Spatial number, multi-dimensional complex

by

H. L. Yang (Beijing)

1. **Definition and expression**

We know that a complex number $z = a + bi$ indicates a point (a, b) in the complex plane, hence the modulus of the vector from $(0, 0)$ to (a, b) i.e $|\vec{z}|$ is r=$\sqrt{a^2 + b^2}$ and it's argument θ= $\arctan\frac{b}{a}$, which is the angle turning counter-clockwise away from the real axis X. Regardless of its periodicity, θ∈ $(-\pi, \ \pi]$. Exponential expressed is it as $\vec{Z} = re^{\theta i}$。

In the same way, we define $\vec{S} = a + bi + cj$ as a point $(a, b, \ c)$ in a 3-dimensional space, where the real part a is on axis X, the first imaginary part b is on axis Y and the second imaginary part c is on axis Z . The modulus of vector $\vec{S}$ is $r = \sqrt{a^2 + b^2 + c^2}$. If let $r_1 = \sqrt{a^2 + b^2}$, called first modulus, indicating the modulus of projective vector in X-Y plane, then $\theta_1 = \arctan\frac{b}{a}$, indicating the angle turning conter-clockwise away from the real axis X , which is ranged in $(-\pi, \ \pi]$, while $\theta_2 = \arctan\frac{c}{r_1}$, indicates the angle leaving the X-Y plane for Z in range $\left(-\frac{\pi}{2}, \ \frac{\pi}{2}\right]$. We call θ_1 the first argument and θ_2 the second argument. So in exponential coordinate the expression is $\vec{S} = re^{\theta_1 i + \theta_2 j}$

Now we generalize the idea above as the following statement:

$$S = a_0 + a_1\bar{\imath} + a_2\acute{\imath} + a_3\check{\imath} + a_4\grave{\imath} + \cdots\cdots$$

represents a point $(a_0, a_1, a_2, \cdots\cdots, a_n)$ in $n + 1$ dimensional space. Such defined number is called $n + 1$ dimensional **spatial number**.

If let $r_0 = a_0$,

$r_1 = \sqrt{a_0^2 + a_1^2}$, (first modulus)

$r_2 = \sqrt{a_0^2 + a_1^2 + a_2^2}$, (second modulus)

$\cdots\cdots$,

$r_{n-1} = \sqrt{a_0^2 + a_1^2 + \cdots + a_{n-1}^2}$,

and $r = \sqrt{a_0^2 + a_1^2 + \cdots + a_n^2}$,

then $\theta_1 = \arctan\frac{a_1}{r_0}$, (first argument)

$\theta_2 = arctan\frac{a_2}{r_1}$, (second argument)

...... ,

and $\theta_n = arctan\frac{a_n}{r_{n-1}}$,

It also can be expressed in exponential form as $S = re^{\theta_1\bar{\imath}+\theta_2\acute{\imath}+\cdots}$.

To simplify the expression, let a represent the sequence $[a_1, a_2, \cdots, a_n]$(called $\mathbf{spatial-position-row}$) and I represent the sequence $[\bar{\imath}, \acute{\imath}, \check{\imath}, \grave{\imath}, \cdots,]$ (called **spatial-axis-column**), so that $S = a_0 + aI$, where aI implies the dot product of a and I. We call it the **spatial-position expression**. If the real part a_0 is inserted into a and 1 into I, i.e. $A = [a_0, a_1, a_2, \cdots, a_n]$ and $\hat{I} = [1, \bar{\imath}, \acute{\imath}, \check{\imath}, \grave{\imath}, \cdots,]$, we can get the simpliest form $S = A\hat{I}$, which is called **all-position expression**.
Parallelly in exponential expression, if $\theta = [\theta_1, \theta_2, \theta_3, \cdots, \theta_n]$, then $S = re^{\theta I}$, which is called **spatial-phrase expression**. Since $r > 0$, so if let $\theta_0 = \ln r$ and $\Theta = [\theta_0, \theta_1, \theta_2, \theta_3, \cdots, \theta_n]$, then $S = e^{\Theta\hat{I}}$, that is called **all-phrase expression**.

2. **Operation rules**

2.1 Addition and subtraction

Spatial numbers follow the same rules of addition and subtraction as applied in complex numbers, that is to add or subtract each part respectively.

$$S = a_0 + a_1\bar{\imath} + a_2\acute{\imath} + a_3\check{\imath} + \cdots = a_0 + aI$$
$$P = b_0 + b_1\bar{\imath} + b_2\acute{\imath} + b_3\check{\imath} + \cdots = b_0 + bI$$

then

$$\begin{aligned} S \pm P &= (a_0 \pm b_0) + (a_1 \pm b_1)\bar{\imath} + (a_2 \pm b_2)\acute{\imath} + (a_3 \pm b_3)\check{\imath} + \cdots \\ &= (a_0 \pm b_0) + (a \pm b)I \end{aligned}$$

While in exponential expression, it has to be transformed in the following way:

If $S = re^{\theta_1\bar{\imath}+\theta_2\acute{\imath}+\cdots+\theta_n\hat{\imath}} = re^{\theta I}$.

and

$P = \rho e^{\varphi_1\bar{\imath}+\varphi_2\acute{\imath}+\cdots+\varphi_n\hat{\imath}} = \rho e^{\varphi I}$.

then

$$a_0 = r\cos\theta_1\cos\theta_2\cos\theta_3 \dots \cos\theta_n$$
$$b_0 = \rho\cos\varphi_1\cos\varphi_2\cos\varphi_3 \dots \cos\varphi_n$$

$$a_1 = r\sin\theta_1\cos\theta_2\cos\theta_3 \dots \cos\theta_n$$
$$b_1 = \rho\sin\varphi_1\cos\varphi_2\cos\varphi_3 \dots \cos\varphi_n$$

$$a_2 = r\sin\theta_2\cos\theta_3 \dots \cos\theta_n$$
$$b_2 = \rho\sin\varphi_2\cos\varphi_3 \dots \cos\varphi_n$$
$$\dots\dots$$
$$a_{n-1} = r\sin\theta_{n-1}\cos\theta_n$$
$$b_{n-1} = \rho\sin\varphi_{n-1}\cos\varphi_n$$
$$a_n = r\sin\theta_n$$
$$b_n = \rho\sin\varphi_n$$

For the sake of simplicity, we initiatively define two functions:
The first is **Recursive-cosine** function, defined as:

$$Recos\theta = \cos\theta_1\cos\theta_2\cos\theta_3 \dots \cos\theta_n$$

where $\theta = [\theta_1, \theta_2, \theta_3, \cdots, \theta_n]$,
and the second is **Recursive-sine** function, defined as:

$$Resin\theta = [\sin\theta_1\cos\theta_2\cos\theta_3 \dots \cos\theta_n, \sin\theta_2\cos\theta_3 \dots \cos\theta_n, \sin\theta_3\cos\theta_4 \dots \cos\theta_n, \cdots\cdots, \sin\theta_{n-1}\cos\theta_n, \sin\theta_n]$$

then we can get the following expression:

$$S = re^{\theta I} = r(Recos\theta + IResin\theta)$$
$$P = \rho e^{\varphi I} = \rho(Recos\varphi + IResin\varphi)$$
$$S \pm P = re^{\theta I} \pm \rho e^{\varphi I} = rRecos\theta \pm \rho Recos\theta + I(rResin\theta \pm \rho Resin\theta)$$

2.2 Multiplication and division

Before inquiring the method of multiplication for spatial numbers, we should review the rule applied in complex numbers:

$$Z_1 = r_1 e^{\theta i}$$
$$Z_2 = r_2 e^{\varphi i}$$
$$Z_1 Z_2 = r_1 r_2 e^{(\theta+\varphi)i}$$

We assume that this rule still holds for spatial numbers, so the multiplication of two spatial numbers can be defined as following statement:
if they are given in exponential expressions, here in **all-phrase-expression**:

$$S = e^{\Theta\hat{I}}$$

$$P = e^{\Phi\hat{I}}$$

Then

$$\mathrm{SP} = \mathrm{e}^{(\Theta+\Phi)\hat{I}}$$

Quite simple!
However, it's not easy to see how to multiply two spatial numbers given in cartesian coordinate expressions:

$$S = a_0 + a_1\bar{\imath} + a_2\acute{\imath} + a_3\check{\imath} + \cdots\cdots = a\hat{I}$$
$$P = b_0 + b_1\bar{\imath} + b_2\acute{\imath} + b_3\check{\imath} + \cdots\cdots = b\hat{I}$$

In fact, it is a recursive procession as follows:

if
$$S_1 = a_0 + a_1\bar{\imath} = (a_0, a_1)$$
$$P_1 = b_0 + b_1\bar{\imath} = (b_0, b_1)$$
$$S_2 = a_0 + a_1\bar{\imath} + a_2\acute{\imath} = (S_1, a_2)$$
$$P_2 = b_0 + b_1\bar{\imath} + b_2\acute{\imath} = (P_1, b_2)$$

......

$$S_n = (S_{n-1}, a_n)$$
$$P_n = (P_{n-1}, b_n)$$

then
$$S_1P_1 = (a_0b_0\left(1 - \frac{a_1b_1}{a_0b_0}\right),\ a_1b_0 + a_0b_1)$$

$$= (a_0b_0 - a_1b_1, a_1b_0 + a_0b_1)$$
(multiplication of two normal complex numbers)

$$S_2P_2 = \left(S_1P_1\left(1 - \frac{a_2b_2}{|S_1||P_1|}\right),\ a_2|P_1| + b_2|S_1|\right)$$
$$S_3P_3 = \left(S_2P_2\left(1 - \frac{a_3b_3}{|S_2||P_2|}\right),\qquad a_3|P_2| + b_3|S_2|\right)$$
$$\cdots\cdots$$
$$S_nP_n = (S_{n-1}P_{n-1}\left(1 - \frac{a_nb_n}{|S_{n-1}||P_{n-1}|}\right),\qquad a_n|P_{n-1}| + b_n|S_{n-1}|)$$
$$\cdots\cdots$$

In the recursive procession above, the $|S_{n-1}||P_{n-1}|$ should be non-zero. Otherwise, when one of the two is zero, if let $|S_{n-1}| \neq 0$ and $|P_{n-1}| = 0$, then $S_nP_n = (-S_{n-1}\frac{a_nb_n}{|S_{n-1}|},\ b_n|S_{n-1}|)$

If $S_{n-1} = P_{n-1} = 0$, then $S_nP_n = -a_nb_n$.

Easily proved is spatial number as a number field.

3. **Discussion on imaginary unit**

3.1 $\bar{\imath}\acute{\imath} = ?$

$\bar{\imath} = (0, 1, 0)$, $\acute{\imath} = (0, 0, 1)$

so $\bar{\imath}\acute{\imath} = \imath$

or

$\bar{\imath} = e^{\frac{\pi}{2}\imath}$ $\acute{\imath} = e^{\frac{\pi}{2}\imath}$

$\bar{\imath}\acute{\imath} = e^{\frac{\pi}{2}\bar{\imath}+\frac{\pi}{2}\imath} = \left(cos\frac{\pi}{2}cos\frac{\pi}{2}, sin\frac{\pi}{2}\ cos\frac{\pi}{2}, sin\frac{\pi}{2}\right) = (0,0,1) = \imath$

Similarly, $\bar{\imath}\check{\imath} = \acute{\imath}\check{\imath} = \bar{\imath}\acute{\imath}\check{\imath} = \check{\imath}$

3.2 $\acute{\imath}^2 = ?$

Just like $i^2 = -1$ in complex number, $\acute{\imath}^2 = -1$ still holds. In fact, $\acute{\imath}^2 = \left(e^{\theta\bar{\imath}+\frac{\pi}{2}\acute{\imath}}\right)^2 = -(cos2\theta,\ sin2\theta)$, where $\theta \in (-\pi, \pi]$. It is an inverse unit circle in the complex plane.

Similarly, $\check{\imath}^2 = \left(e^{\theta\bar{\imath}+\varphi\acute{\imath}+\frac{\pi}{2}\check{\imath}}\right)^2 = -(cos2\theta cos2\varphi,\ sin2\theta cos2\varphi, sin2\varphi)$, where $\theta \in (-\pi, \pi]$ and $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$, representing the inverse unit sphere in 3-D space.

3.3 $\frac{1}{\acute{\imath}} = ?$

$\frac{1}{\acute{\imath}} = e^{\left(0-\frac{\pi}{2}\right)\acute{\imath}} = -\acute{\imath}$

$\frac{1}{\check{\imath}} = e^{\left(0-\frac{\pi}{2}\right)\check{\imath}} = -\check{\imath}$

4. **Extension on complex number**

4.1 On C-R function

Cauchy-Riemann theorem says that for complex number $Z = x + yi$,
if $f(Z) = u(x,y) + v(x,y)i$ is differential at (x_0, y_0),
then exists:

$$u_x = v_y$$
$$u_y = -v_x$$

If spatial number $S = x + y\bar{\imath} + z\acute{\imath}$ and function $f(S) = u(x,y,z) + v(x,y,z)\bar{\imath} + w(x,y,z)\acute{\imath}$ is differential at (x_0, y_0, z_0)

then exists:

$$u_x = v_y = w_z$$
$$v_x = -u_y$$
$$w_x = -w_y = -v_z - u_z$$

If spatial number $S = x + y\bar{\imath} + z\acute{\imath} + t\check{\imath}$ and function $f(S) = u(x,y,z,t) + v(x,y,z,t)\bar{\imath} + w(x,y,z,t)\acute{\imath} + r(x,y,z,t)\check{\imath}$ is differential at (x_0, y_0, z_0, t_0)

then exists:

$$\mathrm{u_x = v_y = w_z = r_t}$$

$$\mathrm{v_x = -u_y = 0}$$
$$\mathrm{w_x = -w_y = -v_z - u_z = 0}$$

$$\mathrm{r_x} =- \mathrm{r_y} =- \mathrm{r_z} =- \mathrm{u_t} - \mathrm{v_t} - \mathrm{w_t}$$

4.2 The angle between two spatial numbers

If two complex numbers $Z_1 = re^{\theta i}$ and $Z_2 = \rho e^{\varphi i}$,
the angle between Z_1 and Z_2 is got from equation:

$$\cos(\theta - \varphi) = cos\theta cos\varphi + sin\theta sin\varphi$$

If two spatial numbers are given as follows:

$$S_1 = re^{\theta I} = a\hat{I} \text{ and } S_2 = \rho e^{\varphi I} = b\hat{I},$$

and the angle between S_1 and S_2 in the plane defined by them is denoted as $\theta \to \varphi$, then we get the equation:

$$\cos(\theta \to \varphi) = Recos\theta Recos\varphi + Resin\theta Resin\varphi = \frac{ab}{|a||b|}$$