# Answers and solutions for quiz-2

Coursera. Stochastic Processes

September 4, 2019

## 2 Week quiz

1. Compute the mathematical expectation of a Poisson process $N_t$ with intensity $\lambda$:

   **Answer:** $\lambda t$

   **Solution:** This is the basic feature of the Poisson process. Keep in mind that $N_t \sim Pois(\lambda t)$

2. Find the probability generating function of a random variable with binomial distribution,

   $$\mathbb{P}\{\xi = k\} = C_n^k p^k (1-p)^{n-k}, \quad k = 0, 1, .., n, \qquad p \in (0,1):$$

   **Answer:** $\varphi(u) = (up + (1-p))^n$

   **Solution:** $PGF = \varphi_\xi(u) = \mathbb{E}[u^\xi] = \sum_{k=0}^n u^k C_n^k p^k (1-p)^{n-k} = \sum_{k=0}^n C_n^k (up)^k (1-p)^{n-k} =$ (Newton binomial) $= (up + (1-p))^n$

3. Let $N_t$ be a (homogeneous) Poisson process with intensity $\lambda$. Find the limit $\lim_{h\to 0} \mathbb{P}\{N_h = 0\}$ :

   **Answer:** 1

   **Solution:** $\lim_{h\to 0} \mathbb{P}\{N_h = 0\} = \lim_{h\to 0} e^{-\lambda h} = 1$

4. Let $N_t$ be a (homogeneous) Poisson process with intensity $\lambda$. Find the limit $\lim_{h\to 0} \mathbb{P}\{N_h = 1\}$:

   **Answer:** 0

   **Solution:** $\lim_{h\to 0} \mathbb{P}\{N_h = 3\} = \lim_{h\to 0} e^{-\lambda h} \frac{\lambda h}{1!} = 1 \cdot 0$

5. Let $N_t$ be a (homogeneous) Poisson process with intensity $\lambda$. Find the limit $\lim_{h\to 0} \mathbb{P}\{N_h = 3\}$::

   **Answer:** 0

   **Solution:** $\lim_{h\to 0} \mathbb{P}\{N_h = 3\} = \lim_{h\to 0} e^{-\lambda h} \frac{(\lambda h)^3}{3!} = 1 \cdot 0$

6. 2 friends are chating: one has a messaging speed equal to 3 messages per minute, another - 2 messages per minute. Assuming that for every person the process of writing the messages is modeled with Poisson process and these processes are independent, find the probability that there will be sent only 2 messages during the first minute:

**Answer:** $e^{-5}\frac{25}{2}$

**Solution:** $\mathbb{P}^* = \mathbb{P}(N_1^A = 2, N_1^B = 0) + \mathbb{P}(N_1^A = 0, N_1^B = 2) + \mathbb{P}(N_1^A = 1, N_1^B = 1) = e^{-5}(\frac{3^2}{2!}\frac{2^0}{0!} + \frac{3^0}{0!}\frac{2^2}{2!} + \frac{3}{1}\frac{2}{1}) = e^{-5}\frac{25}{2}$

7. Purchases in a shop are modelled by the homogeneous Poisson process: 30 purchases are made on average during an hour after the opening of the shop. Find the probability that the interval between $k$ and $k+1$ purchases will be less than **or equal to** 4 minutes, given that the purchase number $k$ was in the time moment $s$:

   **Answer:** $1 - e^{-2}$

   **Solution:**

$$\begin{aligned}\mathbb{P}(S_{k+1} - S_k \le 4 | N_s = k) &= \mathbb{P}(N_{s+4} - N_s \ge 1 | N_s - N_0 = k) \\ &= \mathbb{P}(N_{s+4} - N_s > 0) \\ &= 1 - \mathbb{P}(N_{s+4} - N_s = 0) \\ &= 1 - \mathbb{P}(N_4 = 0) = 1 - e^{-2},\end{aligned}$$

   because $N_4 \sim Pois(4 \cdot 30/60)$

8. The amount of claims to an insurance company is modelled by the Poisson process, and the claim sizes are modelled by an exponential distribution. On average there are 100 claims per day, and the mean value of 1 claim is 5000 USD.

   Find the variance of the process $X_t$, which is equal to the total amount of claims till time $t$ :

   **Answer**: $5t \times 10^9$

   **Solution:** $\xi \sim exp(\mu)$

   $\mathbb{E}\xi = \frac{1}{\mu} = 5 \times 10^3$

   $Var\xi = \frac{1}{\mu^2} = 25 \times 10^6$

   $\mathbb{E}\xi^2 = Var\xi + (\mathbb{E}\xi)^2 = 5 \times 10^7$

   $VarX_t = \lambda t \mathbb{E}\xi^2 = 100t \cdot 5 \times 10^7 = 5t \times 10^9$

9. The amount of claims to an insurance company is modelled by the Poisson process, and the claim sizes are modelled by an exponential distribution. On average there are 100 claims per day, and the mean value of 1 claim is 5000 USD.

   Find the probability that the process $X_t$, which is equal to the total amount of claims till time $t$, is equal to 0 at the moment $t$

   **Answer:** $e^{-100t}$

   **Solution:** $\mathbb{P}(X_t = 0) = \mathbb{P}(N_t = 0) + \mathbb{P}(N_t > 0) \cdot \mathbb{P}(\sum_{k=1}^{N_t} \xi_k = 0) = e^{-100t} + 0$

10. The amount of claims to an insurance company is modelled by the Poisson process, and the claim sizes are modelled by an exponential distribution. On average there are 100 claims per day, and the mean value of 1 claim is 5000 USD.

Find the mean value of the process $X_t$, which is equal to the total amount of claims till time $t$:

**Answer:** 500000

**Solution:** According to the corollary about Compound Poisson processes: $\mathbb{E}(X_t) = \lambda t \cdot \mathbb{E}(\xi) = 100t \cdot 5000 = 500000$

11. Purchases in a shop are modelled with non-homogeneous Poisson process: $30t^{5/4}$ purchases are made on average during $t$ hours after the opening of the shop. Find the probability that the interval between $k$ and $k+1$ purchases will be less or equal than 2 minutes, given that the purchase number $k$ was in the time moment $s$:

    **Answer:** $1 - e^{-30(s+1/30)^{5/4}+30s^{5/4}}$

    **Solution:** $\mathbb{P}(S_{k+1} - S_k \leq 2|N_s = k) = \mathbb{P}(N_{s+2} - N_s \geq 1|N_s = k) = \mathbb{P}(N_{s+2} - N_s \geq 1|N_s - N_0 = k) = \mathbb{P}(N_{s+2} - N_s \geq 1) = 1 - \mathbb{P}(N_{s+2} - N_s = 0) = 1 - e^{-30(s+1/30)^{5/4}+30s^{5/4}}$

12. New: Number of downloads of an app in Google-Play are modelled by a non-homogeneous Poisson process with intensity $\Lambda(t) = t^{13/5}$, where $t$ is measured in hours after app's commencement time. Find the probability that the time between the $1000^{th}$ and $1001^{st}$ downloads is less than or equal to 36 seconds (0.01 hour) given $1000^{th}$ download time being 14 hours after app's launch.

    **Solution:** $\mathbb{P}(S_{1001} - S_{1000} \leq 0.01|S_{1000} = 14) = \mathbb{P}(S_{1001} - S_{1000} \leq 0.01|N_{14} = 1000) = \mathbb{P}(N_{14.01} - N_{14} \geq 1|N_{14} - N_0 = 1000) = \mathbb{P}(N_{14.01} - N_{14} \geq 1) = 1 - \mathbb{P}(N_{14.01} - N_{14} = 0) = 1 - e^{-(\Lambda(14.01)-\Lambda(14))}\frac{(\Lambda(14.01)-\Lambda(14))^0}{0!} = 1 - e^{-14.01^{2.6}+14^{2.6}} = 0.83.$

13. New: Sales of a product are modelled by a non-homogeneous Poisson process with intensity $\Lambda(t), t$ in hours. Find the probability that the time between the $10th$ and $25th$ purchases is less than or equal to 1 hour given $10th$ product purchase time being 18 minutes (0.3 hours) after the shop's opening.

    **Solution:** $\mathbb{P}(S_{25} - S_{10} \leq 1|S_{10} = 0.3) = \mathbb{P}(S_{25} - S_{10} \leq 1|N_{0.3} = 10) = \mathbb{P}(N_{1.3} - N_{0.3} \geq (25 - 10)|N_{0.3} - N_0 = 10) = \mathbb{P}(N_{1.3} - N_{0.3} \geq 15) = 1 - \mathbb{P}(N_{1.3} - N_{0.3} < 15) = 1 - \sum_{k=0}^{14} e^{-(\Lambda(1.3)-\Lambda(0.3))}\frac{(\Lambda(1.3)-\Lambda(0.3))^k}{k!} = 1 - e^{-(\Lambda(1.3)-\Lambda(0.3))}\sum_{k=0}^{14}\frac{(\Lambda(1.3)-\Lambda(0.3))^k}{k!}.$

14. New: Find the probability generating function of the the random variable $N_3$ (where $N_t$ is a homogeneous Poisson process) using the formula $PGF = \varphi_\alpha(u) = \mathbb{E}(u^\alpha)$:

    **Answer:** $e^{-3\lambda(1-u)}$

    **Solution:** $\mathbb{E}(u^{N_3}) = \sum_{k=0}^{\infty} u^k e^{-3\lambda}\frac{(3\lambda)^k}{k!} = e^{-3\lambda}\sum_{k=0}^{\infty}\frac{(3u\lambda)^k}{k!} = e^{-3\lambda}e^{3\lambda u} = e^{(-3\lambda(1-u))}$, because $\sum_{k=0}^{\infty}\frac{\alpha^k}{k!} = e^\alpha$

15. New: There is a speed limit on the street near the secondary school. To keep a lid on traffic violations, local administration decided to put a speed-register. If a car violates the speed limit, the register correctly identifies its ID number with probability 80%. Assume that the number of cars passing

the school and violating the speed limit is modelled by the homogeneous Poisson process $N_t$ with intensity equal to 20. Find the probability that during 2 hours after midday there will be 1 cars registered.

**Answer:** 7%

**Solution:** Denote the number of registered cars till time $t$ by $M_t$. The probability of a correct identification of a car is equal to $p = 0.8$

So, we need to calculate $\mathbb{P}(M_{2p.m.} - M_{12a.m.} = 16)$.

More generally,

$$\begin{aligned}
\mathbb{P}(M_t - M_s = m) &= \mathbb{P}(M_t - M_s = m \cap N_t - N_s = \mathbf{m}) \\
&\quad +\mathbb{P}(M_t - M_s = m \cap N_t - N_s = \mathbf{m+1}) + \\
&\quad +\mathbb{P}(M_t - M_s = m \cap N_t - N_s = \mathbf{m+2}) + \cdots \\
&= \sum_{n=m}^{\infty} \mathbb{P}(M_t - M_s = m \cap N_t - N_s = \mathbf{n}) \\
&= \sum_{n=m}^{\infty} \mathbb{P}(M_t - M_s = m | N_t - N_s = \mathbf{n}) \cdot \mathbb{P}(N_t - N_s = \mathbf{n}) \\
&= \sum_{n=m}^{\infty} C_n^m p^m (1-p)^{n-m} \cdot e^{-\lambda(t-s)} \frac{(\lambda(t-s))^n}{n!} \\
&= \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} p^m (1-p)^{n-m} \cdot e^{-\lambda(t-s)} \frac{(\lambda(t-s))^n}{n!} \\
&= \frac{p^m e^{-\lambda(t-s)}}{m!} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} \frac{(1-p)^n}{(1-p)^m} \cdot \frac{(\lambda(t-s))^n}{n!} \\
&= \left(\frac{p}{1-p}\right)^m \frac{e^{-\lambda(t-s)}}{m!} \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} (1-p)^n \cdot \frac{(\lambda(t-s))^n}{n!} \\
&= \left(\frac{p}{1-p}\right)^m \frac{e^{-\lambda(t-s)}}{m!} \sum_{n=m}^{\infty} \frac{(\lambda(t-s)(1-p))^n}{(n-m)!} \\
&= \left(\frac{p}{1-p}\right)^m \frac{e^{-\lambda(t-s)}}{m!} \sum_{\mathbf{k=0}}^{\infty} \frac{(\lambda(t-s)(1-p))^{\mathbf{k+m}}}{\mathbf{k!}} \\
&= \left(\frac{p}{1-p}\right)^m \frac{e^{-\lambda(t-s)}}{m!} (\lambda(t-s)(1-p))^{\mathbf{k+m}} \sum_{\mathbf{k=0}}^{\infty} \frac{(\lambda(t-s)(1-p))^{\mathbf{k}}}{\mathbf{k!}} \\
&= \left(\frac{p}{1-p}\right)^m \frac{e^{-\lambda(t-s)}}{m!} (\lambda(t-s)(1-p))^{\mathbf{k+m}} e^( \lambda(t-s)(1-p)) \\
&= \frac{(\lambda p(t-s))^m}{m!} e^{-\lambda p(t-s)}.
\end{aligned}$$

Therefore,

$$\mathbb{P}(M_{2p.m.} - M_{12a.m.} = 16) = \frac{(20 \cdot 0.8(2-0))^{16}}{16!} e^{-20 \cdot 0.8(2-0)} = 0.07\%$$