definition 2.11: Two events A and B are **independent** if and only if

$$P(B|A) = P(B) \text{ or } P(A|B) = P(A),$$

assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**. The condition $P(B|A) = P(B)$ implies that $P(A|B) = P(A)$, and conversely.

The Product Rule, or the Multiplicative Rule for Probability:

Multiplicative rule (or **product rule**), enables us to calculate the probability that two events will both occur (in successive trials).

Theorem: If in an experiment the events A and B can both occur, then

$$P(A \cap B) = \begin{cases} P(A)P(B \mid A), & \text{in general} \\ P(A)P(B), & \text{if } A, B \text{ are independent} \end{cases}$$

Since the events $A \cap B$ and $B \cap A$ are equivalent, it follows that we can also write

$$P(A \cap B) = P(B \cap A) = P(B)P(A|B).$$

Example 6.13: Suppose that we have a fuse box containing 20 fuses, of which 5 are defectives. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Solution: We shall let D_1 be the event that the first fuse is defective and D_2 the event that the second fuse is defective; Hence,

without replacing the first $\Rightarrow P(D_1 \cap D_2) = P(D_1)\, P(D_2 \mid D_1) = \left(\frac{5}{20}\right)\left(\frac{4}{19}\right) = \frac{1}{19}.$

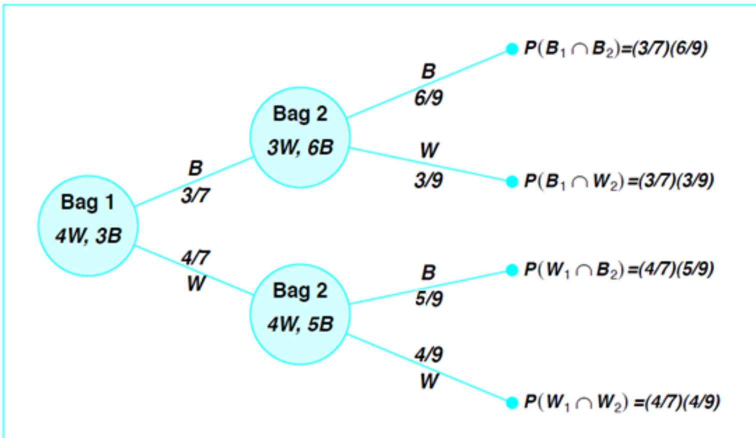
with replacing the first (independent) $\Rightarrow P(D_1 \cap D_2) = P(D_1)\, P(D_2 \mid D_1) = P(D_1)\, P(D_2 \mid D_1)$

$$= \left(\frac{5}{20}\right)\left(\frac{5}{20}\right) = \frac{1}{16}.$$

Example 6.14: One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

Solution: Let B_1, B_2, and W_1 represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1. We are interested in the union of the mutually exclusive events $B_1 \cap B_2$ and $W_1 \cap B_2$. The various possibilities and their probabilities are illustrated in the following figure.

$$\begin{aligned} P[(B_1 \cap B_2) \text{ or } (W_1 \cap B_2)] &= P(B_1 \cap B_2) + P(W_1 \cap B_2) \\ &= P(B_1)P(B_2|B_1) + P(W_1)P(B_2|W_1) \\ &= \left(\frac{3}{7}\right)\left(\frac{6}{9}\right) + \left(\frac{4}{7}\right)\left(\frac{5}{9}\right) = \frac{38}{63}. \end{aligned}$$



Example 6.16: Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that the second card is a 10 or a jack, and A_3 is the event that the third card is greater than 3 but less than 7.

Solution: First we define the events

A_1: the first card is a red ace,

A_2: the second card is a 10 or a jack,

A_3: the third card is greater than 3 but less than 7.

Now

$$P(A_1) = \frac{2}{52}, \quad P(A_2|A_1) = \frac{8}{51}, \quad P(A_3|A_1 \cap A_2) = \frac{12}{50},$$

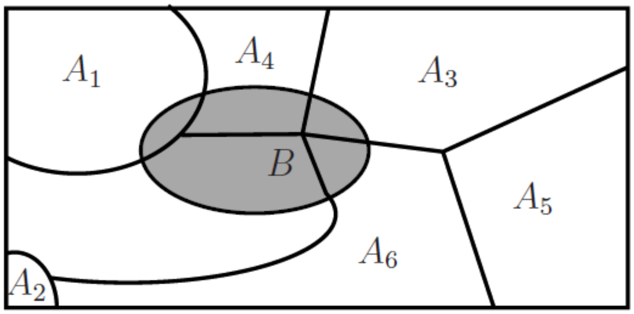
and hence,

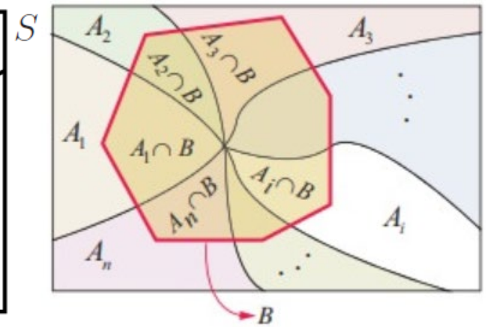
$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \\ &= \left(\frac{2}{52}\right)\left(\frac{8}{51}\right)\left(\frac{12}{50}\right) = \frac{8}{5525}. \end{aligned}$$

Total Probability or (the rule of elimination)

Theorem: If the events $A_1, A_2, \ldots, A_k$ constitute a partition of the sample space S such that $P(A_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event B of S,

$$P(B) = \sum_{i=1}^{k} P(A_i \cap B) = \sum_{i=1}^{k} P(A_i)P(B \mid A_i)$$

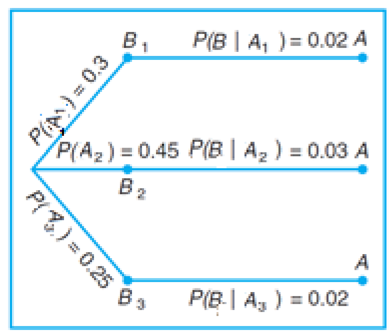




Example 6.17: In a certain assembly plant, three machines, A_1, A_2, and A_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

***Solution*:** Consider the following events:

B: the product is defective,
A_1: the product is made by machine A_1,
A_2: the product is made by machine A_2,
A_3: the product is made by machine A_3.
Applying the rule of elimination, we can write



$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3).$$

Referring to the tree diagram, we find that the three branches give the probabilities

$$P(A_1)P(B|A_1) = (0.3)(0.02) = 0.006,$$
$$P(A_2)P(B|A_2) = (0.45)(0.03) = 0.0135,$$
$$P(A_3)P(B|A_3) = (0.25)(0.02) = 0.005,$$

and hence

$$P(B) = 0.006 + 0.0135 + 0.005 = 0.0245.$$

Bayes' Rule:

Instead of asking for $P(B)$ in the previous example, by the rule of elimination, suppose that we now consider the problem of finding the conditional probability $P(A_i|B)$.

In other words, suppose that a product was randomly selected and it is defective. What is the probability that this product was made by machine A_i? Questions of this type can be answered by using the following theorem, called

Theorem: (Bayes' Rule) If the events $A_1, A_2, \ldots, A_k$ constitute a partition of the sample space S such that $P(Ai) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event B in S such that $P(B) \neq 0$,

$$P(A_r|B) = \frac{P(A_r \cap B)}{\sum_{i=1}^{k} P(A_i \cap B)} = \frac{P(A_r)P(B \mid A_r)}{\sum_{i=1}^{k} P(A_i)P(B \mid A_i)} \quad \text{for } r = 1, 2, \ldots, k.$$

Example 6.18: With reference to the previous example, if a product was chosen randomly and found to be defective, what is the probability that it was made by machine A_3?

***Solution*:** Using Bayes' rule to write

$$P(A_3|B) = \frac{P(A_3)P(B \mid A_3)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)}$$

and then substituting the probabilities calculated in the previous example, we have

$$P(A_3|B) = \frac{0.005}{0.006 + 0.0135 + 0.005} = \frac{0.005}{0.0245} = \frac{10}{49}.$$

In view of the fact that a defective product was selected, this result suggests that it probably was not made by machine A_3.

Example 6.19: A manufacturing firm employs three analytical plans for the design and development of a particular product. For cost reasons, all three are used at varying times. In fact, plans 1, 2, and 3 are used for 30%, 20%, and 50% of the products, respectively. The defect rate is different for the three procedures as follows:

$$P(D|P_1) = 0.01, \qquad P(D|P_2) = 0.03, \qquad P(D|P_3) = 0.02,$$

where $P(D|P_j)$ is the probability of a defective product, given plan j. If a random product was observed and found to be defective, which plan was most likely used and thus responsible?

***Solution*:** From the statement of the problem

$P(P_1) = 0.30, \qquad P(P_2) = 0.20, \qquad$ and $\qquad P(P_3) = 0.50,$

we must find $P(P_j\,|D)$ for $j = 1, 2, 3$. Bayes' rule shows:

$$P(P_1|D) = \frac{P(P_1)P(D \mid P_1)}{P(P_1)P(D \mid P_1) + P(P_2)P(D \mid P_2) + P(P_3)P(D \mid P_3)}$$

$$= \frac{(0.30)(0.01)}{(0.30)(0.01) + (0.20)(0.03) + (0.50)(0.02)} = \frac{0.003}{0.019} = 0.158$$

Similarly,

$$P(P_2|D) = \frac{(0.03)(0.20)}{0.019} = 0.316 \qquad \text{and} \qquad P(P_3|D) = \frac{(0.02)(0.50)}{0.019} = 0.526$$

The conditional probability of a defect given plan 3 is the largest of the three; thus, a defective for a random product is most likely the result of the use of plan 3.