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# Approximating feeder line reliability statistics with partial data collection in assembly systems

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**Abstract**

Due to the difficulty and high cost of data collection, many feeder lines in assembly systems lack full collection of data. However, reliability statistics of feeder lines are important in throughput analysis and continuous improvement of manufacturing systems. In this paper, a simple approximation approach is presented to estimate the reliability statistics of feeder lines from the associated assembly station's collected blocking and starving information. It is shown that the approach is helpful for accurate throughput estimation and sensitivity analysis. We also show how feeder line speed can be used to improve the approximation.


*Keywords:* Reliability; Data collection; Feeder line; Throughput; Sensitivity

## 1. Introduction

Feeder lines are common in many manufacturing plants to build subassemblies that feed the main line. In modern manufacturing environment, more and more feeder lines or subassemblies are introduced to increase the flexibility and agility of the main line. A typical assembly system consists of one or more feeder lines which join a main line at an assembly station. Accurate throughput analysis of assembly systems is important for design, operation and management, and it has attracted a large amount of research efforts (see monographs by Altiok (1997, Chapter 6) and Gershwin (1994, Chapter 5) and, representative papers by Chiang, Kuo, Lim, and Meerkov (2000a,b), Di Mascolo, David, and Dallery (1991), Gershwin and Burman (2000), Helber (1998), Kouikoglou (2002), Liu and Buzacott (1990), and Simon and Hopp (1991)).

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To perform accurate throughput analysis, reliability statistics of all workstations (including feeder lines) are needed. However, on the factory floor, due to the difficulty and high cost of installation and maintenance, data collection mechanisms are not always available in all feeder lines in some assembly systems. But the reliability statistics of the feeder lines can affect system throughput; in particular, a feeder line can be the system bottleneck. Therefore, accurate analysis of feeder line performance is necessary and important. To this end, a method is needed to estimate the feeder line reliability statistics. To our best knowledge, the current literature does not offer such a method to enable us to retrieve the feeder line downtime information in the partial data collection environment. The main contribution of this paper is to provide a simple approach to approximate the feeder line reliability statistics from the starving and blocking data of the associated assembly station.

The remainder of the paper is structured as follows: Section 2 introduces the data collection procedures. The approximation approach and verification examples are presented in Sections 3 and 4, respectively. Section 5 formulates the conclusions. The proof is provided in the Appendix.

## 2. Data collection in assembly systems

Due to the difficulty and high cost of data collection, many assembly systems lack full data collection in feeder lines. In addition, in many automotive assembly plants, the data collection on the assembly station is performed in a sequential manner which does not agree with the general assumptions used in many analytical and simulation models. Fig. 1 illustrates a typical assembly system where the circles represent the work stations and the rectangles are the buffers.

When the assembly station is up, the following steps describe a typical data collection procedure of the assembly station used in many assembly plants:

- *Step 1*. Check blockage of the assembly station. If the station is blocked, i.e. downstream buffer is full, then collect blocking data until there is space available in the downstream buffer.

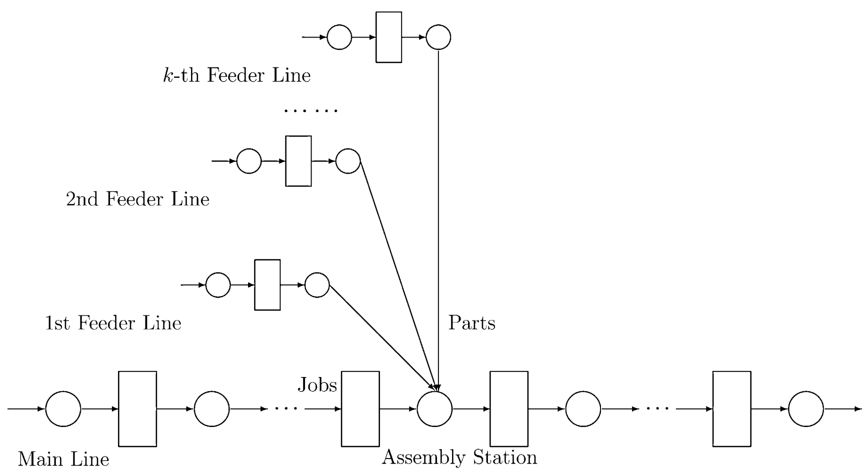


Fig. 1. Assembly system with $k$ feeder lines.

- *Step 2*. Check job availability of the main line upstream buffer. Collect starving data for the main line if the assembly station is starved, i.e. upstream buffer is empty. Load a job from the main line upstream buffer when a job is available.
- *Step 3*. Check part availability of subassemblies (or feeder lines) in the following order:
  - *Step 3.1*. Check feeder line 1 first. If a part is not available, then collect starving data for the first feeder line. Load a part when it is available.
  - *Step 3.2*. Check part availability of the second feeder line. Collect the starving data if the assembly station is starved. Load a part when it is available.
  - *Step 3.3*. Continue this process until parts have been loaded from all feeder lines. Finally, begin operation at the assembly station.

In many manufacturing plants, when the data collection in feeder lines is not available, the collected starving data for each feeder line is often used as its downtime data in throughput analysis. As one can see, the collected starving time is shorter than the actual downtime due to the sequential manner of data collection. For example, the assembly station may already be starved by the $i$th feeder line when it is still awaiting the part availability of $i-1$th feeder line. Thus, the estimate of system throughput by analytical or simulation tools using these collected values directly will be higher than the actual throughput. In addition, most of the analytical models and simulation packages assume synchronous loading of all assemble parts from main and feeder lines (Chiang et al., 2000a,b; Di Mascolo et al., 1991; Chapter 5, Gershwin, 1994; Gershwin & Burman, 2000). Moreover, directly using starving time as a feeder line's downtime may lead to incorrect sensitivity analysis. If the assembly station's starving time due to feeder line's slow speed is assumed to be the feeder line's downtime, then sensitivity analysis of increasing the feeder line speed will underestimate actual throughput increment. This is because the 'downtime' will remain the same under this assumption when, in reality, this 'downtime' disappears with the increment of feeder line speed. Hence, this approach can produce misleading results and even wrong sensitivity analysis. Therefore, more accurate approximation of feeder lines' actual reliability statistics is necessary. In this paper, we present a simple method to approximate the mean time to repair (MTTR) and mean time between failures (MTBF) of a feeder line using the blocking and starving data of the a assembly station. This is illustrated in the next Section.

## 3. Approximation approaches

We discuss two approximation approaches in this Section: feeder line speed is known or unknown. To simplify the discussion and reduce the notation, consider a simplified assembly system shown in Fig. 2.

**Remark 1**. The feeder stations $M_2$ through $M_k$ in Fig. 2 can also be viewed (approximately) as the aggregated stations of feeder lines 1 to $k-1$ in Fig. 1, respectively. Hence, the analysis based on Fig. 2 still applies.

Introduce the following notation for assembly station $M_0$ and feeder stations $M_i$, $i=1, \ldots, k$:

$\tilde{P}_i^s$ fraction of time $M_0$ is starved by $M_i$ in collected data, $i=1, \ldots, k$,
$P_i^s$ fraction of time $M_0$ is starved by $M_i$ in approximation model, $i=1, \ldots, k$,

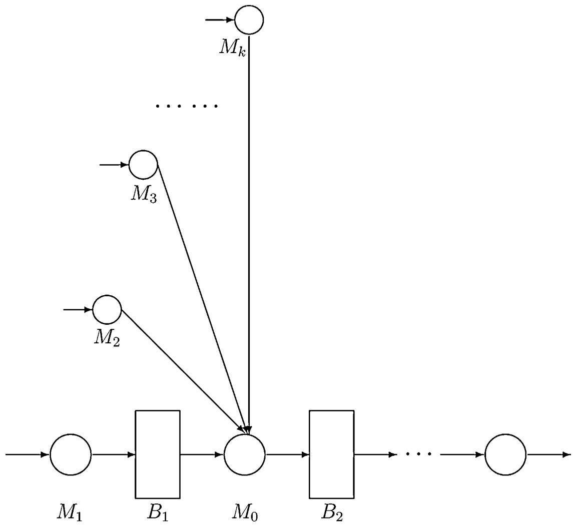


Fig. 2. Simplified assembly system with $k-1$ feeders.

| | |
|---|---|
| $\tilde{P}_0^b$ | fraction of time $M_0$ is blocked in collected data, |
| $P_0^b$ | fraction of time $M_0$ is blocked in approximation model, |
| $e_i$ | efficiency of $M_i$, $i=0, 1,\ldots,k$, |
| $T$ | total planned run time, |
| $S_i$ | speed of $M_i$, |
| $\tilde{n}_i^s(T)$ | number of starvation of $M_0$ by $M_i$ in collected data during $T$, $i=1, \ldots, k$, |
| $\mathrm{MTBS}_i$ | mean (over planned run) time between starving (MTBS) of $M_0$ by $M_i$, i.e. |

$$\mathrm{MTBS}_i = \frac{T}{\tilde{n}_i^s(T)}, i = 1, \ldots, k.$$

**Remark 2**. Based on the current data collection procedure described in Section 2, blockage is always checked first. As a result, the fraction of time $M_0$ is blocked will be same in both collected data and approximation model, i.e.

$$P_0^b = \tilde{P}_0^b. \tag{1}$$

In order to retrieve the downtime data of feeder stations $M_2$ to $M_k$, speed of each feeder must be provided (or assumed). First, we discuss the case where all feeders have the same speed as that of the assembly station. We then extend the results to the case of unequal feeder speeds.

### 3.1. Case 1: feeder speed is unknown

In the worst case, the feeder line speed is unknown. In this case, we assume that feeder line speed is equal to the assembly station speed since this is a reasonable and common line design practice. In order to derive the approximation formula, we introduce the following assumptions (which are similar to the assumptions introduced by Buzacott and Shanthikumar (1993, Chapter 5)):

(i) All stations have identical cycle time to process a part. Hence, $\tilde{S}_i = \tilde{S}_0 = 1$ part/cycle, $\forall i$.
(ii) Parameter $p_i$ is the probability of failure of station $i$ in a cycle, and $r_i$ is the probability of completing repair of station $i$ in a cycle. The up and down times are assumed to follow geometric distributions.
(iii) If buffer is empty, then downstream station is starved during this cycle.
(iv) If buffer is full at the beginning of the cycle, then upstream station is blocked during this cycle.
(v) The change in inventory level is determined by the state at the beginning of the interval.
(vi) In addition, the following assumption is introduced to simplify the analysis. Assume that

$$\Pr\{B_{\text{down}} \text{ is full, } B_{\text{up}} \text{ is empty}\} = \Pr\{B_{\text{down}} \text{ is full}\}\Pr\{B_{\text{up}} \text{ is empty}\},$$

where $B_{\text{up}}$ and $B_{\text{down}}$ are the upstream and downstream buffers of any arbitrary station, respectively.

**Remark 3**. In general, the joint probability (for example, the probability that buffers $B_{\text{up}}$ and $B_{\text{down}}$ are empty and full, respectively) is not close to the product of its marginal distributions. It turns out, however, that for certain values of buffer occupancies (buffer full or empty) which are related to blockages and starvations, they are indeed close. Jacobs and Meerkov (1995) show that results based on assumption (vi) are generally accurate. In addition, Gershwin (p. 76, 1994) claims that this joint probability is very small and can be neglected.

The following adjustment is introduced to approximate a feeder's reliability statistics.

**Proposition 1**. *Under assumptions (i)–(vi), the fraction of time that assembly station $M_0$ is starved by feeder station $M_{\text{i}}$, $i=1,\ldots,k$, is*

$$P_1^s = \frac{e_0 \tilde{P}_1^s}{e_0 - \tilde{P}_0^b}, \tag{2}$$

$$P_i^s = \frac{e_0^i \tilde{P}_i^s}{(e_0 - \tilde{P}_0^b)\prod_{j=1}^{i-1}(e_0 - P_j^s)}, \quad i = 1, \ldots, k. \tag{3}$$

**Proof**. See Appendix

As we introduced before, an approach used in practice is to use $P_i^s$, $i=2, \ldots, k$, to approximate the probability that feeder station $M_i$, $i=2, \ldots, k$, is down. To derive the reliability estimates under this assumption, let $\text{MTTR}_i^{\text{iden}}$ and $\text{MTTR}_i^{\text{iden}}$ denote the estimates of MTTR and MTBF of station $M_i$ when feeder speed is (or assumed to be) identical to the speed of $M_0$, respectively. In this case, the repair time is:

$$\text{MTTR}_i^{\text{iden}} = P_i^s \text{MTBS}_i, \quad i = 2, \ldots, k.$$

The remaining time is the failure time, so that

$$\text{MTBF}_i^{\text{iden}} = \text{MTBS}_i - \text{MTTR}_i, \quad i = 2, \ldots, k.$$

Therefore, the MTTR and MTBF of station $i$, $i=2,\dots,k$, can be estimated as

$$\mathrm{MTTR}_i^{\mathrm{iden}} = \frac{e_0^{i-1}\tilde{P}_i^s \mathrm{MTBS}_i}{(e_0 - \tilde{P}_0^b)\prod_{j=1}^{i-1}(e_0 - P_j^s)}, \tag{4}$$

$$\mathrm{MTBF}_i^{\mathrm{iden}} = \left[1 - \frac{e_0^{i-1}\tilde{P}_i^s}{(e_0 - \tilde{P}_0^b)\prod_{j=1}^{i-1}(e_0 - P_j^s)}\right]\mathrm{MTBS}_i. \tag{5}$$

The accuracy of this approximation is evaluated in Section 4.

### 3.2. Case 2: feeder speed is known

Suppose we know the speed of each feeder line. Denote the speeds of the $i$-1th feeder and assembly station as $S_i$ and $S_0$, respectively. If $S_i = S_0$, adjustment (4) and (5) are used to approximate MTTR and MTBF, respectively. When $S_i \neq S_0$, let $\mathrm{MTTR}_i^{\mathrm{uneq}}$ and $\mathrm{MTBF}_i^{\mathrm{uneq}}$ be the MTTR and MTBF of the feeder station $M_i$ when its speed is known, respectively. The adjustment procedure consists of two steps: first assume the speed of the feeder equals to that of the assembly station, use (4) and (5) to calculate $\mathrm{MTTR}_i^{\mathrm{iden}}$ and $\mathrm{MTBF}_i^{\mathrm{iden}}$. Next, based on the speed of the feeder line, $\mathrm{MTTR}_i^{\mathrm{iden}}$ and $\mathrm{MTBF}_i^{\mathrm{iden}}$ are adjusted to $\mathrm{MTTR}_i^{\mathrm{uneq}}$ and $\mathrm{MTBF}_i^{\mathrm{uneq}}$, respectively.

In most assembly systems, feeder line speeds are often set equal to or greater than main line speed. Therefore, only the case of higher feeder speed is discussed in this paper, i.e. $S_i > S_0$. The case of $S_i < S_0$ can be found in (Li et al., 2002).

When $S_i > S_0$, we assume some downtime of $M_i$ is lost in the data collection due to the higher speed of the feeder. Therefore, we append additional downtime to $\mathrm{MTTR}_i$. More specifically, we set

$$\mathrm{MTTR}_i^{\mathrm{uneq}} = \frac{S_i}{S_0}\mathrm{MTTR}_i^{\mathrm{iden}}, \tag{6}$$

$$\mathrm{MTBF}_i^{\mathrm{uneq}} = \mathrm{MTBS}_i - \frac{S_i}{S_0}\mathrm{MTTR}_i^{\mathrm{iden}}. \tag{7}$$

**Remark 4**. There is no rigorous proof of Eqs. (6) and (7), they are based on intuitive only. Among all the possible adjustments, we try to select a simpler one.

## 4. Verifications

The accuracy of the above approximation is evaluated with simulations using commercial simulation software *Simul8* (http://www.simul8.com) for both one- and two-feeder systems. The verification process is carried out as follows:

At beginning we consider an assembly system where the reliability statistics of all machines (including the feeders) are known. This is used as an baseline model in the verification process. Using *Simul8*, we carry out the simulation experiment and collect the system throughput (denoted as $\mathrm{TP}_{\mathrm{actual}}$) and blockage and starvation data of the assembly station by following the current data collection procedure on the factory floor (described in Section 2). Then using the formulas derived in Section 3, we obtain the approximated MTTR and MTBF of each feeder. Use these data as the feeder's MTTR and MTBF (i.e. to replace the original feeder's reliability data), we carry out the simulation again and obtain the new system throughput $TP_{estimate}$. The accuracy is analyzed by comparing $TP_{actual}$ with $TP_{estimate}$. In all simulation experiments throughout this paper, 400 h simulation time with 17 h warm up period are carried out. All buffers are assumed to have 2 parts at the beginning of each replication. 10 simulation runs are executed for each experiment. The 95% confidence intervals are consistently around $\pm 0.0005$.

### 4.1. Assembly system with single feeder

Consider the assembly system with one feeder shown in Fig. 3. All machines are assumed to have deterministic processing times and exponential up- and downtimes. All buffers have capacity 3. In the baseline model, the main line stations ($M_1$, $M_0$ and $M_3$) operate at 60 jobs per hour. All stations ($M_i$, $i=0$, 1, 2, 3) have MTTR and MTBF equal to 2 and 20 min, respectively. System throughput (denoted as TP) is in units of jobs per hour (jph).

Based on the collected blocking and starving information of station $M_0$, two methods are compared for the cases of actual feeder line speed equal to 60 and 70 jobs per hour, respectively. Method 1 represents the current approach on the factory floor which directly uses the collected starving data as feeder's MTTR. The feeder speed is unknown and assumed to be identical to that of the assembly station. Therefore, the MTTR and MTBF calculated by Method 1, denoted as $\mathrm{MTTR}^{stv}$ and $\mathrm{MTBF}^{stv}$, respectively, are as follows:

$$\mathrm{MTTR}_2^{stv} = \mathrm{MTBS}_2 \tilde{P}_2^s, \tag{8}$$

$$\mathrm{MTBF}_2^{stv} = \mathrm{MTBS}_2 (1 - \tilde{P}_2^s). \tag{9}$$

Method 2 represents the adjustment approach which uses (4) and (5) when feeder speed is unknown or (6) and (7) when feeder speed is known and different with assembly station speed. The identified MTTR and MTBF are denoted as $\mathrm{MTTR}^{\mathrm{iden}}$ and $\mathrm{MTBF}^{\mathrm{iden}}$ for the case of unknown feeder speed, respectively. Analogously, $\mathrm{MTTR}^{\mathrm{uneq}}$, $\mathrm{MTBF}^{\mathrm{uneq}}$ represent the approximated MTTR and MTBF when feeder speed is known, respectively. Both methods are simulated with the same initial conditions. The approximation

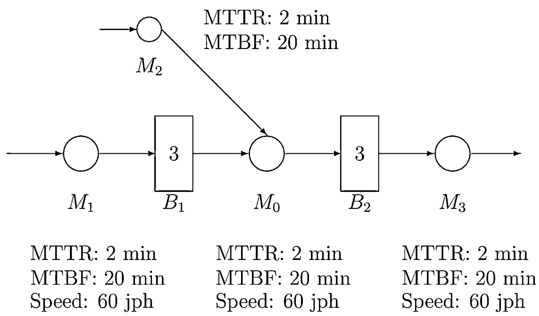


Fig. 3. Station and buffer parameters in assembly system with 1 feeder.

Table 1
Throughput estimate in case 1.1: feeder speed = assembly station speed

| | $MTTR_2$ (min) | $MTBF_2$ (min) | TP (jph) | Error (%) |
|---|---|---|---|---|
| Actual | 2 | 20 | 47.34 | – |
| M 1 FS = LS | 1.72 | 20.28 | 47.94 | 1.27 |
| M 2 FS = LS | 2.0 | 20.0 | 47.35 | 0.02 |

error is defined as

$$\text{error} = \frac{TP_{\text{estimate}} - TP_{\text{actual}}}{TP_{\text{actual}}} 100\%.$$

For convenience, the following notations are used in the subsequent Subsections:

- M1 FS = LS: Feeder speed is unknown, assume feeder speed equals to assembly station speed, use Method 1, i.e. Eqs. (8) and (9).
- M2 FS = LS: Feeder speed is unknown, assume feeder speed equals to assembly station speed, use Method 2, i.e. Eqs. (4) and (5).
- M2 FS > LS: Feeder speed > assembly station speed, use Method 2, i.e. Eqs. (6) and (7).

### 4.1.1. Case 1.1: feeder speed = assembly station speed

In this case, the feeder speed is correctly assumed to be equal to the speed of assembly station. The results of throughput estimation are shown in Table 1. Note that the percent errors are small in this case for both methods, however, Method 2 does produce more accurate estimates in both reliability statistics and system throughput. In addition, as we discussed before, Method 1 underestimates feeder downtime and introduces an overestimation in throughput.

Sensitivity analysis is often necessary in design and continuous improvement of production systems. It can be carried out by adjusting each station's speed, uptime and downtime. In this paper, only the speed sensitivity is within our interests. More specifically, our procedure is to

Table 2
Sensitivity analysis in case 1.1: feeder speed = assembly station speed

| Speed increase (jph) | Actual TP (jph) | M 1 FS = LS TP (jph) | Error (%) | M 2 FS = LS TP (jph) | Error (%) |
|---|---|---|---|---|---|
| −30 | 25.48 | 25.48 | 0 | 25.84 | 1.41 |
| −24 | 30.29 | 30.71 | 1.39 | 30.30 | 0.03 |
| −18 | 34.94 | 35.45 | 1.46 | 34.96 | 0.06 |
| −12 | 39.43 | 39.97 | 1.37 | 39.43 | 0 |
| −6 | 43.64 | 44.23 | 1.35 | 43.65 | 0.02 |
| 0 | 47.34 | 47.94 | 1.27 | 47.35 | 0.02 |
| 6 | 47.54 | 48.15 | 1.28 | 47.55 | 0.02 |
| 12 | 47.72 | 48.32 | 1.26 | 47.74 | 0.04 |
| 18 | 47.86 | 48.49 | 1.32 | 47.89 | 0.06 |
| 24 | 48.02 | 48.60 | 1.21 | 48.03 | 0.02 |
| 30 | 48.16 | 48.72 | 1.16 | 48.15 | −0.02 |

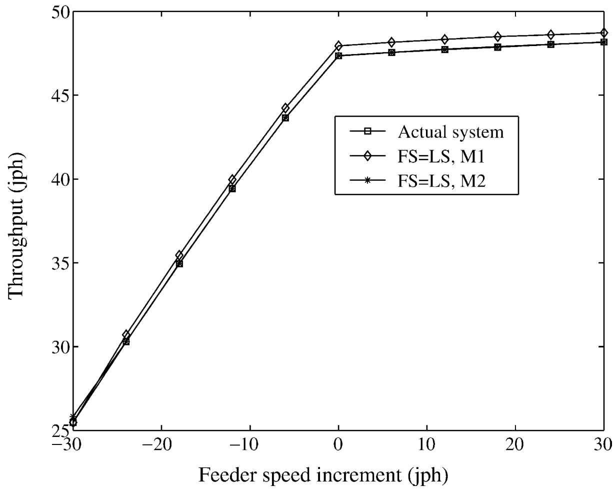


Fig. 4. Sensitivity analysis in case 1.1: feeder speed = assembly station speed.

increase and decrease feeder line speed by 6 jobs per hour, up to 30 jobs per hour and keep all other parameters (MTTR, MTBF) unchanged. The results are shown in Table 2 and illustrated in Fig. 4. Although all errors are small, we see that Method 2 has the best accuracy in estimation of system throughput and sensitivity analysis. Moreover, Method 1 always overestimates system throughput.

### *4.1.2. Case 1.2: feeder speed > assembly station speed*

In the case of fast feeder lines, Method 2 (using (6) and (7)) is used to approximate system performance providing that feeder speed is known (70 jph). The estimates of system throughput are shown in Table 3. Note that all errors of throughput are small with Method 1 still producing overestimate and Method 2 giving the best accuracy. Moreover, concerning about downtime estimates, the error introduced by Method 1 is up to 20% and by Method 2 is less than 1%.

The results of sensitivity analysis are shown in Table 4 and illustrated in Fig. 5. In Method 1 and Method 2 FS = LS, we still assume feeder speed equals to that of the assembly station (60 jph) which is 10 jobs less than that of the baseline system. Therefore, feeder speed is reduced from 60 jobs per hour, and is always 10 jobs less. As a result, the errors in the subsequent throughput analysis are monotonically

Table 3
Throughput estimate in case 1.2: feeder speed > assembly station speed

| | $MTTR_2$ (min) | $MTBF_2$ (min) | TP (jph) | Error (%) |
|---|---|---|---|---|
| Actual | 2 | 20 | 47.66 | – |
| M 1 FS = LS | 1.56 | 20.44 | 48.26 | 1.26 |
| M 2 FS = LS | 1.81 | 20.19 | 47.71 | 0.10 |
| M 2 FS > LS | 2.09 | 19.91 | 47.50 | −0.34 |

Table 4
Sensitivity analysis in case 1.2: feeder speed > assembly station speed

| Speed increase (jph) | Actual TP (jph) | FS = LS M 1 TP (jph) | Error (%) | FS = LS M 2 TP (jph) | Error (%) | FS > LS M 2 TP (jph) | Error (%) |
|---|---|---|---|---|---|---|---|
| −30 | 33.41 | 26.04 | −22.06 | 25.72 | −23.02 | 33.35 | −0.18 |
| −24 | 37.97 | 30.94 | −18.52 | 30.57 | −19.49 | 37.91 | −0.16 |
| −18 | 42.26 | 35.7 | −15.52 | 35.27 | −16.54 | 42.21 | −0.12 |
| −12 | 46.20 | 40.26 | −12.86 | 39.76 | −13.94 | 46.07 | −0.28 |
| −6 | 47.48 | 44.55 | −6.17 | 44 | −7.33 | 47.31 | −0.36 |
| 0 | 47.66 | 48.26 | 1.26 | 47.71 | 0.10 | 47.5 | −0.34 |
| 6 | 47.83 | 48.45 | 1.30 | 47.92 | 0.19 | 47.65 | −0.34 |
| 12 | 47.98 | 48.62 | 1.33 | 48.09 | 0.23 | 47.8 | 0.38 |
| 18 | 48.11 | 48.75 | 1.33 | 48.25 | 0.29 | 47.93 | −0.37 |
| 24 | 48.21 | 48.87 | 1.37 | 48.39 | 0.37 | 48.04 | −0.35 |
| 30 | 48.32 | 48.99 | 1.39 | 48.52 | 0.41 | 48.14 | −0.37 |

increasing, up to over 20%, when feeder speed is decreased. Moreover, when feeder speed is increased, Method 2 still consistently performs better than Method 1. In Method 2 FS > LS, feeder speed is reduced or increased from 70 jobs per hour which is same as that of the baseline model. The errors are consistently small and independent of feeder speed. Therefore, the accuracy of Method 2 FS > LS is not sensitive to feeder speed. Using ((6) and (7)) can significantly improve the accuracy of sensitivity analysis.

### *4.2. Assembly system multiple feeders*

In this Subsection, an assembly system with two feeders, shown in Fig. 6, is discussed.

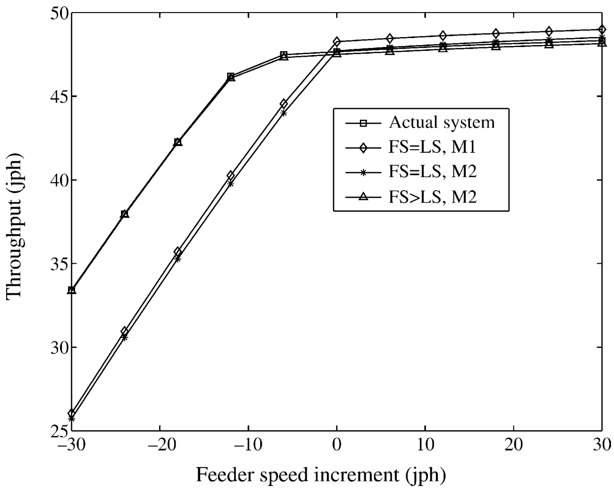


Fig. 5. Sensitivity analysis in case 1.2: feeder speed > assembly station speed.

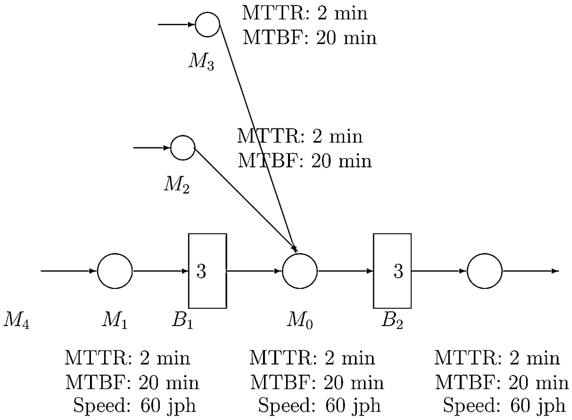


Fig. 6. Station and buffer parameters in assembly system with 2 feeders.

Table 5
Throughput estimate in case 2.1: feeder speed = main line speed

| | $MTTR_2$ (min) | $MTBF_2$ (min) | $MTTR_3$ (min) | $MTBF_3$ (min) | TP (jph) | Error (%) |
|---|---|---|---|---|---|---|
| Actual | 2 | 20 | 2 | 20 | 44.14 | – |
| FS = LS, M 1 | 1.53 | 20.47 | 1.09 | 20.91 | 46.70 | 5.80 |
| FS = LS, M 2 | 1.76 | 20.24 | 1.37 | 20.63 | 45.75 | 3.65 |

### 4.2.1. Case 2.1: feeder speed = main line speed

In this case, all stations in the baseline model operate at 60 jobs per hour and have MTTR and MTBF equal to 2 and 20 min, respectively. All buffers have capacity 3. The throughput estimates for both methods are shown in Table 5.

With more feeder lines merged into an assembly station, both methods show less accuracy. Intuitively, this is because less information is collected for the additional feeders. For instance, feeder

Table 6
Sensitivity analysis with respect to 1st feeder in case 2.1: feeder speed = main line speed

| Speed increase (jph) | Actual TP (jph) | FS = LS, M 1 TP (jph) | Error (%) | FS = LS, M 2 TP (jph) | Error (%) |
|---|---|---|---|---|---|
| −30 | 25.18 | 26.12 | 3.70 | 25.75 | 2.26 |
| −24 | 29.58 | 30.89 | 4.46 | 30.41 | 2.80 |
| −18 | 33.70 | 35.44 | 5.16 | 34.79 | 3.24 |
| −12 | 37.53 | 39.65 | 5.67 | 38.86 | 3.55 |
| −6 | 41.03 | 43.46 | 5.92 | 42.56 | 3.73 |
| 0 | 44.14 | 46.70 | 5.80 | 45.75 | 3.65 |
| 6 | 44.32 | 46.89 | 5.80 | 45.94 | 3.66 |
| 12 | 44.49 | 47.05 | 5.75 | 46.10 | 3.66 |
| 18 | 44.62 | 47.18 | 5.74 | 46.25 | 3.64 |
| 24 | 44.75 | 47.30 | 5.69 | 46.37 | 3.60 |
| 30 | 44.86 | 47.41 | 5.67 | 46.48 | 3.61 |

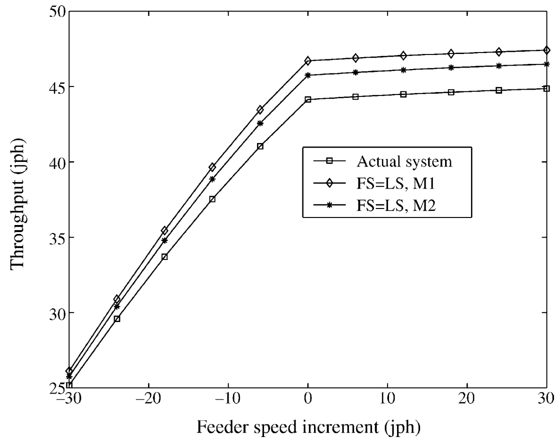


Fig. 7. Sensitivity analysis with respect to 1st feeder in case 2.1: feeder speed = assembly station speed.

$M_k$ may have observed more than one up–down cycles before its part availability is checked by the assembly station due to the breakdown of main line or feeders $M_j$, $j<k$. However, Method 2 is still more accurate than Method 1. In Table 5, Method 2 improves the accuracy of downtime estimate for about 50%.

In the multiple feeder case, the sensitivity analysis is carried out with respect to the 1st feeder, 2nd feeder, and both feeders (i.e. sensitivity with respect to feeders 1 and 2 simultaneously), respectively. Table 6 and Fig. 7 present the results with respect to feeder 1, while Table 7 and Fig. 8 provide the results with respect to feeder 2. Results concerning both feeders are shown in Table 8 and Fig. 9.

As one can see, Method 2 consistently has higher accuracy (almost 40% improvement) compared to Method 1 in sensitivity analysis. In addition, the accuracy is independent of feeders' speeds.

Table 7
Sensitivity analysis with respect to 2nd feeder in case 2.1: feeder speed = main line speed

| Speed increase (jph) | Actual TP (jph) | FS = LS, M 1 TP (jph) | Error (%) | FS = LS, M 2 TP (jph) | Error (%) |
|---|---|---|---|---|---|
| −30 | 25.55 | 26.70 | 4.52 | 26.35 | 3.16 |
| −24 | 30.20 | 31.66 | 4.84 | 31.20 | 3.32 |
| −18 | 34.12 | 36.18 | 6.04 | 35.44 | 3.88 |
| −12 | 37.73 | 40.04 | 6.12 | 39.19 | 3.89 |
| −6 | 41.11 | 43.60 | 6.05 | 42.68 | 3.82 |
| 0 | 44.14 | 46.70 | 5.80 | 45.75 | 3.65 |
| 6 | 44.30 | 46.85 | 5.75 | 45.91 | 3.62 |
| 12 | 44.44 | 46.97 | 5.69 | 46.04 | 3.59 |
| 18 | 44.57 | 47.07 | 5.62 | 46.14 | 3.54 |
| 24 | 44.67 | 47.15 | 5.55 | 46.23 | 3.50 |
| 30 | 44.76 | 47.22 | 5.50 | 46.31 | 3.46 |

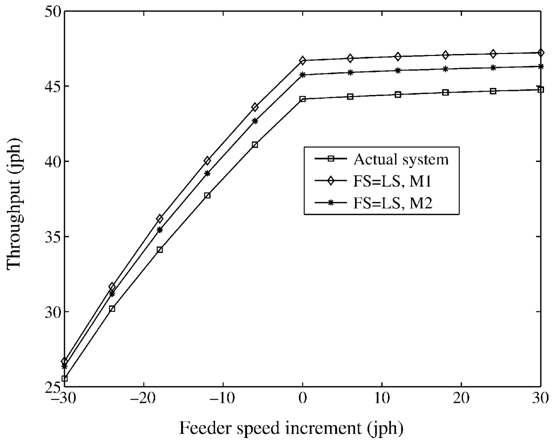


Fig. 8. Sensitivity analysis with respect to 2nd feeder in case 2.1: feeder speed = main line speed.

*4.2.2. Case 2.2: feeders' speeds > main line speed*

All parameters are same as in Case 2.1 except that the feeder stations now operate at 70 jobs per hour. The throughput estimates are shown in Table 9.

As we can see, Method 2 performs much better than Method 1, in particular, in downtime estimates (about 30–70%). In addition, knowing feeders' speeds can relatively improve the accuracy of throughput estimation (about 10%).

In sensitivity analysis, both feeders' speeds are increased or decreased from 60 jobs per hour in Method 1 and Method 2 FS = LS, and from 70 jobs per hour in Method 2 FS > LS. The results are shown in Tables 10–12 and Figs. 10–12, with respect to 1st, 2nd and both feeders, respectively. Similar to single feeder case, using Method 1 and Method 2 FS = LS, the errors become larger when feeders' speeds are

Table 8
Sensitivity analysis with respect to both feeders in case 2.1: feeder speed = main line speed

| Speed increase (jph) | Actual TP (jph) | FS = LS, M 1 TP (jph) | Error (%) | FS = LS, M 2 TP (jph) | Error (%) |
|---|---|---|---|---|---|
| −30 | 24.26 | 25.63 | 5.63 | 25.12 | 3.55 |
| −24 | 28.59 | 30.28 | 5.90 | 29.66 | 3.73 |
| −18 | 32.80 | 34.78 | 6.06 | 34.04 | 5.40 |
| −12 | 36.83 | 39.09 | 6.14 | 38.25 | 3.85 |
| −6 | 41.56 | 43.13 | 3.79 | 42.22 | 1.58 |
| 0 | 44.14 | 46.70 | 5.80 | 45.75 | 3.65 |
| 6 | 44.49 | 47.04 | 5.72 | 46.10 | 3.62 |
| 12 | 44.80 | 47.31 | 5.60 | 46.39 | 3.55 |
| 18 | 45.06 | 47.54 | 5.51 | 46.64 | 3.51 |
| 24 | 45.30 | 47.75 | 5.40 | 46.85 | 3.43 |
| 30 | 45.50 | 47.92 | 5.32 | 47.04 | 3.38 |

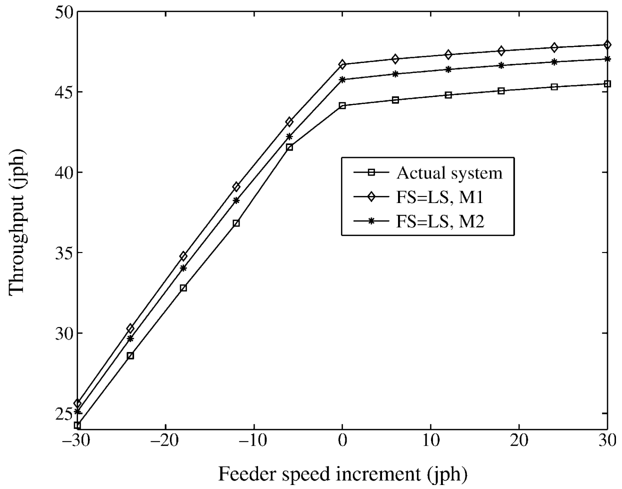


Fig. 9. Sensitivity analysis with respect to both feeders in case 2.1: feeder speed = main line speed.

Table 9
Throughput estimate in case 2.2: feeder speed > main line speed

| | $MTTR_2$ (min) | $MTBF_2$ (min) | $MTTR_3$ (min) | $MTBF_3$ (min) | TP (jph) | Error (%) |
|---|---|---|---|---|---|---|
| Actual | 2 | 20 | 2 | 20 | 44.70 | – |
| FS = LS, M 1 | 1.40 | 20.60 | 0.90 | 21.10 | 47.23 | 5.66 |
| FS = LS, M 2 | 1.62 | 20.38 | 1.13 | 20.87 | 46.47 | 3.96 |
| FS > LS, M 2 | 1.84 | 20.16 | 1.29 | 20.71 | 46.30 | 3.53 |

Table 10
Sensitivity analysis with respect to 1st feeder in case 2.2: feeder speed > main line speed

| Speed increase | Actual TP (jph) | FS = LS, M 1 TP (jph) | Error (%) | FS = LS, M 2 TP (jph) | Error (%) | FS > LS, M 2 TP (jph) | Error (%) |
|---|---|---|---|---|---|---|---|
| −30 | 32.48 | 26.29 | −19.06 | 26.00 | −19.95 | 33.29 | 2.49 |
| −24 | 36.45 | 31.16 | −14.51 | 30.75 | −15.65 | 37.52 | 2.94 |
| −18 | 40.13 | 35.79 | −10.81 | 35.26 | −12.12 | 41.41 | 3.19 |
| −12 | 43.42 | 40.11 | −7.63 | 39.45 | −9.14 | 44.86 | 3.32 |
| −6 | 44.53 | 43.98 | −1.23 | 43.24 | −2.90 | 46.0 | 3.30 |
| 0 | 44.70 | 47.23 | 5.66 | 46.47 | 3.96 | 46.17 | 3.29 |
| 6 | 44.85 | 47.41 | 5.71 | 46.65 | 4.01 | 46.32 | 3.28 |
| 12 | 44.98 | 47.57 | 5.77 | 46.82 | 4.09 | 46.46 | 3.29 |
| 18 | 45.10 | 47.70 | 5.76 | 46.95 | 4.09 | 46.57 | 3.26 |
| 24 | 45.21 | 47.82 | 5.78 | 47.08 | 4.14 | 46.68 | 3.25 |
| 30 | 45.29 | 47.92 | 5.81 | 47.18 | 4.17 | 46.77 | 3.27 |

Table 11
Sensitivity analysis with respect to 2nd feeder in case 2.2: feeder speed > main line speed

| Speed increase | Actual TP (jph) | FS = LS, M 1 TP (jph) | Error (%) | FS = LS, M 2 TP (jph) | Error (%) | FS > LS, M 2 TP (jph) | Error (%) |
|---|---|---|---|---|---|---|---|
| −30 | 33.07 | 26.95 | −18.51 | 26.65 | −19.41 | 34.27 | 3.63 |
| −24 | 36.81 | 31.97 | −13.14 | 31.59 | −14.18 | 38.25 | 3.91 |
| −18 | 40.29 | 36.63 | −9.09 | 36.02 | −10.16 | 41.75 | 3.57 |
| −12 | 43.47 | 40.55 | −6.72 | 39.84 | −8.35 | 44.97 | 3.45 |
| −6 | 44.55 | 44.13 | −0.94 | 43.37 | −2.65 | 46.03 | 3.32 |
| 0 | 44.70 | 47.23 | 5.66 | 46.47 | 3.96 | 46.17 | 3.29 |
| 6 | 44.83 | 47.38 | 5.68 | 46.62 | 3.99 | 46.29 | 3.26 |
| 12 | 44.95 | 47.49 | 5.64 | 46.74 | 3.98 | 46.38 | 3.18 |
| 18 | 45.03 | 47.57 | 5.63 | 46.84 | 4.01 | 46.45 | 3.15 |
| 24 | 45.11 | 47.64 | 5.60 | 46.92 | 4.01 | 46.52 | 3.13 |
| 30 | 45.19 | 47.71 | 5.58 | 47.00 | 4.00 | 46.59 | 3.10 |

decreased. However, the corresponding errors by using Method 2 FS > LS always keep relatively small no matter how feeders' speeds change. Therefore, the accuracy of Method 2 FS > LS is not sensitive to feeders' speeds.

In summary, comparing to current approach which uses the collected starving data as feeder's downtime directly, the proposed method can provide a much more closed approximation in feeders' downtime and lead to more accurate throughput estimation and sensitivity analysis. Moreover, the results obtained are more robust than that of the current approach.

## 5. Conclusions

Due to the difficulty and high cost of data collection, many feeder lines in assembly systems lack full data collection mechanisms. However, accurate data of feeder line reliability is needed in throughput

Table 12
Sensitivity analysis with respect to both feeders in case 2.2: feeder speed > main line speed

| Speed increase | Actual TP (jph) | FS = LS, M 1 TP (jph) | Error (%) | FS = LS, M 2 TP (jph) | Error (%) | FS > LS, M 2 TP (jph) | Error (%) |
|---|---|---|---|---|---|---|---|
| −30 | 31.42 | 25.92 | −17.50 | 25.50 | −18.84 | 32.49 | 3.41 |
| −24 | 35.51 | 30.66 | −13.66 | 30.12 | −15.18 | 36.75 | 3.49 |
| −18 | 39.42 | 35.22 | −10.65 | 34.60 | −12.23 | 40.80 | 3.50 |
| −12 | 43.04 | 39.59 | −8.02 | 38.89 | −9.64 | 44.51 | 3.42 |
| −6 | 44.38 | 43.66 | −1.62 | 42.91 | −3.31 | 45.86 | 3.33 |
| 0 | 44.70 | 47.23 | 5.66 | 46.47 | 3.96 | 46.17 | 3.29 |
| 6 | 44.95 | 47.56 | 5.81 | 46.81 | 4.14 | 46.43 | 3.29 |
| 12 | 45.22 | 47.82 | 5.75 | 47.09 | 4.13 | 46.67 | 3.21 |
| 18 | 45.43 | 48.04 | 5.75 | 47.33 | 4.18 | 46.85 | 3.13 |
| 24 | 45.62 | 48.23 | 5.72 | 47.53 | 4.19 | 47.03 | 3.09 |
| 30 | 45.78 | 48.39 | 5.70 | 47.70 | 4.19 | 47.19 | 3.08 |

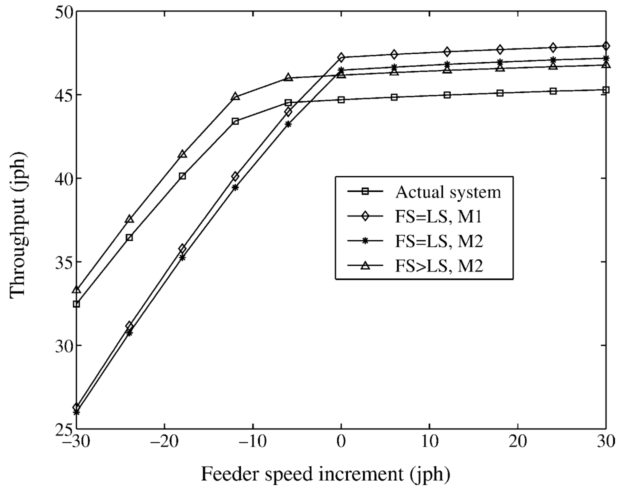


Fig. 10. Sensitivity analysis with respect to 1st feeder in case 2.2: feeder speed > assembly station speed.

analysis. In this paper, a simple approximation method is presented to estimate the reliability statistics of feeder lines. Using blocking and starving information of the associated assembly station, simple approximation formulas are derived corresponding to unknown and known feeder speed, respectively. We show that the approach provides more closed approximation in reliability statistics and is helpful for accurate throughput estimation and sensitivity analysis. Moreover, we find that the use of feeder line speed significantly improves accuracy.

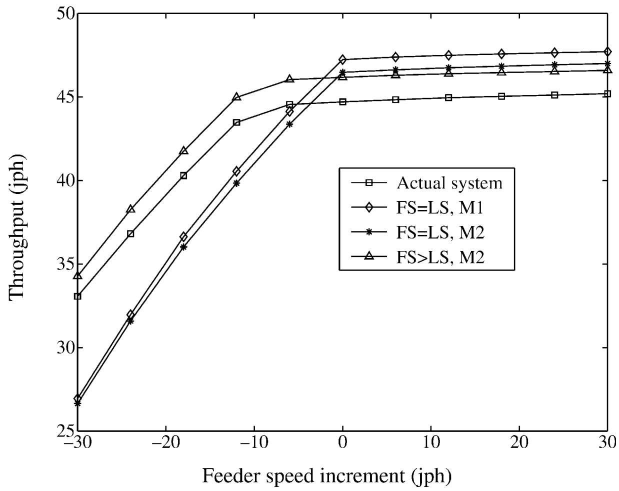


Fig. 11. Sensitivity analysis with respect to 2nd feeder in case 2.2: feeder speed > assembly station speed.

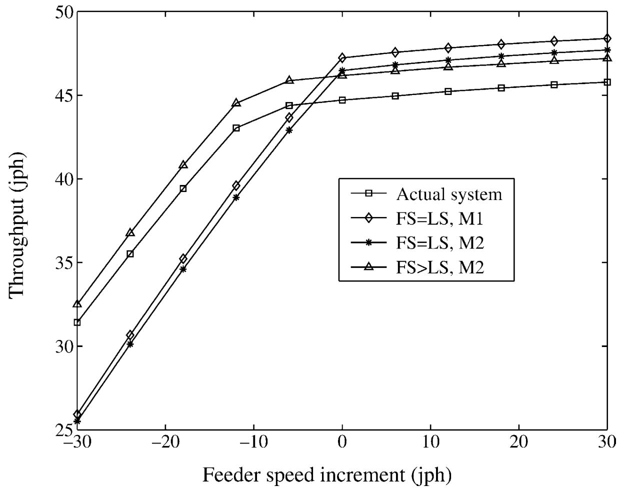


Fig. 12. Sensitivity analysis with respect to both feeders in case 2.2: feeder speed > main line speed.


## Acknowledgements

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## Appendix A

Proof of Proposition 1. : From assumptions (i)–(vi), we have

$$\begin{aligned} P_1^s &= \Pr\{M_0 \text{ is up},\ B_1 \text{ is empty}\} \\ &= \Pr\{M_0 \text{ is up},\ B_2 \text{ is not full},\ B_1 \text{ is empty}\} + \Pr\{M_0 \text{ is up},\ B_2 \text{ is full},\ B_1 \text{ is empty}\} \\ &= \tilde{P}_1^s + \frac{\Pr\{M_0 \text{ is up},\ B_2 \text{ is full}\}\Pr\{M_0 \text{ is up},\ B_1 \text{ is empty}\}}{\Pr\{M_0 \text{ is up}\}} \\ &= \tilde{P}_1^s + \frac{\tilde{P}_0^b P_1^s}{e_0}. \end{aligned}$$

It follows that

$$P_1^s = \frac{\tilde{P}_1^s}{1 - \frac{P_0^b}{e_0}} = \frac{e_0 \tilde{P}_1^s}{e_0 - \tilde{P}_0^b}.$$

For the first feeder $M_2$, we have

$$\begin{aligned}
P_2^s &= \Pr\{M_0 \text{ is up, no part is available from } M_2\} \\
&= \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from } M_2\} \\
&\quad + \Pr\{M_0 \text{ is up, } B_2 \text{ is full, no part is available from } M_2\} \\
&= \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from } M_2\} \\
&\quad + \frac{\Pr\{M_0 \text{ is up, } B_2 \text{ is full}\}\Pr\{M_0 \text{ is up, no part is available from } M_2\}}{\Pr\{M_0 \text{ is up}\}} \\
&= \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from } M_2\} + \frac{\tilde{P}_0^b}{e_0} P_2^s,
\end{aligned}$$

which implies that

$$\begin{aligned}
P_2^s &= \frac{1}{1 - \frac{P_0^b}{e_0}} \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from } M_2\} \\
&= \frac{1}{1 - \frac{P_0^b}{e_0}} [\Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from} \\
&\qquad M_2 \text{ and } B_1 \text{ is empty simultaneously}\} \\
&\quad + \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from } M_2, \text{ but } B_1 \text{ is not empty}\}] \\
&= \frac{1}{1 - \frac{P_0^b}{e_0}} \Bigg[\Pr\{M_0 \text{ is up, } B_2 \text{ is not full, } B_1 \text{ is empty}\} \\
&\quad \times \frac{\Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from } M_2\}}{\Pr\{M_0 \text{ is up, } B_2 \text{ is not full}\}} + \tilde{P}_2^s\Bigg] \\
&= \frac{1}{1 - \frac{P_0^b}{e_0}} \left[\frac{\tilde{P}_1^s P_2^s \left(1 - \frac{P_0^b}{e_0}\right)}{\Pr\{M_0 \text{ is up}\} - \Pr\{M_0 \text{ is up, } B_2 \text{ is not full}\}} + \tilde{P}_2^s\right] \\
&= \frac{1}{1 - \frac{P_0^b}{e_0}} \left[\frac{\tilde{P}_1^s P_2^s \left(1 - \frac{P_0^b}{e_0}\right)}{e_0 - \tilde{P}_0^b} + \tilde{P}_2^s\right] = \frac{1}{1 - \frac{P_0^b}{e_0}} \left[\frac{\tilde{P}_1^s}{e_0} P_2^s + \tilde{P}_2^s\right].
\end{aligned}$$

Thus, we obtain

$$P_2^s = \frac{e_0 \tilde{P}_2^s}{e_0 - \tilde{P}_0^b - \tilde{P}_1^s} = \frac{e_0 \tilde{P}_2^s}{e_0 - \tilde{P}_0^b - \left(1 - \frac{P_0^b}{e_0}\right) P_1^s} = \frac{e_0^2 \tilde{P}_2^s}{(e_0 - \tilde{P}_0^b)(e_0 - P_1^s)}.$$

Repeat this process for the $k-1$th feeder $M_k$, we have

$$\begin{aligned}
P_k^s &= \Pr\{M_0 \text{ is up, no part is available from } M_k\} \\
&= \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from } M_k\} \\
&\quad + \Pr\{M_0 \text{ is up, } B_2 \text{ is full, no part is available from } M_k\} \\
&= \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from } M_k\} \\
&\quad + \frac{\Pr\{M_0 \text{ is up, } B_2 \text{ is full}\}\Pr\{M_0 \text{ is up, no part is available from } M_k\}}{\Pr\{M_0 \text{ is up}\}} \\
&= \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from } M_k\} + \frac{\tilde{P}_0^b}{e_0} S_3.
\end{aligned}$$

It follows that

$$\begin{aligned}
P_k^s &= \frac{1}{1-\frac{P_0^b}{e_0}} \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from } M_k\} \\
&= \frac{1}{1-\frac{P_0^b}{e_0}} \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from } M_k, \\
&\qquad \text{but parts are available from all other } M_i,\ i=1,\cdots,k-1\} \\
&\quad + \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from either } M_k,\ M_{k-1}, \\
&\qquad \text{but part is available from all other } M_i,\ i=1,\cdots,k-2\} \\
&\quad + \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from either } M_k,\ M_{k-2}, \\
&\qquad \text{but part is available from all other } M_i,\ i=1,\cdots,k-2,k-1\} + \cdots \\
&\quad + \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from either } M_k,\ M_1, \\
&\qquad \text{but part is available from all other } M_i,\ i=2,\cdots,k-1\} \\
&\quad + \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from either } M_k,\ M_{k-1},\ M_{k-2}, \\
&\qquad \text{but part is available from all other } M_i,\ i=1,\cdots,k-3\} \\
&\quad + \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from either } M_k,\ M_{k-1},\ M_{k-3}, \\
&\qquad \text{but part is available from all other } M_i,\ i=1,\cdots,k-4,k-2\} + \cdots \\
&\quad + \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from either } M_k,\ M_1,\ M_2, \\
&\qquad \text{but part is available from all other } M_i,\ i=3,\cdots,k-1\} + \cdots \\
&\quad + \Pr\{M_0 \text{ is up, } B_2 \text{ is not full, no part is available from all } M_i,\ i=1,\cdots,k\}]
\end{aligned}$$

$$+\sum_{j_1,j_2=1,j_2>j_1}^{k-1}\Pr\{M_0\text{ is up},\ B_2\text{ is not full},\ \text{no part is available from either }M_k,\ M_{j_1},\ M_{j_2},$$
$$\text{but part is available from all other }M_i,\ i=1,\cdots,k-1,i\neq j_1,i\neq j_2\}+\cdots$$
$$+\sum_{j=1}^{k-1}\Pr\{M_0\text{ is up},\ B_2\text{ is not full},\ \text{no part is available from }M_k,\ M_i,$$
$$\text{but part is only available from }M_j,i=1,\cdots k-1,i\neq j\}$$
$$+\Pr\{M_0\text{ is up},\ B_2\text{ is not full},\ \text{no part is available from }M_i,\ i=1,\cdots,k\}]$$
$$=\frac{1}{1-\frac{P_0^b}{e_0}}\Bigg[\tilde{P}_k^s+\sum_{j=1}^{k-1}\Pr\{M_0\text{ is up},\ B_2\text{ is not full},\ \text{no part is available from }M_k\}$$
$$\times\frac{\Pr\{M_0\text{ is up},\ \text{no part is available from }M_j\}}{\Pr\{M_0\text{ is up}\}}\prod_{i=1,i\neq j}^{k-1}\frac{\Pr\{M_0\text{ is up},\ \text{part is available from }M_i\}}{\Pr\{M_0\text{ is up}\}}$$
$$+\sum_{j_1,j_2=1,j_2>j_1}^{k-1}\Pr\{M_0\text{ is up},\ B_2\text{ is not full},\ \text{no part is available from }M_k\}$$
$$\times\frac{\Pr\{M_0\text{ is up},\ \text{no part is available from }M_{j_1}\}}{\Pr\{M_0\text{ is up}\}}\frac{\Pr\{M_0\text{ is up},\ \text{no part is available from }M_{j_2}\}}{\Pr\{M_0\text{ is up}\}}$$
$$\times\prod_{i=1,\neq j_1,i\neq j_2}^{k-1}\frac{\Pr\{M_0\text{ is up},\ \text{part is available from }M_i\}}{\Pr\{M_0\text{ is up}\}}+\cdots$$
$$+\sum_{j=1}^{k-1}\Pr\{M_0\text{ is up},\ B_2\text{ is not full},\ \text{no part is available from }M_k\}$$
$$\times\frac{\Pr\{M_0\text{ is up},\ \text{part is available from }M_j\}}{\Pr\{M_0\text{ is up}\}}\prod_{i=1,i\neq j}^{k-1}\frac{\Pr\{M_0\text{ is up},\ \text{no part is available from }M_i\}}{\Pr\{M_0\text{ is up}\}}$$
$$+\Pr\{M_0\text{ is up},\ B_2\text{ is not full},\ \text{no part is available from }M_k\}$$
$$\prod_{i=1}^{k-1}\frac{\Pr\{M_0\text{ is up},\ \text{part is available from }M_i\}}{\Pr\{M_0\text{ is up}\}}\Bigg]$$
$$=\frac{1}{1-\frac{P_0^b}{e_0}}\Bigg[\tilde{P}_k^s+\sum_{j=1}^{k-1}P_k^s\left(1-\frac{\tilde{P}_0^b}{e_0}\right)\frac{P_j^s}{e_0}\prod_{i=1,i\neq j}^{k-1}\frac{e_0-P_i^s}{e_0}$$
$$+\sum_{j_1,j_2=1,j_2>j_1}^{k-1}P_k^s\left(1-\frac{\tilde{P}_0^b}{e_0}\right)\frac{P_{j_1}^s}{e_0}\frac{P_{j_2}^s}{e_0}\prod_{i=1,i\neq j_1,i\neq j_2}^{k-1}\frac{e_0-P_i^s}{e_0}+\cdots$$
$$+\sum_{j=1}^{k-1}P_k^s\left(1-\frac{\tilde{P}_0^b}{e_0}\right)\cdot\frac{e_0-P_j^s}{e_0}\cdot\prod_{i=1,i\neq j}^{k-1}P_i^s+P_k^s\left(1-\frac{\tilde{P}_0^b}{e_0}\right)\cdot\prod_{i=1}^{k-1}P_i^s\Bigg]$$

$$= \frac{1}{1-\frac{P_0^b}{e_0}}\left[\tilde{P}_k^s + \frac{P_k^s\left(1-\frac{P_0^b}{e_0}\right)}{e_0^{k-1}}\left(\sum_{j=1}^{k-1}\left[P_j^s\prod_{i=1,i\neq j}^{k-1}(e_0-P_j^s)\right] + \sum_{j_1,j_2=1,j_2>j_1}^{k-1}\left[P_{j_1}^sP_{j_2}^s\prod_{i=1,i\neq j_1,i\neq j_2}^{k-1}(e_0-P_i^s)\right]+\cdots+\sum_{j=1}^{k-1}\left[(e_0-P_j^s)\prod_{i=1,i\neq j}^{k-1}P_i^s\right]+\prod_{i=1}^{k-1}P_i^s\right)\right].$$

It leads to

$$\tilde{P}_k^s e_0^{k-1} = P_k^s\left(1-\frac{\tilde{P}_0^b}{e_0}\right)\left[e_0^{k-1} - \sum_{j=1}^{k-1}\left[P_j^s\prod_{i=1,i\neq j}^{k-1}(e_0-P_i^s)\right] - \sum_{j_1,j_2=1,j_2>j_1}^{k-1}\left[P_{j_1}^sP_{j_2}^s\prod_{i=1,i\neq j_1,i\neq j_2}^{k-1}(e_0-P_i^s)\right]-\cdots-\sum_{j=1}^{k-1}\left[(e_0-P_j^s)\prod_{i=1,i\neq j}^{k-1}P_i^s\right]-\prod_{i=1}^{k-1}P_i^s\right]. \tag{A1}$$

Note that

$$\prod_{i=1}^{k}(x-a_i) = x^k - \sum_{i=1}^{k}a_ix^{k-1} + \sum_{i_1=1}^{k-1}\sum_{i_2=i_1+1}^{k}a_{i_1}a_{i_2}x^{k-2} - \sum_{i_1=1}^{k-2}\sum_{i_2=i_1+1}^{k-1}\sum_{i_3=i_2+1}^{k}a_{i_1}a_{i_2}a_{i_3}x^{k-3}+\cdots + (-1)^{k-2}\sum_{i_1=1}^{k-1}\sum_{i_2=i_1+1}^{k}\left(\prod_{j=1,j\neq i_1,j\neq i_2}a_j\right)^k x^2 + (-1)^{k-1}\sum_{i=1}^{k}\left(\prod_{j=1,j\neq i}^{k}a_j\right)x + (-1)^k\prod_{i=1}^{k}a_i. \tag{A2}$$

It can be shown that

$$x^k - \sum_{j=1}^{k}\left[a_j\prod_{i=1,i\neq j}^{k}(x-a_i)\right] - \sum_{j_1,j_2=1,j_2>j_1}^{k}\left[a_{j_1}a_{j_2}\prod_{i=1,i\neq j_1,i\neq j_2}^{k}(x-a_i)\right]-\cdots - \sum_{j=1}^{k}\left[(x-a_j)\prod_{i=1,i\neq j}^{k}a_i\right] - \prod_{i=1}^{k}a_i = \prod_{i=1}^{k}(x-a_i). \tag{A3}$$

This property is proved by induction in the following. The relationship holds obviously for $k=2$. For $k=3$,

$$x^2 - [a_1(x-a_2)+a_2(x-a_1)] - a_1a_2 = (x-a_1)(x-a_2).$$

Assume that (A3) holds for $k = j-1$, i.e.

$$\prod_{i=1}^{j-1}(x-a_i) = x^{j-1} - \sum_{n=1}^{j-1}\left[a_n \prod_{i=1,\, i\neq n}^{j-1}(x-a_i)\right] - \sum_{i_1,\, i_2=1,\, i_2>i_1}^{j-1}\left[a_{i_1}a_{i_2}\prod_{i=1,\, i\neq i_1,\, i_2}^{j-1}(x-a_i)\right] - \cdots - \sum_{i_1,\, i_2=1,\, i_2>i_1}^{j-1}\left[(x-a_{i_1})(x-a_{i_2})\prod_{i=1,\, i\neq i_1,\, i_2}^{j-1}a_i\right] - \sum_{n=1}^{j-1}\left[(x-a_n)\prod_{i=1,\, i\neq n}^{j-1}a_i\right] - \prod_{i=1}^{j-1}a_i. \tag{A4}$$

Now assume $k=j+1$,

$$\begin{aligned}
&x^j - \sum_{n=1}^{j}\left[a_n \prod_{i=1,\, i\neq n}^{j}(x-a_i)\right] - \prod_{i_1,\, i_2=1,\, i_2>i_1}^{j}\left[a_{i_1}a_{i_2}\prod_{i=1,\, i\neq i_1,\, i_2}^{j}(x-a_i)\right] - \cdots \\
&\quad - \sum_{i_1,\, i_2,\, i_3=1,\, i_3>i_2>i_1}^{j}\left[(x-a_{i_1})(x-a_{i_2})(x-a_{i_3})\prod_{i=1,\, i\neq i_1,\, i_2,\, i_3}^{j}a_i\right] \\
&\quad - \sum_{i_1,\, i_2=1,\, i_2>i_1}^{j-1}\left[(x-a_{i_1})(x-a_{i_2})\prod_{i=1,\, i\neq i_1,\, i_2}^{j-1}a_i\right] - \sum_{n=1}^{j-1}\left[(x-a_n)\prod_{i=1,\, i\neq n}^{j}a_i\right] - \prod_{i=1}^{j}a_i \\
&= x^j - (x-a_j)\left[\sum_{n=1}^{j-1}a_n\prod_{i=1,\, i\neq n}^{j-1}(x-a_i)\right] - a_j\prod_{i=1}^{j-1}(x-a_i) \\
&\quad - (x-a_j)\left[\sum_{i_1,\, i_2=1,\, i_2>i_1}^{j-1}a_{i_1}a_{i_2}\prod_{i=1,\, i\neq i_1,\, i_2}^{j-1}(x-a_i)\right] - a_j\left[\sum_{n=1}^{j-1}a_n\prod_{i=1,\, i\neq n}^{j-1}(x-a_i)\right] \\
&\quad - (x-a_j)\left[\sum_{i_1,\, i_2,\, i_3=1,\, i_3>i_2>i_1}^{j-1}a_{i_1}a_{i_2}a_{i_3}\prod_{i=1,\, i\neq i_1,\, i_2,\, i_3}^{j-1}(x-a_i)\right] \\
&\quad - a_j\left[\sum_{i_1,\, i_2=1,\, i_2>i_1}^{j-1}a_{i_1}a_{i_2}\prod_{i=1,\, i\neq i_1,\, i_2}^{j-1}(x-a_i)\right] - \cdots \\
&\quad - (x-a_j)\left[\sum_{i_1,\, i_2=1,\, i_2>i_1}^{j-1}(x-a_{i_1})(x-a_{i_2})\prod_{i=1,\, i\neq i_1,\, i_2}a_i\right] \\
&\quad - a_j\left[\sum_{i_1,\, i_2,\, i_3=1,\, i_3>i_2>i_1}^{j-1}(x-a_{i_1})(x-a_{i_2})(x-a_{i_3})\prod_{i=1,\, i\neq i_1,\, i_2,\, i_3}^{j-1}a_i\right] \\
&\quad - (x-a_j)\left[\sum_{n=1}^{j-1}(x-a_n)\prod_{i=1,\, i\neq n}^{j-1}a_i\right] - a_j\left[\sum_{i_1,\, i_2=1}^{j-1}(x-a_{i_1})(x-a_{i_2})\prod_{i=1,\, i\neq i_1,\, i_2}^{j-1}a_i\right] \\
&\quad - (x-a_j)\prod_{i=1}^{j-1}a_i - \prod_{i=1}^{j}a_i = x^j - x\left[\sum_{n=1}^{j-1}a_n\prod_{i=1,\, i\neq n}^{j-1}(x-a_i)\right] - a_j\prod_{i=1}^{j-1}(x-a_i)
\end{aligned}$$

$$-x\left[\sum_{i_1,\, i_2=1,\, i_2>i_1}^{j-1} a_{i_1}a_{i_2}\prod_{i=1,\, i\neq i_1,\, i_2}^{j-1}(x-a_i)\right]-x\left[\sum_{i_1,\, i_2,\, i_3=1,\, i_3>i_2>i_1}^{j-1} a_{i_1}a_{i_2}a_{i_3}\prod_{i=1,\, i\neq i_1,\, i_2,\, i_3}^{j-1}(x-a_i)\right]$$

$$-\cdots-x\left[\sum_{i_1,\, i_2=1,\, i_2>i_1}^{j-1}(x-a_{i_1})(x-a_{i_2})\prod_{i=1,\, i\neq i_1,\, i_2}^{j-1}a_i\right]-x\left[\sum_{n=1}^{j-1}(x-a_n)\prod_{i=1,\, i\neq n}^{j-1}a_i\right]$$

$$-x\prod_{i=1}^{j-1}a_i = x\prod_{i=1}^{j-1}(x-a_i)-a_j\prod_{i=1}^{j-1}(x-a_i)=\prod_{i=1}^{j}(x-a_i).$$

This concludes the proof of property (A3).

Then, replace (A2) into (A1), we obtain

$$\tilde{P}_k^s e_0^{k-1} = P_k^s\left(1-\frac{\tilde{P}_0^b}{e_0}\right)\prod_{j=1}^{k-1}(e_0-P_j^s).$$

Therefore, we have

$$P_k^s = \frac{e_0^k\tilde{P}_k^s}{(e_0-\tilde{P}_0^b)\prod_{j=1}^{k-1}(e_0-P_j^s)}.$$

## References


Altiok, T. (1997). *Performance analysis of manufacturing systems*. Berlin: Springer.

Buzacott, J. A., & Shantikumar, J. G. (1993). *Stochastic models of manufacturing systems*. Prentice Hall.

Chiang, S.-Y., Kuo, C.-T., Lim, J.-T., & Meerkov, S. M. (2000a). Improvability theory of assembly systems I: problem formulation and performance evaluation. *Mathematical Problems in Engineering*, *6*, 321–357.

Chiang, S.-Y., Kuo, C.-T., Lim, J.-T., & Meerkov, S. M. (2000b). Improvability theory of assembly systems II: improvability indicators and case study. *Mathematical Problems in Engineering*, *6*, 359–393.

Di Mascolo, M., David, R., & Dallery, Y. (1991). Modeling and analysis of assembly systems with unreliable machines and finite buffers. *IIE Transactions*, *23*, 315–330.

Gershwin, S. B. (1994). *Manufacturing systems engineering*. Prentice Hall.

Gershwin, S. B., & Burman, M. H. (2000). A decomposition method for analyzing inhomogeneous assembly/disassembly systems. *Annals of Operations Research*, *93*, 91–115.

Helber, S. (1998). Decomposition of unreliable assembly/disassembly networks with limited buffer capacity and random processing times. *European Journal of Operational Research*, *109*, 24–42.

Jacobs, D. A., & Meerkov, S. M. (1995). A System-theoretic property of serial production lines: improvability. *International Journal of System Science*, *26*, 95–137.

Kouikoglou, V. 5. (2002). An efficient discrete event model of assembly/disassembly production networks. *International Journal of Production Research*, *40*, 4485–4503.

Li, J., Alden, J. M., & Rabaey, J. R. (2002). Approximating feeder line reliability statistics in assembly systems with patial data collections. *Technical Report R & D-9391*. Warren, MI: General Motors Research &Development center.

Liu, X.-G., & Buzacott, J. A. (1990). Approximate models of assembly systems with finite inventory banks. *European Journal of Operational Research*, *45*, 143–154.

Simon, J. T., & Hopp, W. J. (1991). Availability and average inventory of balanced assembly-like flow systems. *IIE Transactions*, *23*, 161–168.