Lecture 8

Special Probability Distributions:

Discrete Probability Distribution:

An experiment often consists of repeated trials, each with two possible outcomes that may be labeled **success** or **failure**. The most obvious application deals with the testing of items as they come off an assembly line, where each trial may indicate a defective or a nondefective item. We may choose to define either outcome as a success. The process is referred to as a **Bernoulli process**. Each trial is called a **Bernoulli trial**.

The Bernoulli Process

1. Consists of repeated trials
2. Each trial is a success or a failure.
3. The probability of success, denoted by p, remains constant from trial to trial.
4. The repeated trials are independent.

Consider the set of Bernoulli trials where three items are selected at random from a manufacturing process, inspected, and classified as defective or nondefective. A defective item is designated a success. The number of successes is a random variable X assuming integral values from 0 through 3. The eight possible outcomes and the corresponding values of X are

Outcome	*NNN*	*NDN*	*NND*	*DNN*	*NDD*	*DND*	*DDN*	*DDD*
$\boldsymbol{x}$	0	1	1	1	2	2	2	3

Since the items are selected independently and we assume that the process produces 25% defectives, we have

$$P(NDN) = P(N)P(D)P(N) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = \frac{9}{64}.$$

Similar calculations yield the probabilities for the other possible outcomes. The probability distribution of X is therefore

x	0	1	2	3
$f(x)$	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

1. **Binomial Distribution**

The number X of successes in n Bernoulli trials is called a **binomial random variable**. The probability distribution of this discrete random variable is called the **binomial distribution**, and its values will be denoted by $b(x; n, p)$ since they depend on the number of trials and the probability of a success on a given trial.

Binomial Distribution: A Bernoulli trial can result in a success with probability p and a failure with probability $q = 1-p$. Then the probability distribution of the binomial random variable X, the number of successes in n independent trials, is

$$\boxed{b(x;\, n,\, p) = \binom{n}{x} p^x q^{n-x}, \qquad x = 0,\ 1,\ 2,\ \ldots,\ n.}$$

Note that when $n = 3$ and $p = 1/4$, the probability distribution of X, the number of defectives, may be written as

$$b(x;3,\frac{1}{4}) = \binom{3}{x} (\frac{1}{4})^x (\frac{3}{4})^{3-x}, \quad x = 0,1,2,3$$

Theorem: The mean and variance of the binomial distribution $b(x;\, n,\, p)$ are

$$\boxed{\mu = np \text{ and } \sigma^2 = npq.}$$

Example 9.1: Five fair coins are flipped. If the outcomes are assumed independent, find the probability mass function of the number of heads obtained.

Solution: If we let X equal the number of heads (successes) that appear, then X is a binomial random variable with parameters ($n = 5$, $p = \frac{1}{2}$). Hence,

$$P(X=0) = \binom{5}{0} (\frac{1}{2})^0 (\frac{1}{2})^5 = \frac{1}{32}$$

$$P(X=1) = \binom{5}{1} (\frac{1}{2})^1 (\frac{1}{2})^4 = \frac{5}{32}$$

$$P(X=2) = \binom{5}{2} (\frac{1}{2})^2 (\frac{1}{2})^3 = \frac{10}{32}$$

$$P(X=3) = \binom{5}{3} (\frac{1}{2})^3 (\frac{1}{2})^2 = \frac{10}{32}$$

$$P(X=4) = \binom{5}{4} (\frac{1}{2})^4 (\frac{1}{2})^1 = \frac{5}{32}$$

$$P(X=5) = \binom{5}{5} (\frac{1}{2})^5 (\frac{1}{2})^0 = \frac{1}{32}$$

a) P(obtaining 3 heads) $= P(X=3) = \frac{10}{32}$

b) P(number of heads is less than 5) $= P(X<5) = 1 - P(X=5) = \frac{31}{32}$ (or at most 4 heads)

c) P(obtaining at least 2 heads) $=P(X\geq 2) = 1-[P(X=0)+P(X=1)]=1-\frac{6}{32}=\frac{26}{32}$

d) $E(X)=\mu = np = 5(\frac{1}{2}) = \frac{5}{2}$

e) $Var(X) = \sigma^2 = npq = 5(\frac{1}{2})(\frac{1}{2}) = \frac{5}{4}$

2. Poisson Distribution

A discrete probability distribution that is useful when n is large and p is small and when the independent variables occur over a period of time is called the **Poisson distribution.**

Formula for the Poisson Distribution

The probability of X occurrences in an interval of time, volume, area, etc., for a variable where λ (Greek letter lambda) is the mean number of occurrences per unit (time, volume, area, etc.) is

$$\boxed{P(x;\lambda)=\frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0,\ 1,\ 2,\ldots}$$

If X has Poisson distribution, $\boxed{\mu = E(X) = \lambda}$ and $\boxed{\sigma^2 = Var(X) = \lambda}$

Example 9–6 If approximately 2% of the people in a room of 200 people are left-handed, find the probability that exactly 5 people there are left-handed.

Solution

Since $\lambda = np$, then $\lambda =$ (200)(0.02) = 4. Hence,

Poisson

$$P(5;4)=\frac{(2.7183)^{-4}(4)^5}{5!}=0.1563$$

Binomial

which is verified by the formula $\binom{200}{5}(0.02)^5(0.98)^{195}\approx 0.1579$. The difference between the two answers is based on the fact that the Poisson distribution is an approximation and rounding has been used.

3. Hypergeometric Distribution

When sampling is done ***without*** replacement, the binomial distribution does not give exact probabilities, since the trials are not independent. The smaller the size of the population, the less accurate the binomial probabilities will be.

Formula for the Hypergeometric Distribution

Given a population with only two types of objects (females and males, defective and nondefective, successes and failures, etc.), such that there are a items of one kind and b items of another kind and $a + b$ equals the total population, the probability $P(x)$ of selecting without replacement a sample of size n with x items of type a and n-x items of type b is

$$h(x) = \frac{\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}}$$

The mean of the hypergeometric distribution $h(x)$ are $\boxed{\mu = E(X) = \frac{na}{a+b}}$.

Example 9–8 We have a lot of 100 items of which 12 are defective. What is the probability that in a sample of 10, 3 are defective?

Solution

Using the hypergeometric probability function, we have

$$\mathrm{P(3D)} = \frac{\binom{12}{3}\binom{88}{7}}{\binom{100}{10}} = 0.08.$$

Solve using Multiplication rule !!!!!

Example 9–7

If 5 cards are drawn at random, we are interested in the probability of selecting 3 red cards from the 26 available in the deck and 2 black cards from the 26 available in the deck. There are $\binom{26}{3}$ ways of selecting 3 red cards, and for each of these ways we can choose 2 black cards in $\binom{26}{2}$ ways. Therefore, the total number of ways to select 3 red and 2 black cards in 5 draws is the product $\binom{26}{3}\binom{26}{2}$. The total number of ways to select any 5 cards from the 52 that are available is $\binom{52}{5}$.

Hence, the probability of selecting 5 cards without replacement of which 3 are red and 2 are black is given by

$$\frac{\binom{26}{3}\binom{26}{2}}{\binom{52}{5}} = 0.3251.$$