1: The normal distribution and expectation

(a) By definition, the indicator function is equal to 1 if the argument is true, and it is equal to 0 otherwise. So by definition of expectation we have

$$\begin{aligned}\mathbb{E}(I\{X > K\}) &= \int_{-\infty}^{+\infty} I\{x > K\} p(x|\mu, \sigma)\mathrm{d}x \\ &= \int_{K}^{+\infty} p(x|\mu, \sigma)\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} p(x|\mu, \sigma)\mathrm{d}x - \int_{-\infty}^{K} p(x|\mu, \sigma)\mathrm{d}x \\ &= 1 - \Phi(K|\mu, \sigma) \end{aligned} \tag{1}$$

Where, to get to the last line we use the fact that the normal distribution integrates to 1 over the real line, together with the definition of the cumulative distribution function.

(b) (Hint not used). First we write $\max\{K - X, 0\} = I\{K > X\}(K - X)$. Then,

$$\begin{aligned}\mathbb{E}(\max\{K - X, 0\}) &= I\{K > X\}(K - X) \\ &= \int_{-\infty}^{+\infty} I\{K > x\}(K - x) p(x|\mu, \psi)\mathrm{d}x \\ &= K \int_{-\infty}^{K} p(x|\mu, \psi)\mathrm{d}x - \int_{-\infty}^{K} x p(x|\mu, \psi)\mathrm{d}x \\ &= K\Phi(K|\mu, \psi) - \int_{-\infty}^{K} x p(x|\mu, \psi)\mathrm{d}x \end{aligned} \tag{2}$$

The second term can be simplified using integration by parts:

$$\begin{aligned}\int_{-\infty}^{K} x p(x|\mu, \psi) &= x\Phi(x|\mu, \psi)\Big|_{-\infty}^{K} - \int_{-\infty}^{K} \Phi(x|\mu, \psi)\mathrm{d}x \\ &= K\Phi(K|\mu, \psi) - \int_{-\infty}^{K} \Phi(x|\mu, \psi)\mathrm{d}x. \end{aligned} \tag{3}$$

Finally,

$$\mathbb{E}(\max\{K - X, 0\}) = \int_{-\infty}^{K} \Phi(x|\mu, \psi)\mathrm{d}x \tag{4}$$

(c) This simply requires an algebraic manipulation of the terms in the integral:

$$\begin{aligned}
\mathbb{E}(e^{tX}) &= \int_{-\infty}^{+\infty} e^{tx} p(x|\mu,\psi)\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} e^{tx} \frac{1}{\sqrt{2\pi\psi^2}} e^{-\frac{(x-\mu)^2}{2\psi^2}} \mathrm{d}x \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\psi^2}} e^{-\frac{(x-\mu-t\psi^2)^2}{2\psi^2}} e^{\mu t + \frac{t^2\psi^2}{2}} \mathrm{d}x \\
&= e^{\mu t + \frac{t^2\psi^2}{2}} \int_{-\infty}^{+\infty} p(x|\mu + t\psi^2, \psi)\mathrm{d}x \qquad (5)
\end{aligned}$$

And, since the normal distribution integrates to 1 over the real line, we have

$$\mathbb{E}(e^{tX}) = e^{\mu t + \frac{t\psi^2}{2}} \qquad (6)$$

2: The log-normal distribution

(a) If X is normally distributed with mean μ and variance σ^2, then $Y = e^X$ is log-normally distributed. Then,

$$\mathbb{E}_{p_Y}(Y^t) = \mathbb{E}_{p_X}(e^{tX}) = e^{\mu t + \frac{t^2\sigma^2}{2}} \qquad (7)$$

(I've included the subscript to emphasise the distribution, but I'll normally leave it as implicit). Then,

$$\begin{aligned}
\mathbb{E}(Y) &= e^{\mu + \frac{\sigma^2}{2}} \qquad (8) \\
\mathbb{E}(Y^2) &= e^{2\mu + 2\sigma^2} \qquad (9) \\
\mathbb{V}(Y) &= E(Y^2) - E(Y)^2 \\
&= e^{2\mu + \frac{\sigma^2}{2}} - e^{2\mu+\sigma^2} \\
&= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) \qquad (10)
\end{aligned}$$

(b) Let $\mu = \log f - \frac{1}{2}\sigma^2$. By plugging this into the previous result we immediately get:

$$\begin{aligned}
\mathbb{E}(Y) &= f \qquad (11) \\
\mathbb{V}(Y) &= f^2(e^{\sigma^2} - 1) = f^2(1 + \sigma^2 + \frac{1}{2}\sigma^4 + ... - 1) \quad \text{(Taylor expand the exponential)} \\
&= f^2\sigma^2(1 + \frac{1}{2}\sigma^2 + ...) \qquad (12)
\end{aligned}$$

3: Conditional expectation

(a) The variable X_n takes only two possible values once X_{n-1} is given. Therefore,

$$\begin{aligned}\mathbb{E}(X_n|X_{n-1}) &= X_{n-1}(1+u)p + X_{n-1}(1+d)(1-p) \\ &= X_{n-1}((1+u)p + (1+d)(1-p)) \end{aligned} \tag{13}$$

(b) First note that $Y_n = \frac{Y_{n-1}}{1+r}$. So in general we have

$$\mathbb{E}(Y_n|Y_{n-1}) = \frac{Y_{n-1}}{1+r}\left((1+u)p + (1+d)(1-p)\right). \tag{14}$$

By requiring $\mathbb{E}(Y_n|Y_{n-1}) = Y_{n-1}$ we get

$$r = up + d(1-p). \tag{15}$$

And since $p \in [0,1]$ we obtain that $r \in [d,u]$.

(c) By *iterated expectation* we know that

$$\mathbb{E}(\mathbb{E}(Y_n|Y_{n-1})|Y_{n-2}) = \mathbb{E}(Y_n|Y_{n-2}). \tag{16}$$

On the other hand,

$$\mathbb{E}(\mathbb{E}(Y_n|Y_{n-1})|Y_{n-2}) = \mathbb{E}(Y_{n-1}|Y_{n-2}) = Y_{n-2}. \tag{17}$$

It follows that $\mathbb{E}(Y_n|Y_{n-2}) = Y_{n-2}$. The result for general n and m can be obtained by simply using more iterations (or by formal induction).

4: The normal cumulative distribution function

By definition

$$\Phi(x) = \int_{-\infty}^{x} \mathcal{N}(y|0,1)\mathrm{d}y \tag{18}$$

By symmetry of the normal distribution around zero, we have that $\Phi(0) = 1/2$. And the first three derivatives are

$$\Phi'(x) = \mathcal{N}(x|0,1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{19}$$

$$\Phi''(x) = -\frac{x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{20}$$

$$\Phi'''(x) = \frac{x^2}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} - \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{21}$$

Plugging the above into the Taylor expansion we get

$$\begin{aligned}\Phi(x) &= \Phi(0) + x\Phi'(0) + \frac{x^2}{2!}\Phi''(0) + \frac{x^3}{3!}\Phi'''(0) + ... \\ &= \frac{1}{2} + \frac{x}{\sqrt{2\pi}} - \frac{x^3}{6\sqrt{2\pi}} + ... \end{aligned} \tag{22}$$