

Quantum information

We're now prepared to move on to quantum information in the setting of multiple systems. Much like in the previous lesson on single systems, the mathematical description of quantum information for multiple systems is quite similar to the probabilistic case and makes use of similar concepts and techniques.

Quantum states

Multiple systems can be viewed collectively as single, compound systems. We've already observed this in the probabilistic setting, and the quantum setting is analogous. Quantum states of multiple systems are therefore represented by column vectors having complex number entries and Euclidean norm equal to 1, just like quantum states of single systems. In the multiple system case, the entries of these vectors are placed in correspondence with the *Cartesian product* of the classical state sets associated with each of the individual systems, because that's the classical state set of the compound system.

For instance, if **X** and **Y** are qubits, then the classical state set of the pair of qubits (**X**, **Y**), viewed collectively as a single system, is the Cartesian product $\{0, 1\} \times \{0, 1\}$. By representing pairs of binary values as binary strings of length two, we associate this Cartesian product set with the set $\{00, 01, 10, 11\}$. The following vectors are therefore all examples of quantum state vectors of the pair (**X**, **Y**) :

$$\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{6}}|01\rangle + \frac{i}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{6}}|11\rangle, \quad \frac{3}{5}|00\rangle - \frac{4}{5}|11\rangle, \quad \text{and}$$


There are variations on how quantum state vectors of multiple systems are expressed, and we can choose whichever variation suits our preferences. Here are some examples for the first quantum state vector above.

1. We may use the fact that $|ab\rangle = |a\rangle|b\rangle$ (for any classical states a and b) to instead write

$$\frac{1}{\sqrt{2}}|0\rangle|0\rangle - \frac{1}{\sqrt{6}}|0\rangle|1\rangle + \frac{i}{\sqrt{6}}|1\rangle|0\rangle + \frac{1}{\sqrt{6}}|1\rangle|1\rangle.$$

2. We may choose to write the tensor product symbol explicitly like this:

$$\frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle - \frac{1}{\sqrt{6}}|0\rangle \otimes |1\rangle + \frac{i}{\sqrt{6}}|1\rangle \otimes |0\rangle + \frac{1}{\sqrt{6}}|1\rangle \otimes |1\rangle.$$

3. We may subscript the kets to indicate how they correspond to the systems being considered, like this:

$$\frac{1}{\sqrt{2}}|0\rangle_X|0\rangle_Y - \frac{1}{\sqrt{6}}|0\rangle_X|1\rangle_Y + \frac{i}{\sqrt{6}}|1\rangle_X|0\rangle_Y + \frac{1}{\sqrt{6}}|1\rangle_X|1\rangle_Y.$$

Of course, we may also write quantum state vectors explicitly as column vectors:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}.$$

Depending upon the context in which it appears, one of these variations may be preferred — but they are all equivalent in the sense that they describe the same vector.

Tensor products of quantum state vectors

Similar to what we have for probability vectors, tensor products of quantum state vectors are also quantum state vectors — and again they represent *independence* among systems.

In greater detail, and beginning with the case of two systems, suppose that $|\phi\rangle$ is a quantum state vector of a system \mathbf{X} and $|\psi\rangle$ is a quantum state vector of a system \mathbf{Y} . The tensor product $|\phi\rangle \otimes |\psi\rangle$, which may alternatively be written as $|\phi\rangle|\psi\rangle$ or as $|\phi \otimes \psi\rangle$, is then a quantum state vector of the joint system (\mathbf{X}, \mathbf{Y}) . Again we refer to a state of this form as a being a *product state*.

Intuitively speaking, when a pair of systems (\mathbf{X}, \mathbf{Y}) is in a product state $|\phi\rangle \otimes |\psi\rangle$, we may interpret this as meaning that \mathbf{X} is in the quantum

state $|\phi\rangle$, \mathbf{Y} is in the quantum state $|\psi\rangle$, and the states of the two systems have nothing to do with one another.

The fact that the tensor product vector $|\phi\rangle \otimes |\psi\rangle$ is indeed a quantum state vector is consistent with the Euclidean norm being *multiplicative* with respect to tensor products:

$$\begin{aligned}\| |\phi\rangle \otimes |\psi\rangle \| &= \sqrt{\sum_{(a,b) \in \Sigma \times \Gamma} |\langle ab|\phi \otimes \psi\rangle|^2} \\ &= \sqrt{\sum_{a \in \Sigma} \sum_{b \in \Gamma} |\langle a|\phi\rangle \langle b|\psi\rangle|^2} \\ &= \sqrt{\left(\sum_{a \in \Sigma} |\langle a|\phi\rangle|^2\right) \left(\sum_{b \in \Gamma} |\langle b|\psi\rangle|^2\right)} \\ &= \| |\phi\rangle \| \| |\psi\rangle \|.\end{aligned}$$

Because $|\phi\rangle$ and $|\psi\rangle$ are quantum state vectors, we have $\| |\phi\rangle \| = 1$ and $\| |\psi\rangle \| = 1$, and therefore $\| |\phi\rangle \otimes |\psi\rangle \| = 1$, so $|\phi\rangle \otimes |\psi\rangle$ is also a quantum state vector.

This generalizes to more than two systems. If $|\psi_0\rangle, \dots, |\psi_{n-1}\rangle$ are quantum state vectors of systems $\mathbf{X}_0, \dots, \mathbf{X}_{n-1}$, then $|\psi_{n-1}\rangle \otimes \dots \otimes |\psi_0\rangle$ is a quantum state vector representing a *product state* of the joint system $(\mathbf{X}_{n-1}, \dots, \mathbf{X}_0)$. Again, we know that this is a quantum state vector because

$$\| |\psi_{n-1}\rangle \otimes \dots \otimes |\psi_0\rangle \| = \| |\psi_{n-1}\rangle \| \dots \| |\psi_0\rangle \| = 1^n = 1.$$

Entangled states

Not all quantum state vectors of multiple systems are product states. For example, the quantum state vector

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \tag{1}$$

of two qubits is not a product state. To reason this, we may follow exactly the same argument that we used in the previous section for a probabilistic state. That is, if (1) were a product state, there would exist quantum state vectors $|\phi\rangle$ and $|\psi\rangle$ for which

$$|\phi\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$$

But then it would necessarily be the case that

$$\langle 0|\phi\rangle\langle 1|\psi\rangle = \langle 01|\phi\otimes\psi\rangle = 0$$

implying that $\langle 0|\phi\rangle = 0$ or $\langle 1|\psi\rangle = 0$ (or both). That contradicts the fact that

$$\langle 0|\phi\rangle\langle 0|\psi\rangle = \langle 00|\phi\otimes\psi\rangle = \frac{1}{\sqrt{2}}$$

and

$$\langle 1|\phi\rangle\langle 1|\psi\rangle = \langle 11|\phi\otimes\psi\rangle = \frac{1}{\sqrt{2}}$$

are both nonzero. Thus, the quantum state vector (1) represents a *correlation* between two systems, and specifically we say that the systems are *entangled*.

Notice that the specific value $1/\sqrt{2}$ is not important to this argument — all that is important is that this value is nonzero. Thus, for instance, the quantum state

$$\frac{3}{5}|00\rangle + \frac{4}{5}|11\rangle$$

is also not a product state, by the same argument.

Entanglement is a quintessential feature of quantum information that will be discussed in greater detail in a later lesson. Entanglement can be complicated, particularly for the sorts of noisy quantum states that can be described by density matrices (which are discussed in the *General formulation of quantum information* course, which is the third course in the *Understanding Quantum Information and Computation* series). For quantum state vectors, however, entanglement is equivalent to correlation: any quantum state vector that is not a product state represents an entangled state.

In contrast, the quantum state vector

$$\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle$$

is an example of a product state.

$$\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle \right)$$

Hence, this state is not entangled.

Bell states

We'll now take a look at some important examples of multiple-qubit quantum states, beginning with the *Bell states*. These are the following four two-qubit states:

$$\begin{aligned} |\phi^+\rangle &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \\ |\phi^-\rangle &= \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle \\ |\psi^+\rangle &= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \\ |\psi^-\rangle &= \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle \end{aligned}$$

The Bell states are so-named in honor of John Bell. Notice that the same argument that establishes that $|\phi^+\rangle$ is not a product state reveals that none of the other Bell states are product states either: all four of the Bell states represent entanglement between two qubits.

The collection of all four Bell states

$$\{|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle\}$$

is known as the *Bell basis*. True to its name, this is a basis; any quantum state vector of two qubits, or indeed any complex vector at all having entries corresponding to the four classical states of two bits, can be expressed as a linear combination of the four Bell states. For example,

$$|00\rangle = \frac{1}{\sqrt{2}}|\phi^+\rangle + \frac{1}{\sqrt{2}}|\phi^-\rangle.$$

GHZ and W states

Next we will consider two interesting examples of states of three qubits. The first example is the *GHZ state* (so named in honor of Daniel Greenberger, Michael Horne, and Anton Zeilinger, who first studied some of its properties):

$$\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle.$$

The second example is the so-called W state:

$$\frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle.$$

Neither of these states is a product state, meaning that they cannot be written as a tensor product of three qubit quantum state vectors. We'll examine both of these states later when we discuss partial measurements of quantum states of multiple systems.

Additional examples

The examples of quantum states of multiple systems we've seen so far are states of two or three qubits, but we can also consider quantum states of multiple systems having different classical state sets.

For example, here's a quantum state of three systems, X , Y , and Z , where the classical state set of X is the binary alphabet (so X is a qubit) and the classical state set of Y and Z is $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$:

$$\frac{1}{2}|0\rangle|\heartsuit\rangle|\heartsuit\rangle + \frac{1}{2}|1\rangle|\spadesuit\rangle|\heartsuit\rangle - \frac{1}{\sqrt{2}}|0\rangle|\heartsuit\rangle|\diamondsuit\rangle.$$

And here's an example of a quantum state of three systems, X , Y , and Z , that all share the same classical state set $\{0, 1, 2\}$:

$$\frac{|012\rangle - |021\rangle + |120\rangle - |102\rangle + |201\rangle - |210\rangle}{\sqrt{6}}.$$

Systems having the classical state set $\{0, 1, 2\}$ are often called *trits* or (assuming that they can be in a quantum state) *qutrits*. The term *qudit* refers to a system having classical state set $\{0, \dots, d - 1\}$ for an arbitrary choice of d .

Measurements of quantum states

Standard basis measurements of quantum states of single systems were discussed in the previous lesson: if a system having classical state set Σ is in a quantum state represented by the vector $|\psi\rangle$, and that system is measured (with respect to a standard basis measurement), then each classical state $a \in \Sigma$ appears with probability $|\langle a|\psi\rangle|^2$. This tells us what happens when we have a quantum state of multiple systems and choose to measure the entire compound system, which is equivalent to measuring *all* of the systems.

To state this precisely, let us suppose that X_0, \dots, X_{n-1} are systems having classical state sets $\Sigma_0, \dots, \Sigma_{n-1}$, respectively. We may then view (X_{n-1}, \dots, X_0) collectively as a single system whose classical state set is the Cartesian product $\Sigma_{n-1} \times \dots \times \Sigma_0$. If a quantum state of this system is represented by the quantum state vector $|\psi\rangle$, and all of the systems are measured, then each possible outcome $(a_{n-1}, \dots, a_0) \in \Sigma_{n-1} \times \dots \times \Sigma_0$ appears with probability $|\langle a_{n-1} \cdots a_0 | \psi \rangle|^2$.

For example, if systems X and Y are jointly in the quantum state

$$\frac{3}{5}|0\rangle|\heartsuit\rangle - \frac{4i}{5}|1\rangle|\spadesuit\rangle,$$

then measuring both systems with standard basis measurements yields the outcome $(0, \heartsuit)$ with probability $9/25$ and the outcome $(1, \spadesuit)$ with probability $16/25$.

Partial measurements

Now let us consider the situation in which we have multiple systems in some quantum state, and we measure a proper subset of the systems. As before, we will begin with two systems X and Y having classical state sets Σ and Γ , respectively.

In general, a quantum state vector of (X, Y) takes the form

$$|\psi\rangle = \sum_{(a,b) \in \Sigma \times \Gamma} \alpha_{ab}|ab\rangle,$$

where $\{\alpha_{ab} : (a, b) \in \Sigma \times \Gamma\}$ is a collection of complex numbers satisfying

$$\sum_{(a,b) \in \Sigma \times \Gamma} |\alpha_{ab}|^2 = 1,$$

which is equivalent to $|\psi\rangle$ being a unit vector.

We already know, from the discussion above, that if both X and Y are measured, then each possible outcome $(a, b) \in \Sigma \times \Gamma$ appears with probability

$$|\langle ab | \psi \rangle|^2 = |\alpha_{ab}|^2.$$

If we suppose instead that just the first system X is measured, the probability for each outcome $a \in \Sigma$ to appear must therefore be equal to

$$\sum_{b \in \Gamma} |\langle ab | \psi \rangle|^2 = \sum_{b \in \Gamma} |\alpha_{ab}|^2.$$

This is consistent with what we already saw in the probabilistic setting, as well as our current understanding of physics: the probability for each outcome to appear when \mathbf{X} is measured can't possibly depend on whether or not \mathbf{Y} was also measured, as that would allow for faster-than-light communication.

Having obtained a particular outcome $a \in \Sigma$ of a standard basis measurement of \mathbf{X} , we naturally expect that the quantum state of \mathbf{X} changes so that it is equal to $|a\rangle$, just like we had for single systems. But what happens to the quantum state of \mathbf{Y} ?

To answer this question, we can first express the vector $|\psi\rangle$ as

$$|\psi\rangle = \sum_{a \in \Sigma} |a\rangle \otimes |\phi_a\rangle,$$

where

$$|\phi_a\rangle = \sum_{b \in \Gamma} \alpha_{ab} |b\rangle$$

for each $a \in \Sigma$. Here we're following the same methodology as in the probabilistic case, of isolating the standard basis states of the system being measured. The probability for the standard basis measurement of \mathbf{X} to give each outcome a is as follows:

$$\sum_{b \in \Gamma} |\alpha_{ab}|^2 = \|\phi_a\|^2.$$

And, as a result of the standard basis measurement of \mathbf{X} giving the outcome a , the quantum state of the pair (\mathbf{X}, \mathbf{Y}) together becomes

$$|a\rangle \otimes \frac{|\phi_a\rangle}{\|\phi_a\|}.$$

That is, the state "collapses" like in the single-system case, but only as far as is required for the state to be consistent with the measurement of \mathbf{X} having produced the outcome a .

Informally speaking, $|a\rangle \otimes |\phi_a\rangle$ represents the component of $|\psi\rangle$ that is consistent with the measurement of \mathbf{X} resulting in the outcome a . We then *normalize* this vector — by dividing it by its Euclidean norm, which is equal to $\|\phi_a\|$ — to obtain a valid quantum state vector having Euclidean norm equal to 1. This normalization step is analogous to what

we did in the probabilistic setting when we divided vectors by the sum of their entries to obtain a probability vector.

As an example, consider the state of two qubits (X, Y) from the beginning of the section:

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{6}}|01\rangle + \frac{i}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{6}}|11\rangle.$$

To understand what happens when the first system X is measured, we begin by writing

$$|\psi\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{6}}|1\rangle \right) + |1\rangle \otimes \left(\frac{i}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle \right).$$

We now see, based on the description above, that the probability for the measurement to result in the outcome 0 is

$$\left\| \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{6}}|1\rangle \right\|^2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3},$$

in which case the state of (X, Y) becomes

$$|0\rangle \otimes \frac{\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{6}}|1\rangle}{\sqrt{\frac{2}{3}}} = |0\rangle \otimes \left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle \right);$$

and the probability for the measurement to result in the outcome 1 is

$$\left\| \frac{i}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle \right\|^2 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3},$$

in which case the state of (X, Y) becomes

$$|1\rangle \otimes \frac{\frac{i}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle}{\sqrt{\frac{1}{3}}} = |1\rangle \otimes \left(\frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right).$$

The same technique, used in a symmetric way, describes what happens if the second system Y is measured rather than the first. This time we rewrite the vector $|\psi\rangle$ as

$$|\psi\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{6}}|1\rangle \right) \otimes |0\rangle + \left(-\frac{1}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle \right) \otimes |1\rangle.$$

The probability that the measurement of Y gives the outcome 0 is

$$\left\| \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{6}}|1\rangle \right\|^2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3},$$

in which case the state of (X, Y) becomes

$$\frac{\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{6}}|1\rangle}{\sqrt{\frac{2}{3}}} \otimes |0\rangle = \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle \right) \otimes |0\rangle;$$

and the probability that the measurement outcome is 1 is

$$\left\| -\frac{1}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle \right\|^2 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3},$$

in which case the state of (X, Y) becomes

$$\frac{-\frac{1}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle}{\frac{1}{\sqrt{3}}} \otimes |1\rangle = \left(-\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right) \otimes |1\rangle.$$

Remark on reduced quantum states

The previous example shows a limitation of the simplified description of quantum information, which is that it does not offer us a way to describe the reduced (or marginal) quantum state of just one of two systems (or of a proper subset of any number of systems) like in the probabilistic case.

Specifically, for a probabilistic state of two systems (X, Y) described by a probability vector

$$\sum_{(a,b) \in \Sigma \times \Gamma} p_{ab}|ab\rangle,$$

we can write the *reduced* or *marginal* probabilistic state of X alone as

$$\sum_{a \in \Sigma} \left(\sum_{b \in \Gamma} p_{ab} \right) |a\rangle = \sum_{(a,b) \in \Sigma \times \Gamma} p_{ab} |a\rangle.$$

For quantum state vectors, there isn't an analogous way to do this. In particular, for a quantum state vector

$$|\psi\rangle = \sum_{(a,b) \in \Sigma \times \Gamma} \alpha_{ab} |ab\rangle,$$

the vector

$$\sum_{(a,b) \in \Sigma \times \Gamma} \alpha_{ab} |a\rangle$$

is not a quantum state vector in general, and does not properly represent the concept of a reduced or marginal state.

What we may do instead is turn to the notion of a *density matrix*, which is discussed in the *General formulation of quantum information* course. Density matrices provide us with a meaningful way to define reduced quantum states that is analogous to the probabilistic setting.

Partial measurements for three or more systems

Partial measurements for three or more systems, where some proper subset of the systems are measured, can be reduced to the case of two systems by dividing the systems into two collections, those that are measured and those that are not. Here is a specific example that illustrates how this can be done. It demonstrates specifically how subscripting kets by the names of the systems they represent can be useful — in this case because it gives us a simple way to describe permutations of the systems.

For this example, we'll consider a quantum state of a 5-tuple of systems (X_4, \dots, X_0) , where all five of these systems share the same classical state set $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$:

$$\sqrt{\frac{1}{7}}|\heartsuit\rangle|\clubsuit\rangle|\diamondsuit\rangle|\spadesuit\rangle|\spadesuit\rangle + \sqrt{\frac{2}{7}}|\diamondsuit\rangle|\clubsuit\rangle|\diamondsuit\rangle|\spadesuit\rangle|\clubsuit\rangle + \sqrt{\frac{1}{7}}|\spadesuit\rangle|\spadesuit\rangle|\clubsuit\rangle|\langle$$

$$-i\sqrt{\frac{2}{7}}|\heartsuit\rangle|\clubsuit\rangle|\diamondsuit\rangle|\heartsuit\rangle|\heartsuit\rangle - \sqrt{\frac{1}{7}}|\spadesuit\rangle|\heartsuit\rangle|\clubsuit\rangle|\spadesuit\rangle|\clubsuit\rangle.$$

We'll consider the situation in which the first and third systems are measured, and the remaining systems are left alone.

Conceptually speaking, there's no fundamental difference between this situation and one in which one of two systems is measured.

Unfortunately, because the measured systems are interspersed with the unmeasured systems, we face a hurdle in writing down the expressions needed to perform these calculations.

One way to proceed, as suggested above, is to subscript the kets to indicate which systems they refer to. This gives us a way to keep track of the systems as we permute the ordering of the kets, which makes the mathematics simpler.

First, the quantum state vector above can alternatively be written as

$$\begin{aligned} & \sqrt{\frac{1}{7}}|\heartsuit\rangle_4|\clubsuit\rangle_3|\diamondsuit\rangle_2|\spadesuit\rangle_1|\spadesuit\rangle_0 + \sqrt{\frac{2}{7}}|\diamondsuit\rangle_4|\clubsuit\rangle_3|\diamondsuit\rangle_2|\spadesuit\rangle_1|\clubsuit\rangle_0 \\ & + \sqrt{\frac{1}{7}}|\spadesuit\rangle_4|\spadesuit\rangle_3|\clubsuit\rangle_2|\diamondsuit\rangle_1|\clubsuit\rangle_0 - i\sqrt{\frac{2}{7}}|\heartsuit\rangle_4|\clubsuit\rangle_3|\diamondsuit\rangle_2|\heartsuit\rangle_1|\heartsuit\rangle_0 \\ & - \sqrt{\frac{1}{7}}|\spadesuit\rangle_4|\heartsuit\rangle_3|\clubsuit\rangle_2|\spadesuit\rangle_1|\clubsuit\rangle_0. \end{aligned}$$

Nothing has changed, except that each ket now has a subscript indicating which system it corresponds to. Here we've used the subscripts $0, \dots, 4$, but the names of the systems themselves could also be used (in a situation where we have system names such as X , Y , and Z , for instance).

We can now re-order the kets and collect terms as follows:

$$\begin{aligned} & \sqrt{\frac{1}{7}}|\heartsuit\rangle_4|\diamondsuit\rangle_2|\clubsuit\rangle_3|\spadesuit\rangle_1|\spadesuit\rangle_0 + \sqrt{\frac{2}{7}}|\diamondsuit\rangle_4|\diamondsuit\rangle_2|\clubsuit\rangle_3|\spadesuit\rangle_1|\clubsuit\rangle_0 \\ & + \sqrt{\frac{1}{7}}|\spadesuit\rangle_4|\clubsuit\rangle_2|\spadesuit\rangle_3|\diamondsuit\rangle_1|\clubsuit\rangle_0 - i\sqrt{\frac{2}{7}}|\heartsuit\rangle_4|\diamondsuit\rangle_2|\clubsuit\rangle_3|\heartsuit\rangle_1|\heartsuit\rangle_0 \\ & - \sqrt{\frac{1}{7}}|\spadesuit\rangle_4|\clubsuit\rangle_2|\heartsuit\rangle_3|\spadesuit\rangle_1|\clubsuit\rangle_0 \\ & = |\heartsuit\rangle_4|\diamondsuit\rangle_2 \left(\sqrt{\frac{1}{7}}|\clubsuit\rangle_3|\spadesuit\rangle_1|\spadesuit\rangle_0 - i\sqrt{\frac{2}{7}}|\clubsuit\rangle_3|\heartsuit\rangle_1|\heartsuit\rangle_0 \right) \\ & + |\diamondsuit\rangle_4|\diamondsuit\rangle_2 \left(\sqrt{\frac{2}{7}}|\clubsuit\rangle_3|\spadesuit\rangle_1|\clubsuit\rangle_0 \right) \\ & + |\spadesuit\rangle_4|\clubsuit\rangle_2 \left(\sqrt{\frac{1}{7}}|\spadesuit\rangle_3|\diamondsuit\rangle_1|\clubsuit\rangle_0 - \sqrt{\frac{1}{7}}|\heartsuit\rangle_3|\spadesuit\rangle_1|\clubsuit\rangle_0 \right) \end{aligned}$$

The tensor products are still implicit, even when parentheses are used, as in this example.

To be clear about permuting the kets, tensor products are not commutative: if $|\phi\rangle$ and $|\pi\rangle$ are vectors, then, in general, $|\phi\rangle \otimes |\pi\rangle$ is different from $|\pi\rangle \otimes |\phi\rangle$, and likewise for tensor products of three or more vectors. For instance, $|\heartsuit\rangle|\clubsuit\rangle|\diamondsuit\rangle|\spadesuit\rangle|\spadesuit\rangle$ is a different vector than $|\heartsuit\rangle|\diamondsuit\rangle|\clubsuit\rangle|\spadesuit\rangle|\spadesuit\rangle$. Re-ordering the kets as we have just done should not be interpreted as suggesting otherwise.

Rather, for the sake of performing calculations, we're simply making a decision that it's more convenient to collect the systems together as $(X_4, X_2, X_3, X_1, X_0)$ rather than $(X_4, X_3, X_2, X_1, X_0)$. The subscripts

on the kets serve to keep this all straight, and we're free to revert back to the original ordering later if we wish to do that.

We now see that, if the systems X_4 and X_2 are measured, the (nonzero) probabilities of the different outcomes are as follow:

- The measurement outcome $(\heartsuit, \diamondsuit)$ occurs with probability

$$\left\| \sqrt{\frac{1}{7}}|\clubsuit\rangle_3|\spadesuit\rangle_1|\spadesuit\rangle_0 - i\sqrt{\frac{2}{7}}|\clubsuit\rangle_3|\heartsuit\rangle_1|\heartsuit\rangle_0 \right\|^2 = \frac{1}{7} + \frac{2}{7} = \frac{3}{7}$$

- The measurement outcome $(\diamondsuit, \diamondsuit)$ occurs with probability

$$\left\| \sqrt{\frac{2}{7}}|\clubsuit\rangle_3|\spadesuit\rangle_1|\clubsuit\rangle_0 \right\|^2 = \frac{2}{7}$$

- The measurement outcome (\spadesuit, \clubsuit) occurs with probability

$$\left\| \sqrt{\frac{1}{7}}|\spadesuit\rangle_3|\diamondsuit\rangle_1|\clubsuit\rangle_0 - \sqrt{\frac{1}{7}}|\heartsuit\rangle_3|\spadesuit\rangle_1|\clubsuit\rangle_0 \right\|^2 = \frac{1}{7} + \frac{1}{7} = \frac{2}{7}.$$

If the measurement outcome is $(\heartsuit, \diamondsuit)$, for instance, the resulting state of our five systems becomes

$$\begin{aligned} |\heartsuit\rangle_4|\diamondsuit\rangle_2 &\otimes \frac{\sqrt{\frac{1}{7}}|\clubsuit\rangle_3|\spadesuit\rangle_1|\spadesuit\rangle_0 - i\sqrt{\frac{2}{7}}|\clubsuit\rangle_3|\heartsuit\rangle_1|\heartsuit\rangle_0}{\sqrt{\frac{3}{7}}} \\ &= \sqrt{\frac{1}{3}}|\heartsuit\rangle_4|\clubsuit\rangle_3|\diamondsuit\rangle_2|\spadesuit\rangle_1|\spadesuit\rangle_0 - i\sqrt{\frac{2}{3}}|\heartsuit\rangle_4|\clubsuit\rangle_3|\diamondsuit\rangle_2|\heartsuit\rangle_1|\heartsuit\rangle_0. \end{aligned}$$

Here, for the final answer, we've reverted back to our original ordering of the systems, just to illustrate that we can do this. For the other possible measurement outcomes, the state can be determined in a similar way.

Finally, here are two examples promised earlier, beginning with the GHZ state

$$\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle.$$

If just the first system is measured, we obtain the outcome 0 with probability $1/2$, in which case the state of the three qubits becomes $|000\rangle$; and we also obtain the outcome 1 with probability $1/2$, in which case the state of the three qubits becomes $|111\rangle$.

For a W state, on the other hand, assuming again that just the first system is measured, we begin by writing this state like this:

$$\begin{aligned} & \frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle \\ &= |0\rangle \left(\frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle \right) + |1\rangle \left(\frac{1}{\sqrt{3}}|00\rangle \right). \end{aligned}$$

The probability that a measurement of the first qubit results in the outcome 0 is therefore equal to

$$\left\| \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle \right\|^2 = \frac{2}{3},$$

and conditioned upon the measurement producing this outcome, the quantum state of the three qubits becomes

$$|0\rangle \otimes \frac{\frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle}{\sqrt{\frac{2}{3}}} = |0\rangle \left(\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \right) = |0\rangle|\psi^+\rangle.$$

The probability that the measurement outcome is 1 is $1/3$, in which case the state of the three qubits becomes $|100\rangle$.

The W state is symmetric, in the sense that it does not change if we permute the qubits. We therefore obtain a similar description for measuring the second or third qubit rather than the first.

Unitary operations

In principle, any unitary matrix whose rows and columns correspond to the classical states of a system represents a valid quantum operation on that system. This, of course, remains true for compound systems, whose classical state sets happen to be Cartesian products of the classical state sets of the individual systems.

Focusing in on two systems, if \mathbf{X} is a system having classical state set Σ , and \mathbf{Y} is a system having classical state set Γ , then the classical state set of the joint system (\mathbf{X}, \mathbf{Y}) is $\Sigma \times \Gamma$. Therefore, quantum operations on this joint system are represented by unitary matrices whose rows and columns are placed in correspondence with the set $\Sigma \times \Gamma$. The ordering of the rows and columns of these matrices is the same as the ordering used for quantum state vectors of the system (\mathbf{X}, \mathbf{Y}) .

For example, let us suppose that $\Sigma = \{1, 2, 3\}$ and $\Gamma = \{0, 1\}$, and recall that the standard convention for ordering the elements of the

Cartesian product $\{1, 2, 3\} \times \{0, 1\}$ is this:

$$(1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1).$$

Here's an example of a unitary matrix representing an operation on (X, Y) :

$$U = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{i}{2} & -\frac{1}{2} & 0 & 0 & -\frac{i}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} & 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

This unitary matrix isn't special, it's just an example. To check that U is unitary, it suffices to compute and check that $U^\dagger U = \mathbb{I}$, for instance. Alternatively, we can check that the rows (or the columns) are orthonormal, which is made simpler in this case given the particular form of the matrix U .

The action of U on the standard basis vector $|1, 1\rangle$, for instance, is

$$U|1, 1\rangle = \frac{1}{2}|1, 0\rangle + \frac{i}{2}|1, 1\rangle - \frac{1}{2}|2, 0\rangle - \frac{i}{2}|3, 0\rangle,$$

which we can see by examining the second column of U , considering our ordering of the set $\{1, 2, 3\} \times \{0, 1\}$.

As with any matrix, it is possible to express U using Dirac notation, which would require 20 terms for the 20 nonzero entries of U . If we did write down all of these terms, however, rather than writing a 6×6 matrix, it would be messy and the patterns that are evident from the matrix expression would not likely be as clear. Simply put, Dirac notation is not always the best choice.

Unitary operations on three or more systems work in a similar way, with the unitary matrices having rows and columns corresponding to the Cartesian product of the classical state sets of the systems. We've already seen one example in this lesson: the three-qubit operation

$$\sum_{k=0}^7 |(k+1) \bmod 8\rangle \langle k|,$$

where numbers in bras and kets mean their 3-bit binary encodings. In addition to being a deterministic operation, this is also a unitary operation. Operations that are both deterministic and unitary are called *reversible* operations. The conjugate transpose of this matrix can be written like this:

$$\sum_{k=0}^7 |k\rangle\langle(k+1) \bmod 8| = \sum_{k=0}^7 |(k-1) \bmod 8\rangle\langle k|.$$

This represents the *reverse*, or in mathematical terms the *inverse*, of the original operation — which is what we expect from the conjugate transpose of a unitary matrix. We'll see other examples of unitary operations on multiple systems as the lesson continues.

Unitary operations performed independently on individual systems

When unitary operations are performed independently on a collection of individual systems, the combined action of these independent operations is described by the tensor product of the unitary matrices that represent them. That is, if X_0, \dots, X_{n-1} are quantum systems, U_0, \dots, U_{n-1} are unitary matrices representing operations on these systems, and the operations are performed independently on the systems, the combined action on (X_{n-1}, \dots, X_0) is represented by the matrix $U_{n-1} \otimes \dots \otimes U_0$. Once again, we find that the probabilistic and quantum settings are analogous in this regard.

One would naturally expect, from reading the previous paragraph, that the tensor product of any collection of unitary matrices is unitary. Indeed this is true, and we can verify it as follows.

Notice first that the conjugate transpose operation satisfies

$$(M_{n-1} \otimes \dots \otimes M_0)^\dagger = M_{n-1}^\dagger \otimes \dots \otimes M_0^\dagger$$

for any chosen matrices M_0, \dots, M_{n-1} . This can be checked by going back to the definition of the tensor product and of the conjugate transpose, and checking that each entry of the two sides of the equation are in agreement. This means that

$$(U_{n-1} \otimes \dots \otimes U_0)^\dagger (U_{n-1} \otimes \dots \otimes U_0) = (U_{n-1}^\dagger \otimes \dots \otimes U_0^\dagger) (U_{n-1} \otimes \dots \otimes U_0)$$

Because the tensor product of matrices is multiplicative, we find that

$$(U_{n-1}^\dagger \otimes \cdots \otimes U_0^\dagger)(U_{n-1} \otimes \cdots \otimes U_0) = (U_{n-1}^\dagger U_{n-1}) \otimes \cdots \otimes (U_0^\dagger U_0)$$

Here we have written $\mathbb{I}_0, \dots, \mathbb{I}_{n-1}$ to refer to the matrices representing the identity operation on the systems X_0, \dots, X_{n-1} , which is to say that these are identity matrices whose sizes agree with the number of classical states of X_0, \dots, X_{n-1} .

Finally, the tensor product $\mathbb{I}_{n-1} \otimes \cdots \otimes \mathbb{I}_0$ is equal to the identity matrix for which we have a number of rows and columns that agrees with the product of the number of rows and columns of the matrices $\mathbb{I}_{n-1}, \dots, \mathbb{I}_0$. This larger identity matrix represents the identity operation on the joint system (X_{n-1}, \dots, X_0) .

In summary, we have the following sequence of equalities:

$$\begin{aligned} & (U_{n-1} \otimes \cdots \otimes U_0)^\dagger (U_{n-1} \otimes \cdots \otimes U_0) \\ &= (U_{n-1}^\dagger \otimes \cdots \otimes U_0^\dagger)(U_{n-1} \otimes \cdots \otimes U_0) \\ &= (U_{n-1}^\dagger U_{n-1}) \otimes \cdots \otimes (U_0^\dagger U_0) \\ &= \mathbb{I}_{n-1} \otimes \cdots \otimes \mathbb{I}_0 \\ &= \mathbb{I}. \end{aligned}$$

We therefore conclude that $U_{n-1} \otimes \cdots \otimes U_0$ is unitary.

An important situation that often arises is one in which a unitary operation is applied to just one system — or a proper subset of systems — within a larger joint system. For instance, suppose that X and Y are systems that we can view together as forming a single, compound system (X, Y) , and we perform an operation just on the system X . To be precise, let us suppose that U is a unitary matrix representing an operation on X , so that its rows and columns have been placed in correspondence with the classical states of X .

To say that we perform the operation represented by U just on the system X implies that we do nothing to Y , meaning that we independently perform U on X and the *identity operation* on Y . That is, "doing nothing" to Y is equivalent to performing the identity operation on Y , which is represented by the identity matrix \mathbb{I}_Y . (Here, by the way, the subscript Y tells us that \mathbb{I}_Y refers to the identity matrix having a number of rows and columns in agreement with the classical state set of Y .) The operation on (X, Y) that is obtained when we perform U on X and do nothing to Y is therefore represented by the unitary matrix

$$U \otimes \mathbb{I}_Y.$$

For example, if X and Y are qubits, performing a Hadamard operation on X and doing nothing to Y is equivalent to performing the operation

$$H \otimes I_Y = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$


on the joint system (X, Y) .

Along similar lines, if an operation represented by a unitary matrix U is applied to Y and nothing is done to X , the resulting operation on (X, Y) is represented by the unitary matrix

$$I_X \otimes U.$$

For example, if we again consider the situation in which both X and Y are qubits and U is a Hadamard operation, the resulting operation on (X, Y) is represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Not every unitary operation on a collection of systems can be written as a tensor product of unitary operations like this, just as not every quantum state vector of these systems is a product state. For example, neither the swap operation nor the controlled-NOT operation on two qubits, which are described below, can be expressed as a tensor product of unitary operations.

The swap operation

To conclude the lesson, let's take a look at two classes of examples of unitary operations on multiple systems, beginning with the *swap operation*.

Suppose that X and Y are systems that share the same classical state set Σ . The *swap* operation on the pair (X, Y) is the operation that exchanges the contents of the two systems, but otherwise leaves the

systems alone — so that X remains on the left and Y remains on the right. We'll denote this operation as **SWAP**, and it operates like this for every choice of classical states $a, b \in \Sigma$:

$$\text{SWAP } |a\rangle|b\rangle = |b\rangle|a\rangle.$$

One way to write the matrix associated with this operation using the Dirac notation is as follows:

$$\text{SWAP} = \sum_{c,d \in \Sigma} |c\rangle\langle d| \otimes |d\rangle\langle c|.$$

It may not be immediately clear that this matrix represents **SWAP**, but we can check it satisfies the condition $\text{SWAP } |a\rangle|b\rangle = |b\rangle|a\rangle$ for every choice of classical states $a, b \in \Sigma$. As a simple example, when X and Y are qubits, we find that

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Controlled-unitary operations

Now let us suppose that Q is a qubit and R is an arbitrary system, having whatever classical state set we wish. For every unitary operation U acting on the system R , a *controlled-U* operation is a unitary operation on the pair (Q, R) defined as follows:

$$CU = |0\rangle\langle 0| \otimes \mathbb{I}_R + |1\rangle\langle 1| \otimes U.$$

For example, if R is also a qubit, and we consider the Pauli X operation on R , then a controlled- X operation is given by

$$CX = |0\rangle\langle 0| \otimes \mathbb{I}_R + |1\rangle\langle 1| \otimes X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We already encountered this operation in the context of classical information and probabilistic operations earlier in the lesson. Replacing the Pauli X operation on R with a Z operation gives this operation:

$$CZ = |0\rangle\langle 0| \otimes \mathbb{I}_R + |1\rangle\langle 1| \otimes Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

If instead we take R to be two qubits, and we take U to be the *swap operation* between these two qubits, we obtain this operation:

$$\text{CSWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

This operation is also known as a *Fredkin operation*, or more commonly, a *Fredkin gate*. Its action on standard basis states can be described as follows:

$$\text{CSWAP } |0bc\rangle = |0bc\rangle$$

$$\text{CSWAP } |1bc\rangle = |1cb\rangle$$

Finally, a *controlled-controlled-NOT operation*, which we may denote as CCX , is called a *Toffoli operation* or *Toffoli gate*. Its matrix representation looks like this:

$$CCX = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We may alternatively express it using the Dirac notation as follows:

$$CCX = (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|) \otimes \mathbb{I} + |11\rangle\langle 11| \otimes X.$$

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