Assignment 4

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COMP 5107

In this assignment we have two classes X_1 and X_2 with means same as the previous assignments M_1 and M_2 where:

$$M_1 = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -3 & 1 & -4 \end{bmatrix}$$

and covariance matrices as follows:

$$\Sigma_{X_1} = \begin{bmatrix} a^2 & \beta ab & \alpha ac \\ \beta ab & b^2 & \beta bc \\ \alpha ac & \beta bc & c^2 \end{bmatrix}, \quad \Sigma_{X_2} = \begin{bmatrix} c^2 & \alpha bc & \beta ac \\ \alpha bc & b^2 & \alpha ab \\ \beta ac & \alpha ab & a^2 \end{bmatrix}$$

The parameters used in this assignment is as follows:

$$a = 2, \quad b = 3, \quad c = 4, \quad \alpha = 0.1, \quad \beta = 0.2, \quad \#points = 200$$

This resulted the covariance matrices to have the following values:

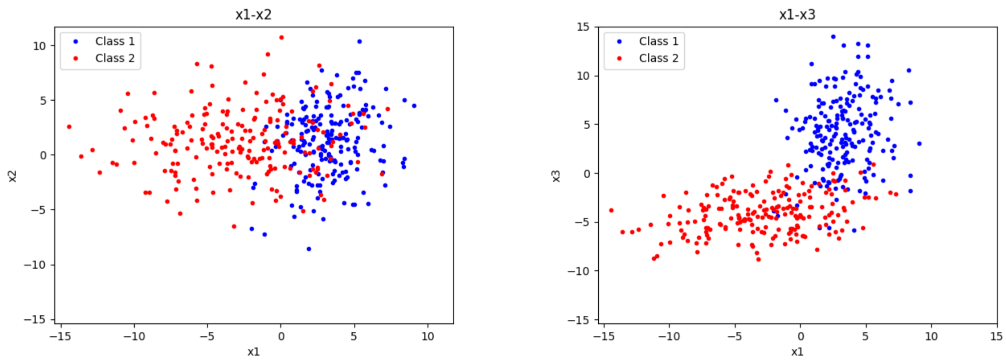
$$\Sigma_{X_1} = \begin{bmatrix} 4 & 1.2 & 0.8 \\ 1.2 & 9 & 2.4 \\ 0.8 & 2.4 & 16 \end{bmatrix}, \quad \Sigma_{X_2} = \begin{bmatrix} 16 & 1.2 & 1.6 \\ 1.2 & 9 & 0.6 \\ 1.6 & 0.6 & 4 \end{bmatrix}$$

a. Create points for each distribution:

Here we used the same exact methods for creating the points from the previous assignment. And the plots of the points are available below.

Listing 1: points generation

```
# create point matrices for the two classes X1 and X2
z1_training_points, x1_training_points = h.generate_point_matrix(v_x1, lambda_x1, m1, number_of_points)
z2_training_points, x2_training_points = h.generate_point_matrix(v_x2, lambda_x2, m2, number_of_points)
```



(a) x1-x2 (b) x1-x3

Figure 1: Training points

b. Estimate the ML and BL:

To estimate the parameters for both data sets using ML:

$$\hat{M} = \frac{1}{N}\sum_{i=1}^{N} x_i, \qquad \hat{\Sigma} = frac1N \sum_{i=1}^{N}[X_i - \hat{M}][X_i - \hat{M}]^T$$

Listing 2: ML and BL mean and covariance

```
def estimate_mean_ml(points, n):
    points = np.array(points)
    points = points[:, :n]
    mean = np.sum(points, axis=1)
    mean = mean / n
    mean = np.array(mean)[np.newaxis]
    return mean.transpose()

def estimate_cov_ml(points, mean, n):
    points = np.array(points)
    mean = np.array(mean)
    cov = (points - mean) @ (points - mean).transpose()
    cov = cov / n
    return cov

def estimate_mean_bl(points, mean0, cov_initial, cov_actual, n):
    points = np.array(points)
    points = points[:, :n]
```

```
    mean0 = np.array(mean0)
    cov_initial = np.array(cov_initial)
    cov_actual = np.array(cov_actual)
    points_sum = np.sum(points, axis=1) / n
    points_sum = np.array(points_sum)[np.newaxis]
    points_sum = points_sum.transpose()

    m = cov_actual / n @ np.linalg.inv(
        cov_actual / n + cov_initial) @ mean0 + cov_initial @ np.linalg.inv(
        cov_actual / n + cov_initial) @ points_sum
    return m
```

And these were the results of ML:

$$\hat{M}_1 = \begin{bmatrix} 3.22 \\ 1.23 \\ 4.16 \end{bmatrix}, \quad \hat{M}_2 = \begin{bmatrix} -2.85 \\ 1.23 \\ -4.09 \end{bmatrix}$$

$$\hat{\sum_{X_1}} = \begin{bmatrix} 4.30 & 1.22 & 1.29 \\ 1.22 & 9.81 & 2.35 \\ 1.29 & 2.35 & 14.49 \end{bmatrix}, \quad \hat{\sum_{X_2}} = \begin{bmatrix} 17.75 & 0.97 & 2.57 \\ 0.97 & 8.48 & 1.21 \\ 2.57 & 1.21 & 4.13 \end{bmatrix}$$

Then we used BL to estimate the mean is using the following equation:

$$(m)_n = \frac{1}{n}\Sigma\left[\frac{1}{n}\Sigma + \Sigma_0\right]^{-1} m_0 + \Sigma_0 \left[\frac{1}{n}\Sigma + \Sigma_0\right]^{-1} \left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)$$

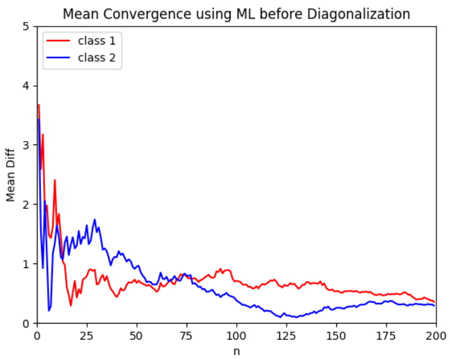
The resulting means are as below:

$$\hat{M}_1 = \begin{bmatrix} 3.22 \\ 1.24 \\ 4.13 \end{bmatrix}, \quad \hat{M}_2 = \begin{bmatrix} -2.85 \\ 1.23 \\ -4.10 \end{bmatrix}$$

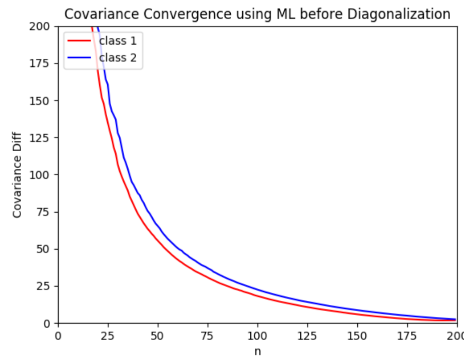
c. Use Parzen window approach:

In this part we are going to use Parzen window, gaussian kernels, to estimate the means and density functions of each dimension. The kernel function used is:

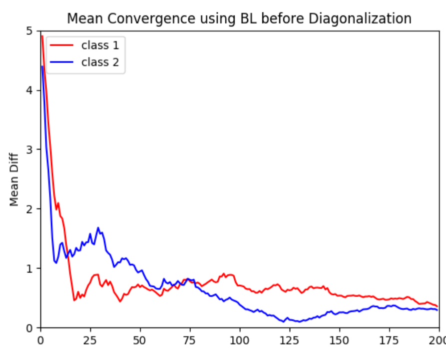
$$\hat{f}_i(x) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-x_i)^2}{2\sigma^2}}$$



(a) ML Mean



(b) ML Covariance



(c) BL Mean

Figure 2: Mean and covariance Convergances

The estimated mean:

$$\hat{m} = \sum_x x\hat{f}(x)\Delta x$$

The estimated covariance:

$$\hat{\sigma}^2 = \sum_x (x - \hat{m})^2 \hat{f}(x)\Delta x$$

Listing 3: Parzen window Code

```
def kernel_function(x, xi, cov):
    result = (1 / (math.sqrt(2 * math.pi) * cov)) * math.exp(-math↩
        .pow(x - xi, 2) / (2 * math.pow(cov, 2)))
    return result

def parzen_expected_mean(x, f_x, delta_x):
    return x * f_x * delta_x

def parzen_expected_covariance(x, f_x, delta_x, mean):
    return math.pow(x - mean, 2) * f_x * delta_x
```

And these were the results of Parzen:

$$\hat{M}_1 = \begin{bmatrix} 3.19 \\ 1.21 \\ 4.12 \end{bmatrix}, \quad \hat{M}_2 = \begin{bmatrix} -2.87 \\ 1.20 \\ -4.09 \end{bmatrix}$$

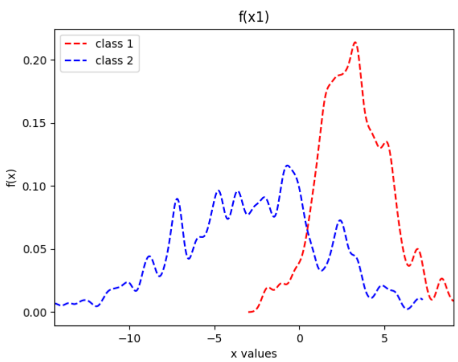
$$\hat{\sum_{X_1}} = \begin{bmatrix} 4.28 & 0 & 0 \\ 0 & 9.69 & 0 \\ 0 & 0 & 14.32 \end{bmatrix}, \quad \hat{\sum_{X_2}} = \begin{bmatrix} 17.51 & 0 & 0 \\ 0 & 8.33 & 0 \\ 0 & 0 & 4.09 \end{bmatrix}$$

d. Compute Bayes discriminant function for ML, BL and Parzen:

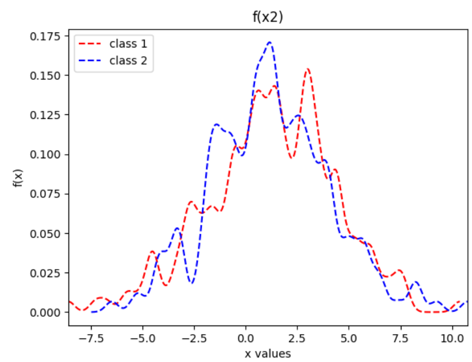
Here I used the same code from the previous assignment to calculate the discriminant function for the three methods. And the results are in the figure below.

e. Use 10-Cross validation to test the classifiers:

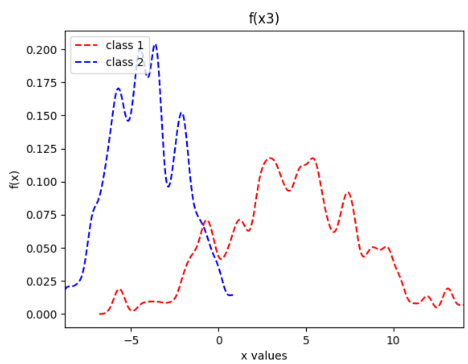
We created 200 more points for each class so we have 400 points total for each class.



(a) x_1 estimated $\hat{f}(x)$

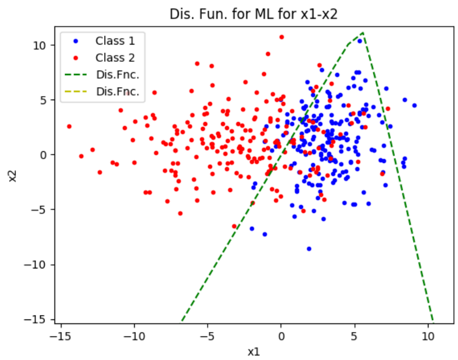


(b) x_2 estimated $\hat{f}(x)$

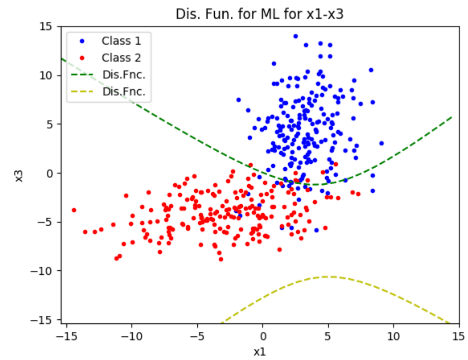


(c) x_3 estimated $\hat{f}(x)$

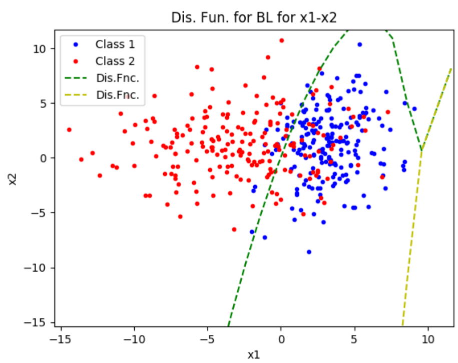
Figure 3: Density Functions Estimation using Parzen Window



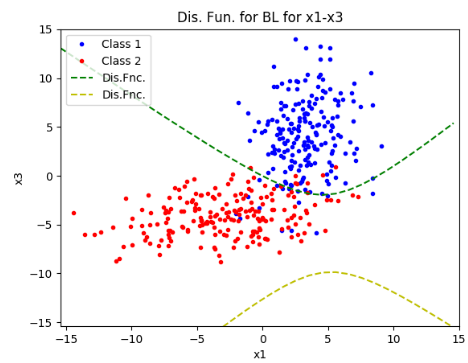
(a) x_1 estimated $\hat{f}(x)$



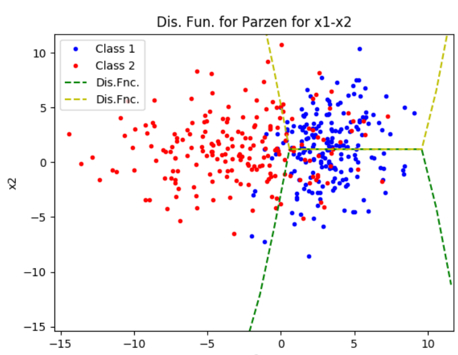
(b) x_2 estimated $\hat{f}(x)$



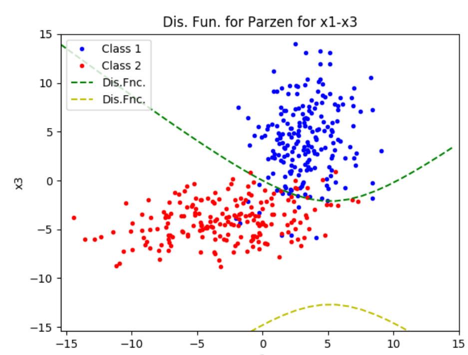
(c) x_3 estimated $\hat{f}(x)$



(d) x_1 estimated $\hat{f}(x)$



(e) x_2 estimated $\hat{f}(x)$



(f) x_3 estimated $\hat{f}(x)$

Figure 4: Density Functions Estimation using Parzen Window

Listing 4: K-cross validation

```
test_results_ml_class1 = []
test_results_ml_class2 = []

test_results_bl_class1 = []
test_results_bl_class2 = []

test_results_parzen_class1 = []
test_results_parzen_class2 = []

k = 10

class1_total_points = v1_training_points
class1_total_points = np.append(class1_total_points, v1_test_points, axis=1)

class2_total_points = v2_training_points
class2_total_points = np.append(class2_total_points, v2_test_points, axis=1)

print(class1_total_points[:, 399])
n = number_of_points + test_points_count
for i in range(0, k, 1):
    print('Cross:' + str(i+1))
    number_of_testing_points = int(n / k)
    number_of_training_points = int(n - n / k)
    start = int(n * i / k)
    end = int((i + 1) * n / k - 1)

    class1_test_points = class1_total_points[:, start: end]
    class1_train_points = class1_total_points[:, 0:start]
    class1_train_points = np.append(class1_train_points, class1_total_points[:, end:], axis=1)

    class2_test_points = class2_total_points[:, start: end]
    class2_train_points = class2_total_points[:, 0:start]
    class2_train_points = np.append(class2_train_points, class2_total_points[:, end:], axis=1)

    # estimated mean using ML
    x1_ml_estimated_mean = h.estimate_mean_ml(class1_train_points, number_of_training_points)
    x1_ml_estimated_cov = h.estimate_cov_ml(class1_train_points, x1_ml_estimated_mean, number_of_training_points)

    x2_ml_estimated_mean = h.estimate_mean_ml(class2_train_points,
```

```
        number_of_points)
    x2_ml_estimated_cov = h.estimate_cov_ml(class2_train_points, x2_ml_estimated_mean, number_of_training_points)

    # Estimating the means using BL
    x1_bl_estimated_mean, x2_bl_estimated_mean = h.bl_expected_mean(class1_train_points, class2_train_points, sigma_v1, sigma_v2, v1_mean, v2_mean, number_of_training_points)

    # estimated mean and cov using parzen window
    x1_parzen_estimated_mean, x1_parzen_estimated_covariance, x2_parzen_estimated_mean, x2_parzen_estimated_covariance = h.estimated_mean_parzen(class1_train_points, class2_train_points, kernel_covariance, step_size)

    ml_class1_accuracy, ml_class2_accuracy = h.test_classifier(class1_test_points, class2_test_points, x1_ml_estimated_cov, x2_ml_estimated_cov, x1_ml_estimated_mean, x2_ml_estimated_mean, number_of_testing_points)
    test_results_ml_class1 = np.append(test_results_ml_class1, ml_class1_accuracy)
    test_results_ml_class2 = np.append(test_results_ml_class2, ml_class2_accuracy)

    bl_class1_accuracy, bl_class2_accuracy = h.test_classifier(class1_test_points, class2_test_points, sigma_v1, sigma_v2, x1_bl_estimated_mean, x2_bl_estimated_mean, number_of_testing_points)
    test_results_bl_class1 = np.append(test_results_bl_class1, bl_class1_accuracy)
    test_results_bl_class2 = np.append(test_results_bl_class2, bl_class2_accuracy)

    parzen_class1_accuracy, parzen_class2_accuracy = h.test_classifier(class1_test_points, class2_test_points, x1_parzen_estimated_covariance, x2_parzen_estimated_covariance, x1_parzen_estimated_mean, x2_parzen_estimated_mean, number_of_testing_points)
    test_results_parzen_class1 = np.append(test_results_parzen_class1, parzen_class1_accuracy)
    test_results_parzen_class2 = np.append(test_results_parzen_class2, parzen_class2_accuracy)
```

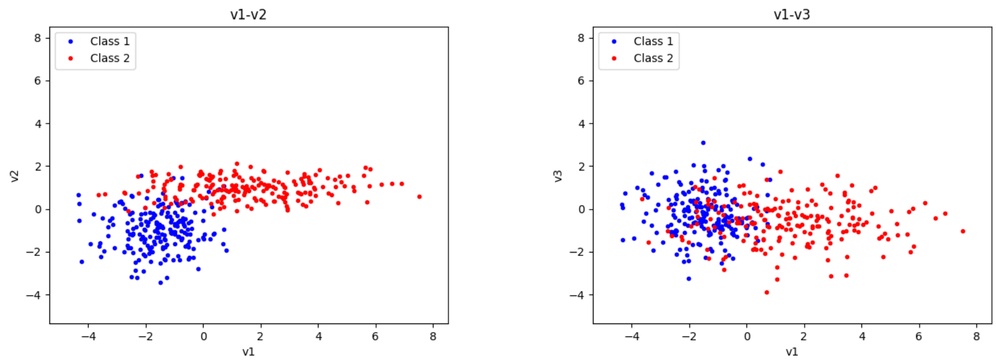
And the resulting testing accuracy is as follows:

Listing 5: Accuracy before Diagonalization

```
ML Accuracy before Diagonalization:
+---------+----------+
|         | Accuracy |
+---------+----------+
| class 1 |  88.75   |
| class 2 |  93.75   |
+---------+----------+

BL Accuracy before Diagonalization:
+---------+----------+
|         | Accuracy |
+---------+----------+
| class 1 |   89.5   |
| class 2 |  94.25   |
+---------+----------+

Parzen Accuracy before Diagonalization:
+---------+----------+
|         | Accuracy |
+---------+----------+
| class 1 |  91.75   |
| class 2 |   90.5   |
+---------+----------+
```



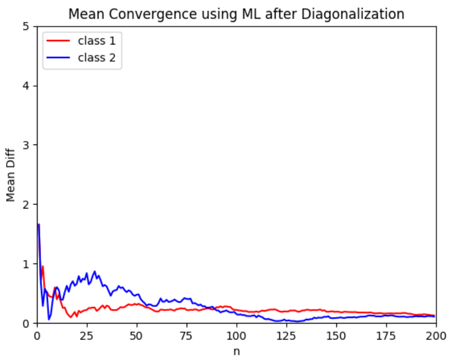
(a) v1-v2

(b) v1-v3

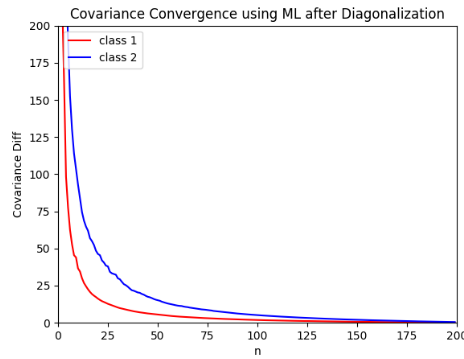
Figure 5: Training points after diagonalization

f. Diagonalize the points and redo everything:

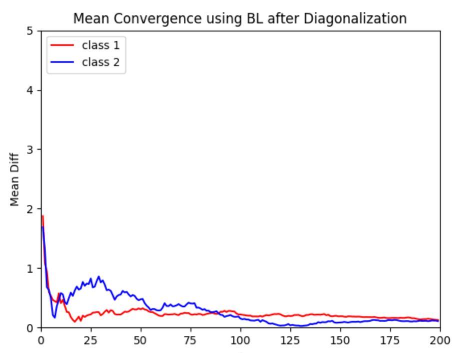
After diagonalizing the data, I will only include the results obtained using the same methods.



(a) ML Mean

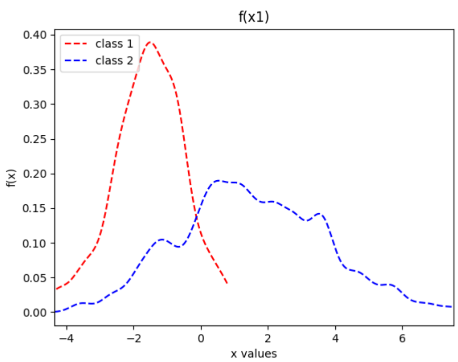


(b) ML Covariance

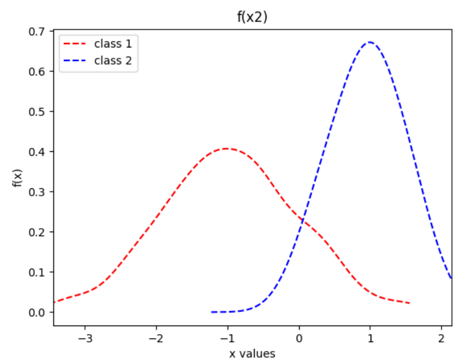


(c) BL Mean

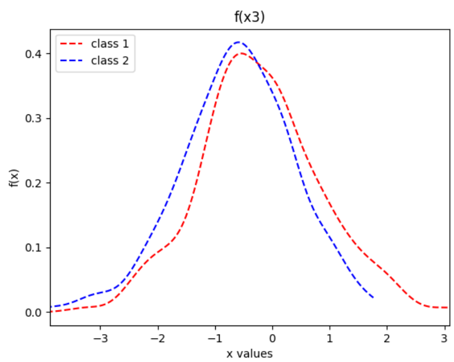
Figure 6: Mean and covariance Convergances after diagonalization



(a) x_1 estimated $\hat{f}(x)$

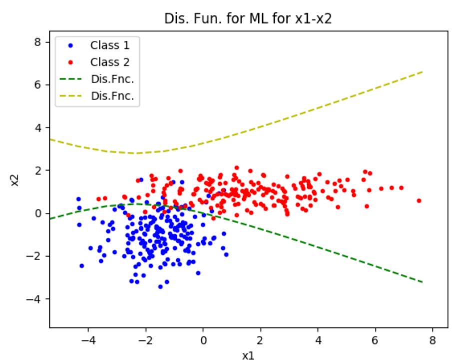


(b) x_2 estimated $\hat{f}(x)$

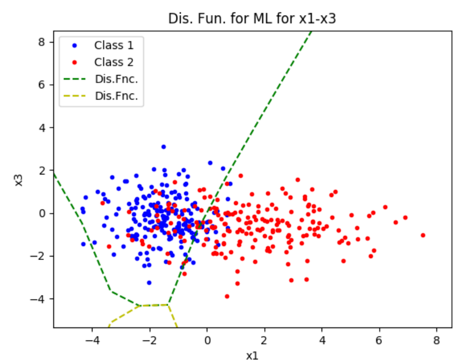


(c) x_3 estimated $\hat{f}(x)$

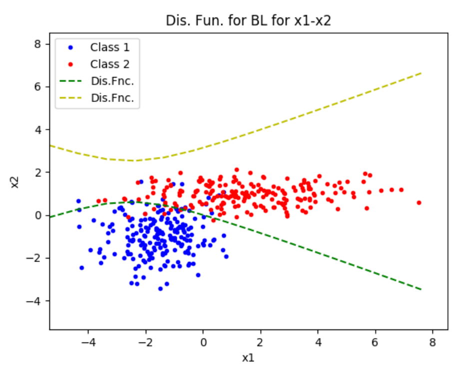
Figure 7: Density Functions Estimation using Parzen Window after diagonalization



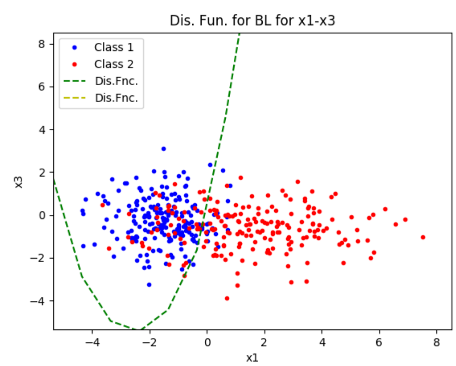
(a) x_1 estimated $\hat{f}(x)$



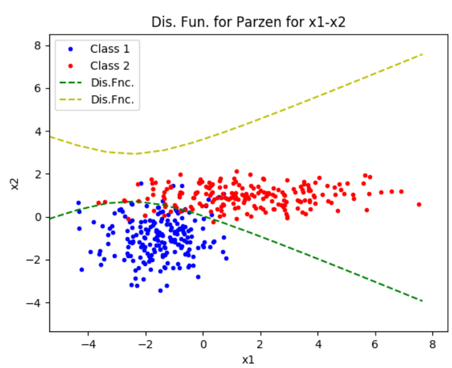
(b) x_2 estimated $\hat{f}(x)$



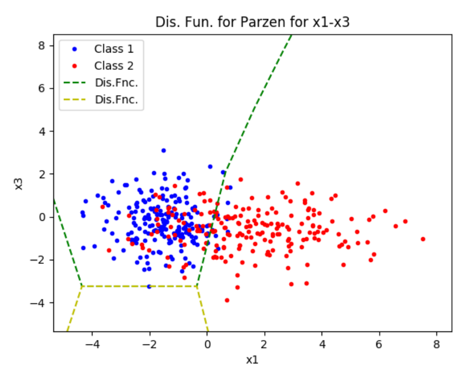
(c) x_3 estimated $\hat{f}(x)$



(d) x_1 estimated $\hat{f}(x)$



(e) x_2 estimated $\hat{f}(x)$



(f) x_3 estimated $\hat{f}(x)$

Figure 8: Discriminant Functions after diagonalization

And the resulting testing accuracy is as follows:

Listing 6: Accuracy after Diagonalization

```
ML Accuracy After Diagonalization:
+---------+----------+
|         | Accuracy |
+---------+----------+
| class 1 |  88.75   |
| class 2 |  93.75   |
+---------+----------+

BL Accuracy after Diagonalization:
+---------+----------+
|         | Accuracy |
+---------+----------+
| class 1 |   89.5   |
| class 2 |   94.5   |
+---------+----------+

Parzen Accuracy after Diagonalization:
+---------+----------+
|         | Accuracy |
+---------+----------+
| class 1 |   91.5   |
| class 2 |  92.25   |
+---------+----------+
```