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ORIGINAL PAPER

Dynamic portfolio allocation in goals-based wealth management

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Abstract

We report a dynamic programming algorithm which, given a set of efficient (or even inefficient) portfolios, constructs an optimal portfolio trading strategy that maximizes the probability of attaining an investor's specified target wealth at the end of a designated time horizon. Our algorithm also accommodates periodic infusions or withdrawals of cash with no degradation to the dynamic portfolio's performance or runtime. We explore the sensitivity of the terminal wealth distribution to restricting the segment of the efficient frontier available to the investor. Since our algorithm's optimal strategy can be on the efficient frontier and is driven by an investor's wealth and goals, it soundly beats the performance of target date funds in attaining investors' goals. These optimal goals-based wealth management strategies are useful for independent financial advisors to implement behavioral-based FinTech offerings and for robo-advisors.

Keywords Goals · Wealth management · Behavioral portfolio theory · Dynamic portfolios · Efficient portfolios

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1 Introduction

Goals-based wealth management (GBWM) refers to the management of an investor's portfolios with a view to meeting long-term financial goals, as opposed to only optimizing a risk-return tradeoff. In this paper, we present and analyze a dynamic programming approach for GBWM that is fully optimal in meeting long-term goals, while also optimizing risk-return tradeoff. We will show that our approach has several advantageous features in terms of speed and adaptability, and that it soundly outperforms portfolios based on current widespread approaches, such as the target-date paradigm.

GBWM is a modern implementation of behavioral ideas from early work by Shefrin and Statman (2000) espousing behavioral portfolio theory that were further discussed by Nevins (2004) from a practitioner viewpoint. A specific framework for GBWM was initially proposed by Chabbra (2005) and is now followed by many practitioners (see Brunel 2015). Our approach in this paper is also cognizant of prospect theory (Kahneman and Tversky 1979), which is aimed at modeling how people realistically make decisions, rather than an optimization framework, which usually does not account for all the criteria used by people when making tradeoffs between gains and losses. Our framework also accommodates mental accounting (Thaler 1985, 1999), which is the paradigm where people behave as if they have different risk-return preferences depending on the goal being achieved, being risk averse in some settings and even risk seeking in others. In our context, mental accounting is the idea that financial goals may be sharply defined and better managed in separate portfolios. We also include paradigms for how investors form both upside and downside goals, such as aspirational goal-setting (Lopes 1987), a safety-first criterion (Roy 1952), and loss aversion (Shefrin and Statman 1985).

We can combine these seminal GBWM ideas for achieving long term goals with modern portfolio theory (Markowitz 1952) to achieve the short term goal of minimizing the risk associated with any specific expected return. GBWM is fully consistent with Markowitz's mean-variance theory. This was developed in Das et al. (2010) as a static model for mental accounts, and a full analysis of the static model in a GBWM context is provided in Das et al. (2018).

These static models are not fully optimal, because they can only be implemented period by period in a myopic manner. Previous fully optimal, dynamic approaches to GBWM include solving continuous-time partial differential equations, as shown in the early work by Merton (1969, 1971), Browne (1995, 1997), or using continuous-time martingale methods based on seminal ideas in Cox and Huang (1989), as shown in more recent work by Wang et al. (2011) and Deguest et al. (2015). In this paper we introduce a discrete time fully optimal dynamic programming approach that has many advantages, including being adaptable to many financial situations, being simple to implement, and being fast to run. For example, our approach handles periodic infusions and withdrawals of varying amounts, as well as bankruptcy, which is not always easy to do with the aforementioned continuous-time approaches. Also, our approach employs a polynomial time algorithm that corresponds in general to a runtime of only a few seconds. These features give our algorithm wide applicability, enabling either traditional financial advisors or robo-advisors to quickly and optimally determine the best strategy for investors to pursue to meet their individual wealth goals.

The essence of GBWM lies in choosing a different objective function for dynamic optimization than that chosen in utility-based portfolio problems. The simplest form of the GBWM objective function is probabilistic: that is, define a portfolio goal wealth G at the defined horizon T and find the dynamic portfolio strategy that maximizes the probability of achieving the goal, i.e.,

$$\max_{\{A(0),A(1),\ldots,A(T-1)\}} Pr[W(T) \geq G],$$

where $W(T)$ is the terminal portfolio wealth value and $A(t)$ are the possible allocations among the funds available to the investor at each time $t = 0, 1, 2, \ldots, T-1$. The optimal allocation at any time t is a function of $W(t)$, the level of wealth at time t; G, the goal wealth; T, the timeframe; and other investor specific parameters.

This approach may be contrasted with a utility-based approach where the dynamic strategy is chosen to maximize the expected utility from any consumption over time, as well as the expected utility of the final wealth. This approach, as well as GBWM, are attractive since they focus on long-run outcomes and are not myopic one-period optimization models. Single-period optimization changes the focus from achieving long-term goals to short-term risk-return tradeoffs. We see that strategies that are aimed at meeting an upside goal with a lower threshold might result in very different strategies than dynamic asset allocation based on utility function maximization, such as taking on more risk when not reaching goals close to the investment horizon, see Browne (1999b).

Hybrid approaches, where the objective function to be maximized is expected utility subject to goals as constraints, have also been attempted as in Browne (1999a, 2000), Deguest et al. (2015). However, this approach requires choosing a utility function for the investor which is hard to determine, and therefore an ad-hoc choice is often required. In this paper, we do not require a utility function to be specified. We provide an optimal solution procedure to the GBWM problem and examine the properties of the trading strategy dictated by the algorithm.

The following briefly characterizes our solution: (i) In our numerical experiments, the backward recursion algorithm that solves the dynamic problem is approximately quadratic in its dependence on the timeframe T, between linear and quadratic with a growth exponent of 1.5 in its dependence on the granularity of the wealth grid, and approximately linear in its dependence on the number of portfolio choices. Our results are quite robust to the granularity of the wealth grid and the number of portfolio choices, which suggests that expanding the scale of the problem with additional features, such as tax optimization, can be performed without significantly degrading run times or the solution's accuracy. (ii) In our base case timeframe of 10 years with annual rebalancing, having a wealth grid granularity of at least three wealth nodes per yearly standard deviation in portfolio performance, and allowing for 15 portfolio choices, our algorithm runs in under 5 s. (iii) As the markets move up or down through the investment tenure, we rebalance the portfolio such that the probability of reaching the goal wealth continues to remain the highest. The optimal allocation and rebalancing is intuitive in the sense that when the portfolio is far from its goal due to underperformance, risk is increased in order to enhance the probability of reaching the goal, and when the portfolio is outperforming, risk is dialed back to reduce the risk of missing the goal. Therefore, the strategy depends on both time and state (wealth), unlike target-date fund strategies, which only depend on time and cannot accommodate investor-specific goals. (iv) The dynamic program uses a collection of exogenously provided portfolios, that may or may not be chosen to be on the efficient frontier, so an asset management team can develop the set of model portfolios, while, acting separately, the optimization team can tune the dynamic programming algorithm. This offers a plug-and-play approach to GBWM, where portfolio construction is separated from portfolio allocation.

Our GBWM strategy allows for a variety of important features and results: (i) The final wealth distribution may be modulated by limiting the range of available portfolios (i.e., controls), which exogenously alters the risk that is taken. Raising the minimum risk of the portfolios that are available to the investor increases the right tail of the wealth distribution more than the left tail. Conversely, raising the maximum risk available to the investor, increases the left tail more than the right tail. This suggest that investors in a GBWM environment who wish to have higher returns are better off raising their minimum risk, instead of the more intuitive move of raising their maximum risk. That said, raising the minimum risk will decrease the chance of attaining the investor's goal wealth, while raising the maximum risk will increase this chance. (ii) The algorithm is flexible: (a) If desired, the algorithm can optimize for multiple wealth goals at the end of the portfolio horizon instead of a single wealth goal, with weights for the relative importance of these multiple goals that the investor can decide. (b) The algorithm can include an investor's specified portfolio infusions and withdrawals. We will see that even small infusions can increase the probability of reaching the goal wealth substantively. When there are withdrawals, there is a chance that the investor will run out money (i.e., go bankrupt) during the investment timeframe. Our algorithm can determine this chance of bankruptcy, and much more importantly, show how to minimize it. (c) The algorithm can accommodate any desired time period between rebalancing and between infusions or withdrawals. (iii) Our algorithm allows us to determine how to optimize retirement savings. For example, our algorithm will allow us to explore the effect of infusions on the minimized probability of going bankrupt in retirement. In particular, we will consider a 50 year old investor who currently has 100 thousand dollars in their retirement account and intends to take out 50 thousand present day dollars every year after they turn 65 through the age of 80. We show that to attain a 58.6% probability of maintaining this income stream would require the investor to make annual inflation-adjusted infusions of 15 thousand dollars a year until retiring at age 65. (iv) Finally, we compare our GBWM optimal strategy to the performance of target date funds (TDFs), by considering a TDF with a typical glide path for three index funds representing total domestic bond, total international stock, and total domestic stock. Because our GBWM strategy is on the efficient frontier, uses a wealth-dependent strategy, and accounts for investor-specific goals, we will see that our GBWM strategy, which uses the same three index funds, shows a much higher probability of reaching an investor's goals. For example, we will show that using our TDF in the case presented in the previous point reduces the probability of maintaining the desired income stream from 58.6% down to 26.6%.

The rest of this paper is organized as follows. In Sect. 2 we describe the algorithm and in Sect. 3 the extensions of the algorithm to deal with other cases are presented. The results of our study are set out in Sect. 4 and our conclusions are summarized in Sect. 5.

2 Algorithm

2.1 Notation

We describe the notation used in the paper here.

- Let $t = 0, h, 2h, \ldots, Nh = T$ be the N time periods in the model. Here, h is the time step, usually taken to be 1 year. Therefore, the time frame of the model is $0 \le t \le T$.
- Initial wealth: $W(0)$.
- Target wealth (goal G) at the horizon: $W(T) \equiv G$.
- We have n equity assets indexed by $i = 1, 2, \ldots, n$.
- $C(t)$: known cash flows $C(t)$ of capital into the portfolio each year. When $C(t) > 0$, we have an infusion into the portfolio. When $C(t) < 0$, we have a withdrawal from the portfolio. All of these cash flows are assumed to be determined at $t = 0$, so they are pre-committed by the investor.

Our goal is to dynamically allocate the portfolio among these n assets so that we maximize the probability at $t = T$ of attaining a final portfolio worth at least $W(T) = G$, our goal wealth.

2.2 The efficient frontier for stock portfolios

For any given portfolio volatility, σ, it is always optimal to maximize the portfolio expected return, μ. Modern portfolio theory, which was developed by Markowitz, see Markowitz (1952), gives a method for the exact allocation among the n assets that gives the maximum μ. If, for every value of σ, we plot the point (σ, μ) where μ is the maximum portfolio expected return from modern portfolio theory, we sketch out a hyperbola in the (σ, μ) plane, which is called the efficient frontier. It is always optimal to maintain the portfolio on the efficient frontier. The question that remains is how to optimally adjust ourselves along the efficient frontier over time.

As shown in Das et al. (2018), the specific hyperbola for the efficient frontier is the equation

$$\sigma = \sqrt{a\mu^2 + b\mu + c}. \tag{1}$$

The constants, a, b, and c are defined by $\mathbf{m}$, which is a vector of the n expected returns; $\mathbf{o}$, which is a vector of n ones; and $\mathbf{\Sigma}$, which is the $n \times n$ covariance matrix of the n assets, via the following equations:

$$a = \mathbf{h}^\top \boldsymbol{\Sigma} \mathbf{h}$$
$$b = 2\mathbf{g}^\top \boldsymbol{\Sigma} \mathbf{h}$$
$$c = \mathbf{g}^\top \boldsymbol{\Sigma} \mathbf{g},$$

where the vectors $\mathbf{g}$ and $\mathbf{h}$ are defined by

$$\mathbf{g} = \frac{l\boldsymbol{\Sigma}^{-1}\mathbf{o} - k\boldsymbol{\Sigma}^{-1}\mathbf{m}}{lp - k^2}$$
$$\mathbf{h} = \frac{p\boldsymbol{\Sigma}^{-1}\mathbf{m} - k\boldsymbol{\Sigma}^{-1}\mathbf{o}}{lp - k^2},$$

and the scalars k, l, and p are defined by

$$k = \mathbf{m}^\top \boldsymbol{\Sigma}^{-\mathbf{1}} \mathbf{o}$$
$$l = \mathbf{m}^\top \boldsymbol{\Sigma}^{-\mathbf{1}} \mathbf{m}$$
$$p = \mathbf{o}^\top \boldsymbol{\Sigma}^{-\mathbf{1}} \mathbf{o}.$$

We allow for restrictions on the segment of the efficient frontier available to the investor, due, for example, to restrictions such as disallowing short positions. We will consider this truncated efficient frontier as the set of potential portfolios from which we optimize the probability of attaining our goal wealth G. More specifically, we define $\mu_{\min}$ and $\mu_{\max}$ to be the smallest and largest values of μ in this truncated efficient frontier. The optimal policy or "control" in our problem is a value of $\mu \in [\mu_{\min}, \mu_{\max}]$, where the corresponding value of σ, the volatility, is given by Eq. (1).

Because it is optimal, we stay on the efficient frontier in this paper. However, we note that the method presented in this paper also applies to any fixed group of allowable portfolios on or off the efficient frontier. So, for example, if there are limitations on the fractions that can be devoted to specific assets within the portfolio, our method can still be applied.

2.3 The state space gridpoints

The state space consists of time values, t, and wealth values, $W(t)$, which are discretized, so that we consider annual rebalancing (and, later, non-annual rebalancing in Sect. 3.1) at times $t = 0, 1, 2, \ldots, T$ and, at each of these times, we use wealth grid points, W_i, where the index $i \in \{0, 1, 2, \ldots, i_{\max}\}$. The smallest wealth grid point, W_0, should correspond to the smallest possible wealth, $W_{\min}$, attainable, and the largest wealth grid point, $W_{i_{\max}}$, should correspond to the largest possible wealth, $W_{\max}$, attainable. We next determine values for $i_{\max}$, $W_{\min}$, and $W_{\max}$.

To move forward in time, we require a stochastic model for the evolution of the initial portfolio wealth, $W(0)$. We have chosen to use geometric Brownian motion for this paper. That is,

$$W(t) = W(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z}, \tag{2}$$

where Z is a standard normal random variable, however, we could have just as easily worked with other stochastic models, including those containing tails that are fatter than normal distributions to more accurately reflect observed volatility. The only restriction on the evolution model is that it is Markovian. That is, the evolution model is not affected by previous events.

For our geometric Brownian motion model, we will assume that Z realistically takes values between -3 and 3, so the smallest realistic value for $W(t)$ corresponds to computing Eq. (2) after setting $Z = -3$, $\mu = \mu_{\text{min}}$, and σ equal to σ_{max}, which is the value of σ from Eq. (1) when $\mu = \mu_{\text{max}}$. The largest realistic value is computed by again using $\sigma = \sigma_{\text{max}}$, but replacing Z with 3 and μ with μ_{max}.

Since we must also include the effects of the cashflows, $C(t)$, as well as the initial investment, $W(0)$, and both are affected by the same geometric Brownian motion model, we have that

$$\hat{W}_{\text{min}} = \min_{\tau \in \{0,1,2,\ldots,T\}} \left[W(0) e^{\left(\mu_{\text{min}} - \frac{\sigma_{\text{max}}^2}{2}\right)\tau - 3\sigma_{\text{max}}\sqrt{\tau}} + \sum_{t=0}^{\tau} C(t) e^{\left(\mu_{\text{min}} - \frac{\sigma_{\text{max}}^2}{2}\right)(\tau - t) - 3\sigma_{\text{max}}\sqrt{\tau - t}} \right] \tag{3}$$

$$\hat{W}_{\text{max}} = \max_{\tau \in \{0,1,2,\ldots,T\}} \left[W(0) e^{\left(\mu_{\text{max}} - \frac{\sigma_{\text{max}}^2}{2}\right)\tau + 3\sigma_{\text{max}}\sqrt{\tau}} + \sum_{t=0}^{\tau} C(t) e^{\left(\mu_{\text{max}} - \frac{\sigma_{\text{max}}^2}{2}\right)(\tau - t) + 3\sigma_{\text{max}}\sqrt{\tau - t}} \right], \tag{4}$$

where $C(0) = 0$, since any cash flow at $t = 0$ is incorporated into $W(0)$. For the moment, we assume that $\hat{W}_{\text{min}} > 0$, because the $C(t)$ are not sufficiently negative that bankruptcy is a realistic possibility; in Sect. 3.2 we will explore the case where $C(t)$ are sufficiently negative that bankruptcy is a realistic possibility.

We next look to fill in the grid points between $\hat{W}_{\text{min}}$ and $\hat{W}_{\text{max}}$. First, analogous to our σ_{max} definition, we define σ_{min} to be the value of σ from Eq. (1) when $\mu = \mu_{\text{min}}$. We then define the exogenously selected wealth grid density parameter, ρ_{grid}, which corresponds to the number of wealth grid points chosen per each σ_{min} in the following sense: After setting t and Z to one in Eq. (2) and ignoring the drift term $\left(\mu - \frac{\sigma^2}{2}\right) t$, we see that σ is proportional to the logarithm of wealth. Therefore, we compute $\ln(\hat{W}_{\text{min}})$ and $\ln(\hat{W}_{\text{max}})$ and then, starting with $\ln(\hat{W}_{\text{min}})$, we add a grid point every $\frac{\sigma_{\text{min}}}{\rho_{\text{grid}}}$ units, stopping once we reach or surpass $\ln(\hat{W}_{\text{max}})$. This yields a total of $i_{\text{max}} + 1$ grid points where i_{max} equals $\frac{(\ln(\hat{W}_{\text{max}}) - \ln(\hat{W}_{\text{min}}))\rho_{\text{grid}}}{\sigma_{\text{min}}}$ after rounding up to the nearest integer.

Next, we equally shift all of these $i_{\text{max}} + 1$ values downward by the smallest amount necessary to match one of these values to $\ln(W(0))$, the logarithm of the initial portfolio wealth. Finally, we exponentiate all $i_{\text{max}} + 1$ values to obtain our wealth grid values,

W_0 through $W_{i_{\max}}$, noting the following: (1) One of the grid points will equal $W(0)$, which will be important for computing the probability distribution for the wealth in Sect. 2.5. (2) We define $W_{\min} = W_0$ and $W_{\max} = W_{i_{\max}}$, observing that the small downward shift will have created a small difference between $\hat{W}_{\min}$ and $W_{\min}$ and between $\hat{W}_{\max}$ and $W_{\max}$. (3) While the values of $\ln(W_i)$ where $i = 0, 1, 2, \ldots, i_{\max}$ are equally spaced, the values of W_i are not. The spacing between W_i values increases exponentially as the wealth increases.

2.4 Dynamic programming for optimizing the chance of obtaining the investor's goal

The value function, $V(W(t))$, is the probability that the investor will attain their goal wealth, G, or more at the time horizon T, given they have a worth $W(t)$ at time t. This means that at time T,

$$V(W_i(T)) = \begin{cases} 0 & \text{if } W_i(T) < G \\ 1 & \text{if } W_i(T) \geq G. \end{cases} \tag{5}$$

We next determine the Bellman equation so that we can determine V at year $t = T - 1$, then $t = T - 2$, etc., iterating backwards in time until we finish at $t = 0$. We begin by determining the transition probabilities, $p(W_j(t+1)|W_i(t), \mu)$. The transition probability is the normalized relative probability that we will be at the wealth node W_j at time $t + 1$ if we start at the wealth node W_i at time t and, between times t and $t + 1$, our portfolio is run with an expected return of μ and its corresponding volatility, σ, from Eq. (1). Defining $\phi(z)$ to be the value of the probability density function of the standard normal random variable at $Z = z$, we have from Eq. (2) the following probability density function values

$$\tilde{p}(W_j(t+1)|W_i(t), \mu) = \phi\left(\frac{1}{\sigma}\left(\ln\left(\frac{W_j}{W_i + C(t)}\right) - \left(\mu - \frac{\sigma^2}{2}\right)\right)\right). \tag{6}$$

Normalizing these probability density function values yields the desired transition probabilities:

$$p(W_j(t+1)|W_i(t), \mu) = \frac{\tilde{p}(W_j(t+1)|W_i(t), \mu)}{\sum_{k=0}^{i_{\max}} \tilde{p}(W_k(t+1)|W_i(t), \mu)}.$$

Since $V(W(t))$ is the expected value of $V(W(T))$, our Bellman recursion equation is simply

$$V(W_i(t)) = \max_{\mu \in [\mu_{\min}, \mu_{\max}]} \left[\sum_{j=0}^{i_{\max}} V(W_j(t+1))\, p(W_j(t+1)|W_i(t), \mu)\right]. \tag{7}$$

We denote $\mu_{i,t}$ as the value of μ at which the maximum is attained in the Bellman equation, and $\sigma_{i,t}$ is, of course, its corresponding volatility on the efficient frontier. As a computational matter, we select an integer m, divide the interval $[\mu_{\min}, \mu_{\max}]$ into an array of m equally spaced values, and let $\mu_{i,t}$ and $V(W_i(t))$ be determined from the μ value within this array that optimizes the sum in the right-hand side of the Bellman recursion equations. In practice $m = 15$ was generally sufficient to maintain accuracy in our results, as we further discuss in Sect. 4.2.1.

First setting $t = T - 1$, we solve the Bellman equation (7) to determine $\mu_{i,T-1}$ and $V(W_i(T-1))$ for each $i \in [0, i_{\max}]$. We then continue backwards in time to $t = T-2, t = T-3$, etc., until we reach $t = 0$. The value of $V(W(0))$ is the optimal probability of the investor attaining their wealth goal G from their initial wealth $W(0)$.

2.5 Probability distribution for the investor's wealth at future times

To determine the probability distribution for the investor's wealth at future times, we use the optimal strategy information, $\mu_{i,t}$ and $\sigma_{i,t}$, determined previously from dynamic programming to evolve the probability distribution forward in time, starting with $t = 0$, then $t = 1$, ending at $t = T - 1$. At any given value of t, we determine for each $j \in [0, i_{\max}]$

$$p(W_j(t+1)) = \sum_{i=0}^{i_{\max}} p(W_j(t+1)|W_i(t), \mu_{i,t}) \cdot p(W_i(t)). \tag{8}$$

Define i_0 so that W_{i_0} is the wealth node that equals $W(0)$. We then start at $t = 0$ with $p(W_{i_0}(0)) = 1$ and $p(W_i(0)) = 0$ for all $i \neq i_0$ to generate the entire set of probabilities for $t = 1$, i.e., $p(W_j(1))$ for each $j \in [0, i_{\max}]$. After that, moving forward in time, we recursively apply Eq. (8), until we obtain the probability distribution for the wealth nodes in every year of the lifetime of the portfolio.

2.6 Summary

We summarize the flow and meaning of the dynamic procedure from the investor point of view in four broad steps:

1. The investor determines an initial investment wealth, a goal wealth and a timeframe by which they hope to grow their initial investment into their goal wealth. The investor may also specify annual cash flows for the portfolio, i.e., infusions or withdrawals, if desired.
2. Lower and upper bounds on the mean along the efficient frontier are chosen. (Alternatively, these can be specified through lower and upper bounds on the risk, given by the corresponding standard deviations on the frontier.) These bounds, which determine the specific range of the efficient frontier to which we restrict portfolio choice, can depend on the investor's goal wealth and timeframe, as well as the desire to limit downside or increase upside, as we will explore in Sect. 4.3.3.
3. As the markets move up or down through the investment tenure, our program rebalances the portfolio so that the probability of reaching the goal continues to remain the highest.

4. At any given point in time, we keep track of the portfolio's current wealth, the conditional probabilities of transitioning to different wealth levels at future points in time, and also the probability of meeting the goal under the optimal strategy. This is information that enables the investor to keep track of the performance of the portfolio strategy.

An alternative solution approach is multi-stage stochastic linear programming using an asset-liability management context [see Birge and Louveaux (1997) for a good reference; examples in financial optimization, including wealth goals, are in Wallace and Ziemba (2005)]. This approach can be adapted to an objective function similar to our GBWM objective function. These optimization models are also flexible in allowing a multitude of possible constraints such as risk exposure limits, and they have the advantage of using discrete scenarios that do not require an assumption about the distribution of returns. On the other hand, they are computationally very hard and sometimes require heuristic solutions and small scenario trees.

3 Expanding our dynamic programming algorithm to other cases

As stated earlier, we can easily extend our results to cases where the available portfolios are not on the efficient frontier, and we can also extend our model from geometric Brownian motion to any other Markovian portfolio evolution model. Here are three additional extensions of our model:

3.1 Non-annual updates

Our main algorithm is written for annual updates. Annual updates particularly make sense in taxable accounts where short-term capital gains rates are levied on gains realized from stocks held less than a year. For other accounts, however, it might make more sense to update the portfolio every h years, where $h \neq 1$. For example, we might choose $h = 0.25$ if we want quarterly updates. In this context, the integer t now becomes the index of the update, so if $h = 0.25$, then $t = 4$ corresponds to the state of the portfolio after 1 year and, say, $T = 40$ means that we look to see if we have attained our goal, G, in the 40th update at the end of 10 years.

The main change needed to accommodate these cases where $h \neq 1$ is to note that since ht now represents the actual time represented by index t, Eq. (2) becomes

$$W(t) = W(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)ht + \sigma\sqrt{ht}Z},$$

where we note that $W(t)$ is the wealth at index t, which is time ht. This means that the Eqs. (3) and (4) for $\hat{W}_{\min}$ and $\hat{W}_{\max}$ become

$$\hat{W}_{\min} = \min_{\tau \in \{0,1,2,\ldots,T\}} \Bigg[W(0) e^{\left(\mu_{\min} - \frac{\sigma_{\max}^2}{2}\right) h\tau - 3\sigma_{\max}\sqrt{h\tau}} + \sum_{t=0}^{\tau} C(t) e^{\left(\mu_{\min} - \frac{\sigma_{\max}^2}{2}\right) h(\tau - t) - 3\sigma_{\max}\sqrt{h(\tau - t)}} \Bigg]$$

$$\hat{W}_{\max} = \max_{\tau \in \{0,1,2,\ldots,T\}} \Bigg[W(0) e^{\left(\mu_{\max} - \frac{\sigma_{\max}^2}{2}\right) h\tau + 3\sigma_{\max}\sqrt{h\tau}} + \sum_{t=0}^{\tau} C(t) e^{\left(\mu_{\max} - \frac{\sigma_{\max}^2}{2}\right) h(\tau - t) + 3\sigma_{\max}\sqrt{h(\tau - t)}} \Bigg].$$

It also means that we replace $\sigma_{\min}$ by $\sigma_{\min}\sqrt{h}$ if we continue to want to have ρ_{grid} wealth grid points for every period's minimum standard deviation, so $i_{\max}$ becomes $\frac{(\ln(\hat{W}_{\max}) - \ln(\hat{W}_{\min}))\rho_{\text{grid}}}{\sigma_{\min}\sqrt{h}}$ after rounding up to the nearest integer. Finally, it means that Eq. (6) becomes

$$\tilde{p}(W_j(t+1)|W_i(t), \mu) = \phi\left(\frac{1}{\sigma\sqrt{h}}\left(\ln\left(\frac{W_j}{W_i + C(t)}\right) - \left(\mu - \frac{\sigma^2}{2}\right)h\right)\right).$$

Otherwise, our main algorithm is the same.

3.2 Incorporating bankruptcy

When investor withdrawals, $C(t) < 0$, are sufficiently negative, they may cause the investor to go bankrupt. That is, we must consider how to alter our algorithm for these bankruptcy cases where $W_i + C(t) \leq 0$.

To do this, for each time t, we define $i_{pos}(t)$ to be the smallest index i such that $W_i + C(t) > 0$. (The notation "pos" reflects the fact that $W_i + C(t)$ is positive.) This means that for each $i < i_{pos}(t)$, we have a state of bankruptcy after the time t withdrawal, since $W_i + C(t) \leq 0$, while for each $i \geq i_{pos}(t)$, the investor still has money after the time t withdrawal. We note that an investor, once bankrupt, cannot attain their goal wealth, therefore $V(W_i(t)) = 0$ for any $i < i_{pos}(t)$.

If $i_{pos}(t)$ fails to exist at any time t, it means that $W_{i_{\max}} + C(t) \leq 0$ at this time, so, from the point of view of our algorithm, the investor is guaranteed to be bankrupt by time t. If $i_{pos}(t)$ exists for each t, we only need to make the following adjustments to our algorithm:

- In Sect. 2.3, the potential for bankruptcy means that Eq. (3) would make $\hat{W}_{\min} \leq 0$, which is undesirable. We therefore replace Eq. (3) with a value for $\hat{W}_{\min}$ that is positive, but near bankruptcy, like ten dollars, $\hat{W}_{\min} = 10$, or one dollar, $\hat{W}_{\min} = 1$, or even a penny, $\hat{W}_{\min} = 0.01$.

- In Sect. 2.4, we still determine the transition probabilities, $p(W_j(t+1)|W_i(t), \mu)$, for all j values, but now only for i values where $i \geq i_{pos}(t)$. After that, the Bellman equation (7) is only used to compute $V(W_i(t))$ for $i \geq i_{pos}(t)$. For $i < i_{pos}(t)$, we have that $V(W_i(t)) = 0$, as stated before.
- Finally, in Sect. 2.5, we alter Eq. (8) for the probability of being at a wealth node so that the summation over i is from $i_{pos}(t)$ to $i_{\max}$ instead of from 0 to $i_{\max}$.

We note that the probability of going bankrupt due to the withdrawal at a given time t is $\sum_{i=0}^{i_{pos}(t)-1} p(W_i(t))$, while the probability of being bankrupt prior to this time is $1 - \sum_{i=0}^{i_{\max}} p(W_i(t))$. In particular, since there is no cash flow at the final time T, the probability of the investor going broke by time T is $1 - \sum_{i=0}^{i_{\max}} p(W_i(T))$.

3.3 Multiple weighted goals

An investor may hope to attain a wealth goal, while at the same time also valuing the goal of not falling below a lower wealth threshold. Further, the investor may want to emphasize the relative importance of one of these wealth goals over the other. Mathematically, this corresponds to letting the lower wealth goal value, G_1, have a weight of w_1 and the higher wealth goal value, G_2, have a weight of w_2, where $w_1 + w_2 = 1$ and the higher w_1 is, the more important goal G_1 is relative to goal G_2. To accommodate these two goals, we simply replace the terminal value Eq. (5) with

$$V(W_i(T)) = \begin{cases} 0 & \text{if } W_i(T) < G_1 \\ w_1 & \text{if } G_1 \leq W_i(T) < G_2 \\ 1 & \text{if } W_i(T) \geq G_2, \end{cases}$$

and run the algorithm as before, noting that the value function V no longer represents the probability of attaining the goal wealth, since there is no longer a single goal wealth. If desired, we can easily extend this to k wealth goals $G_1 < G_2 < \cdots < G_k$ with weights $w_1, w_2, \ldots, w_k$ that sum to one, by replacing the terminal value Eq. (5) with

$$V(W_i(T)) = \begin{cases} 0 & \text{if } W_i(T) < G_1 \\ \sum\limits_{l=1}^{j} w_l & \text{if } G_j \leq W_i(T) < G_{j+1} \text{ for } j = 1, 2, \ldots, k-1 \\ 1 & \text{if } W_i(T) \geq G_k. \end{cases}$$

4 Results

In this section, we begin by describing a base case, which we then fine tune by adjusting our algorithm's parameters so as to optimize the algorithm's speed vs. accuracy trade-offs. We then demonstrate the results of the algorithm for a number of cases, both with and without periodic infusions and withdrawals. At the end of the section we demonstrate how our algorithm can be used to minimize the probability of an investor going bankrupt during retirement, how an investor can understand the effect of periodic

Table 1 Summary statistics on returns from January 1998 to December 2017 for our three index funds

Index fund category	Mean return	Covariance of returns		
U.S. Bonds	0.0493	0.0017	– 0.0017	– 0.0021
International Stocks	0.0770	– 0.0017	0.0396	0.03086
U.S. Stocks	0.0886	– 0.0021	0.0309	0.0392

infusions on the minimum probability of going bankrupt, and how our algorithm compares to target date fund performance for achieving goals.

4.1 Base case

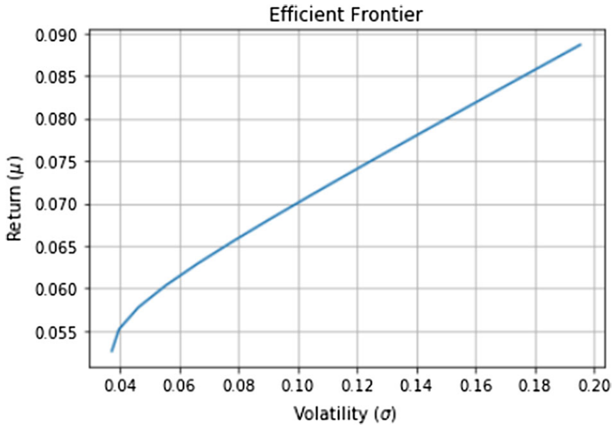
We create a base case from which we will later compute comparative statics. For our base case, we assume the investor begins with a \$100 investment at $t = 0$, so $W(0) = 100$. The investor's goal is to maximize their probability of reaching a goal wealth of $G =$ \$200 at the end of year $T = 10$. We note that the ratio of these wealth values is all that is important here. That is, the maximum probability of going from a wealth of \$100 to at least \$200 at $T = 10$ in our base case is the same as the maximum probability of doubling any initial investment at $T = 10$. So, for example, we can think of $W(0) = 100$ thousand dollars and $G = 200$ thousand dollars. The calculations are not changed.

No cash flows are present in the base case, although we will certainly explore their effect later. The base case value for m, the number of potential portfolios we consider along the efficient frontier, is 15. The base case value for ρ_{grid}, the grid point density per minimum annual standard deviation in the portfolio's performance, is 3.0. These values for m and ρ_{grid} have been chosen for reasons that will be explained in Sect. 4.2.

The efficient frontier arising from the investments available to the investor in our base case is exogenously determined. For illustrative purposes, we have generated this efficient frontier using historical returns from the 20 year period between January 1998 to December 2017 for index funds representing U.S. Bonds, International Stocks, and U.S. Stocks.[1] The mean and covariance of returns for these indexes are given in Table 1.

The data from this table is used in conjunction with the mathematics in Sect. 2.2 to generate the efficient frontier, shown in Fig. 1. The range for μ is restricted so that $\mu_{\min} \leq \mu \leq \mu_{\max}$. The bounds, $\mu_{\min}$ and $\mu_{\max}$, can be chosen through a variety of methods, including bounds on the investor's tolerance for risk via the standard deviation on the efficient portfolio or, as we will explore in Sect. 4.3.3, the investor's interest in limiting the downside or increasing the potential upside in the portfolio's final wealth distribution. For our base case, we have selected $\mu_{\min} = 0.0526$ to

[1] The three index funds used are (i) Vanguard Total Bond Market II Index Fund Investor Shares (VTBIX), representative of U.S. Fixed Income (Intermediate-Term Bond), (ii) Vanguard Total International Stock Index Fund Investor Shares (VGTSX), representative of Global Equity (Large Cap Blend), (iii) Vanguard Total Stock Market Index Fund Investor Shares (VTSMX), representative of U.S. Equity (Large Cap Blend). These three funds have been chosen only as representatives of their respective asset categories for illustrative purposes.



	Portfolio Weights		
Portfolio number	U.S. Bonds	International Stocks	U.S. Stocks
0	0.9098	0.0225	0.0677
1	0.8500	0.0033	0.1467
2	0.7903	-0.0160	0.2257
3	0.7305	-0.0352	0.3047
4	0.6707	-0.0545	0.3837
5	0.6110	-0.0737	0.4628
6	0.5512	-0.0930	0.5418
7	0.4915	-0.1122	0.6208
8	0.4317	-0.1315	0.6998
9	0.3719	-0.1507	0.7788
10	0.3122	-0.1700	0.8578
11	0.2524	-0.1892	0.9368
12	0.1927	-0.2085	1.0158
13	0.1329	-0.2277	1.0948
14	0.0731	-0.2470	1.1738

Fig. 1 Top: The efficient frontier generated from the data shown in Table 1 for the returns of our three indexes. Bottom: The table gives the portfolio weights in the three indexes for 15 consecutive portfolios on the efficient frontier, starting from the low point to the high point. Both long-only and long-short portfolios are permissible

correspond to the lowest possible portfolio standard deviation on the efficient frontier, which is $\sigma = 0.0374$. We have selected $\mu_{\max} = 0.0886$, the highest mean return of the three index funds, so as to avoid long-short portfolios. This value of $\mu_{\max}$ corresponds to $\sigma = 0.1954$. These numbers are realistic and match those in related research, see for example Exhibit 5 in Wang et al. (2011). From Sect. 2.3, we can now compute that there are 327 nodes in our wealth grid (i.e., $i_{\max} = 326$) ranging between $W_{\min} = 21.72$ and $W_{\max} = 1269$. We consider $m = 15$ portfolios on the efficient frontier whose μ values are equally spaced over the interval $[\mu_{\min}, \mu_{\max}]$. At each year and wealth node in the state space, the dynamic strategy determines which of these 15 portfolios is optimal.

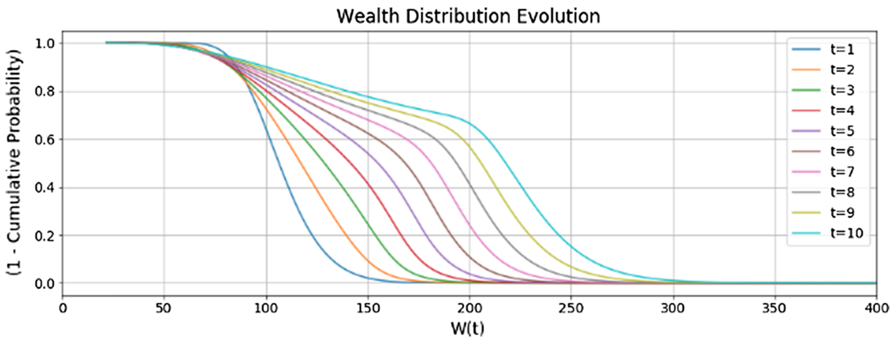


Fig. 2 Annual evolution of the probability distribution for wealth under the optimal dynamic strategy. The plot shows the probability of having at least the corresponding wealth on the curve at each time

Under the optimal dynamic strategy, we find that the highest achievable probability of reaching the goal wealth or more is $Pr[W(T) \geq G = 200] = 0.669$. The initial portfolio has $\mu = 0.0835$ and $\sigma = 0.1686$. Figure 2 shows the plot of one minus the cumulative distribution of wealth for all $t = 1, 2, \ldots, 10$ years. Using one minus the cumulative distribution is more natural than using the cumulative distribution in the context of this paper, because one minus the cumulative probability depicts the optimal probability of meeting a given level of wealth at each horizon, so higher values on the graph are better. For example, the point (100, 0.63) for the $t = 1$ curve means the investor has a 63% chance of having at least \$100 after a year, and, of course, the investor would have been happier if the graph had been higher since that would mean a higher chance of having at least \$100 after a year.

The evolution of the distribution in Fig. 2 shows the shift of these wealth distributions to the right as time proceeds, but, more interestingly, it also shows the adjustments the distribution shape makes so as to maximize the probability of exceeding the value of $G = 200$ by the final year $T = 10$. In the earlier years, the distribution has a slight positive skew, as is the case for a lognormal distribution, but the adjustments to attain the goal wealth eventually reverse this and create a negative skew to the distribution as it progresses to its time horizon. In Fig. 2, this corresponds to the development of a hump that evolves to finally lift the curve higher at $G = 200$ when $T = 10$, so as to maximize the probability of meeting the investor's goal wealth.

Figure 3 shows the optimal probability of reaching our goal wealth $G = 200$ (i.e., the value function) at each point in time and for any level of wealth in the state space; that is, at each $\{t, W_i(t)\}$ grid point. As is expected, higher wealth levels are associated with higher probabilities of reaching the goal wealth. The figure also reflects the fact that as we reach the final time $T = 10$, we have more certainty about whether we will attain the goal wealth.

In Fig. 4, we show the optimal portfolio strategy at each $\{t, W_i(t)\}$ grid point. That is, we show which of the 15 μ values are optimal, where portfolio number 0 corresponds to $\mu_{\min}$ (and the lightest color) and portfolio number 14 corresponds to $\mu_{\max}$ (and the darkest color). When the portfolio has a lot of money, it moves towards lower portfolio numbers, since the corresponding decrease in volatility makes it less likely to incur

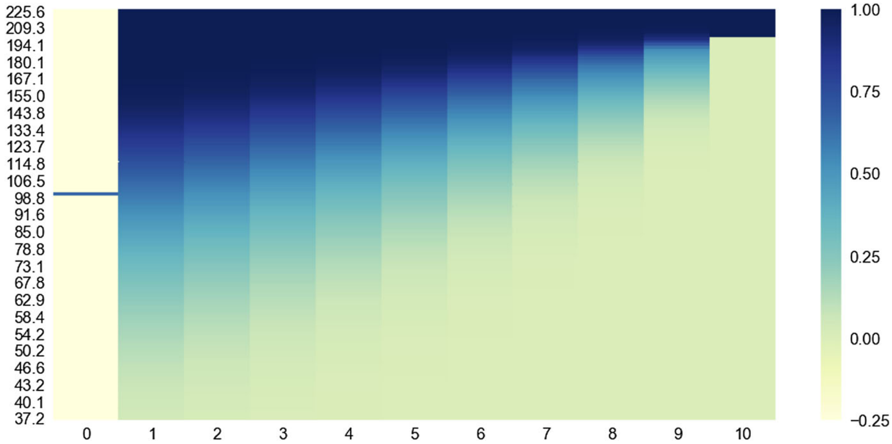


Fig. 3 The value function, which is the optimal probability of reaching the goal wealth $G = 200$ at each wealth node and time found by our dynamic strategy. The darker the color on the graph, the higher the value function is. The x-axis shows the time in years and the y-axis depicts the level of wealth. Note that the y-axis is an exponentially increasing scale. The figure truncates the displayed wealth values from its full range between $W_{\min} = \$21.72$ and $W_{\max} = \$1269$ to the range of wealths between \$37 and \$226 where the optimal probability varies

big losses that could remove investors from the path to attaining the goal wealth that they are currently on. When the portfolio has less money, it moves towards higher portfolio numbers, since the increase in both expected return and volatility makes it more likely to attain the goal wealth.

4.2 Fine tuning the parameters to balance algorithm speed vs. accuracy

4.2.1 Effect of changing, *m*, the number of portfolio strategies

In Table 2, we consider the effect on the base case of changing the number of intermediate strategies we consider on the efficient frontier between $\mu = \mu_{\min}$ and $\mu = \mu_{\max}$. Because the efficient frontier is part of a hyperbola, it becomes progressively linear as μ increases. This means the intermediate strategies matter more near $\mu_{\min}$. From Table 2, we see that $m = 15$, our base case value (denoted by an asterisk in this table, as will all base case values in later tables), is more than sufficient for providing enough accuracy in determining the probability of the initial investment $W(0) = 100$ gaining at least 50% and at least 100% of its initial worth after 10 years. The table also shows that the rate of growth in the run time as m increases is initially slightly sub-linear, but then becomes linear for larger m values.

4.2.2 Effect of changing the grid point density, ρ_{grid}

In Table 3, we consider the effect on the base case of changing the grid point density, ρ_{grid}, recalling from Sect. 2.3 that ρ_{grid} represents the number of grid points (i.e., wealth nodes) per $\sigma_{\min}$, the minimum annual portfolio volatility. Analyzing the run time data

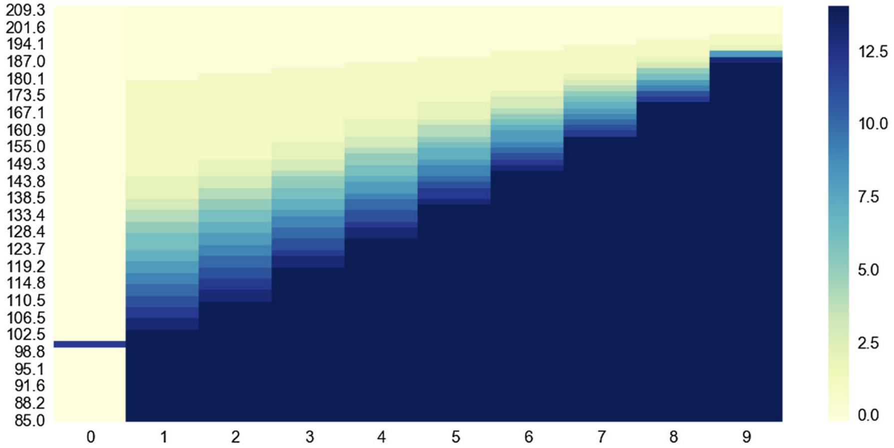


Fig. 4 Optimal portfolio strategy at each wealth node and time. Note that portfolio numbers run from 0, the lightest color corresponding to $\mu_{\min}$, to 14, the darkest color corresponding to $\mu_{\max}$, for the $m = 15$ model portfolios on the efficient frontier. The x-axis shows the time in years and the y-axis depicts the level of wealth on an exponentially increasing scale. The figure truncates the displayed wealth values from its full range between $W_{\min} = \$21.72$ and $W_{\max} = \$1269$ to the range of wealths between \$37 and \$226 where the portfolio choice varies

Table 2 Effect of changing m, the number of strategies

Value of m	Run time (s)	$Pr[W(T) \geq 150]$	$Pr[W(T) \geq G = 200]$
5	1.8	.775	.665
10	2.4	.776	.668
15*	2.9	.777	.669
20	3.4	.777	.669
40	8.5	.777	.669
60	12	.777	.669
80	15	.777	.669
100	18	.777	.669

*denote the base case

in Table 3 shows that the rate of growth in the run time as ρ_{grid} increases has an exponent of 1.5, directly between linear and quadratic growth. The optimal probabilities of the initial investment gaining at least 50% and at least 100% of its initial worth have some noise that dies down rather slowly as ρ_{grid} increases. We choose $\rho_{\text{grid}} = 3$ for the base case (again, denoted with an asterisk), since the noise is within reason while the run time has not grown too large.

4.3 The effect of the investor making changes

4.3.1 Effect of changing $W(0)$, the initial investment

At the end of Sect. 2.4, our algorithm has determined the value function V at time $t = 0$ for the investor's specified initial investment, $W(0)$, but it has also determined V

Table 3 Effect of changing ρ_{grid}, the grid point density

Value of ρ_{grid}	Run time (s)	$Pr[W(T) \geq 150]$	$Pr[W(T) \geq G = 200]$
1.0	0.64	.777	.662
1.5	1.1	.776	.673
2.0	1.6	.780	.676
2.5	2.2	.773	.666
3.0*	2.9	.777	.669
3.5	3.7	.780	.673
4.0	4.6	.779	.674
4.5	5.5	.778	.669
5.0	6.5	.777	.670
5.5	7.7	.779	.673
6.0	9.5	.778	.668

*denote the base case

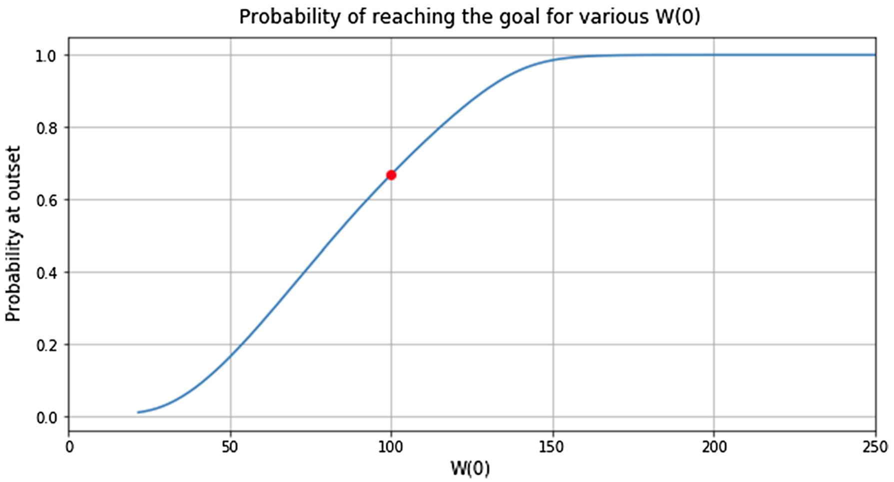


Fig. 5 Initial optimal probability of reaching the goal wealth $G = 200$ if we start at different levels of initial wealth $W(0)$. The red dot corresponds to our base case of $W(0) = 100$, where the optimal probability is 66.9%

at time $t = 0$ for all of the other wealth nodes between W_{min} and W_{max}. This enables an investor to determine the effect that increasing or decreasing their initial investment will have on V, the optimal probability of their attaining their wealth goal G at time T. In Fig. 5, we can see precisely how increasing the initial investment, $W(0)$, from its base case value of \$100, increases the optimal probability of attaining the goal of \$200 at $T = 10$.

We can also see the effect of decreasing the initial investment, seemingly until we reach the cutoff in the figure's graph at $W_{min} = 21.72$. However, near this cutoff, our results are questionable, since the accuracy of the graph near W_{min} requires that the value function at W_{min} be very near zero, which is not quite the case here. The remedy

Table 4 Effect of changing T, the portfolio's time horizon

Time horizon T	Run time (s)	Goal wealth, G	$Pr[W(T) \geq G]$
5	0.86	150	.574
10*	2.9	200*	.669
15	6.1	250	.763
20	10	300	.843
25	16	350	.913
30	23	400	.957
40	44	500	.995

*denote the base case

for this problem is easy: simply reduce $W_{\min}$ so that the full range of potential $W(0)$ values stays well above the newly chosen value of $W_{\min}$. Similarly, if $W_{\max}$ creates a cutoff, it should be increased.

4.3.2 Effect of changing T, the portfolio's time horizon

In Table 4, we look at the effect of changing the time horizon T for the portfolio. In this table, we let the goal wealth G increase linearly with T, although G actually scales exponentially with T. As a result, we see the associated probability of attaining G grow as T increases in the table. We also note from the table's data that the growth rate in the run time as T increases starts out being below quadratic growth for small T, but increases to quadratic growth for large T.

4.3.3 Effect of changing $\mu_{\min}$ or $\mu_{\max}$ for the efficient frontier truncation

Our algorithm can accommodate any given set of funds from which it then forms allowable portfolios along the efficient frontier. This has two benefits. First, because the funds are selected independently from the mechanics of the algorithm, the determination of the funds and the efficient frontier can be determined by a different operating team in the fund management business from the team running the dynamic programming algorithm. Further, if different sectors of the fund management business need to work with different funds, our algorithm can easily accommodate each sector separately. Second, the spread in the wealth distribution at time T can be controlled to some degree by changing the endpoints of the interval $\mu \in [\mu_{\min}, \mu_{\max}]$ that restrict the (σ, μ) pairs on the efficient frontier available to our algorithm. In this subsection we explore this second benefit by altering $\mu_{\min}$ and then $\mu_{\max}$ in our base case.

Recall that in our base case we consider 15 (σ, μ) pairs on the efficient frontier, with the lower end of the frontier at (0.0374, 0.0526) and the upper end at (0.1954, 0.0886), so $\mu_{\min} = 0.0526$ and $\mu_{\max} = 0.0886$. In the top panel of Fig. 6, we see the effect on the terminal distribution of wealth when we chose four different values for $\mu_{\min}$: 0.0526, 0.06, 0.065, and 0.07. As $\mu_{\min}$ increases, the probability of attaining the goal wealth $G = 200$ goes down since the interval of available controls shrinks. Also, the wealth distribution has a higher variance, as is to be expected. But these higher risk

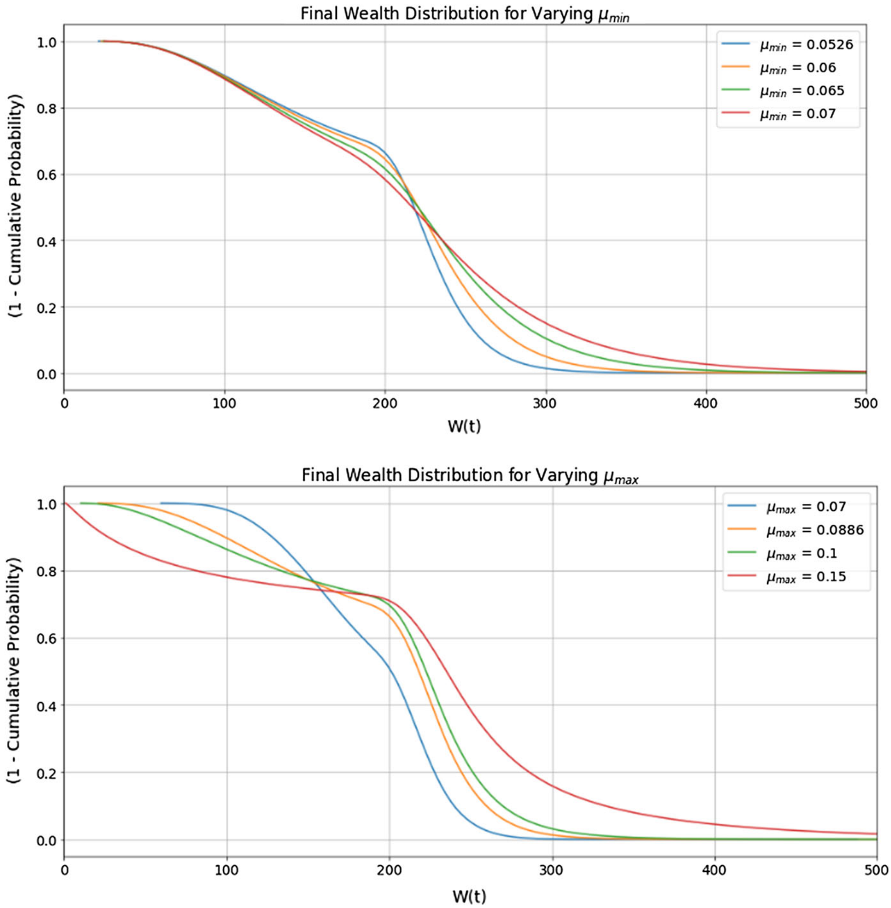


Fig. 6 Graphs for the probability distribution of the terminal wealth when $\mu_{\min}$ and $\mu_{\max}$ are varied. All other parameters from the base case remain the same. The ranges are: $\mu_{\min} = \{0.0526, 0.06, 0.065, 0.07\}$ and $\mu_{\max} = \{0.07, 0.0886, 0.10, 0.15\}$

distributions also have higher positive skewness, evident from the longer right tails. Therefore, choosing the value of $\mu_{\min}$ corresponds to choosing a trade-off between variance and skewness. More notably, the left tails of all three distributions are very similar, indicating that modulating $\mu_{\min}$ has a much stronger effect on the right side of the wealth distribution.

The lower panel of Fig. 6 shows the effect of varying $\mu_{\max}$ by considering four different $\mu_{\max}$ values: 0.07, 0.0886, 0.10, and 0.15. Again, the probability of attaining the goal wealth $G = 200$ increases as $\mu_{\max}$ increases, because the interval of available controls grows. However, in this case we notice that both the left and right sides of the probability distribution are affected, although the effect on the downside is more pronounced than that on the upside, suggesting that varying $\mu_{\max}$ has a greater effect on the left tail.

Table 5 Effect of changing annual infusions, $C(t) \geq 0$

Value of $C(t)$	$Pr[W(T) \geq 150]$	$Pr[W(T) \geq G = 200]$
0*	.777	.669
1	.832	.730
2	.881	.789
3	.926	.848
4	.961	.901
5	.984	.944
6	.996	.976
7	.999	.992
8	.999	.998
9	.999	.999

*denote the base case

An investor that is more accepting of risk would tend to first want to increase $\mu_{\max}$, but Fig. 6 suggests that increasing $\mu_{\min}$ might be the wiser course of action, since we can see in this case that increasing $\mu_{\min}$ appears to have a stronger influence on the upside potential whereas increasing $\mu_{\max}$ appears to have a stronger influence on the downside, risking significant losses without that much compensating gains.

The reason for this is actually straightforward: The algorithm is only interested in attaining the goal wealth, so the optimal strategy for a well-off investor is to move μ to $\mu_{\min}$ so as to reduce volatility and the chance of major losses resulting in no longer being on track to attain the goal wealth. Because the well-off investor uses $\mu_{\min}$, increasing $\mu_{\max}$ has little effect on the right tail, while increasing $\mu_{\min}$ has a significant effect. Similarly, when the investor is worse off, they select $\mu_{\max}$ because that increases both μ and σ, which increase the chance of big gains and attaining the goal wealth. Of course this also increases the chance of big losses, which inflates the left tail of the wealth distribution.

4.3.4 The effect of cash flows: infusions or withdrawals

1. Annual infusions: $C(t) > 0$

   We continue to work with our base case where we have an initial investment of $W(0) = 100$ thousand dollars and a goal of having at least $W(T) = G = 200$ thousand dollars at the end of $T = 10$ years. In Table 5, we look at how constant annual infusions of $C(t) = 1, 2, \ldots, 9$ thousand dollars affect the maximum probability of reaching this goal, as well as the probability of reaching at least 150 thousand dollars. We note from the table that even small infusions can have a significant effect on increasing these probabilities.
2. Annual withdrawals: $C(t) < 0$ and the probability of going bankrupt.

   We now look at the same situation, but with constant annual withdrawals instead of infusions, so $C(t)$ is now a negative constant. Should the annual withdrawal

Table 6 Effect of changing annual withdrawals, $C(t) \leq 0$

Value of $C(t)$	$Pr[W(T) = 0]$	$Pr[W(T) \geq 150]$	$Pr[W(T) \geq G = 200]$
0*	0	.777	.669
− 1	0	.720	.609
− 5	.002	.491	.387
− 10	.124	.246	.182
− 15	.492	.099	.072
− 20	.796	.034	.021
− 25	.937	.004	.001

*denote the base case

amount become significant, the investor will now risk bankruptcy (i.e., $W(T) = 0$). In Table 6, we see how increasing the withdrawal rate increases the chance of the investor going bankrupt, while decreasing both the probability of reaching 150 thousand dollars and the probability of reaching the goal wealth of 200 thousand dollars at time $T = 10$.

4.3.5 Attaining retirement goals: our algorithm versus a target date approach

Our algorithm can be used to solve a variety of important problems for optimizing retirement savings. Here, for example, we consider the effect of infusions on the maximized probability of not running out of money in retirement. In particular, we consider at $t = 0$ a 50 year old investor who currently has 100 thousand dollars in their retirement account and intends to take out 50 thousand present day dollars every year after they turn 65 through the age of 80, where $t = T = 30$. We assume that the annual rate of inflation is 3%, so, in thousands of dollars, that means that at age 66, $C(t) = -50 \cdot 1.03^{16}$, at age 67, $C(t) = -50 \cdot 1.03^{17}$, up through age 79 where $C(t) = -50 \cdot 1.03^{29}$. Because $C(t)$ isn't defined at time T, which corresponds to when the investor is 80, we need to set the goal wealth G equal, in thousands of dollars, to $G = 50 \cdot 1.03^{30} = 121.4$, so that the investor can make their final withdrawal without going bankrupt.

Our algorithm now optimizes the chance that the investor does not go bankrupt, but it finds, unfortunately, that this optimal probability is only 12.8%. The investor, therefore, considers making infusions of c thousand present day dollars each year until retiring at age 65, starting with an infusion of $c \cdot 1.03$ at the age of 51. The effect of increasing c on the maximized probability of remaining solvent at age 80 is given in the first two columns of Table 7.

Target Date Funds (TDFs) play an important role in the space of retirement investing. They provide a logical investment strategy that has the advantage of being customized to the age of an investor. Our GBWM approach allows the investment strategy to be further customized to the investor's needs by considering their goals, timeframe, cash flows, and state of wealth over time, in addition to their age.

To quantify the advantages this additional customization provides, we have, for comparison to our GBWM results, created a hypothetical TDF using the same three

Table 7 Effect of changing pre-retirement infusions, c, for our algorithm with the goal of staying solvent and for our Target Date Fund

Value of c	Our GBWM algorithm: probability of staying solvent (%)	Our Target Date Fund: probability of staying solvent (%)
0	12.8	0.7
5	25.8	3.9
10	42.0	11.9
15	58.6	26.6
20	73.5	45.0
25	85.4	62.7
30	93.8	77.0
35	98.4	86.6
40	99.8	92.8

Table 8 Our Target Date Fund glide path

Age range	50–54 (%)	55–59 (%)	60–64 (%)	65–69 (%)	70–74 (%)	75–80 (%)
1. U.S. Stock	44	40	35	28	20	18
2. International Stock	29	26	23	19	13	12
3. U.S. Bond	27	34	42	53	67	70

index funds representative of U.S. Bonds, International Stocks, and U.S. Stocks used by our GBWM algorithm, along with a typical glide path, shown in Table 8, for an investor transitioning from age 50 to age 80. Using the allocations from the glide path in this table, we use simulation to determine the probabilities of an investor remaining solvent at age 80. This gives us the results shown in the Target Date Fund column in Table 7.

We see from Table 7 that our algorithm shows a significantly higher ability to achieve the investor's goal of staying solvent. More specifically, the advantage in using our GBWM algorithm is greater than 30 percentage points when $c = 10$ or 15 and remains high for the other values of c as well.

There are a number of reasons for our algorithm's considerable outperformance. One reason is that the TDF retirement allocations are not generally on the efficient frontier, so they incur some additional volatility that is not compensated by increased expected returns. A second reason is that the allocation within TDFs is time dependent, but does not depend on the level of the investor's wealth, whereas our algorithm's allocation strategy depends on both time and the level of wealth. Finally, our algorithm takes into account the investor's stated goals and decisions, specifically the infusions and the withdrawals the investor has pre-determined, as well as the investor's timeframe and the goal of staying solvent at the end of this timeframe. By allowing our optimization to be customizable to an individual investor's circumstances, specifications, and goals, our algorithm gains a considerable advantage over the one-size-fits-all nature of target date funds. Table 7 shows that these differences are not small.

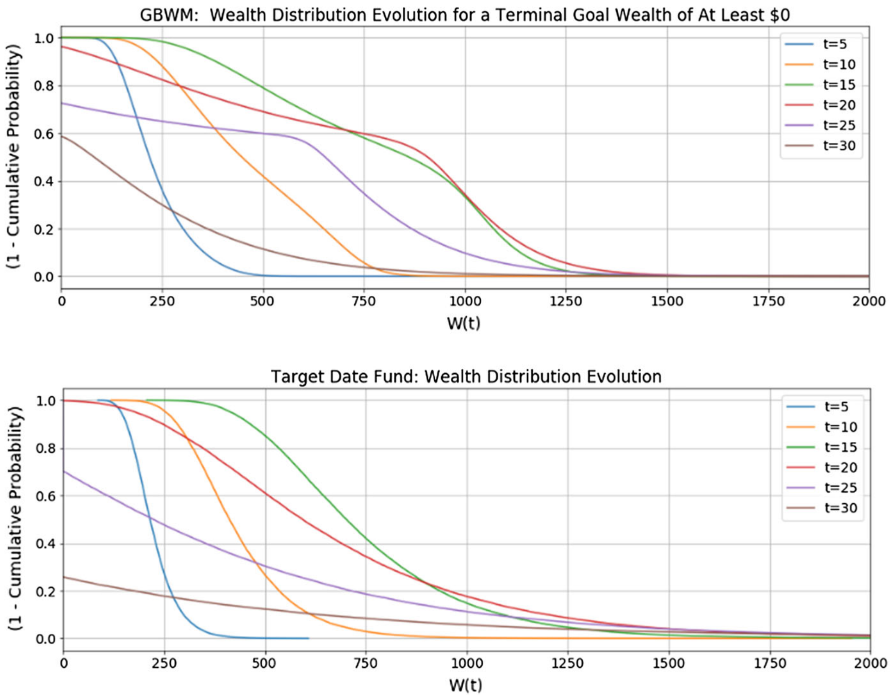


Fig. 7 The distribution of wealth, given as one minus the cumulative distribution function, as it evolves across $t = \{5, 10, 15, 20, 25, 30\}$ years when the contribution level, c, is 15 from $t = 1$ to $t = 15$. The top panel depicts the wealth evolution when the goal of our GBWM methodology is to remain solvent at $T = 30$. We note that wealth curve presented for $t = T = 30$ already has the final goal wealth, $G = 121.4$, subtracted from it, since G is to be withdrawn at $T = 30$. The bottom panel shows the wealth evolution for our Target Date Fund. Comparing the two $t = 30$ curves at $W = 0$, we see the jump from 26.6 to 58.6% in the probability of remaining solvent that our GBWM strategy provides, as shown in Table 7

We can better understand the trade-offs between our GBWM algorithm and our Target Date Fund by comparing the two panels in Fig. 7, where we present one minus the cumulative probability distributions at $t = \{5, 10, 15, 20, 25, 30\}$ years for our GBWM methodology and our Target Date Fund. For this figure we have chosen $c = 15$. We note that the right tail of our Target Date Fund at $t = 30$ is higher. We also note at times like $t = 20$ that the chance of not going bankrupt, which is the value of the graph at $W = 0$, is slightly lower for our GBWM methodology than our TDF. These are the trade-offs that lead to our GBWM methodology attaining its goal of a much higher probability of not going bankrupt than our TDF (58.6% versus 26.6%) at $t = 30$.

In Table 9, we change the goal of our algorithm from staying solvent to ending with a balance at or above $500,000. This has no effect on our Target Date Fund numbers, of course, but for our algorithm, as expected, it lowers the probability of staying solvent, while increasing the probability of ending above $500,000. Again, we see that the advantage in using our GBWM methodology is greater than 30 percentage points, this time when $c = 15, 20$, or 25, and, again, it remains high for the other values of c as well. The effect of changing the GBWM goal to having a balance at or above $500,000

Table 9 Effect of changing pre-retirement infusions, c, for our algorithm with the goal of ending with at least \$500,000 and for our Target Date Fund

Value of c	Our algorithm: probability of ending with $\geq$ \$500,000 (%)	Our Target Date Fund: probability of ending with $\geq$ \$500,000 (%)
0	8.9	0.2
5	18.5	1.2
10	31.2	4.2
15	45.1	12.0
20	58.7	24.5
25	70.9	39.9
30	81.2	55.6
35	89.2	69.5
40	94.9	80.1

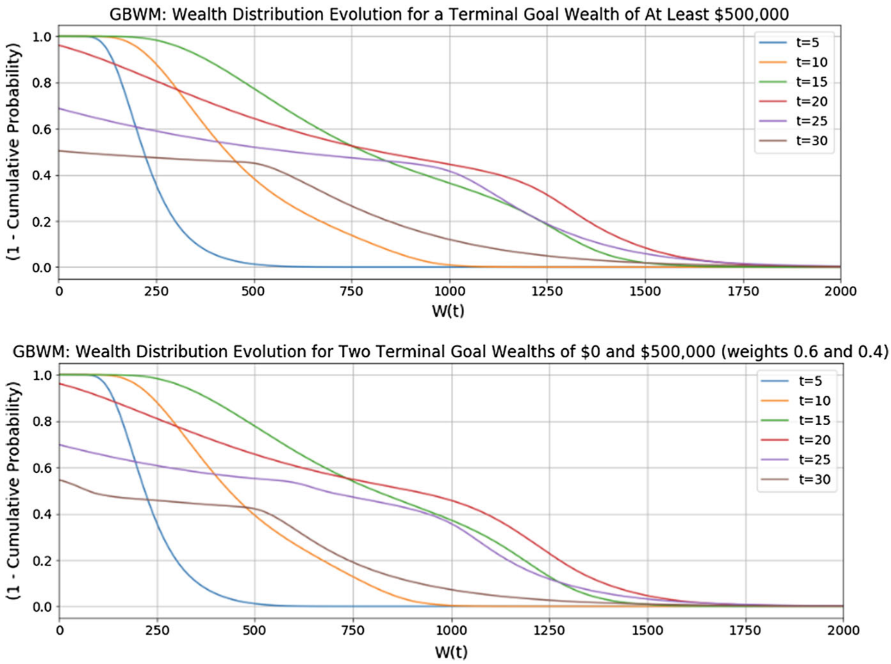


Fig. 8 The distribution of wealth, given as one minus the cumulative distribution function, as it evolves across $t = \{5, 10, 15, 20, 25, 30\}$ years when the contribution level, c, is 15 from $t = 1$ to $t = 15$. The top panel depicts the wealth evolution when the goal of our GBWM methodology is to deliver an amount greater than \$500,000 at $T = 30$. The bottom panel displays the wealth evolution when we split the goal for our GBWM methodology between staying solvent with a 60% weight and delivering an amount greater than \$500,000 with a 40% weight at $T = 30$. In both panels, the wealth curve presented for $t = T = 30$ already has the final goal wealth, $G = 121.4$, subtracted from it, since G is to be withdrawn at $T = 30$

Table 10 Effect of changing pre-retirement infusions, c, for our algorithm with the split goal of staying solvent with a 60% weight and ending with at least \$500,000 with a 40% weight

Value of c	Our algorithm: probability of staying solvent (%)	Our algorithm: probability of ending with ≥ \$500,000 (%)
0	11.8	8.1
5	23.8	16.9
10	38.9	28.8
15	54.5	42.2
20	68.7	55.5
25	80.5	67.8
30	89.5	78.5
35	95.6	87.2
40	98.8	93.8

on the wealth distribution over time can be seen in the top panel of Fig. 8. Note in this panel the evolution of the advantageous kink that finally lands at \$500 thousand in the $t = T = 30$ curve.

Finally, in Table 10, we show the results of dividing our goal, as discussed in Sect. 3.3, between staying solvent with a 60% weight and ending with a balance at or above \$500,000 with a 40% weight. Comparing the results in Table 10 with the results in Table 7, we see that having a 60% weight, as opposed to the full weight, on the goal of staying solvent leads to losses in the probability for attaining this goal of less than 5 percentage points, while comparing Table 10 with Table 9 for the goal of attaining at least \$500,000, we only observe losses of less than 4 percentage points. The evolution of the distribution for this mixed GBWM goal when $c = 15$ can be seen in the bottom panel of Fig. 8. Note that, as one would expect, we now see two advantageous kinks at \$0 and \$500 thousand in the $t = T = 30$ curve.

5 Concluding discussion

In this paper, we have developed an algorithm that can quickly determine (generally within a few seconds to a minute) an optimal dynamic trading strategy for goals-based wealth management (GBWM). The objective function we maximize in our GBWM framework is the probability of reaching a given goal wealth at the end of a given investment horizon, in contrast to maximizing the expected utility of the wealth at the end of a given time horizon. Without any sacrifice in runtime, our algorithm can optimally allocate from among any given set, large or small, of chosen portfolios, which can be determined outside of our algorithm. Also, without any sacrifice in runtime, our algorithm can accommodate periodic specified infusions or withdrawals. Further, we can easily compute the probability of running out of money at each period due to any of these withdrawals.

The dynamic programming approach, whose computation works backwards in time, has important advantages over approaches that work forwards in time. Dynamic programming addresses the fact that the optimal portfolio allocation will shift over time.

Attempting to consider this with a forwards in time algorithm is computationally infeasible, so these algorithms are reduced to myopic approaches, where the allocation chosen at a point in time is only optimal under the assumption that no further reallocation will occur. That is, the forwards in time algorithms are restricted to the inferior approach of repeated static optimization. Interestingly, despite this, the results from the repeated static optimization method described in Das et al. (2018) end up being very similar to the results obtained from our dynamic programming approach, however, that paper's method, which relies on the solution to a stochastic differential equation (SDE), cannot be extended to infusions or withdrawals and it has a longer computational time. Most forwards in time approaches rely on simulation instead of solving an SDE, which leads to even longer computational times to be reasonably accurate.

We can modulate the left and right tails of the terminal wealth distribution by varying the chosen range of allowable portfolios on the efficient frontier. If we remove either the most risky investments at the top of this range or the least risky investments at the bottom of this range, we lower the probability of reaching our goal wealth, since we are restricting the set of controls available to the algorithm. However, removing the most risky investments has the benefit of lowering the left tail in the distribution, as it lowers the chance of significant losses, while removing the least risky investments has the benefit of raising the right tail in the distribution, making it more likely to attain wealth values that are significantly higher than the goal wealth.

Target date (or life-cycle) funds have the important feature of being able to reallocate funds based on time, specifically on the age of the investor. They cannot, however, reallocate based on the investor's wealth or the investor's specific goals. Nor are the allocations underlying a target date fund necessarily on the efficient frontier. The natural question arises as to weather incorporating these GBWM features and efficiency constraints, which require individualized attention through automation and/or human advisors, adds much value to the investor. Our paper concludes that they can actually add considerable value. We explored the case of a 50 year old saving for retirement at age 65, who then takes out inflation adjusted withdrawals until the age of 80. Working with just three index funds representative of U.S. Bonds, International Stocks, and U.S. Stocks, we show examples where our algorithm's optimal dynamic reallocation strategy increases the chance of this investor remaining solvent at age 80 by more than 30 percentage points over using a Target Date Fund during this period.

The approach in this paper can be expanded to answer questions arising in other important financial situations. These include incorporating the effect of taxes or incorporating stochastic efficient frontier models. Because our algorithm runs so quickly, it is unlikely that these additional features will degrade the runtime particularly significantly.

References

Birge JR, Louveaux FV (1997) Introduction to stochastic programming. Springer, New York

Browne S (1995) Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin. Math Oper Res 20(04):937–958

Browne S (1997) Survival and growth with a liability: optimal portfolio strategies in continuous time. Math Oper Res 22(02):468–493

Browne S (1999a) Reaching goals by a deadline: digital options and continuous-time active portfolio management. Adv Appl Probab 31(02):551–577

Browne S (1999b) The risk and rewards of minimizing shortfall probability. J Portfolio Manag 25(04):76–85

Browne S (2000) Risk-constrained dynamic active portfolio management. Manag Sci 46(09):1188–1199

Brunel JLP (2015) Goals-based wealth management: an integrated and practical approach to changing the structure of wealth advisory practices. Wiley, Hoboken

Chhabra A (2005) Beyond Markowitz: a comprehensive wealth allocation framework for individual investors. J Wealth Manag 7(4):8–34

Cox JC, Huang C (1989) Optimal consumption and portfolio policies when asset prices follow a diffusion process. J Econ Theory 49(1):33–83

Das SR, Markowitz H, Scheid J, Statman M (2010) Portfolio optimization with mental accounts. J Financ Quant Anal 45(2):311–334

Das SR, Ostrov D, Radhakrishnan A, Srivastav D (2018) Goals-based wealth management: a new approach. J Invest Manag 16(3):1–27

Deguest R, Martellini L, Milhau V, Suri A, Wang H (2015) Introducing a comprehensive risk allocation framework for goals-based wealth management. EDHEC-Risk Institute, Lille

Kahneman D, Tversky A (1979) Prospect theory: an analysis of decision under risk. Econometrica 47:263–291

Lopes L (1987) Between hope and fear: the psychology of risk. Adv Exp Soc Psychol 20:255–295

Markowitz H (1952) Portfolio selection. J Finance 6:77–91

Merton RC (1969) Lifetime portfolio selection under uncertainty: the continuous-time case. Rev Econ Stat 51(3):247–257

Merton RC (1971) Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory 3(4):373–413

Nevins D (2004) Goals-based investing: integrating traditional behavioral finance. J Wealth Manag 6(4):8–23

Roy AD (1952) Safety-first and the holding of assets. Econometrica 20:431–449

Shefrin HM, Statman M (1985) The disposition to sell winners too early and ride losers too long: theory and evidence. J Finance 40:777–790

Shefrin HM, Statman M (2000) Behavioral portfolio theory. J Financ Quant Anal 35(2):127–151

Thaler RH (1985) Mental accounting and consumer choice. Mark Sci 4:199–214

Thaler RH (1999) Mental accounting matters. J Behav Decis Mak 12:183–206

Wallace SW, Ziemba WT (eds) (2005) Applications of stochastic programming. MPS-SIAM mathematical series on optimization, SIAM and MPS, Philadelphia

Wang H, Suri A, Laster D, Almadi H (2011) Portfolio selection in goals-based wealth management. J Wealth Manag 14(1):55–65