COMPSCI/SFWRENG 2FA3
Discrete Mathematics with Applications II
Winter 2021

Assignment 2

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Assignment 2 consists of two problems. You must write your solutions to the problems using LaTeX.

Please submit Assignment 2 as two files, `Assignment_2_`*`YourMacID`*`.tex` and `Assignment_2_`*`YourMacID`*`.pdf`, to the Assignment 2 folder on Avenue under Assessments/Assignments. *`YourMacID`* must be your personal MacID (written without capitalization). The `Assignment_2_`*`YourMacID`*`.tex` file is a copy of the LaTeX source file for this assignment (`Assignment_2.tex` found on Avenue under Contents/Assignments) with your solution entered after each problem. The `Assignment_2_`*`YourMacID`*`.pdf` is the PDF output produced by executing

```
pdflatex Assignment_2_YourMacID
```

This assignment is due **Sunday, February 7, 2020 before midnight.** You are allow to submit the assignment multiple times, but only the last submission will be marked. **Late submissions and files that are not named exactly as specified above will not be accepted!** It is suggested that you submit your preliminary `Assignment_2_`*`YourMacID`*`.tex` and `Assignment_2_`*`YourMacID`*`.pdf` files well before the deadline so that your mark is not zero if, e.g., your computer fails at 11:50 PM on February 7.

Although you are allowed to receive help from the instructional staff and other students, your submission must be your own work. Copying will be treated as academic dishonesty! If any of the ideas used in your submission were obtained from other students or sources outside of the lectures and tutorials, you must acknowledge where or from whom these ideas were obtained.

Problems

1. **[10 points]**

   Let $\mathsf{SimpleTree}$ be the inductive set defined by the following constructors:

   a. $\mathsf{Leaf} : \mathbb{N} \to \mathsf{SimpleTree}$.
   b. $\mathsf{Branch1} : \mathsf{SimpleTree} \to \mathsf{SimpleTree}$.
   c. $\mathsf{Branch2} : \mathsf{SimpleTree} \times \mathsf{SimpleTree} \to \mathsf{SimpleTree}$.

   The function $\mathsf{leaves} : \mathsf{SimpleTree} \to \mathbb{N}$ is defined by recursion and pattern matching as:

   a. $\mathsf{leaves}(\mathsf{Leaf}(n)) = 1$.
   b. $\mathsf{leaves}(\mathsf{Branch1}(t)) = \mathsf{leaves}(t)$.
   c. $\mathsf{leaves}(\mathsf{Branch2}(t_1, t_2)) = \mathsf{leaves}(t_1) + \mathsf{leaves}(t_2)$.

   The function $\mathsf{branches} : \mathsf{SimpleTree} \to \mathbb{N}$ is defined by recursion and pattern matching as:

   a. $\mathsf{branches}(\mathsf{Leaf}(n)) = 0$.
   b. $\mathsf{branches}(\mathsf{Branch1}(t)) = 1 + \mathsf{branches}(t)$.
   c. $\mathsf{branches}(\mathsf{Branch2}(t_1, t_2)) = 1 + \mathsf{branches}(t_1) + \mathsf{branches}(t_2)$.

   Prove that, for all $t \in \mathsf{SimpleTree}$,

   $$\mathsf{leaves}(t) \leq \mathsf{branches}(t) + 1.$$

   Mohammad Omar Zahir, zahirm1, Feb 4, 2021

   Proof. Let $P(t)$ hold iff

   $$\mathsf{leaves}(t) \leq \mathsf{branches}(t) + 1.$$

   We will prove $P(t)$ for all $t \in \mathsf{SimpleTree}$ by structural induction.

   Base case: $t = \mathsf{Leaf}(n)$.

   $$\begin{aligned} & \mathsf{leaves}(\mathsf{Leaf}(n)) & \\ &= \mathsf{1} & \langle \text{definition of leaves} \rangle \\ &= \mathsf{0{+}1} & \langle \text{arithmetic} \rangle \\ &\leq \mathsf{branches}(\mathsf{Leaf}(n)){+}\mathsf{1} & \langle \text{definition of branches, R.H.S.} \rangle \end{aligned}$$

   So $P(t)$ holds for $t = \mathsf{Leaf}(n)$.

Induction Step. Having proven the base case, we must show that $P(n)$ recursively holds for both other SimpleTree constructors Branch1(n) and Branch2$(t1, t2)$.

Case 1: $t = \mathsf{Branch1}(n)$. Assume $P(n)$ holds.

$$\begin{aligned}
&\mathsf{leaves}(\mathsf{Branch1}(n)) && \\
&= \mathsf{leaves}(n) && \langle\text{definition of leaves}\rangle \\
&\leq \mathsf{branches}(n)\mathsf{+1} && \langle\text{induction hypothesis}\rangle \\
&= \mathsf{branches}(\mathsf{Branch1}(n)) && \langle\text{definition of branches}\rangle \\
&\leq \mathsf{branches}(\mathsf{Branch1}(n))\mathsf{+1} && \langle\text{definition of inequality: } 0 \leq 1\rangle
\end{aligned}$$

So $P(t)$ holds for $t = \mathsf{Branch1}(n)$.

Case 2: $t = \mathsf{Branch2}(t1, t2)$. Assume $P(t1)$ and $P(t2)$ hold.

$$\begin{aligned}
&\mathsf{leaves}(\mathsf{Branch2}(t1,t2)) && \\
&= \mathsf{leaves}(t1) + \mathsf{leaves}(t2) && \langle\text{definition of leaves}\rangle \\
&\leq \mathsf{branches}(t1) + \mathsf{leaves}(t2) + \mathsf{1} && \langle\text{induction hypothesis}\rangle \\
&\leq \mathsf{branches}(t1) + \mathsf{branches}(t2) + \mathsf{1} + \mathsf{1} && \langle\text{induction hypothesis}\rangle \\
&= \mathsf{branches}(\mathsf{Branch2}(t1,t2))\mathsf{+1} && \langle\text{definition of branches}\rangle
\end{aligned}$$

So $P(t)$ holds for $t = \mathsf{Branch2}(t1,t2)$.

Therefore, $P(t)$ holds for all $t \in \mathsf{SimpleTree}$ by structural induction.

$\square$

2. **[10 points]**

Let BinNum be the inductive set defined by the following constructors:

Zero : BinNum.

One : BinNum.

JoinZero : BinNum $\to$ BinNum.

JoinOne : BinNum $\to$ BinNum.

The members of BinNum represent binary numerals like 1011 and 010. Zero represents 0; One represents 1; and if u represents U, then JoinZero(u) represents U0 and JoinOne(u) represents U1. For example,

JoinOne(JoinZero(JoinOne(One)))

represents the binary number 1101.

The function

$$\mathsf{len} : \mathsf{BinNum} \to \mathbb{N}$$

maps a member of BinNum to its length. len is defined by the following equations using recursion and pattern matching:

$$\mathsf{len}(\mathsf{Zero}) = 1.$$
$$\mathsf{len}(\mathsf{One}) = 1.$$
$$\mathsf{len}(\mathsf{JoinZero}(u)) = \mathsf{len}(u) + 1.$$
$$\mathsf{len}(\mathsf{JoinOne}(u)) = \mathsf{len}(u) + 1.$$

The function

$$\mathsf{val} : \mathsf{BinNum} \to \mathbb{N}$$

maps a member of BinNum to the value of the binary numeral it represents. For example,

$$\mathsf{val}(\mathsf{JoinOne}(\mathsf{JoinZero}(\mathsf{JoinOne}(\mathsf{One})))) = (1101)_2 = 13.$$

val is defined by the following equations using recursion and pattern matching:

$$\mathsf{val}(\mathsf{Zero}) = 0.$$
$$\mathsf{val}(\mathsf{One}) = 1.$$
$$\mathsf{val}(\mathsf{JoinZero}(u) = 2 * \mathsf{val}(u).$$
$$\mathsf{val}(\mathsf{JoinOne}(u) = (2 * \mathsf{val}(u)) + 1.$$

The function

$$\mathsf{add} : \mathsf{BinNum} \times \mathsf{BinNum} \to \mathsf{BinNum}$$

is intended to implement addition on members of BinNum. It is defined by the following equations using recursion and pattern matching:

$$\mathsf{add}(u, \mathsf{Zero}) = u.$$
$$\mathsf{add}(\mathsf{Zero}, u) = u.$$
$$\mathsf{add}(\mathsf{One}, \mathsf{One}) = \mathsf{JoinZero}(\mathsf{One}).$$
$$\mathsf{add}(\mathsf{JoinZero}(u), \mathsf{One}) = \mathsf{JoinOne}(u).$$
$$\mathsf{add}(\mathsf{One}, \mathsf{JoinZero}(u)) = \mathsf{JoinOne}(u).$$
$$\mathsf{add}(\mathsf{JoinOne}(u), \mathsf{One}) = \mathsf{JoinZero}(\mathsf{add}(u, \mathsf{One}).$$
$$\mathsf{add}(\mathsf{One}, \mathsf{JoinOne}(u)) = \mathsf{JoinZero}(\mathsf{add}(u, \mathsf{One}).$$

$$\mathsf{add}(\mathsf{JoinZero}(u), \mathsf{JoinZero}(v)) = \mathsf{JoinZero}(\mathsf{add}(u, v).$$
$$\mathsf{add}(\mathsf{JoinOne}(u), \mathsf{JoinZero}(v)) = \mathsf{JoinOne}(\mathsf{add}(u, v).$$
$$\mathsf{add}(\mathsf{JoinZero}(u), \mathsf{JoinOne}(v)) = \mathsf{JoinOne}(\mathsf{add}(u, v).$$
$$\mathsf{add}(\mathsf{JoinOne}(u), \mathsf{JoinOne}(v)) = \mathsf{JoinZero}(\mathsf{add}(\mathsf{add}(u, v), \mathsf{One})).$$

Notice that the algorithm behind the definition is essentially the same algorithm that children learn to add numbers represented as decimal numerals. The last equation is a bit complicated because it involves a carry of 1.

Lemma 1. For all $u, v \in \mathsf{BinNum}$,

$$\mathsf{len}(\mathsf{add}(u, v)) \leq \mathsf{len}(u) + \mathsf{len}(v).$$

Theorem 1. For all $u, v \in \mathsf{BinNum}$,

$$\mathsf{val}(\mathsf{add}(u, v)) = \mathsf{val}(u) + \mathsf{val}(v).$$

Theorem 1 states that add correctly implements addition on the members of BinNum.

Prove Theorem 1 assuming Lemma 1. (You are not required to prove Lemma 1.) Hint: Use strong induction with $P(n) \equiv \mathsf{val}(\mathsf{add}(u, v)) = \mathsf{val}(u) + \mathsf{val}(v)$ for all $u, v \in \mathsf{BinNum}$ such that $n = \mathsf{len}(u) + \mathsf{len}(v)$.

Mohammad Omar Zahir, zahirm1, Feb 7, 2021

Proof. Assuming Lemma 1, let $P(n)$ hold iff $\mathsf{val}(\mathsf{add}(u, v)) = \mathsf{val}(u) + \mathsf{val}(v)$. We will prove $P(n)$ for all $u, v \in \mathsf{BinNum}$ such that $n = \mathsf{len}(u) + \mathsf{len}(v)$ using strong induction.

To prove the base case for $P(n)$, there are three distinct cases which serve as the basis for our induction steps. These base cases were determined by finding the lowest possible value for $n = \mathsf{len}(u) + \mathsf{len}(v)$, which is 2 from the cases below.

Base case 1 $P(2)$: $u, v = \mathsf{Zero}$, Zero.

$$\begin{aligned}
&\mathsf{val}(\mathsf{add}(\mathsf{Zero}, \mathsf{Zero})) && \\
&= \mathsf{val}(\mathsf{Zero}) && \langle \text{definition of add: } u, \text{ Zero} \rangle \\
&= \mathsf{0} && \langle \text{definition of val: Zero} \rangle \\
&= \mathsf{0} + \mathsf{0} && \langle \text{arithmetic} \rangle \\
&= \mathsf{val}(\mathsf{Zero}) + \mathsf{val}(\mathsf{Zero}) && \langle \text{definition of val: Zero} \rangle
\end{aligned}$$

So $P(n)$ holds for $n = len(u) + len(v)$ where $u, v = \mathsf{Zero},\ \mathsf{Zero}$.

Base case 2 $P(2)$: $u, v = \mathsf{One},\ \mathsf{One}$.

$$\begin{aligned}
&\mathsf{val(add(One,\ One))} \\
&= \mathsf{val(JoinZero(One))} && \langle\text{definition of add: One, One}\rangle \\
&= 2 * \mathsf{val(One)} && \langle\text{definition of val: JoinZero}\rangle \\
&= 2 * 1 && \langle\text{definition of val: One}\rangle \\
&= 1 + 1 && \langle\text{arithmetic}\rangle \\
&= \mathsf{val(One)} + \mathsf{val(One)} && \langle\text{definition of val: One}\rangle
\end{aligned}$$

So $P(n)$ holds for $n = len(u) + len(v)$ where $u, v = \mathsf{One},\ \mathsf{One}$.

Base case 3 $P(3)$: $u, v = \mathsf{Zero},\ \mathsf{One}$.

$$\begin{aligned}
&\mathsf{val(add(Zero,\ One))} \\
&= \mathsf{val(One)} && \langle\text{definition of add: Zero, One}\rangle \\
&= 1 && \langle\text{definition of val: One}\rangle \\
&= 0 + 1 && \langle\text{arithmetic}\rangle \\
&= \mathsf{val(Zero)} + \mathsf{val(One)} && \langle\text{definition of val: Zero and One}\rangle
\end{aligned}$$

So $P(n)$ holds for $n = len(u) + len(v)$ where $u, v = \mathsf{Zero},\ \mathsf{One}$.

Induction Step

With the base cases proven, we must show that $P(n)$ holds for all $n > 2$. We will thus prove 7 distinct cases below that hold for $n = len(u) + len(v)$ that have $n > 2$ with the underlying assumption of Lemma 1.

Case 1: u, v = $\mathsf{JoinZero}(u)$, Zero:

$$\begin{aligned}
&\mathsf{val(add(JoinZero}(u)\mathsf{,\ Zero))} \\
&= \mathsf{val(JoinZero}(u)\mathsf{)} && \langle\text{definition of add: JoinZero, Zero}\rangle \\
&= \mathsf{val(JoinZero}(u)\mathsf{)} + 0 && \langle\text{arithmetic}\rangle \\
&= \mathsf{val(JoinZero}(u)\mathsf{)} + \mathsf{val(Zero)} && \langle\text{definition of val: Zero}\rangle
\end{aligned}$$

Thus $P(n)$ holds for $n = \mathsf{len}(u)+\mathsf{len}(v)$, where $u, v = \mathsf{JoinZero}(u),\ \mathsf{Zero}$, respectively.

Case 2: u, v = $\mathsf{JoinZero}(u)$, One:

$$\begin{aligned}
&\mathsf{val}(\mathsf{add}(\mathsf{JoinZero}(u),\ \mathsf{One})) \\
&= \mathsf{val}(\mathsf{JoinOne}(u)) && \langle\text{definition of add: JoinZero, One}\rangle \\
&= 2 * \mathsf{val}((u)) + 1 && \langle\text{definition of val: JoinOne}\rangle \\
&= \mathsf{val}(\mathsf{JoinZero}(u)) + 1 && \langle\text{definition of val: JoinZero}\rangle \\
&= \mathsf{val}(\mathsf{JoinZero}(u)) + \mathsf{val}(\mathsf{One}) && \langle\text{definition of val: One}\rangle
\end{aligned}$$

Thus $P(n)$ holds for $n = \mathsf{len}(u)+\mathsf{len}(v)$, where $u, v = \mathsf{JoinZero}(u),\ \mathsf{One}$, respectively.

Case 3: u, v = $\mathsf{JoinZero}(u)$, $\mathsf{JoinZero}(v)$: Because strong induction is being used in this proof, we can assume that $P(n)$ holds for all n up to $n+1$. This is because the current values of u, v are JoinZero and JoinZero which show that $P(n+2) = P(len(u)+1+len(v)+1)$, which represent $P(n+2)$, allowing us to use our induction hypothesis.

$$\begin{aligned}
&\mathsf{val}(\mathsf{add}(\mathsf{JoinZero}(u),\ \mathsf{JoinZero}(v))) \\
&= \mathsf{val}(\mathsf{JoinZero}(\mathsf{add}(u, v))) && \langle\text{definition of add: JoinZero, JoinZero}\rangle \\
&= 2 * \mathsf{val}(\mathsf{add}(u, v)) && \langle\text{definition of val: JoinZero}\rangle \\
&= 2 * (\mathsf{val}(u) + \mathsf{val}(v)) && \langle\text{induction hypothesis: } P(n)\rangle \\
&= 2 * \mathsf{val}(u) + 2 * \mathsf{val}(v) && \langle\text{arithmetic}\rangle \\
&= \mathsf{val}(\mathsf{JoinZero}(u)) + \mathsf{val}(\mathsf{JoinZero}(v)) && \langle\text{definition of val: JoinZero}\rangle
\end{aligned}$$

Thus $P(n)$ holds for $n = \mathsf{len}(u)+\mathsf{len}(v)$, where $u, v = \mathsf{JoinZero}(u),\ \mathsf{JoinZero}(v)$, respectively.

Case 4: u, v = $\mathsf{JoinOne}(u)$, Zero:

$$\begin{aligned}
&\mathsf{val}(\mathsf{add}(\mathsf{JoinOne}(u),\ \mathsf{Zero})) \\
&= \mathsf{val}(\mathsf{JoinOne}(u)) && \langle\text{definition of add: JoinOne, Zero}\rangle \\
&= \mathsf{val}(\mathsf{JoinOne}(u)) + 0 && \langle\text{arithmetic}\rangle \\
&= \mathsf{val}(\mathsf{JoinOne}(u)) + \mathsf{val}(\mathsf{Zero}) && \langle\text{definition of val: Zero}\rangle
\end{aligned}$$

Thus $P(n)$ holds for $n = \mathsf{len}(u)+\mathsf{len}(v)$, where $u, v = \mathsf{JoinOne}(u),\ \mathsf{Zero}$, respectively.

Case 5: u, v = $\mathsf{JoinOne}(u)$, One: Because strong induction is being used in this proof, we can assume that $P(n)$ holds for all values up to n. This is because the current values of u, v are JoinOne and One which show that $P(n{+}1) = P(len(u){+}1{+}1)$, which represent $P(n{+}1)$, allowing us to use our induction hypothesis.

$$
\begin{aligned}
& \mathsf{val(add(JoinOne(}u\mathsf{),\ One))} \\
&= \mathsf{val(JoinZero(add((}u\mathsf{,\ One)))} && \langle \text{definition of add: JoinOne, One} \rangle \\
&= 2 * \mathsf{val(add(}u\mathsf{,\ One))} && \langle \text{definition of val: JoinZero} \rangle \\
&= 2 * (\mathsf{val(}u\mathsf{)} + \mathsf{val\ (One)}) && \langle \text{induction hypothesis: } P(n) \rangle \\
&= 2 * (\mathsf{val(}u\mathsf{)} + 1) && \langle \text{definition of val: One} \rangle \\
&= 2 * \mathsf{val(}u\mathsf{)} + 1 + 1 && \langle \text{arithmetic} \rangle \\
&= 2 * \mathsf{val(}u\mathsf{)} + 1 + \mathsf{val(One)} && \langle \text{definition of val: One} \rangle \\
&= \mathsf{val(JoinOne(}u\mathsf{))} + \mathsf{val(One)} && \langle \text{definition of val: JoinOne} \rangle
\end{aligned}
$$

Thus $P(n)$ holds for $n = \mathsf{len}(u){+}\mathsf{len}(v)$, where $u, v = \mathsf{JoinOne}(u),\ \mathsf{One}$, respectively.

Case 6: u, v = $\mathsf{JoinOne}(u)$, $\mathsf{JoinOne}(v)$: Because strong induction is being used in this proof, we can assume that $P(n)$ holds for all n up to $n+1$. This is because the current values of u, v are JoinOne and JoinOne which show that $P(n{+}2) = P(len(u){+}1{+}len(v){+}1)$, which represent $P(n+2)$, allowing us to use our induction hypothesis. The first induction hypothesis that was used in this proof, $P(n{+}1)$ is done so as len(add(u,v)) can be written as len(u) + len(v) based on the definition of lemma 1.

$$
\begin{aligned}
& \mathsf{val(add(JoinOne(}u\mathsf{),\ JoinOne(}v\mathsf{)))} \\
&= \mathsf{val(JoinZero(add(add(}u, v\mathsf{), One))))} && \langle \text{definition of add: JoinOne, JoinOne} \rangle \\
&= 2 * \mathsf{val(add\ (add\ (u,v),One))} && \langle \text{definition of val: JoinZero} \rangle \\
&= 2 * (\mathsf{val(add(}u, v\mathsf{))} + \mathsf{val(One)}) && \langle \text{induction hypothesis: } \geq P(n+1) \rangle \\
&= 2 * (\mathsf{val(}u\mathsf{)}{+}\mathsf{val(}v\mathsf{)}) + \mathsf{val(One)}) && \langle \text{induction hypothesis: } P(n) \rangle \\
&= 2 * (\mathsf{val(}u\mathsf{)}{+}\mathsf{val(}v\mathsf{)}) + 2 * \mathsf{val(One)} && \langle \text{arithmetic} \rangle \\
&= 2 * (\mathsf{val(}u\mathsf{)}{+}\mathsf{val(}v\mathsf{)}) + 2 * 1 && \langle \text{definition of val: One} \rangle \\
&= 2 * \mathsf{val(}u\mathsf{)} + 2 * \mathsf{val(}v\mathsf{)} + 1 + 1 && \langle \text{arithmetic} \rangle \\
&= \mathsf{val(JoinOne(}u\mathsf{))} + \mathsf{val(JoinOne(}v\mathsf{))} && \langle \text{definition of val: JoinOne} \rangle
\end{aligned}
$$

Thus $P(n)$ holds for $n = \mathsf{len}(u){+}\mathsf{len}(v)$, where $u, v = \mathsf{JoinOne}(u),\ \mathsf{JoinOne}(v)$, respectively.

Case 7: u, v = $\mathsf{JoinOne}(u)$, $\mathsf{JoinZero}(v)$: Because strong induction is being used in this proof, we can assume that $P(n)$ holds for all n

up to $n+1$. This is because the current values of u, v are JoinOne and JoinZero which show that $P(n+2) = P(len(u)+1+len(v)+1)$, which represent $P(n+2)$, allowing us to use our induction hypothesis.

$$
\begin{aligned}
&\mathsf{val}(\mathsf{add}(\mathsf{JoinOne}(u),\ \mathsf{JoinZero}(v))) & \\
&= \mathsf{val}(\mathsf{JoinOne}(\mathsf{add}(u,v))) & \langle\text{definition of add: JoinOne, JoinZero}\rangle \\
&= 2 * \mathsf{val}(\mathsf{add}(u,v)) + 1 & \langle\text{definition of val: JoinOne}\rangle \\
&= 2 * (\mathsf{val}(u) + \mathsf{val}(v)) + 1 & \langle\text{induction hypothesis: } P(n)\rangle \\
&= 2 * \mathsf{val}(u) + 2 * \mathsf{val}(v) + 1 & \langle\text{arithmetic}\rangle \\
&= \mathsf{val}(\mathsf{JoinOne}(u)) + 2 * \mathsf{val}(v) & \langle\text{definition of val: JoinOne}\rangle \\
&= \mathsf{val}(\mathsf{JoinOne}(u)) + \mathsf{val}(\mathsf{JoinZero}(v)) & \langle\text{definition of val: JoinZero}\rangle
\end{aligned}
$$

Thus $P(n)$ holds for $n = \mathsf{len}(u)+\mathsf{len}(v)$, where $u, v = \mathsf{JoinOne}(u),\ \mathsf{JoinZero}(v)$, respectively.

Therefore, $P(n)$ holds for all $n = \mathsf{len}(u) + \mathsf{len}(v)$ by strong induction.

$\square$