Quiz-1 answers and solutions

Coursera. Stochastic Processes

June 7, 2019

1 week quiz

1. Let η be a random variable with distribution function F_η. Define a stochastic process $X_t = \eta + t$. Compute the distribution function of a finite-dimensional distribution $(X_{t_1}, ...X_{t_n})$, where $t_1, ..., t_n \in \mathbb{R}_+$:

   Answer: $\mathbb{F}_\eta\{min(x_1 - t_1, ..., x_n - t_n)\}$

   Solution: $\mathbb{F}_{\vec{X}}(\vec{x}) = \mathbb{P}(X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n) = \mathbb{P}(\eta + t_1 \leq x_1, \ldots, \eta + t_n \leq x_n) = \mathbb{P}(\eta \leq x_1 - t_1, \ldots, \eta \leq x_n - t_n) = \mathbb{F}_\eta\{min(x_1 - t_1, ..., x_n - t_n)\}$

2. Let S_n be a renewal process such that $\xi_n = S_n - S_{n-1}$ takes the values 1 or 2 with equal probabilities $p = 1/2$. Find the mathematical expectation of the counting process N_t at t=3:

   Answer: 15/8

   Solution:

   We can calculate it without using the Laplase transform in this case:

$$
\begin{aligned}
\mathbb{E}(N_3) &= \sum_{k=0}^{\infty} k \cdot \mathbb{P}(N_3 = k) \\
&= 0 + 1 \cdot \mathbb{P}(N_3 = 1) + 2 \cdot \mathbb{P}(N_3 = 2) + 3 \cdot \mathbb{P}(N_3 = 3) + 4 \cdot \mathbb{P}(N_3 = 4) \\
&= 1 \cdot \mathbb{P}(\xi_1 = 2; \xi_2 = 2) + 2 \cdot (\mathbb{P}(\xi_1 = 1; \xi_2 = 2) + \mathbb{P}(\xi_1 = 2; \xi_2 = 1) + \\
&\quad + \mathbb{P}(\xi_1 = 1; \xi_2 = 1, \xi_3 = 2)) + 3 \cdot \mathbb{P}(\xi_1 = 1; \xi_2 = 1; \xi_3 = 2) \\
&= 1 \cdot \frac{1}{4} + 2 \cdot (\frac{1}{4} + \frac{1}{4} + \frac{1}{8}) + 3 \cdot \frac{1}{8} \quad \frac{1}{4} + 1\frac{1}{4} + \frac{3}{8} \\
&= 15/8
\end{aligned}
$$

3. Let $S_n = S_{n-1} + \xi_n$ be a renewal process and $p_\xi(x) = \lambda e^{-\lambda x}$. Find the mathematical expectation of the corresponding counting process N_t:

   Answer: none of the above

   Solution:

   step 1. $p \to \mathcal{L}_p$

$$\mathcal{L}_p(s) = \int_0^\infty e^{-sx} \cdot \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-(s+\lambda)x} dx = -\frac{\lambda}{s+\lambda}(0 - 1) = \frac{\lambda}{s+\lambda}$$

   step 2. $\mathcal{L}_p \to \mathcal{L}_u$

$$\mathcal{L}_u = \frac{\frac{\lambda}{s+\lambda}}{s\left(1-\frac{\lambda}{s+\lambda}\right)} = \frac{\lambda}{s^2}$$

step 3. $\mathcal{L}_u \to u(t)$

Since $\mathcal{L}_u = \lambda \cdot \frac{1!}{s^{1+1}}$, then $u(t) = \lambda t$.

4. Let η be a random variable with distribution function F_η. Define a stochastic process $X_t = e^\eta t^2$. What is the distribution function of $(X_{t_1}, ..., X_{t_n})$ for positive $t_1, ..., t_n$?

Answer: $\mathbb{F}_\eta\{\min(\ln\left(x_1/t_1^2\right), ..., \ln\left(x_n/t_n^2\right)\}$

Solution:

$$\begin{aligned}
\mathbb{F}_{\vec{X}}(\vec{x}) &= \mathbb{P}(X_{t_1} \le x_1, \ldots, X_{t_n} \le x_n) \\
&= \mathbb{P}(e^\eta t_1^2 \le x_1, \ldots, e^\eta t_n^2 \le x_n) \\
&= \mathbb{P}(\eta \le x_1/t_1^2, \ldots, \eta \le x_n/t_n^2) \\
&= \mathbb{F}_\eta\{min(x_1/t_1^2, ..., x_n/t_n^2\}.
\end{aligned}$$

5. Let N_t be a counting process of a renewal process $S_n = S_{n-1} + \xi_n$ such that the i.i.d. random variables $\xi_1, \xi_2, ...$ have a probability density function $p_\xi(x) = \begin{cases} \frac{1}{2}e^{-x}(x+1), & x \ge 00, \\ x < 0. \end{cases}$

Find the mean of N_t.

Answer: $-\dfrac{1}{9} + \dfrac{2}{3}t + \dfrac{1}{9}e^{-(3/2)t}$

Solution:

Step 1. $p \to \mathcal{L}_p(s)$

$$\begin{aligned}
\mathcal{L}_p(s) &= \int_0^\infty e^{-sx} \cdot \frac{1}{2}e^{-x}(x+1)dx \\
&= \frac{1}{2}\int_0^\infty e^{-(s+1)x} \cdot xdx + \frac{1}{2}\int_0^\infty e^{-(s+1)x}dx \\
&= -\frac{1}{2(s+1)}\int_0^\infty xd(e^{-(s+1)x}) - \frac{1}{2(s+1)}(0-1) \\
&= -\frac{1}{2(s+1)}(xe^{-(s+1)x}|_0^\infty - \int_0^\infty e^{-(s+1)x}dx) + \frac{1}{2(s+1)} \\
&= -\frac{1}{2(s+1)}(0 + \frac{e^{-(s+1)x}}{s+1}|_0^\infty) + \frac{1}{2(s+1)} \\
&= -\frac{1}{2(s+1)}(0 + \frac{0-1}{s+1}) + \frac{1}{2(s+1)} \\
&= \frac{1}{2(s+1)^2} + \frac{1}{2(s+1)} \\
&= \frac{s+2}{2(s+1)^2}.
\end{aligned}$$

Step 2. $\mathcal{L}_p(s) \to \mathcal{L}_u(s)$

$$\begin{aligned}\mathcal{L}_u &= \frac{\frac{s+2}{2(s+1)^2}}{s\left(1-\frac{s+2}{2(s+1)^2}\right)} \\ &= \frac{s+2}{s(2(s+1)^2-s-2)} \\ &= \frac{s+2}{s^2(2s+3)}.\end{aligned}$$

Step 3. $\mathcal{L}_u \to u$

$$\begin{aligned}\frac{s+2}{s^2(2s+3)} &= \frac{A}{s}+\frac{B}{s^2}+\frac{C}{2s+3} \\ &= \frac{A\cdot s(2s+3)+B\cdot(2s+3)+C\cdot s^2}{s^2(2s+3)} \\ &= \frac{s^2(2A+C)+s(3A+2B)+3B}{s^2(2s+3)}.\end{aligned}$$

$$\begin{cases}2A+C=0\\ 3A+2B=1\\ 3B=2\end{cases} \to \begin{cases}C=2/9\\ A=-1/9\\ B=2/3\end{cases}$$

$$L_u = -\frac{1}{9s}+\frac{2}{3s^2}+\frac{1}{9(s+\frac{3}{2})}$$

Therefore,

$$\mathbb{E}N_t = u(t) = -\frac{1}{9}+\frac{2}{3}t+\frac{1}{9}e^{-(3/2)t}.$$

6. Let ξ and η be 2 random variables. It is known that the distribution of η is symmetric, that is, $\mathbb{P}\{\eta > x\} = \mathbb{P}\{\eta < -x\}$ for any $x > 0$, and moreover $\mathbb{P}\{\eta = 0\} = 0$. Find the probability of the event that the trajectories of stochastic process $X_t = \xi^2 + t(\eta + t)$, $t \geq 0$ increase:

   Answer: $\frac{1}{2}$

   Solution:

$$X_t = t^2 + \eta t + \xi^2$$

$$\mathbb{P}(\frac{d}{dt}X_t > 0 \forall t \geq 0) = \mathbb{P}(2t+\eta > 0 \forall t \geq 0) = \mathbb{P}(\eta > 0) = \frac{1}{2}$$