Problem Set on Credit Risk

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1. Default risk models using Merton (1974), Vassalou and Xing (2004) and Bharath and Shumway (2008).

The total value of a firm follows a geometric Brownian motion:

$$dV_t = \mu V_t dt + \sigma_V V_t dW_t$$

Where V is the total value of the firm, μ is the expected continuously compounded return on V, σ_V is the volatility of the firm value and W is a Brownian motion.

The Equity is given by the following formula:

$$E_t\left(V, F, T\right) = V_t \mathcal{N}\left(d_1\right) - e^{-\mu(T-t)} F_t \mathcal{N}\left(d_2\right) \tag{0.1}$$

The formula above expresses the value of a firm's equity as a function of the value of the firm.
Where

$$d_1 = \frac{ln\left(\frac{V_t}{F_t}\right) + \left(\mu + \frac{\sigma_V^2}{2}\right)(T-t)}{\sigma_V \sqrt{T-t}}$$

And

$$d_2 = d_1 - \sigma_V \sqrt{T-t}$$

Using Ito lemma we derive the following formula:

$$\sigma_E = \left(\frac{V}{E}\right) \mathcal{N}\left(d_1\right) \sigma_V \tag{0.2}$$

In order to calculate the probability of default it is necessary of determine the values of V and σ_V, we then need to calculate the distance-to-default (DD) given by:

$$DD = \frac{ln\left(\frac{V_t}{F_t}\right) + \left(\mu - \frac{\sigma_V^2}{2}\right)(T-t)}{\sigma_V \sqrt{T-t}}$$

Finally we can estimate the theoretical probability of default using the formula below:

$$\mathbb{P}\left(t, T\right) = \mathcal{N}\left(-DD\right)$$

Question a:

Following the guidelines of Vassalou and Xing (2004) and Bharath and Shumway (2008), collect data from an individual firm of your choice for at least 1 year (more years is better). You can use Bloomberg, Datastream, etc. Draw some graphs showing the evolution of your variables, e.g. market value of equity, book value of equity, book-to-market ratio, debt value, risk-free rate, etc. Provide also a table reporting summary statistics for all the variables used in your sample see, for example, Bharath and Shumway (2008, Table 1).

We selected the American firm IBM (International Business Machines Corporation) and collected among others the following data using Bloomberg:

- "^TNX.csv" contains the risk free rates.

- “IBM_stdebt.xlsx”: contains the short term debt values.
- “IBM_ltdebt.xslx”: contains the long term debt values.
- “IBM_mktcap.xlsx” contains the market values of equity.
- “IBM_moneyness.xlsx” contains the values of the implied volatilities.

Due to Bloomberg limitations we cannot collect more than one year of data, we also couldn’t have access to the values book value of equity. Merging the data frames allows us to construct the data set below.

	Date	Equity	Debt	Rate	implied_vol
0	2021-01-04	110437.6190	36619.5	0.00917	0.274845
1	2021-01-05	112397.9446	36619.5	0.00955	0.268459
2	2021-01-06	115204.7745	36619.5	0.01042	0.252178
3	2021-01-07	114937.4574	36619.5	0.01071	0.223382
4	2021-01-08	114527.5711	36619.5	0.01105	0.217142
...	...	...	...	...	...
247	2021-12-27	118036.8621	33919.0	0.01481	0.185847
248	2021-12-28	118942.6304	33919.0	0.01481	0.181878
249	2021-12-29	119588.3267	33919.0	0.01543	0.181130
250	2021-12-30	120090.5349	33919.0	0.01515	0.181097
251	2021-12-31	119866.3348	31450.5	0.01512	0.178102

252 rows × 5 columns

Figure 0.1: The collected data.

The variable debt is obtained using the following formula:

$$D^* := STD + \frac{LTD}{2}$$

Where STD corresponds to quarterly short term debt and LTD to quarterly long term liabilities.

We obtained the following figures by plotting the variables time evolution:

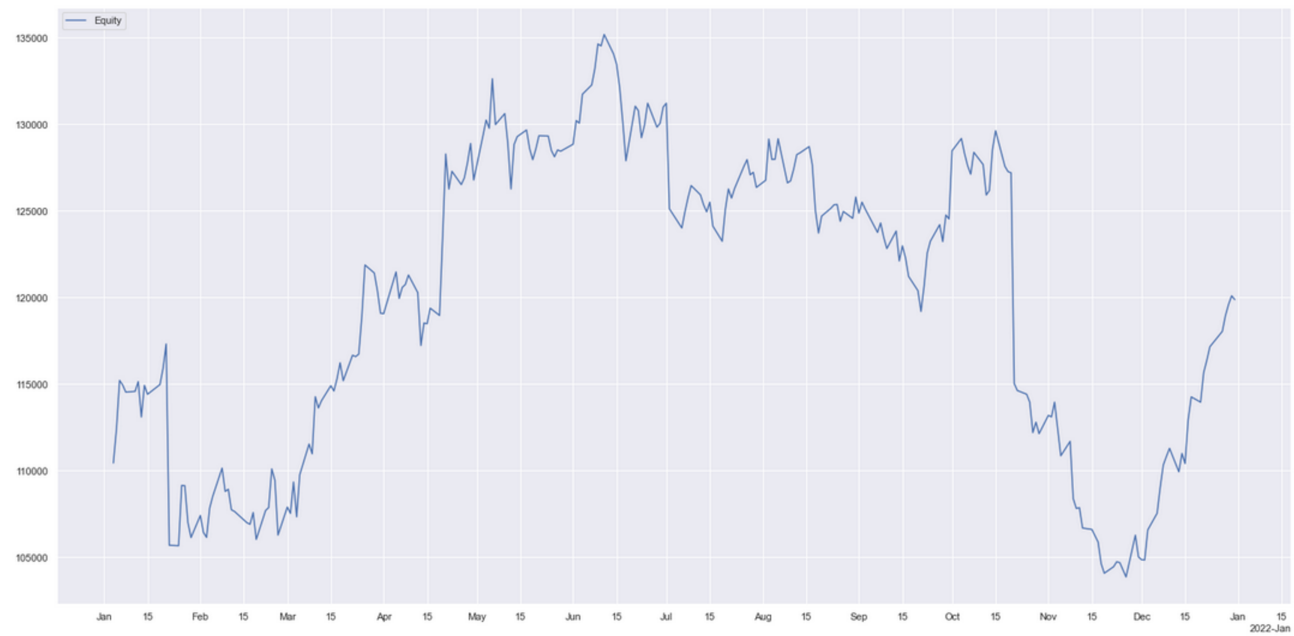

Figure 0.2: Equity values.

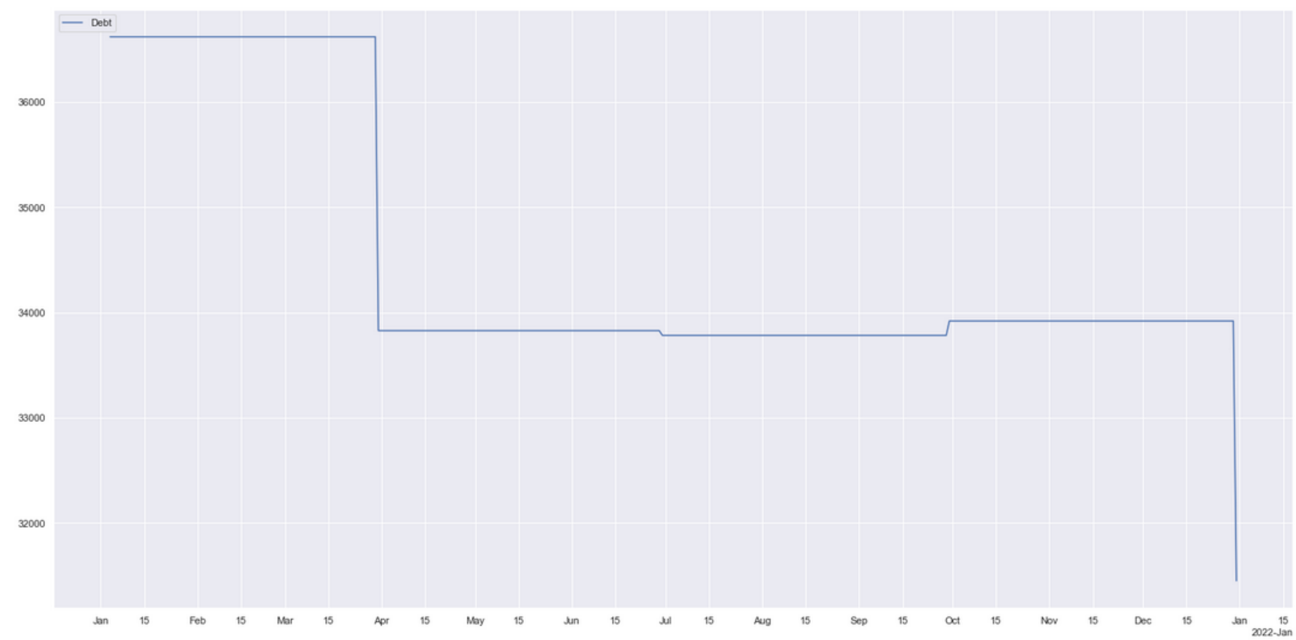

Figure 0.3: Debt values.

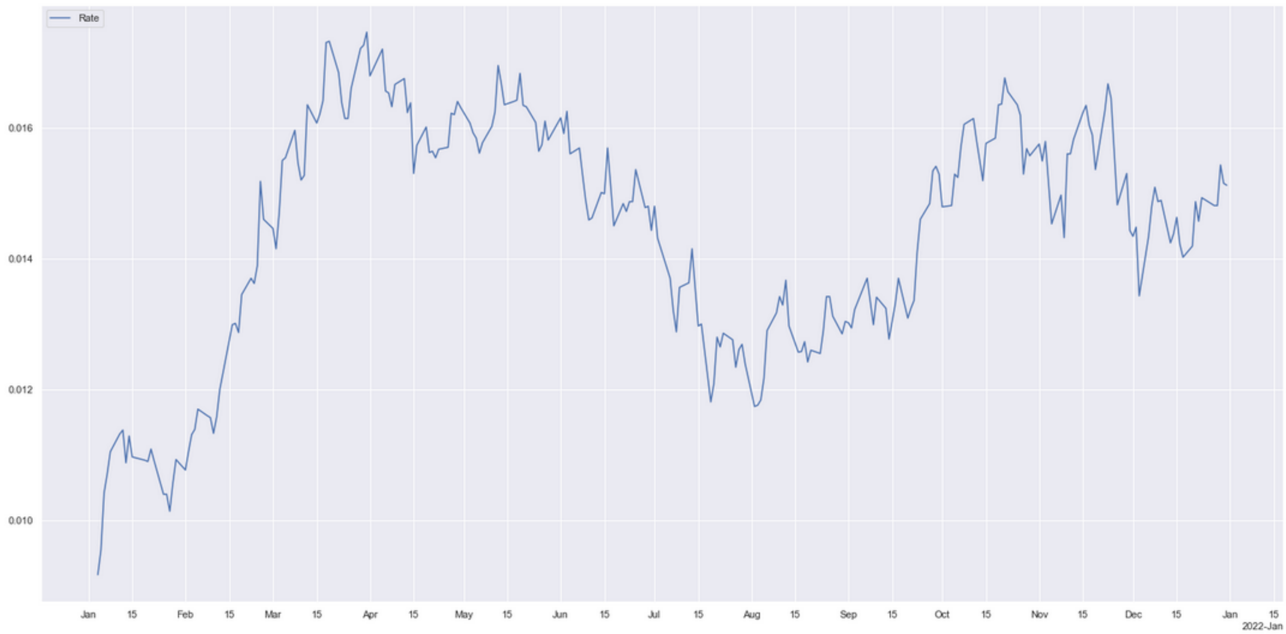

Figure 0.4: Risk free rates values.

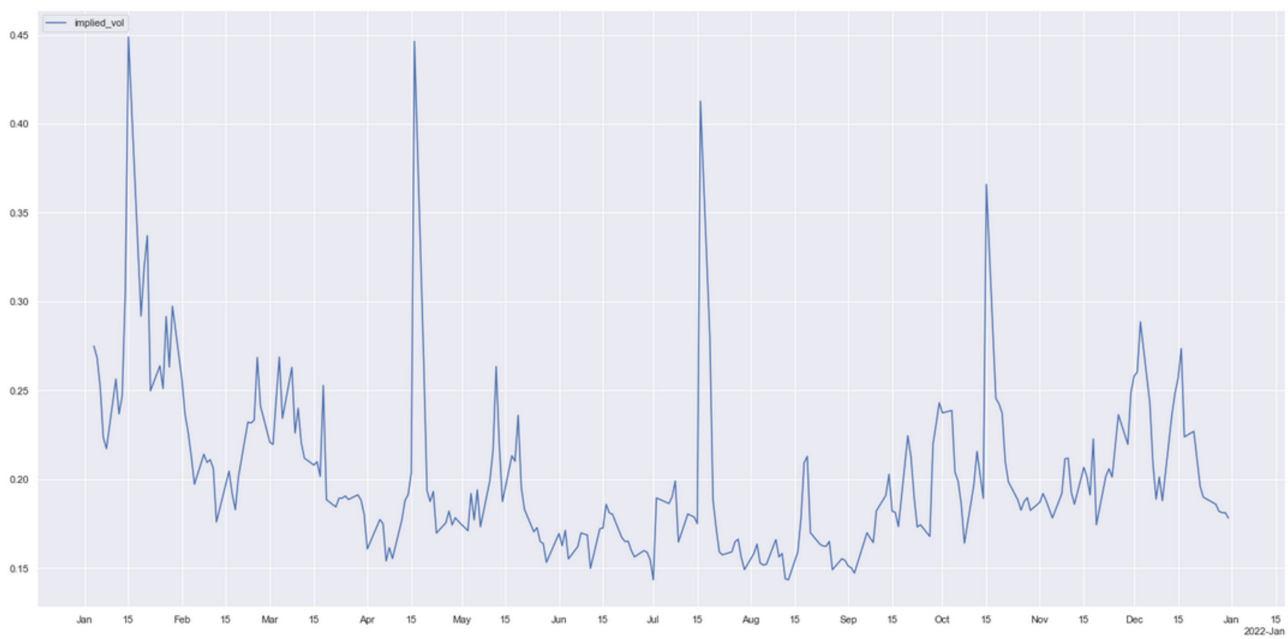

Figure 0.5: Implied volatility values.

The summary statistics are grouped in the following table:

	count	mean	std	min	25%	50%	75%	max
Equity	251.0	119907.027726	8642.998122	103858.448500	112295.164050	121868.362000	127566.263500	135172.13340
Debt	251.0	34496.932271	1202.563911	31450.500000	33782.500000	33827.500000	33919.000000	36619.50000
Rate	251.0	0.014401	0.001822	0.009170	0.013030	0.014810	0.015830	0.01746
implied_vol	251.0	0.201911	0.045878	0.143321	0.171632	0.189577	0.219926	0.44889

Figure 0.6: Summary statistics.

Question b:

Compute the distance to default and the probability of default at the end of each month using Bharath and Shumway (2008, Equations 6 and 7) except that the expected return on assets μ is replaced by the risk-free rate r and with the market value of assets and asset volatility being computed via the non-linear system of equations approach. To estimate equity volatility use 1 year of historical returns data.

The asset value V_t and volatility σ_V are the solution to the following non linear system of equations:

$$\begin{cases} E_t & = V_t \mathcal{N}(d_1) - e^{-r\tau} F_t \mathcal{N}(d_2) \\ \sigma_E & = \frac{V_t}{E_t} \mathcal{N}(d_1) \sigma_V \end{cases}$$

The initial guesses are set to:

$$\begin{cases} V_0 & = E + e^{-rT} F \\ \sigma_{V_0} & = \sigma_E \frac{E}{E + e^{-rT} F} \end{cases}$$

The distances to default and the probabilities of default at the end of each month are as follow:

	Date	Qb_V	Qb_sigma_V	Qb_DD	Qb_EDF
18	2021-01-29	142355.224748	0.305532	4.326884	7.561693e-06
37	2021-02-26	142363.852655	0.315369	4.194060	1.370026e-05
60	2021-03-31	152322.340638	0.324065	4.535132	2.878373e-06
81	2021-04-30	160053.267243	0.245522	6.273986	1.759604e-10
101	2021-05-28	161731.864028	0.227075	6.846604	3.781194e-12
123	2021-06-30	164280.030247	0.222710	7.055179	8.618951e-13
144	2021-07-30	159711.794660	0.198896	7.773098	3.829485e-15
166	2021-08-31	159134.394971	0.198830	7.760799	4.219784e-15
187	2021-09-30	157930.070921	0.198313	7.734260	5.200334e-15
208	2021-10-29	145524.612420	0.188528	7.713274	6.131503e-15
229	2021-11-30	138448.384295	0.183931	7.633482	1.142486e-14
251	2021-12-31	150844.880208	0.192612	8.122082	2.291269e-16

Figure 0.7: Question b summary.

The figure above shows a relatively high stable values of σ_V at the end of each month. The probabilities of default are very low which is the expected result.

Question c:

Compute the distance to default and the probability of default at the end of each month using Bharath and Shumway (2008, Equations 6 and 7) expect that the expected return on assets μ is replaced by the risk-free rate r and with the market value of assets and asset volatility being computed via the non-linear system of equations approach. The equity volatility to be used when solving the two equations is the option-implied volatility of firm equity at the end of each month.

The distances to default and the probabilities of default at the end of each month are as follow:

	Date	Qc_V	Qc_sigma_V	Qc_DD	Qc_EDF
18	2021-01-29	142355.241389	0.221638	6.064448	6.620397e-10
37	2021-02-26	142363.884381	0.180018	7.533710	2.465923e-14
60	2021-03-31	152322.346555	0.140826	10.738589	3.353067e-27
81	2021-04-30	160053.267243	0.139555	11.184140	2.438062e-29
101	2021-05-28	161731.864028	0.121633	12.932946	1.466799e-38
123	2021-06-30	164280.030247	0.123390	12.873350	3.179163e-38
144	2021-07-30	159711.794660	0.117923	13.219372	3.391392e-40
166	2021-08-31	159134.394971	0.121978	12.751490	1.529210e-37
187	2021-09-30	157930.070921	0.191602	8.011969	5.644309e-16
208	2021-10-29	145524.612420	0.140565	10.401298	1.222899e-25
229	2021-11-30	138448.384295	0.188830	7.430608	5.405000e-14
251	2021-12-31	150844.880208	0.141526	11.114178	5.351562e-29

Figure 0.8: Question c summary.

The probabilities of default are lower than the ones obtained in the previous question.

Question d:

Compute the distance to default and the probability of default at the end of each month using Bharath and Shumway (2008, Equations 6 and 7) expect that the expected return on assets μ is replaced by the risk-free rate r and with the market value of assets and asset volatility being computed via the iterative approach.

Rearranging the Black-Scholes formula we have:

$$V_t = \frac{E_t + e^{-r(T-t)} F_t \mathcal{N}(d_2)}{\mathcal{N}(d_1)}$$

The formula above allows to iteratively determinate the values of V and σ_V.

The distances to default and the probabilities of default at the end of each month are as follow:

	Date	Qd_V_i	Qd_V_i+1	Qd_sigma_V	Qd_DD	Qd_EDF
18	2021-01-29	142355.175123	142355.175123	0.325739	4.038876	2.685400e-05
37	2021-02-26	142363.884381	142363.884381	0.142293	9.573773	5.153679e-22
60	2021-03-31	152322.346555	152322.346555	0.172557	8.735048	1.217766e-18
81	2021-04-30	160053.267243	160053.267243	0.191483	8.106236	2.610593e-16
101	2021-05-28	161731.864028	161731.864028	0.132922	11.823772	1.470861e-32
123	2021-06-30	164280.030247	164280.030247	0.126892	12.514657	3.103769e-36
144	2021-07-30	159711.794660	159711.794660	0.154721	10.042896	4.936610e-24
166	2021-08-31	159134.394971	159134.394971	0.115705	13.449331	1.553341e-41
187	2021-09-30	157930.070921	157930.070921	0.105981	14.604964	1.305597e-48
208	2021-10-29	145524.612139	145524.612139	0.280974	5.098209	1.714407e-07
229	2021-11-30	138448.384295	138448.384295	0.136756	10.322033	2.801194e-25
251	2021-12-31	150844.880208	150844.880208	0.117537	13.409021	2.677034e-41

Figure 0.9: Question d summary.

The probabilities of default values are very low which is the expected result.

Question e:

Compute the distance to default and the probability of default at the end of each month using Bharath and Shumway (2008, Equations 6 and 7) and with the market value of assets and asset volatility being computed via the iterative approach.

To obtain the value of μ the expected continuously compounded return, we need to use the CAPM formula.

$$\mathbb{E}\left[R_i\right] = R_f + \beta_i\left(\mathbb{E}\left[R_m\right] - R_f\right)$$

- $\mathbb{E}\left[R_i\right]$: capital asset expected return.
- R_f: risk-free rate of interest.
- β_i: sensitivity.
- $\mathbb{E}\left[R_m\right]$: expected return of the market.

Meaning that we need to perform a linear regression between the excess of returns of the S&P index and the excess of returns of the IBM asset values.

The excess returns formula as expressed on the provided excel file is as follow:

$$\frac{P_{t+1}}{P_t} - \left(1 + \frac{r_t}{T}\right)$$

After determining the value of β, we can easily get the value of μ.

The distances to default and the probabilities of default at the end of each month are as follow:

	Date	Qe_V_i	Qe_V_i+1	Qe_sigma_V	Qe_DD	Qe_EDF
18	2021-01-29	142371.113939	142371.113939	0.326130	4.032649	2.757585e-05
37	2021-02-26	142512.514176	142512.514176	0.142198	9.558715	5.961414e-22
60	2021-03-31	152554.852487	152554.852487	0.172462	8.708389	1.541114e-18
81	2021-04-30	160247.522887	160247.522887	0.191562	8.078771	3.271140e-16
101	2021-05-28	161909.475387	161909.475387	0.132258	11.851889	1.051973e-32
123	2021-06-30	164411.485070	164411.485070	0.126886	12.490503	4.206026e-36
144	2021-07-30	159775.251170	159775.251170	0.155080	10.009623	6.913899e-24
166	2021-08-31	159219.532670	159219.532670	0.115733	13.428670	2.053606e-41
187	2021-09-30	158090.796968	158090.796968	0.106125	14.549325	2.949093e-48
208	2021-10-29	145694.690079	145694.690079	0.280482	5.093703	1.755686e-07
229	2021-11-30	138580.370268	138580.370268	0.138060	10.201612	9.750415e-25
251	2021-12-31	150988.643300	150988.643300	0.117641	13.365694	4.796829e-41

Figure 0.10: Question e summary.

The probabilities of default values are very low which is the expected result.

The second approach stated in the class note, suggests calculating μ using the mean of log returns of V, the distances to default and the probabilities of default at the end of each month are as follow:

	Date	Qe_V_i_WithDrift	Qe_V_i+1_WithDrift	Qe_sigma_V_WithDrift	Qe_DD_WithDrift	Qe_EDF_WithDrift
18	2021-01-29	142810.509956	142810.509956	0.325161	4.018041	2.934204e-05
37	2021-02-26	142909.866511	142909.866511	0.141807	9.528172	8.003887e-22
60	2021-03-31	152824.776689	152824.776689	0.172396	8.675468	2.059257e-18
81	2021-04-30	160521.222177	160521.222177	0.191228	8.059587	3.827631e-16
101	2021-05-28	162271.852947	162271.852947	0.131964	11.813898	1.654298e-32
123	2021-06-30	164744.322288	164744.322288	0.126628	12.453988	6.651390e-36
144	2021-07-30	160175.892443	160175.892443	0.154694	9.974152	9.889164e-24
166	2021-08-31	159581.346944	159581.346944	0.115472	13.385660	3.667162e-41
187	2021-09-30	158446.735675	158446.735675	0.105884	14.504339	5.686421e-48
208	2021-10-29	146220.087534	146220.087534	0.279535	5.069227	1.997174e-07
229	2021-11-30	139031.324266	139031.324266	0.137625	10.160960	1.480724e-24
251	2021-12-31	151193.162091	151193.162091	0.117687	13.316315	9.303353e-41

Figure 0.11: Question e summary.

Both approaches give very similar results which indicates the robustness of our solution.

Question f:

Compute the distance to default and the probability of default at the end of each month using the naive approach of Bharath and Shumway (2008).

The naive volatility is given by:

$$\sigma_V = \frac{E_t}{E_t + F_t}\sigma_E + \frac{F_t}{E_t + F_t}\left(0.05 + \frac{\sigma_E}{4}\right)$$

Where σ_E is the annualized stock volatility estimated over the previous year.
The default probability is estimated using the classical formula:

$$\mathbb{P}\left(t, T\right) = \mathcal{N}\left(-DD\right)$$

Where:

$$DD = \frac{ln\left(\frac{E_t + F_t}{F_t}\right) + \left(r_{it-1} - \frac{\sigma_V^2}{2}\right)(T - t)}{\sigma_V\sqrt{T - t}}$$

Where r_{it-1} is the stock return over the previous year (proxy for the expected return on the firm's assets).
The distances to default and the probabilities of default at the end of each month are as follow:

	Date	Qf_naive_sigma	Qf_naive_DD	Qf_naive_EDF
10	2021-01-29	0.343406	3.751112	0.000088
21	2021-02-26	0.343471	3.751733	0.000088
39	2021-03-31	0.352477	4.052951	0.000025
56	2021-04-30	0.355200	4.138340	0.000017
71	2021-05-28	0.355755	4.206783	0.000013
89	2021-06-30	0.356638	4.288540	0.000009
107	2021-07-30	0.355113	4.149323	0.000017
124	2021-08-31	0.354924	4.156934	0.000016
141	2021-09-30	0.354315	4.190276	0.000014
158	2021-10-29	0.349646	4.060705	0.000024
172	2021-11-30	0.346589	3.875674	0.000053
191	2021-12-31	0.355916	4.270475	0.000010

Figure 0.12: Question f summary.

The probabilities of default values are very low which is the expected result.

Question g:

Plot a figure of the default likelihood indicator with the five series of monthly default probabilities - see, for example, Vassalou and Xing (2004, Figure 1).

The obtained results are as follow:

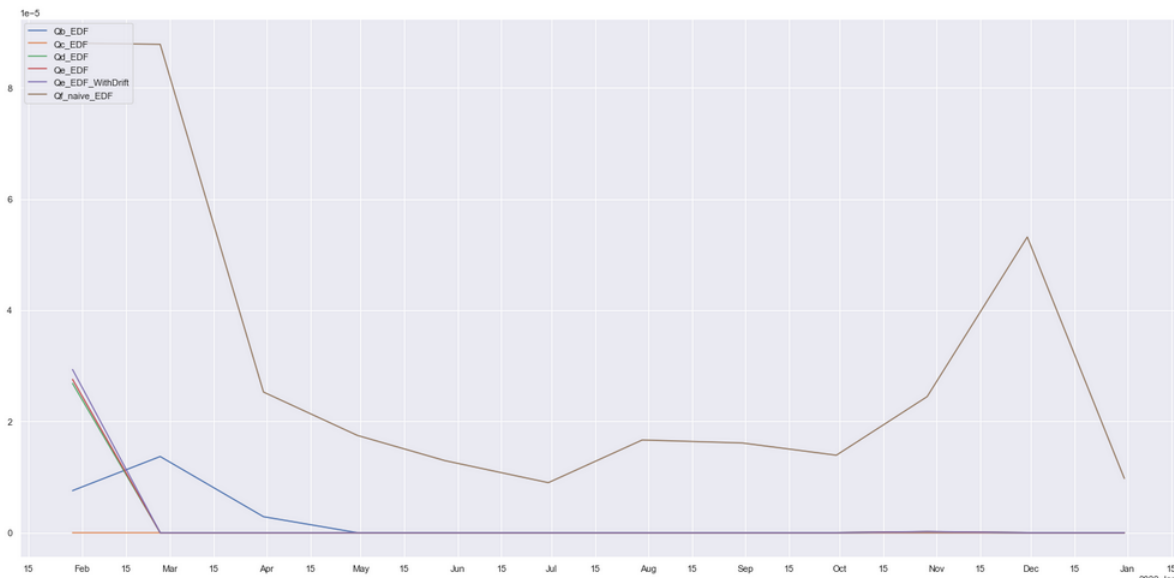

Figure 0.13: EDFs variation.

As we can see from the figure above the naive approach have the highest monthly default probabilities.
The iterative approach default probabilities obtained in questions d and e seems to converge to the same values.
The monthly default probabilities obtained in question c using the implied volatilities are the lowest.

Question h:

What are your main conclusions?

The values of V obtained in questions b, c, d and e are as follow:

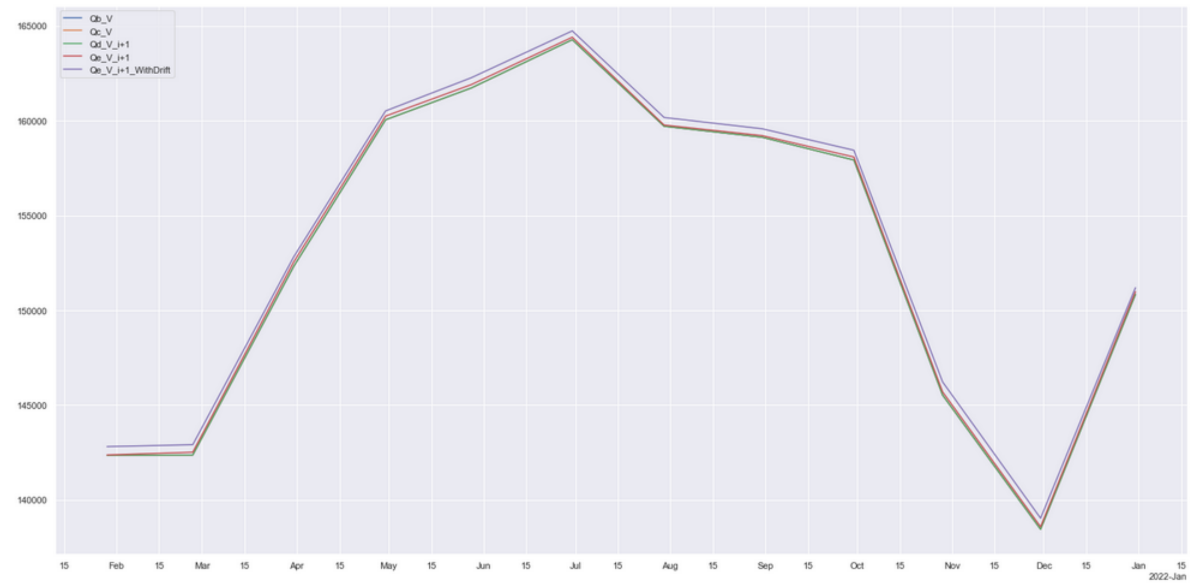

Figure 0.14: V summary.

The values of V seem to converge.

The values of σ_V obtained in the previous questions are as follow:

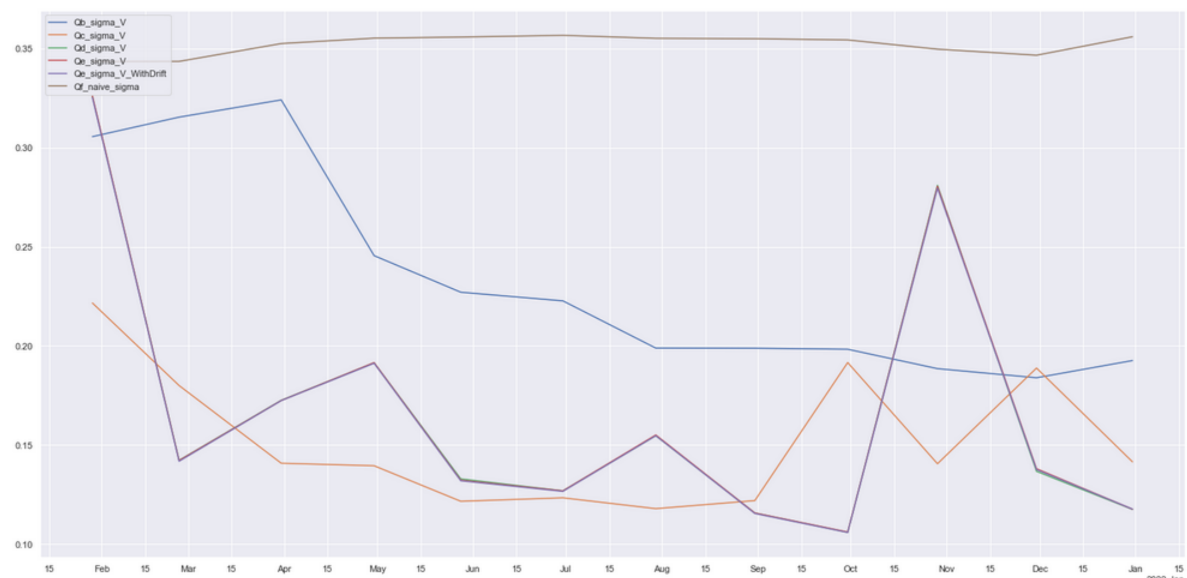

Figure 0.15: σ_V summary.

The values of σ_V obtained using the iterative approach in questions d and e, seem to converge to the same values, which is not the case for the other variables of σ_V, which implies that changing the expected return on assets μ by the risk-free or vice versa did not have a substantial effect in that case, this could be the cases just due to the fact that IBM's expected return on assets using CAPM is already very similar to risk-free rates.

	Date	Qb_V	Qb_sigma_V	Qc_V	Qc_sigma_V	Qd_V_i+1	Qd_sigma_V	Qe_V_i+1	Qe_sigma_V	Qf_naive_sigma
0	2021-01-29	142355.226228	0.304292	142355.241389	0.221638	142355.175123	0.325739	142371.113939	0.326130	0.343406
1	2021-02-26	142363.869705	0.304678	142363.884381	0.180018	142363.884381	0.142293	142512.514176	0.142198	0.343471
2	2021-03-31	152322.342554	0.319070	152322.346555	0.140826	152322.346555	0.172557	152554.852487	0.172462	0.352477
3	2021-04-30	160053.264603	0.323274	160053.267243	0.139555	160053.267243	0.191483	160247.522887	0.191562	0.355200
4	2021-05-28	161731.861607	0.324113	161731.864028	0.121633	161731.864028	0.132922	161909.475387	0.132258	0.355755
5	2021-06-30	164280.028138	0.325412	164280.030247	0.123390	164280.030247	0.126892	164411.485070	0.126886	0.356638
6	2021-07-30	159711.791902	0.322873	159711.794660	0.117923	159711.794660	0.154721	159775.251170	0.155080	0.355113
7	2021-08-31	159134.392143	0.322619	159134.394971	0.121978	159134.394971	0.115705	159219.532670	0.115733	0.354924
8	2021-09-30	157930.067847	0.321813	157930.070921	0.191602	157930.070921	0.105981	158090.796968	0.106125	0.354315
9	2021-10-29	145524.606296	0.314480	145524.612420	0.140565	145524.612139	0.280974	145694.690079	0.280482	0.349646
10	2021-11-30	138448.375018	0.309581	138448.384295	0.188830	138448.384295	0.136756	138580.370268	0.138060	0.346589
11	2021-12-31	150844.878005	0.324321	150844.880208	0.141526	150844.880208	0.117537	150988.643300	0.117641	0.355916

Figure 0.16: Final data frame.

The above figure shows that the obtained results are very similar to each others.

Conclusion: After calculating the probability of default using multiple approaches, in which some of them converge when the approach is consistent, we conclude that generally the probability of default of IBM company is very low and almost impossible, this shall be driven by the company's strong financial position and relatively stable debt to equity. The results seem consistent to what was initially expected, however, there are multiple considerations that could improve the results:

- Using longer periods of historical data, this can improve the accuracy of all derived variables that rely on market data. For example, volatility values can be aggravated when looking at 1 year of historical data due to market volatility and not firm-specific risk. With a large cap as IBM that happens to be a large constituent of many market indices and Exchange
- Traded Funds, the volatility can be strongly distorted.
- When using (CAPM), one could modify and use different time ranges for the Beta estimations
- The implied volatility values used were derived via Bloomberg aggregations of multiple options, other tests could be implemented selecting specific options
- There were several instances where padding was applied, to quarterly values (eg. long-term debt, book-value of equity etc.), other methodologies of padding could be applied. More importantly, there is a forward-looking bias that the analysis was prone to. Most fundamental metrics are typically delayed in reporting (eg. Q1 results appear by Q2), let alone the audited results. This bias is difficult to mitigate but some interpolation with historical data could perform better, or at least be more prudent than considering quarterly values that would have not been publicly available at each point-in-time of each calculation using such values

References

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