OVERVIEW OF THE INTEGER HOKEY POKEY

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The Integer Hokey Pokey is a research paper by Dr. Cornelia Van Cott that begins with a focus on numbers that can "turn themselves around." For example, the number 1089 "turns itself around" when multiplied by 9 to become 9801. Similarly, $2178 \times 4 = 8712$. Here, we are seeing a reflection: the n-th digit and nth-to-last digit switch places throughout the number.

Definition 1. *An integer A is an n-flip if multiplying it by a certain value n causes the order of the digits to reverse.*

From this, it should hopefully be clear that no n-flips exist for $n \geq 10$. The reason for this is because multiplying a number by at least 10 means that it gains digits, and this means that the product will not be a reversal of the initial number. 8-flips do not exist for a similar reason; any number that is an 8-flip must start with the number 1 so that it does not gain digits when multiplied by 8. This means that the last digit of a supposed 8-flip would be 1, which is a contradiction since multiplying any number by 8 should yield an even number. Using this same reasoning, we can rule out n-flips for all positive integer values of n except for 4 and 9. This means that, in base 10, *only* 4 and 9-flips exist!

One way to find new n-flips is to make palindromes from existing n-flips. For example, we currently know the set of 9-flips consists of at least 2 numbers: 0 (trivial) and 1089. To find a new 9-flip, we can concatenate these number to form a palindrome such as $1089\underline{\mathbf{0}}1089$. Multiply this number by 9 and you will indeed find that it reverses itself. What about n-flips that cannot be made from concatenations?

Definition 2. *A basic n-flip is a n-flip that is not a palindrome of other n-flips.*

To try and find the set of all basic 9-flips, note that the first digit of A must be 1 and the second must be 0 in order to preserve the number of integers. Through further investigation, we find that all basic 9-flips must begin with 10, end with 89, and have the rest of the digits in the middle be 9. So, the set is $\{0, 1089, 10989, 109989, 1099989, \dots\}$. The set of basic 4-flips is $\{0, 2178, 21978, 219978, 2199978, \dots\}$.

Something very interesting happens when we try to find the number of n-flips with a certain number of digits. Let a_n be the number of n-flips with n digits. For 9-flips,

$$\begin{array}{cccccc} a_4 = 1 & a_6 = 1 & a_8 = 2 & a_{10} = 3 & a_{12} = 5 & a_{14} = 8 \\ a_5 = 1 & a_7 = 1 & a_9 = 2 & a_{11} = 3 & a_{13} = 5 & a_{15} = 8 \end{array}$$

As we can see, this pattern is producing a Fibonacci sequence repeated twice! The same pattern shows up in 4-flips. To understand why this happens, we need to define some terms to distinguish different types of 9-flips.

Definition 3. *A straddler is a 9-flip that is a palindrome using an odd number of basic 9-flips (eg 1089**109989**1089).*

Definition 4. *A non-straddler is a 9-flip that is a palindrome using an even number of basic 9-flips.*

Inserting a 0 into the middle of a non-straddler with $2k$ digits makes it have $2k + 1$ digits. The same happens when we add 9 to the middle of a straddler containing $2k$ digits. Every 9-flip with an odd number of digits must be a straddler with the middle digit as 0 or another basic 9-flip with an odd number of digits. When we remove the middle 9 or 0 from one of these numbers, it has even digits. This establishes a one-to-one correspondence between 9-flips with $2k$ digits and those with $2k + 1$ digits, and this is why we see the repetition in pairs.

To see why the Fibonacci numbers come up, let S_{2k} be the number of straddler 9-flips with $2k$ digits and N_{2k} be the number of non-straddler 9-flips with $2k$ digits. So, $a_{2k} = N_{2k} + S_{2k}$. Through some manipulation, we find that $a_{2k} = N_{2k} + S_{2k} = N_{2k-2} + S_{2k-2} + N_{2k-4} + S_{2k-2} = a_{2k-2} + a_{2k-4}$. This produces the Fibonacci sequence in the even terms and a similar process produces it in the odd terms. We are thus left with the following theorem:

Theorem 5. *Let a_m denote the number of 9-flips with m digits. Then we have $a_{2k} = a_{2k+1}$ for all $k \geq 1$. Moreover, the sequence a_{2k} satisfies the recursive relationship $a_{2k} = a_{2k-2} + a_{2k-4}$ for all $k \geq 3$, while $a_2 = 0$ and $a_4 = 1$. That is, a_{2k} and a_{2k+1} are both given by the $(k-1)$st Fibonacci number.*

We arrived at this theorem by showing that the numbers a_{2n} satisfy the same recursive relationship as the Fibonacci numbers.

As it turns out, n-flips are also present in number systems with a base other than 10. Furthermore, every base b contains a $(b-1)$-flip. For example, base 3 contains a 2-flip, base 4 contains a 3-flip, and so on. The Fibonacci sequence comes up again in some of these bases, but not all of them. In some bases like base 8 for example, the number of 5-flips follows the powers of 2.

Incidentally, n-flips are not the only numbers with a capacity to dance. Other numbers known as *cyclical numbers* are able to "slide to the left" when multiplied by a given integer. For example, $142857 \times 3 = 428571$ and $142857 \times 2 = 285714$. Cyclic numbers are very hard to find, with the next two after 142,857 having 16 and 18 digits respectively. Cyclic numbers also correspond to what are known as *full period primes*; for example, 142,857 corresponds to 7 because $1/7 = .\overline{142857}$. The next two cyclic numbers correspond to 17 and 19. It is currently unknown whether there are infinitely many full period primes.