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1) a)

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$$

$\det(A - \lambda I) = 0 \Rightarrow \det\left(\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0 \Rightarrow \det\left(\begin{pmatrix} 1-\lambda & 3 \\ 0 & 2-\lambda \end{pmatrix}\right) = 0$

$(1-\lambda)(2-\lambda) = 0$

$\lambda_1 = 1 \quad \lambda_2 = 2$

For $\lambda_1 = 1$:

$(A - \lambda_1 I)v = 0$

$\left(\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} - 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)v = 0$

$\begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$3x_2 = 0 \quad x_2 = 0$

$x_2 = 0 \quad x_1 = x_1$

$(\lambda_1, v) = (1, \begin{pmatrix} 1 \\ 0 \end{pmatrix})$

For $\lambda_2 = 2$:

$(A - \lambda_2 I)v = 0$

$\left(\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)v = 0$

$\begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$-x_1 + 3x_2 = 0 \quad x_1 = 3 \quad x_2 = 1$

$v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for $\|\cdot\|_\infty = 1$ $v = \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix}$

$(\lambda_2, v) = (2, \begin{pmatrix} 1 \\ 1/3 \end{pmatrix})$

$$B = \begin{pmatrix} 0 & 8 & 0 \\ 3 & -2 & -3 \\ 0 & 0 & 4 \end{pmatrix}$$

$\det(B - \lambda I) = 0$

$\det\left(\begin{pmatrix} -\lambda & 8 & 0 \\ 3 & -2-\lambda & -3 \\ 0 & 0 & 4-\lambda \end{pmatrix}\right) = 0$

$-\lambda((-2-\lambda)(4-\lambda) - (-3)(0)) - 8((3)(4-\lambda) - (-3)(0)) - 0 = 0$

$-\lambda^3 + 2\lambda^2 + 32\lambda - 96 = 0$

$\lambda_1 = 4, \lambda_2 = -6$

For $\lambda = 4$

$(B - \lambda_1 I)v = 0$

$\left(\begin{array}{ccc|c} -4 & 8 & 0 & 0 \\ 3 & -6 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{R_1}{-4}} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 3 & -6 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 = -3R_1 + R_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$x_1 - 2x_2 = 0$

$-3x_3 = 0$

$x_1 = 2$

$x_2 = 1$

$x_3 = 0$

$v = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$(\lambda_1, v) = (4, \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix})$

$(B-\lambda_2 I)=0$

$$\left(\begin{array}{ccc|c} 6 & 8 & 0 & 0 \\ 3 & 4 & -3 & 0 \\ 0 & 0 & 10 & 0 \end{array}\right) \xrightarrow{\frac{R_1}{6}} \left(\begin{array}{ccc|c} 1 & 4/3 & 0 & 0 \\ 3 & 4 & -3 & 0 \\ 0 & 0 & -10 & 0 \end{array}\right) \xrightarrow{R_2=-3R_1+R_2} \left(\begin{array}{ccc|c} 1 & 4/3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 10 & 0 \end{array}\right)$$

$x_1 + \frac{4}{3}x_2 = 0$ $\quad x_1 = -\frac{4}{3}$ $\quad (\lambda_2, v) = \left(-6, \begin{pmatrix} 1 \\ -3/4 \\ 0 \end{pmatrix}\right)$

$-3x_3 = 0$ $\quad x_2 = 1$

$10x_3 = 0$ $\quad x_3 = 0$

$C = \begin{pmatrix} -3 & 0 \\ -1 & 0 \end{pmatrix}$

$\det(C - \lambda I) = 0$

$\det\begin{pmatrix} -3-\lambda & 0 \\ -1 & -\lambda \end{pmatrix} = 0$

$(-3-\lambda)(-\lambda) = 0$

$\lambda_1 = 0, \lambda_2 = -3$

For $\lambda_1 = 0$:

$(C - \lambda_1 I)v = 0$

$$\left(\begin{array}{cc|c} -3 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right) \xrightarrow{\frac{R_1}{-3}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right) \xrightarrow{R_2=R_2+R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 = 0$ $\quad v = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$

$x_2 = \mathbb{R}$

$(\lambda_1, v) = \left(0, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

For $\lambda = -3$

$(C - \lambda_1 I) = 0$

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 3 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{R_1}{-1}} \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 - 3x_2 = 0$ $\quad x_1 = 3$

$x_2 = 1$

$(\lambda_2, v) = \left(-3, \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix}\right)$

b) $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\det(D - \lambda I) = 0$

$\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$

$(-\lambda)(-\lambda) - (-1)(1) = 0$

$\lambda^2 + 1 = 0$

$E = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 3 \end{pmatrix}$

$\det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 1 & 2-\lambda & 3 \\ 0 & 1 & 3-\lambda \end{pmatrix} = 0$

$(1-\lambda)((2-\lambda)(3-\lambda) - (3)(1)) - 0 - ((1)(1) - (2-\lambda)(0)) = 0$

$(1-\lambda)(6 - 5\lambda + \lambda^2 - 3) - 1 = 0$

$-\lambda^3 + 6\lambda^2 - 8\lambda + 4 = 0$