COMPSCI/SFWRENG 2FA3
Discrete Mathematics with Applications II
Winter 2021

Assignment 4

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Assignment 4 consists of two problems. You must write your solutions to the problems using LaTeX.

Please submit Assignment 4 as two files, `Assignment_4_`*`YourMacID`*`.tex` and `Assignment_4_`*`YourMacID`*`.pdf`, to the Assignment 4 folder on Avenue under Assessments/Assignments. *`YourMacID`* must be your personal MacID (written without capitalization). The `Assignment_4_`*`YourMacID`*`.tex` file is a copy of the LaTeX source file for this assignment (`Assignment_4.tex` found on Avenue under Contents/Assignments) with your solution entered after each problem. The `Assignment_4_`*`YourMacID`*`.pdf` is the PDF output produced by executing

```
pdflatex Assignment_4_YourMacID
```

This assignment is due **Sunday, February 28, 2021 before midnight.** You are allow to submit the assignment multiple times, but only the last submission will be marked. **Late submissions and files that are not named exactly as specified above will not be accepted!** It is suggested that you submit your preliminary `Assignment_4_`*`YourMacID`*`.tex` and `Assignment_4_`*`YourMacID`*`.pdf` files well before the deadline so that your mark is not zero if, e.g., your computer fails at 11:50 PM on February 28.

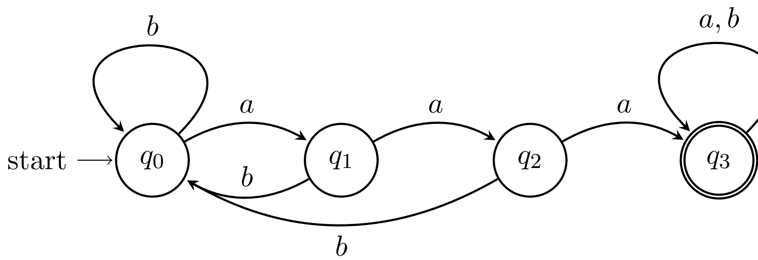
Although you are allowed to receive help from the instructional staff and other students, your submission must be your own work. Copying will be treated as academic dishonesty! If any of the ideas used in your submission were obtained from other students or sources outside of the lectures and tutorials, you must acknowledge where or from whom these ideas were obtained.

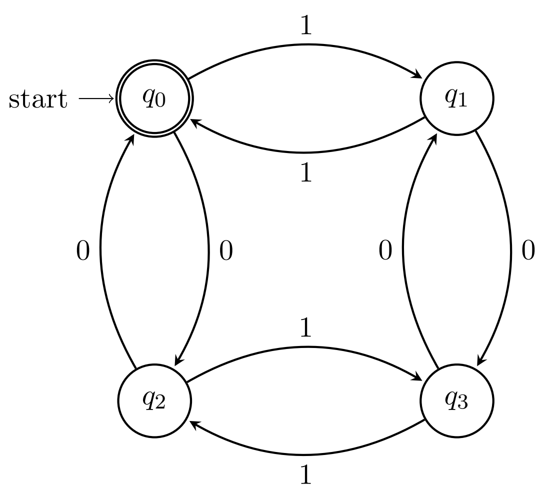
Presenting DFAs and NFAs Transition Diagrams

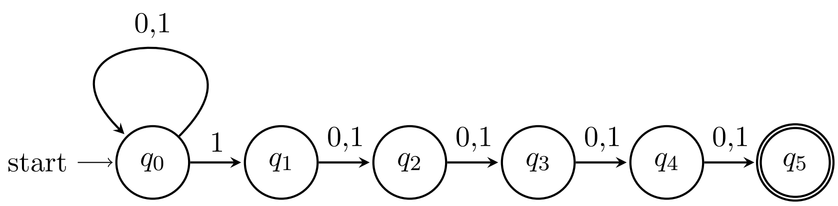
In this assignment you are asked to present DFAs as transition diagrams. You are can do this in one of two ways.

The first way is to present the diagram using the LaTeX graphics package TikZ. The TikZ code can either be written by hand or automatically generated using the finsm system available at `http:finsm.io`.

Here are some examples of how it can be used to create DFA and NFA transition diagrams that appear in the lectures slides:







The second way is to take a picture of a hand-written transition diagram and then embed it into your assignment using the following LaTeX code:

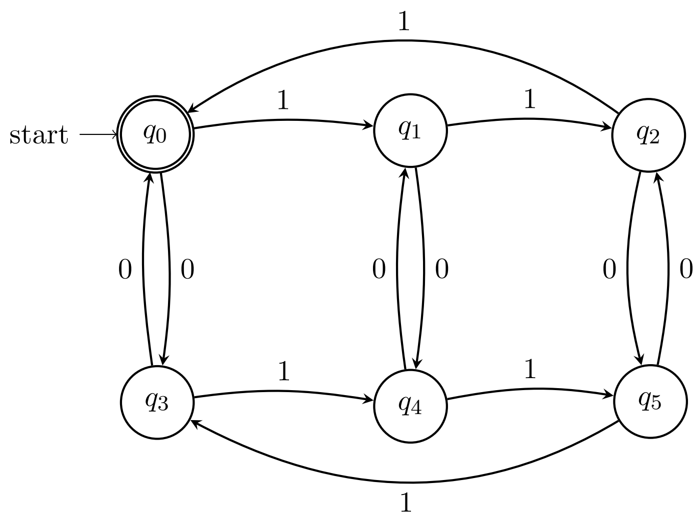
```
\begin{center}
\includegraphics[scale = 0.5]{diagram.jpg}
\end{center}
```

Please make sure your diagram is legible.

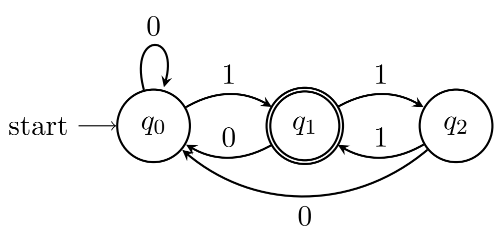
Problems

1. **[10 points]** Construct a deterministic finite automaton (DFA) A for the alphabet $\Sigma = \{0, 1\}$ such that $L(A)$ is the set of all strings x in Σ^* for which $\#0(x)$ is even and $\#1(x)$ is divisible by 3. Present A as a transition diagram.

   Mohammad Omar Zahir, zahirm1, Feb 28, 2021



2. **[10 points]** Let B be the DFA given by the following transition diagram:



   Prove that $L(B)$ is the language of all binary strings that end with an odd number of 1s. Hint: Use weak induction on the length of the input string, and let $P(n)$ be the statement that, for all input strings w with $|w| = n$, the following conditions hold:

   a. If $\delta^*(q_0, w) = q_0$, then w is ϵ or ends with 0.
   b. If $\delta^*(q_0, w) = q_1$, then w ends with an odd number of 1s.
   c. If $\delta^*(q_0, w) = q_2$, then w ends with an even number of 1s.

Proof. We will prove that $L(B)$ is the language of all binary strings that end with an odd number of 1s. We will prove $P(n)$ for the the three cases P_1, P_2, P_3, corresponding to a, b, c in the question respectively, where $n = |w|$ the length of the word, by weak induction.

Base Case: n = 0
We will first prove P_1, P_2, P_3 for the base case, where $n = |w| = 0$, or the empty word ϵ, and prove $P(0)$.

P1: We know this to be trivially true as this is what is stated in the condition for $w = \epsilon$, which is the empty word. Thus it holds for $P1$.

$\delta^*(q_0, \epsilon) = q_0$

P2: We know this to be true because the condition states that when the state is at q_1, the word must end with an odd number of 1s. Because we have the empty word $|w| = 0$, that means that our word does not end with anything, meaning our state never leaves q_0. This can be modelled as an implication like $\delta^*(q_0, w) = q_1 \implies w$. We know that the left side of our implication must be False since we stay at q_0 for the empty word, meaning that we get $False \implies w$ which is equivalent to $True$ by 'Ex-Falso Quodlibet'. Thus P_2 holds.

P3: This is proved identically to P_2. We know this to be true because the condition states that when the state is at q_2, the word must end with an even number of 1s. Because we have the empty word $|w| = 0$, that means that our word does not end with anything, meaning our state never leaves q_0. This can be modelled as an implication like $\delta^*(q_0, w) = q_2 \implies w$. We know that the left side of our implication must be False since we stay at q_0 for the empty word, meaning that we get $False \implies w$ which is equivalent to $True$ by 'Ex-Falso Quodlibet'. Thus P_3 holds.
Having proved the three base case properties, we can say P(0) holds.

Induction Step: n > 0
With the single base cases proven for each P_1, P_2, P_3, we will thus prove $P(n + 1)$ holds for all $n = |w| > 0$. Specifically, $w = xi$ which represents the concatenation of x and i where $x \in \Sigma^*$ and i is the end of the word specified by the transition to the state. Additionally, $|w| = n + 1$ and $|x| = n$, and as we are using weak induction, we assume $P(n)$ holds and prove for $P(n + 1)$.

P1: To prove this, we must show that w must end with 0, as w can no longer be ϵ. Since we know that the we must end in the q_0 state for this property, we can analyse the transitions that are followed to get to q_0.

$$\begin{aligned}
&\delta^*(q_0, w) && \\
&= \delta^*(q_0, xi) && \langle\text{Substitution}\rangle \\
&= \delta(\delta^*(q_0, x), i) && \langle\text{Definition of } \delta^*\rangle \\
&= q_0 && \langle\text{R.H.S. of } P_1\rangle
\end{aligned}$$

We can see that all the transitions, represented by the arrows, that are pointing into or ending at q_0 are ending with a 0, meaning that if the state is at q_0 the last number of the word must be 0. This can formally be written as $\delta^*(q, 0)$ for all $q \in Q$, where Q is all possible states. Thus $P1$ holds, as we have satisfied the condition for all cases for each state.

P2: To prove this, we can model the situation as follows.

$$\begin{aligned}
&\delta^*(q_0, w) && \\
&= \delta^*(q_0, xi) && \langle\text{Substitution}\rangle \\
&= \delta(\delta^*(q_0, x), i) && \langle\text{Definition of } \delta^*\rangle \\
&= q_1 && \langle\text{R.H.S. of } P_2 \text{ (For Induction Step)}\rangle
\end{aligned}$$

Here we know that i must represent 1 for both cases as both the transitions to q_1 end with a 1. Therefore to prove this we have to analyse the possibilities of the two cases for q_1.

Case 1: $\delta(q_0, 1)$
Here we can see that q_0 represents $\delta^*(q_0, x) = q_0$. Since we are using weak induction and assuming $P(n)$, we can apply our induction hypothesis for P_1 on x which has length of n, giving us an empty word or a word that ends with 0. We will then concatenate 1, as we know that $i = 1$, and in either case, we will end the word with a single 1, which is an odd number of 1s. Thus, we have proved that *Case* 1 holds.

Case 2: $\delta(q_2, 1)$
Here we can see that q_2 represents $\delta^*(q_0, x) = q_2$. Since we are using weak induction and assuming $P(n)$, we can apply our induction hypothesis for P_3 on x which has length of n, giving us a word that ends with an even number of 1s. We will then concatenate 1, as we know that $i = 1$, and the word will end with adding a single 1, which is an odd number of 1s, after an even number of 1s, which is an odd number of 1s since we trivially know that even + odd = odd. Thus, we have proved that *Case* 2 holds.

Having proven that the two distinct Cases for P_2 hold, we can say that P_2 holds.

P3: To prove this, we can model the situation as follows.

$$
\begin{aligned}
&\delta^*(q_0, w) \\
&= \delta^*(q_0, xi) && \langle \text{Substitution} \rangle \\
&= \delta(\delta^*(q_0, x), i) && \langle \text{Definition of } \delta^* \rangle \\
&= q_2 && \langle \text{R.H.S. of } P_3 \text{ (For Induction Step)} \rangle
\end{aligned}
$$

Here we know that i must represent 1 for the single transition case for the q_2 state which can be proved as a case below.

Case 1: $\delta(q_1, 1)$
Here we can see that q_1 represents $\delta^*(q_0, x) = q_1$. Since we are using weak induction and assuming $P(n)$, we can apply our induction hypothesis for P_2 on x which has length of n. This gives us a word that ends with an odd number of 1s based on the induction hypothesis. We will then concatenate 1, as we know that $i = 1$ from the transition, which is also an odd number of 1s. Their concatenation will give us an even number of 1s as we trivially know that odd + odd = even. Thus, we have proved that *Case* 1 holds.

Having proven the single distinct case for P_3, we can say that P_3 holds.

We have proven $P(n+1)$ holds for the three cases P_1, P_2, P_3, where $n = |w|$ the length of the word. Thus we have proved that $P(n)$ holds by weak induction.

□