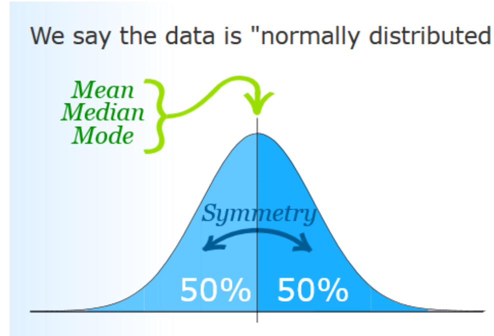
Lecture 11

Continuous Probability Distribution:

1- Normal Distribution



Its graph, called the **normal curve**, is the bell-shaped curve of the following figure. The normal distribution is often referred to as the **Gaussian distribution**.

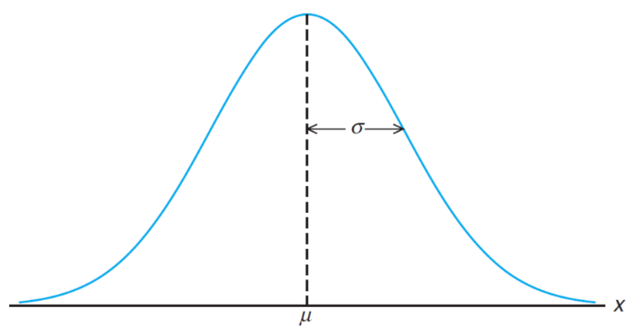
A continuous random variable X having the bell-shaped distribution is called a **normal random variable**. The mathematical equation for the probability distribution of the normal variable depends on the two parameters μ and σ, its mean and standard deviation, respectively. Hence, we denote the values of the density of X by $n(x; \mu, \sigma)$.

Normal Distribution: The density of the normal random variable X, with mean μ and variance σ^2, is

$$n(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \qquad -\infty < x < \infty,$$

where $\pi = 3.14159 \ldots$ and $e = 2.71828 \ldots$.

Once μ and σ are specified, the normal curve is completely determined. For example, if $\mu = 50$ and $\sigma = 5$, then the ordinates $n(x; 50, 5)$ can be computed for various values of x and the curve drawn.



Properties of the normal curve:

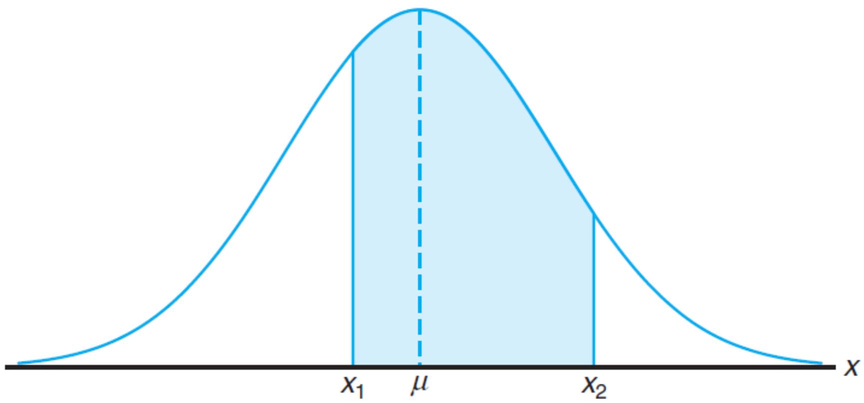
1. The mode occurs at $x = \mu$.
2. The curve is symmetric about the mean μ.
3. The curve has its points of inflection at $x = \mu \pm \sigma$; it is concave downward if $\mu - \sigma < X < \mu + \sigma$ and is concave upward otherwise.
4. The normal curve approaches the horizontal axis asymptotically as we proceed in either direction away from the mean.
5. The total area under the curve and above the horizontal axis is equal to 1. $\int_{-\infty}^{\infty} n(x; \mu, \sigma)\,dx = 1$

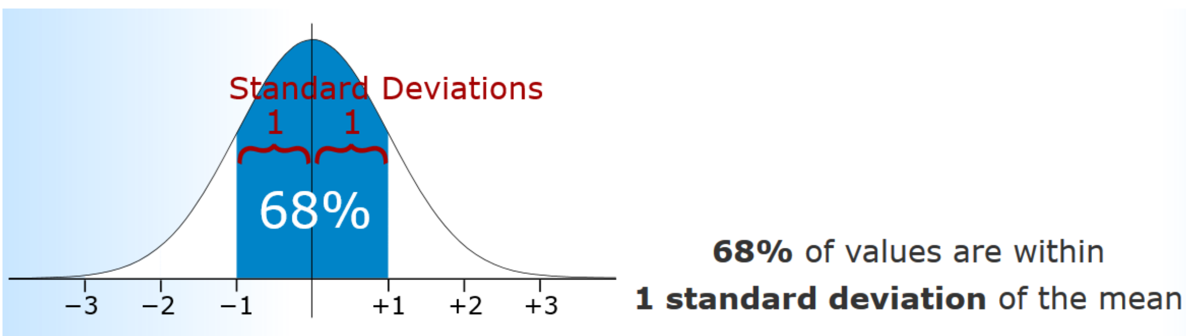
Theorem: The mean $E(X) = \mu$ and variance $Var(X) = \sigma^2$, Hence, the standard deviation $= \sigma$.

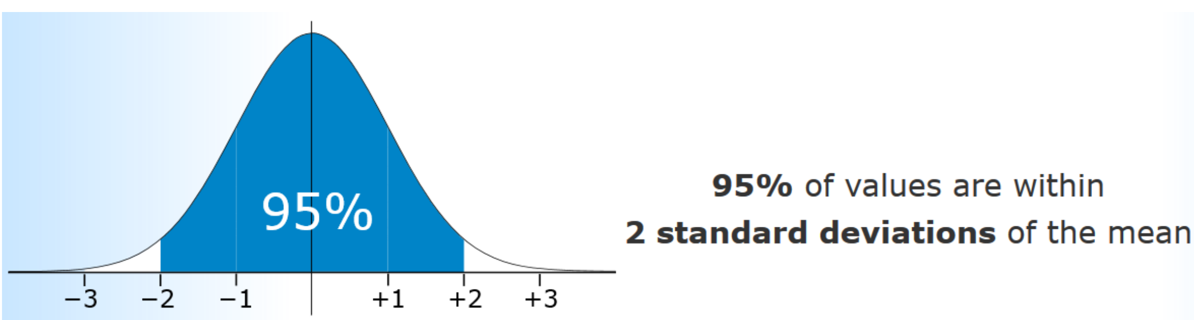
Areas under the Normal Curve

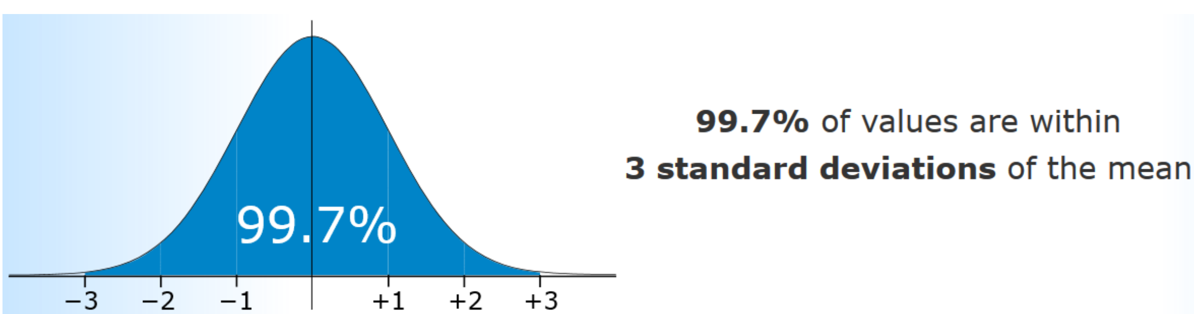
$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} n(x;\mu,\sigma)\,dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{x_1}^{x_2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx$$

is represented by the area of the shaded region.









standard normal distribution:

Consider the transformation

$$\boxed{Z = \frac{X-\mu}{\sigma}.}$$

where Z is the normal random variable with mean 0 and variance 1.

Whenever X assumes a value x, the corresponding value of Z is given by $z = \frac{x-\mu}{\sigma}$. Therefore, if X falls between the values $x = x_1$ and $x = x_2$, the random variable Z will fall between the corresponding values $z_1 = (x_1 - \mu)/\sigma$ and $z_2 = (x_2 - \mu)/\sigma$. Consequently, we may write

$$P(x_1 < X < x_2) = \frac{1}{\sigma\sqrt{2\pi}}\int_{x_1}^{x_2} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}\,dx = \frac{1}{\sqrt{2\pi}}\int_{z_1}^{z_2} e^{-\frac{1}{2}z^2}\,dz$$

$$\Rightarrow \int_{z_1}^{z_2} n(z;0,1)\,dz = P(z_1 < Z < z_2),$$

$$n(z;\,0,\,1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}, \qquad -\infty < z < \infty,$$

Definition: The distribution of a normal random variable with mean 0 and variance 1 is called a **standard normal distribution**.

The following table indicates the area under the standard normal curve corresponding to $\Phi(Z) = \underline{P(Z \le z)} = \int_{-\infty}^{z} f(z)\,dz = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}z^2}\,dz$ [Cumulative distribution function] for values of z ranging from −3.49 to 3.49.

$$\boxed{\Phi(0) = 0.5, \quad \Phi(-\infty) = 0 \quad \text{and} \quad \Phi(\infty) = 1}$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

e.g. $P(Z < 1.74) = \Phi(1.74) = \underline{0.9591}$,

The z value leaving an area of 0.2148 under the curve to the left of z is seen to be $\underline{-0.79}$.

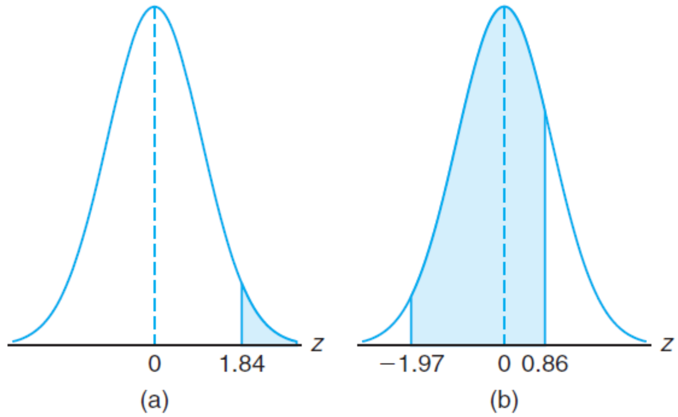
Example 11.1: Given a standard normal distribution, find the area under the curve that lies

(a) to the right of $z = 1.84$ and

(b) between $z = -1.97$ and $z = 0.86$.

Solution**:** (a) The area to the right of $z = 1.84$ is equal to 1 minus the area in the table to the left of $z = 1.84$, namely, $1 - 0.9671 = \underline{0.0329}$.

(b) The area between $z = -1.97$ and $z = 0.86$ is equal to the area to the left of $z = 0.86$ minus the area to the left of $z = -1.97$. From the table, we find the desired area to be $0.8051 - 0.0244 = \underline{0.7807}$.



Example 11.2: Given a standard normal distribution, find the value of k such that

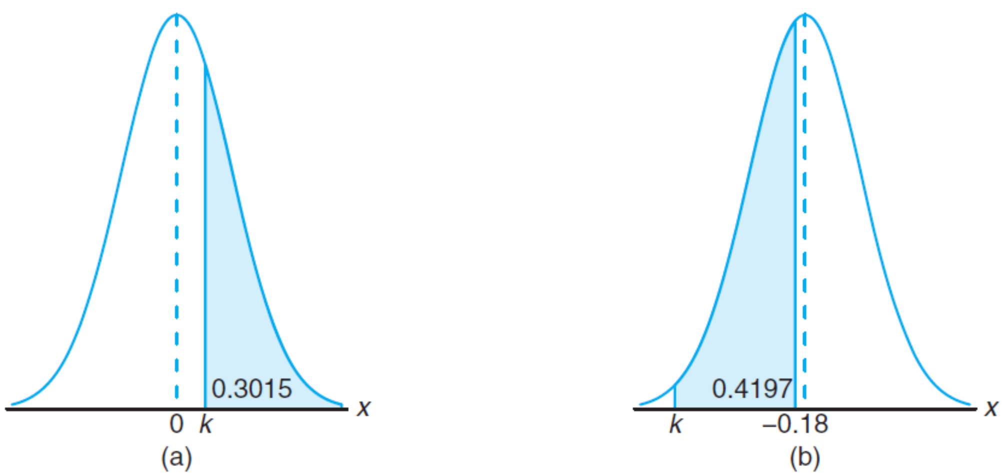
(a) $P(Z > k) = 0.3015$ and

(b) $P(k < Z < -0.18) = 0.4197$.

Solution:

(a) $1 - \Phi(k) = 0.3015 \Rightarrow \Phi(k) = 0.6985$. From the table, it follows that $\underline{k = 0.52}$.

(b) $\Phi(-0.18) - \Phi(k) = 0.4197 \Rightarrow \Phi(k) = 0.4286 - 0.4197 = 0.0089$. Hence, from the table, we have $\underline{k = -2.37}$.



Example 11.3: Given a random variable X having a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X assumes a value between 45 and 62.

Solution: The z values corresponding to $\underline{x_1 = 45}$ and $\underline{x_2 = 62}$ are

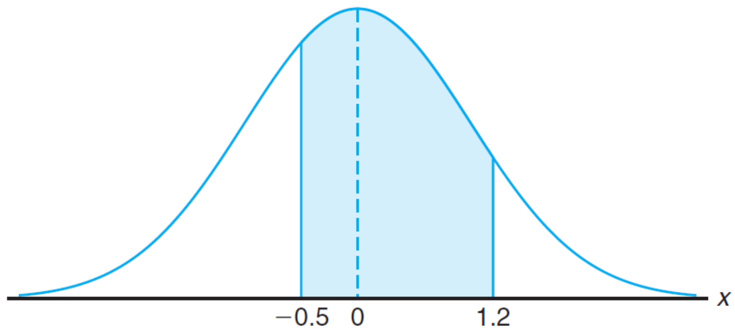
$$\boxed{Z_1 = \frac{45-50}{10} = -0.5 \text{ and } Z_2 = \frac{62-50}{10} = 1.2.}$$

Therefore,

$$P(45 < X < 62) = P(-0.5 < Z < 1.2).$$

$P(-0.5 < Z < 1.2)$ is shown by the area of the shaded region in the following figure. This area may be found by subtracting the area to the left of the ordinate $z = -0.5$ from the entire area to the left of $z = 1.2$. Using the table, we have

$$P(45 < X < 62) = P(-0.5 < Z < 1.2) = \Phi(1.2) - \Phi(-0.5)$$
$$= 0.8849 - 0.3085 = \underline{\underline{0.5764}}.$$



Using the Normal Curve in Reverse

Sometimes, we are required to find the value of z corresponding to a specified probability that falls between values listed in the table.

Example 11.4: Given a normal distribution with $\mu = 40$ and $\sigma = 6$, find the value of x that has

(a) 45% of the area to the left and

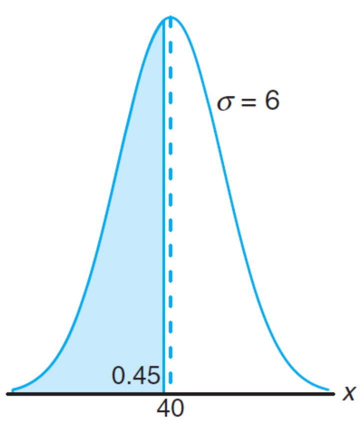
(b) 14% of the area to the right.

***Solution*:** (a) For an area of 0.45 to the left of the desired x value, we require a z value that leaves an area of 0.45 to the left. From the table, we find $\Phi(-0.13) = 0.45$, so the desired z value is -0.13. Hence,

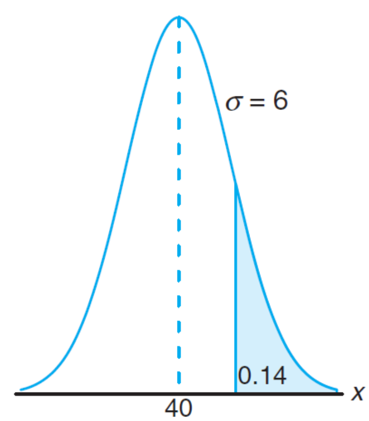
$$Z = \frac{x-\mu}{\sigma} \Rightarrow x = \sigma z + \mu.$$
$$\Rightarrow x = (6)(-0.13) + 40 = \underline{\underline{39.22}}.$$

(b) For an area equal to 0.14 to the right of the desired x value, we require a z value that leaves 0.14 of the area to the right and hence an area of 0.86 to the left. Again, from the table, we find $\Phi(1.08) = 0.86$, so the desired z value is 1.08 and

$$x = (6)(1.08) + 40 = \underline{\underline{46.48}}.$$



(a)



(b)

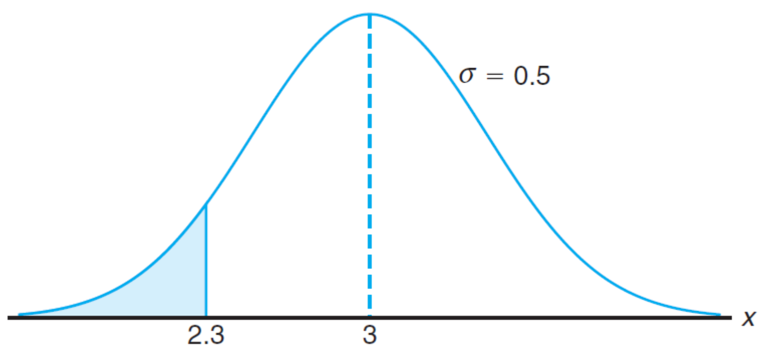
Example 11.5: A certain type of storage battery lasts, on average, 3.0 years with a standard deviation of 0.5 year. Assuming that battery life is normally distributed, find the probability that a given battery will last less than 2.3 years.

***Solution*:** To find $P(X < 2.3)$, we need to evaluate the area under the normal curve to the left of 2.3. This is accomplished by finding the area to the left of the corresponding z value. Hence, we find that

$$Z = \frac{2.3-3}{0.5} = -1.4,$$

and then, using the table, we have

$$P(X < 2.3) = P(Z < -1.4) = 0.0808.$$



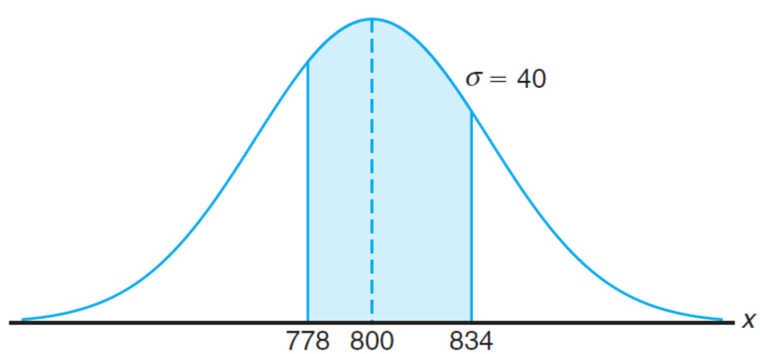
Example 11.6: An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a bulb burns between 778 and 834 hours.

***Solution*:** The z values corresponding to $x_1 = 778$ and $x_2 = 834$ are

$$Z_1 = \frac{778-800}{40} = -0.55 \text{ and } Z_2 = \frac{834-800}{40} = 0.85.$$

Hence,

$$P(778 < X < 834) = P(-0.55 < Z < 0.85) = P(Z < 0.85) - P(Z < -0.55)$$
$$= 0.8023 - 0.2912 = 0.5111.$$



2- Exponential Distribution

To predict the amount of waiting time until the next event

- **The amount of time** you need to **wait until** the bus arrives.
- **The amount of time until** the hardware component fails.
- **………**

The continuous random variable X has an **exponential distribution**, with parameter β, if its density function is given by

$$f(x;\beta)=\begin{cases}\dfrac{1}{\beta}e^{-x/\beta}, & x\geq 0,\\ 0 & \text{elsewhere,}\end{cases} \qquad \text{where } \beta > 0.$$

The mean and variance of the exponential distribution are $\mu = \beta$ and $\sigma^2 = \beta^2$.

Example: Let the random variable X has an **exponential distribution**, with parameter $\beta = \dfrac{100}{3}$, if its density function is given by

$$f(y)=\begin{cases}0.03e^{-0.03x}, & x>0,\\ 0 & \text{elsewhere,}\end{cases}$$

a) Find mean, variance and standard deviation
b) Find the probability that X is
   1- At least 5
   2- At most 7
   3- Between 3 and 6

Solution: a) $\mu = \beta = \dfrac{100}{3}$, Var(X) $= \sigma^2 = \beta^2 = (\dfrac{100}{3})^2$ and Standard deviation (X) $= \sigma =$ $(\dfrac{100}{3})$.

b) 1. $P(X\geq 5) = \int_5^\infty 0.03e^{-0.03x}dx = -e^{-0.03x}\,|_5^\infty = e^{-0.15}$

2. $P(X\leq 7) = \int_0^7 0.03e^{-0.03x}dx = -e^{-0.03x}\,|_0^7 = 1-e^{-0.21}$

3. $P(3\leq X\leq 6) = \int_3^6 0.03e^{-0.03x}dx = -e^{-0.03x}\,|_3^6 = e^{-0.09} - e^{-0.18}$

Example 11.12: Based on extensive testing, it is determined that the time Y in years before a major repair is required for a certain washing machine is characterized by the density function

$$f(y) = \begin{cases} \frac{1}{4} e^{-y/4}, & y > 0, \\ 0 & \text{elsewhere,} \end{cases}$$

Note that Y is an exponential random variable with $\mu = \beta = 4$ years. The machine is considered a bargain if it is unlikely to require a major repair before the sixth year. What is the probability $P(Y > 6)$? What is the probability that a major repair is required in the first year?

Solution: Consider the cumulative distribution function $F(y)$ for the exponential distribution,

$$\boxed{F(y) = \int_{-\infty}^{y} f(t)dt = \frac{1}{\beta} \int_{0}^{y} e^{-t/\beta} dt = 1 - e^{-y/\beta}.}$$

Then $$P(Y > 6) = 1 - F(6) = e^{-3/2} = 0.2231$$

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The probability that a major repair is necessary in the first year is:

$$P(Y < 1) = F(1) = 1 - e^{-1/4} = 1 - 0.779 = 0.221.$$