18CS54

Automata, Computability, & Complexity

Why Study the Theory of Computation?

Implementations come and go.

Chapter 1

Why study this?

Science of Computing

- Mathematical Properties (problems & algorithms) having nothing to do with current technology or languages
- Provides Abstract Structures
- Defines Provable Limits

Goals

Study Principles of Problems themselves

- Does a solution exist?
  - If not, is there a restricted variation?
- Can solution be implemented in fixed memory?
- Is Solution efficient?
  - Growth of time & memory with problem size?
- Are there equivalent groups of problems?

Applications of the Theory

- Programming languages, compilers, & context-free grammars.
- FSMs (finite state machines) for parity checkers, vending machines, communication protocols, & building security devices.
- Interactive games as nondeterministic FSMs.
- Natural languages are mostly context-free. Speech understanding systems use probabilistic FSMs.
- Computational Biology: DNA & proteins are strings.
- The undecidability of a simple security model.
- Artificial Intelligence: the undecidability of first-order logic.

Languages and Strings

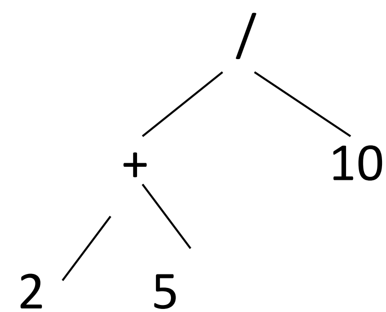
This is one of MOST important chapters.
It includes the TERMINOLOGY required to be successful in this course.

KNOW this chapter & ALL DEFINITIONS!!

Let's Look at Some Problems

```
int alpha, beta;
alpha = 3;
beta = (2 + 5) / 10;
```

(1) **Lexical analysis**: Scan the program; break it into variable names, numbers, etc.
(2) **Parsing**: Create a tree that corresponds to the sequence of operations to be executed, e.g.,



(3) **Optimization**: Recognize, can skip first assignment since value is never used; can precompute the arithmetic expression, since it contains only constants.
(4) **Termination**: Determine if program is guaranteed to halt.
(5) **Interpretation**: Figure out what (if anything) useful program does.

A Framework for Analyzing Problems

We need a single framework in which we can analyze a very diverse set of problems.

The framework is

Language Recognition

*A ***language*** is a (possibly *infinite*) set of *finite* length strings over a *finite* alphabet.

NOTE: Pay particular attention to use of *finite* & *infinite* in all definitions!

Alphabet - Σ

- An alphabet is a non-empty, finite set of characters/symbols
  - Use Σ to denote an alphabet

- Examples

  $\Sigma = \{ a, b \}$

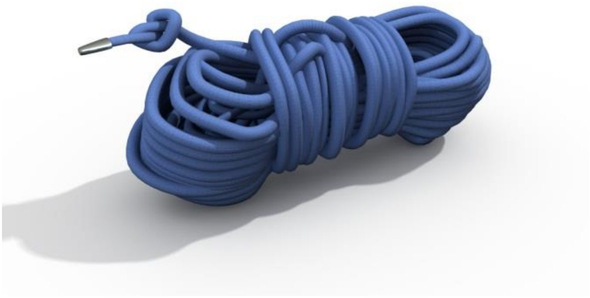
  $\Sigma = \{ 0, 1, 2 \}$

  $\Sigma = \{ a, b, c, \ldots z, A, B, \ldots Z \}$

  $\Sigma = \{ \#, \$, *, @, \& \}$

Strings

- A ***string*** is a finite sequence, possibly empty, of characters drawn from some alphabet Σ.
- ε is the empty string
- Σ^* is the set of all possible strings over an alphabet Σ.



Example Alphabets & Strings

Alphabet name	*Alphabet symbols*	*Example strings*
The lower case English alphabet	{a, b, c, …, z}	ε, aabbcg, aaaaa
The binary alphabet	{0, 1}	ε, 0, 001100,11
A star alphabet	{□ , □ , □ , □, □, □}	ε, □□, □□□□□□
A music alphabet	{w, h, q, e, x, r, ●}	ε, q w , w w r

Functions on Strings

Length:

- $|s|$ is the length of string s
- $|s|$ is the number of characters in string s.

$|\varepsilon| = 0$
$|1001101| = 7$

$\#_c(s)$ is defined as the number of times that c occurs in s.

$\#_a(abbaaa) = 4$.

More Functions on Strings

Concatenation: the ***concatenation*** of 2 strings *s* and *t* is the string formed by appending t to s; written as s||t or more commonly, *st*

Example:

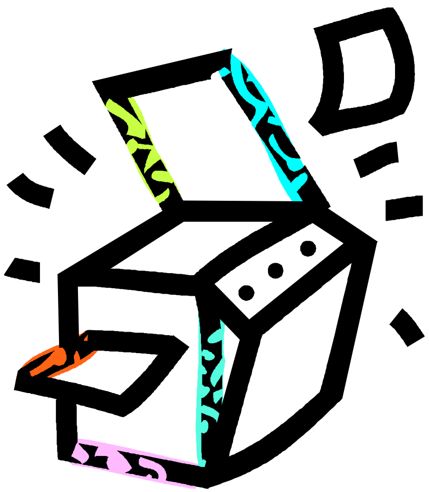
If x = `good` and y = `bye`, then xy = `goodbye` and yx = `byegood`

- Note that $|xy| = |x| + |y|$ -- Is it always??
- ε is the identity for concatenation of strings. So,
  $\forall x\ (x\,\varepsilon = \varepsilon\, x = x)$
- Concatenation is associative. So,
  $\forall s, t, w\ ((st)w = s(tw))$

More Functions on Strings

Replication: For each string w and each natural number k, the string w^k is:

$$w^0 = \varepsilon$$
$$w^{k+1} = w^k w$$



Examples:

$\texttt{a}^3 = \texttt{aaa}$

$(\texttt{bye})^2 = \texttt{byebye}$

$\texttt{a}^0\texttt{b}^3 = \texttt{bbb}$

$\texttt{b}^2\texttt{y}^2\texttt{e}^2 = ??$

Natural Numbers {0,1,2,…}

More Functions on Strings

Reverse: For each string w, w^R is defined as:

if $|w| = 0$ then $w^R = w = \varepsilon$

if $|w| = 1$ then $w^R = w$

if $|w| > 1$ then:

$\exists a \in \Sigma\ (\exists u \in \Sigma^*\ (w = ua))$

So define $w^R = a\ u^R$

OR

if $|w| > 1$ then:

$\exists a \in \Sigma\ \&\ \exists u \in \Sigma^*\ \ni w = ua$

So define $w^R = a\ u^R$

Proof is by simple induction

Concatenation & Reverse of Strings

Theorem: If w and x are strings, then $(w\,x)^R = x^R\,w^R$.

Example:

$(\texttt{nametag})^R = (\texttt{tag})^R\,(\texttt{name})^R = \texttt{gateman}$

In this course, will not use this too much!
Not responsible for inductive proof on next slide.

Concatenation & Reverse of Strings

Proof: By induction on $|x|$:

$|x| = 0$: Then $x = \varepsilon$, and $(wx)^R = (w\,\varepsilon)^R = (w)^R = \varepsilon\, w^R = \varepsilon^R\, w^R = x^R\, w^R$.

$\forall n \geq 0\ (((|x| = n) \rightarrow ((w\,x)^R = x^R\, w^R)) \rightarrow$
$((|x| = n + 1) \rightarrow ((w\,x)^R = x^R\, w^R)))$:

Consider any string x, where $|x| = n + 1$. Then $x = u\,a$ for some character a and $|u| = n$. So:

$(w\,x)^R$	$= (w\,(u\,a))^R$	rewrite x as ua
	$= ((w\,u)\,a)^R$	associativity of concatenation
	$= a\,(w\,u)^R$	definition of reversal
	$= a\,(u^R\,w^R)$	induction hypothesis
	$= (a\,u^R)\,w^R$	associativity of concatenation
	$= (ua)^R\,w^R$	definition of reversal
	$= x^R\,w^R$	rewrite ua as x

Relations on Strings - Substrings

- Substring: string s is a *substring* of string t if s occurs contiguously in t
  - Every string is a substring of itself
  - ε is a substring of every string
- Proper Substring: s is a proper substring of t iff s ≠ t
- Suppose t = aabbcc.
  - Substrings: ε, a, aa, ab, bbcc, b, c, aabbcc
  - Proper substrings?
  - Others?

The Prefix Relations

s is a ***prefix*** of t iff $\exists x \in \Sigma^* \, (t = sx)$.

s is a ***proper prefix*** of t iff s is a prefix of t and $s \neq t$.

Examples:

The ***prefixes*** of `abba` are: ε, `a`, `ab`, `abb`, `abba`.
The ***proper prefixes*** of `abba` are: ε, `a`, `ab`, `abb`.

- Every string is a prefix of itself.
- ε is a prefix of every string.

The Suffix Relations

s is a ***suffix*** of t iff $\exists x \in \Sigma^* \, (t = xs)$.

s is a ***proper suffix*** of t iff s is a suffix of t and $s \neq t$.

Examples:

The ***suffixes*** of `abba` are: ε, `a`, `ba`, `bba`, `abba`.
The ***proper suffixes*** of `abba` are: ε, `a`, `ba`, `bba`.

- Every string is a suffix of itself.
- ε is a suffix of every string.

Defining a Language

A ***language*** is a (finite or infinite) set of strings over a (finite) alphabet Σ.

Examples: Let Σ = {`a`, `b`}

Some languages over Σ:

$\varnothing$ = { } // the empty language, no strings

{ε} // language contains only the empty string

{`a`, `b`}

{ε, `a`, `aa`, `aaa`, `aaaa`, `aaaaa`}

Defining a Language

Two ways to define a language via a Machine = Automaton

AKA – Computer Program

- Recognizer
- Generator

Which do we want? Why?

Σ^*

- Σ^* is defined as the set of all possible strings that can be formed from the alphabet Σ^*
  - Σ^* is a language

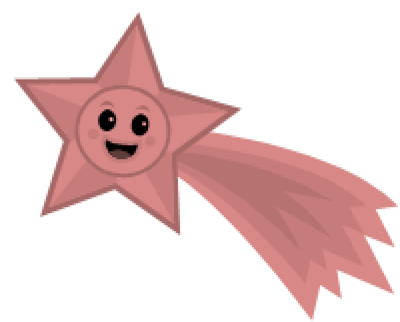
- Σ^* contains an *infinite* number of strings
  - Σ^* is *countably infinite*

Σ^* Example

Let $\Sigma = \{$a, b$\}$

$\Sigma^* = \{\varepsilon$, a, b, aa, ab, ba, bb, aaa, aab, … $\}$

Later, we will spend some more time studying Σ^*.



Defining Languages

Remember we are defining a set

Set Notation:

$$L = \{ w \in \Sigma^* \mid \text{description of } w\}$$

$$L = \{ w \in \{a,b,c\}^* \mid \text{description of } w\}$$

- "description of w" can take many forms but must be precise
- Notation can vary, but must precisely define

Example Language Definitions

$L = \{x \in \{\texttt{a}, \texttt{b}\}^* \mid \text{all } \texttt{a}\text{'s precede all } \texttt{b}\text{'s}\}$

- `aab`, `aaabb`, and `aabbb` are in L.
- `aba`, `ba`, and `abc` are not in L.
- What about ε, `a`, `aa`, and `bb`?

$L = \{x : \exists y \in \{\texttt{a}, \texttt{b}\}^* \mid x = y\texttt{a}\}$

- Give an English description.

Example Language Definitions

Let $\Sigma = \{a, b\}$

- $L = \{ w \in \Sigma^* : |w| < 5\}$
- $L = \{ w \in \Sigma^*$ | w begins with b}
- $L = \{ w \in \Sigma^* \mid \#_b(w) = 2\}$
- $L = \{ w \in \Sigma^*$ | each a is followed by exactly 2 b's}
- $L = \{ w \in \Sigma^*$ | w does not begin with a}

The Perils of Using English

$L = \{x\#y\text{: } x, y \in \{$`0, 1, 2, 3, 4, 5, 6, 7, 8, 9`$\}^*$ and, when x & y are viewed as decimal representations of natural numbers, $square(x) = y\}$.

Examples:

`3#9`, `12#144`

`3#8`, `12`, `12#12#12`

`#`

A Halting Problem Language

L = {w | w is a C++ program that halts on all inputs}

- Well specified.
- Can we decide what strings it contains?
- Do we want a generator or recognizer?

More Examples

What strings are in the following languages?

$L = \{w \in \{\mathtt{a}, \mathtt{b}\}^*$: no prefix of w contains $\mathtt{b}\}$

$L = \{w \in \{\mathtt{a}, \mathtt{b}\}^*$: no prefix of w starts with $\mathtt{a}\}$

$L = \{w \in \{\mathtt{a}, \mathtt{b}\}^*$: every prefix of w starts with $\mathtt{a}\}$

$L = \{\mathtt{a}^n : n \geq 0\}$

$L = \{\mathtt{b}\mathtt{a}^{2n} : n \geq 0\}$

$L = \{\mathtt{b}^n\mathtt{a}^n : n \geq 0\}$

Enumeration

Enumeration: to list all strings in a language (set)

- Arbitrary order

- More useful: ***lexicographic order***
  - Shortest first
  - Within a length, dictionary order
  - Define linear order of arbitrary symbols

Lexicographic Enumeration

$\{w \in \{\mathtt{a}, \mathtt{b}\}^* : |w| \text{ is even}\}$

$\{\varepsilon$, aa, ab, bb, aaaa, aaab, …}

What string is next?

How many strings of length 4?

How many strings of length 6?

Cardinality of a Language

- Cardinality of a Language: the number of strings in the language
- | L |
- Smallest language over any Σ is ∅, with cardinality 0.
- The largest is Σ*.
  - Is this true?
  - How big is it?
- Can a language be uncountable?

Countably Infinite

Theorem: If $\Sigma \neq \varnothing$ then Σ^* is countably infinite.

Proof: The elements of Σ^* can be lexicographically enumerated by the following procedure:

- Enumerate all strings of length 0, then length 1, then length 2, and so forth.
- Within the strings of a given length, enumerate them in dictionary order.

This enumeration is infinite since there is no longest string in Σ^*. Since there exists an infinite enumeration of Σ^*, it is countably infinite.

How Many Languages Are There?

Theorem: If $\Sigma \neq \varnothing$ then the set of languages over Σ is uncountably infinite (uncountable).

Proof: The set of languages defined on Σ is $\mathcal{P}(\Sigma^*)$. Σ^* is countably infinite. By Theorem A.4, if S is a countably infinite set, $\mathcal{P}(S)$ is uncountably infinite. So $\mathcal{P}(\Sigma^*)$ is uncountably infinite.

What does this mean?!?!?!

Functions on Languages

Set (Language) functions
Have the traditional meaning

- Union
- Intersection
- Complement
- Difference

Language functions

- Concatenation
- Kleene star

Concatenation of Languages

If L_1 and L_2 are languages over Σ:

$$L_1L_2 = \{w : \exists s \in L_1 \ \& \ \exists t \in L_2 \ni w = st\}$$

Examples:

$L_1 = \{$`cat`, `dog`$\}$

$L_2 = \{$`apple`, `pear`$\}$

$L_1 L_2 = \{$`catapple`, `catpear`, `dogapple`, `dogpear`$\}$

$L_2 L_1 = \{$`applecat,appledog,pearcat,peardog`$\}$

Concatenation of Languages

$\{\varepsilon\}$ is the identity for concatenation:

$$L\{\varepsilon\} = \{\varepsilon\}L = L$$

$\varnothing$ is a zero for concatenation:

$$L\,\varnothing = \varnothing\,L = \varnothing$$

Concatenating Languages Defined Using Variables

The scope of any variable used in an expression that invokes replication will be taken to be the entire expression.

$L_1 = \{\mathtt{a}^n: \; n \geq 0\}$
$L_2 = \{\mathtt{b}^n : n \geq 0\}$

$L_1 \, L_2 = \{\mathtt{a}^n\mathtt{b}^m : n, \, m \geq 0\}$

$L_1 L_2 \neq \{\mathtt{a}^n\mathtt{b}^n : n \geq 0\}$

Kleene Star *

L - language consisting of 0 or more concatenations of strings from L*

$L^* = \{\varepsilon\} \cup \{w \in \Sigma^* : w = w_1 w_2 \ldots w_k,\ k \geq 1\ \&$
$w_1, w_2, \ldots w_k \in L\}$

Examples:

L = {`dog`, `cat`, `fish`}

L^* = {ε, `dog, cat, fish, dogdog, dogcat, dogfish, fishcatfish, fishdogdogfishcat,` …}

L_1 = `a`* L_2 = `b`*

What is a? b*?*

$L_1 L_2 =$

$L_2 L_1 =$

$L_1 L_1 =$

The + Operator

L^+ = *language consisting of 1 or more concatenations of strings from L*

$L^+ = L\, L^*$

$L^+ = L^* - \{\varepsilon\}$ iff $\varepsilon \notin L$

Explain this definition!!

When is $\varepsilon \in L^+$?

Closure

- A set S is closed under the operation @ if for every element x & y in S, x@y is also an element of S
- A set S is closed under the operation @ if for every element $x \in S$ & $y \in S$, $x@y \in S$
- Examples

Semantics: Assigning Meaning to Strings

When is the meaning of a string important?

A semantic interpretation function assigns meanings to the strings of a language.

Can be very complex.

Example from English:

I brogelled the yourtish.
He’s all thumbs.

Uniqueness???

- Chocolate, please.
- I’d like chocolate.
- I’ll have chocolate today.
- I guess I’ll have chocolate.

They all have the same meaning!

Uniqueness???

Hand

- Give me a hand.
- I smashed my hand in the door.
- Please hand me that book.

These all have different meanings!!!

Uniqueness in CS???

- int x = 4; x++;
- int x = 4; ++x;
- int x = 4; x = x + 1;
- int x = 4; x=x - -1;
- int x = 5;

These all have the same result/meaning!

Semantic Interpretation Functions

For formal languages:

- Programming languages
- Network protocol languages
- Database query languages
- HTML
- BNF

For other kinds of "natural" languages:

- DNA

Other computing

- Genetic algorithm solutions
- Other problem solutions