## 1: Daycount and frequency

**(a)** Using either a semi-bond or annual-money, the accrued value after one year (365 days) should be the same. Therefore

$$\begin{aligned}(1+y_{SB}\frac{180}{360})^2 &= (1+y_{AM}\frac{365}{360}) \\ \implies y_{AM} &= \frac{360}{365}\left[(1+\frac{y_{SB}}{2})^2-1\right] \qquad (1)\end{aligned}$$

**(b)** We simply need to plug in the different values in the above result:

$$y_{AM}\Big|_{y_{SB}=0.05} \approx 0.04993 \qquad (2)$$

$$y_{AM}\Big|_{y_{SB}=0.06} \approx 0.06007 \qquad (3)$$

$$y_{AM}\Big|_{y_{SB}=0.07} \approx 0.07024 \qquad (4)$$

We see that when $y_{SB} \in [0.05, 0.07]$ then $y_{AM} \in [0.04993, 0.07024]$ so we conclude that $y_{AM}$ has higher standard deviation.

**(c)**From part **(a)** we have that

$$\begin{aligned}\frac{\mathrm{d}y_{AM}}{y_{AM}} &= \frac{360}{365y_{AM}}\left[\left(1+\frac{y_{SB}}{2}\right)\mathrm{d}y_{SB}\right] \\ &= \frac{\left(1+\frac{y_{SB}}{2}\right)\mathrm{d}y_{SB}}{\left(1+\frac{y_{SB}}{2}\right)^2-1} \\ &= \frac{\left(1+\frac{y_{SB}}{2}\right)}{\left(1+\frac{y_{SB}}{4}\right)}\frac{\mathrm{d}y_{SB}}{y_{SB}} \qquad (5) \\ & \qquad (6)\end{aligned}$$

Then, at $y_{SB} = 0.06$, the ratio of the volatilities is $1.03/1.025 \approx 1.0049$.

## 2: Simple interest

Let $r, r_A, r_Q$ and $r_C$ be the simple, annually compouded, quarterly compounded, and continuous interest rates respectively. After $T$ years, they should yield the same accrued values and therefore

$$1+rT = (1+r_A)^T = (1+\frac{r_Q}{4})^{4T} = e^{r_C T}. \qquad (7)$$

So that,

$$r_A = (1+rT)^{1/T}-1 \qquad (8)$$

$$r_Q = 4\left[(1+rT)^{1/4T}-1\right] \qquad (9)$$

$$r_C = \frac{1}{T}\log(1+rT) \qquad (10)$$

Finally, by plugging in $r = 0.05$ and $T = 10$, we get $r_A \approx 0.0413$, $r_Q \approx 0.0407$ and $r_C = 0.0405$.
**(b)** We take the limit as $T \to \infty$ of equation (8):

$$\begin{aligned} r_A \to \lim_{T\to\infty} (1 + rT)^{1/T} - 1 &= \lim_{T\to\infty} e^{\frac{1}{T}\log(1+rT)} - 1 \\ &= e^{\lim_{T\to\infty} \frac{1}{T}\log(1+rT)} - 1 \\ &= e^{\lim_{T\to\infty} \frac{r}{1+rT}} - 1 \\ &= e^0 - 1 \\ &= 0 \end{aligned} \tag{11}$$

## 3: Non-standard annuity

**(a)** Let today be denoted by $t = 0$ and the maturity date by $T$. The phrasing of the problem is a bit vague, but let's assume that the payments are made each year. Then, the value today is

$$\begin{aligned} V &= \sum_{n=1}^{T} \frac{n}{(1+r)^n} \\ \implies \frac{V_T}{(1+r)} &= \sum_{n=1}^{T} \frac{n}{(1+r)^{n+1}} \\ &= \sum_{n=1}^{T} \frac{n+1}{(1+r)^{n+1}} - \sum_{n=1}^{T} \frac{1}{(1+r)^{n+1}} \\ &= \sum_{n=2}^{T+1} \frac{k}{(1+r)^k} - \frac{1}{r(1+r)}\left(1 - \frac{1}{(1+r)^T}\right) \\ &= -\frac{1}{1+r} + \frac{T+1}{(1+r)^{T+1}} + \sum_{n=1}^{T} \frac{k}{(1+r)^k} - \frac{1}{r(1+r)}\left(1 - \frac{1}{(1+r)^T}\right) \\ &= -\frac{1}{1+r} + \frac{T+1}{(1+r)^{T+1}} + V - \frac{1}{r(1+r)}\left(1 - \frac{1}{(1+r)^T}\right) \\ \implies V &= \frac{1}{r}\left(1 - \frac{T+1}{(1+r)^T}\right) + \frac{1}{r^2}\left(1 - \frac{1}{(1+r)^T}\right) \end{aligned} \tag{12}$$

**(b)** Taking the limit of the above as $T \to \infty$ we get that $V \to (r+1)/r^2$.