

FROM  
SCRATCH

BUILD A  
**Reasoning  
Model**

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**MEAP Edition**  
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**Build a Reasoning Model (From Scratch)**  
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# welcome

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Thank you for purchasing the MEAP for *Build A Reasoning Model (From Scratch)*.

If you are like most people these days, LLMs are already part of your everyday toolkit. Maybe you have asked an LLM to proofread an email, debug a tricky piece of code, or explain a concept that sent you down an unexpected rabbit hole. Since 2022, and the launch of ChatGPT, these models have moved from experimental novelties to essential tools in our daily work and learning.

It's been quite a journey to get here. The earliest GPT models, introduced in 2018, could generate text that was more or less human-like, but they were primarily text-completion models, largely unable to answer even simple queries, and the response quality was nowhere near that of the LLMs we use today.

Then came instruction fine-tuning and alignment with human preferences, which ChatGPT popularized in 2022. The techniques behind ChatGPT transformed LLMs into the everyday problem solvers we use today. Currently, we are in the latest phase: developing reasoning models. Reasoning is the ability for an LLM to tackle more complex problems step-by-step.

Reasoning is one of the most exciting and important recent advances in improving LLMs, but it's also one of the easiest to misunderstand if you only hear the term reasoning and read about it in theory. That's why this book takes a hands-on approach. We'll start with a pre-trained base LLM and then add reasoning capabilities ourselves, step by step in code, so you can see exactly how it works.

This book isn't a "production deployment" manual, and we won't use any third-party LLM libraries. Instead, think of it as a behind-the-scenes tour where you get to develop the machinery yourself.

By the end, you will not only understand what reasoning is and how it works, but you will also have built it from scratch. That's a perspective that will serve you well whether you are using, developing, or planning to deploy LLMs in the future.

Please be sure to post any questions, comments, or suggestions you have about the book in the [liveBook discussion forum](#).

— Sebastian Raschka, PhD

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# 1 *Understanding reasoning models*

## This chapter covers

- What "reasoning" means specifically in the context of LLMs
- The conventional pre-training and post-training stages of LLMs
- Key approaches to improving reasoning abilities in LLMs
- How reasoning differs from pattern matching
- Why we should build reasoning models from scratch

Welcome to the next stage of large language models (LLMs): *reasoning*. LLMs have transformed how we process and generate text, but their success has been largely driven by statistical pattern recognition. However, new advances in reasoning methodologies now enable LLMs to tackle more complex tasks, such as solving logical puzzles and advanced math problems involving multi-step arithmetic. Moreover, reasoning is an essential technique for making "agentic" AI practical. Understanding reasoning methodologies is the central focus of this book.

In *Build a Reasoning Model (From Scratch)*, you will learn the inner workings of LLM reasoning methods through a hands-on, code-first approach. We will start from a pre-trained LLM and extend it step by step with reasoning capabilities. We implement these reasoning components ourselves, from scratch, to see how these methods work in practice.

If you are curious about how LLMs themselves are built and trained, my earlier book *Build A Large Language Model (From Scratch)* published by Manning (<http://mng.bz/orYv>) provides a detailed coverage of these foundations, but it is not required for following along here. By the end of this book, you will understand how reasoning models work and be equipped to design, prototype, and evaluate the main methods for improving reasoning in LLMs.

With its focus on practical applications and explanations, this book is written to speak to LLM engineers, machine learning researchers, applied scientists, and software developers alike.

## 1.1 Defining reasoning in the context of LLMs

What is LLM-based reasoning? The answer and discussion of this question itself would provide enough content to fill a book. However, this would be a different kind of book than this practical, hands-on coding focused book that implements LLM reasoning methods from scratch rather than arguing about reasoning on a conceptual or philosophical level. Nonetheless, I think it's important to briefly define what we mean by reasoning in the context of LLMs.

So, before we get to the coding portions of this book in the upcoming chapters, I want to kick off this book with this section that defines reasoning in the context of LLMs, what the main techniques are that improve the so-called reasoning behaviors of LLMs, and briefly explain how it relates to pattern matching and logical reasoning. This will lay the groundwork for further discussions on how LLMs are currently built, how they handle reasoning tasks, and what they are good and not so good at.

In the context of LLMs, reasoning means the model shows how it got to an answer by explaining the intermediate steps before giving the final response.

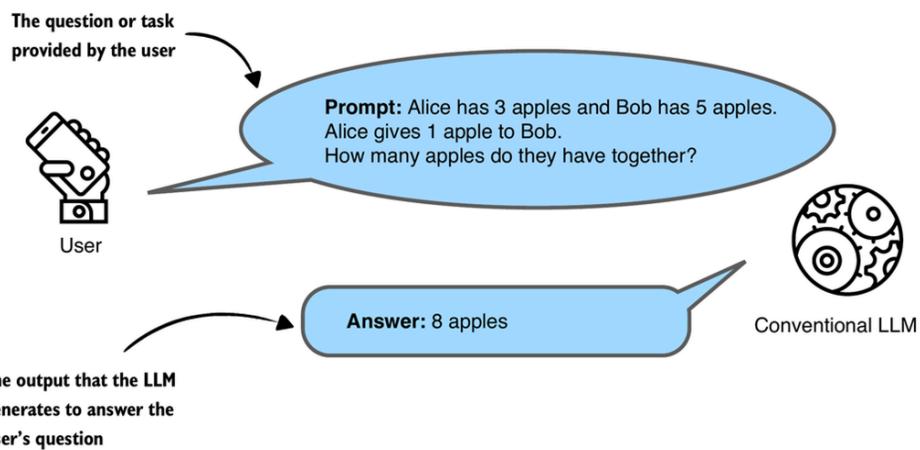
Furthermore, the current definition of a reasoning model is an LLM that has been improved to explain its steps before giving an answer, where explaining the intermediate steps often increases the accuracy on complex tasks such as coding, logical puzzles, and math problems.

### CHAIN-OF-THOUGHT (CoT)

Reasoning in LLMs is often referred to as chain-of-thought (CoT). In this process, the model generates a series of intermediate steps or explanations that show how it arrives at its final answer. Instead of jumping straight to the result, researchers and engineers often say the model "thinks" through the problem step by step. In other words, the model makes its reasoning process explicit and easier to follow.

Note that in this book, we use the terms "reasoning" and "thinking" as they are commonly used in the context of LLMs by researchers and engineers working on LLMs and the so-called reasoning models. However, this does not imply that LLMs actually reason or think in the same way humans do.

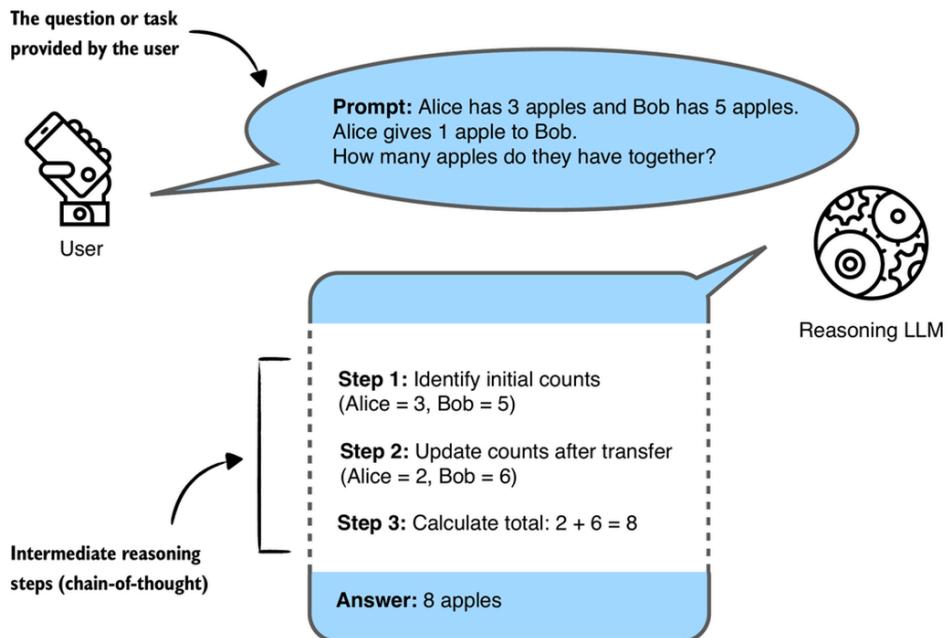
Figure 1.1 illustrates how a conventional LLM generates the answer to a user's question.



**Figure 1.1 A simplified illustration of how a conventional, non-reasoning LLM might respond to a question with a short answer.**

As shown in figure 1.1, a conventional might not show how it came up with its answer. While the answer might be correct it doesn't help the user understand the process behind the answer.

Figure 1.2 illustrates a simple example of multi-step (CoT) reasoning in an LLM.



**Figure 1.2** A simplified illustration of how a reasoning LLM might tackle a multi-step reasoning task. Rather than just recalling a fact, the model combines several intermediate reasoning steps to arrive at the correct conclusion. The intermediate reasoning steps may or may not be shown to the user, depending on the implementation.

LLM-produced intermediate reasoning steps, as shown in figure 1.2, look very much like a person articulating internal thoughts aloud. Yet how closely these methods (and the resulting reasoning processes) mirror human reasoning remains an open research question, one this book does not attempt to answer. It's not even clear that such a question can be definitively answered.

While figure 1.1 is a typical example of chain-of-thought reasoning, it is important to emphasize that LLM reasoning differs from traditional, deterministic reasoning.

For instance, a symbolic logic engine or theorem prover follows strict, rule-based steps that guarantee consistency and correctness. (A symbolic logic engine works like following a recipe where every step is fixed and must produce the same result each time. To use a cooking analogy, if you follow the steps exactly, you always end up with the same dish.)

In contrast, an LLM generates reasoning *autoregressively*, which means that it is predicting one token at a time based on statistical patterns in its training data. As a result, the LLM's "reasoning steps" are not guaranteed to be logically sound, even if they look convincing.

Instead, this book focuses on explaining and implementing the fundamental techniques that improve LLM-based reasoning and thus make LLMs better at handling complex tasks. My hope is that by gaining hands-on experience with these methods, you will be better prepared to understand and improve those reasoning methods being developed and maybe even explore how they compare to human reasoning.

### LLM VERSUS HUMAN REASONING

Reasoning processes in LLMs may superficially resemble human thought, particularly in how intermediate steps are articulated. However, it is important to recognize a key difference: humans can engage in deterministic reasoning by deliberately applying rules of logic or by reasoning over an internal model of the world. Deterministic here means that if we start with the same facts and follow the same steps, we will always reach the same conclusion. In contrast, current LLM reasoning is probabilistic, meaning that it generates one token at a time based on statistical patterns in training data, without guarantees of logical consistency.

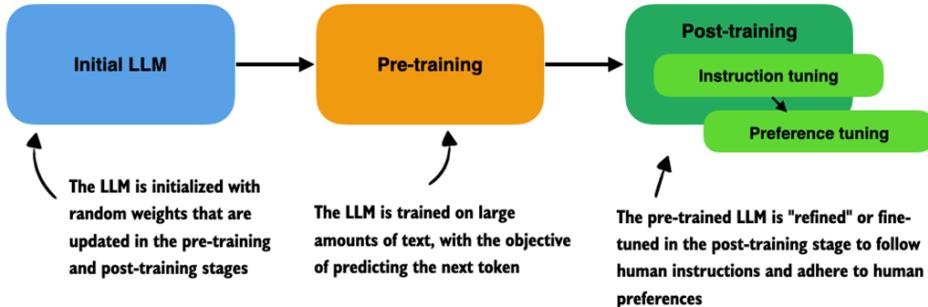
Humans often reason by consciously manipulating concepts, intuitively understanding abstract relationships, or generalizing from a few examples. LLMs, by contrast, work by picking up patterns from huge amounts of text rather than relying on built-in reasoning rules or any kind of conscious thought.

In short, although the outputs of reasoning-enhanced LLMs can appear human-like, the underlying mechanisms differ substantially and remain an active area of exploration.

## 1.2 Understanding the standard LLM training pipeline

This section briefly summarizes how conventional (non-reasoning) LLMs are typically trained so that we can understand where their limitations lie. This background will also help frame our upcoming discussions on the differences between pattern matching and logical reasoning.

Before applying any specific reasoning methodology, conventional LLM training is usually structured into two stages: *pre-training* and *post-training*, which are illustrated in figure 1.3.



**Figure 1.3 Overview of a typical LLM training pipeline. The process begins with an initial model initialized with random weights, followed by pre-training on large-scale text data to learn language patterns by predicting the next token. Post-training then refines the model through instruction fine-tuning and preference fine-tuning, which enables the LLM to follow human instructions better and align with human preferences.**

In the pre-training stage of a typical LLM training pipeline, as shown in figure 1.3, LLMs are trained on massive amounts (many terabytes) of unlabeled text, which includes books, websites, research articles, and many other sources. The pre-training objective (goal) for the LLM is to learn to predict the next word (i.e., *token*) in these texts.

#### WORDS AND TOKENS

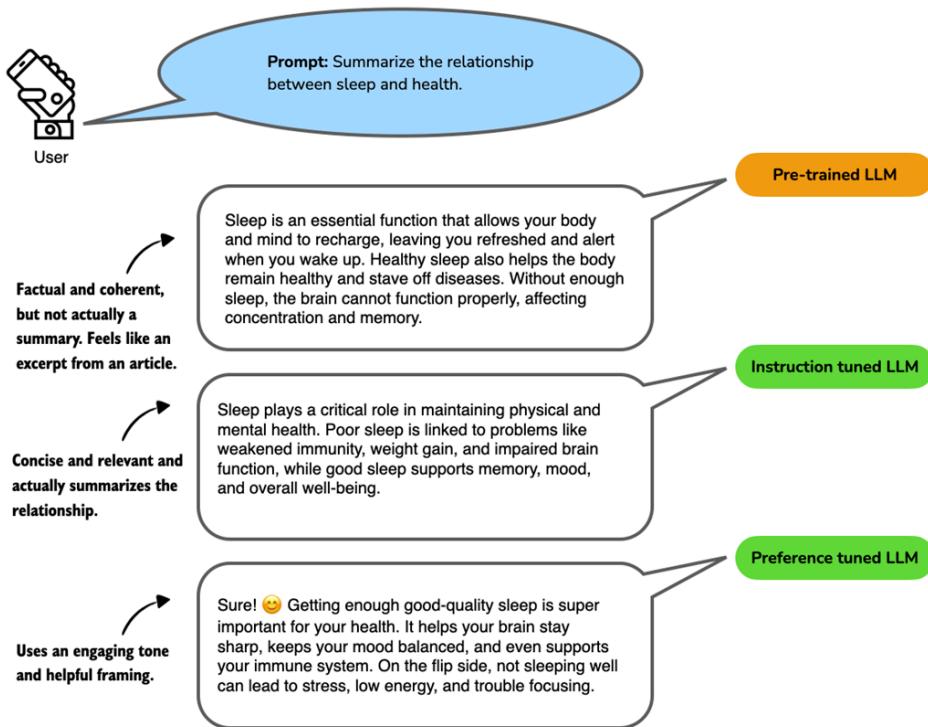
A token is a small unit of text that a language model processes. A token can be a full word, part of a word, or even punctuation, depending on how the text is split by a so-called tokenizer.

For example, the sentence "An LLM can be useful." might be broken into tokens like "An", " L", "LM", " can", " be", " useful", and "." by a common tokenizer. These tokens are then converted into numerical IDs that the model can ingest.

A tokenizer is a model or tool that is not directly part of the LLM itself but is nonetheless a critical component of the LLM text processing and generation pipeline. We will see how tokenization works in practice in the next chapter.

LLMs become highly capable when pre-trained on massive datasets, which typically involves several terabytes of text (equivalent to around 300 to 400 billion tokens). This training requires thousands of GPUs running for many months and can cost millions of dollars. Here, "capable" means that the LLMs begin to generate text that closely resembles human writing. Also, to some extent, pre-trained LLMs will begin to exhibit so-called *emergent properties*, which means that they will be able to perform tasks that they were not explicitly trained to do, including translation, code generation, and so on.

However, these pre-trained models merely serve as base models for the post-training stage, which uses two key techniques: *supervised fine-tuning* (often abbreviated as *SFT* in the literature and also known as *instruction tuning*) and *preference tuning* (often implemented via a technique called *Reinforcement Learning with Human Feedback*) to teach LLMs to respond to user queries, which are illustrated in figure 1.4.



**Figure 1.4 Example responses from a language model at different training stages. The prompt asks for a summary of the relationship between sleep and health. The pre-trained LLM produces a relevant but unfocused answer without directly following the instructions. The instruction-tuned LLM generates a concise and accurate summary aligned with the prompt. The preference-tuned LLM further improves the response by using a friendly tone and engaging language, which makes the answer more relatable and user-centered.**

As shown in figure 1.4, instruction tuning improves an LLM's capabilities of personal assistance-like tasks such as question-answering, summarizing and translating text, and many more. The preference tuning stage then refines these capabilities. As the term implies, preference tuning helps tailor responses to user preferences. (Some readers may be familiar with terms like *Reinforcement Learning with Human Feedback* or *RLHF*, which are specific techniques to implement preference tuning.)

In short, we can think of pre-training as "raw language prediction" (via next-token prediction) that gives the LLM some basic properties and capabilities to produce coherent texts. The post-training stage then improves the task understanding of LLMs via instruction tuning and refines the LLM to create answers with preferred stylistic choices via preference tuning.

However, it is worth noting that even an instruction-tuned model is not yet a "chatbot." A chat interface adds another layer that guides the model's responses in an interactive, multi-turn setting. This typically involves a system prompt, conversation history management, and other orchestration (an example of this is implemented in appendix G).

**NOTE** These pre-training and post-training stages mentioned above are covered in my book "Build A Large Language Model (From Scratch)" (<http://mng.bz/orYv>) published by Manning. The book you are reading now does not require detailed knowledge of these stages. Concretely, in the next chapter, we will load a model that has already undergone the expensive pre-training and post-training stages mentioned above, so that we can focus on the methodology that is specific to reasoning models in the subsequent chapters.

## 1.3 Improving LLM reasoning with training and inference techniques

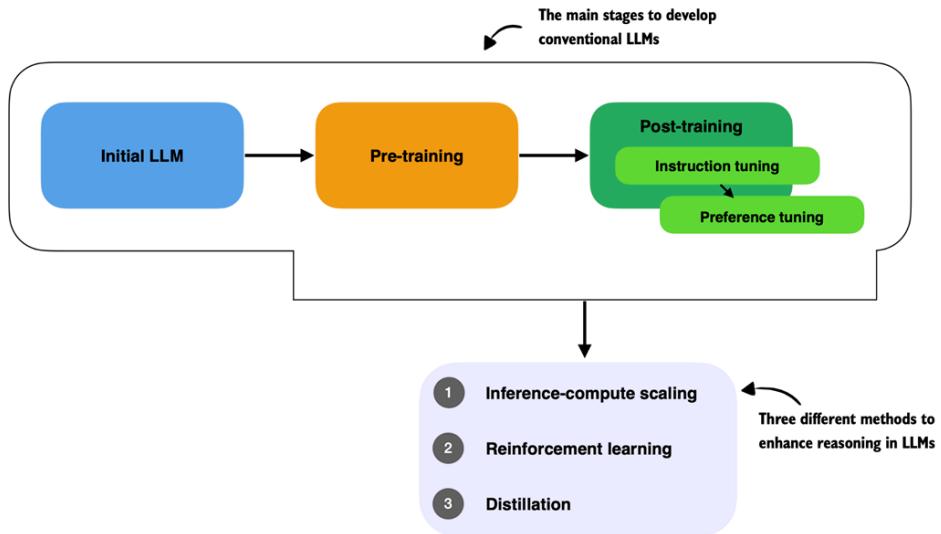
Reasoning in the context of LLMs became popular in the public eye with the announcement of OpenAI's o1 in ChatGPT on September 12, 2024, which popularized the concept of reasoning in LLMs. In the announcement article (<https://openai.com/index/introducing-openai-o1-preview/>), OpenAI mentioned that "We've developed a new series of AI models designed to spend more time thinking before they respond."

Furthermore, OpenAI wrote: "These enhanced reasoning capabilities may be particularly useful if you're tackling complex problems in science, coding, math, and similar fields."

A few months later, in January 2025, DeepSeek released the DeepSeek-R1 model and technical report (<https://arxiv.org/abs/2501.12948>), which details training methodologies to develop reasoning models, which made big waves as they not only made freely and openly available a model that competes with and exceeds the performance of the proprietary o1 model but also shared a blueprint on how to train such a model.

This book aims to explain how these methodologies used to develop reasoning models work by implementing similar methods from scratch.

The different approaches to developing and improving an LLM's reasoning capabilities can be grouped into three broad categories, as illustrated in figure 1.4.



**Figure 1.5 Three approaches commonly used to improve reasoning capabilities in LLMs. These methods (inference-compute scaling, reinforcement learning, and distillation) are typically applied after the conventional training stages (initial model training, pre-training, and post-training with instruction and preference tuning).**

As illustrated in figure 1.5, these methods are applied to LLMs that have undergone the conventional pre-training and post-training phases, including instruction and preference tuning.

1. *Inference-time compute scaling* (also often called *inference compute scaling*, *test-time scaling*, or other variations) includes methods that improve model reasoning capabilities at inference time (when a user prompts the model) without training or modifying the underlying model weights. The core idea is to trade off increased computational resources for improved performance, which helps make even fixed models more capable through techniques such as chain-of-thought reasoning, and various sampling procedures. This topic will be the focus of chapters 4 and 5.
2. *Reinforcement learning (RL)* refers to training methods that improve a model's reasoning capabilities by encouraging it to take actions that lead to high reward signals. These rewards can be broad, such as task success or heuristic scores, or they can be narrowly defined and verifiable, such as correct answers in math problems or coding tasks.

Unlike Inference-time compute scaling, which can improve reasoning performance without modifying the model, RL updates the model's weights during training. This enables the model to learn and refine reasoning strategies through trial and error, based on the feedback it receives from the environment. We will explore RL in more detail in chapters 6 and 7.

### REINFORCEMENT LEARNING FOR REASONING AND PREFERENCE TUNING

In the context of developing reasoning models, it is important to distinguish the RL approach here from reinforcement learning with human feedback (RLHF), which is used during preference tuning when developing a conventional LLM as illustrated previously in figure 1.5.

Both settings use the same underlying process (RL) but they differ primarily in how the reward is obtained and validated (human judgments for RLHF versus automated verifiers or environments for reasoning RL).

RLHF incorporates explicit human evaluations or rankings of model outputs as reward signals, directly guiding the model toward human-preferred behaviors. In contrast, RL in the context of reasoning models typically relies on automated or environment-based reward signals, which can be more objective but potentially less aligned with human preferences. For instance, RL in a reasoning model development pipeline might train a model to excel at mathematical proofs by providing explicit rewards for correctness. In contrast, RLHF would involve human evaluators ranking various responses to encourage outputs that align closely with human standards and subjective preferences.

3. *Distillation* involves transferring complex reasoning patterns learned by powerful, larger models into smaller or more efficient models. Within the context of LLMs, this typically means performing *supervised fine-tuning (SFT)* using high-quality labeled instruction datasets generated by a larger, more capable model. This technique is commonly referred to as *knowledge distillation* or simply *distillation* in LLM literature. However, it's important to note that this differs slightly from traditional knowledge distillation in deep learning, where a smaller ("student") model typically learns from both the outputs and the logits produced by a larger ("teacher") model. This topic is discussed further in chapter 8.

## 1.4 Modeling language through pattern matching

As mentioned in the previous section, during pre-training, LLMs are exposed to vast quantities of text and learn to predict the next token by identifying and reproducing statistical associations in that data. This process enables them to generate fluent and coherent text, but it is fundamentally rooted in surface-level correlations rather than deep understanding.

LLMs respond to prompts by generating text continuations that are statistically consistent with the patterns seen during training. In essence, they match patterns between input and output, rather than deducing answers through logical inference.

Consider the following example:

### Prompt

The capital of Germany is...

### Response

Berlin.

An LLM producing the answer "Berlin" is not logically deducing the answer. Instead, it is recalling a strong statistical association learned from training data. This behavior reflects what we refer to as *pattern matching*, which means that the model completes text based on learned correlations and not by applying structured reasoning steps.

But what about tasks that go beyond pattern recognition, i.e., tasks where a correct answer depends on drawing conclusions from given facts? This brings us to a different kind of capability: logical reasoning.

Logical reasoning involves systematically coming up with conclusions using rules. Unlike pattern matching, it depends on intermediate reasoning steps and the ability to recognize contradictions or draw implications based on formal relationships.

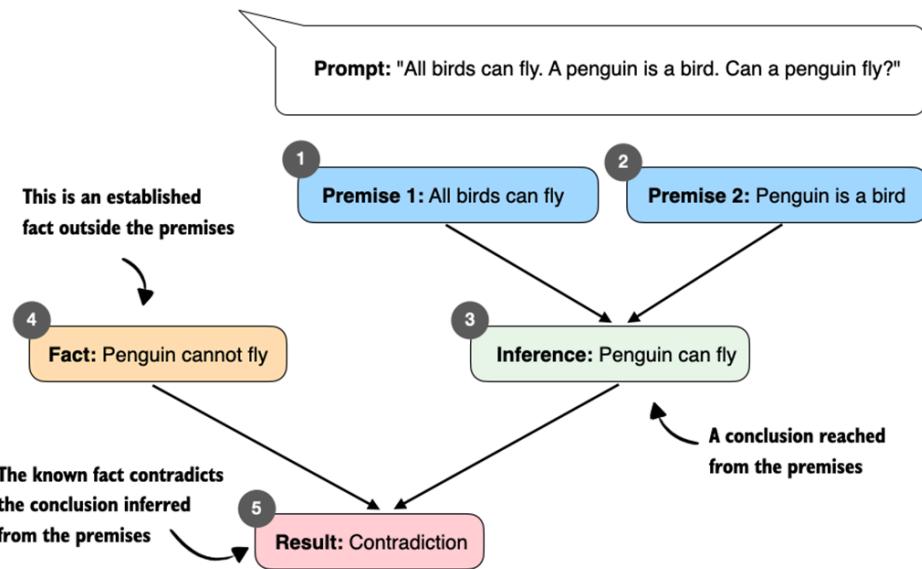
Consider the following prompt as an example:

- Prompt: "All birds can fly. A penguin is a bird. Can a penguin fly?"

There are two ways to evaluate this.

First, in a *closed-world* (prompt only) setting, from the two premises (claims, assumptions) in the prompt ("All birds can fly" and "A penguin is a bird"), the valid answer is "Yes, a penguin can fly."

Second, in an *open-world* (with background knowledge) setting, if we also allow background knowledge not included in the prompt (for example, that penguins cannot fly), this external fact conflicts with the conclusion derived from the premises, as shown in figure 1.6. A reasoning system should notice the inconsistency and either ask for clarification or weaken the first statement (for example, "Most birds can fly, with exceptions such as penguins").



**Figure 1.6 Illustration of how contradictory premises lead to a logical inconsistency. From "All birds can fly" and "A penguin is a bird," we infer "Penguin can fly." This conclusion conflicts with the established fact "Penguin cannot fly," which results in a contradiction.**

Figure 1.6 shows how a system based on logical reasoning could process the previously introduced "All birds can fly..." prompt.

In contrast, a statistical (pattern-matching) LLM does *not* explicitly track contradictions, such as the one shown in figure 1.6, but instead predicts based on learned text distributions. For instance, if information such as "All birds can fly" is reinforced strongly in training data, the model may confidently answer: "Yes, penguins can fly."

In the next section, we will look at a concrete example of how an LLM handles this "All birds can fly..." prompt.

### LOGICAL REASONING AND RULE-BASED SYSTEMS

Why are explicit rule-based systems not more popular? Rule-based systems were used widely in the '80s and '90s for medical diagnosis, legal decisions, and engineering. They are still used in critical domains (medicine, law, aerospace), which often require explicit inference and transparent decision processes. However, they are hard to implement as they largely rely on human-crafted heuristics. (Heuristics are simple decision rules that give a good-enough answer quickly without guaranteeing an optimal one.) In contrast, deep neural networks, including LLMs, do not implement hand-written rules; they learn decision patterns from data and can be highly flexible when trained at scale.

## 1.5 Simulating reasoning without explicit rules

In the previous section, we saw how contradictory premises can lead to logical inconsistencies. A conventional LLM does not explicitly track contradictions but generates responses based on learned text distributions.

Let's see a concrete example, shown in figure 1.7, of how a non-reasoning-enhanced LLM like GPT-4o in OpenAI's ChatGPT responds to the "All birds can fly..." prompt discussed in the previous section.

ChatGPT 4o ▾

 Share

Consider that all birds fly. And a penguin is a bird. Can penguins fly?

If we assume the statement "**all birds fly**" to be absolutely true, then logically, since a penguin is a bird, it should be able to fly. However, in reality, penguins are a known exception—they are birds that cannot fly.

This highlights a flaw in the initial premise. The assumption that *all* birds fly is incorrect because exceptions exist. This is an example of a logical syllogism where a universal premise is actually false.

Would you like me to formalize this reasoning using symbolic logic?



Is this conversation helpful so far?   

**Figure 1.7 An illustrative example of how a language model (GPT-4o in ChatGPT) appears to "reason" about a contradictory premise.**

The example in figure 1.7 shows that GPT-4o appears to answer correctly even though this model is not considered a reasoning model, unlike OpenAI's other offerings like o1, o3, o4-mini, and more recent GPT-5, which have been explicitly developed with reasoning methodology.

So, how did the 4o model generate its answer? Does this mean GPT-4o explicitly reasons logically? No, not necessarily. However, at a minimum 4o is highly effective at simulating logical reasoning in familiar contexts.

GPT-4o does not implement explicit contradiction-checking and instead generates answers based on probability-weighted patterns. This approach works well enough if training data includes many instances correcting the contradiction (e.g., text like "penguins cannot fly") so that the model learns a statistical association between "penguins" and "not flying." As we see in figure 1.7, this allows the model to answer correctly without explicitly implementing rule-based or explicit logical reasoning methodologies.

In other words, the model recognizes the contradiction implicitly because it has frequently encountered this exact reasoning scenario during training. This effectiveness relies heavily on statistical associations built from a lot of exposure to reasoning-like patterns in training data.

So, even when a conventional LLM seems to perform logical deduction as shown in figure 1.7, it's not executing explicit, rule-based logic but is instead leveraging patterns from its vast training data.

Nonetheless, GPT-4o's success here is a great illustration of how powerful implicit pattern matching can become when trained at a massive scale. However, these types of pattern-based reasoning models usually struggle in scenarios where:

- The logical scenario is novel (not previously encountered in training data).
- Reasoning complexity is high, involving intricate, multi-step logical relationships.

### **LOGICAL REASONING AND CURRENT REASONING LLM OFFERINGS**

While GPT-4o is not officially labeled as a reasoning model, OpenAI offers several dedicated reasoning models, including o1, o3, o4-mini, and GPT-5. Moreover, other companies have been developing LLMs with explicit reasoning capabilities. As of this writing, popular examples include Anthropic's Claude 4, xAI's Grok 4, Google's Gemini 2.5, DeepSeek's R1, Alibaba's Qwen3, and many more. The techniques employed by these models are the focus of this book. As we will see, this is achieved without implementing a rule-based reasoning pipeline (figure 1.6 illustrates the general idea of rule-based reasoning). Instead, the LLM learns or improves its reasoning capabilities as a result of the modified inference and training methodologies.

We might say that LLMs simulate logical reasoning through learned patterns, and we can improve it further with specific reasoning methods that include inference-compute scaling and post-training strategies like reinforcement learning, but they are not explicitly executing any rule-based logic internally.

Moreover, it's worth mentioning that reasoning in LLMs exists on a spectrum. This means that even before the advent of dedicated reasoning models such as OpenAI's o1 and DeepSeek-R1, LLMs were capable of simulating reasoning behavior. For instance, these models exhibited behaviors aligning with our earlier definition, such as generating intermediate steps to arrive at correct conclusions. What we now explicitly label a "reasoning model" is essentially a more refined version of this capability. And these improved reasoning capabilities are achieved by leveraging specific inference-compute scaling techniques (chapters 4 and 5) and targeted post-training methods, such as reinforcement learning (chapters 6 and 7), which are designed to improve and reinforce reasoning-like behavior.

## 1.6 Why build reasoning models from scratch?

Following the release of DeepSeek-R1 in January 2025, improving the reasoning abilities of LLMs has become one of the hottest topics in AI, and for good reason. Stronger reasoning skills allow LLMs to tackle more complex problems, making them more capable across various tasks users care about.

This shift is also reflected in a February 12, 2025, statement from OpenAI's CEO:

"We will next ship GPT-4.5, the model we called Orion internally, as our last non-chain-of-thought model. After that, a top goal for us is to unify o-series models and GPT-series models by creating systems that can use all our tools, know when to think for a long time or not, and generally be useful for a very wide range of tasks."

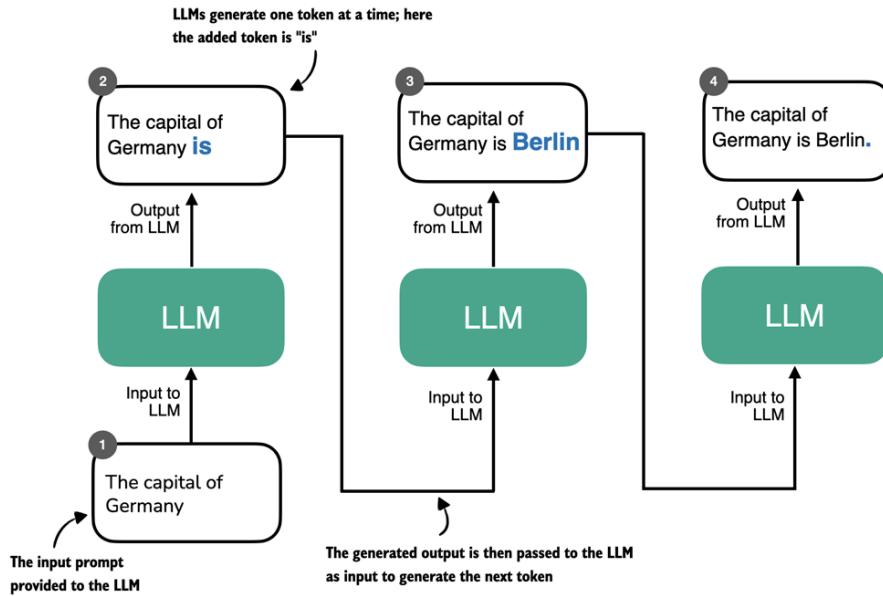
The quote above underlines the major shift from leading LLM providers towards reasoning models, where "chain-of-thought" refers to a prompting technique that guides language models to reason step-by-step to improve their reasoning capabilities, which we will cover in more detail in chapters 4 and 5.

Also noteworthy is the mention of knowing "when to think for a long time or not." This hints at an important design consideration: reasoning is not always necessary or desirable.

For instance, reasoning models are designed to be good at complex tasks such as solving puzzles, advanced math problems, and challenging coding tasks. However, they are not necessary for simpler tasks like summarization, translation, or knowledge-based question answering. In fact, using reasoning models for everything can be inefficient and expensive. For instance, reasoning models are typically more expensive to use, more verbose, and sometimes more prone to errors due to "overthinking." Also, here, the simple rule applies: use the right tool (or type of LLM) for the task.

Reasoning models are often more expensive than non-reasoning models for two reasons.

First, they tend to produce longer outputs because they include intermediate steps that explain how an answer is derived. As figure 1.8 illustrates, LLMs generate text one token at a time and each token requires a *full forward pass*. If a reasoning model's answer is twice as long, generation involves roughly twice as many forward passes, which increases compute costs.



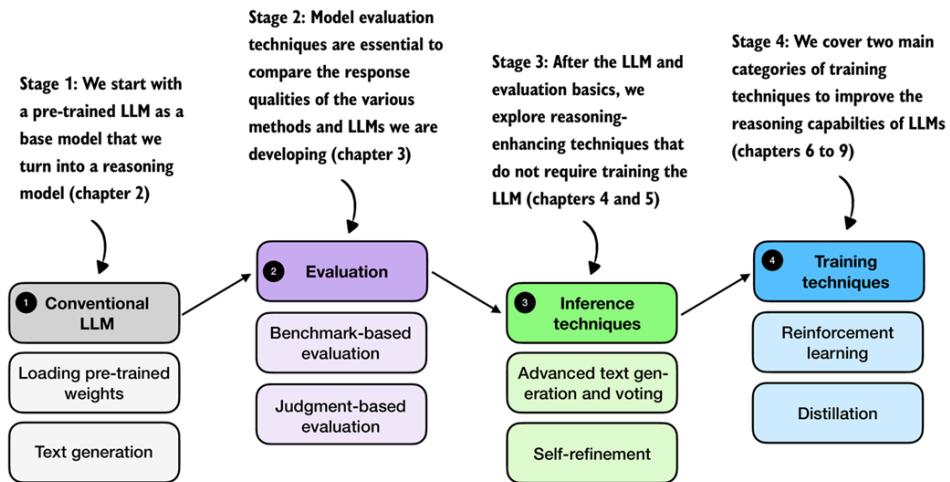
**Figure 1.8 Token-by-token generation in an LLM.** At each step, the LLM takes the full sequence generated so far and predicts the next token, which may represent a word, subword, or punctuation mark depending on the tokenizer. The newly generated token is appended to the sequence and used as input for the next step. This iterative decoding process is used in both standard language models and reasoning-focused models.

Second, many reasoning workflows require running the model several times for a single task, for example to sample multiple candidate solutions, call tools, or run a verifier. These additional calls multiply the total number of tokens processed and further increase cost beyond the single-call behavior shown in figure 1.8.

That's exactly why it helps to implement these models and methods from scratch. It's one of the best ways to understand how they work. And if we understand how LLMs and these reasoning models work, we can better understand these trade-offs.

## 1.7 A roadmap to reasoning models from scratch

Now that we have discussed reasoning in LLMs from a bird's-eye view, the subsequent chapters will guide you through the process of coding and applying reasoning methods from scratch. We will tackle this in multiple stages, as outlined in figure 1.9.



**Figure 1.9 A mental model of the main reasoning model development stages covered in this book. We start with a conventional LLM as base model (stage 1). In stage 2, we cover evaluation strategies to track the reasoning improvements introduced via the reasoning methods in stages 3 and 4.**

As shown in figure 1.9, we cover the reasoning model development in several stages. In stage 1 (next chapter), we load a conventional LLM that has already undergone the basic pre-training and instruction fine-tuning stages. Then, in stage 2, we cover common methods for evaluating LLMs and reasoning capabilities, so that we can measure the improvements we make when we apply reasoning-enhancing methods in stages 3 and 4.

Stage 3 covers inference techniques that can improve the response quality and reasoning behavior of LLMs. Note that these techniques can be applied to improve any LLM, conventional LLMs and LLMs that have been trained as reasoning models. Stage 4 will introduce training methods to develop reasoning models.

I am looking forward to the journey ahead and hope you are as well.

## 1.8 Summary

- Conventional LLM training occurs in several stages:
  - Pre-training, where the model learns language patterns from vast amounts of text.
  - Instruction fine-tuning, which improves the model's responses to user prompts.
  - Preference tuning, which aligns model outputs with human preferences.
- Reasoning methods are applied on top of a conventional LLM.

- Reasoning in LLMs refers to improving a model so that it explicitly generates intermediate steps (chain-of-thought) before producing a final answer, which often increases accuracy on multi-step tasks such as math problems, logical puzzles, and coding challenges.
- Reasoning in LLMs is different from rule-based reasoning and it also likely works differently than human reasoning; currently, the common consensus is that reasoning in LLMs relies on statistical pattern matching.
- Pattern matching in LLMs relies purely on statistical associations learned from data, which enables fluent text generation but lacks explicit logical inference.
- Improving reasoning in LLMs can be achieved through:
  - Inference-time compute scaling, enhancing reasoning without retraining (e.g., chain-of-thought prompting).
  - Reinforcement learning, training models explicitly with reward signals.
  - Supervised fine-tuning and distillation, using examples from stronger reasoning models.
- Building reasoning models from scratch provides practical insights into LLM capabilities, limitations, and computational trade-offs.

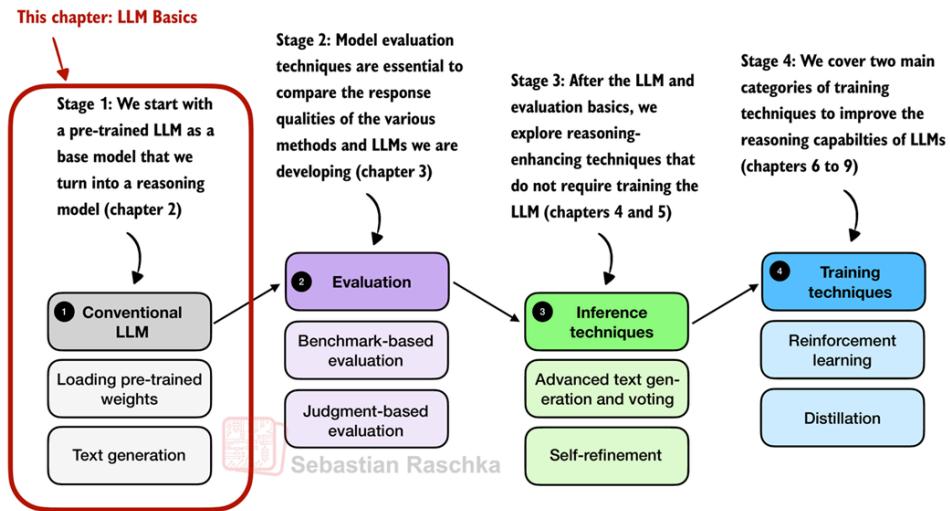
## 2 Generating text with a pre-trained LLM

### This chapter covers

- Setting up the code environment for working with LLMs
- How to use a tokenizer to prepare input text for an LLM
- The step-by-step process of text generation using a pre-trained LLM
- Caching and compilation techniques for speeding up LLM text generation

In the previous chapter, we discussed the difference between *conventional large language models (LLMs)* and *reasoning models*. Also, we introduced several techniques to improve the reasoning capabilities of LLMs. These reasoning techniques are usually applied on top of a conventional (base) LLM.

In this chapter, we will lay the groundwork for the upcoming chapters and load such a conventional LLM on top of which we can apply reasoning techniques in subsequent chapters, as illustrated in figure 2.1. This conventional LLM is an LLM that has already been pre-trained to generate general texts (but it has not been specifically trained or enhanced for reasoning).



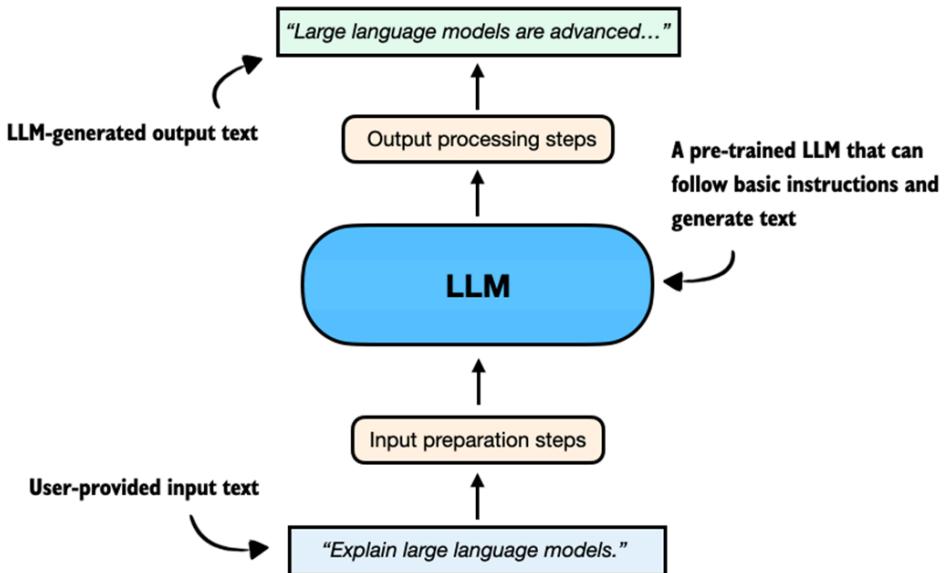
**Figure 2.1** A mental model depicting the four main stages of developing a reasoning model. This chapter focuses on stage 1, loading a conventional LLM and implementing the text generation functionality.

In addition to setting up the coding environment and loading a pre-trained LLM, you will learn how to use a *tokenizer* to prepare text input for the model. As illustrated in figure 2.1, you will also implement a text generation function, enabling practical use of the LLM to generate text. This functionality will be used and further improved in later chapters.

## 2.1 Introduction to LLMs for text generation

In this chapter, we implement all the necessary LLM essentials, from setting up our coding environment and loading a pre-trained LLM to generating text that we will reuse and build upon in this book. In this sense, this chapter can be understood as a setup chapter.

This LLM will be capable of following basic instructions and generating coherent text, as illustrated in figure 2.2.



**Figure 2.2 An overview depicting an LLM generating a response (output text) given a user query (input text)**

Figure 2.2 summarizes the components of an LLM text generation pipeline, and we will discuss and implement these steps in more detail later in this chapter.

**NOTE** By convention, diagrams involving neural networks such as LLMs are drawn and read vertically from bottom (inputs) to top (outputs). Arrows indicate the flow of information upward through the model.

If you have not coded an LLM or used LLMs programmatically before, this chapter will teach you how the *text generation* process works. However, in this chapter, we will not go deep into the internals of an LLM, such as the attention mechanism and other architecture components; this is the topic of my other book, *Build a Large Language Model (from Scratch)*. Note that understanding these internals are not required for this book, and, if you are curious, you can learn about them after you finish reading this book.

This chapter will also be helpful if you have already read the earlier *Build a Large Language Model (From Scratch)* book, since it adds new material on speeding up inference and other practical optimizations.

Before we begin implementing the components shown in figure 2.2, including input preparation, loading the LLM, and generating text, we first need to set up our coding environment. This is the focus of the next section.

## 2.2 Setting up the coding environment

This section provides instructions and recommendations for setting up your Python coding environment to follow along with the examples in this book. I recommend reading this section in its entirety before deciding which way is for you.

If you are reading this book, you have probably coded in Python before. In this case, the simplest way to install dependencies, if you already have a Python environment set up (with Python 3.10 or newer), is to use Python's package installer (`pip`) in your terminal.

If you have downloaded the code from the publisher's website, use the `requirements.txt` file to install the required Python libraries used throughout this book:

```
pip install -r requirements.txt
```

Alternatively, to install the required packages directly without downloading the `requirements.txt` file, use:

```
pip install -r https://raw.githubusercontent.com/\nrasbt/reasoning-from-scratch/refs/heads/main/requirements.txt
```

### PYTHON PACKAGES USED IN THIS CHAPTER

If you prefer to install only the packages used in this chapter, you can do this with the following command:

```
pip install torch>=2.7.1 tokenizers>=0.21.2 reasoning-from-scratch
```

- `torch` refers to PyTorch, a widely used deep learning library that provides tools for building and training neural networks.
- `tokenizers` is a library that provides efficient tokenization algorithms, used to prepare input data for LLMs.
- `reasoning-from-scratch` is a custom library that I developed for this book. It includes all the code examples implemented throughout the chapters, along with additional utility functions we will be using.

While `pip` is the canonical way to install Python packages, my preferred way to use Python is via the widely recommended `uv` Python package and project manager instead. Like many others, I now recommend `uv` because it is significantly faster, and it handles dependency resolution more reliably than `pip`. It also creates isolated environments automatically and comes with its own Python executable (but will use the system Python if a compatible version is already installed) so it is also a great option if you do not have Python installed on your system yet.

Figure 2.3 outlines the 4-step process from installing `uv` to getting ready to execute the code in this chapter, which we will cover in the remainder of this section.

```

Developer curl -Lsf https://astral.sh/uv/install.sh | sh
downloading uv 0.8.2 aarch64-apple-darwin
no checksums to verify
installing to /Users/Author/.local/bin
uv
uvx
everything's installed!
Developer git clone --depth 1 https://github.com/rasbt/reasoning-from-scratch.git
Cloning into 'reasoning-from-scratch'...
remote: Enumerating objects: 36, done.
remote: Counting objects: 100% (36/36), done.
remote: Compressing objects: 100% (32/32), done.
remote: Total 36 (delta 5), reused 18 (delta 1), pack-reused 0 (from 0)
Receiving objects: 100% (36/36), 2.07 MiB | 4.35 MiB/s, done.
Resolving deltas: 100% (5/5), done.
Developer cd reasoning-from-scratch
reasoning-from-scratch git:(main)
reasoning-from-scratch git:(main)
reasoning-from-scratch git:(main)
reasoning-from-scratch git:(main) uv run jupyter lab
Using CPython 3.13.5 interpreter at: /opt/homebrew/opt/python@3.13/bin/python3.13
Creating virtual environment at: .venv
Built reasoning-from-scratch @ file:///Users/Author/Developer/reasoning-from-scratch
Installed 114 packages in 447ms
[I 2025-07-23 19:01:38.714 ServerApp] jupyter_lsp | extension was successfully linked.

```

**Figure 2.3 Installing and using the `uv` Python package and project manager via the macOS terminal**

Note that figure 2.3 steps through the `uv` installation and usage on a macOS terminal, but `uv` is supported by Linux and Windows as well.

- 1) To install `uv`, run the installation for your OS from the official website: <https://docs.astral.sh/uv/getting-started/installation/>
- 2) Next, clone the GitHub repo:

```
git clone --depth 1 https://github.com/rasbt/reasoning-from-scratch.git
```

Here, the `--depth 1` option tells `git` to perform a shallow clone, which means it only downloads the latest version of the code without the full version history. This makes the download faster and uses less space.

If you don't have `git` installed, you can also manually download the source code repository from the publisher's website or by opening this link in your browser: <https://github.com/rasbt/reasoning-from-scratch/archive/refs/heads/main.zip> (unzip it after downloading).

3) Next, in the terminal, navigate to the `reasoning-from-scratch` folder.

4) Inside the `reasoning-from-scratch` folder, execute:

```
uv run jupyter lab
```

The command above will launch JupyterLab, where you can open a blank Jupyter notebook to type and execute code or open the chapter 2 notebook that contains all the code covered in this chapter.

---

**TIP** Python script files can be executed via `uv run script-name.py`.

---

The above `uv run...` command also sets up a local virtual environment (usually inside an invisible `.venv/` folder) and installs all dependencies from the `pyproject.toml` file inside the `reasoning-from-scratch` folder automatically. So, the manual installation of code dependencies via the requirements file is not needed. However, if you plan to install additional packages, you can use the following command:

```
uv add packagename
```

The supplementary code repository contains additional installation instructions and details inside the `ch02` subfolder if needed.

## 2.3 Understanding hardware needs and recommendations

You may have heard that training LLMs is very expensive. For leading LLM companies, it is not uncommon to spend anywhere between 1-10 million Dollars on the small end and >50 million Dollars on the high end in terms of compute costs to train a new base model LLM before even adding any reasoning techniques.

For example, the DeepSeek V3 model, which serves as the base checkpoint for the DeepSeek R1 reasoning system, was trained on 2,048 Nvidia H800 GPUs for about 11 weeks with an estimated cost of 5.5 million USD. (DeepSeek V3 is one of the few more recent models with a fully transparent compute disclosure.)

Furthermore, according to the technical report, the final training run used 14.8 million GPU-hours of compute. The energy usage was roughly 620 MWh, which is roughly the amount of electricity that an average American household uses in about 55 years.

This resource requirements and high price tag would make the development of an LLM unfeasible for me and most readers. So, we are going to use a relatively small (but capable) pre-trained LLM on top of which we implement reasoning techniques.

Note that this smaller LLM is a scaled-down version that otherwise follows the same architecture as contemporary state-of-the-art models. And the reasoning methods that we will apply are the same as those used by larger LLMs. The difference is that the smaller LLM allows us to explore these methods in a budget-friendly way.

As an analogy, imagine you are curious to learn how cars work. If you are new to cars, as a learning exercise, you probably wouldn't start out building an expensive Ferrari right away. Instead, you would, for example, create a smaller car like a Volkswagen Beetle to start with, which still teaches you a lot about how engines and the transmission work. On the contrary, I would even say that working on a smaller car helps you *better* understand how the engine and transmission work because it gets complicated refinements and other details out of the way.

However, while we will use a relatively small model for these educational purposes in this book, the usage, development, and application of the reasoning techniques are still computationally intensive, and later chapters, such as chapters 5-8, will benefit from using a GPU.

If you followed the previous section, you should have PyTorch installed, which you can use to see if your computer has a PyTorch-supported GPU by executing the following PyTorch code in Python:

```
import torch

print(f"PyTorch version {torch.__version__}")
if torch.cuda.is_available():
    print("CUDA GPU")
elif torch.mps.is_available():
    print("Apple Silicon GPU")
else:
    print("Only CPU")
```

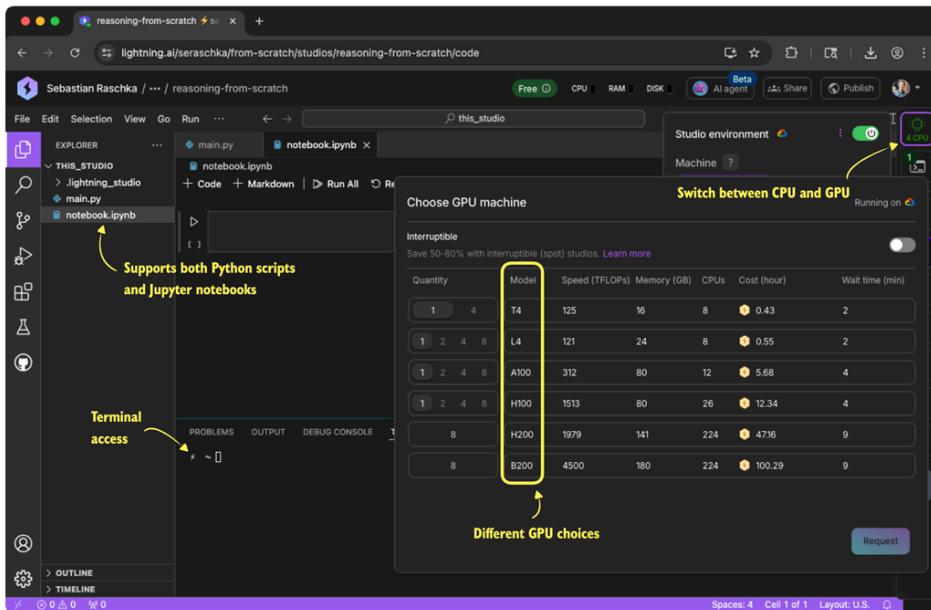
Depending on your machine, the code may return:

```
PyTorch version 2.7.1
Only CPU
```

Don't worry if your machine does not have a GPU to run the code. Chapters 2-5 can be executed in a reasonable time on a CPU.

Depending on the chapter, the code will automatically use an NVIDIA GPU if available, otherwise run on the CPU (or Apple Silicon GPU if recommended for a particular section or chapter). However, I will provide more information in the respective sections and chapters.

Like many other AI researchers who work on and with LLMs daily, I don't have a machine with the necessary GPU hardware to train LLMs at home and use cloud resources instead. If you are looking for cloud provider options, my personal preference is Lightning AI Studio (<https://lightning.ai/>), due to its ease of use and feature support, as shown in figure 2.4. Alternatively, Google Colab (<https://colab.research.google.com/>) is another good choice.



**Figure 2.4 An overview of the Lightning AI GPU cloud platform in a web browser. The interface supports Python scripts, Jupyter notebooks, terminal access, and lets users switch between CPU and various GPU types based on their compute needs.**

As of this writing, Lightning AI also offers users free compute credits after the sign-up and verification process, which can be used for the different GPU choices shown in figure 2.4. (As mentioned before, a GPU is not needed for this chapter; however, if you want to use a GPU, the L4 GPU is more than sufficient for this chapter.)

---

**NOTE** For disclosure, I helped build and launch the Lightning AI platform in 2023 and still hold a small stake. I am not sponsored to recommend it and pay for it myself. I use it because I simply find it the most convenient. It supports multiple types of GPUs, allows easy switching between them and back to CPU to save costs, and lets me pause or resume environments without redoing the setup.

---

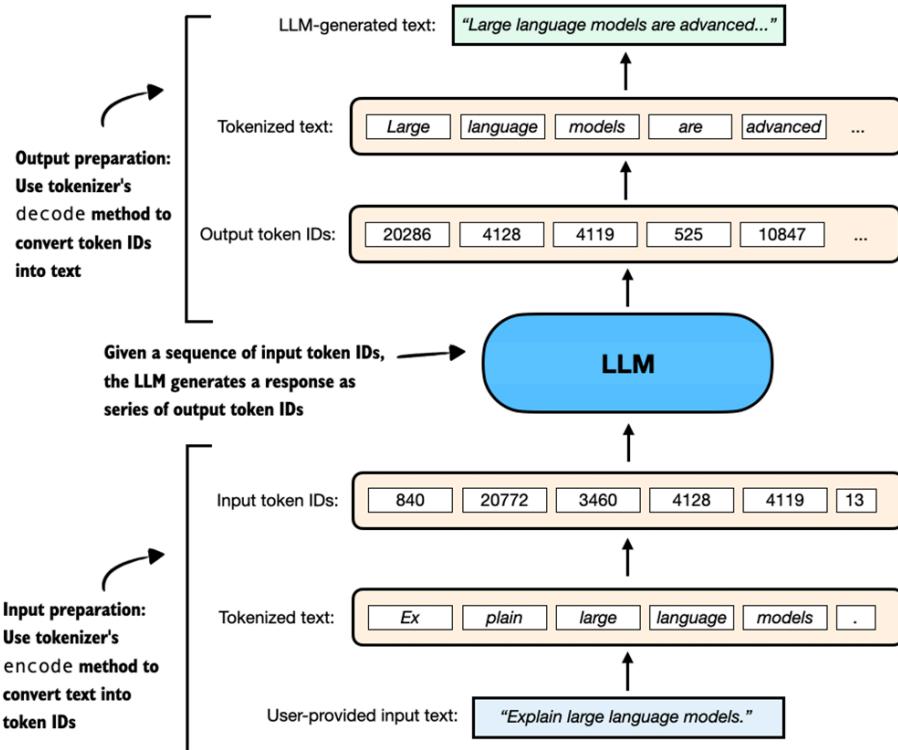
The supplementary code repository contains additional GPU platform recommendations inside the `ch02` subfolder if needed.

### USING PYTORCH

In this section, we imported and used the PyTorch library, which is currently the most widely used general-purpose library. We will use it throughout this book to run and train LLMs, including the reasoning methods we will develop. If you are new to PyTorch, to get the most out of this book, I recommend reading through my *PyTorch in One Hour: From Tensors to Training Neural Networks on Multiple GPUs* tutorial, which is freely available on my website at <https://sebastianraschka.com/teaching/pytorch-1h>.

## 2.4 Preparing input texts for LLMs

In this section, we explore how to use a tokenizer to process input and output text for an LLM, as shown in figure 2.5, which expands on the input and output preparation steps shown earlier in figure 2.2 to provide a more detailed view of the tokenization pipeline.



**Figure 2.5** A simplified illustration of how an LLM receives input data and generates output. The user-provided text is tokenized into IDs using the tokenizer's encode method, which are then processed by the LLM to generate output token IDs. These are decoded back into human-readable text using the tokenizer's decode method.

To see how this works in practice, we will begin by loading a tokenizer from this book's reasoning-from-scratch Python package, which should have been installed according to the instructions in section 2.2.

To download the tokenizer files (corresponding to the *Qwen3* base LLM, which we will introduce in the next section), run:

```
from reasoning_from_scratch.qwen3 import download_qwen3_small
download_qwen3_small(kind="base", tokenizer_only=True, out_dir="qwen3")
```

This will display a progress bar similar to:

```
tokenizer-base.json: 100% (6 MiB / 6 MiB)
```

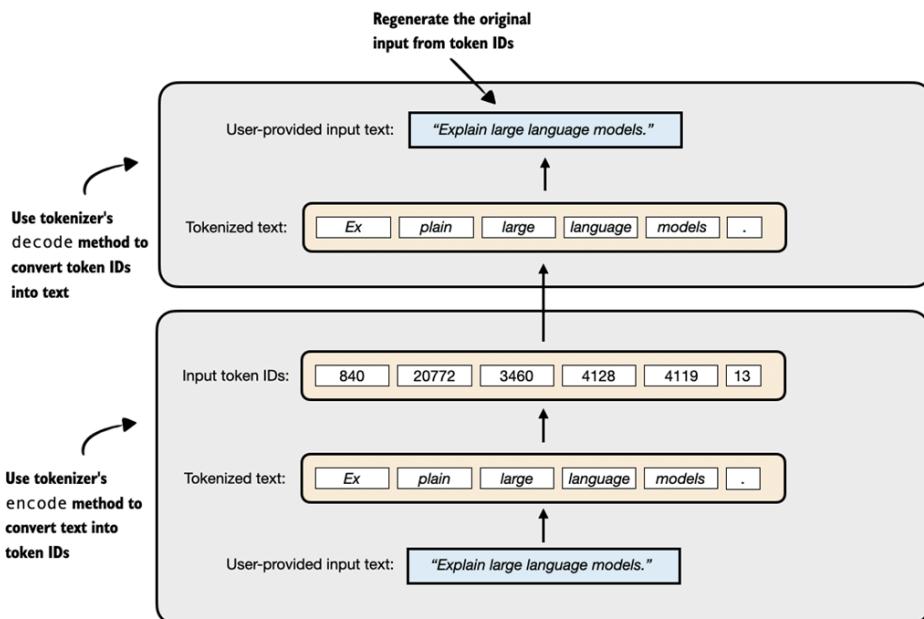
The command downloads the `tokenizer-base.json` file (approximately 6 megabytes in size) and saves it in a `qwen3` subdirectory.

Now, we can load the tokenizer settings from the tokenizer file into the `Qwen3Tokenizer`:

```
from pathlib import Path
from reasoning_from_scratch.qwen3 import Qwen3Tokenizer

tokenizer_path = Path("qwen3") / "tokenizer-base.json"
tokenizer = Qwen3Tokenizer(tokenizer_file_path=tokenizer_path)
```

Since we have not loaded the LLM yet (the central component shown in figure 2.5), we will first do a simpler dry run using just the tokenizer. Specifically, we will do a tokenization round-trip, that is, we will encode a text into *token IDs* and then decode those IDs back into text, as illustrated in figure 2.6.



**Figure 2.6** A demonstration of the round-trip tokenization process using a tokenizer. The user-provided input text is first converted into token IDs using the `encode` method, and then accurately reconstructed back into the original text using the `decode` method.

The following code snippet implements the encoding process shown at the bottom of figure 2.6:

```
prompt = "Explain large language models."
input_token_ids_list = tokenizer.encode(prompt)
```

And the following code implements the decoding process, converting the token IDs back into text, shown at the top of figure 2.6:

```
text = tokenizer.decode(input_token_ids_list)
print(text)
```

Based on the printed results, we can see that the tokenizer reconstructed the original input prompt from the token IDs:

```
'Explain large language models.'
```

Before we move on to the LLM, let's take a look at the token IDs that were generated by the `encode` method. The following code prints each token ID and its corresponding decoded string to help illustrate how the tokenizer works:

```
for i in input_token_ids_list:
    print(f"[{i}] --> {tokenizer.decode([i])}")
```

The output is as follows:

```
840 --> Ex
20772 --> plain
3460 --> large
4128 --> language
4119 --> models
13 --> .
```

As shown in the output, the original text is split into six token IDs. Each token represents a word or subword, depending on how the tokenizer segments the input.

For example, "Explain" was split into two separate tokens, "Ex" and "plain". This is because the tokenizer algorithm uses a subword-based method based on *Byte Pair Encoding (BPE)*. BPE can represent both common and rare words using a mix of full words and subword units. Spaces are also often included in tokens (for example, " large"), which helps the LLM detect word boundaries.

The `Qwen3Tokenizer` has a *vocabulary* of about 151,000 tokens, which is considered relatively large as of this writing (for comparison, the early GPT-2 has a vocabulary size of approximately 50,000 tokens, and Llama 3 has a vocabulary size of approximately 128,000 tokens).

A larger vocabulary in a language model increases its size and computational cost for each individually generated token, but it also allows more words to be represented as single tokens rather than being split into subword components. This is beneficial because splitting a word (like breaking "Explain" into "Ex" and "plain") results in more input tokens. More tokens lead to longer input sequences, which increases processing time and resource usage. For instance, doubling the number of tokens can roughly double the computational cost of running the model as it needs to generate more tokens to complete the response.

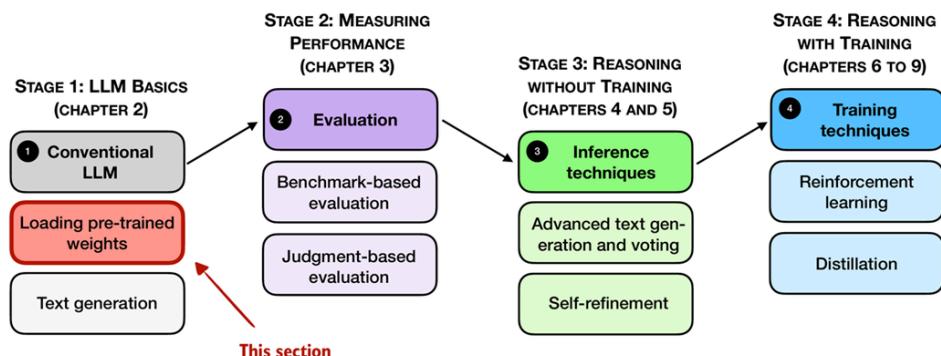
Unfortunately, a detailed coverage and from-scratch implementation of a tokenizer is outside the scope of this book. However, interested readers can find additional resources, including my from-scratch implementation, in the further resources and reading sections in appendix A.

#### EXERCISE 2.1: ENCODING UNKNOWN WORDS

Experiment with the tokenizer to see if and how it handles unknown words. For this, get creative and make up words that don't exist. Also, if you speak multiple languages, try to encode words in a different language than English.

## 2.5 Loading pre-trained models

In the previous section, we loaded and familiarized ourselves with the tokenizer that prepares the input data for an LLM and converts LLM outputs back into a human-readable text representation. In this section, we will load the LLM itself, as shown in the overview in figure 2.7.



**Figure 2.7** An overview of the four key stages in developing a reasoning model in this book. This section focuses on loading pre-trained LLM in Stage 1.

As mentioned in the previous section, this book uses Qwen3 0.6B as a pre-trained base model. In this section, we load its pre-trained weights, as shown in figure 2.7. The "0.6B" in the model name indicates that the model contains approximately 0.6 billion weight parameters.

Why Qwen3? After carefully evaluating several open-weight base models, I chose Qwen3 0.6B for the following reasons:

- For this book, we want a small yet capable open-weight model that can run on consumer hardware.
- The larger variants of the Qwen3 model family are, as of this writing, the leading open-weight models in terms of modeling performance.
- Qwen3 0.6B is more memory-efficient compared to Llama 3 1B and OLMo 2 1B.
- Qwen3 offers both a base model (the focus of our reasoning model development) and an official reasoning variant that we can use as a reference for evaluation purposes.

**NOTE** The canonical spelling of "Qwen3" does not include whitespace, whereas "Llama 3" does.

In line with the spirit of building things "from scratch," this book uses a custom reimplementation of Qwen3 that I wrote in pure PyTorch, which is entirely independent of external LLM libraries. The emphasis of this reimplementation is on code readability and tweakability, in case you want to modify it later for your own experiments. Despite being built from scratch, this implementation remains fully compatible with the original pre-trained Qwen3 model weights.

However, this book does not cover the Qwen3 code implementation in depth. This topic alone would fill an entire separate book, similar to my other book, *Build A Large Language Model (From Scratch)*. Instead, this *Build A Reasoning Model (From Scratch)* book specifically focuses on implementing reasoning methods on top of a base model, in this case, Qwen3.

**NOTE** This reimplemented Qwen3 LLM runs entirely locally, just like any other neural network implemented in PyTorch. There are no server-side components or external API calls involved. All model usage happens on your own machine, and your data stays on your device. If you are concerned about privacy, the setup we are using ensures full control over both the LLM inputs and outputs.

For those interested in additional details about Qwen3, as well as the model code, please see appendix C.

Before we load the model, we can specify the device we are going to use, namely, CPU or GPU. The following code in listing 2.1 will select the best-available device automatically:

#### **Listing 2.1 Get device automatically**

```
def get_device(enable_tensor_cores=True):
    if torch.cuda.is_available():
        device = torch.device("cuda")
        print("Using NVIDIA CUDA GPU")

        if enable_tensor_cores:
            major, minor = map(int, torch.__version__.split(".")[:2])
            if (major, minor) >= (2, 9):
                torch.backends.cuda.matmul.fp32_precision = "tf32"
                torch.backends.cudnn.conv.fp32_precision = "tf32"
            else:
                torch.backends.cuda.matmul.allow_tf32 = True
                torch.backends.cudnn.allow_tf32 = True

    elif torch.backends.mps.is_available():
        device = torch.device("mps")
        print("Using Apple Silicon GPU (MPS)")

    elif torch.xpu.is_available():
        device = torch.device("xpu")
        print("Using Intel GPU")

    else:
        device = torch.device("cpu")
        print("Using CPU")

    return device
```

Note that if you have a modern NVIDIA GPU (based on the Ampere architecture or newer), the `get_device()` function automatically enables Tensor Cores for faster matrix multiplications when `enable_tensor_cores=True`. This can slightly change floating-point rounding but does not noticeably affect results in this book, and the speed advantage on newer cards is worth it. On non-NVIDIA devices, these settings are ignored.

Using the code from listing 2.1, we can then obtain the device as follows:

```
device = get_device()
```

While GPUs generally provide substantial speed and performance improvements, it can be helpful to initially run the remaining code in this chapter using the CPU for compatibility and debugging purposes. You can temporarily override the automatic selection by explicitly setting:

```
device = torch.device("cpu")
```

After finishing the chapter and verifying the code works properly on the CPU, remove or comment out the manual override and rerun the code. If your system has a GPU, you should then observe improved performance.

**NOTE** The code in the remainder of this chapter was executed on a Mac Mini with an Apple M4 CPU. Performance comparisons with the Apple Silicon M4 GPU and the NVIDIA H100 GPU are included at the end of the chapter.

However, before we load the model and put it onto the selected `device`, we first need to download the weights for Qwen3 0.6B. These files are required to initialize the pre-trained model correctly:

```
download_qwen3_small(kind="base", tokenizer_only=False, out_dir="qwen3")
```

The output is as follows:

```
qwen3-0.6B-base.pth: 100% (1433 MiB / 1433 MiB)
✓ qwen3/tokenizer-base.json already up-to-date
```

(There is a checkmark in front of the tokenizer because we already downloaded it in the previous section.)

After downloading the model weights via the previous step, we can now instantiate a `Qwen3Model` class into which we load the pre-trained weights via PyTorch's `load_state_dict` method:

```

from reasoning_from_scratch.qwen3 import Qwen3Model, QWEN_CONFIG_06_B

model_path = Path("qwen3") / "qwen3-0.6B-base.pth"
model = Qwen3Model(QWEN_CONFIG_06_B) #A
model.load_state_dict(torch.load(model_path)) #B
model.to(device) #C

#A Instantiate a Qwen3 model with random weights as placeholders
#B Load the pre-trained weights into the model
#C Transfer the model to the designated device (e.g., "cuda")

```

Note that if the device setting is "cpu", the `model.to(device)` operation will be skipped because the model already sits in CPU memory by default.

After executing the code above, you should see the following output (if you are not running the code in an interactive environment like a Jupyter notebook, you have to run `print(model)` to see the output):

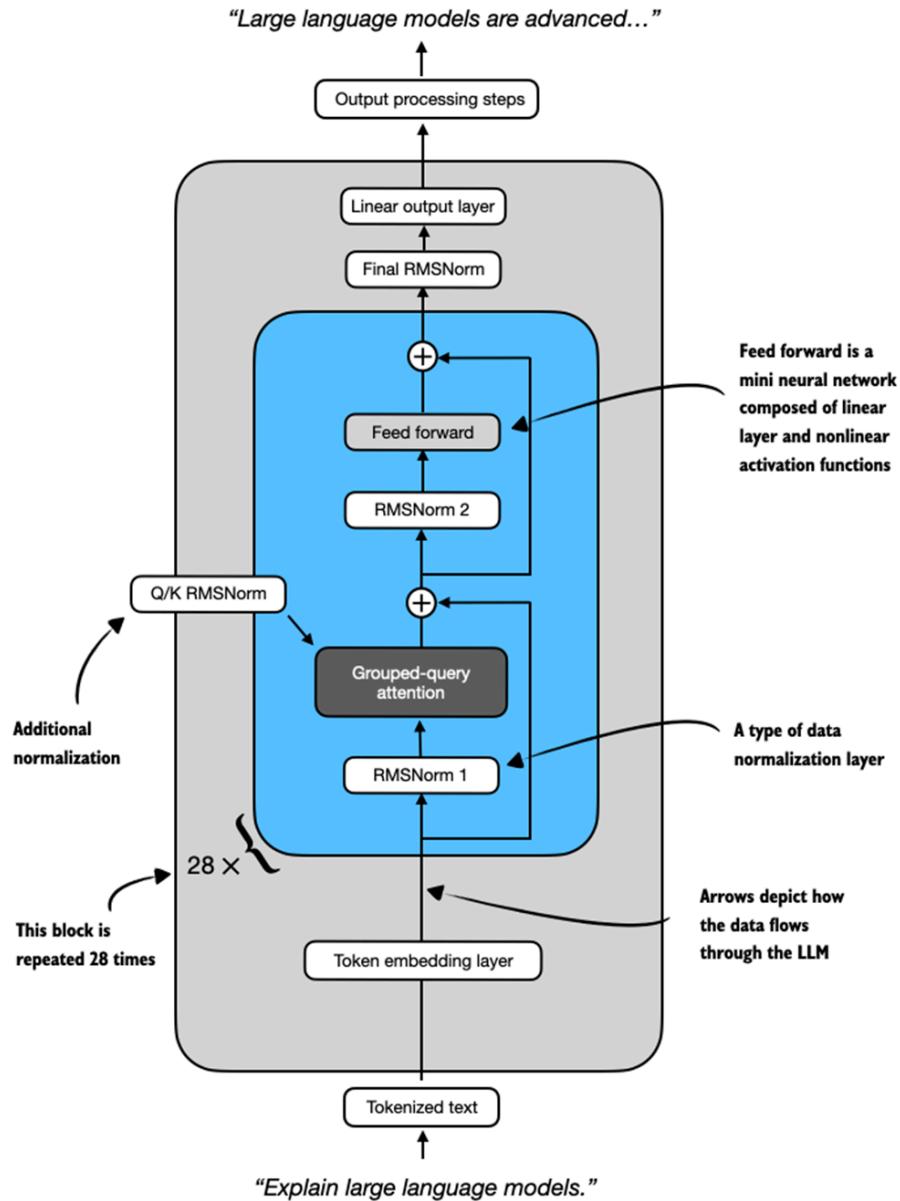
```

Qwen3Model(
  (tok_emb): Embedding(151936, 1024)
  (trf_blocks): ModuleList(
    (0-27): 28 x TransformerBlock(
      (att): GroupedQueryAttention(
        (W_query): Linear(in_features=1024, out_features=2048, bias=False)
        (W_key): Linear(in_features=1024, out_features=1024, bias=False)
        (W_value): Linear(in_features=1024, out_features=1024, bias=False)
        (out_proj): Linear(in_features=2048, out_features=1024, bias=False)
        (q_norm): RMSNorm()
        (k_norm): RMSNorm()
      )
      (ff): FeedForward(
        (fc1): Linear(in_features=1024, out_features=3072, bias=False)
        (fc2): Linear(in_features=1024, out_features=3072, bias=False)
        (fc3): Linear(in_features=3072, out_features=1024, bias=False)
      )
      (norm1): RMSNorm()
      (norm2): RMSNorm()
    )
  )
  (final_norm): RMSNorm()
  (out_head): Linear(in_features=1024, out_features=151936, bias=False)
)

```

This output is a summary of the Qwen3 0.6B base model architecture, as printed by PyTorch. It highlights the model's core components: an embedding layer, a stack of 28 transformer blocks, and a final linear projection head. Each transformer block includes a grouped-query attention mechanism and a multi-layer feedforward network, along with normalization layers throughout.

These components are also illustrated visually in figure 2.8 for readers familiar with LLM architectures. However, a detailed understanding of this architecture is not required for this book. Since we are not modifying the base model itself, but rather building reasoning methods on top of it, you can safely treat the architecture as a black box for now. However, interested readers can optionally find more information on these components, such as RMSNorm, in appendix C.

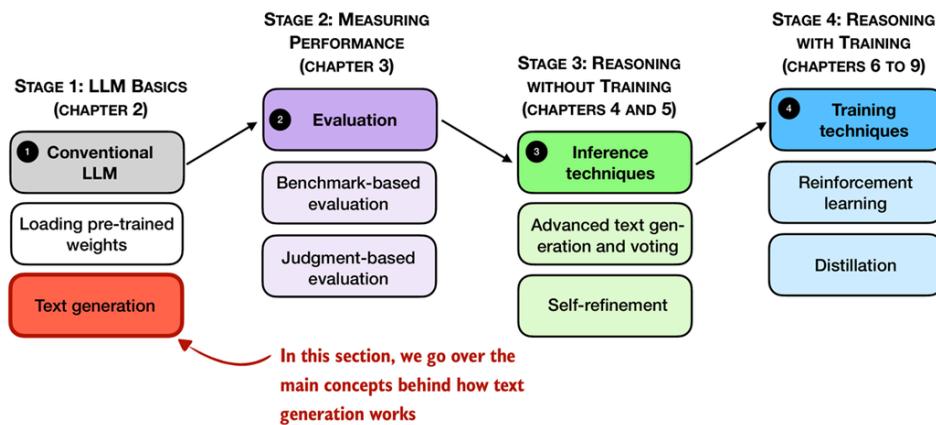


**Figure 2.8 Overview of the Qwen3 0.6B model architecture, as instantiated earlier. Input text is tokenized and passed through an embedding layer, followed by 28 repeated transformer blocks. Each block contains grouped-query attention, feedforward layers, and RMS normalization. The model ends with a final normalization and linear output layer. Arrows show the data flow through the model.**

The key takeaway from this section is that we have now loaded a pre-trained model, with its architecture shown in figure 2.8, that should be capable of generating coherent text. In the next section, we will code a text generation function that feeds tokenized data into the model and returns the response in a human-readable format.

## 2.6 Understanding the sequential LLM text generation process

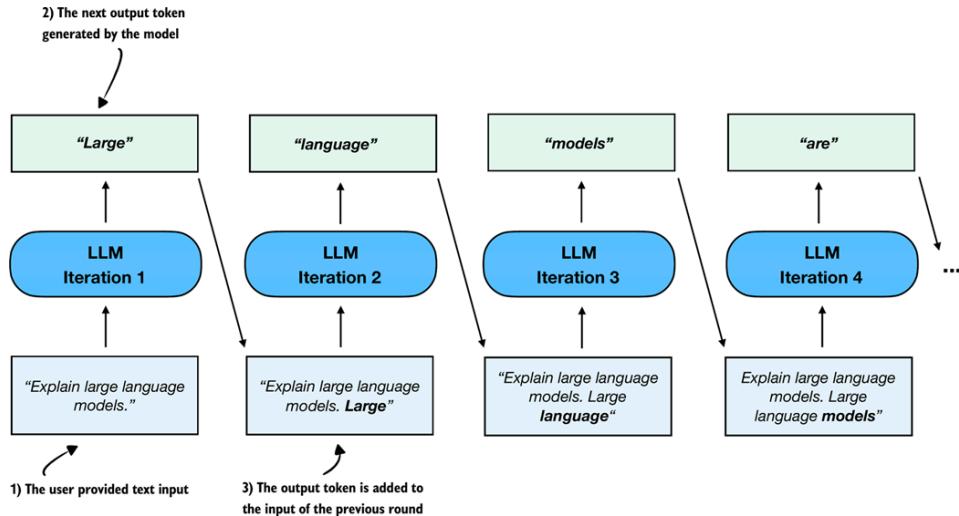
After loading a pre-trained LLM, our goal is to write a function that leverages the LLM to generate text. This function forms the foundation for reasoning-improving methods that we will implement later in the book, as shown in figure 2.9.



**Figure 2.9** An overview of the four key stages in developing a reasoning model in this book. This section explains the main concept behind text generation in LLMs, which allows us to implement a text generation function for using the pre-trained LLM in the remainder of this chapter.

However, before we get to implement this text generation function that we will use in this and upcoming chapters (as shown in figure 2.9), let's go over the basic concepts behind text generation in LLMs.

You may already know that text generation in LLMs is a sequential process where LLMs generate one word at a time. This is often also called *autoregressive* text generation and is shown in figure 2.10



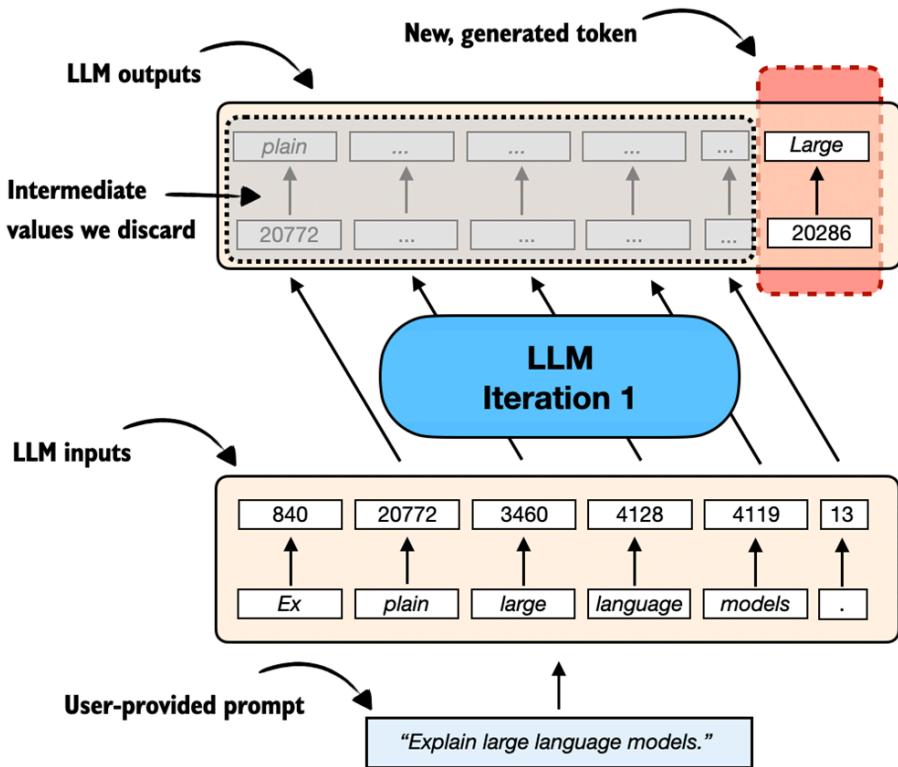
**Figure 2.10 An illustration of the sequential (autoregressive) text generation in LLMs. At each iteration, the model generates the next token based on the input and previously generated tokens, which are cumulatively fed back into the model to produce coherent output.**

Note that the sequential text generation process shown in figure 2.10 is a broad overview. The figure shows one generated output token (top row) at each step, when feeding it with an input prompt. This is done for simplicity to explain the main concept behind LLM-based text generation.

### CHATBOT INTERFACES

While the diagram in figure 2.10 shows the underlying next-token prediction process, a chat interface simply wraps this mechanism in a conversational loop. The model still predicts one token at a time, but the system feeds the entire dialogue history back into the model so that each new reply feels context-aware and coherent from turn to turn. This book focuses on single-turn conversations, where the current answer is independent of previous answers. However, interested readers can find an implementation of a chat interface with answer history in appendix G.

Now, if we look at one of these iterations more closely, an LLM generates one output token for each input token. This means that if we have six input tokens, the LLM returns six output tokens, as illustrated in figure 2.11. However, it is important to note that we only care about the last generated token in each iteration.



**Figure 2.1.1** A closer look at a single iteration of the autoregressive text generation process. The LLM generates an output sequence that mirrors the input but is shifted one position to the right. At each iteration, the model predicts the next token in the sequence. The LLM effectively learns to continue the input prompt one token at a time.

Before implementing a text-generation function that uses the concept shown in figure 2.11 for each iteration to implement the autoregressive text generation process shown in figure 2.10, let's take a look at a code example to illustrate figure 2.11 further by reusing the "Explain large language models." example prompt from section 2.4:

```

prompt = "Explain large language models."
input_token_ids_list = tokenizer.encode(prompt)
print(f"Number of input tokens: {len(input_token_ids_list)}")

input_tensor = torch.tensor(input_token_ids_list) #A
input_tensor_fmt = input_tensor.unsqueeze(0) #B
input_tensor_fmt = input_tensor_fmt.to(device)

output_tensor = model(input_tensor_fmt) #C
output_tensor_fmt = output_tensor.squeeze(0) #D
print(f"Formatted Output tensor shape: {output_tensor_fmt.shape}")

#A Convert Python list into PyTorch tensor
#B Add an additional dimension
#C Generate the output
#D Remove the extra dimension

```

## SQUEEZING AND UNSQUEEZING TENSORS

Tensors are generalized matrices of  $n$  dimensions. Many PyTorch functions and model components expect tensors with specific dimensions, so being able to add or remove dimensions is important for making data compatible with these operations.

The `.squeeze()` and `.unsqueeze()` operations in PyTorch are used to change the shape of a tensor by removing or adding dimensions of size 1. This is often useful for reshaping a tensor to match what a model expects. For example, a model might expect input tensors with two dimensions (e.g., rows and columns) so it can process batches of inputs (see appendix E). But if the input is just a row vector, we can use `.unsqueeze(0)` to add an extra dimension and make it compatible:

```

example = torch.tensor([1, 2, 3])
print(example)
print(example.unsqueeze(0))

```

This returns:

```

tensor([1, 2, 3])
tensor([[1, 2, 3]])

```

Here, `.unsqueeze(0)` adds a new dimension at position 0, turning a 1D tensor into a 2D tensor with shape  $(1, 3)$ . Conversely, `.squeeze(0)` removes a dimension of size 1 from position 0:

```
example = torch.tensor([[1, 2, 3]])
print(example)
print(example.squeeze(0))
```

This returns:

```
tensor([[1, 2, 3]])
tensor([1, 2, 3])
```

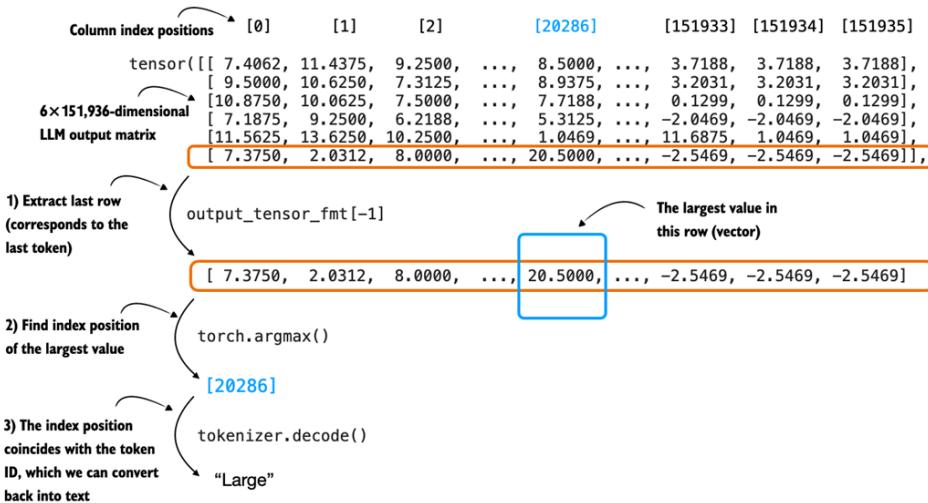
This is useful when you want to remove extra dimensions that are not needed.

The output from the previous code example is follows:

```
Number of input tokens: 6
Formatted Output tensor shape: torch.Size([6, 151936])
```

As we can see, we feed six input tokens into the model, which returns a  $6 \times 151,936$ -dimensional matrix. The 6 in this matrix corresponds to the six input tokens. The second dimension, 151,936, corresponds to the vocabulary size that the model supports. For instance, each of the six tokens is represented by a vector with 151,936 values. We can think of the values in these vectors as scores for each possible word in the vocabulary, where the highest score corresponds to the most likely word or subword (in the 151,936-entry vocabulary) to be chosen as the generated token.

So, to get the next generated word, we extract the last row of this  $6 \times 151,936$ -dimensional matrix, find the token ID corresponding to the largest score value in this row, and convert this token ID back into text via the tokenizer, as illustrated in figure 2.12.



**Figure 2.12** A closer look at how the raw scores output by an LLM, in a single text generation iteration, are converted into a token ID and its corresponding text representation.

Let's see how we can convert the LLM output matrix into the generated text token (shown in figure 2.12) in code:

Note that LLMs are trained with a next-word prediction task, and as shown in figure 2.11, we are only interested in the last token, which we can obtain via the `[-1]` index:

```
last_token = output_tensor_fmt[-1].detach()
print(last_token)
```

Here, `.detach()` separates the tensor from the part of the system that tracks how the model learns. In simple terms, it lets us take the last token from the model's output and use it for the next step without keeping extra information we don't need during generation. This saves memory and can make things run faster.

This prints the 151,936 values corresponding to the last token:

```
tensor([ 7.3750, 2.0312, 8.0000, ..., -2.5469, -2.5469, -2.5469],
      dtype=torch.bfloat16)
```

Then, we can use the `argmax` function to obtain the position with the largest value score (value) in this tensor:

```
print(last_token.argmax(dim=-1, keepdim=True))
```

The result is:

```
tensor([20286])
```

This returned integer value is the position of the largest value in this vector, and it also corresponds to the token ID of the generated token (`last_token`), which we can translate back into text via the tokenizer:

```
print(tokenizer.decode([20286]))
```

This prints the generated token:

```
Large
```

### MAX VERSUS ARGMAX

It is helpful to briefly recall how `max` and `argmax` work and how they differ, since we will use `torch.argmax()` later when we select the next token when implementing the text generation function.

The `torch.max()` and `torch.argmax()` functions in PyTorch are used to find the largest value in a tensor and the index of that value. For example:

```
example = torch.tensor([-2, 1, 3, 1])
print(torch.max(example))
print(torch.argmax(example))
```

This returns:

```
tensor(3)
tensor(2)
```

The maximum value is 3, and it first appears at index 2.

We can also use `keepdim=True` with `torch.argmax()` to keep the output shape consistent by retaining the reduced dimension:

```
print(torch.argmax(example, keepdim=True))
```

This returns:

```
tensor([2])
```

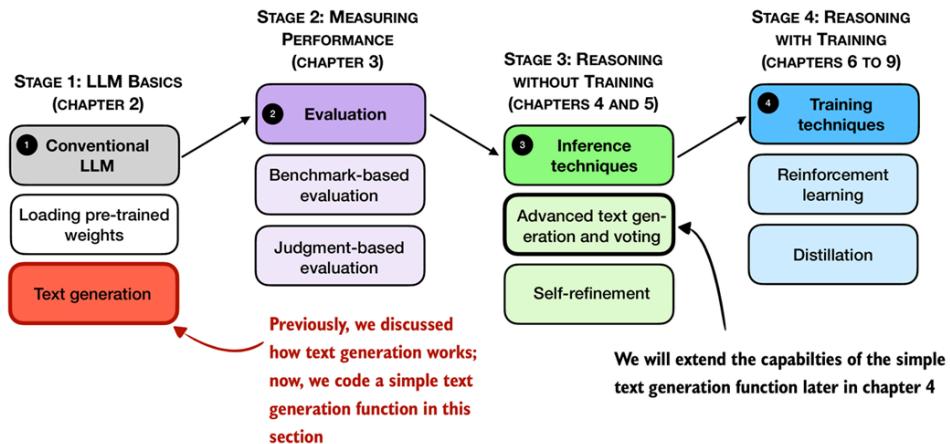
Here, `keepdim=True` keeps the result as a 1D tensor with the same number of dimensions as the input, which can be helpful for keeping the shape required by the tokenizer and for concatenation later on in our text generation function.

To recap, figure 2.10 illustrated the iterative (autoregressive) text generation process in an LLM. Then, figure 2.11 zooms in on one of the iterations in this process. Figure 2.12 then further zoomed into this one iteration and shows how the score matrix (output by an LLM), gets converted into a token ID (and its corresponding text representation).

While we have seen how to use the LLM to generate a single token, in the next section, we will put these concepts to action and implement a function that applies this concept sequentially to generate coherent output text.

## 2.7 Coding a minimal text generation function

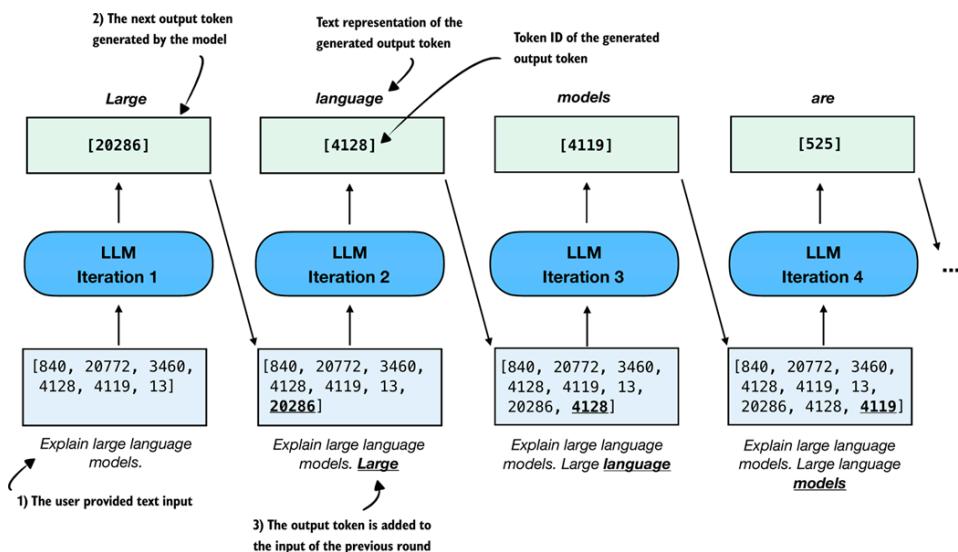
The previous section explained a single iteration in the basic, sequential text generation process in LLMs. In this section, building on that concept, we will implement a text generation function that uses the pre-trained LLM to generate coherent text following a user prompt, as illustrated in Figure 2.13 in the chapter overview.



**Figure 2.13** An overview of the four key stages in developing a reasoning model in this book. In this section we implement a text generation function for the pre-trained LLM.

This text generation function, mentioned in figure 2.13, works by first converting the input prompt into token IDs that the model can process. The model then predicts the next most likely token, appends it to the sequence, and reprocesses the extended sequence to generate the next token. This iterative process continues until a stopping condition is met, and the generated token IDs are then decoded back into text.

Figure 2.14 shows this process step by step, with both the generated token IDs and their corresponding text at each stage. (This figure is similar to figure 2.10 shown at the beginning of the previous section, except it shows the generated token ID alongside their text representation.)



**Figure 2.14 An illustration of sequential (autoregressive) text generation in large language models (LLMs), with token IDs shown explicitly. At each iteration, the model generates the next token based on the original input and all previously generated tokens. The predicted token is added to the sequence in both its textual and token ID form.**

The `generate_text_basic` function in listing 2.2 below implements the sequential text generation process (figure 2.14) using the `argmax` function introduced in the previous section:

**Listing 2.2 A basic text generation function**

```

@torch.inference_mode()                                     #A
def generate_text_basic(
    model,
    token_ids,
    max_new_tokens,
    eos_token_id=None
):
    input_length = token_ids.shape[1]
    model.eval()                                         #B

    for _ in range(max_new_tokens):
        out = model(token_ids)[:, -1]                      #C
        next_token = torch.argmax(out, dim=-1, keepdim=True)

        if (eos_token_id is not None                         #D
            and next_token.item() == eos_token_id):
            break

        token_ids = torch.cat(                                #E
            [token_ids, next_token], dim=1)                  #E
    return token_ids[:, input_length:]                      #F

```

#A Disable gradient tracking for speed and memory efficiency

#B Switch model to evaluation mode to enable deterministic behavior (best practice)

#C Get the scores of the last token

#D Stop if all sequences in the batch have generated EOS

#E Append the newly predicted token to the sequence

#F Return only the generated tokens (excluding the original input)

In essence, the `generate_text_basic` function listing 2.2 applies the `argmax`-based token ID extraction via a for-loop for a user-specified number of iterations (`max_new_tokens`). It returns the generated token IDs, similar to what's shown in figure 2.14, which we can then convert back into text.

Let's use the function to generate a 100-token response to a simple "Explain large language models in a single sentence." prompt to make sure that the `Qwen3Model` and `generate_text_basic` function work (we get to the reasoning task examples in later chapters).

Please note that the following code will be slow and can take 1-3 minutes to complete, depending on your computer (we will speed it up in later sections):

```

prompt = "Explain large language models in a single sentence."
input_token_ids_tensor = torch.tensor(
    tokenizer.encode(prompt),
    device=device
).unsqueeze(0)

max_new_tokens = 100 #B
output_token_ids_tensor = generate_text_basic(
    model=model,
    token_ids=input_token_ids_tensor,
    max_new_tokens=max_new_tokens,
)
output_text = tokenizer.decode(
    output_token_ids_tensor.squeeze(0).tolist() #C
)
print(output_text)

#A Transfer the input token IDs onto the same device (CPU, GPU) where the model is located
#B Let the model generate up to 100 new tokens
#C Convert output token IDs from PyTorch tensor to Python list

```

The generated output text is as follows:

Large language models are artificial intelligence systems that can understand, generate, and process human language, enabling them to perform a wide range of tasks, from answering questions to writing articles, and even creating creative content.<|endoftext|>Human language is a complex and dynamic system that has evolved over millions of years to enable effective communication and social interaction. It is composed of a vast array of symbols, including letters, numbers, and words, which are used to convey meaning and express thoughts and ideas. The evolution of language has

Note that the output above was generated on a CPU. Depending on the device (e.g., CPU versus GPU), the exact wording may vary slightly due to differences in floating-point behavior on different hardware.

As we can see based on the output above, the model follows the instruction quite well by producing a single, clear sentence in response to the prompt. However, it continues generating additional, off-topic text after the special token <|endoftext|>. This token is used during training to mark the end of a document and separate different samples.

---

**TIP** The leading whitespace in " Large" (the first output word) appears because the model continued the text based on the input prompt but we sliced off the original prompt with `token_ids[:, input_length:]` in the return line in listing 2.2. If this leading whitespace bothers you, you can remove it via `token_ids[:, input_length:].lstrip()` or `output_text.lstrip()`.

---

When using the model for inference (generating text after training), we typically want it to stop as soon as it produces the special token `<|endoftext|>`. This token is represented by the ID 151643, which we can confirm using:

```
print(tokenizer.encode("<|endoftext|>"))
```

For convenience, this token ID is also saved via the `tokenizer.eos_token_id` attribute. We can pass this ID to the `generate_text_basic` function to signal when generation should stop:

```
output_token_ids_tensor = generate_text_basic(
    model=model,
    token_ids=input_token_ids_tensor,
    max_new_tokens=max_new_tokens,
    eos_token_id=tokenizer.eos_token_id  #A
)
output_text = tokenizer.decode(
    output_token_ids_tensor.squeeze(0).tolist()
)
print(output_text)

#A Pass end-of-sequence (eos) token ID
```

The output looks like this:

Large language models are artificial intelligence systems that can understand, generate, and process human language, enabling them to perform a wide range of tasks, from answering questions to writing articles, and even creating creative content.

If we compare the response to the previous response, we can see that the text generation stopped once the end-of-sequence token was encountered.

You may have noticed that generating the response is relatively slow and might take several seconds up to multiple minutes, depending on the hardware.

### EXERCISE 2.2: STREAMING TOKEN GENERATION

Write a modified version of the `generate_text_basic` function that returns each token as it is generated and prints it, which is also known as *streaming* token generation.

The goal of this exercise is to understand how to implement token-by-token text generation, a technique often used in real-time applications like chatbots and interactive assistants.

Tip 1: Use `yield` instead of `return` to turn the function into a generator.

Tip 2: Then, outside the function, decode each token using a tokenizer and print it as it's generated (`for token in generate_text_basic_stream(...): ...`) to simulate streaming output.

Before we wrap up and learn how to speed up this function substantially, let's implement a simple utility function in listing 2.3 that measures the runtime of the text generation process:

#### **Listing 2.3 Token generation speed and memory usage**

```
def generate_stats(output_token_ids, tokenizer, start_time,
                   end_time, print_tokens=True):
    total_time = end_time - start_time
    print(f"Time: {total_time:.2f} sec")
    print(f"{int(output_token_ids.numel() / total_time)} tokens/sec")

    for name, backend in (("CUDA", getattr(torch, "cuda", None)),
                          ("XPU", getattr(torch, "xpu", None))):
        if backend is not None and backend.is_available():
            max_mem_bytes = backend.max_memory_allocated()
            max_mem_gb = max_mem_bytes / (1024 ** 3)
            print(f"Max {name} memory allocated: {max_mem_gb:.2f} GB")
            backend.reset_peak_memory_stats()

    if print_tokens:
        output_text = tokenizer.decode(output_token_ids.squeeze(0).tolist())
        print(f"\n{output_text}")
```

The `generate_stats` function in listing 2.3 will calculate the total runtime, given a start and end time stamp, the generation speed in terms of tokens per second (tokens/sec), and the GPU memory used. Note that the GPU memory usage is currently only computed for CUDA-supported GPUs, as PyTorch lacks similar utility functions for CPUs and Apple Silicon GPUs.

To apply the `generate_stats` function, we obtain a `start_time` and `end_time` stamp immediately before and after running the `generate_text_basic` function via Python's `time` module:

```
import time

start_time = time.time()
output_token_ids_tensor = generate_text_basic(
    model=model,
    token_ids=input_token_ids_tensor,
    max_new_tokens=max_new_tokens,
    eos_token_id=tokenizer.eos_token_id
)
end_time = time.time()
generate_stats(output_token_ids_tensor, tokenizer, start_time, end_time)
```

The output, on a Mac Mini M4 CPU, is as follows:

```
Time: 7.94 sec
5 tokens/sec
Large language models are artificial intelligence systems that can
understand, generate, and process human language, enabling them to
perform a wide range of tasks, from answering questions to writing
articles, and even creating creative content.
```

At 5 tokens per second, the generation speed is relatively slow. In the next section, we will implement a caching technique that speeds up the generation process 5-6 fold.

### TEXT GENERATION AND INFERENCE TERMINOLOGY

When reading LLM literature or software documentation, you will often see the term *inference* used where you might expect text generation. In a neural network context, inference means something very specific, namely, taking a model whose parameters are already learned and fixed, running a forward pass, and producing a prediction (for example, generating the next token). Nothing is being estimated or learned during this stage. In the forward pass, we are simply applying a function.

This is different from inference in statistics, where the goal is to learn unknown information from data. *Statistical inference* involves estimating parameters, quantifying uncertainty, or testing hypotheses about a population or data-generating process.

Note, that while in large language model contexts, the model is computing the next-token distribution, and people sometimes say the model is "estimating" or "inferring" the next token, this is not estimation in the statistical sense. At this stage, the model is a fixed function and its parameters are already learned.

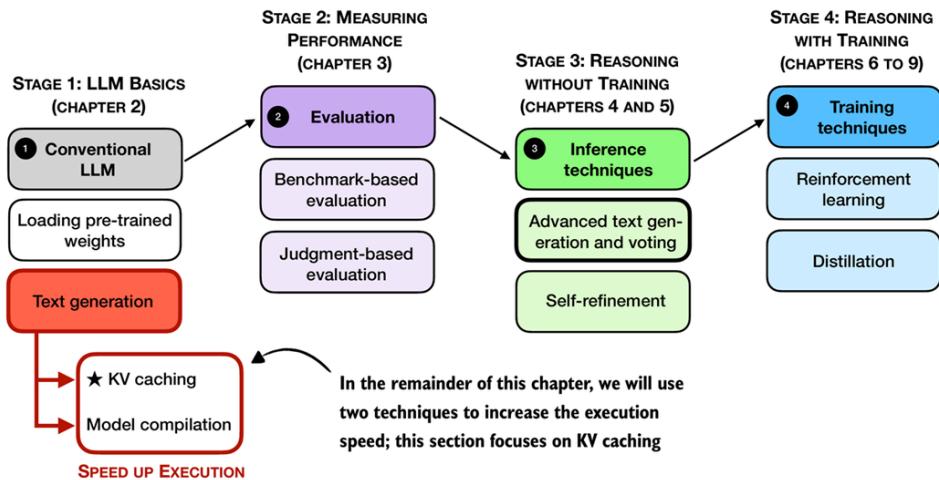
So when we call the `generate_text_basic` function, we are not performing statistical inference. We are performing neural network inference, which is just the forward application of a trained model to produce the next-token distribution and select the next generated token from this distribution.

## 2.8 Faster inference via KV caching

So now that we have a basic text generation function in place, we can turn our attention to what happens when we actually run it in practice. As you may have noticed, the text generation in the previous section can be a bit slow. That slowdown points us to a key concern: performance during inference.

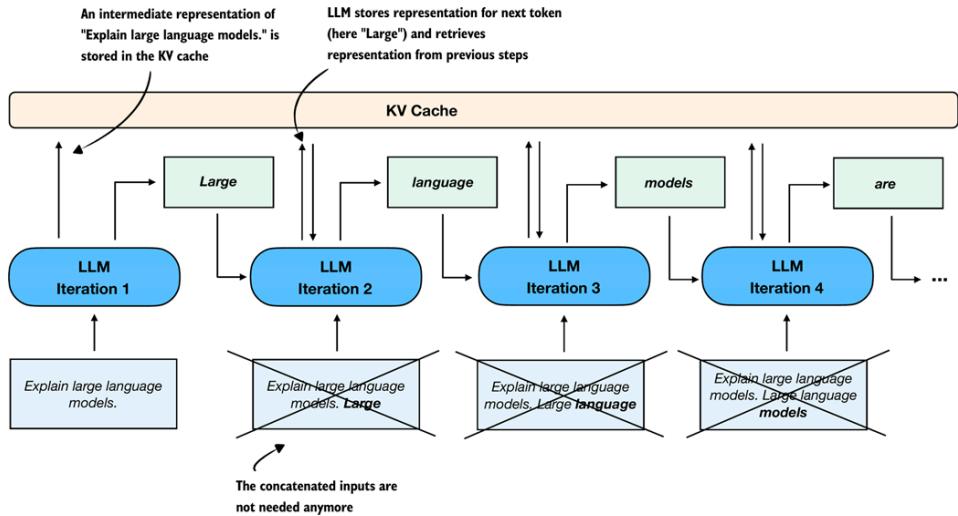
When running inference with LLMs, which in this context means generating text from a prompt, runtime performance (efficiency) quickly becomes important, especially for long sequences. While the code in this book emphasizes clarity over speed, real-world systems often use engineering tricks to make inference more efficient.

In the remaining two sections, we will cover two fundamental techniques, KV caching and model compilation, as shown in the overview in figure 2.15, to speed up the text generation.



**Figure 2.15** An overview of the four key stages in developing a reasoning model in this book. This section builds on pre-trained LLM and the basic text generation function we coded earlier and applies KV caching to speed up execution.

As shown in figure 2.15, One engineering trick that increases the text generation speed is *KV caching*, where KV refers to the keys and values used in the model's attention mechanism. If you are not familiar with these terms, that's okay. The key idea is that we can cache certain intermediate values and reuse them at each step of text generation, as shown in figure 2.16, which helps speed up inference.



**Figure 2.16 Illustration of how a KV cache improves efficiency during autoregressive text generation. Instead of reprocessing the entire input sequence at each step, the KV cache stores intermediate representations so that the LLM can reuse them to generate the next token. This eliminates the need to concatenate the generated token with prior inputs in each subsequent iteration.**

The key idea of KV caching, as shown in figure 2.16, is to store intermediate values computed in each iteration in a cache. Previously, each new token generated by the network was concatenated to the entire input sequence and fed back into the model repeatedly (indicated by crossed-out boxes in the diagram). This approach was inefficient because all tokens, except the newly generated one, remain identical in subsequent iterations. By using a KV cache, we avoid redundant computation and instead directly retrieve stored intermediate representations.

As mentioned earlier, the non-reasoning focused LLM details like KV caching, which we used to improve the token generation speed, are outside the scope of this book, and they are not required for the topics covered later in this book. However, interested readers can find more information on the mechanics of KV caching in my freely available article: *Understanding and Coding the KV Cache in LLMs from Scratch* (<https://magazine.sebastianraschka.com/p/coding-the-kv-cache-in-langs>).

Below is a modified version of the `generate_text_basic` function that incorporates a KV cache, which is almost identical to the basic text generation function in listing 2.2, except for the KV cache-related change highlighted via the comments:

**Listing 2.4 A basic text generation function with KV cache**

```

from reasoning_from_scratch.qwen3 import KVCache

@torch.inference_mode()
def generate_text_basic_cache(
    model,
    token_ids,
    max_new_tokens,
    eos_token_id=None
):

    input_length = token_ids.shape[1]
    model.eval()
    cache = KVCache(n_layers=model.cfg["n_layers"])           #A
    model.reset_kv_cache()
    out = model(token_ids, cache=cache)[:, -1]      #B

    for _ in range(max_new_tokens):
        next_token = torch.argmax(out, dim=-1, keepdim=True)

        if (eos_token_id is not None
            and next_token.item() == eos_token_id):
            break

        token_ids = torch.cat([token_ids, next_token], dim=1)
        out = model(next_token, cache=cache)[:, -1]          #C

    return token_ids[:, input_length:]

```

#A Initialize the KV cache

#B In the first round, the whole input is provided to the model as before

#C Consequent iterations only feed the next\_token to the input

The `generate_text_basic_cache` function in listing 2.4 differs only slightly from the `generate_text_basic` function in listing 2.2. The main difference is the introduction of a `KVCache` object.

During the first iteration, the model is given the full input token sequence as before, using `model(token_ids, cache=cache)`. Behind the scenes, the KV cache stores intermediate values for all these input tokens.

In the following iterations, we no longer need to pass the entire sequence. Instead, we only provide the `next_token` to the model using `model(next_token, cache=cache)`. The model then retrieves the necessary context from the previously stored KV cache.

Let's time this function to see whether it provides any performance benefits:

```
start_time = time.time()
output_token_ids_tensor = generate_text_basic_cache(
    model=model,
    token_ids=input_token_ids_tensor,
    max_new_tokens=max_new_tokens,
    eos_token_id=tokenizer.eos_token_id,
)
end_time = time.time()
generate_stats(output_token_ids_tensor, tokenizer, start_time, end_time)
```

The output is:

```
Time: 1.40 sec
29 tokens/sec
```

Large language models are artificial intelligence systems that can understand, generate, and process human language, enabling them to perform a wide range of tasks, from answering questions to writing articles, and even creating creative content.

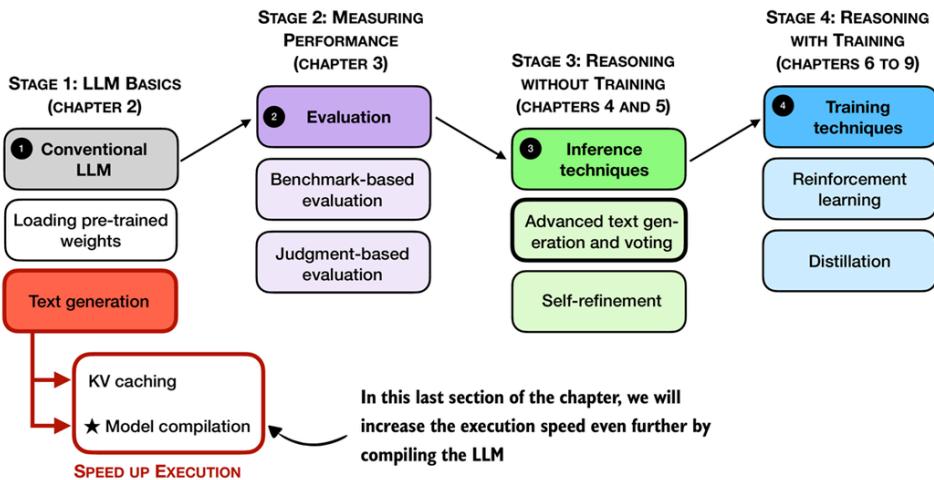
As we can see, this approach is significantly faster, generating 29 tokens per second compared to just 5 tokens per second previously (measured on a Mac Mini M4 CPU).

Importantly, we also see that the generated text is the same as before, which is an important sanity check to ensure that the KV cache is implemented and used correctly.

In the next section, we will learn about another technique we can use to further improve the generation speed, which will come in handy when we evaluate the model in the upcoming chapters. Faster generation allows us to run more evaluations in less time and makes it easier to compare different models or settings efficiently.

## 2.9 Faster inference via PyTorch model compilation

In the previous section, we covered KV caching as a technique to improve runtime efficiency as shown in the overview in figure 2.17.



**Figure 2.17 An overview of the four key stages in developing a reasoning model in this book. This section builds on pre-trained LLM and the basic text generation function we coded earlier, including KV caching, and adds model compilation to speed up the execution speed even further.**

As shown in figure 2.17, in this remaining section of this chapter, we will apply another technique that can substantially speed up model inference: model compilation using `torch.compile`. This feature allows the model to be compiled ahead of time, which reduces overhead and improves runtime performance during text generation.

```
major, minor = map(int, torch.__version__.split(".")[:2])
if (major, minor) >= (2, 8):
    # This avoids retriggering model recompilations
    # in PyTorch 2.8 and newer
    # if the model contains code like self.pos = self.pos + 1
    torch._dynamo.config.allow_unspec_int_on_nn_module = True

model_compiled = torch.compile(model)
```

If you are using a Mac with Apple Silicon and encounter an `InductorError`, please make sure to use PyTorch 2.9 or newer.

It is worth noting that the first execution using the compiled model may be slower than usual due to the initial compilation and optimization steps. To better measure the performance improvement, we will repeat the text generation process multiple times.

To begin, we will test this using the non-cached version of the generation function. The code in listing 2.5 is similar to what we used before except that we run it three times in a row. The code execution may take a few minutes to finish, depending on the system:

**Listing 2.5 Generating text with the compiled model**

```
for i in range(3): #A
    start_time = time.time()
    output_token_ids_tensor = generate_text_basic(
        model=model_compiled,
        token_ids=input_token_ids_tensor,
        max_new_tokens=max_new_tokens,
        eos_token_id=tokenizer.eos_token_id
    )
    end_time = time.time()

    if i == 0: #B
        print("Warm-up run") #B
    else:
        print(f"Timed run {i}:")
    generate_stats(output_token_ids_tensor, tokenizer, start_time, end_time)

    print(f"\n{30*'-'}\n")

#A We run the token generation three times
#B The first run is labeled as "Warm-up run"
```

The output is as follows:

```
Warm-up run
Time: 11.68 sec
3 tokens/sec
```

Large language models are artificial intelligence systems that can understand, generate, and process human language, enabling them to perform a wide range of tasks, from answering questions to writing articles, and even creating creative content.

-----

```
Timed run 1:
Time: 6.78 sec
6 tokens/sec
Output text:
```

Large language models are artificial intelligence systems that can understand, generate, and process human language, enabling them to perform a wide range of tasks, from answering questions to writing articles, and even creating creative content.

-----

```
Timed run 2:
Time: 6.80 sec
6 tokens/sec
Output text:
```

Large language models are artificial intelligence systems that can understand, generate, and process human language, enabling them to perform a wide range of tasks, from answering questions to writing articles, and even creating creative content.

-----

As we can see from the results above, the compiled model achieves a slight improvement in speed, with around 6 tokens per second compared to the previous 5 tokens per second.

Next, let's see how the KV cache version performs in comparison, using the same code as before except for swapping `generate_text_basic` with `generate_text_basic_cache`:

**Listing 2.6 Generating text with the compiled model using a KV cache**

```
for i in range(3):
    start_time = time.time()
    output_token_ids_tensor = generate_text_basic_cache(
        model=model_compiled,
        token_ids=input_token_ids_tensor,
        max_new_tokens=max_new_tokens,
        eos_token_id=tokenizer.eos_token_id
    )
    end_time = time.time()

    if i == 0:
        print("Warm-up run")
        generate_stats(
            output_token_ids_tensor, tokenizer, start_time, end_time
        )
    else:
        print(f"Timed run {i}:")
        generate_stats(output_token_ids, tokenizer, start_time, end_time)

print(f"\n{30*'-'}\n")
```

The output is as follows:

```
Warm-up run
Time: 8.07 sec
5 tokens/sec
```

Large language models are artificial intelligence systems that can understand, generate, and process human language, enabling them to perform a wide range of tasks, from answering questions to writing articles, and even creating creative content.

-----

```
Timed run 1:
Time: 0.60 sec
68 tokens/sec
Output text:
```

Large language models are artificial intelligence systems that can understand, generate, and process human language, enabling them to perform a wide range of tasks, from answering questions to writing articles, and even creating creative content.

-----

```
Timed run 2:
Time: 0.60 sec
68 tokens/sec
Output text:
```

Large language models are artificial intelligence systems that can understand, generate, and process human language, enabling them to perform a wide range of tasks, from answering questions to writing articles, and even creating creative content.

-----

As we can see based on the outputs above, the model generation speed improved from 29 tokens per second for the uncompiled model with KV cache to 68 tokens per second when the same model is compiled (on a Mac Mini M4 CPU), which is more than a 2-fold speed-up.

If you don't see any improvement, try running `torch.compile` with the "max-autotune" mode instead of the default settings. For instance, replace

```
model = torch.compile(model)
```

With

```
model = torch.compile(model, mode="max-autotune")
```

**EXERCISE 2.3: RERUN CODE ON NON-CPU DEVICES**

If you have access to a GPU, rerun the code in this chapter on a GPU device and compare the runtimes to the CPU runtimes.

In case you are curious, how the different model configurations compare on an Apple Silicon GPU and a high-end NVIDIA GPU, see table 2.1.

**Table 2.1 Token generation speeds and GPU memory usage for different model configurations on different hardware**

Mode	Hardware	Tokens/sec	GPU memory
Regular	Mac Mini M4 CPU	5	-
Regular compiled	Mac Mini M4 CPU	6	-
KV cache	Mac Mini M4 CPU	28	-
KV cache compiled	Mac Mini M4 CPU	68	-
Regular	Mac Mini M4 GPU	27	-
Regular compiled	Mac Mini M4 GPU	43	-
KV cache	Mac Mini M4 GPU	41	-
KV cache compiled	Mac Mini M4 GPU	71	-
Regular	NVIDIA H100 GPU	51	1.55 GB
Regular compiled	NVIDIA H100 GPU	164	1.81 GB
KV cache	NVIDIA H100 GPU	48	1.52 GB
KV cache compiled	NVIDIA H100 GPU	141	1.81 GB

As shown in the table above, the NVIDIA GPU delivers the best performance, which is expected. The CPU still performs surprisingly well once a KV cache and a compiled model are enabled. However, there are a few important details that explain why the GPU gains are not larger for this particular setup.

First, the model in this chapter is not optimized for GPUs. The implementation aims for a good balance between memory usage and compute speed because memory is the main bottleneck for most readers. A GPU-optimized variant would pre-allocate the full K and V tensors up to the maximum context length (in this case, 40 thousand tokens). This pre-allocation avoids repeated concatenation via `torch.cat`, but it also increases memory consumption.

With a large context size, pre-allocating, for example, 40 thousand entries for both K and V adds a noticeable footprint. Another approach would be to let users specify a fixed context size at construction time, but that introduces extra configuration overhead. The KV cache here grows on demand via `torch.cat`, which is simpler and more memory friendly, although concatenating non-preallocated tensors is a bit slower on GPUs.

I included a GPU-optimized version in the bonus materials, where the KV cache variant is slightly faster than the regular version, but it uses more memory (see [https://github.com/rasbt/reasoning-from-scratch/tree/main/ch02/03\\_optimized-LLM](https://github.com/rasbt/reasoning-from-scratch/tree/main/ch02/03_optimized-LLM)).

Second, the model used for benchmarking is small. Larger models benefit much more from KV caching and from GPU-optimized memory layouts. The same holds for batched inference, where GPUs can better saturate their compute units. With a larger model or a GPU-focused implementation, the performance gap in favor of the NVIDIA GPU would be more pronounced.

All examples were run using a single prompt (i.e., a batch size of 1). For readers interested in how performance scales with multiple inputs, batched inference is discussed in appendix E.

## 2.10 Summary

- Using LLMs to generate text involves multiple key steps:
  - Setting up the coding environment to run LLM code and install necessary dependencies.
  - Loading a pre-trained base LLM (such as Qwen3 0.6B), which will be extended with reasoning capabilities in later chapters.
  - Initializing and using a tokenizer, which converts text input into token IDs and decodes output back to human-readable form.
- Text generation in LLMs follows a sequential (autoregressive) process, where the model generates one token at a time by predicting the next most likely token.
- The speed and efficiency of text generation can be improved through:
  - KV caching, which stores intermediate states to avoid recomputing previously encountered input tokens at each step.
  - Model compilation using `torch.compile`, which optimizes runtime performance.
- This chapter lays the technical foundation for reasoning capabilities in upcoming chapters by implementing a functional, efficient text generation pipeline using a pre-trained base LLM.

## 3 Evaluating reasoning models

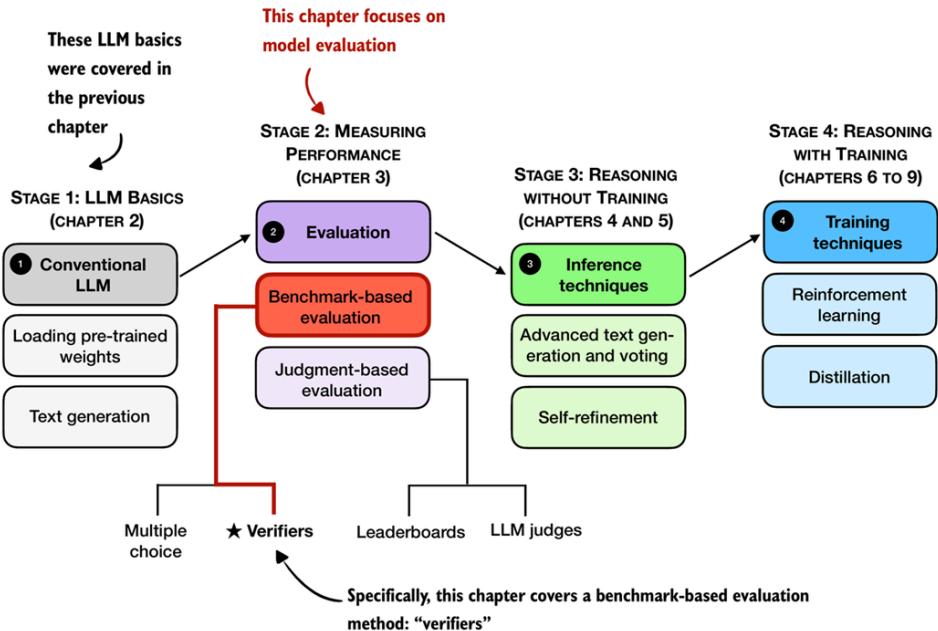
### This chapter covers

- Extracting final answers reliably from an LLM response
- Verifying answer correctness by comparing an LLM's output to the reference solution using a symbolic math solver
- Running a full evaluation pipeline by loading a pre-trained model, generating outputs, and grading them against a dataset

Evaluation is what lets us distinguish between LLMs that merely sound convincing and those that can solve problems correctly. LLM evaluation techniques span a broad range of approaches, from measuring task accuracy to making sure that LLMs adhere to specific safety standards.

In this chapter, we focus on implementing a *verification-based* method that checks whether an LLM can solve math problems accurately by comparing its own answers against reference solutions using a calculator-like implementation.

This verifier is particularly useful because it not only evaluates performance on math tasks but also introduces the principle of *verifiable rewards*, which is the foundation of the reinforcement learning approach to reasoning models that we will implement later in chapter 6. (Interested readers can find additional evaluation methods in appendix F.)



**Figure 3.1** A mental model of the topics covered in this book. This chapter covers evaluation methods (stage 2), with a special focus on implementing verifiers.

### 3.1 Building a math verifier

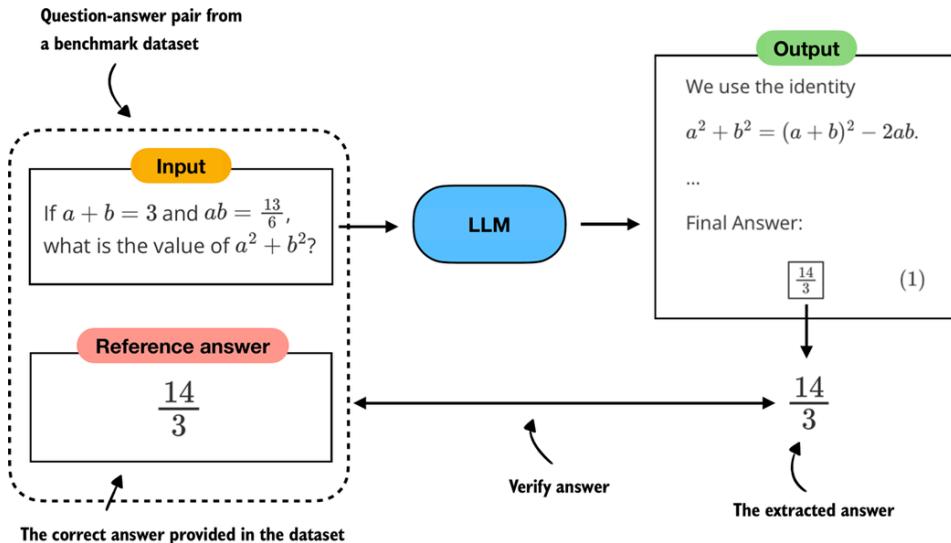
There are four common ways of evaluating trained LLMs in practice: *multiple choice*, *verifiers*, *leaderboards*, and *LLM judges*, as shown in figure 3.1. These methods are widely used across research papers, technical reports, marketing materials, and model cards, and results often draw from more than one category.

As figure 3.1 illustrates, these evaluation approaches can be grouped into two broader types: *benchmark-based evaluation* and *judgment-based evaluation*. All four evaluation methods are useful in different contexts, but verifiers are especially relevant for reasoning models.

Math problems provide a natural example: depending on the problem complexity, math problems benefit from step-by-step reasoning to solve, yet evaluation is straightforward because the final answer can be checked against a correct answer. In this setting, the verifier approach provides a simple and reliable way to measure whether a model's reasoning steps lead to the correct outcome.

In this chapter, we focus on verifiers as a benchmark-based approach for measuring answer correctness in math problems, as illustrated in figure 3.2.

**NOTE** For readers interested in going further, appendix F covers other evaluation methods such as multiple-choice benchmarks, preference-based leaderboards, and LLM-as-a-judge approaches.



**Figure 3.2 Evaluating an LLM with a verification-based method in free-form question answering. The model generates a free-form answer (which may include multiple steps) and a final boxed answer, which is extracted and compared against the correct answer from the dataset.**

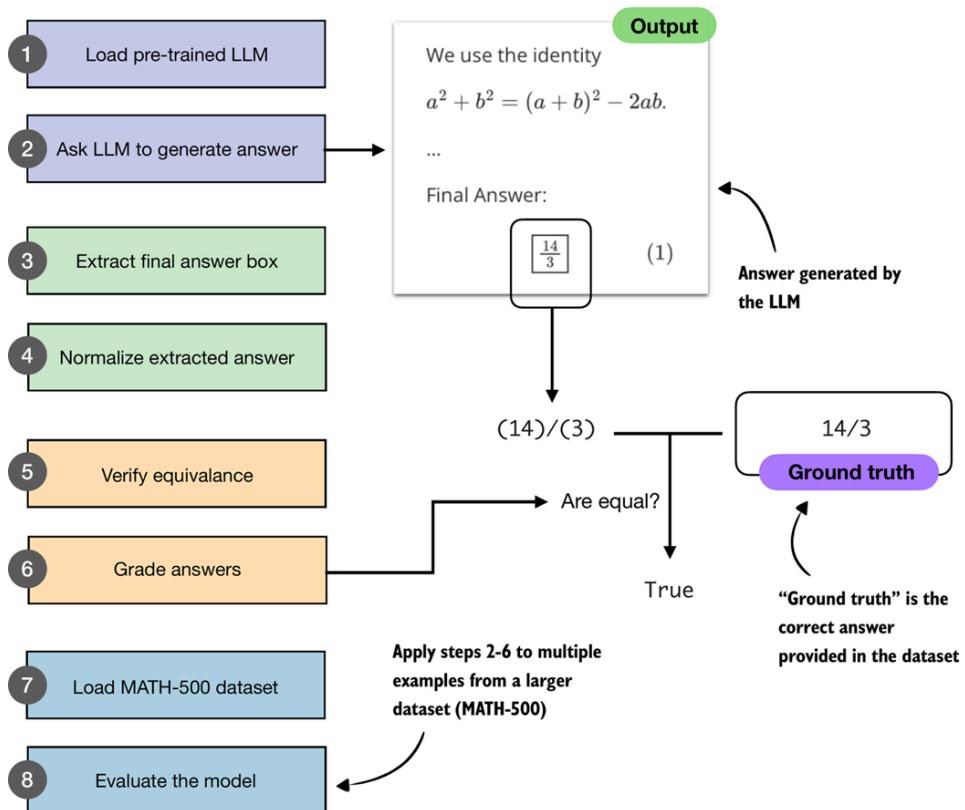
Verifiers compare the extracted answer with the reference solution, as shown in figure 3.2, often by relying on external tools such as code interpreters or calculator programs.

While our immediate focus is evaluation, verifiers will reappear later in this book. They not only serve as a way to measure performance but also provide the feedback signal used in reinforcement learning methods for training reasoning models, which we will explore in chapter 6.

The downside is that verifier methods can only be applied to domains that can be easily (and ideally deterministically) verified, such as math and code. Also, this approach can introduce additional complexity and dependencies, and it may shift part of the evaluation burden from the model itself to the external tool.

However, because math problem solving can be generated in unlimited variations programmatically and benefits from step-by-step reasoning, this task has become a cornerstone of reasoning model evaluation and development.

In the remainder of this chapter, we will build a math verifier step by step, following the 8 steps shown in figure 3.3.



**Figure 3.3 A step-by-step workflow for building and applying a math verifier. Starting with a pre-trained LLM, we generate answers, extract and normalize them, and then compare them against the ground-truth solutions. Verified answers are then graded, and the process is repeated across a dataset (MATH-500) to evaluate overall model performance.**

The next section will start with steps 1 and 2 shown in figure 3.3, namely, loading the pre-trained LLM introduced in the previous chapter and setting it up to generate answers.

## 3.2 Loading a pre-trained model to generate text

In this section, we begin implementing the verifier by following steps 1 and 2 of the workflow in figure 3.3. Specifically, we will load the pre-trained LLM introduced in the previous chapter and configure it to generate answers. This provides the foundation for the later steps, where we will extract, normalize, and verify these answers.

Specifically, we use the same pre-trained base model that we used in chapter 2. However, for our convenience, and for reuse in future chapters, we wrap the model loading logic in a `load_model_and_tokenizer` function call as shown in listing 3.1:

**Listing 3.1 Loading a pre-trained model**

```

from pathlib import Path
import torch

from reasoning_from_scratch.ch02 import (
    get_device
)
from reasoning_from_scratch.qwen3 import (
    download_qwen3_small,
    Qwen3Tokenizer,
    Qwen3Model,
    QWEN_CONFIG_06_B
)

def load_model_and_tokenizer(
    which_model, device, use_compile, local_dir="qwen3"
):
    if which_model == "base":

        download_qwen3_small(
            kind="base", tokenizer_only=False, out_dir=local_dir
        )

        tokenizer_path = Path(local_dir) / "tokenizer-base.json"
        model_path = Path(local_dir) / "qwen3-0.6B-base.pth"
        tokenizer = Qwen3Tokenizer(tokenizer_file_path=tokenizer_path)

    elif which_model == "reasoning":

        download_qwen3_small(
            kind="reasoning", tokenizer_only=False, out_dir=local_dir
        )

        tokenizer_path = Path(local_dir) / "tokenizer-reasoning.json"
        model_path = Path(local_dir) / "qwen3-0.6B-reasoning.pth"
        tokenizer = Qwen3Tokenizer(
            tokenizer_file_path=tokenizer_path,
            apply_chat_template=True,
            add_generation_prompt=True,
            add_thinking=True,
        )

```

```

else:
    raise ValueError(f"Invalid choice: which_model={which_model}")

model = Qwen3Model(QWEN_CONFIG_06_B)
model.load_state_dict(torch.load(model_path))

model.to(device)

if use_compile: #A
    torch._dynamo.config.allow_unspec_int_on_nn_module = True
    model = torch.compile(model)

return model, tokenizer

WHICH_MODEL = "base" #B
device = get_device()
# device = torch.device("cpu") #C

model, tokenizer = load_model_and_tokenizer(
    which_model=WHICH_MODEL,
    device=device,
    use_compile=False
)

#A Optionally set to true to enable model compilation
#B Uses the base model, similar to chapter 2, by default
#C Uncomment this line if you have compatibility issues with your device

```

By default, listing 3.1 loads the base model, just as in chapter 2. An optional variant is the reasoning model, which the Qwen3 team trained on top of the base model using reasoning-specific methods. We will cover these training methods in chapter 6. Here, the reasoning model, which can be loaded by setting `WHICH_MODEL = "reasoning"` in listing 3.1, is included as an option so that we can later compare its evaluation results with those of the base model.

Now that we have loaded the model, we can use the text generation function from chapter 2 to generate text. However, instead of the `generate_text_basic_stream` function introduced in the main chapter, we use the slightly modified `generate_text_basic_stream_cache` version from exercise 2.2 (see the solution in appendix B for the source code) as it prints the tokens as soon as they are generated, which can be useful for debugging purposes (so it doesn't appear the LLM is stuck when generating a longer response). The usage of this function is shown in listing 3.2.

**Listing 3.2 Generating model outputs**

```

from reasoning_from_scratch.ch02_ex import (
    generate_text_basic_stream_cache
)

prompt = ( #A
    r"If $a+b=3$ and $ab=\tfrac{13}{6}$, "
    r"what is the value of $a^2+b^2$?" #B
)

input_token_ids_tensor = torch.tensor( #B
    tokenizer.encode(prompt),
    device=device
).unsqueeze(0) #C

all_token_ids = []

for token in generate_text_basic_stream_cache( #D
    model=model,
    token_ids=input_token_ids_tensor,
    max_new_tokens=2048,
    eos_token_id=tokenizer.eos_token_id
):
    token_id = token.squeeze(0) #E
    decoded_id = tokenizer.decode(token_id.tolist())
    print( #F
        decoded_id,
        end="",
        flush=True
    )
    all_token_ids.append(token_id)

all_tokens = tokenizer.decode(all_token_ids) #G

```

#A Define the math problem as a string prompt  
#B Convert the prompt into token IDs that the model can process  
#C Add batch dimension  
#D Generate output tokens from the model, one at a time  
#E Remove batch dimension  
#F Print token as it is generated  
#G Decode the full generated sequence into text

In listing 3.2, we start by encoding a simple math problem into token IDs that the model can process. The model then generates tokens one by one in a streaming fashion, which we print immediately as they appear so we can read the output while it's being generated. At the same time, we collect the generated tokens into a list so that we can later decode them into the complete final answer string. This pattern of both streaming and collecting tokens is handy because it lets us monitor the generation live while still storing the full answer text (`all_tokens`) that we can process later.

The response, generated by the code in listing 3.2, is as follows:

```
To find the value of \(\( a^2 + b^2 \) given that \( a + b = 3 \)
and \( ab = \frac{13}{6} \), we can use the following algebraic identity:

\[ 
a^2 + b^2 = (a + b)^2 - 2ab
\]

**Step 1:** Substitute the given values into the equation.

\[ 
a^2 + b^2 = (3)^2 - 2 \left( \frac{13}{6} \right)
\]

[...] #A

**Final Answer:**

\[ 
\boxed{\frac{14}{3}}
\]

#A Shortened for brevity
```

As we can see, based on this answer, even though it is a base model, it provides a reasoning model-like explanation. This is likely because the Qwen3 team included chain-of-thought data during the pre-training stages, as stated in their technical report. However, even though the model has some reasoning-model-like behavior, adding additional reasoning methods can further improve these capabilities. (Note that the response may differ depending on whether you executed the code on a CPU, CUDA, or MPS device.)

Furthermore, if you are unfamiliar with the LaTeX syntax that is commonly used for mathematics, the response above can be very hard to decipher. If this is the case, you can use IPython's `Latex` class to render it, as shown below:

```
from IPython.display import Latex, display
display(Latex(all_tokens))
```

Executing the code above in a code notebook will render the response as shown in figure 3.4.

**“\*\*” symbols, used for bold font in Markdown syntax, are not part of the Latex syntax and are thus not rendered**

\*\*Step 1:\*\* Substitute the given values into the equation.

$$a^2 + b^2 = (3)^2 - 2 \left( \frac{13}{6} \right)$$

\*\*Step 2:\*\* Calculate  $(3)^2$ .

$$(3)^2 = 9$$

\*\*Step 3:\*\* Calculate  $2 \times \frac{13}{6}$ .

$$2 \times \frac{13}{6} = \frac{26}{6} = \frac{13}{3}$$

Latex code is properly rendered

\*\*Step 4:\*\* Subtract the second result from the first.

$$a^2 + b^2 = 9 - \frac{13}{3}$$

\*\*Step 5:\*\* Convert 9 to a fraction with a denominator of 3 to perform the subtraction.

$$9 = \frac{27}{3}$$

$$a^2 + b^2 = \frac{27}{3} - \frac{13}{3} = \frac{14}{3}$$

\*\*Final Answer:\*\*

The final answer is given in an answer box

$\frac{14}{3}$

**Figure 3.4 Rendered response with step-by-step calculations and the final boxed answer.**

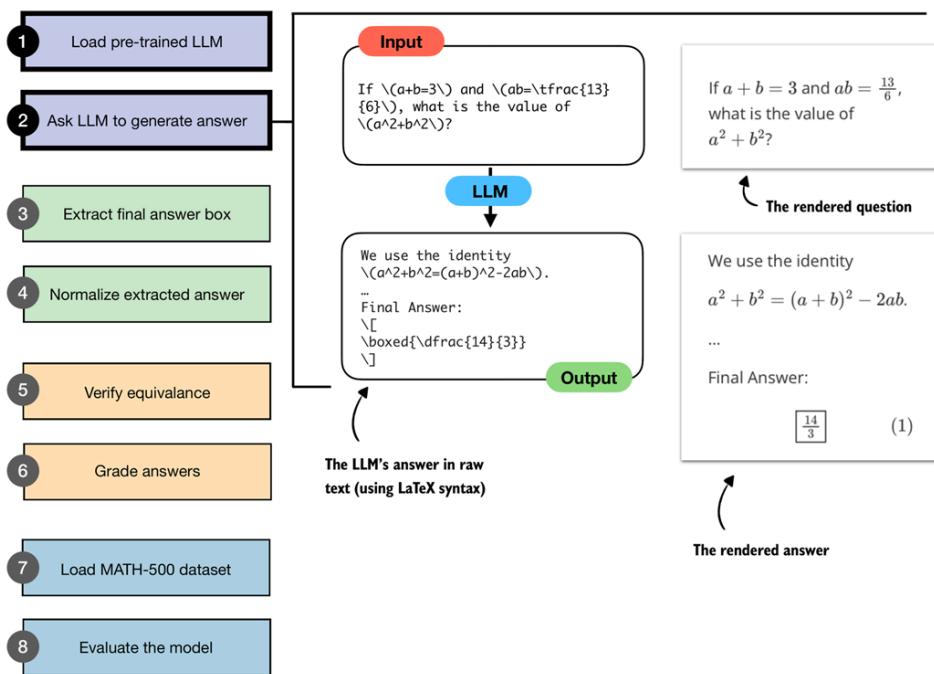
Note that the final answer given in figure 3.4,

$$\frac{14}{3},$$

is indeed the correct answer to this problem.

### 3.3 Implementing a wrapper for easier text generation

In the previous section, we loaded the pre-trained LLM and set up the text generation functionality (as illustrated in figure 3.5), which are the first two steps of the evaluation process covered in the remainder of this chapter.



**Figure 3.5 Illustration of steps 1 and 2 from the verifier workflow. A pre-trained LLM is loaded and prompted with a math problem, producing an output in raw LaTeX syntax. The answer is also shown in the rendered and more readable form.**

For additional convenience in later sections, we create a wrapper (listing 3.3) for the text generation function so that we only have to pass in the model, tokenizer, and prompt, along with some additional settings instead of repeating the tokenization and input preparation steps each time:

**Listing 3.3 A wrapper for streamed text generation**

```

def generate_text_stream_concat(
    model, tokenizer, prompt, device, max_new_tokens,
    verbose=False,
):
    input_ids = torch.tensor(                                     #A
        tokenizer.encode(prompt), device=device
    ).unsqueeze(0)

    generated_ids = []
    for token in generate_text_basic_stream_cache( #B
        model=model,
        token_ids=input_ids,
        max_new_tokens=max_new_tokens,
        eos_token_id=tokenizer.eos_token_id,
    ):
        next_token_id = token.squeeze(0)
        generated_ids.append(next_token_id.item())

        if verbose: #C
            print(
                tokenizer.decode(next_token_id.tolist()),
                end="",
                flush=True
            )

    return tokenizer.decode(generated_ids) #D

```

#A Encode prompt text into token IDs and place on device  
#B Stream tokens one by one using cached generation  
#C Optionally print tokens as they are generated  
#D Decode all generated IDs into final text string

This wrapper function in listing 3.3 handles the full cycle of text generation: it tokenizes the input prompt, streams new tokens from the model, and then decodes the results into a final string. And, as mentioned before, the optional verbose flag allows us to see tokens as they are generated in real time. The function can be used as follows:

```
generated_text = generate_text_stream_concat(
    model, tokenizer, prompt, device,
    max_new_tokens=2048,
    verbose=True #A
)
#A Using False will suppress the live token-by-token printing
```

This prints the exact same response as in section 3.2:

```
[...] #A

**Final Answer:**

\[[
\boxed{\frac{14}{3}}
\]
```

#A Shortened for brevity

### 3.4 Extracting the final answer box

Now that we have the model loaded and ready, we can get to the chapter-specific and interesting parts: evaluating the model.

Before we get started, it is worth recalling that this book takes a from-scratch approach, which naturally includes some detailed, sometimes tedious steps. This is intentional, since we want to build the evaluation pipeline ourselves to better understand how it works, rather than just calling predefined wrapper functions and building blocks.

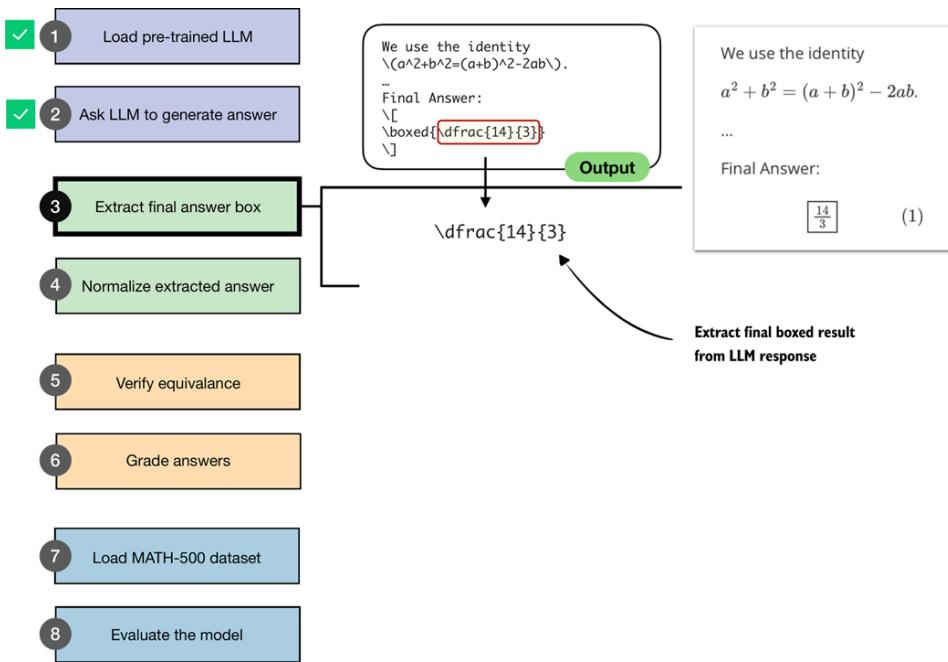
In the previous section, we saw that the model returned the final answer in an answer box (written as `r"\boxed{\frac{14}{3}}"` in raw text), even though we hadn't specifically asked for this format.

The reason the model answered in this specific format is likely because the model has seen examples from benchmark datasets (including MATH-500) that were similarly formatted during pretraining. This behavior is not necessarily proof of overfitting to the evaluation tasks we will run later. But it reflects that pre-trained models often encounter many problem formats online and learn to reproduce those stylistic conventions.

However, it's also true that as a general rule, it is fair to assume that any information available on the internet when a model was trained has been part of the training data.

Although it was not necessary here, when we evaluate the model in the MATH-500 dataset later on, we will add a specific prompt that instructs the model to return answers in this boxed form, as it is a common convention that makes the evaluation more consistent across different models and makes data extraction easier.

We will now write code that performs this extraction of the boxed answer content as illustrated in figure 3.6.



**Figure 3.6 An illustration of how the boxed result from the LLM output is extracted.**

Specifically, this section implements step 3 shown in figure 3.6. The next section will implement the normalization method for step 4.

Since your model may produce slightly different responses (depending on your hardware) than those shown above, we will work with a hard-coded answer for the time being (pretending that this answer was generated by the model). In later sections, we will revisit the model and have it generate answers for the tasks in the MATH-500 dataset.

```

model_answer = (
    """... some explanation...
**Final Answer:**

\[
\boxed{\frac{14}{3}}
\]
"""

)

```

#A The answer box we want to extract

**NOTE** We are using a raw string (`="""\dots"""` instead of a regular string `"""...\"""`). Raw strings make it easier to handle the `\` characters, which would otherwise be treated as escape sequences and require doubling each backslash.

Next, let's define a function in listing 3.4 to extract the answer box from the model answer.

**Listing 3.4 Extracting answer boxes**

```

def get_last_boxed(text):
    boxed_start_idx = text.rfind(r"\boxed") #A
    if boxed_start_idx == -1:
        return None

    current_idx = boxed_start_idx + len(r"\boxed") #B

    #C
    while current_idx < len(text) and text[current_idx].isspace():
        current_idx += 1

    #D
    if current_idx >= len(text) or text[current_idx] != "{":
        return None

    current_idx += 1
    brace_depth = 1
    content_start_idx = current_idx

    #E
    while current_idx < len(text) and brace_depth > 0:
        char = text[current_idx]
        if char == "{":
            brace_depth += 1
        elif char == "}":
            brace_depth -= 1
        current_idx += 1

    if brace_depth != 0: #F
        return None

    return text[content_start_idx:current_idx-1] #G

#A Find the last occurrence of "\boxed"
#B Get position after "\boxed"
#C Skip any whitespace after "\boxed"
#D Expect an opening brace "{"
#E Parse the braces with nesting
#F Account for unbalanced braces
#G Extract content inside the outermost braces

```

The `get_last_boxed` helper utility function in listing 3.4 extracts out the content of the last `\boxed{...}` expression from a model's output. More specifically, it scans for the final `\boxed`, skips over whitespace, checks for braces, and handles any nesting so that we capture the intended answer string.

While it may look a bit tedious, having this parser in place will pay off when we run evaluations on datasets like MATH-500, where extracting the correct final answer is the first step toward measuring a model's reasoning ability. (MATH-500 is a curated collection of 500 problems that is widely used as a reasoning model benchmark dataset, which we will use later in this chapter.)

Now, let's test it on the model answer:

```
extracted_answer = get_last_boxed(model_answer)
print(extracted_answer)
```

The output of this function call is "`\frac{14}{3}`", which is the boxed answer we wanted to extract.

## RENDERING MATH FORMULAS

We can render math formulas via the `Latex` class we introduced earlier. Alternatively, for single math formulas that are not accompanied by answer text, we can also use the simpler `Math` class:

```
from IPython.display import Math
display(Math(r"\frac{14}{3}))
```

This renders the fraction as

$$\frac{14}{3}$$

While the previous `get_last_boxed` function correctly extracted the text, we will make the answer extraction a bit more robust to account for cases where a final answer box is either missing or incomplete via the `extract_final_candidate` function in listing 3.5:

**Listing 3.5 Extracting the final answer candidate**

```

import re

RE_NUMBER = re.compile( #A
    r"-?(?:\d+/\d+|\d+(?:\.\d+)?(?:[eE][+-]?\d+)?)"
)

def extract_final_candidate(text, fallback="number_then_full"):

    result = "" #B

    if text: #C
        boxed = get_last_boxed(text.strip())
        if boxed:
            result = boxed.strip().strip("$ ")

        #D
        elif fallback in ("number_then_full", "number_only"):
            m = RE_NUMBER.findall(text)
            if m:
                result = m[-1] #E
            elif fallback == "number_then_full":

                result = text #F

    return result

```

#A Regular expression for extracting numeric values from the text

#B Default return value if nothing matches

#C Prefer the last boxed expression if present

#D If no boxed expression, try fallback

#E Use last number

#F Else return full text if no number found

The `extract_final_candidate` function in listing 3.5 provides `fallback` settings in case no boxed answer can be found, which are as follows:

- `"number_then_full"` (default): pick the last simple number, else the whole text;
- `"number_only"`: pick the last simple number, else return an empty string `" "`;
- `"none"`: extract only boxed content, else return empty string `" "`.

For the fallback setting, the code in listing 3.5 uses *regular expressions* (regex for short) via Python's `re` library. Regexes are a way to search for patterns in text. In our case, the regex pattern is designed to recognize numbers, including fractions, decimals, and scientific notation. While the regex syntax looks intimidating, you don't need to worry about the exact syntax here. What matters is that this gives us a reliable tool to extract the last numeric candidate from the model's output when no boxed answer is available.

Let's try it on our model answer:

```
print(extract_final_candidate(model_answer))
```

This correctly returns "`\dfrac{14}{3}`". Next, let's try some additional examples. First, another boxed candidate:

```
print(extract_final_candidate(r"\boxed{ 14/3. }"))
```

This correctly returns "14/3.", stripping the extra whitespace but not the punctuation. However, the punctuation character will be handled correctly by the equality check we implement later.

Next, let's try a candidate without a box, which should trigger the fallback setting, and see what happens:

```
print(extract_final_candidate("abc < > 14/3 abc"))
```

Thanks to the default fallback setting, it will find the last number in the answer and also correctly return "14/3".

In this section, we defined utility functions to extract the LLM's answer from within its answer text context. This brings us one step closer to achieving the overall goal of verifying whether this answer is indeed correct. In the next section, we will normalize the response into a more general, canonical form before we implement the checking functionality.

### WHY NOT USE AN LLM FOR THE ANSWER EXTRACTION?

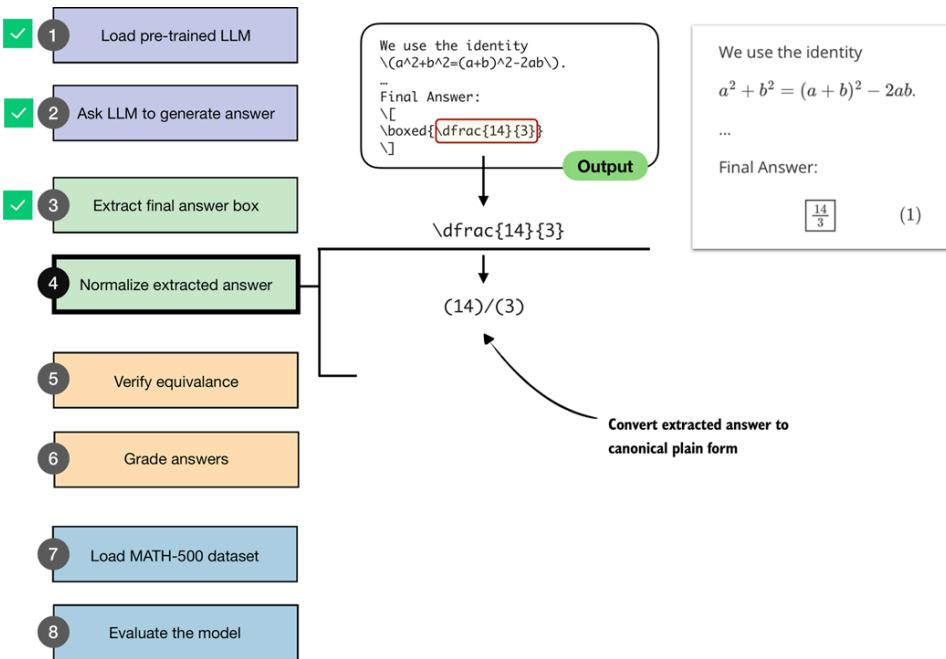
We could use an LLM itself to extract the boxed answer. However, this would introduce unnecessary complexity and potential errors. Extraction is a simple, mechanical task, assuming that the LLM outputs the answer in a specific format: we just need to locate the last boxed expression or, if that is missing, fall back to a number or the raw text.

Regular expressions may look complicated at first, but in the end, we have a small, reusable utility function that is cheap to execute and handles the extraction deterministically and reproducibly, without depending on the variability of another model's output.

## 3.5 Normalizing the extracted answer

Previously, we extracted the boxed answer " $\frac{14}{3}$ " from the model's response. However, models may print the same value in many ways, such as " $\frac{14}{3}$ ", " $14/3$ ", " $\$14/3\$$ ", or " $(14) / (3)$ ". To implement and use a robust checking system that can check whether the answer is correct, we first need a consistent method of comparing results.

In this section, we implement a normalization (or canonicalization) pass (step 4 in figure 3.7) that strips formatting and standardizes the answer.



**Figure 3.7 An illustration of how the boxed result from the LLM output is extracted and converted into a canonical plain form. This normalized answer is then later used for verification against the correct answer.**

The normalization step shown in figure 3.7 is implemented via the `normalize_text` function in listing 3.6.

**Listing 3.6 Normalizing extracted answers**

```
LATEX_FIXES = [ #A
    (r"\\" Left\s*", ""),
    (r"\\" right\s*", ""),
    (r"\\", |\\"!|\\";|\\":, ""),
    (r"\\" cdot", "*"),
    (r"\u00B7|\u00D7", "*"),
    (r"\\"^"\\" circ", ""),
    (r"\\" dfra", r"\\" frac"),
    (r"\\" tfrac", r"\\" frac"),
    (r"\\" o", ""),
]
RE_SPECIAL = re.compile(r"<\|[^>]+?\|>") #B
SUPERSCRIPT_MAP = {
    "0": "0", "1": "1", "2": "2", "3": "3", "4": "4", #C
    "5": "5", "6": "6", "7": "7", "8": "8", "9": "9", #C
    "+": "+", "-": "-", "(": "(", ")": ")"
}
def normalize_text(text):
    if not text:
        return ""
    text = RE_SPECIAL.sub("", text).strip()

    #D
    match = re.match(r"^[A-Za-z]\s*[.:\"]\s*(.+)$", text)
    if match:
        text = match.group(1)

    text = re.sub(r"\\" \s*\{\\" \circ", "", text) #D
    text = re.sub(r"\\" \circ", "", text) #E
    text = text.replace("\", "") #E

    match = re.match(r"^\\" text\{(?P<x>.+)\}\$", text) #F
    if match:
        text = match.group("x")

    text = re.sub(r"\\"(|\\)|\\[|\\]", "", text) #G

for pat, rep in LATEX_FIXES: #H
```

```

text = re.sub(pat, rep, text)

def convert_superscripts(s, base=None):
    converted = "".join(
        SUPERSCRIPT_MAP[ch] if ch in SUPERSCRIPT_MAP else ch
        for ch in s
    )
    if base is None:
        return converted
    return f"{base}**{converted}"

text = re.sub(
    r"([0-9A-Za-z\])\]\})\([0123456789+-]+\)",
    lambda m: convert_superscripts(m.group(2), base=m.group(1)),
    text,
)
text = convert_superscripts(text)

#I
text = text.replace("\%\%", "%").replace("$", "").replace("%", "")
text = re.sub(
    r"\sqrt\s*\{([^\}]*\}",
    lambda match: f"sqrt({match.group(1)})",
    text,
)
text = re.sub(
    r"\sqrt\s+([^\sqrt\s{}]+)",
    lambda match: f"sqrt({match.group(1)})",
    text,
)

#J
text = re.sub(
    r"\frac\s*\{([^\}]+)\}\s*\{([^\}]+)\}",
    lambda match: f"({match.group(1)})/({match.group(2)})",
    text,
)
text = re.sub(
    r"\frac\s+([^\sqrt\s{}]+)\s+([^\sqrt\s{}]+)",
    lambda match: f"({match.group(1)})/({match.group(2)})",
    text,
)

```

```

#K
text = text.replace("^", "**")
text = re.sub(
    r"(?<=\d)\s+(\d+/\d+)",
    lambda match: "+" + match.group(1),
    text,
)

#L
text = re.sub(
    r"(?<=\d),(?= \d\d\d(\D|\$))",
    "",
    text,
)

return text.replace("{", "").replace("}", "").strip().lower()

#A LaTeX formatting to be replaced (left: original value, right: new value)
#B Strip chat special tokens like "<|assistant|>"
#C Dictionary mapping to convert unicode superscripts to plaintext superscripts
#D Strip leading multiple-choice labels (e.g., like "c. 3" -> 3)
#E Remove angle-degree markers
#F Unwrap "\text{...}" if the whole string is wrapped
#G Strip inline/display math wrappers: \(\) \(\)
#H LaTeX canonicalization
#I Normalize number and root expressions
#J Convert LaTeX fractions into division form
#K Handle exponents and mixed numbers
#L Remove thousands separators in numbers

```

The `normalize_text` function in listing 3.6 takes an extracted answer string and rewrites it into a standardized format that we can reliably compare against reference solutions. It first strips away special tokens and unnecessary LaTeX clutter, such as `\left`, `\right`, or degree symbols. It then unwraps cases like `\text{...}`, removes inline math markers, and rewrites common structures into a calculator-style form. For example, it turns `\sqrt{a}` into `sqrt(a)` and `\frac{a}{b}` into `(a)/(b)`. Finally, it normalizes exponents, mixed numbers, and thousands separators and cleans up braces and casing. In short, the function transforms differently formatted LaTeX outputs into a clean, standardized string representation.

Let's now try the `normalize_text` function on our model answer:

```
print(normalize_text(extract_final_candidate(model_answer)))
```

As a result, instead of printing the answer with LaTeX formatting (`r"\frac{14}{3}"`), it returns the answer in a standardized, LaTeX-free form:

```
"(14)/(3)"
```

Next, let's try a differently formatted answer:

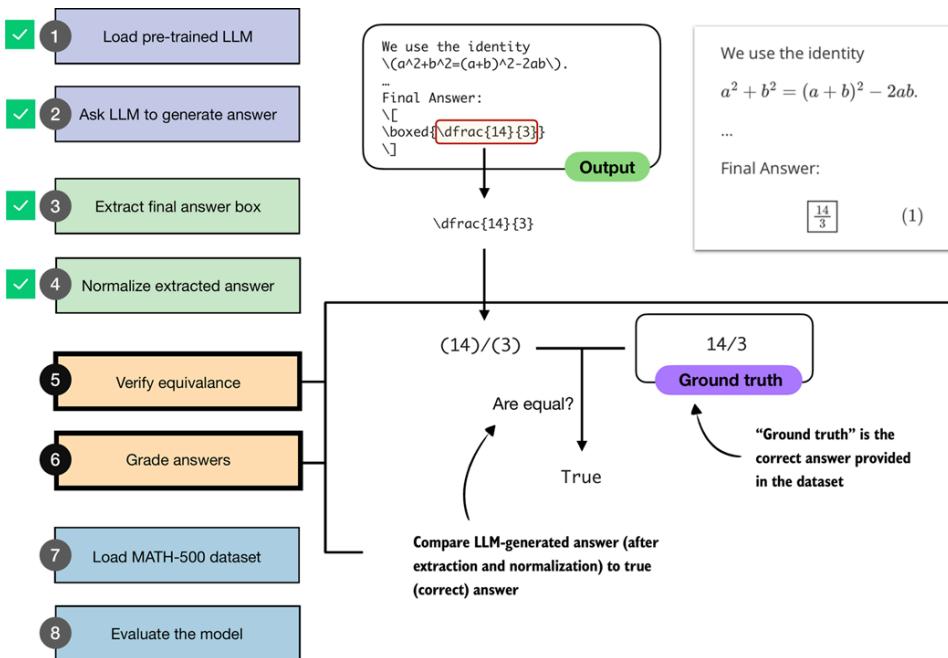
```
print(normalize_text(r"\text{[\frac{14}{3}]})")
```

This also returns, "`(14) / (3)`", as intended.

We now have a robust method to extract answer texts from an LLM's response. The next task, covered in the next section, is to implement a function to compare the LLM answer to a correct reference answer.

### 3.6 Verifying mathematical equivalence

So far, in this chapter, we implemented steps to ask an LLM to generate an answer, extract the relevant portion, and normalize it. The next step, as illustrated in figure 3.8, is to compare the extracted answer to a correct reference answer, which is, in technical contexts, referred to as *ground truth*.



**Figure 3.8 An illustration of how an LLM-generated answer is checked against the correct reference answer (ground truth). The final boxed answer is extracted and normalized, then compared to the correct answer provided in the dataset. If both match, the response is graded as correct.**

Note that if we want to implement the equality check shown in figure 3.8, a direct comparison using Python's `==` operator is not sufficient, since expressions like "`14/3`" and "`(14) / (3)`" would not match, and equivalent but unnormalized fractions such as "`(28) / (6)`" and "`(14) / (3)`" would also be treated as unequal.

As part of our equality check, we implement an additional intermediate step: parsing the extracted and normalized answer using a symbolic math engine.

For this, we use the SymPy open-source math library (<https://sympy.org>), which has been developed and tested for two decades and has become a staple of scientific computing in Python. The parsing function is implemented in listing 3.7.

**NOTE** If you haven't installed the dependencies in chapter 2, you can manually install SymPy via `uv pip install sympy` (or `uv add sympy`).

**Listing 3.7 SymPy parser for mathematical equality check**

```

from sympy.parsing import sympy_parser as spp
from sympy.core.sympify import SympifyError
from sympy.polys.polyerrors import PolynomialError
from tokenize import TokenError

def sympy_parser(expr):
    try:
        return spp.parse_expr(
            expr,
            transformations=(
                *spp.standard_transformations, #A
                #B
                spp.implicit_multiplication_application,
            ),
            evaluate=True, #C
        )
    except (SympifyError, SyntaxError, TypeError, AttributeError,
            IndexError, TokenError, ValueError, PolynomialError):
        return None

```

#A Standard transformations like handling parentheses  
#B Allow omitted multiplication symbols (e.g.,  $2y \rightarrow 2*y$ )  
#C Evaluate during parsing so simple constants simplify (e.g.,  $2+3 \rightarrow 5$ )

The `sympy_parser` function in listing 3.7 takes an input expression, such as the normalized answers we extract from the LLM response, and converts it into a SymPy object that can be reliably compared for mathematical equivalence. To do so, it applies SymPy's standard parsing rules, supports implicit multiplication like ( $2y$  instead of  $2*y$ ), and also simplifies basic arithmetic (so  $2+3$  becomes 5).

---

**NOTE** The `sympy_parser` takes into account what looks like an excessive amount of error cases, but these are all errors that I encountered when evaluating the model on all 500 MATH-500 problems, as the model does not always generate perfectly formatted outputs.

---

Let's see it in action and apply it to the normalized answer candidate:

```

print(sympy_parser(normalize_text(
    extract_final_candidate(model_answer)
)))

```

This returns the fraction  $14/3$ . Next, let's try an unnormalized fraction:

```
print(sympy_parser("28/6"))
```

Similarly, this returns  $14/3$ .

Using the `sympy_parser`, we can now implement the equality check function in listing 3.8:

#### Listing 3.8 Equality check function using SymPy

```
from sympy import simplify

def equality_check(expr_gtruth, expr_pred):
    if expr_gtruth == expr_pred: #A
        return True

    #B
    gtruth, pred = sympy_parser(expr_gtruth), sympy_parser(expr_pred)

    if gtruth is not None and pred is not None: #C
        try:
            return simplify(gtruth - pred) == 0 #D
        except (SympifyError, TypeError):
            pass

    return False

#A First, check if the two expressions are exactly the same string
#B Parse both expressions into SymPy objects (returns None if parsing fails)
#C If both expressions were parsed successfully, try symbolic comparison
#D If the difference is 0, they are equivalent
```

The `equality_check` function in listing 3.8 determines whether a model's answer matches the ground-truth solution. It first looks for an exact string match, which is the simplest case. If the strings differ, it parses both expressions into SymPy objects (via the `sympy_parser` function we implemented in listing 3.7) and checks whether their difference simplifies to zero. This allows us to recognize answers that may look different on the surface (for example,  $14/3$  and  $28/6$ ) but are mathematically the same.

Let's try the equality checker from listing 3.8 on an example:

```
print(equality_check(
    normalize_text("13/4."),
    normalize_text(r"(13)/(4)"))
))
```

As intended, this ignores the formatting and returns `True`. Next, let's try a more challenging example and see whether the symbolic math parser recognizes that 0.5 is the same as  $1/2$ :

```
print(equality_check(
    normalize_text("0.5"),
    normalize_text(r"(1)/(2)"))
))
```

This also returns `True`. Now, let's try a negative example:

```
print(equality_check(
    normalize_text("14/3"),
    normalize_text("15/3"))
))
```

This returns `False` since the expressions are different.

So far, so good. Based on the encouraging results above, we may conclude that we now have a robust equality checker that we can use to evaluate the LLM on a math benchmark dataset. However, to make sure that it's ready for prime time, let's try one more example:

```
print(equality_check(
    normalize_text("(14/3, 2/3)"),
    normalize_text("(14/3, 4/6)"))
))
```

In this case, we are comparing two tuples. Since  $2/3$  and  $4/6$  are mathematically equivalent, we would expect the result to be `True`. Instead, the function returns `False`, because it currently only handles simple expressions, not tuples. We will address this limitation in the next section.

### 3.7 Grading answers

Now, we will build upon the mathematical equality checking function from the previous section to implement a robust grading function that can also handle tuple-like expressions, such as correctly comparing expressions like " $(14/3, 2/3)$ " and " $(14/3, 4/6)$ ".

First, we implement a Python helper function that splits such tuple-like expressions into individual subparts via listing 3.9.

#### **Listing 3.9 Helper function to split tuple-like expressions**

```
def split_into_parts(text):
    result = [text]

    if text: #A
        if (
            len(text) >= 2
            and text[0] in "(["
            and text[-1] in "])"
            and "," in text[1:-1]
        ):
            items = [p.strip() for p in text[1:-1].split(",")]
            #B
            if all(items):
                result = items
        else: #C
            result = []

    return result

#A Check if text looks like a tuple or list, e.g. "(a, b)" or "[a, b]"
#B Split on commas inside brackets and strip whitespace
#C If text is empty, return an empty list
```

The `split_into_parts` function in listing 3.9 helps us handle answers with multiple components. If the input looks like a tuple or list, such as `(a, b)` or `[a, b]`, it splits the content on commas and returns the individual pieces. (If the string is empty, it simply returns an empty list.) In essence, this function breaks down multi-part answers into smaller parts that can be checked one by one.

Before we implement the grading function next, let's take the `split_into_parts` for a test drive and try it on the tuple-like expression from earlier:

```
split_into_parts(normalize_text(r"(14/3, 2/3)"))
```

This returns `['14/3', '2/3']`, as desired.

Now, we can implement the `grade_answer` function (listing 3.10), which splits tuple-like expressions (if present) into subparts, and then uses the `equality_check` function from the previous section to compare a generated answer to a reference (ground truth) answer.

**Listing 3.10 Function to grade predicted answers against ground truth**

```
def grade_answer(pred_text, gt_text):
    result = False #A
    if pred_text is not None and gt_text is not None: #B
        gt_parts = split_into_parts(
            normalize_text(gt_text)
        )
        pred_parts = split_into_parts(
            normalize_text(pred_text)
        )

        if (gt_parts and pred_parts) #C
            and len(gt_parts) == len(pred_parts)): #C
            result = all(
                equality_check(gt, pred)
                for gt, pred in zip(gt_parts, pred_parts)
            ) #D

    return result #E
```

#A Default outcome if checks fail

#B Only continue if both inputs are non-empty strings

#C Ensure both sides have same number of valid parts

#D Check each part for mathematical equivalence

#E True only if all checks passed

The implementation of the `grade_answer` function in listing 3.10 first assumes the prediction is incorrect (`False`) and only continues if both prediction and ground truth are non-empty. It then normalizes each side and splits them into subparts (for example, breaking "(14/3, 2/3)" into ["14/3", "2/3"]). If the number of subparts matches, it compares them one by one using `equality_check`. The result is returned as correct (`True`) only if all pairs match mathematically.

We can think of the `grade_answer` function as an advanced version of the `equality_check` function from the previous section. The `grade_answer` function can split tuple-like expressions and normalize the answers before applying the `equality_check` function.

On simple expressions, it works similarly to the `equality_check`, returning `True` if two expressions are mathematically equivalent:

```
grade_answer("14/3", r"\frac{14}{3}")
```

In addition, as described above, it now also returns `True` in case of two mathematically equivalent tuple-like expressions:

```
grade_answer(r"(14/3, 2/3)", "(14/3, 4/6)")
```

To check the `grade_answer` function more comprehensively, the code in listing 3.11 contains more diverse test cases.

**Listing 3.11 Test cases and demo function to test the grader**

```

tests = [ #A
    ("check_1", "3/4", r"\frac{3}{4}", True),
    ("check_2", "(3)/(4)", r"3/4", True),
    ("check_3", r"\frac{\sqrt{8}}{2}", "sqrt(2)", True),
    ("check_4", r"\left(\frac{1}{2} + \frac{1}{6}\right)", "2/3", True),
    ("check_5", "(1, 2)", r"(1,2)", True),
    ("check_6", "(2, 1)", "(1, 2)", False),
    ("check_7", "(1, 2, 3)", "(1, 2)", False),
    ("check_8", "0.5", "1/2", True),
    ("check_9", "0.3333333333", "1/3", False),
    ("check_10", "1,234/2", "617", True),
    ("check_11", r"\text{2/3}", "2/3", True),
    ("check_12", "50%", "1/2", False),
    ("check_13", r"2\cdot 3/4", "3/2", True),
    ("check_14", r"90^\circ", "90", True),
    ("check_15", r"\left(\frac{3}{4}\right)", "3/4", True),
    ("check_16", r"2^2", "2**2", True),
]

def run_demos_table(tests):
    header = ("Test", "Expect", "Got", "Status")
    rows = []
    for name, pred, gtruth, expect in tests:
        got = grade_answer(pred, gtruth) #B
        status = "PASS" if got == expect else "FAIL"
        rows.append((name, str(expect), str(got), status))

    data = [header] + rows

    col_widths = [ #C
        max(len(row[i]) for row in data)
        for i in range(len(header))
    ]

    for row in data: #D
        line = " | ".join(
            row[i].ljust(col_widths[i])
            for i in range(len(header)))
    )
    print(line)

```

```

passed = sum(r[3] == "PASS" for r in rows) #E
print(f"\nPassed {passed}/{len(rows)}") #E

#A Define test cases: (name, prediction, ground truth, expected result)
#B Run equality check
#C Compute max width for each column to align table nicely
#D Print table row by row
#E Print summary of passed tests

```

The code in listing 3.11 is a simple test suite that takes in a selection of `tests` to check whether the `grade_answer` function works as intended. The `tests` list contains tuples that cover a selection of fractions, LaTeX notations, tuple inputs, decimals, percentages, and other tricky formats.

The `run_demos_table` function then runs each test by calling `grade_answer`, collects the outcomes, and organizes the results into a formatted table.

Calling the `run_demos_table(test)` function in listing 3.11 prints the following:

Test	Expect	Got	Status
check_1	True	True	PASS
check_2	True	True	PASS
check_3	True	True	PASS
check_4	True	True	PASS
check_5	True	True	PASS
check_6	False	False	PASS
check_7	False	False	PASS
check_8	True	True	PASS
check_9	False	False	PASS
check_10	True	True	PASS
check_11	True	True	PASS
check_12	False	False	PASS
check_13	True	True	PASS
check_14	True	True	PASS
check_16	True	True	PASS

Passed 16/16

As we can see based on the `PASS` results above, the `grade_answer` function is relatively robust and capable of handling a variety of differently formatted expressions.

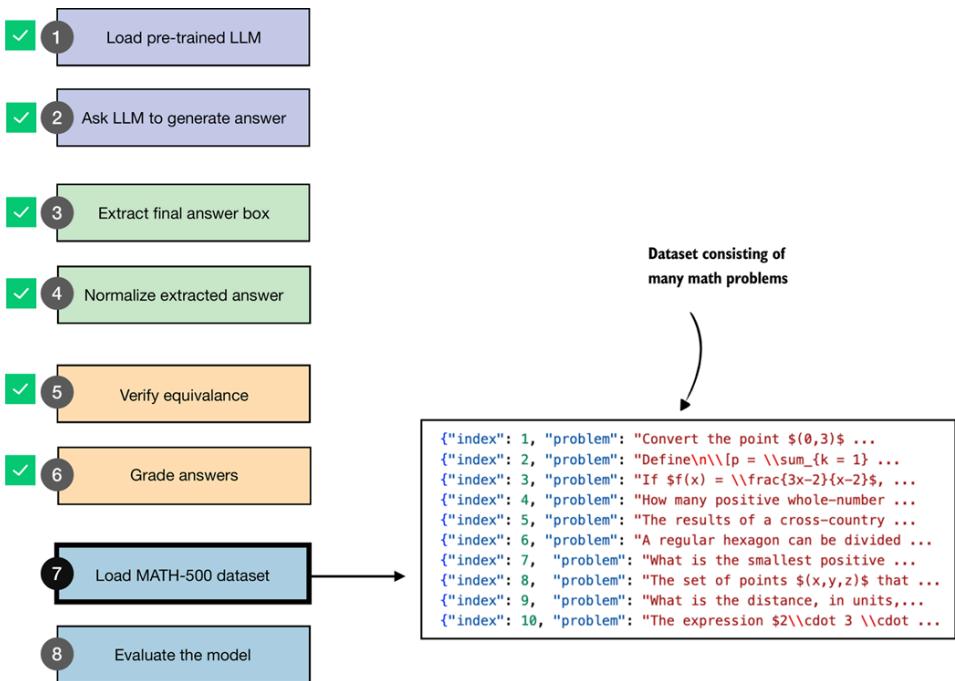
**EXERCISE 3.1: ADDING MORE TEST CASES**

Try to think of additional test cases, ideally challenging ones, and add them to the `run_demos_table()` function. Can you find cases where the check fails incorrectly?

With the `grade_function` implemented, we now have the building blocks in place to evaluate the LLM. In the next section, we will load a math dataset on which we will evaluate the LLM.

### 3.8 Loading the evaluation dataset

As we have seen in the chapter so far, implementing a robust verification pipeline can be a tedious task. Fortunately, we now have all the pieces in place, from answer extraction to grading, and are ready to evaluate the LLM on a benchmark dataset. For this, as illustrated in figure 3.9, we will use the MATH-500 dataset (<https://huggingface.co/datasets/HuggingFaceH4/MATH-500>), a widely used benchmark for reasoning models. It is a curated collection of 500 problems sampled from the original MATH dataset.



**Figure 3.9 Loading the evaluation dataset.** After completing steps 2–6 on individual problems (generate, extract, normalize, verify, and grade answers) in the previous sections, the two remaining steps are to load the full dataset (step 7) and apply the same procedure across all problems to evaluate the model (step 8).

We will load the MATH-500 dataset (step 7 in figure 3.9) using the following code:

**Listing 3.12 Loading the MATH-500 dataset**

```

import json
import requests

def load_math500_test(local_path="math500_test.json", save_copy=True):
    local_path = Path(local_path)
    url = (
        "https://raw.githubusercontent.com/rasbt/reasoning-from-scratch/"
        "main/ch03/01_main-chapter-code/math500_test.json"
    )

    if local_path.exists():
        with local_path.open("r", encoding="utf-8") as f:
            data = json.load(f)
    else:
        r = requests.get(url, timeout=30)
        r.raise_for_status()
        data = r.json()

        if save_copy: # Saves a local copy
            with local_path.open("w", encoding="utf-8") as f:
                json.dump(data, f, indent=2)

    return data

math_data = load_math500_test()
print("Number of entries:", len(math_data))

```

This prints:

Number of entries: 500

**LOADING THE DATASET FROM HUGGING FACE MODEL HUB**

The following information and code example is optional and provided for reference, and you don't need to run the code below.

The MATH-500 dataset split was originally proposed in the PRM800K repository (<https://github.com/openai/prm800k/tree/main?tab=readme-ov-file#math-splits>) and is also available on the Hugging Face Hub (<https://huggingface.co/datasets/HuggingFaceH4/MATH-500>). However, in this book, we load a copy from the code repository to ensure reproducibility in case the external sources change.

If you prefer to download the dataset directly from Hugging Face, you can use the following code. Note that this requires the `datasets` library, which can be installed via `pip install datasets` or `uv add datasets`:

```
from datasets import load_dataset
dset = load_dataset("HuggingFaceH4/MATH-500", split="test")
```

Before we jump to the next section to implement the model evaluation pipeline, let's take a closer look at the structure of the dataset by printing its first entry (we use the built-in `pprint` library for nicer formatting):

```
from pprint import pprint
pprint(math_data[0])
```

This produces the following output:

```
{
  'answer': '\\left( 3, \\frac{\\pi}{2} \\right)',
  'level': 2,
  'problem': 'Convert the point $(0,3)$ in rectangular coordinates to polar '
    'coordinates. Enter your answer in the form $(r,\\theta)$, where '
    '$r > 0$ and $0 \\leq \\theta < 2 \\pi$',
  'solution': 'We have that $r = \\sqrt{0^2 + 3^2} = 3$. Also, if we draw the '
    'line connecting the origin and $(0,3)$, this line makes an angle '
    'of $\\frac{\\pi}{2}$ with the positive $x$-axis.\n'
  '\n'
  '[asy]\n'
  'unitsize(0.8 cm);\n'
  '\n'
  'draw((-0.5,0)--(3.5,0));\n'
  'draw((0,-0.5)--(0,3.5));\n'
  'draw(arc((0,0),3,0,90),red,Arrow(6));\n'
  '\n'
  'dot((0,3), red);\n'
  'label("$\\left( 0,3 \\right)", (0,3), W);\n'
  'dot((3,0), red);\n'
  '[/asy]\n'
  '\n'
  'Therefore, the polar coordinates are $\\boxed{\\left( 3, '
  '\\frac{\\pi}{2} \\right)}$.',
  'subject': 'Precalculus',
  'unique_id': 'test/precalculus/807.json'}
}
```

As we can see, the dataset entry is formatted as a Python dictionary with keys and values. The relevant keys are

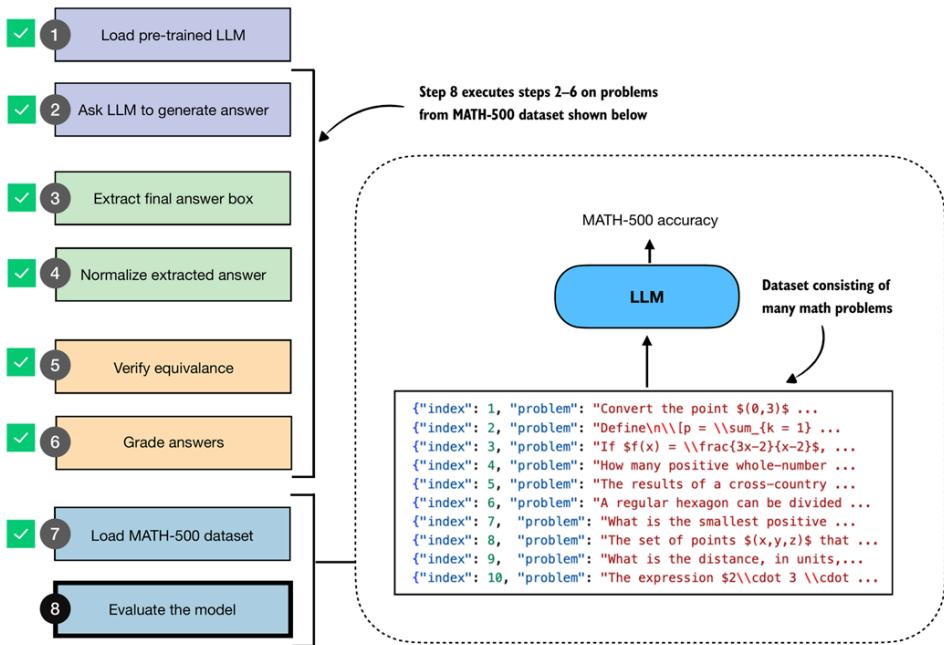
- "problem": the math question or problem for the LLM to solve,
- "answer": the correct (ground truth) answer we want to compare the LLM answer against,
- "solution": a worked-out, step-by-step explanation of the problem (not used in this chapter, but useful for training or analysis).

(Note that the output contains the keys sorted in alphabetical order: "answer", "problem", "solution", but the bulleted uses the more logical ordering for readability.)

Now that we have a pre-trained LLM, evaluation functions, and a benchmark dataset to work with, we can implement the model evaluation.

### 3.9 Evaluating the model

In this section, we put the LLM text generation and evaluation tools from steps 2–6 in figure 3.10 into practice and apply them to the MATH-500 dataset (step 8 in figure 3.10), which we loaded in the previous section.



**Figure 3.10 The complete evaluation pipeline on the MATH-500 dataset. After loading the dataset (step 7), steps 2–6 are applied systematically across all problems to obtain the final model evaluation (step 8).**

As you may recall from section 3.4 (*Extracting the final answer box*), our answer checking pipeline expects that the model returns the answer in boxed form, which is a common convention when evaluating reasoning models on math problems. To increase the likelihood that the model adheres to this format, we can format the prompt as shown in listing 3.13:

**Listing 3.13 Function to render a prompt template for math evaluation**

```
def render_prompt(prompt):
    template = (
        "You are a helpful math assistant.\n"
        "Answer the question and write the final result on a new line as:\n"
        "\boxed{ANSWER}\n\n"
        f"Question:\n{prompt}\n\nAnswer:"
    )
    return template
```

Let's now apply the prompt template from listing 3.13 to the example prompt we introduced earlier in this chapter (section 3.2). For convenience, we redefine the example prompt here:

```
prompt = (
    r"If $a+b=3$ and $ab=\tfrac{13}{6}$, "
    r"What is the value of $a^2+b^2$?"
)
prompt_fmt = render_prompt(prompt)
print(prompt_fmt)
```

The formatted prompt is now as follows:

```
You are a helpful math assistant.
Answer the question and write the final result on a new line as:
\boxed{ANSWER}
```

```
Question:
If $a+b=3$ and $ab=\tfrac{13}{6}$, what is the value of $a^2+b^2$?
```

Answer:

Next, we pass the prompt to the text generation wrapper function we defined in listing 3.3 in section 3.3 to recap the text generation process before we construct the model evaluation function:

```
generated_text = generate_text_stream_concat(
    model, tokenizer, prompt_fmt, device,
    max_new_tokens=2048,
    verbose=True
)
```

Using this prompt example, the model responds with a relatively brief answer: "\boxed{10}". (Note that the generated response may differ depending on whether you executed the code on a CPU, CUDA, or MPS device.)

While brevity can speed up generation by reducing the number of tokens, the response is incorrect. In contrast, in section 3.3, without a prompt template, the model produced a longer response, which led to the correct answer, 14/3.

However, whether a prompt template is well-suited for a given model and task ideally needs to be determined on a larger set of examples before we can draw any conclusions, for instance, the MATH-500 dataset we will evaluate the model on later in this section.

### PROMPT TEMPLATE CHOICES

The prompt template in listing 3.13 is used here to demonstrate how a model evaluation pipeline can be implemented with answers that are automatically checked for correctness. The chosen template encourages short outputs, which lets you work through this chapter efficiently on a first read. Afterward, I recommend revisiting the chapter with alternative settings to optimize accuracy on the reasoning model variant.

As it turns out, using no prompt template boosts the base model performance by 50%, but it reduces the accuracy of the reasoning model by 40%.

Additionally, we may also experiment with alternative prompt templates. For instance, the common standard prompt for the MATH-500 benchmark is the following variant that swaps "Question:" with "Problem:" in listing 3.13. This seemingly minor change improves the base model's accuracy by approximately 20%, likely because it better matches the memorized training data (assuming the MATH-500 test set was included in the training corpus). However, while the base model benefits from this change, the accuracy of the reasoning model variant drops by 30%.

Earlier, we mentioned that answer extraction is a simple, mechanical task that can be solved with deterministic code rather than employing another LLM for the answer extraction. Based on the accuracy change due to different prompting templates, it looks like our extraction method may not be reliable. However, this is not necessarily the case.

Also, it's not necessarily the case that the LLM generates misformatted answers. It can just be incorrect, and switching to another LLM for extraction would not solve that. Smaller base models are often quite sensitive to prompt phrasing. In the next chapter, we will see how this becomes even more apparent once we introduce additional prompt variations.

Next, before we implement the final model evaluation function, let us test our model evaluation pipeline end to end on a smaller example via the demo function in listing 3.14:

#### **Listing 3.14 Demo function to run the evaluation pipeline**

```
def mini_eval_demo(model, tokenizer, device):
    ex = { #A
        "problem": "Compute 1/2 + 1/6.",
        "answer": "2/3"
    }
    prompt = render_prompt(ex["problem"])      #B
    gen_text = generate_text_stream_concat( #C
        model, tokenizer, prompt, device,
        max_new_tokens=64,
    )                                         #C
    pred_answer = extract_final_candidate(gen_text) #D
    is_correct = grade_answer(                 #E
        pred_answer, ex["answer"]             #E
    )                                         #E

    print(f"Device: {device}")
    print(f"Prediction: {pred_answer}")
    print(f"Ground truth: {ex['answer']}")
    print(f"Correct: {is_correct}")

#A Test example with "problem" and "answer" fields
#B 1. Apply prompt template
#C 2. Generate response
#D 3. Extract and normalize answer
#E 4. Grade answer
```

The `mini_eval_demo` function in listing 3.14 combines all the aspects we have covered so far in this chapter:

1. Applying a prompt template
2. Feeding the formatted prompt to the LLM to generate an answer
3. Extracting and normalizing the answer
4. Grading the answer

This `mini_eval_demo` function essentially connects the evaluation components together into a small function that we can use to test the code before coding the final evaluation pipeline for the MATH-500 dataset. The code starts from a toy example (`ex`), renders the problem into the prompt template (`prompt`), and streams a response from the model (`generate_text_stream_concat`). It then parses the model output to a final candidate answer (`pred_answer`) and grades it against the ground truth with `grade_answer`. Lastly, it prints the results for us to evaluate.

Calling the `mini_eval_demo(model, tokenizer, device)` function results in the following output:

```
Device: mps
Prediction: 1/3
Ground truth: 2/3
Correct: False
```

We can see that the generated answer ("1/3") was correctly extracted, but it doesn't match the correct answer ("2/3"), and hence the check returns `False`. (Note that the results may differ depending on whether you execute the code on a CPU, CUDA, or MPS device.)

Now that we have tested our workflow on a simpler example, let's implement it to run on the MATH-500 dataset.

**Listing 3.15 End-to-end model evaluation pipeline for MATH-500 dataset**

```

import time


def eta_progress_message( #A
    processed,
    total,
    start_time,
    show_eta=False,
    label="Progress",
):
    progress = f"{label}: {processed}/{total}"
    if not show_eta or processed <= 0:
        return progress

    elapsed = time.time() - start_time
    if elapsed <= 0:
        return progress

    remaining = max(total - processed, 0)

    if processed:
        avg_time = elapsed / processed
        eta_seconds = avg_time * remaining
    else:
        eta_seconds = 0

    eta_seconds = max(int(round(eta_seconds)), 0)
    minutes, rem_seconds = divmod(eta_seconds, 60)
    hours, minutes = divmod(minutes, 60)
    if hours:
        eta = f"{hours}h {minutes:02d}m {rem_seconds:02d}s"
    elif minutes:
        eta = f"{minutes:02d}m {rem_seconds:02d}s"
    else:
        eta = f"{rem_seconds:02d}s"

    return f"{progress} | ETA: {eta}"


def evaluate_math500_stream(
    model,
    tokenizer,

```

```

device,
math_data,
out_path=None,
max_new_tokens=512,
verbose=False,
):

    if out_path is None:
        dev_name = str(device).replace(":", "-")      #B
        out_path = Path(f"math500-{dev_name}.jsonl")

    num_examples = len(math_data)
    num_correct = 0
    start_time = time.time()

    with open(out_path, "w", encoding="utf-8") as f:    #C
        for i, row in enumerate(math_data, start=1):
            prompt = render_prompt(row["problem"])      #D
            gen_text = generate_text_stream_concat(      #E
                model, tokenizer, prompt, device,
                max_new_tokens=max_new_tokens,
                verbose=verbose,
            )

            extracted = extract_final_candidate(        #F
                gen_text
            )
            is_correct = grade_answer(                  #G
                extracted, row["answer"]
            )
            num_correct += int(is_correct)

            record = {                                #H
                "index": i,
                "problem": row["problem"],
                "gtruth_answer": row["answer"],
                "generated_text": gen_text,
                "extracted": extracted,
                "correct": bool(is_correct),
            }
            f.write(json.dumps(record, ensure_ascii=False) + "\n")

    progress_msg = eta_progress_message(
        processed=i,

```

```

        total=num_examples,
        start_time=start_time,
        show_eta=True,
        label="MATH-500",
    )
    print(progress_msg, end="\r", flush=True)

    if verbose:                      #I
        print(
            f"\n\n{'='*50}\n{progress_msg}\n"
            f"{'='*50}\nExtracted: {extracted}\n"
            f"Expected: {row['answer']}\n"
            f"Correct so far: {num_correct}\n{'='*50}"
        )

seconds_elapsed = time.time() - start_time
acc = num_correct / num_examples if num_examples else 0.0
print(f"\nAccuracy: {acc*100:.1f}% ({num_correct}/{num_examples})")
print(f"Total time: {seconds_elapsed/60:.1f} min")
print(f"Logs written to: {out_path}")
return num_correct, num_examples, acc

#A Helper function to print progress with optional ETA (estimated time to arrival)
#B Make filename compatible with Windows
#C Save results for inspection
#D 1. Apply prompt template
#E 2. Generate response
#F 3. Extract and normalize answer
#G 4. Grade answer
#H Record to be saved for inspection
#I Print responses during generation

```

The `evaluate_math500_stream` function in listing 3.15 uses the same main steps as the smaller `demo` function from listing 3.14: for each problem, it renders the prompt, streams a model response, extracts the answer candidate, and grades it against the reference answer.

In addition to iterating over a dataset with multiple entries, it adds some additional bells and whistles. For instance, it saves the generated responses to a JSON file in a Python dictionary-like format for record keeping and closer inspection.

Let's now run this function on a subset, the first 10 examples of MATH-500, which takes about 0.7 minutes on a Mac Mini with an M4 chip. (Evaluating the reasoning model variant takes about 7 min as it generates longer responses.)

```

print("Model:", WHICH_MODEL)
num_correct, num_examples, acc = evaluate_math500_stream(
    model, tokenizer, device,
    math_data=math_data[:10], #A
    max_new_tokens=2048,
    verbose=False #B
)

#A Only evaluate on the first 10 examples
#B Set to true to read the responses as they are being generated

```

In the code example above, we set `max_new_tokens` to a generous 2048, since the reasoning model variant, per design, tends to generate much longer responses, and we don't want to cut it off prematurely. This, however, leads to much longer evaluation times, where it may appear that the generation is stuck. Optionally, you could set `verbose=True` to see the response being generated live, token by token.

The result of running the `evaluate_math500_stream` function is as follows:

```

Model: base
Device: mps
MATH-500: 10/10 | ETA: 00s
Accuracy: 30.0% (3/10)
Total time: 0.4 min

```

(Note that the results may differ depending on whether you execute the code on a CPU, CUDA, or MPS device.)

As we can see, the model achieves a relatively low accuracy of 30%. We can open the `math500_base-mps.jsonl` file in a text editor to analyze the results, together with the generated response. For instance, we find that the answers, in all cases, have been successfully extracted, but they are plain wrong, which indicates that the model does not have very strong math problem-solving capabilities (yet). This is expected since it's merely a base model.

### LOADING THE .JSONL FILE PROGRAMMATICALLY

The `.jsonl` file suffix is a convention used for JSON files with one data entry per row. You can view it in your favorite text editor. Optionally, we can load the `.jsonl` file created during the evaluation in Python using the following code:

```
dev_name = str(device).replace(":", "-")
local_path = f"math500_{WHICH_MODEL}-{dev_name}.jsonl"
results = []
with open(local_path, "r") as f:
    for line in f:
        if line.strip():
            results.append(json.loads(line))
```

The reasoning model variant, which you can enable by setting `WHICH_MODEL = "reasoning"` in listing 3.1 in section 3.2, performs much better and achieves a 90% accuracy on the same 10-sample subset and 50.8% on the complete 500-sample dataset, as shown in table 3.1.

**Table 3.1 MATH-500 task accuracy on different devices**

Mode	Device	Accuracy	MATH-500 size
Base	CPU	30%	10
Base	CUDA	30%	10
Base	MPS	30%	10
Reasoning	CPU	90%	10
Reasoning	CUDA	90%	10
Reasoning	MPS	80%	10
Base	CUDA	15.3%	500
Reasoning	CUDA	50.8%	500

As shown in table 3.1, the reasoning variant, with its longer responses, has a drastically improved accuracy, but also increases the compute intensity and answer generation time substantially (from 0.4 min for the base model to 7 min for the reasoning model on a Mac Mini with M4 chip on the 10-sample subset, and from 13.3 min to 185.4 min on an H100 on the 500-sample dataset), which highlights one of the trade-offs of using reasoning models. Please note that these numbers were obtained in PyTorch 2.8 and can differ in different versions of PyTorch.

---

**TIP** The code repository contains a bonus script ([https://github.com/rasbt/reasoning-from-scratch/blob/main/ch03/02\\_math500-verifier-scripts/evaluate\\_math500\\_batched.py](https://github.com/rasbt/reasoning-from-scratch/blob/main/ch03/02_math500-verifier-scripts/evaluate_math500_batched.py)) that runs the code in this chapter in batched mode. This means it processes multiple examples per forward pass to accelerate the evaluation while requiring more RAM. With a batch size of 128, this reduces the runtime of the base model, when evaluating all 500 samples, from 13.3 min to 3.3 min on an H100 GPU. Similarly, it reduces the runtime of the reasoning model from 185.4 min to 14.6 min. Note that the H100 is used as an example, and the script is compatible with other GPUs as well.

---

### EXERCISE 3.2: CALCULATING THE AVERAGE RESPONSE LENGTH

Try to modify the code in this chapter to also report the average response length in the evaluation function in listing 3.14. Instead of modifying the function directly, you could also compute the response length from the generated JSON report files.

### EXERCISE 3.3: EXTENDING OR CHANGING THE EVALUATION DATASET

We choose a subset of only 10 examples for computational efficiency. However, readers are encouraged to consider running the code on larger or different portions of the dataset to observe whether the 10-sample subset is representative. Ideally, you could also experiment with your own data. (For reference, evaluating the base model on the complete MATH-500 dataset takes about 13.3 min for the base model and 185.4 min for the reasoning model on an H100.)

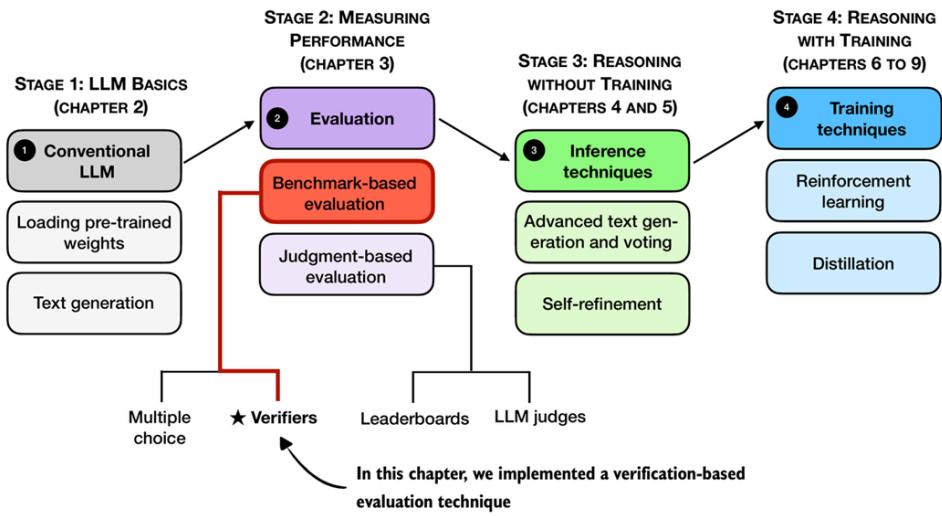
#### EXERCISE 3.4: EXPERIMENTING WITH DIFFERENT PROMPT TEMPLATES

Models can be sensitive to different prompt templates. Experiment with different prompt templates in listing 3.13 to see how it affects the results. Also, while the Qwen3 team recommends using the base model without an additional chat template, you can additionally enable the `apply_chat_template=True` setting in the tokenizer (listing 3.1) and observe whether it improves the base model performance.

Note that this concludes our chapter on implementing a verification-based approach for math tasks (figure 3.11). We chose math because it is both natural to implement and widely used in reasoning-specific training, particularly reinforcement learning with verifiable rewards, which we will cover in chapter 6. The same concept can be extended to other domains, such as code, although we did not explore that here since executing code would require additional setup of a secure virtual environment.

Before we move on, it is worth noting that evaluation also comes in many other flavors. In this chapter we focused on verification-based accuracy for math problems, as it is a popular method, and because we will reuse the same verifier as part of the reinforcement learning pipeline in chapters 6 and 7.

For a broader overview, appendix F walks through other common evaluation strategies, such as multiple-choice benchmarks, verifiers, leaderboards, and LLM-as-judge setups. This appendix provides an overview with hands-on examples if you want a quick tour of how these methods work in practice.



**Figure 3.11** Mental model of the topics covered in this book. This chapter implemented a verifier-based evaluation pipeline. In the next chapter, we will improve the reasoning capabilities of the LLM via more advanced inference techniques.

Now, with an evaluation framework in place, the next chapter, as shown in figure 3.11, focuses on improving the reasoning capabilities through more advanced inference (text generation) techniques.

## 3.10 Summary

- There are four main evaluation methods for LLMs: multiple choice, verifiers, leaderboards, and LLM judges
- Verification-based evaluation methods allow free-form answers and use external tools to check correctness
- This chapter focuses on verification-based evaluation by building a math verifier that extracts, normalizes, and checks answers with SymPy
  - The verification pipeline involves several core steps from loading the LLM to running the evaluation on a dataset
  - As part of the verification pipeline, answer extraction uses string parsing to locate boxed content (with fallback mechanisms for missing boxes)
  - Another step implements normalization, which standardizes diverse answer formats by stripping LaTeX and converting mathematical notation
- Finally, the pipeline uses mathematical equivalence checking (via SymPy) to compare expressions symbolically

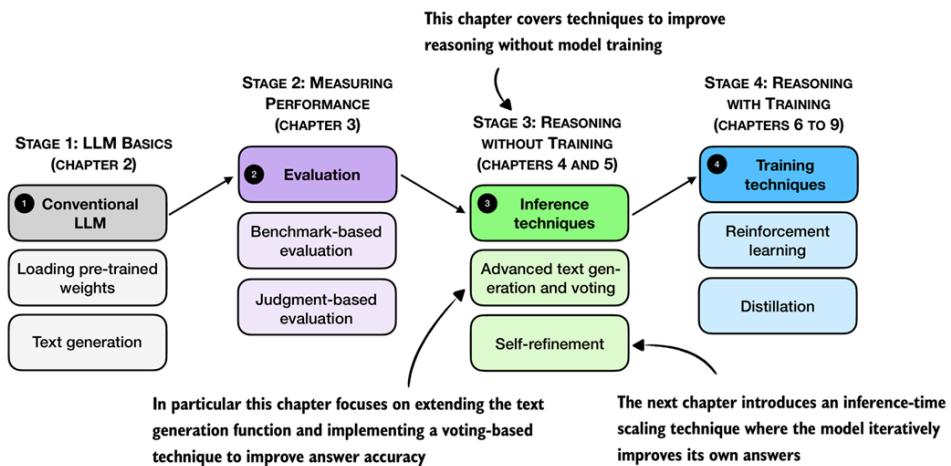
- The MATH-500 dataset provides 500 curated math problems for evaluation
- Prompt templates significantly impact model performance
- The reasoning model achieves higher accuracy than the base model, but requires much longer runtime

## ***4 Improving reasoning with inference-time scaling***

### **This chapter covers**

- Prompting an LLM to explain its reasoning to improve answer accuracy
- Modifying the text generation function to produce diverse responses
- Improving reasoning reliability by sampling multiple responses

Reasoning performance and answer accuracy can be improved without retraining or modifying the model itself. These methods operate during inference, when the model generates text. As shown in the overview in figure 4.1, in this chapter, we cover two inference-time scaling methods. As we will see later in this chapter, both methods more than double the accuracy of the base model we used in previous chapters.



**Figure 4.1** A mental model of the topics covered in this book. This chapter focuses on techniques that improve reasoning without additional training (stage 3). In particular, it extends the text-generation function and implements a voting-based method to improve answer accuracy. The next chapter then introduces an inference-time scaling approach where the model iteratively refines its own answers.

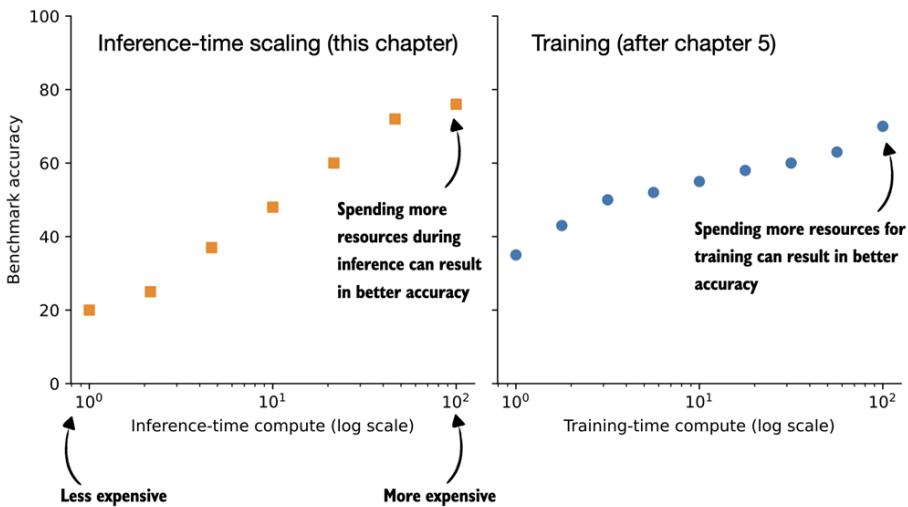
The next section provides a general introduction to inference-time scaling before discussing the inference methods that are shown in figure 4.1 in more detail.

## 4.1 Introduction to inference-time scaling

In general, there are two main strategies to improve reasoning:

1. Increasing training compute and
2. increasing inference compute (also known as inference-time scaling or test-time scaling).

(In machine learning and AI, "compute" refers to the computational resources required to train or run a model.) These two approaches are illustrated in figure 4.2.



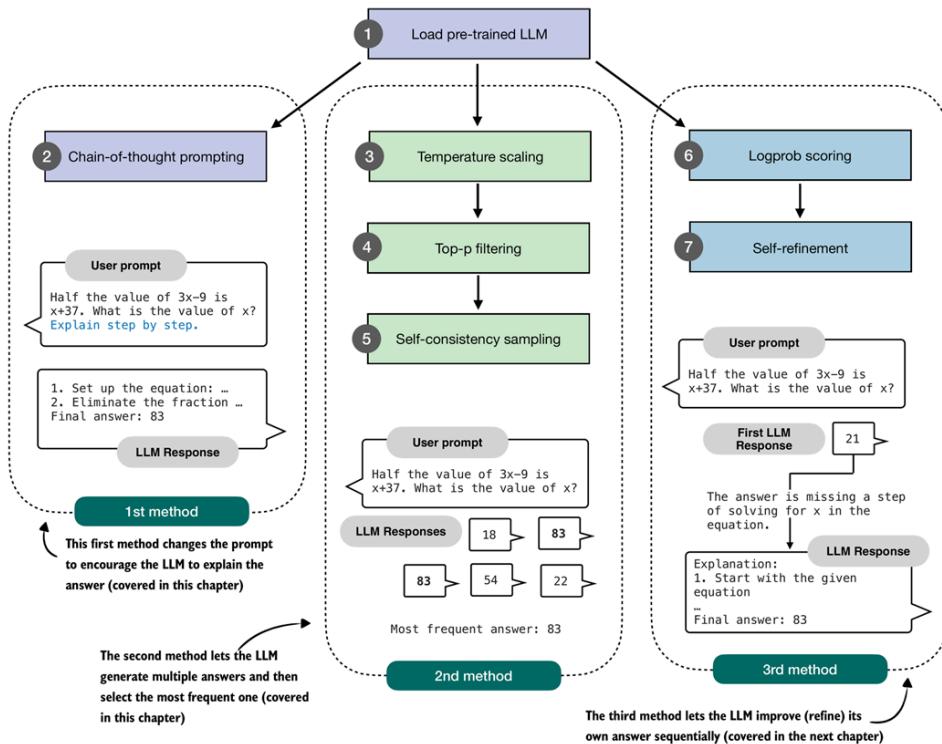
**Figure 4.2 Comparison of inference-time scaling (this chapter) and training-time scaling (after chapter 5).** Both improve accuracy by using more compute, but inference-time scaling does this on the fly, without changing the model's weight parameters. The plots are inspired by OpenAI's article introducing their first reasoning model (<https://openai.com/index/learning-to-reason-with-langs/>).

The plots shown in figure 4.2 make it look like we improve reasoning either by increasing training-time compute or inference-time compute. However, LLMs are usually designed to improve reasoning by combining heavy training-time compute (a topic of future chapters) and increased *inference-time compute* (the topic of this chapter).

*Inference-time compute scaling* (also called *inference-time scaling* and *test-time compute scaling*) is a broad category that describes any technique that expends more resources to get better answers from the model. This includes making the model generate more tokens and "think" longer, sample multiple answers, or successively refine its answer.

In this book, we focus on three practical and foundational inference-time techniques (figure 4.3):

- Method 1: Extending the *chain-of-thought* response to prompt the model to explain its reasoning. This is a simple technique that can substantially improve accuracy.
- Method 2: Parallel sampling via *self-consistency*, where the model generates multiple responses and selects the most frequent one.
- Method 3: Iterative *self-refinement*, where the model reviews and improves its own reasoning and answers across multiple steps. (This topic is implemented and covered in more detail in the next chapter.)



**Figure 4.3 Overview of three inference-time methods to improve reasoning covered in this book. The first modifies the prompt to encourage step-by-step reasoning, and the second samples multiple answers and selects the most frequent one. Both are discussed in this chapter. The third method, in which the model iteratively refines its own response, is introduced in the next chapter.**

The methods shown in figure 4.3 fall under the category of inference-time scaling because they cause the model to generate more tokens, which increases compute resources during inference. In other words, these methods achieve better accuracy while making the inference process more expensive, which is a common theme of inference-time scaling techniques.

The first method in this chapter improves answer accuracy by prompting the model to explain its reasoning, a simple yet highly effective approach.

Next, we extend the text generation function introduced earlier to enable sampling multiple responses for the same input (the 2nd method shown in figure 4.3). Using this modified function, we implement *self-consistency*, a voting-based *inference-time scaling* technique that increases answer accuracy by generating multiple answers and selecting the most frequent one.

In total, I evaluated more than ten different inference-time scaling techniques across thousands of experiments. The three methods illustrated in figure 4.3 were chosen because they offer strong improvements, but also because they are popular and representative of the main paradigms: generating longer responses and generating multiple responses.

Longer responses that include reasoning and explanations (methods 1 and 3) are what we typically expect from reasoning models, as discussed in previous chapters. And parallel sampling (method 2) is also widely used in production systems, such as Claude 4 (as described in <https://www.anthropic.com/news/clause-4>).

## 4.2 Loading a pre-trained model

Before we begin implementing the inference-time scaling methods described in the previous section, we load the pre-trained base model we used in the previous sections.

**Listing 4.1 Load tokenizer and base model**

```
import torch
from reasoning_from_scratch.ch02 import get_device
from reasoning_from_scratch.ch03 import (
    load_model_and_tokenizer
)

device = get_device()
device = torch.device("cpu") #A

model, tokenizer = load_model_and_tokenizer(
    which_model="base",
    device=device,
    use_compile=False
)

#A Delete this line to run the code on a GPU (if supported by your machine)
```

The code in listing 4.1 loads the model and tokenizer we are using in this chapter. It is similar to the code we used in previous chapters.

Since the code in this chapter is cheap to run, I recommend running this chapter on the "cpu" device for the first time to get the same results as shown in this chapter (running this chapter on a GPU can subtly alter the results.)

Let's try the model on a prompt from the MATH-500 dataset, which we worked with in the previous chapter:

```
from reasoning_from_scratch.ch03 import render_prompt

raw_prompt = (
    "Half the value of $3x-9$ is $x+37$. "
    "What is the value of $x$?"
)
prompt = render_prompt(raw_prompt)
print(prompt)
```

The formatted prompt is as follows:

You are a helpful math assistant.

Answer the question and write the final result on a new line as:

\boxed{ANSWER}

Question:

Half the value of  $3x-9$  is  $x+37$ . What is the value of  $x$ ?

Answer:

We can use the prompt above as input to the `generate_text_stream_concat` we defined in the previous chapter. However, we will add a subtle modification to this text generation function that will allow us to plug in the advanced text generation function later. The changes are highlighted via the code comments in listing 4.2:

**Listing 4.2 Modified generate\_text\_stream\_concat function**

```
from reasoning_from_scratch.ch02_ex import generate_text_basic_stream_cache

def generate_text_stream_concat_flex(
    model, tokenizer, prompt, device, max_new_tokens,
    verbose=False,
    generate_func=None, #A
    **generate_kwargs #A
):
    if generate_func is None: #B
        generate_func = generate_text_basic_stream_cache

    input_ids = torch.tensor(
        tokenizer.encode(prompt), device=device
    ).unsqueeze(0)

    generated_ids = []
    for token in generate_func(
        model=model,
        token_ids=input_ids,
        max_new_tokens=max_new_tokens,
        eos_token_id=tokenizer.eos_token_id,
        **generate_kwargs, #C
    ):
        next_token_id = token.squeeze(0)
        generated_ids.append(next_token_id.item())

        if verbose:
            print(
                tokenizer.decode(next_token_id.tolist()),
                end="",
                flush=True
            )
    return tokenizer.decode(generated_ids)
```

#A We add parameters to accept a text generation function and additional arguments

#B If the text generation function is undefined, we use generate\_text\_basic\_stream\_cache similar to chapter 3

#C We can pass additional arguments to the text generation function if needed

In short, the `generate_text_stream_concat_flex` function above is similar to the `generate_text_stream_concat` function from the previous chapter, except that we can now pass in the text generation function (like `generate_text_basic_stream_cache`) as a function argument instead of hard-coding it. In future sections, we will swap the `generate_text_basic_stream_cache` function with more advanced functions.

The usage is also similar to before, except that we can now pass the text generator function (for example, `generate_text_basic_stream_cache`) explicitly:

```
response = generate_text_stream_concat_flex(
    model, tokenizer, prompt, device,
    max_new_tokens=2048, verbose=True,
    generate_func=generate_text_basic_stream_cache
)
```

The generated output is:

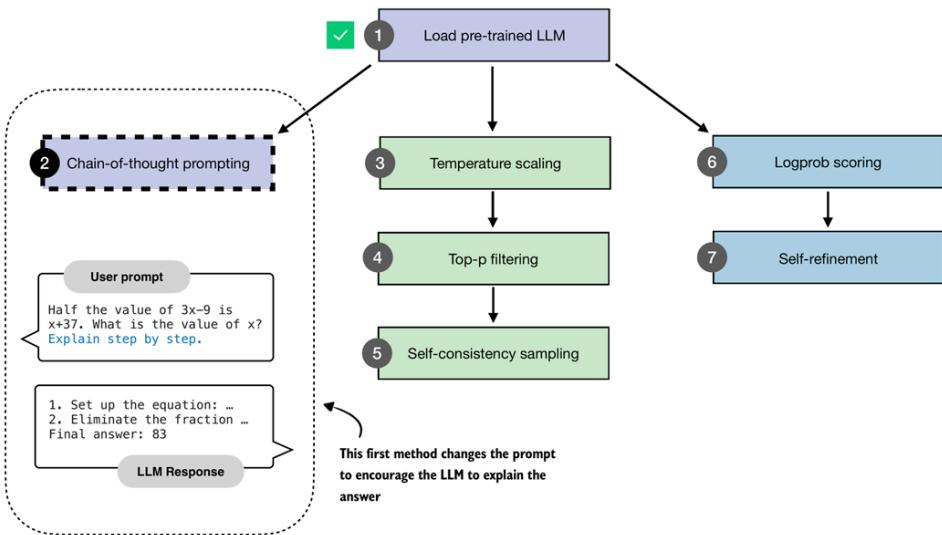
```
\boxed{20}
```

Note that the answer is wrong, and the correct solution is 83. In the remainder of this chapter, and the next chapter, we will implement inference-time scaling methods to get the model to generate the correct answer.

### 4.3 Generating better responses with chain-of-thought prompting

After loading the pre-trained base model in the previous section and setting up the text generation function, this section focuses on improving the model output via so-called *chain-of-thought* prompting.

Chain-of-thought prompting is a classic but simple and effective technique that modifies the input prompt to encourage the LLM to generate an explanation or so-called chain-of-thought (also called reasoning chain), as illustrated in figure 4.4.



**Figure 4.4 The first inference-time method, chain-of-thought prompting, modifies the prompt to encourage the model to explain its reasoning step by step before producing a final answer.**

The simplest way to try chain-of-thought prompting is to append a phrase like "Explain step by step." to the prompt, as shown below:

```

prompt_cot = prompt + "\n\nExplain step by step."

response_cot = generate_text_stream_concat_flex(
    model, tokenizer, prompt_cot, device,
    max_new_tokens=2048, verbose=True,
)

```

The response is now as follows:

To solve the problem, we need to find the value of  $\lfloor x \rfloor$  such that half the value of  $\lfloor 3x - 9 \rfloor$  is equal to  $\lfloor x + 37 \rfloor$ .

```
# ... #A

### Step 3: Solve for \lfloor x \rfloor
Subtract \lfloor 2x \rfloor from both sides to isolate \lfloor x \rfloor:
\[
3x - 2x - 9 = 74
\]
Simplify:
\[
x - 9 = 74
\]
Add 9 to both sides to solve for \lfloor x \rfloor:
\[
x = 74 + 9
\]
\[
x = 83
\]

### Final Answer:
\boxed{83}
```

#A The response was truncated to save space.

As we can see, the model now writes a lengthy step-by-step explanation and, in this case, now arrives at the correct answer.

This simple chain-of-thought prompting is a good demonstration of the inference-time scaling trade-off. While the model now answers correctly, it expends many more tokens than before, which makes the model much more compute-intensive or costly in practice.

Note that while the model generates the correct answer in this case, not all problems benefit from chain-of-thought prompting. On simple problems, it can even sometimes degrade the model's performance, as the model might sometimes generate erroneous explanations and mislead itself. This phenomenon is also known as "overthinking."

Lastly, not every model benefits from the "Explain step by step" (or similar) instruction. In this case, we use a simple base model that doesn't always generate explanations, so chain-of-thought prompting can clearly help. However, trained reasoning models, such as the "reasoning" variant of the Qwen3 model we used in the exercises of the previous chapter, already write explanations alongside their responses and don't need or benefit from this type of chain-of-thought prompting.

### WHY CHAIN-OF-THOUGHT CAN IMPROVE ACCURACY

Chain-of-thought prompting asks the model to write out the intermediate steps that lead to a final answer. This helps in two practical ways.

First, walking through the steps gives the model more opportunities to correct itself.

Second, step-by-step reasoning matches how many training examples are written. For instance, large math and logic datasets often contain detailed solutions, so asking for a chain of thought aligns the model with patterns it has already learned.

At the same time, chains-of-thought are not a guarantee for correctness. It can still produce wrong reasoning, and for very simple problems it may even introduce unnecessary steps that lead to more mistakes. In other words, chains-of-thought can improve accuracy on many reasoning tasks, but it is not universally beneficial.

Overall, chain-of-thought answering does not provide the model with new knowledge, but it changes how the model uses its existing knowledge. Often, this shift can lead to more reliable answers. This is especially true for math, code, logic problems, and other sorts of multi-step problems.

### EXERCISE 4.1: USE CHAIN-OF-THOUGHT PROMPTING ON MATH-500

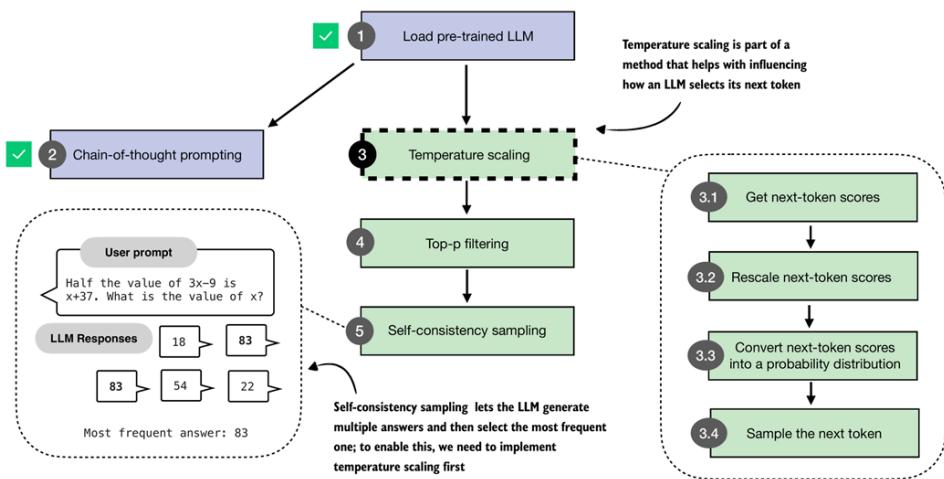
Modify the `evaluate_math500_stream` function in section 3.9 of chapter 3 to see if chain-of-thought prompting improves the MATH-500 accuracy of the base model.

## 4.4 Controlling output diversity with temperature scaling

The previous section gave a brief taste of inference-time scaling by extending the model's answer via chain-of-thought prompting. Chain-of-thought prompting can be seen as a sequential technique as we extend the number of next-token prediction steps.

In this section, we will implement a technique that generates multiple answers, as illustrated in figure 4.5. Since the answers are independent of each other, this can be implemented as a parallel sampling (if we have the necessary resources, for example, using multiple GPUs), which in this case wouldn't increase the wait time for a user to get the answer.

(Note that chain-of-thought prompting can also be combined with this technique, but more on that in a later section.)



**Figure 4.5 The second inference-time method, self-consistency sampling, generates multiple answers and selects the most frequent one. This method relies on temperature scaling, covered in this section, which influences how the model samples its next token.**

The *self-consistency technique*, illustrated in figure 4.5 and covered in this chapter, is also called *self-consistency sampling* and was formally described in the Google Research paper *Self-Consistency Improves Chain-of-Thought Reasoning in Language Models* (<https://arxiv.org/abs/2203.11171>).

Before implementing self-consistency sampling (step 5 in figure 4.5), we first need to extend the text generation function so that it can produce multiple different answers for the same prompt. To achieve this, we will implement two techniques, *temperature scaling* (step 3) and *top-p filtering* (step 4), which allow the model to sample different responses.

Temperature scaling is the main focus of this section. Before we get to temperature scaling, the next subsection provides a brief overview of how the next token is sampled in an LLM.

#### 4.4.1 Understanding the process of selecting the next token

This subsection gives a closer look at the text generation process we implemented so far and explains how the next-token selection process works under the hood. This information will help you understand the motivation behind temperature scaling.

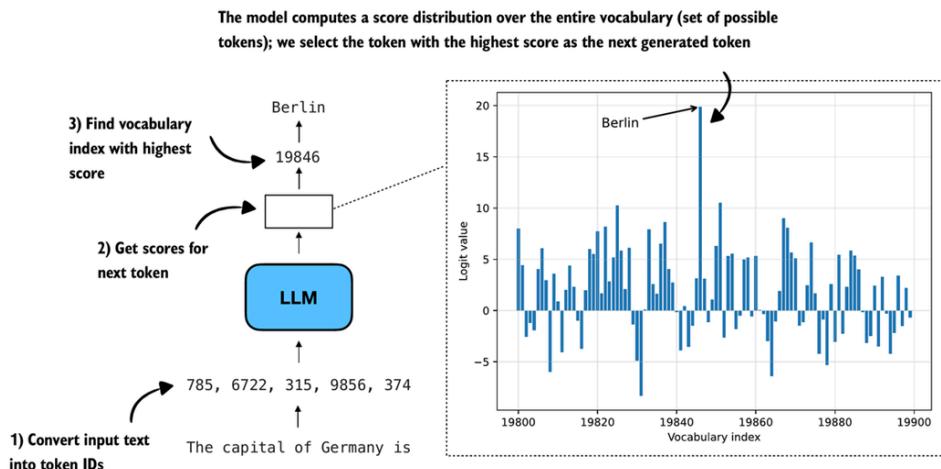
For instance, suppose we have the following simple prompt:

```
ex_prompt = "The capital of Germany is"

response = generate_text_stream_concat_flex(
    model, tokenizer, ex_prompt, device,
    max_new_tokens=1, verbose=True
)
```

The model's response is " Berlin".

While this looks relatively simple, there are multiple steps happening under the hood, as illustrated in figure 4.6.



**Figure 4.6 Illustration of how an LLM generates the next token. The model converts the input into token IDs, computes scores for all possible next tokens, and selects the one with the highest score as the next output.**

When generating text, as discussed in chapter 2, the inputs are first converted into token IDs:

```
input_token_ids = torch.tensor(
    tokenizer.encode(ex_prompt), device=device
).unsqueeze(0)
print(input_token_ids)
```

In this case, the token IDs are

```
tensor([[ 785, 6722, 315, 9856, 374]])
```

In the second step (step 2 in figure 4.6), we get the scores for the output token we want to generate. These model output scores are also called *logits*. Note that an LLM generates one output token for each input token, but we are only interested in the last token, which we select via the `[:, -1]` tensor indexing. This last token corresponds to the token we want to generate:

```
with torch.inference_mode():
    next_token_logits = model(input_token_ids)[:, -1]
print(next_token_logits.shape)
```

The printed output shape is `[1, 151936]`, where 151936 is the vocabulary size of this tokenizer and model. The vocabulary size contains all the unique tokens the tokenizer can handle and the LLM can generate.

To actually obtain the next generated token (here: "Berlin"), we have to find a vocabulary entry that is associated with the largest score (step 3 in figure 4.6):

```
max_token_id = torch.argmax(next_token_logits)
print(f"Token ID: {max_token_id}")
print(f"Decoded token: '{tokenizer.decode([max_token_id])}'")
```

The output is:

```
Token ID: 19846
Decoded token: ' Berlin'
```

Above, we covered the three main steps in generating the next token. Before we go to the next section, let us take a closer look at the score distribution of the `next_token_logits` tensor we passed into `torch.argmax` function to obtain the token IDs, and plot them in matplotlib:

**Listing 4.3 Plotting the next-token logit scores**

```

import matplotlib.pyplot as plt

def plot_scores_bar(
    next_token_logits, start=19_800, end=19_900,
    arrow=True, ylabel="Logit value"
):
    x = torch.arange(start, end) #A
    logits_section = next_token_logits[0, start:end].float().cpu() #B

    plt.bar(x, logits_section) #C
    plt.xlabel("Vocabulary index")
    plt.ylabel(ylabel)

    if arrow: #D
        max_idx = torch.argmax(logits_section)
        plt.annotate(
            "Berlin",
            xy=(x[max_idx], logits_section[max_idx]),
            xytext=(x[max_idx] - 25, logits_section[max_idx] - 2),
            arrowprops={
                "facecolor": "black", "arrowstyle": "->", "lw": 1.5
            },
            fontsize=10,
        )

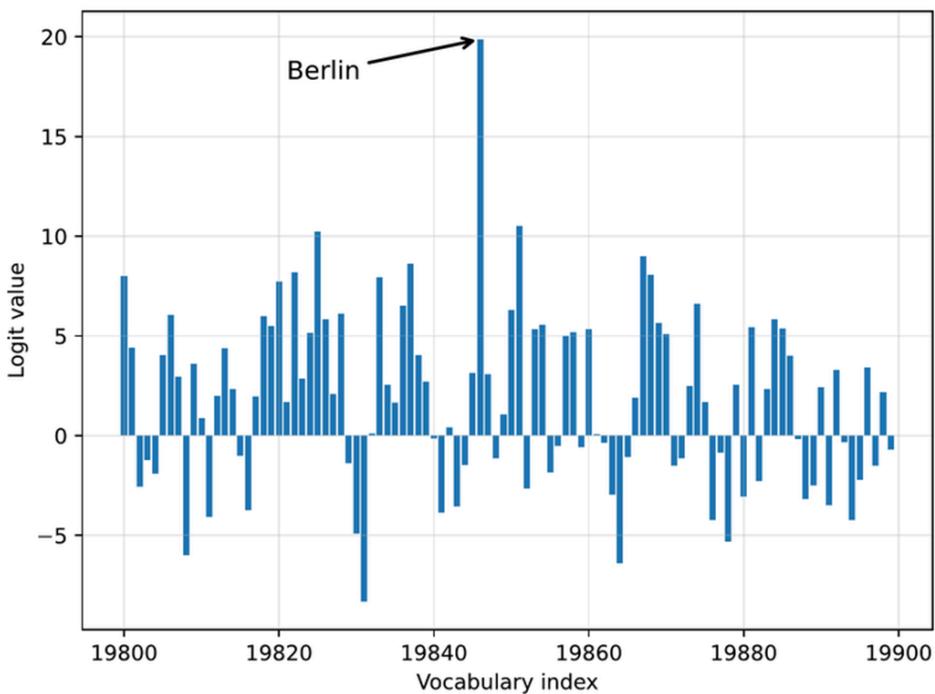
    plt.grid(alpha=0.3)
    plt.tight_layout()
    plt.show()

plot_scores_bar(next_token_logits)

```

#A Select vocabulary subsection  
#B .cpu() is a shortcut for to(torch.device("cpu"))  
#C Plot the logits (scores) within the selected range  
#D Draw an arrow to highlight the largest score

Note that we restrict the plot to the 100 vocabulary index tokens between 19,800 and 19,900, rather than plotting the scores for all 151,936 entries, which would make the plot too crowded. (I selected this specific range such that it contains the entry with the highest score.) The resulting plot is shown in figure 4.7.



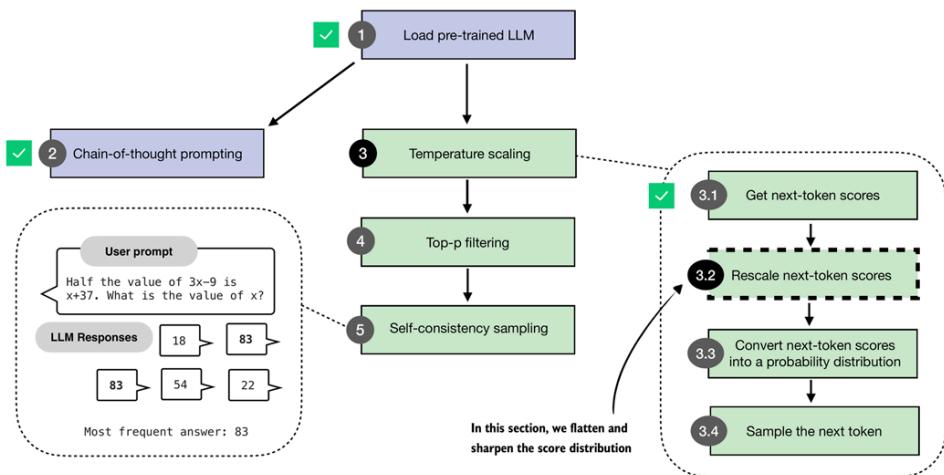
**Figure 4.7 Example of next-token logits for a language model. Each bar represents a possible token's score, with "Berlin" having the highest logit value and being selected as the next token.**

The plot in figure 4.7 shows all 100 logit values (scores) for vocabulary indices 19,800–19,899. The values range approximately from -8 to 20, where 20 corresponds to the vocabulary index for the token " Berlin".

#### 4.4.2 Rescaling token scores (logits) via a temperature parameter

Now that we have walked through how the model selects its next token, we can introduce the concept of temperature. Temperature, or rather a chosen temperature parameter, changes how sharp or spread out the logits (token scores) are, which in turn affects how the next token is selected.

As shown in figure 4.8, this section focuses on rescaling the next-token logits with a temperature parameter before using them for sampling. Rescaling here means adjusting the magnitude of the scores so the sampling step becomes more or less sensitive to the score differences.



**Figure 4.8** In this section, we implement the core part of temperature scaling (step 3.2), which adjusts the next-token scores. This allows us to control how confidently the model selects its next token in later steps.

The code for implementing the temperature rescaling step, step 3.2 in figure 4.8, is relatively short and simple, as shown in listing 4.4 below.

#### Listing 4.4 Rescaling next-token scores via temperature scaling

```
def scale_logits_by_temperature(logits, temperature):
    if temperature <= 0:
        raise ValueError("Temperature must be positive")
    return logits / temperature
```

In essence, the code in listing 4.4 rescales the logit values before converting them to probabilities in the next section.

In practice, temperature values are expected to be positive numbers. A temperature of 1.0 means no change, since dividing a number by 1 is the number itself.

We will add additional temperature scaling safeguards later when we add temperature scaling to our text generation function.

For all other values, the logits are divided by the temperature. A temperature lower than 1.0 makes the distribution sharper (which will make the model more confident when we select the next token in the upcoming sections). Temperatures higher than 1.0 flatten the logits, which can make the sampling (step 3.4 in figure 4.8) more diverse.

Let's see this `scale_logits_by_temperature` function in action and try it out with relatively extreme temperature values 0.5 and 5.0 for a stronger effect:

**Listing 4.5 Plotting the temperature-rescaled logits**

```

def plot_logits_with_temperature(
    next_token_logits, start=19_800, end=19_900,
    temps=(0.5, 5.0),
):
    x = torch.arange(start, end)
    logits_orig = next_token_logits[0, start:end].float().cpu()

    logits_scaled = [    #A
        scale_logits_by_temperature(logits_orig, T) for T in temps #A
    ] #A

    plt.plot(x, logits_orig, label="Original logits", lw=2) #B
    plt.plot(
        x, logits_scaled[0], #B
        label=f"T={temps[0]} (sharper)", ls="--", lw=1 #B
    )
    plt.plot(
        x, logits_scaled[1], #B
        label=f"T={temps[1]} (flatter)", ls=":", lw=3 #B
    )

    # Highlight max logit
    max_idx = torch.argmax(logits_orig) #C
    plt.annotate(
        "Berlin", #C
        xy=(x[max_idx], logits_orig[max_idx]), #C
        xytext=(x[max_idx] - 25, logits_orig[max_idx] + 2), #C
        arrowprops={"facecolor": "black", "arrowstyle": "->", "lw": 1.5}, #C
        fontsize=12, #C
    )

    plt.xlabel("Vocabulary index")
    plt.ylabel("Logit value")
    plt.legend()
    plt.grid(alpha=0.3)
    plt.tight_layout()
    plt.show()

plot_logits_with_temperature(
    next_token_logits,
)

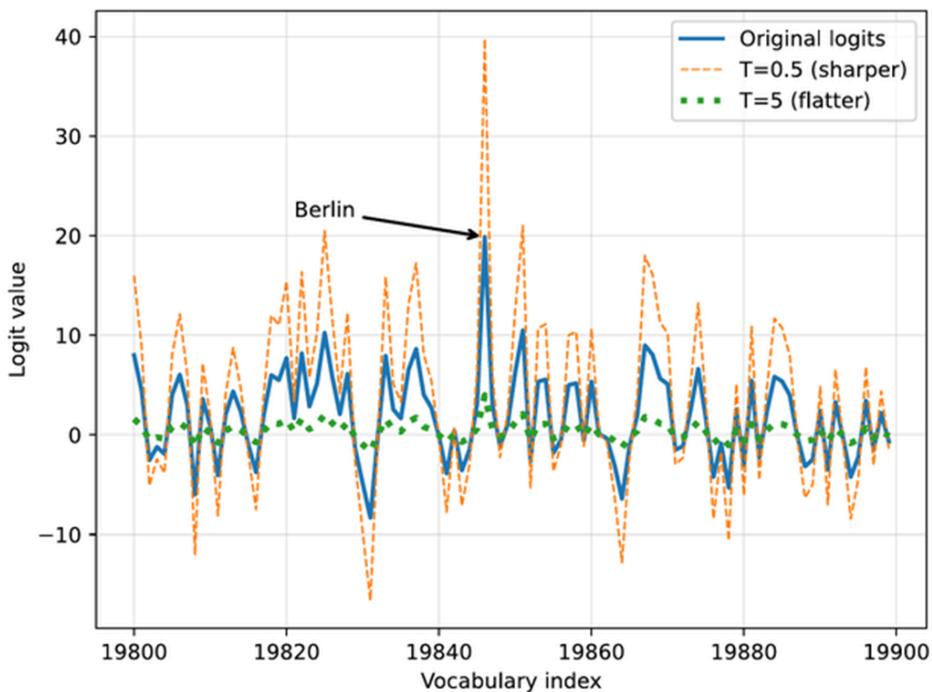
```

```

    temps=(0.5, 5.0) #D
)
#A Apply temperature scaling
#B Plot the logits (scores) within the selected range
#C Draw an arrow to highlight the largest score
#D Run the rescaling and plotting with temperatures 0.5 and 5

```

As shown in the plot in figure 4.9, the larger temperature (5.0) yields a much flatter score distribution, whereas the smaller temperature (0.5) yields a much sharper one.



**Figure 4.9 The effect of temperature scaling on logits. Lower temperatures make the distribution sharper, while higher temperatures flatten it. (Please note that this visualization is shown as a line plot for readability, though a bar plot would more accurately represent the discrete vocabulary scores.)**

Note that the plotting code in listing 4.5 above looks very similar to the plotting code we used in the previous section (listing 4.3). However, in addition to applying the temperature scaling, we now use a line plot (`plt.plot`) instead of a bar plot (`plt.bar`). While a bar plot is technically a better choice for the discrete vocabulary indices on the x-axis, the line plot makes it easier to visualize and compare the rescaled logits in this case.

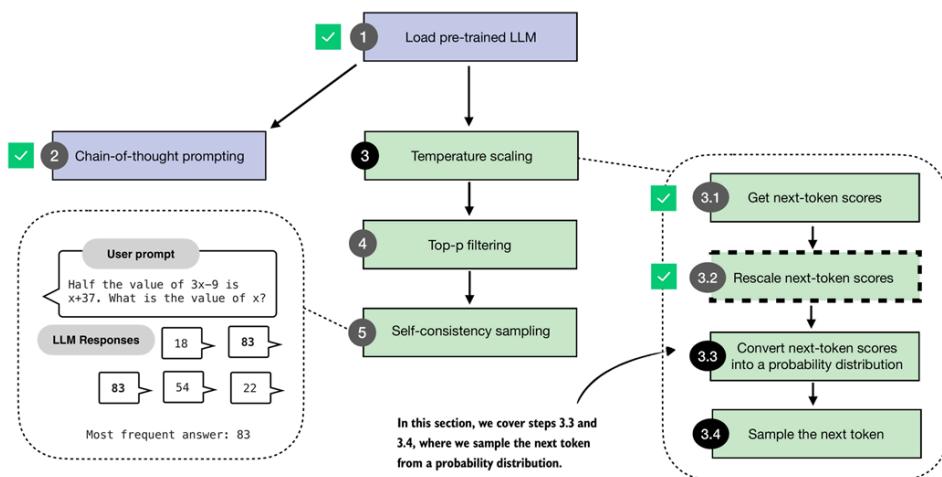
## WHY TEMPERATURE?

The term temperature comes from physics, where temperature controls how much randomness or movement there is in a system. In LLMs, we use the same idea to control how confidently or creatively the model chooses its next token, as we will see in the next two sections.

### 4.4.3 Sampling the next token from a probability distribution

The previous section rescaled the logit values using different temperature values. The purpose of this is that it lets us (later) influence how the model selects the next token.

However, before we get to the next-token sampling portion in the next section, this section adds one more intermediate step: converting the rescaled logits into probability scores, as shown in figure 4.10.



**Figure 4.10 Overview of the sampling process for generating tokens. In this section, we focus on steps 3.3 and 3.4, where the next-token scores are converted into a probability distribution, and the next token is sampled based on that distribution.**

To demonstrate how rescaled logits are converted into probability scores, as described in step 3.3 of figure 4.10, we will use a temperature of 5.0. This makes it easier to visualize the resulting probabilities in a plot. The " Berlin" token, for example, has such a high logit value that it would otherwise dominate the scale and make it difficult to see the probabilities of the surrounding tokens.

The conversion from rescaled logits to probability scores can be done with a single function call, `torch.softmax`, as shown in listing 4.6 below.

**Listing 4.6 Sampling the next-token from a probability distribution**

```
rescaled_logits = scale_logits_by_temperature(next_token_logits, 5.0) #A

next_token_probas = torch.softmax( #B
    rescaled_logits, dim=-1
)

#A Step 3.2: Rescale next-token scores
#B Step 3.3 Convert rescaled logits into probability scores
```

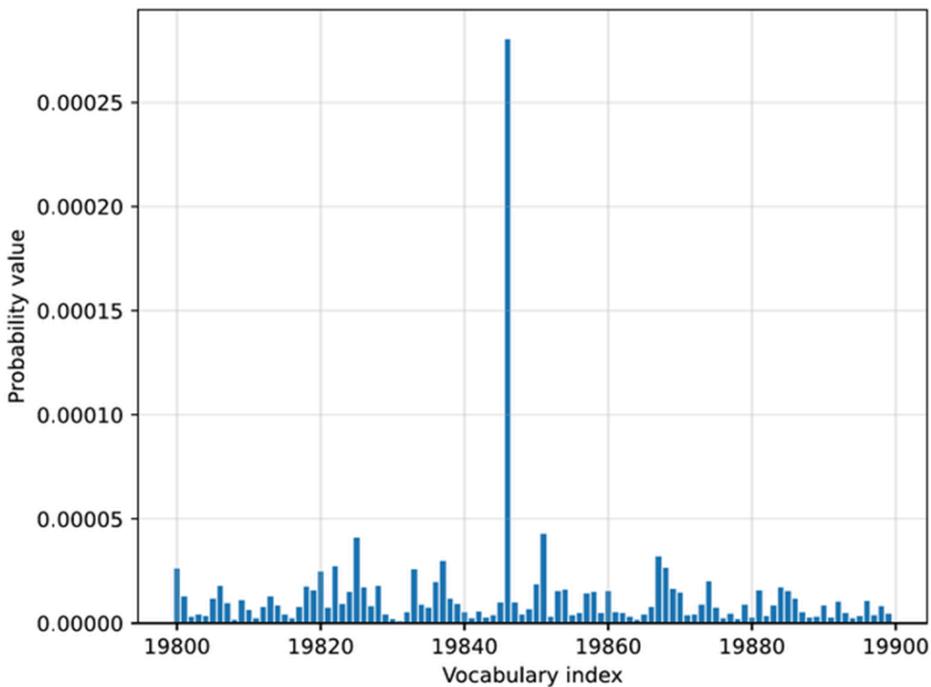
The `torch.softmax` function in listing 4.6 normalizes the logit values into values in the range between 0 and 1, and such that the values sum to 1, which we can confirm via the following code:

```
print("Probability sum:", torch.sum(next_token_probas))
```

Additionally, let's visualize the converted scores by reusing the `plot_scores_bar` function from listing 4.3 earlier:

```
plot_scores_bar(
    next_token_probas, arrow=False, ylabel="Probability value"
)
```

The resulting plot, now with the probability values on the y-axis, is shown in figure 4.11.



**Figure 4.11 Token probabilities obtained by applying the softmax function to the rescaled logits. The token of the highest probability (corresponding to " Berlin", but with the label omitted for code simplicity) is selected as the next output.**

In the plot in figure 4.11, we can see that the token ID 19,846 (" Berlin") has the highest value in this selected vocabulary range. The probability is 0.0003, which we can confirm via the following code:

```
print("Token ID 19,846 probability:", next_token_probas[:, 19846])
```

Note that while this 0.0003 value is relatively small, there is no other value larger than this outside the selected vocabulary range in this plot. For instance, using the following code, we can confirm that 0.0003 is indeed the largest value:

```
print("Highest probability:", max(next_token_probas.squeeze(0)))
```

We can interpret this score as the model's confidence. This means that the model is more confident in " Berlin" (19,846) as the next token than in other tokens.

The reason the value is so small is that we are using a large temperature, which makes it easier to plot this value next to the other token values. However, if we change the temperature from 5 to 0.5, the probability score increases from 0.0003 to 0.3398, and the probability scores of the other tokens will be even closer to 0.

### THE SOFTMAX FUNCTION UNDER THE HOOD

The softmax function converts a vector of raw scores (logits) into a probability distribution where each value lies between 0 and 1, and all values sum to 1. This conversion makes it easier to interpret them and sample from them later.

If you are familiar with mathematical notation, the formula behind the softmax function is

$$\text{softmax}(z_i) = \exp(z_i) / \sum_j \exp(z_j)$$

Here,  $z$  is a vector of real-valued inputs

$$z = [z_1, z_2, \dots, z_n],$$

where

- $n$  is the number of elements in the vector,
- $i$  is the index of the current element ( $1 \leq i \leq n$ ),
- $j$  is the index used to sum over all elements ( $1 \leq j \leq n$ ),

This produces a normalized probability for each element  $z_i$ , such that

$$\sum_i \text{softmax}(z_i) = 1$$

In code, the softmax function can be implemented with a simple 3-liner:

```
def softmax_with_temperature(logits, temperature):
    scaled_logits = logits / temperature
    return torch.softmax(scaled_logits, dim=0)
```

However, in practice, we prefer the `torch.softmax` function because it has some additional numerical stability improvements to handle very small and very large values more reliably.

The purpose of converting the logits into these probability scores is that the probability scores are somewhat more interpretable, and we can now sample from them using a `torch.multinomial` function.

For instance, if we draw one sample from this probability distribution, we have a 0.03% chance of getting " Berlin" with our temperature setting of 5.0 (and a 33.98% chance of getting " Berlin" with a temperature of 0.5).

The sampling process, corresponding to step 3.4 in the figure at the beginning of this section, can be implemented as follows:

```
torch.manual_seed(123)
print(
    "Sampled token:",
    torch.multinomial(next_token_probas.cpu(), num_samples=1)
)
```

This code returns token ID 65,094, which corresponds to the word " mistress". Note that the word doesn't make sense in our context, "The capital of Germany is", and it was selected randomly in this case and influenced by the high-temperature setting, which encourages tokens other than " Berlin" to be sampled.

The `torch.multinomial` function samples the vocabulary indices proportional to their probabilities. In other words, vocabulary indices with higher probabilities are more likely to be sampled. If we repeated the sampling a very large number of times, we would sample the vocabulary index corresponding to the token " Berlin" with 0.03% probability, given a temperature of 5.

Before we look at some additional examples, note that we specified a random seed above to make the code in this chapter reproducible. However, the `torch.multinomial()` function may still yield different results on "cuda" and "mps" devices, and may even crash when we draw larger numbers of samples (I observed this issue in PyTorch 2.9 on both "cuda" and "mps" devices), which is why we run the sampling on the CPU via `.cpu()`.

Let's now sample more next-token candidates to get a more representative sample.

**Listing 4.7 Sampling multiple next-token candidates**

```

def count_samples(probas, num_samples=1000, threshold=1, tokenizer=None):
    samples = torch.multinomial( #A
        probas.cpu(), num_samples=num_samples, replacement=True
    )
    counts = torch.bincount(samples.squeeze(0), minlength=1) #B

    for i, c in enumerate(counts):
        if c > threshold: #C
            if tokenizer is None:
                print(f"Vocab index {i}: {c.item()}x")
            else:
                print(f'{tokenizer.decode([i])}: {c.item()}x')

#A Draw samples according to probabilities
#B Count how often each index was selected
#C Print frequently sampled vocabulary indices (entries)

```

This `count_samples` function in listing 4.7 samples token indices from a probability distribution and counts how often each token is drawn. It uses `torch.multinomial` to randomly select `num_samples` indices proportional to their probabilities. The `replacement=True` setting allows us to draw the same token multiple times.

Then, `torch.bincount` counts how often each index appears. Finally, it prints only tokens that occur more than a specified threshold, so that it doesn't clutter the output with more infrequently drawn tokens.

### MULTINOMIAL SAMPLING

Multinomial sampling is the procedure used to pick the next token given a probability distribution over the vocabulary, like the softmax probability scores. In multinomial sampling, instead of always selecting the most likely token (known as *greedy decoding*), we draw one token at random, where tokens with higher probability are more likely to be selected. This randomness is important for generating diverse responses, which we later use in self-consistency.

To illustrate it further we can think of the probability scores as a set of weighted choices. For example, a token with probability 0.40 is four times as likely to appear as one with probability 0.10, but both remain possible outcomes.

Suppose the model assigns the following probabilities:

- "Berlin": 0.70
- "Munich": 0.20

- "Hamburg": 0.10

Greedy decoding would always return "Berlin."

Multinomial sampling instead draws one token according to these weights. Across a very large number of draws, "Berlin" will appear most often (70% of the time), "Munich" sometimes (20% of the time), and "Hamburg" only occasionally (10% of the time).

This variability is what allows us to generate multiple candidate answers for self-consistency in later sections.

Please note that the `count_samples` function is meant for illustration only. It draws many samples from the distribution so you can see how often each token is selected. In real text generation we only draw one token at a time, but taking a large number of samples here makes the underlying probabilities easier to visualize and understand.

First, let's run the `count_samples` function on the probability scores that we obtained by applying a temperature of 5:

```
torch.manual_seed(123)
count_samples(next_token_probas, tokenizer=tokenizer)
```

The output is as follows:

```
'}': 2x
' </'': 2x
' represent': 2x
' Inf': 2x
' ()*': 2x
' beside': 2x
' Kob': 2x
' ♦': 2x
```

As we can see, even though the default sample size is 1000, none of the sampled tokens appear more than 2 times. Also, these are all nonsense tokens in the context of the "The capital of Germany is" query. The reason for these nonsensical results is that we used a temperature value that is much too high.

Let's try a lower temperature value of 0.35, which makes the score distribution sharper, and which makes it more likely to select a meaningful next token:

```

torch.manual_seed(123)
probas_lowT = torch.softmax(
    scale_logits_by_temperature(next_token_logits, 0.35), dim=-1
)

```

In this case, we see the following output:

```

' __': 158x
' Berlin': 435x
' ____': 169x
' _____': 209x
' Munich': 3x
' Hamburg': 3x
' ____': 18x

```

The output makes a lot more sense as next-token candidates for our "The capital of Germany is" query. Out of the 1000 samples, the token " Berlin" was drawn 435 times. You can check via `print(probas_lowT[0, 19_846])` that the probability of drawing this token is approximately 42% given the temperature value 0.35. To make it even more likely that " Berlin" is sampled, we could further reduce the temperature.

Notice that some of the other rarer candidates also make sense, for example, both "Munich" and " Hamburg" are big cities in Germany, so they are not completely unrelated to the query. The underscore responses (" \_\_\_\_") are likely due to the model having seen text in the form of a quiz with a placeholder, e.g., "The capital of Germany is \_\_\_\_".

You may wonder, if " Berlin" is the correct answer, what's the point in making the model occasionally give the wrong answer by adding this temperature-rescaling and multinomial sampling?

In general, for different kinds of queries, introducing randomness during sampling helps the model explore alternative responses instead of always choosing the single most likely token. This variability can be useful for creative or open-ended tasks, where there may be multiple valid completions.

Specifically, in reasoning tasks, we can leverage this sampling diversity through techniques such as self-consistency (section 4.6), which generate multiple candidate answers and compare them to improve answer accuracy.

#### 4.4.4 Adding temperature scaling to the text generation function

Before we move to the next section and introduce another improvement to the text generation process by adding a token-probability selection filter, let's add the temperature scaling modification to the text generation function so we can use it more readily when generating new tokens via the model.

**Listing 4.8 Text generation with temperature scaling**

```

from reasoning_from_scratch.qwen3 import KVCache

@torch.inference_mode()
def generate_text_temp_stream_cache(
    model,
    token_ids,
    max_new_tokens,
    eos_token_id=None,
    temperature=0.
):
    model.eval()
    cache = KVCache(n_layers=model.cfg["n_layers"])
    model.reset_kv_cache()

    out = model(token_ids, cache=cache)[:, -1]  #A
    for _ in range(max_new_tokens):

        ######
        # NEW:
        orig_device = token_ids.device

        if temperature is None or temperature == 0.0:
            next_token = torch.argmax(out, dim=-1, keepdim=True)

        else:
            logits = scale_logits_by_temperature(out, temperature)  #B
            probas = torch.softmax(logits, dim=-1)  #C
            next_token = torch.multinomial(probas.cpu(), num_samples=1)  #D
            next_token = next_token.to(orig_device)

        #####
        if (eos_token_id is not None
            and next_token.item() == eos_token_id):
            break

        yield next_token
        out = model(next_token, cache=cache)[:, -1]

#A Step 3.1: Get logits
#B Step 3.2: Apply temperature scaling on logits
#C Step 3.3: Convert to probabilities
#D Step 3.4: Sample token according to probabilities

```

The `generate_text_temp_stream_cache` in listing 4.8 is similar to the `generate_text_stream_cache` function from the chapter 2 exercises, which we also used in chapter 3. What's new is that we have now inserted the temperature rescaling and sampling. The new parts are below the `# New` comment indicator in the code.

```
torch.manual_seed(123)
response = generate_text_stream_concat_flex(
    model, tokenizer, prompt, device,
    max_new_tokens=2048, verbose=True,
    generate_func=generate_text_temp_stream_cache, #A
    temperature=1.1
)
#A Use the new temperature scaling-based text generation function
```

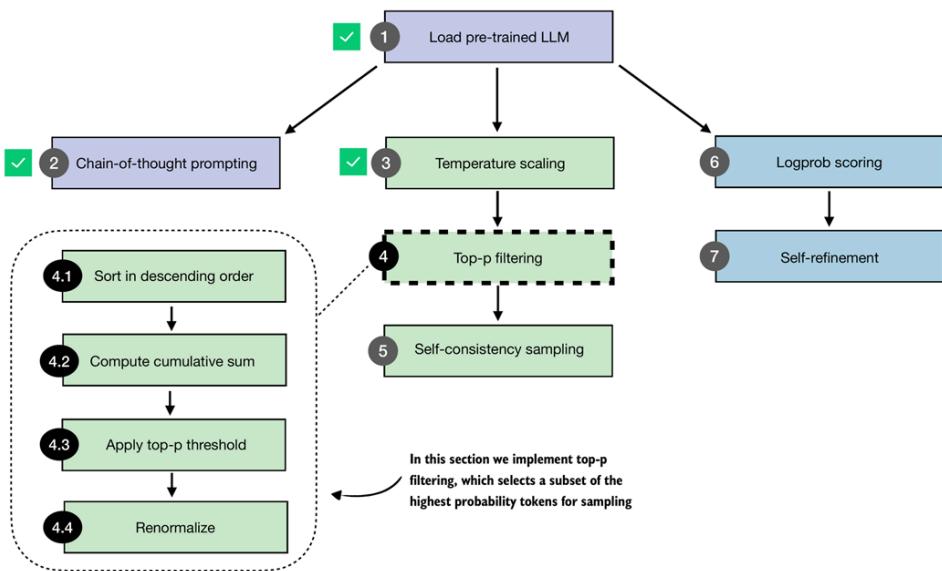
The output is `\boxed{x = \frac{90}{7}}`. The correct answer is 83, so the model is still wrong, but this was more meant as a demonstration that we can tweak the answer by using temperature scaling and sampling.

In the next section, we will learn how to improve the sampling process.

## 4.5 Balancing diversity and coherence with top-p sampling

In the previous section, we saw how temperature scaling and sampling (via `torch.multinomial`) can increase the diversity of the LLM responses, for better or for worse. Specifically, we saw that using the approach described in the previous section, we may end up sampling "weird" tokens that are unrelated to the user query.

In this section, we improve the sampling process by adding a *top-p filter* (figure 4.12) such that very low-confidence tokens are not sampled by accident. The top-p sampling process described in this section is also known as *nucleus sampling*.



**Figure 4.12 Overview of the top-p filtering process. The filter keeps only the highest-probability tokens by sorting them, applying a cumulative cutoff, selecting the top-p subset, and renormalizing the result.**

Figure 4.12 summarizes the four steps that make up the top-p filter: sorting the token probabilities (4.1), computing their cumulative sum (4.2), selecting the subset that satisfies the top-p cutoff (4.3), and renormalizing the remaining probabilities so that they again form a valid distribution (4.4).

The renormalization step is necessary because, once we remove all low-probability tokens, the remaining probabilities no longer sum to one. To sample correctly, we rescale these remaining values so they represent a proper probability distribution.

Having outlined the full top-p filtering pipeline in figure 4.12, the next subsections walk through each step in detail. We will start with a brief recap of temperature scaling and then implement steps 4.1 to 4.4 that are shown in figure 4.12.

#### 4.5.1 Selecting a subset of top-p tokens

In this section, we implement the top-p filter illustrated in figure 4.12 that we will use to improve the text generation function, which we plan to use for self-consistency sampling.

---

**NOTE** The purpose of top-p filtering in this section is to drop low-probability tokens so that only the most plausible options remain during sampling. This reduces the chance of producing tokens that do not fit the context.

---

Before we implement the top-p filter, let us briefly recap the temperature-scaling and sampling process with a simpler toy dataset, which makes it easier to illustrate the process. For instance, let's assume the model's and tokenizer's vocabularies have only 10 entries, rather than 151,936.

**Listing 4.9 Recap of temperature scaling and sampling with toy data**

```
toy_logits = torch.tensor( #A
    [-0.7, -3.0, 0.1, -1.2, 2.0, -1.0, -0.5, -2.0, 0.3, 1.5]
)

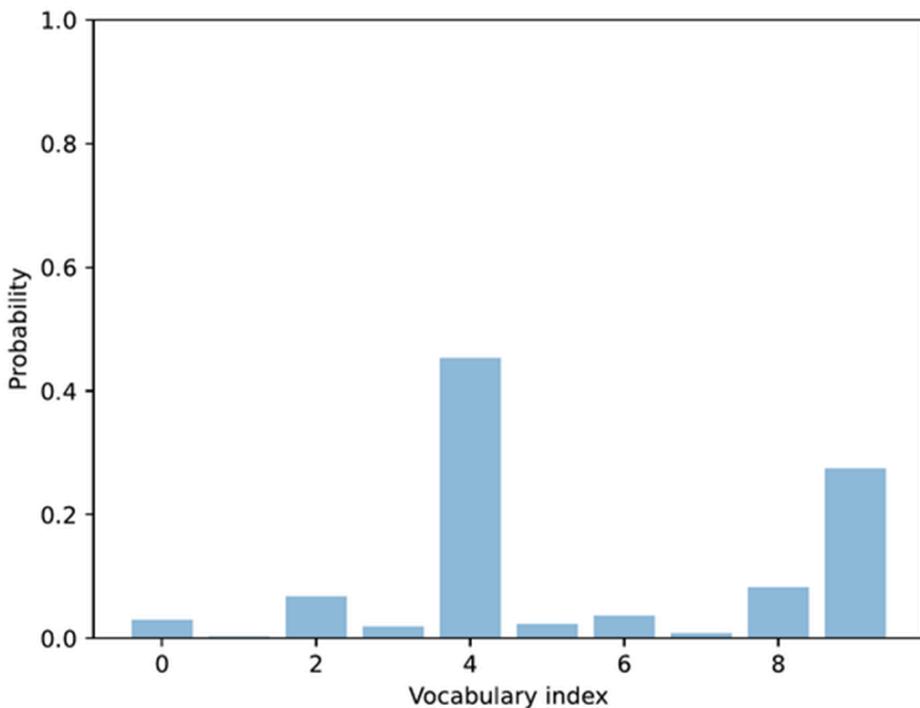
toy_logits_scaled = scale_logits_by_temperature(toy_logits, 1.0) #B
toy_probas = torch.softmax(toy_logits_scaled, dim=-1) #C

plt.bar( #D
    torch.arange(len(toy_probas)), toy_probas,
    alpha=0.5
)

plt.ylim([0, 1])
plt.xlabel("Vocabulary index")
plt.ylabel("Probability")
plt.show()

#A Step 3.1: Get logits (here: use toy logits for 10 tokens)
#B Step 3.2: Apply temperature scaling (we use 1.0 as a placeholder)
#C Step 3.3: Convert to probabilities
#D Plot probabilities in a bar plot
```

Please note that the `toy_logits` variable holds the example values that we would get for the next-token logit scores via the `model(token_ids, cache=cache)[:, -1]` call in the previous section, assuming that the vocabulary size is 10. The resulting plot is shown in figure 4.13.



**Figure 4.13 Example of token probabilities before top-p filtering. The distribution includes many low-probability tokens, which will later be truncated by applying a cumulative probability threshold.**

The bar plot in figure 4.13 shows the next-token logit scores after rescaling them to a probability score. So far, this is a recap of the previous section. Next, we will add the first two top-p filtering steps (steps 4.1 and 4.2) illustrated in figure 4.12. This involves sorting the probability scores in descending order and computing the cumulative sum.

**Listing 4.10 Compute cumulative probability sum**

```
sorted_probas, sorted_idx = torch.sort(toy_probas, descending=True) #A
cumsum = torch.cumsum(sorted_probas, dim=-1) #B

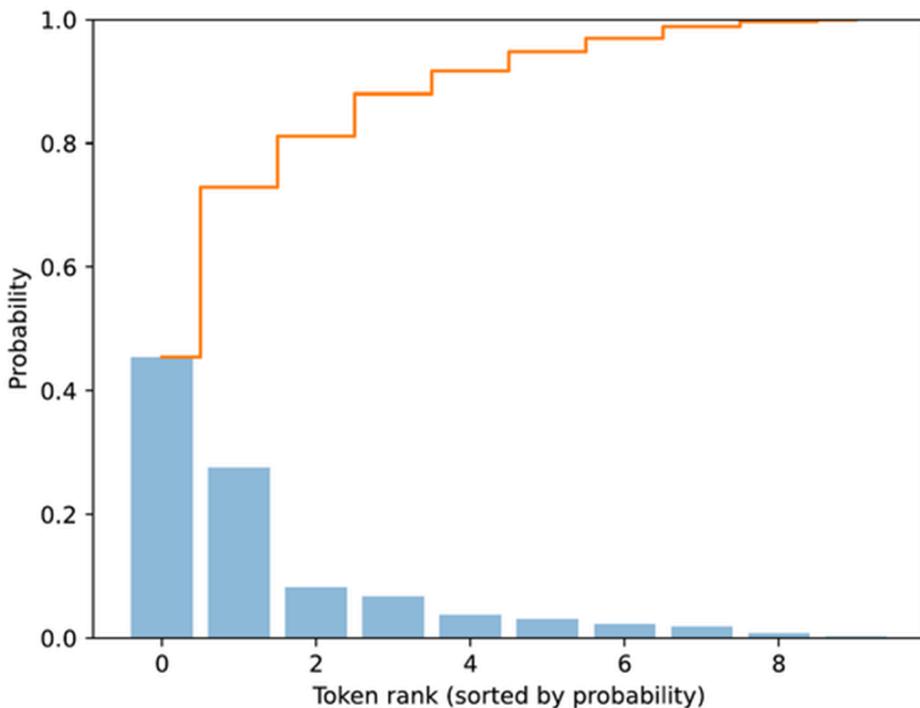
plt.bar(
    torch.arange(len(sorted_probas)), sorted_probas,
    alpha=0.5
)
plt.step(
    torch.arange(len(cumsum)), cumsum,
    where="mid", color="C1", label="Cumulative sum"
)

plt.ylim([0, 1])
plt.xlabel("Token rank (sorted by probability)")
plt.ylabel("Probability")
plt.show()

#A Step 4.1: Sort by descending probability
#B Step 4.2: Compute cumulative sum
```

The `torch.cumsum` function used in listing 4.10 above computes the cumulative sum of elements along a given dimension. In this example, it takes the sorted token probabilities and adds them up step by step, so each position in `torch.cumsum` represents the total probability accumulated up to that token.

For instance, the first element equals the highest probability, the second equals the sum of the top two, and so on, until the final value reaches 1. This is best explained by looking at the cumulative step plot, produced by the code in listing 4.10 and shown in figure 4.14.



**Figure 4.14 Visualization of sorted token probabilities and their cumulative sum. This step prepares for top-p filtering by showing how probabilities accumulate when ordered from highest to lowest, which helps determine where to set the cutoff threshold.**

Now that we have the cumulative probability sum, we can implement the core top-p filtering step. The "p" in top-p stands for probability, and top-p translates to "keep the smallest set of tokens whose cumulative probability stays below or equal to p."

**Listing 4.11 Determining how many tokens are kept in top-p filtering**

```
top_p = 0.8
keep_mask = cumsum <= top_p
n_kept = torch.sum(keep_mask).item()
```

In code, this is implemented as `keep_mask = cumsum <= top_p`, which marks all tokens whose cumulative probability mass does not yet exceed the threshold  $p$  (here, defined via `top_p=0.8`). The number of retained tokens is then computed and assigned to the variable `n_kept`.

Let's illustrate this top filtering with a plot using the code below.

**Listing 4.12 Visualizing top-p filtering**

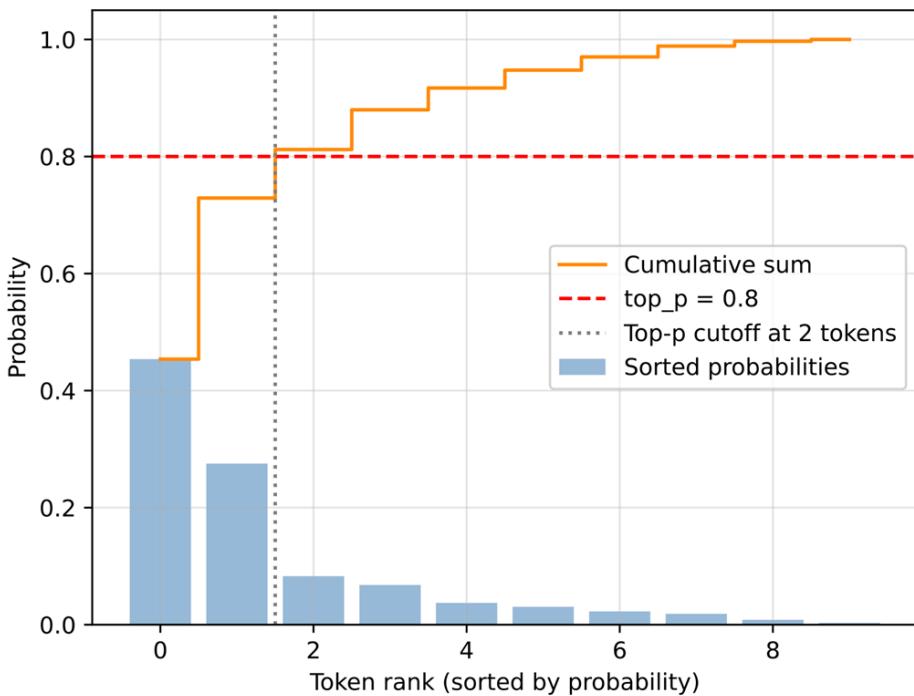
```
plt.bar(
    torch.arange(len(sorted_probas)), sorted_probas,
    alpha=0.5, label="Sorted probabilities"
)
plt.step(
    torch.arange(len(cumsum)), cumsum, where="mid",
    color="darkorange", label="Cumulative sum"
)

#A
plt.axhline(
    top_p, color="red", linestyle="--",
    label=f"top_p = {top_p}"
)
plt.axvline(
    n_kept - 0.5, color="gray", linestyle=":",
    label=f"Top-p cutoff at {n_kept} tokens"
)

plt.xlabel("Token rank (sorted by probability)")
plt.ylabel("Probability")
plt.legend()
plt.grid(alpha=0.3)
plt.ylim(0, 1.05)
plt.show()
```

#A Highlight cutoff

The plot produced from executing the code in listing 4.12 results is shown in figure 4.15.



**Figure 4.15 Illustration of top-p (nucleus) filtering. Tokens are sorted by probability, and the smallest subset whose cumulative probability exceeds the threshold ( $p = 0.8$ ) is kept for sampling.**

The plot in figure 4.15 shows the `top_p = 0.8` threshold value (dashed horizontal line) that defines which tokens are kept. In this case, the cumulative sum of the first two tokens is below the threshold, hence we exclude all other tokens to the right of the cut-off (vertical dotted line).

To implement this threshold cutoff, we can use the following code, which first zeroes out all values to the right side of the cutoff (vertical dashed line in figure 4.15) and restores the original sorting order.

#### Listing 4.13 Applying top-p filtering

```
kept_sorted = torch.where(
    keep_mask, sorted_probas,
    torch.zeros_like(sorted_probas)
)
filtered = torch.zeros_like(toy_probas).scatter(0, sorted_idx, kept_sorted)
print(filtered)
```

The resulting tensor looks as follows:

```
tensor([0.0000, 0.0000, 0.0000, 0.0000, 0.4538, 0.0000, 0.0000, 0.0000, 0.0000, 0.2752])
```

We can see that all values except the ones at index positions 4 and 9 are zeroed out. This means that if we now use the multinomial function only those two tokens will be considered. Finally, we renormalize these values so that they sum up to one again:

```
denom = torch.sum(filtered).clamp_min(1e-12)
renormalized = filtered / denom
print(renormalized)
```

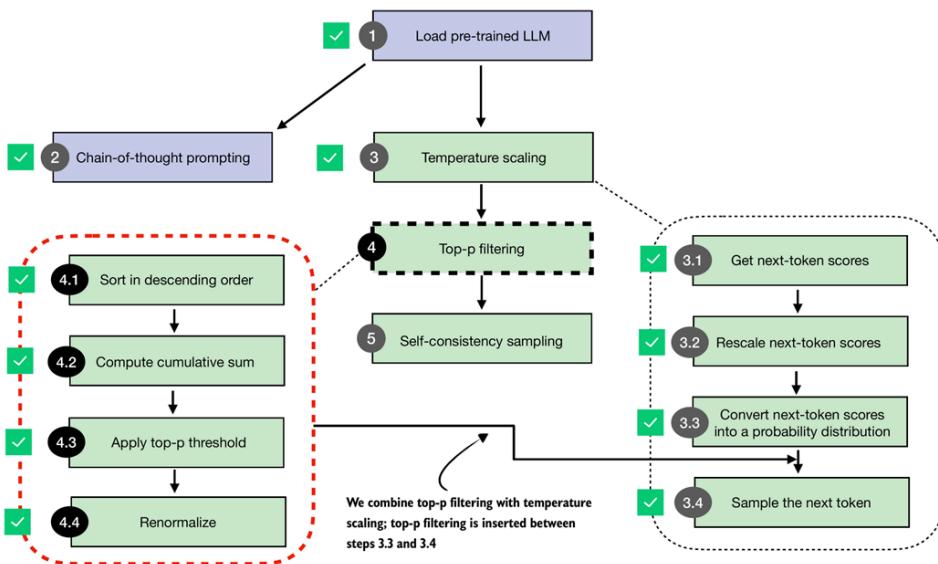
The resulting, normalized tensor is:

```
tensor([0.0000, 0.0000, 0.0000, 0.0000, 0.6225, 0.0000, 0.0000, 0.0000, 0.0000, 0.3775])
```

The goal of top-p filtering, which we implemented in this section, is to remove tokens with low probabilities to avoid them from being sampled later. This helps reduce nonsensical token responses in given contexts.

#### 4.5.2 Adding a top-p filter to the text generation function

In the previous section, we walked through the top-p filtering steps using a simple toy example with a vocabulary size 10 to be able to visualize the procedure in a bar plot. In this section, we are adding the four main top-p filtering steps (steps 4.1-4.4 in figure 4.16) to the existing text generation function.



**Figure 4.16 Integration of top-p filtering with temperature scaling. After rescaling the next-token scores, top-p filtering is applied between steps 3.3 and 3.4 to limit sampling to the most probable tokens.**

Given our previous text generation function, as illustrated in figure 4.16, we add the top-p filtering between the probability conversion and the sampling we implemented earlier in the temperature scaling section.

For this, we will first put together the top-p filtering steps from the previous section into a single, convenient function we can call.

**Listing 4.14 Top-p filtering function**

```

def top_p_filter(probas, top_p):
    if top_p is None or top_p >= 1.0:
        return probas

    sorted_probas, sorted_idx = torch.sort(probas, dim=1, descending=True) #A
    cumprobas = torch.cumsum(sorted_probas, dim=1) #B

    keep = cumprobas <= top_p #C
    keep[:, 0] = True #D

    kept_sorted = torch.where( #E
        keep, sorted_probas, #E
        torch.zeros_like(sorted_probas) #E
    )
    filtered = torch.zeros_like(probas).scatter(1, sorted_idx, kept_sorted) #F

    denom = torch.sum(filtered, dim=1).clamp_min(1e-12) #G
    return filtered / denom

#A Step 4.1: Sort by descending probability
#B Step 4.2: Cumulative sum
#C Step 4.3.1: Keep tokens where cumprob <= top_p
#D For top_p <= 0, only the highest-probability token is guaranteed to be kept as a fallback
#E Step 4.3.2: Zero out beyond cutoff
#F Step 4.3.3: Map back to original order
#G Step 4.4: Renormalize to sum to 1

```

To briefly recap what's happening in top-p filtering, the `top_p_filter` function in listing 4.13 first sorts token probabilities and computes their cumulative sum. It then keeps all tokens whose cumulative probability is below the `top_p` threshold, zeroes out the rest, and maps the kept values back to their original order. Finally, it renormalizes the remaining probabilities so they sum to 1 again.

Before we add this `top_p_filter` function to our text generation functions, let's give it a try and see how it works with the previous temperature-scaling approach. First, we get the logits:

```

with torch.inference_mode():
    next_token_logits = model(input_token_ids)[:, -1]
print(next_token_logits.shape)

```

The code above prints the dimensions [1, 151936] for the `next_token_logits` tensor, since we are now working with the real data and full vocabulary, which has 151,936 entries.

Next, we rescale the logits into probability scores and apply the temperature scaling with a temperature of 0.35, similar to before:

```
torch.manual_seed(123)
probas_lowT = torch.softmax(
    scale_logits_by_temperature(next_token_logits, 0.35), dim=-1
)
count_samples(probas_lowT, threshold=1, tokenizer=tokenizer)
```

This code is, so far, similar to what we used in the previous temperature scaling examples and prints the following sampled outputs:

```
' __': 158x
' Berlin': 435x
' ____': 169x
' _____': 209x
' Munich': 3x
' Hamburg': 3x
' _____': 18x
```

Now, let's add the top-p filter and see how it changes the results:

```
torch.manual_seed(123)
probas_lowT = torch.softmax(
    scale_logits_by_temperature(next_token_logits, 0.35), dim=-1
)
probas_lowT_filtered = top_p_filter(probas_lowT, top_p=0.8)
count_samples(probas_lowT_filtered, threshold=1, tokenizer=tokenizer)
```

With a `top_p` threshold of 0.8, which is a typical value, we get rid of the 20% lowest probability tokens, and the sampled outputs now look as follows:

```
' Berlin': 683x
' _____': 317x
```

As we can see, we now either select the correct city (and remove "Munich" and "Hamburg" as sampled options) or print the underscore token, which the model might use to format the text as a quiz question with the ' \_\_\_\_\_ ' as a placeholder.

Now that we walked through the top-p filtering process using a simple toy example, let's return to our math query and add the top-p filter to the `generate_text_temp_stream_cache` function we coded earlier. The updated function, now called `generate_text_top_p_stream_cache`, is shown in listing 4.15 below.

**Listing 4.15 Text generation with top-p filtering**

```

@torch.inference_mode()
def generate_text_top_p_stream_cache(
    model,
    token_ids,
    max_new_tokens,
    eos_token_id=None,
    temperature=0.,
    top_p=None
):
    model.eval()
    cache = KVCache(n_layers=model.cfg["n_layers"])
    model.reset_kv_cache()

    out = model(token_ids, cache=cache)[:, -1] #A
    for _ in range(max_new_tokens):

        orig_device = token_ids.device

        if temperature is None or temperature == 0.0:
            next_token = torch.argmax(out, dim=-1, keepdim=True)

        else:
            logits = scale_logits_by_temperature(out, temperature) #B
            probas = torch.softmax(logits, dim=-1) #C

            probas = top_p_filter(probas, top_p) #D

            next_token = torch.multinomial(probas.cpu(), num_samples=1) #E
            next_token = next_token.to(orig_device)

        if (eos_token_id is not None
            and next_token.item() == eos_token_id):
            break

        yield next_token
        out = model(next_token, cache=cache)[:, -1]

#A Step 3.1: Get logits
#B Step 3.2: Apply temperature scaling on logits
#C Step 3.3: Convert to probabilities
#D (New) Step 4: Apply top-p filter to probabilities
#E Step 3.4: Sample token according to probabilities

```

To conclude this section, let's plug this new text generation function into the `generate_text_stream_concat_flex`, similar to what we have done with the temperature-scaling version of the text generation function before:

```
torch.manual_seed(123)
response = generate_text_stream_concat_flex(
    model, tokenizer, prompt, device,
    max_new_tokens=2048, verbose=True,
    generate_func=generate_text_top_p_stream_cache,
    temperature=0.5,
    top_p=0.8,
)
```

The output is "`\boxed{18}`", which is still not correct. However, note that everything we have done up to this point is mainly for the purpose of being able to sample different outputs. In the next section, we will use the `generate_text_stream_concat_flex` with our augmented text generation function (`generate_text_top_p_stream_cache`) to sample different outputs when implementing the self-consistency inference-time scaling technique.

## TOP-K FILTERING

Top-k filtering is another way to limit the set of candidate next tokens during sampling.

Instead of keeping all tokens whose cumulative probability stays below a threshold (as in top-p), top-k keeps only the k most likely tokens based on the model's logits.

After sorting the vocabulary by probability, everything past the first k entries is removed.

The remaining k tokens are then renormalized and sampled from.

In short, top-k keeps a fixed number of the most likely tokens, whereas top-p keeps a variable number of tokens depending on their cumulative mass.

Top-k is simpler to implement, and I covered it in my Build a Large Language Model (From Scratch) book. However, top-p sampling has become more popular recently.

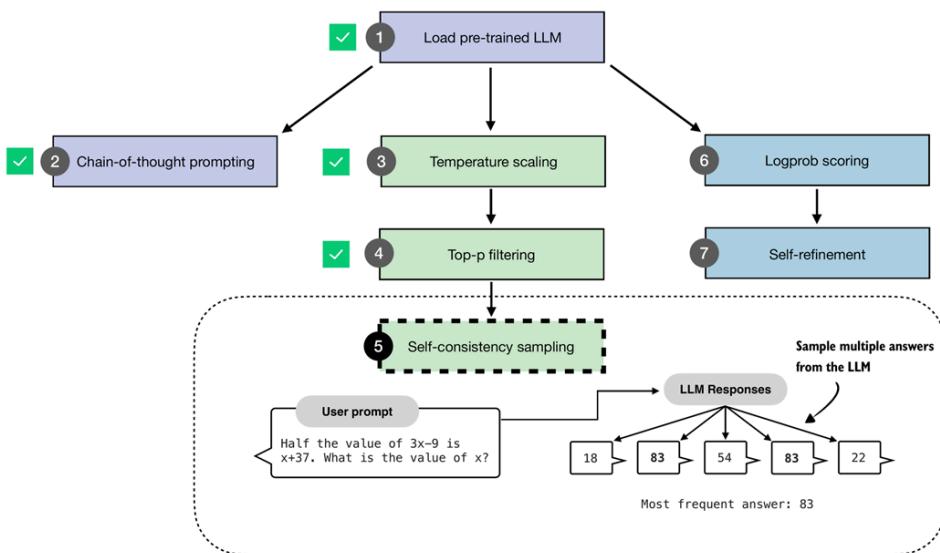
#### EXERCISE 4.2: USE TEMPERATURE SCALING AND TOP-P FILTERING ON MATH-500

Modify the `evaluate_math500_stream` function in section 3.9 of chapter 3 to see if adding temperature scaling and top p sampling changes the MATH-500 accuracy of the base model. You can use a setting of 0.9 for both temperature scaling and top-p.

## 4.6 Improving response accuracy with self-consistency

After all the previous work on coding temperature scaling and top-p filtering, we are now ready to implement the second inference-time scaling technique in this chapter: self-consistency sampling.

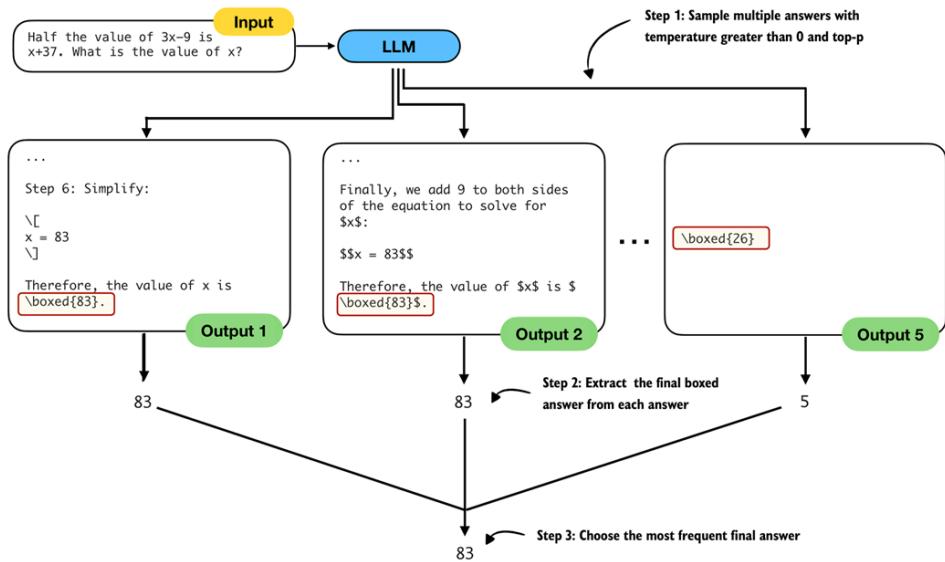
The idea behind self-consistency sampling was formally introduced in the Google Research paper *Self-Consistency Improves Chain-of-Thought Reasoning in Language Models* (<https://arxiv.org/abs/2203.11171>). However, despite the fancy name, it's essentially a form of simple majority voting, where we use temperature scaling and top-p filtering to generate multiple answers and then select the most frequent one, as shown in figure 4.17.



**Figure 4.17** The self-consistency sampling method generates multiple responses from the LLM and selects the most frequent answer, which improves answer accuracy through majority voting across these sampled responses.

The self-consistency sampling technique shown in figure 4.17 counts as an inference-time scaling technique, since we don't update the model itself and we expend more compute resources to improve response accuracy (more on the accuracy later, after we implement and try this technique).

The self-consistency code implementation, thanks to the `generate_text_stream_concat_flex` function, is relatively straightforward, and the main procedure can be summarized in three main steps as illustrated in figure 4.18.



**Figure 4.18 The three main steps for implementing self-consistency sampling.** First, we generate multiple answers for the same prompt using a temperature greater than zero and top-p filtering to generate different answers. Second, we extract the final boxed answer from each generated solution. Third, we select the most frequently extracted answer as the final prediction.

To implement the three steps illustrated in figure 4.18, in listing 4.14 below, we are simply calling `generate_text_stream_concat_flex` repeatedly to generate multiple answers and reuse the `extract_final_candidate` function from chapter 3. The only new code is the majority voting based on the answer frequency.

**Listing 4.16 Text generation with top-p filtering**

```

from reasoning_from_scratch.ch03 import extract_final_candidate
from collections import Counter

def self_consistency_vote(
    model, tokenizer, prompt, device,
    num_samples=10, temperature=0.8, top_p=0.9, max_new_tokens=2048,
    show_progress=True, show_long_answer=False, seed=None,
):
    full_answers, short_answers = [], []

    for i in range(num_samples): #A
        if seed is not None:
            torch.manual_seed(seed + i + 1)

        answer = generate_text_stream_concat_flex(
            model=model, tokenizer=tokenizer, prompt=prompt, device=device,
            max_new_tokens=max_new_tokens, verbose=show_long_answer,
            generate_func=generate_text_top_p_stream_cache,
            temperature=temperature, top_p=top_p,
        )

        short = extract_final_candidate( #B
            answer, fallback="number_then_full" #B
        )
        full_answers.append(answer)
        short_answers.append(short)
        if show_progress:
            print(f"[Sample {i+1}/{num_samples}] → {short!r}")

    counts = Counter(short_answers)
    groups = {s: [] for s in counts}
    for idx, s in enumerate(short_answers):
        groups[s].append(idx)

    mc = counts.most_common() #C
    if not mc:
        majority_winners, final_answer = [], None
    else:
        top_freq = mc[0][1]
        majority_winners = [s for s, f in mc if f == top_freq]
        final_answer = mc[0][0] if len(majority_winners) == 1 else None

```

```

    return {
        "full_answers": full_answers,
        "short_answers": short_answers,
        "counts": dict(counts),
        "groups": groups,
        "majority_winners": majority_winners,
        "final_answer": final_answer,
    }
}

```

#A 1) Sample multiple answers

#B 2) Extract the final (short) answer from each answer

#C 3) Choose the most frequent final answer (self-consistency vote)

In short, the self-consistency method in listing 4.15 above

1. Samples multiple answers with temperature greater than 0 and top-p
2. Extracts the final boxed answer from each answer
3. Chooses the most frequent final answer

Note that in our implementation, we use a for-loop to generate these answers sequentially. In practice, it is also common to generate the answers on different devices so that the sampling can be parallelized.

Furthermore, in the code above, we set the random seed for each round individually:

```

if seed is not None:
    torch.manual_seed(seed + i + 1)

```

Technically, this is not necessary and seeing the random seed once should be sufficient to generate diverse samples. However, this explicit seeding is useful if we want to individually rerun some of the rounds.

Let's try out this function in practice and see what we get:

```

results = self_consistency_vote(
    model,
    tokenizer,
    prompt,
    device=device,
    num_samples=5,
    temperature=0.8,
    top_p=0.9,
    max_new_tokens=2048,
    seed=123,
    show_progress=True,
)

```

The printed results are shown below:

```
[Sample 1/5] → '83'
[Sample 2/5] → '22'
[Sample 3/5] → '54'
[Sample 4/5] → '83'
[Sample 5/5] → '26'
```

And the final answer, since 83 is the most frequent answer, is 83 (we can access this number programmatically via `results["final_answer"]`). If you want to read the explanation and full answer, you can access one of the correct solutions returning 83, for example `print(results["full_answers"][3])`, which prints:

```
To solve the equation, we first need to find the value of $3x-9$ by setting
it equal to twice the value of $x+37$:
$$3x-9 = 2(x+37)$$
Next, we distribute the 2 on the right side of the equation:
$$3x-9 = 2x+74$$
Now, we can subtract $2x$ from both sides of the equation to
isolate the variable:
$$x-9 = 74$$
Finally, we add 9 to both sides of the equation to solve for $x$:
$$x = 83$$
Therefore, the value of $x$ is $\boxed{83}$.
```

Finally, with this self-consistency approach, we were able to generate the correct answer. (Note that the results may vary when executing the code on an "mps" or "cuda" device.)

The `self_consistency_vote` function currently also does not handle ties, so if multiple samples have the same frequency, it returns `None` as the final answer. We will implement a scoring method in the next chapter to calculate the confidence of a given answer, which can be used as a tie-breaker.

### EXERCISE 4.3: USE SELF-CONSISTENCY SAMPLING ON MATH-500

Modify the `evaluate_math500_stream` function in section 3.9 of chapter 3 to evaluate whether self-consistency sampling improves the MATH-500 accuracy of the base model. Use a sample size of 3 and set both the temperature and top-p value to 0.9.

As part of this exercise, implement tie-breaking so that ties are resolved by the first answer appearing in the sample list. For example, if the samples answers are 13, 15, 13, 15, 16, the selected answers should be 13.

Note that you don't have to modify the `self_consistency_vote` itself to implement this tie-breaking, but you can work with the `results` dictionary returned by the function to implement this simple tie-breaking rule.

### EXERCISE 4.4: EARLY STOPPING IN SELF-CONSISTENCY SAMPLING

To improve computational efficiency, implement an early-stopping version of self-consistency that ends sampling once more than half of the answers agree.

### CHOOSING TEMPERATURE AND TOP-P SETTINGS

For reasonable results, temperature settings are usually between 0.5 and 0.9, and a top-p value of 0.7 to 0.9 is typical.

As a rule of thumb, if all (long) answers look nearly identical, it's a good idea to increase the temperature gently to increase diversity. Here, "long" answer means the full answer before extracting the final boxed result. You can inspect the long answers in the `results` dictionary returned by the `self_consistency_vote` function or run the function with the `show_long_answer=True` setting.

If (long) answers look off or nonsensical, it usually means the temperature is too high and should be decreased.

You may recall that the aforementioned paper title that described this technique was *Self-Consistency Improves Chain-of-Thought Reasoning in Language Models*. Where or how does the "chain-of-thought reasoning" aspect factor into all of this?

Chain-of-thought reasoning here simply refers to the chain-of-thought prompting we introduced earlier in this chapter, when we modified the prompt via "\n\n Explain step by step." so that the LLM generates longer responses.

So, let's combine chain-of-thought prompting with self-consistency sampling:

```
results = self_consistency_vote(  
    model,  
    tokenizer,  
    prompt + "\n\nExplain step by step.",  
    device=device,  
    num_samples=5,  
    temperature=0.8,  
    top_p=0.9,  
    max_new_tokens=2048,  
    seed=123,  
    show_progress=True,  
)
```

In this case, all 5 answers are 83 (correct), likely because the question is relatively simple for the LLM when using a chain-of-thought.

Instead, it would be more interesting to see how well the model performs on the entire MATH-500 dataset from the previous chapter. The results from various experiments are shown in table 4.1.

**Table 4.1 MATH-500 task accuracy for different methods**

	<b>Method</b>	<b>Model</b>	<b>Accuracy</b>	<b>Time</b>
1	Baseline (chapter 3), greedy decoding	Base	15.2%	10.1 min
2	Baseline (chapter 3), greedy decoding	Reasoning	48.2%	182.1 min
3	Chain-of-thought prompting ("CoT")	Base	40.6%	84.5 min
4	Temperature and top-p ("Top-p")	Base	17.8%	30.7 min
5	"Top-p" + Self-consistency (n=3)	Base	29.6%	97.6 min
6	"Top-p" + Self-consistency (n=5)	Base	27.8%	116.8 min
7	"Top-p" + Self-consistency (n=10)	Base	31.6%	300.4 min
8	"Top-p" + "CoT"	Base	33.4%	129.2 min
9	Self-consistency (n=3) + "Top-p" + "CoT"	Base	42.2%	211.6 min
10	Self-consistency (n=5) + "Top-p" + "CoT"	Base	48.0%	452.9 min
11	Self-consistency (n=10) + "Top-p" + "CoT"	Base	52.0%	862.6 min
12	Self-consistency (n=3) + "Top-p" + "CoT"	Reasoning	55.2%	544.4 min

The accuracy values shown in table 4.1 were computed on all 500 samples in the MATH-500 test set using a "cuda" GPU (DGX Spark). Let's go through the results one by one. The n=3 abbreviation means that we used a sample size of 3 in self-consistency sampling.

Rows 1 and 2 show the results of the base and reasoning variants using the code from chapter 3, meaning we use the text generation function without temperature scaling or top-p filtering. This text function simply selects the token with the highest score at each step, which is also known as greedy decoding. We can see that the reasoning variant has approximately 3 times the accuracy but also substantially increased runtime, since it generates more tokens.

Next, in row 3, we see the results for the chain-of-thought prompting approach via the "\n\nExplain step by step." prompt modification. As we can see, this boosts the base model's accuracy from approximately 15% to 40%.

Row 4 shows what happens when we add temperature-scaling and top-p sampling to the base model. (All experiments involving temperature scaling and top-p sampling in table 4.1 used a setting of 0.9 for both.) As we can see, the accuracy over the base model row 1 is only moderately improved from 15.2% to 17.8%. This is expected because temperature-scaling and top-p scaling merely help us to control the sampling diversity but are not inference-time scaling techniques themselves. We can also see that the runtime increased from 10.1 min to 30.7 min. This is not due to the sampling code overhead, but rather because the model now generates longer responses in some cases.

Rows 5-7 show the self-consistency scaling results. We can see that increasing the number of samples from 3 to 10 further improves accuracy to 31.6%, but it also significantly increases runtime. In this case, there is almost no accuracy advantage to using 10 instead of 3 samples.

Row 8 shows temperature-scaling and top-p sampling combined with chain-of-thought prompting. In this case, the sampling makes the chain-of-thought results worse (33.4% compared to the 40.6% in row 3). However, when combining chain-of-thought prompting with self-consistency, we can see that accuracy improves to 52% with a sample size of 10, but this also substantially increases runtime to a staggering 862.9 min.

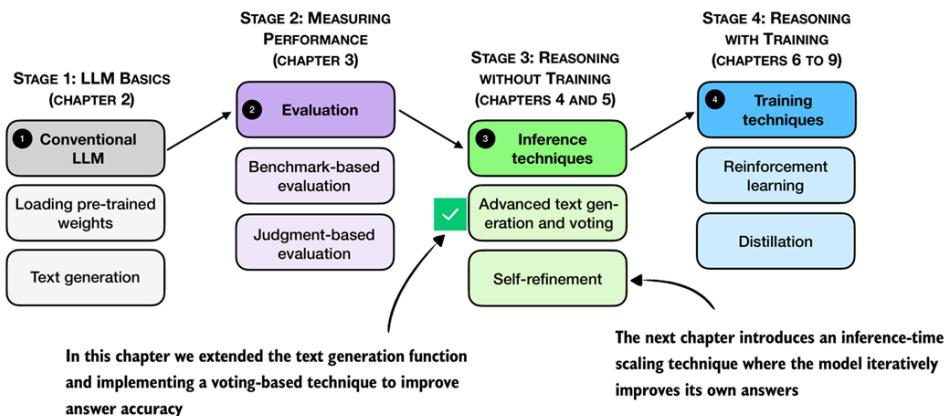
However, note that in practice, if you have access to multiple GPUs, the different samples in self-consistency sampling can be computed in parallel rather than sequentially. This would still use the same amount of compute, but it could be distributed and parallelized to generate the results faster.

Lastly, in row 12, we can see that the reasoning variant benefits from self-consistency sampling as well, taking the performance of the reasoning variant (row 2) from 48.2% to 55.2% accuracy. Again, this comes at an increased runtime.

The takeaway is that the results in table 4.1 nicely highlight the trade-off in inference-time scaling, where we trade better accuracy for more compute.

Note that one major downside of the self-consistency sampling approach in this chapter is that it relies on a final boxed answer that we can extract for majority voting. This method is trickier to apply to problems that don't have numeric or short final answers.

In the next chapter, as illustrated in figure 4.19, we will implement a different and more versatile inference-time scaling method called self-refinement, in which the model iteratively improves its own answers.



**Figure 4.19 Summary of this chapter's focus on inference-time techniques. Here, the text generation function was extended with a voting-based method to improve answer accuracy. The next chapter introduces self-refinement, in which the model iteratively improves its responses.**

## 4.7 Summary

- Reasoning abilities and answer accuracy can be improved without retraining the model by increasing compute at inference time (inference-time scaling).
- This chapter focuses on two such techniques: chain-of-thought prompting and self-consistency; a third method, self-refinement, which was briefly described, will be covered in for the next chapter.
- A flexible text generation wrapper (`generate_text_stream_concat_flex`) that uses different sampling strategies that can be plugged in without changing the surrounding code.
- Next tokens are produced from logits via softmax
- Temperature scaling changes logits to control the diversity of the generated text.
- Top-p (nucleus) sampling filters out low-probability tokens to reduce the chance of generating nonsensical answers
- Chain-of-thought prompting (like "Explain step by step." or similar) often yields more accurate answers by encouraging the model to write out intermediate reasoning, though it increases the number of generated tokens and thus increases the runtime cost.
- Self-consistency sampling generates multiple answers, extracts the final boxed result from each, and selects the most frequent answer via majority vote to improve the answer accuracy.

- Experiments on the MATH-500 dataset show that combining chain-of-thought prompting with self-consistency can substantially boost accuracy compared to the baseline without sampling, at the cost of much longer runtimes.
- The central trade-off of inference-time scaling: higher accuracy in exchange for more compute.

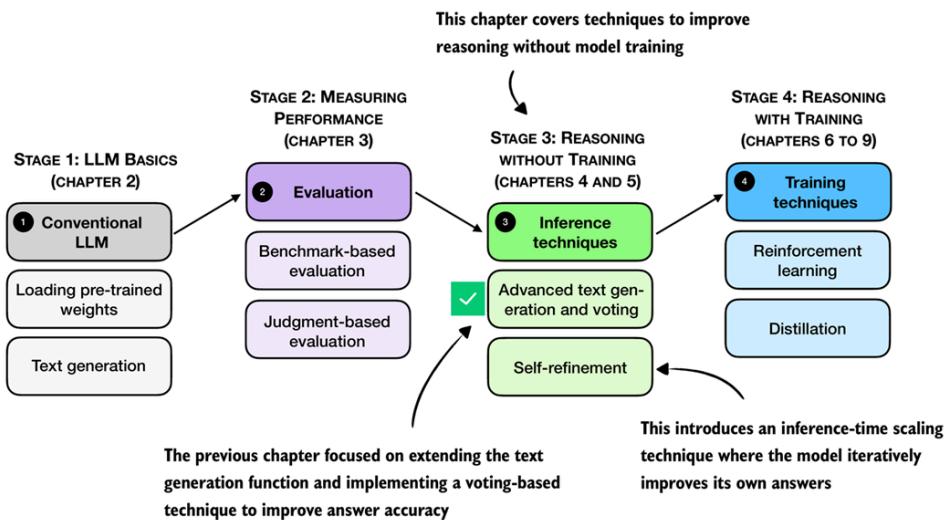
## 5 *Inference-time scaling via self-refinement*

### This chapter covers

- Scoring LLM answers with a simple rule-based scorer
- Computing an LLM's own confidence in its answers
- Coding a self-refinement loop where the LLM iteratively improves its answers

The previous chapter introduced the concept of *inference-time scaling* (*inference scaling* for short), which improves the model response accuracy without further training the model. In particular, the focus of the previous chapter was on *self-consistency*, where the model generates multiple answers, and the final answer is chosen by *majority vote*.

As outlined in figure 5.1, this chapter moves beyond the simple majority voting for inference scaling and covers another popular and useful inference-scaling technique, *self-refinement*. Instead of generating multiple answers to choose from, self-refinement focuses on iteratively refining a single answer to correct potential mistakes.

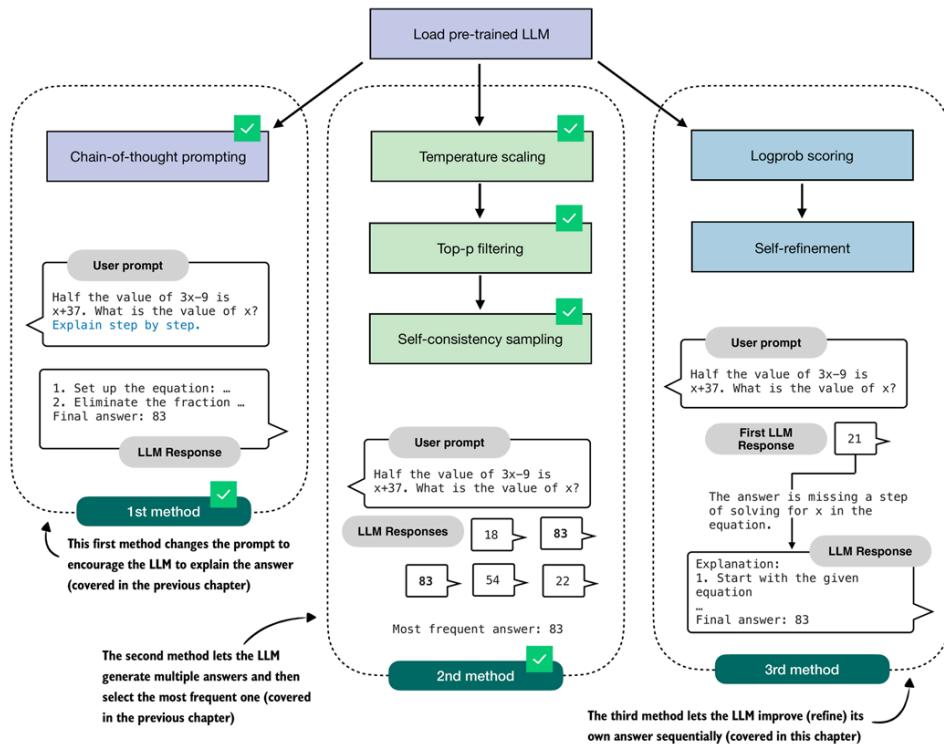


**Figure 5.1** A mental model of the topics covered in this book. This chapter continues stage 3 and focuses on inference-time techniques for improving reasoning without additional training. This chapter introduces self-refinement, where the model iteratively critiques and improves its own answers.

## 5.1 Scoring and iteratively improving model responses

As discussed in the previous chapter, inference scaling provides a way to trade additional compute for better accuracy. We also covered two inference scaling techniques, *chain-of-thought prompting* and *self-consistency*.

Chain-of-thought prompting, as illustrated in figure 5.2, modifies the prompt, for example, by adding the phrase "Explain step by step.", which can trigger a base model to write longer explanations which can in turn result in better answer accuracy. This method is particularly useful for base models that don't naturally provide reasoning-like explanations. However, models trained as reasoning models usually don't benefit from this type of inference scaling, since they already explain their answers.



**Figure 5.2 Overview of three inference-time methods to improve reasoning covered in this book. The first two methods were covered in the previous chapter. This chapter covers the third method, self-refinement, where the model iteratively improves its own answers.**

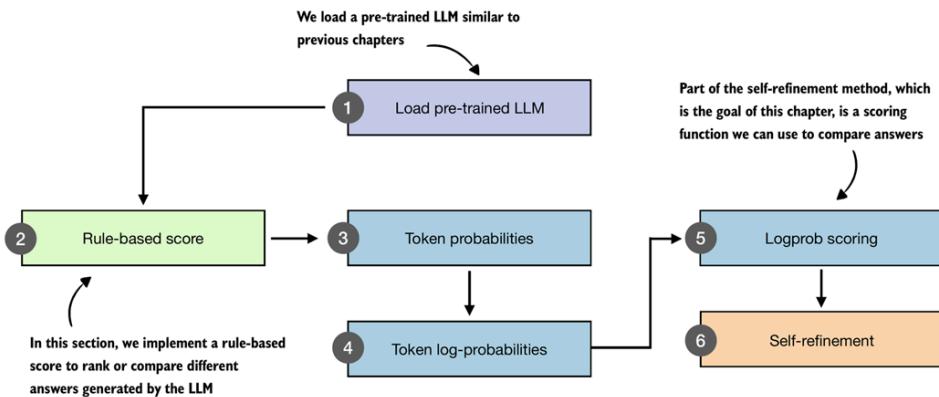
Self-consistency, the second method shown in figure 5.2, lets the model produce several answers in parallel. We then pick the final answer by taking a simple majority vote over these candidates.

Even though self-consistency is quite simple, it often yields large gains in answer accuracy, which is why it has become a common choice in LLM applications where accuracy is a higher priority than latency. Recent examples include DeepSeekMath-V2 and Google's Gemini 3 Deep Think mode (see appendix A for references).

However, one downside of self-consistency is that majority voting requires short answers that can be compared.

In this chapter, we implement a more versatile technique, *self-refinement*, in which the LLM learns to improve its own answers iteratively (method 3 in figure 5.2).

But before we implement the self-refinement technique, we will first implement scoring functions that we will use to compare and rank different answers, as outlined in figure 5.3.



**Figure 5.3 Overview of the components introduced in this chapter. We build a simple rule-based score, compute token probabilities and log-probabilities, and then use these scores as part of a self-refinement method where the model iteratively improves its own answers.**

After loading the pre-trained LLM, as in previous chapters (step 1 in figure 5.3), we will start this chapter with a simple *rule-based scoring* function to illustrate the concept of scoring (step 2). Then, we will go over the concepts of *token probabilities* (step 3) and *token log-probabilities* (step 4), which we will need to implement the *logprob scoring* method (step 5) for our self-refinement loop (step 6).

We will primarily use the logprob scoring function (step 5 in figure 5.3) to track the progress within the self-refinement loop. However, these scoring functions can also be used to break ties in self-consistency or select the best response instead of relying on a majority vote.

The chapter overview in figure 5.3, leading up to self-refinement, looks relatively short and straightforward. However, the topics of token probability and log-probability are somewhat complex and will also be relevant in the next chapter, where we implement *reinforcement learning* methods to train the LLM. So, this chapter will spend a significant portion on explaining the concept of log-probability scoring.

## 5.2 Loading a pre-trained model

As in the previous chapter, we begin by loading the model used throughout this chapter.

**Listing 5.1 Load tokenizer and base model**

```

import torch
from reasoning_from_scratch.ch02 import get_device
from reasoning_from_scratch.ch03 import (
    load_model_and_tokenizer
)

device = get_device()
device = torch.device("cpu") #A

model, tokenizer = load_model_and_tokenizer(
    which_model="base",
    device=device,
    use_compile=False
)

```

#A Delete this line to run the code on a GPU (if supported by your machine)

As in previous chapters, the code in listing 5.1 loads the model and tokenizer used throughout this chapter.

Note that the code above runs on the CPU by default to ensure results that are generally more consistent with those shown in this chapter. While small numerical differences can still occur on the CPU depending on the operating system and machine, these differences are typically smaller and more predictable than those observed across different accelerator backends.

Later sections (sections 5.4 and 5.5) use computations involving very small numbers with many digits after the decimal point, which are more sensitive to device choice and low-level implementation details. This is not an issue in practice, but such mismatches can be confusing on a first read-through. For that reason, I recommend starting with the CPU device and considering MPS or CUDA devices later.

Next, to ensure that the model is loaded correctly, let's use it together with the *temperature* and *top-p* sampler code from the previous chapter on a *MATH-500* prompt:

**Listing 5.2 Generating text with temperature scaling and top-p sampling**

```

from reasoning_from_scratch.ch03 import render_prompt
from reasoning_from_scratch.ch04 import (
    generate_text_stream_concat_flex,
    generate_text_top_p_stream_cache
)

raw_prompt = (
    "Half the value of  $3x - 9$  is  $x + 37$ . "
    "What is the value of  $x$ ?"
)
prompt = render_prompt(raw_prompt)
prompt_cot = prompt + "\n\nExplain step by step."

torch.manual_seed(0)
response_1 = generate_text_stream_concat_flex(
    model, tokenizer, prompt_cot, device,
    max_new_tokens=2048, verbose=True,
    generate_func=generate_text_top_p_stream_cache,
    temperature=0.9,
    top_p=0.9
)

```

The model produces the following response:

```

### Step 1: Understand the Problem

The problem states that half the value of  $(3x - 9)$  is  $(x + 37)$ . We need
to find the value of  $(x)$ .
... #A

### Final Answer

\[
\boxed{83}
\]

#A Response truncated to preserve space

```

Because we are using temperature scaling and top-p sampling, changing the random seed produces a different response:

```

torch.manual_seed(3)
response_2 = generate_text_stream_concat_flex(
    model, tokenizer, prompt_cot, device,
    max_new_tokens=2048, verbose=True,
    generate_func=generate_text_top_p_stream_cache,
    temperature=0.9,
    top_p=0.9,
)

```

This time, the model responds as follows:

```

We start with the given equation:
\[ 
\frac{1}{2}(3x - 9) = x + 37
\]
... #A
**Final Answer:** 
\[ 
\boxed{83}
\]

```

#A Response truncated to preserve space

In both cases, the model produces the correct final answer (83). However, the second response is much shorter, which we can confirm by printing the number of characters or tokens in each response:

```

print("Response 1 characters:", len(response_1))
print("Response 1 tokens:", len(tokenizer.encode(response_1)))
print("\nResponse 2 characters:", len(response_2))
print("Response 2 tokens:", len(tokenizer.encode(response_2)))

```

The result is:

```

Response 1 characters: 1422
Response 1 tokens: 537

Response 2 characters: 651
Response 2 tokens: 284

```

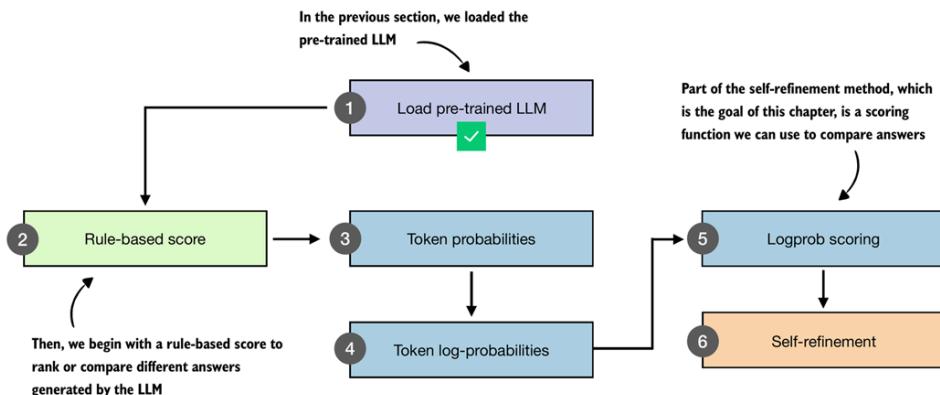
A shorter response is not necessarily better. If two responses reach the same correct answer, judging which one is better is not straightforward. It often depends on human preferences regarding clarity, usefulness, and the partial correctness of the intermediate steps.

Scoring intermediate steps of a response remains an active research area (see appendix A), and methods such as process reward models, which evaluate the reasoning itself, do not always yield better outputs in practice.

If the qualitative value of two responses is comparable, one thing is certain: shorter responses are cheaper because they require generating fewer tokens and are therefore preferred.

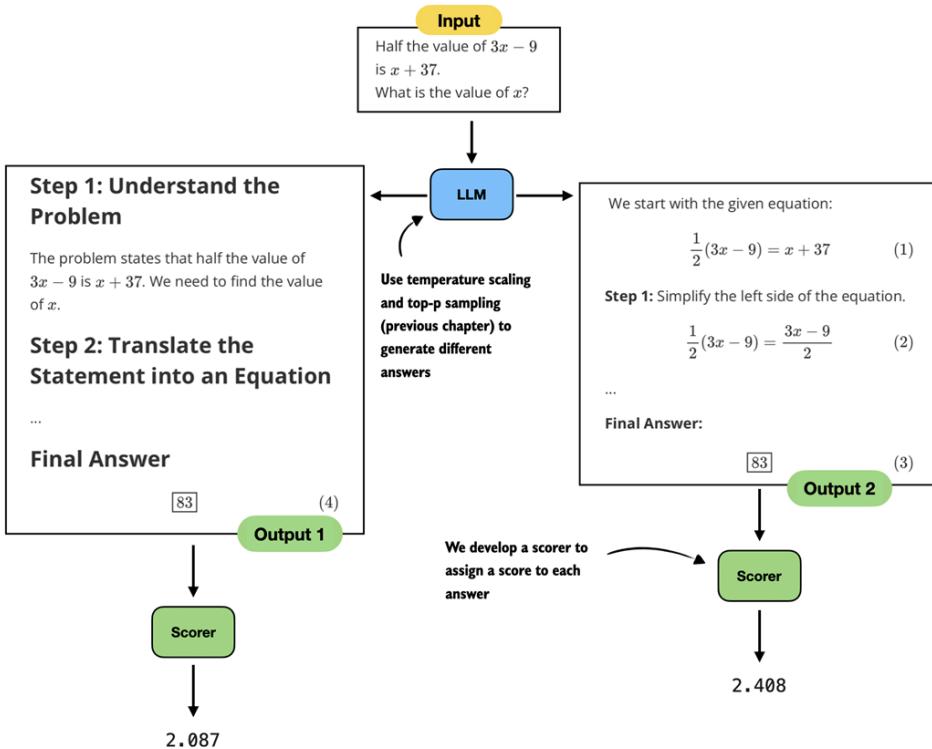
### 5.3 Scoring LLM responses with a rule-based score

In the previous section, the LLM generated two correct responses (figure 5.4). In this section, we develop a simple rule-based scoring function to compare them.



**Figure 5.4 This section implements a rule-based scorer to rank different answers generated by the pre-trained LLM.**

The rule-based scoring function assigns a score to each of two LLM responses (figure 5.5), which allows us to rank them and select the better one. Here, "better" refers to format and brevity, not correctness.



**Figure 5.5 Two generated responses reach the same correct answer but differ in their explanations. A scorer evaluates the responses and assigns a score to each response.**

There are several ways to implement a scoring function (scorer) for evaluating responses, as shown in figure 5.5. The scorer can be a *heuristic*, meaning a simple rule-based method. It can also be another LLM that rates the answers, often referred to as *LLM-as-a-judge* (see appendix F.5). Or it can rely on internal *probability scores* or *likelihoods*, which we will explore later in this chapter.

In this section, we begin with a heuristic, that is, a rule-based scorer, which we implement in listing 5.3 below:

**Listing 5.3 A simple rule-based scorer**

```

from reasoning_from_scratch.ch03 import extract_final_candidate
import math

def heuristic_score(
    answer,
    prompt=None, #A
    brevity_bonus=500.0,
    boxed_bonus=2.0,
    extract_bonus=1.0,
    fulltext_bonus=0.0,
):
    score = 0.0

    #B
    cand = extract_final_candidate(answer, fallback="none")
    if cand:
        score += boxed_bonus

    #C
    else:
        cand = extract_final_candidate(answer, fallback="number_only")
        if cand:
            score += extract_bonus
        else:
            cand = extract_final_candidate(
                answer, fallback="number_then_full"
            )
            if cand:
                score += fulltext_bonus

    #D
    score += 1.5 * math.exp(-len(answer) / brevity_bonus)
return score

```

#A A placeholder that we ignore in this section

#B Reward answers that have a final boxed value

#C Give weaker rewards if answer doesn't have a boxed value

#D Add a brevity reward that decays with text length

This `heuristic_score` assigns a numerical score to an LLM answer based on how cleanly the answer can be extracted and the length of the answer. For instance, we award a bonus if the answer has a `\boxed{}` response (`boxed_bonus`). Otherwise, we give a smaller bonus if it at least contains a number (`extract_bonus`). The `brevity_bonus` assigns points based on how short the answer is.

**NOTE** We develop a scorer and do not use the verifier from chapter 3 here, because we assume we don't know the true answer to the question. The verifier is used only for evaluation purposes when we evaluate the model on an existing test set.

The `prompt` argument is a placeholder that does not serve a purpose in the function itself, but it will make our lives easier (and the code simpler) when we develop the self-refinement function with swappable scorer plugins later in the chapter.

Although `brevity_bonus=500.0` may appear high, note that it is used as an exponentially decaying term via `1.5 * math.exp(-len(answer) / brevity_bonus)`. The plot below illustrates its effect on the score.

#### Listing 5.4 Plotting the brevity penalty curve

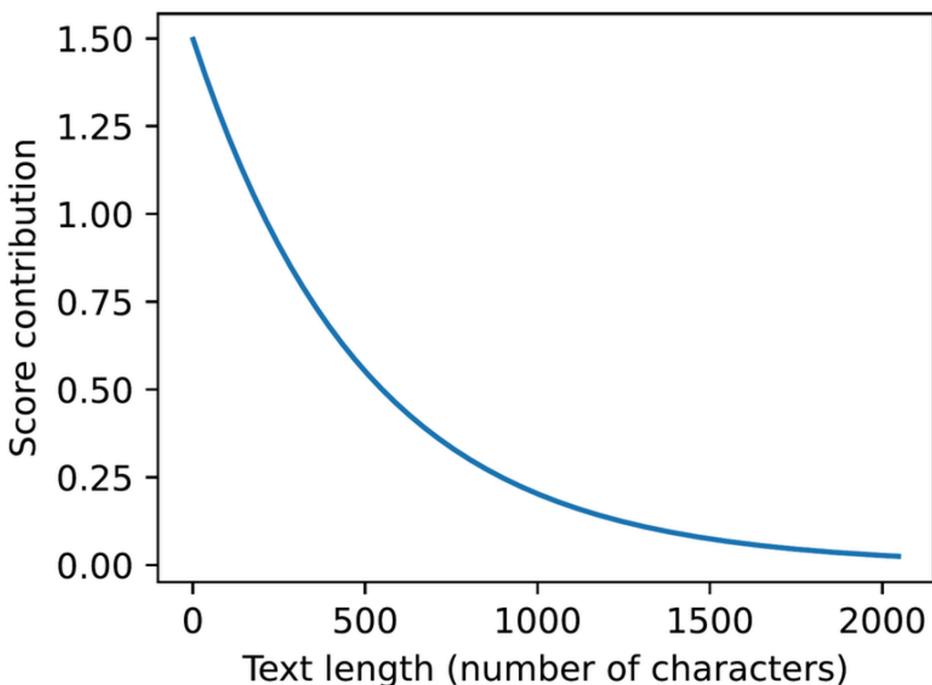
```
import matplotlib.pyplot as plt

def plot_brevity_curve(brevity_bonus, max_len=2048):
    lengths = torch.arange(1, max_len)
    scores = 1.5 * torch.exp(-lengths / brevity_bonus)

    plt.figure(figsize=(4, 3))
    plt.plot(lengths, scores)
    plt.xlabel("Text length (number of characters)")
    plt.ylabel("Score contribution")
    plt.tight_layout()
    #plt.savefig("brevity_curve.pdf")
    plt.show()

plot_brevity_curve(500)
```

The resulting plot is shown below:



**Figure 5.6 A simple rule-based length penalty used by our scorer. Longer explanations receive a smaller score contribution.**

The score bonus approaches 1.5 for shorter answers, while answers longer than 1,000 characters receive a bonus of 0.2 or less.

For simplicity, the brevity bonus is computed using the number of characters rather than tokens, which avoids passing the `tokenizer` to the scoring function. Using token counts would also be reasonable (and may even be preferable).

We now apply the heuristic scorer to the first (longer) response from the previous section:

```
print(round(heuristic_score(response_1), 3))
```

The resulting score is 2.087. Next, let's try the second (shorter) response:

```
print(round(heuristic_score(response_2), 3))
```

The computed score is 2.408, which means the second (shorter) response would be the preferred answer.

This section introduced the basic idea of scoring methods. Later in the chapter, we develop another scorer and return to the heuristic scorer when using it as part of the self-refinement method.

#### EXERCISE 5.1: USING THE HEURISTIC SCORER AS A TIE-BREAKER IN SELF-CONSISTENCY

Extend the self-consistency implementation (`self_consistency_vote`) in the previous chapter so that it can handle ties among the candidate answers. For instance, when two or more answers receive the same number of votes, apply the heuristic scorer (`heuristic_score`) from this section to the tied candidates and select the one with the highest score. Tip: You don't need to modify the `self_consistency_vote` function itself to apply the tie-breaking, but you can apply the tie-breaking to the `self_consistency_vote` function's returned `results` dictionary.

#### EXERCISE 5.2: USING THE HEURISTIC SCORER IN A BEST-OF-N SETUP

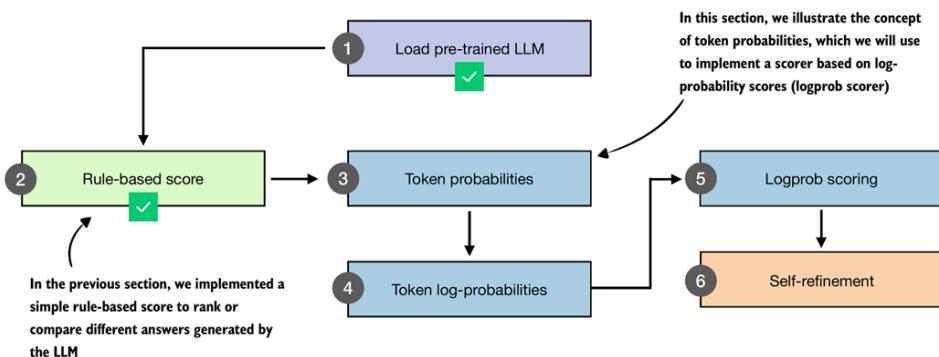
Modify the self-consistency implementation so that the final answer is chosen using the heuristic scorer rather than majority voting. Generate  $N$  candidate answers for each problem (where  $N=2$  or higher), score each candidate with the heuristic scorer, and select the one with the highest score as the final prediction. (If we use a scorer instead of majority vote, the method is called *Best-of-N* in the literature as opposed to self-consistency.)

Apply this method to a small subset of MATH-500 and compare the results to both plain Best-of-N and the self-consistency tie-breaker from the previous exercise.

Note that the scorer includes several parameters chosen by intuition, which is why we refer to it as a heuristic score. To optimize the extraction and brevity reward settings, we could plug this scorer into the self-consistency or *Best-of-N* methods from exercises 5.1 and 5.2 and evaluate which settings yield higher accuracy on a benchmark dataset such as MATH-500.

## 5.4 Understanding token probability scores

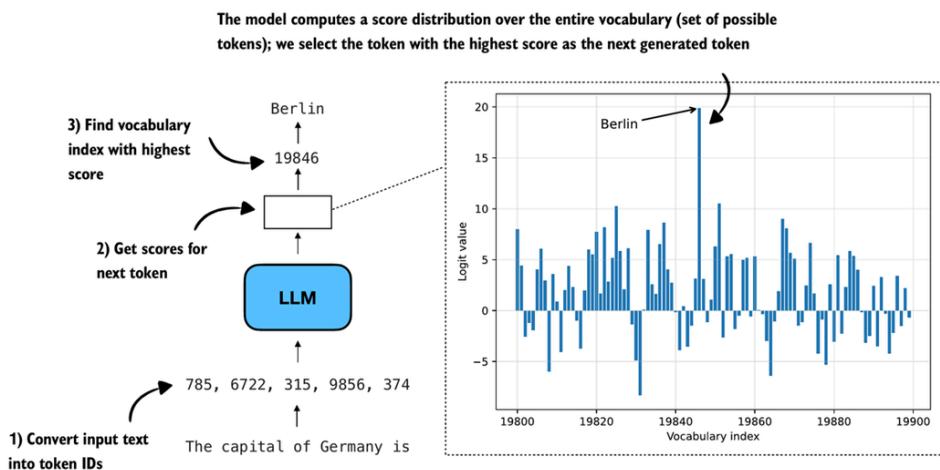
In this section, we take the first step toward building a scorer based on the model's own confidence (step 5 in figure 5.7), where confidence means model-assigned probability. Specifically, in this section, we start by computing the token probability scores for a proposed answer and use them to estimate how likely the model considers that answer to be (step 3 in figure 5.7).



**Figure 5.7** In this section, we move from a simple rule-based scorer to token-level probabilities. These probabilities form the basis for the logprob scoring method that we will use later in the self-refinement approach.

The token probability scores computed in this section are also known as *next-token probabilities*, *per-token probabilities*, *sequence likelihoods*, or loosely *token-level likelihoods* in the literature. They represent the probability the model assigns to each possible next token, expressed as a normalized (softmax) distribution over the vocabulary. Although this may sound complicated at first, it relies on the same mechanism used for text generation, which we covered in the previous chapter.

For instance, in the previous chapter, we saw that at each step the model assigns a so-called logit score to each token in the vocabulary and then selects the next token based on these scores. This process, which we discussed in section 4.4 of chapter 4, is illustrated again in figure 5.8 for reference.



**Figure 5.8 Illustration of how an LLM selects the next token. The model converts the input text into token IDs, computes a score for every vocabulary token, and chooses the token with the highest score. The plot on the right shows the logit values for a subset of the vocabulary, with the token for Berlin having the highest score.**

In the previous chapter, we looked up the vocabulary entry corresponding to the highest score (vocabulary index 19846, corresponding to the token "Berlin" in figure 5.8), to get the next generated token.

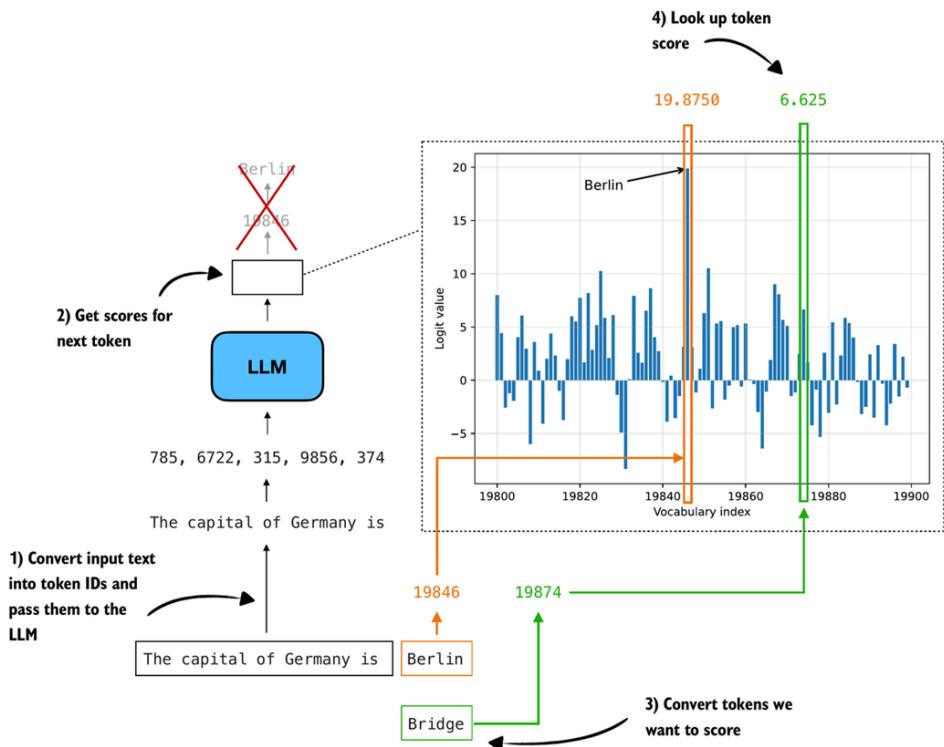
Here, we revisit the logits with a different goal in mind. Instead of generating the next token as shown in figure 5.8, we want to use the scores to quantify the model's confidence in a particular answer. In other words, we are not generating anything here based on the scores but are just inspecting the scores.

For example, consider two candidate answers we want to score:

1. "The capital of Germany is Berlin"
2. "The capital of Germany is Bridge"

The goal is to quantify how confident the model is in each answer.

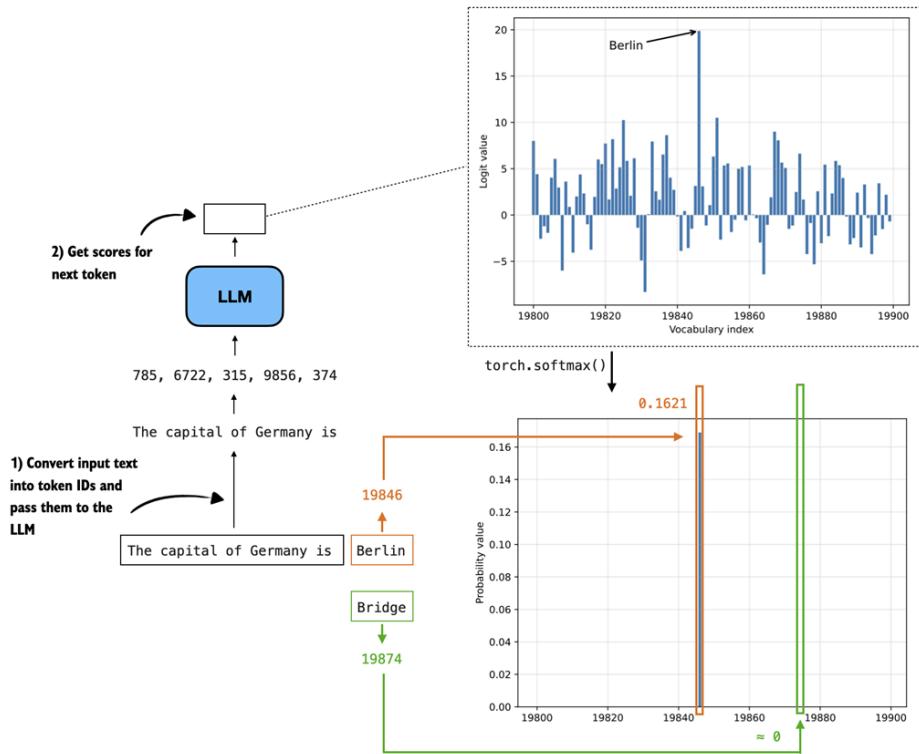
In figure 5.8, the input text "The capital of Germany is" is fed to the model, and the next token with the highest score ("Berlin") is selected. Here, instead of selecting a token, we compare the scores assigned to two candidate next tokens, "Berlin" and "Bridge", as shown in figure 5.9. ("Bridge" is used as an alternative because it appears within the vocabulary range shown in figure 5.8.)



**Figure 5.9 Illustration of how we look up the logit scores for specific tokens. After passing the input text through the model, we obtain a logit value for every vocabulary token. We then convert the candidate tokens we want to score into token IDs and read off their corresponding logit values from the distribution.**

Rather than working with the raw logits shown in figure 5.9, we convert them to probabilities. Probabilities are easier to interpret, comparable across inputs, and form the basis for the logprob scoring method used later in the chapter.

As discussed in chapter 4, applying `torch.softmax` converts logits into probability values. Figure 5.10 shows the resulting probability distribution.



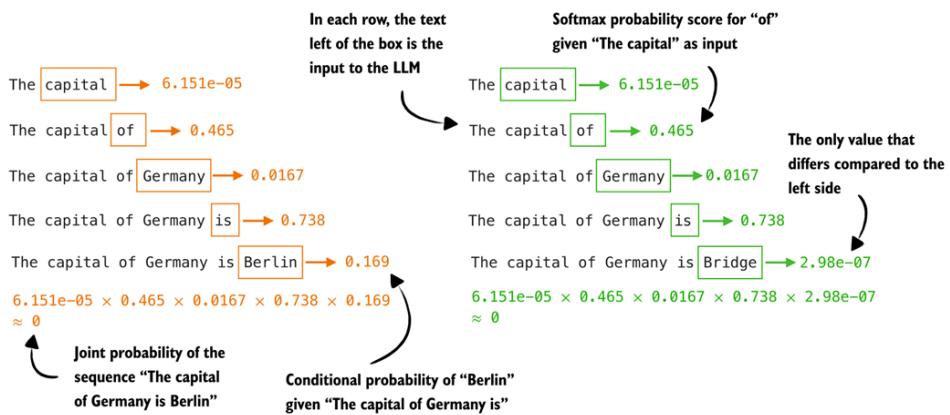
**Figure 5.10 Illustration of next-token scoring.** The input text is converted into token IDs and fed to the LLM, which outputs logits for the next token. After applying a softmax, these logits become probabilities, where tokens like "Berlin" receive high probability and unlikely alternatives such as "Bridge" receive values near zero.

The token probabilities shown in figure 5.10 are computed in the same way as in the previous chapter, using `torch.softmax`, as part of the multinomial sampling procedure described in section 4.4.3. The difference here is that we do not sample from the distribution. Instead, we simply look up the probability assigned to specific tokens.

In figure 5.10, only the bar for "Berlin" is visible, with a probability of 0.1621. The probabilities for other tokens in the plotted range are near zero. The bars shown do not sum to 1 because the full vocabulary contains 151,000 tokens, and the figure displays only a small slice of the distribution (token indices 19,800–19,900).

This indicates that the model assigns higher confidence to "Berlin" than to "Bridge" as the next token given the input text "The capital of Germany is".

In practice, however, we usually want to compare complete answers rather than individual tokens. This extension from single-token to sequence-level scoring is illustrated in figure 5.11.



**Figure 5.11 Illustration of how we compute token probability scores for a given sequence. For each position, we feed the preceding text into the model and read off the softmax probability of the next token. Multiplying these conditional probabilities yields the joint probability of the full sequence.**

As shown in figure 5.11, we compute the probability of each token given its preceding tokens. This is the same procedure as in figure 5.10, except that we repeat it for every token in the sequence. Unlike text generation, we are not producing new tokens here. Instead, we retrospectively look up the probability assigned to each token in the already generated sequence.

We then multiply these probabilities to obtain the probability of the full sequence, also called the *joint probability*.

### JOINT PROBABILITY OF A TOKEN SEQUENCE

For those comfortable with mathematical notation, the joint probability of a token sequence can be written compactly as a product of *conditional probabilities*. For a sequence  $x_1, x_2, \dots, x_T$  and model weights  $W$ , this is:

$$P(x_1, x_2, \dots, x_T | W) = \prod_{t=1}^T P(x_t | x_{1:t-1}, W)$$

Expanded, this becomes:

$$P(x_1 | W) \cdot P(x_2 | x_1, W) \cdot P(x_3 | x_{1:2}, W) \cdot \dots \cdot P(x_T | x_{1:t-1}, W)$$

This matches exactly what we compute in figure 5.11. At each position we feed the context into the model, obtain the probability of the next token, and multiply these scores across the sequence.

Ideally, the probability assigned to the sequence "The capital of Germany is Berlin" should be much higher than that of the nonsensical answer "The capital of Germany is Bridge". However, because the joint probability is obtained by multiplying many small values, the resulting probabilities for both sequences in figure 5.11 are close to zero. We address this issue in the next section.

The two answers in figure 5.11 are nearly identical, differing only in the final token ("Berlin" versus "Bridge"), which we use here for simplicity and illustration. The same method can also be applied to sequences that differ more substantially, such as the answers generated earlier in this chapter for the

MATH-500 prompt ("Half the value of  $\$3x-9\$$  is  $\$x+37\$$ . What is the value of  $\$x\$$ ?").

### HOW THIS SCORING DIFFERS FROM GREEDY AND TEMPERATURE PLUS TOP-P SAMPLING

When we compute token probability scores, we are not generating text. The full sequence already exists, and we simply query the model for the probability of each next token given the fixed context as input. So, nothing about these probabilities influences later inputs. It is a retrospective scoring procedure, not a generation step.

Generation methods like the `generate_text_stream_cache` (*greedy sampling*) and `generate_text_top_p_stream_cache` functions that we coded in chapter 4, behave very differently. Greedy sampling always selects the most likely next token, while temperature sampling and top-p sampling draw from a reshaped probability distribution. These procedures modify the distribution at each step and then commit to a single token, which becomes the input for the next position.

An important consequence is that none of these sampling strategies, including greedy sampling, is guaranteed to produce the sequence with the highest overall probability under the model. A token that looks suboptimal at one step can lead to a future step where the following tokens are much more likely. On the other hand, a locally optimal choice may steer the model towards low-probability sequences later.

Temperature and top-p sampling add even more variability by flattening or truncating the distribution before drawing samples, which may further disconnect the generated text from the globally most likely sequence.

By scoring an existing answer, we avoid all these complications. We do not run a sampling algorithm and the model does not choose any tokens. We only evaluate how probable the given sequence would have been if it had already been written.

After so much conceptual explanation, let's now see the token probability calculation in action:

#### **Listing 5.5 Calculating next token probabilities**

```
@torch.inference_mode()
def calc_next_token_probas(model, tokenizer, prompt, device):

    token_ids = torch.tensor(tokenizer.encode(prompt), device=device)

    logits = model(token_ids.unsqueeze(0)).squeeze(0)      #A
    all_probas = torch.softmax(logits, dim=-1)             #A

    #B
    t_idx = torch.arange(0, token_ids.shape[0] - 1, device=device)

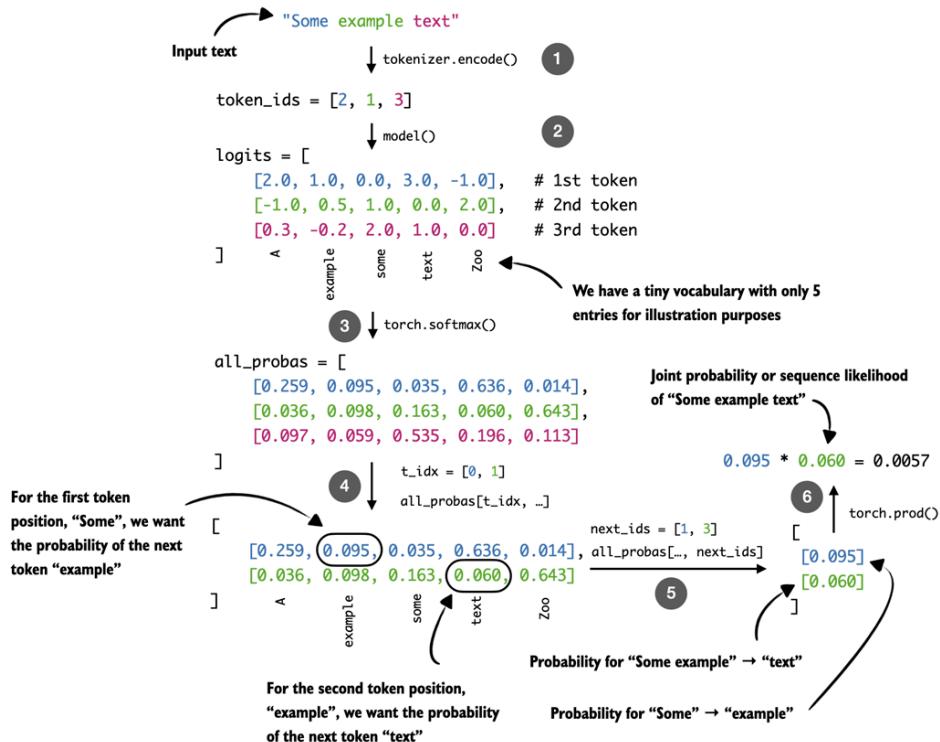
    next_ids = token_ids[1:] #C

    next_token_probas = all_probas[t_idx, next_ids] #D

    print(
        "Next-token probabilities:",
        [p.item() for p in next_token_probas]
    )
    print(
        "Joint probability:",
        torch.prod(next_token_probas) #E
    )

#A Get logits and probabilities similar to text generation functions
#B Select positions we score (here: all)
#C Since we have the text, we know the true next tokens
#D Get probabilities for each next token
#E Likelihood of the sequence is the product of the probability scores
```

As shown above, the `calc_next_token_probas` function, which computes the next-token probabilities as shown in figure 5.11, is relatively short and appears very simple at first glance. However, there's a lot happening in this function to carry out the computation, as illustrated by a simpler input text example and a tiny five-word vocabulary in figure 5.12.



**Figure 5.12 Illustration of how we extract next-token probabilities.** After converting the input text into token IDs, the model computes logits that are transformed into probabilities with a softmax function. Using index tensors for positions and true next tokens, we then get the model's computed probability for each next token.

The `calc_next_token_probas` function, illustrated with a simple example in figure 5.12, computes next-token probabilities in a few steps. First, it computes the logits in the same way as the text-generation functions introduced earlier (for example, the `out = model(token_ids)` line in `generate_text_basic` from chapter 2).

The resulting logits (step 2 in figure 5.12) have shape `[sequence_length, vocab_size]` and are then converted into normalized probability distributions using `torch.softmax` (step 3), so that the probabilities at each position sum to 1.

To extract the probability assigned to the actual next token at each position, we construct an index tensor `t_idx` for the positions we want to score and another tensor, `next_ids`, containing the corresponding target tokens. For example, if the input text is "The capital of Germany is Berlin", then `t_idx` refers to the positions corresponding to "The capital of Germany is", and `next_ids` contains the tokens shifted by one position: "capital of Germany is Berlin".

These target tokens are simply the input tokens shifted by one position, since an LLM is trained to predict the next token in a sequence. For example, given the input "The capital of Germany is Berlin", the model is asked at position 2 (the token for "capital") to predict position 3 (the token for "of"). Using `all_probas[t_idx, next_ids]` (steps 4 and 5) retrieves exactly these probability values for each position in the sequence.

Finally, the function uses `torch.prod` to compute the sequence likelihood as the product of the per-token probabilities (step 6).

Let's finally see this function in action:

```
calc_next_token_probas(
    model, tokenizer, device=device,
    prompt="The capital of Germany is Berlin"
)
```

The resulting output is:

```
Next-token probabilities: [6.151199340820312e-05, 0.46484375,
0.0167236328125, 0.73828125, 0.1689453125]
Joint probability: tensor(5.9372e-08, dtype=torch.bfloat16)
```

Next, let's try another text:

```
calc_next_token_probas(
    model, tokenizer, device=device,
    prompt="The capital of Germany is Bridge"
)
```

This results in:

```
Next-token probabilities: [6.151199340820312e-05, 0.46484375,
0.0167236328125, 0.73828125, 2.980232238769531e-07]
Joint probability: tensor(1.0481e-13, dtype=torch.bfloat16)
```

Analyzing the results above, the first response (ending in "Berlin") gave us the joint probability (sequence likelihood) of  $5.9372e-08$ , which is larger than the joint probability of the second response,  $1.0481e-13$ . (In decimal form,  $5.9372e-08$  is  $0.000000059372$ , and  $1.0481e-13$  is  $0.0000000000010481$ .)

These results make sense since the "Berlin" answer receives a higher sequence likelihood than the "Bridge" answer, as expected. However, both values are extremely small because multiplying many probabilities less than one quickly produces numbers that approach zero. This makes raw likelihoods difficult to work with in practice, especially for longer sequences where underflow becomes a problem.

**NOTE** These scores reflect the model's internal likelihood, not a calibrated probability of correctness.

In the next section, we will introduce a modification to this calculation, log-probabilities, which avoid these numerical issues and give us a more stable way to score entire sequences.

## PROBABILITIES VERSUS LIKELIHOODS

You may have noticed that this section used both terms, "probability" and "likelihood." Is there a difference? In the field of statistics, probabilities describe how likely an event is before we observe any data and must sum to one across all possible outcomes. Likelihoods, in contrast, measure how well a specific model explains observed data and are viewed as a function of the model's weights or parameters, not the data. In short, probabilities predict data given a model, while likelihoods evaluate models given data. (For a more detailed example, please see my article on probabilities versus likelihoods: <https://sebastianraschka.com/faq/docs/probability-vs-likelihood.html>)

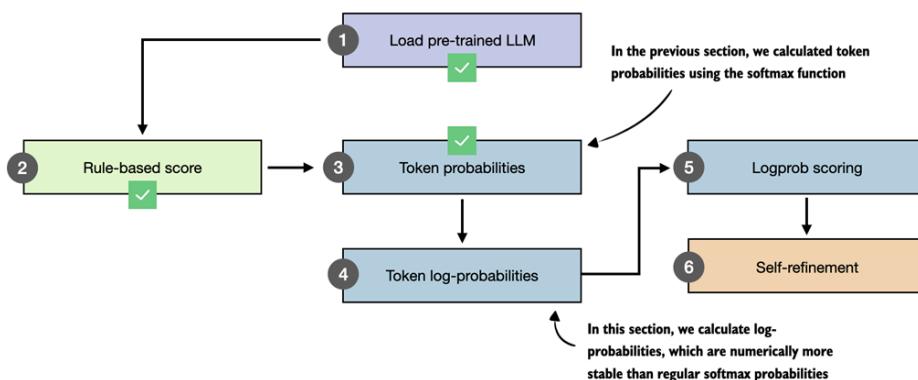
In LLMs, the next-token probability is technically a probability, not a likelihood, because it comes from a normalized distribution over all possible next tokens that sums to one. In other words, the values in `next_token_probas` are probabilities, not likelihoods. They come directly from the model's softmax over the vocabulary for each position, which is a normalized probability distribution.

The product, which is computed as the joint probability (`torch.prod(next_token_probas)`), is the model's probability assigned to the sequence. This quantity is often referred to as the sequence likelihood (especially when viewed as a function of the model parameters).

## 5.5 From token probability scores to log-probabilities

The token probabilities computed in the previous section can be used as a scoring function to rank different responses. However, as we saw, the probability scores can be very small, especially when multiplied to obtain the joint probability or sequence likelihood.

In this section, to avoid numerical stability issues that often arise when working with such small values, we apply a *logarithmic scaling* (also called *log scaling*) to these probabilities, as shown in the overview in figure 5.13.



**Figure 5.13** Overview of how we move from token probabilities to token log-probabilities, which provide a numerically more stable basis for log-probability scoring used later in self-refinement.

Note that this section is simply about applying a scaling transformation to the probabilities in the previous section, and the overall goal is still to compute the token probabilities, or, in this case, log-probabilities, as illustrated in figure 5.14.

For instance, consider this simple example computing the probabilities for a selection of logits values:

```
logits = torch.linspace(-2, 2, steps=7)
probas = torch.softmax(logits, dim=-1)
print(probas)
```

The probability values are:

```
tensor([0.0090, 0.0175, 0.0341, 0.0665, 0.1295, 0.2522, 0.4912])
```

We can turn them into log-probability scores via the `torch.log` function:

```
print(torch.log(probas))
```

The log-probability values are:

```
tensor([-4.7109, -4.0442, -3.3776, -2.7109, -2.0442, -1.3776, -0.7109])
```

Here, `torch.log` applies the mathematical, natural logarithm, where  $\log(0.0090) = -4.7109$  and vice versa  $e^{-4.7109} = 0.0090$ .

Instead of chaining `torch.log(torch.softmax(...))`, PyTorch also has an optimized `torch.log_softmax` function that combines these two operations:

```
log_probas = torch.log_softmax(logits, dim=-1)
print(log_probas)
```

Similar to before, this returns:

```
tensor([-4.7109, -4.0442, -3.3776, -2.7109, -2.0442, -1.3776, -0.7109])
```

Note that the log-scaling only changes the magnitude of the values, not the ordering, so a sequence with a higher likelihood will always have a higher log-likelihood as well. To see this visually, let's plot the logits, probabilities, and log-probabilities side by side:

**Listing 5.6 Plotting logits, softmax probabilities, and log-softmax values**

```
plt.figure(figsize=(9, 4))

#A
plt.subplot(1, 3, 1)
plt.bar(range(len(logits)), logits, color="C0", alpha=0.7)
plt.title("Logits")
plt.xlabel("Token index")
plt.ylabel("Value")
plt.grid(alpha=0.3)

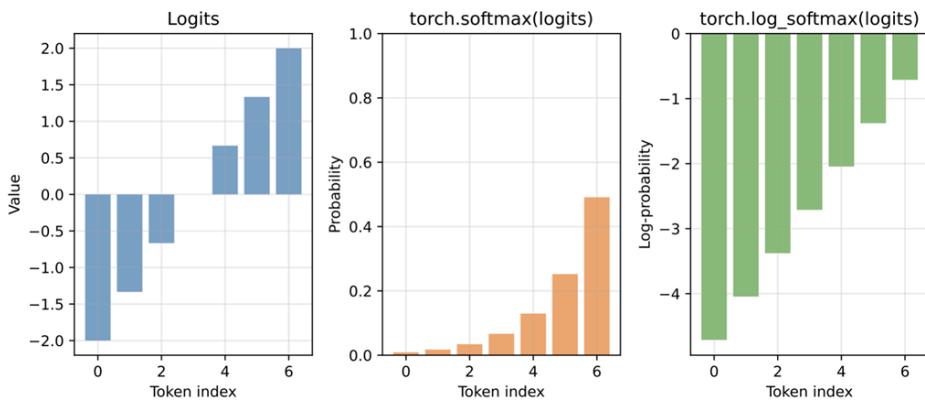
#B
plt.subplot(1, 3, 2)
plt.bar(range(len(probas)), probas, color="C1", alpha=0.7)
plt.title("torch.softmax(logits)")
plt.xlabel("Token index")
plt.ylabel("Probability")
plt.ylim(0, 1)
plt.grid(alpha=0.3)

#C
plt.subplot(1, 3, 3)
plt.bar(range(len(log_probas)), log_probas, color="C2", alpha=0.7)
plt.title("torch.log_softmax(logits)")
plt.xlabel("Token index")
plt.ylabel("Log-probability")
plt.grid(alpha=0.3)

plt.tight_layout()
plt.savefig("logits_softmax_log_softmax.pdf")
plt.show()

#A Plotting logits
#B Plotting softmax probabilities
#C Plotting log-softmax values
```

The resulting plot is shown in figure 5.14.

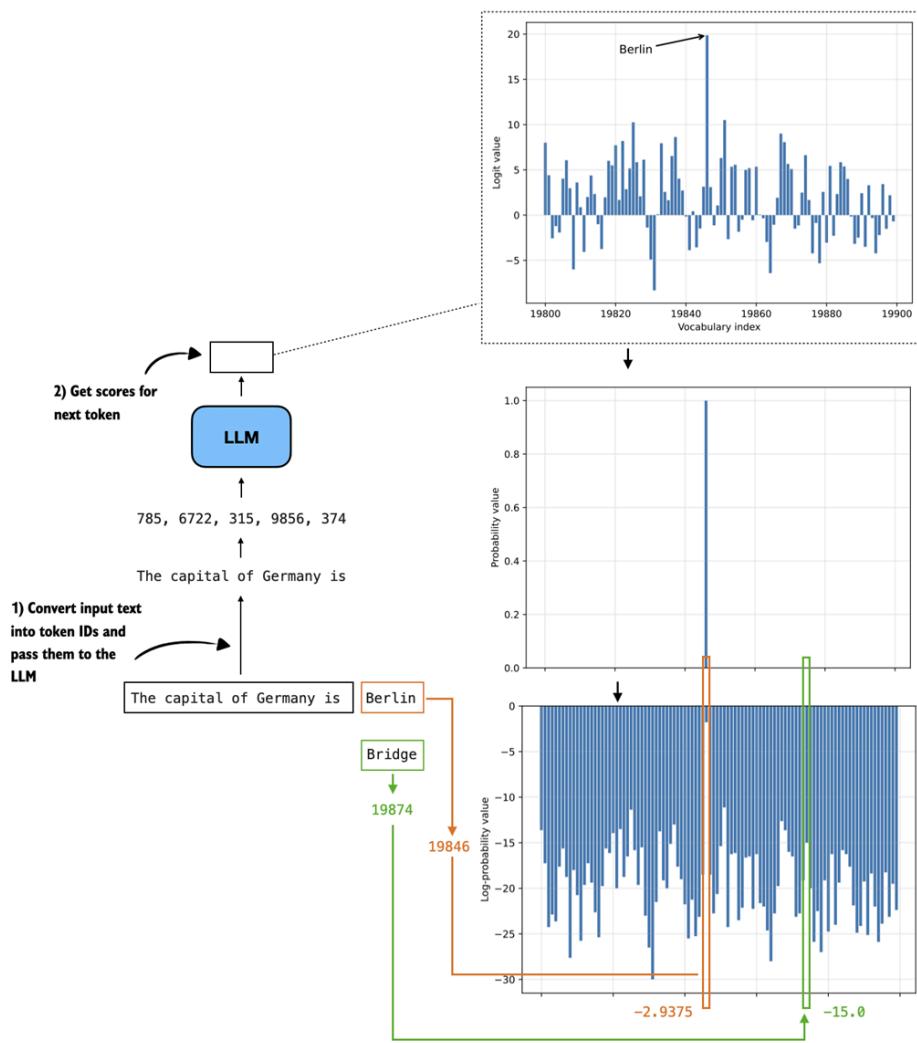


**Figure 5.14 Comparison of logits, softmax probabilities, and log-probabilities for a simple example. The log-probabilities preserve the ordering of the probabilities.**

As we can see in the resulting plot (figure 5.14), the log-probabilities have the same sorting order as the logits and probability values. A larger logit corresponds to a larger probability and a less negative log-probability. On the other hand, smaller logits produce smaller probabilities and more negative log-probabilities.

In practical terms, higher probabilities are better because they indicate the model assigns more confidence to a token. For log-probabilities, values closer to zero are better, since zero corresponds to a probability of one, while very negative values correspond to extremely small probabilities. This makes it easy to compare tokens: the least negative log-probability is the most likely one, and the most negative log-probability is the least likely.

While figure 5.14 illustrates the relationship between logits, probabilities, and log-probabilities for a simpler toy example, we can apply it to the probability values for the "The capital of Germany is" prompt example from earlier, as shown in figure 5.15.

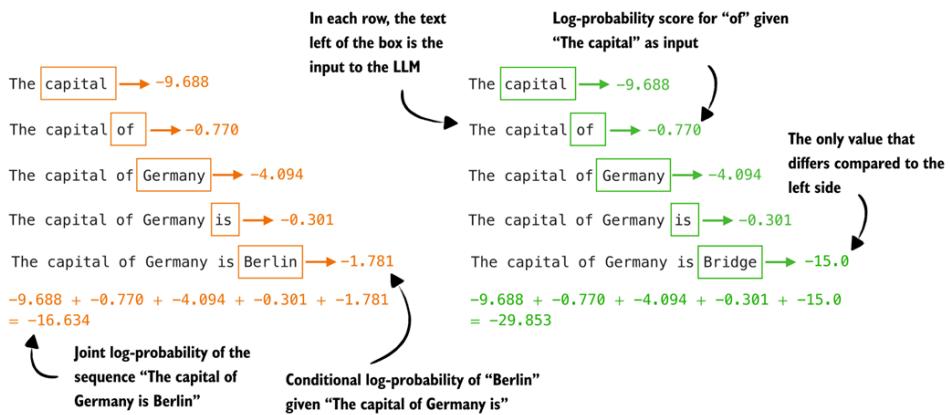


**Figure 5.15 Illustration of how logits are converted to probabilities and log-probabilities for next-token scoring. The correct next token ("Berlin") receives a high logit, which becomes a high probability and a less negative log-probability, while unlikely candidates like "Bridge" map to very small probabilities and large negative log-probabilities.**

We use probabilities instead of logits because probabilities are normalized and easier to interpret as confidence values. Earlier we also noted that log-probabilities offer an additional benefit, since they allow for numerically more stable calculations than working with raw probabilities.

The numerical stability comes from the fact that even very small probability scores map to log-probabilities in a reasonable numeric range. The other reason is that multiplying many probabilities quickly drives the result toward zero, while taking the log turns this multiplication into addition, which avoids underflow and is much more stable for long sequences.

Figure 5.16 shows how the joint log-probability (or sequence log-likelihood) is computed for the two example texts we used earlier.



**Figure 5.16 Illustration of how token-level log-probabilities accumulate to form sequence log-probabilities. Each row shows the log-probability of the next token given the preceding text. Summing these values results in the joint log-probability of the full sequence**

Earlier, when we computed the regular joint probability score, we saw that both sequences had values close to zero. Now, with the joint log-probability, we can see in figure 5.16 that even changing just the final token (for example, "Berlin" versus "Bridge") results in a noticeably different total log-probability (-16.808 versus -29.988).

### WORKING WITH LOG-PROBABILITIES

When we compute the joint probability of a sequence, we multiply many numbers between 0 and 1:

$$P(x_1, x_2, \dots, x_T | W) = \prod_{t=1}^T P(x_t | x_{1:t-1}, W)$$

In the expanded form, we can write it as:

$$P(x_1 | W) \cdot P(x_2 | x_1, W) \cdot P(x_3 | x_{1:2}, W) \cdot \dots \cdot P(x_T | x_{1:t-1}, W)$$

These products quickly become extremely small, which is inconvenient and can cause numerical underflow. To avoid this, we often work in log space. Taking the logarithm of the joint probability gives

$$\log(P(x_1, x_2, \dots, x_T | W)) = \log(\prod_{t=1}^T P(x_t | x_{1:t-1} | W))$$

Using the fact that the logarithm of a product is the sum of the logarithms, this expands to

$$\begin{aligned} & \log P(x_1 | W) + \log P(x_2 | x_1, W) + \\ & \log P(x_3 | x_{1:2}, W) + \dots + \log P(x_T | x_{1:t-1}, W) \end{aligned}$$

In other words, the log-probability of a sequence is just the sum of the log-probabilities of its individual tokens. This is both numerically more stable and easier to work with. It also matches the values returned by `torch.log_softmax`, which provides the log-probabilities directly.

Summing log-probabilities is therefore the standard approach in machine learning and AI.

Reusing the code from the previous section on calculating the token probabilities (listing 5.6), it is straightforward to implement the token log-probability function as it only requires two small changes: changing `torch.softmax` to `torch.log_softmax` and changing `torch.prod` to `torch.sum`.

---

**WARNING** Using mismatched combinations, such as `torch.softmax` with `torch.sum` or `torch.log_softmax` with `torch.prod`, is technically possible but mathematically incorrect.

---

**Listing 5.7 Calculating next token log-probabilities**

```

@torch.inference_mode()
def calc_next_token_logprobas(model, tokenizer, prompt, device):

    token_ids = torch.tensor(tokenizer.encode(prompt), device=device)

    logits = model(token_ids.unsqueeze(0)).squeeze(0)
    #A
    all_logprobas = torch.log_softmax(logits, dim=-1)

    t_idx = torch.arange(0, token_ids.shape[0] - 1, device=device)
    next_ids = token_ids[1:]
    next_token_logprobas = all_logprobas[t_idx, next_ids]

    print(
        "Next-token log-probabilities:",
        [p.item() for p in next_token_logprobas]
    )
    #B
    print(
        "Joint log-probability:",
        torch.sum(next_token_logprobas)
    )

#A We now use log_softmax
#B We replace the product with a sum

```

Let's now try the updated function on the example prompts from earlier:

```

calc_next_token_logprobas(
    model, tokenizer, device=device,
    prompt="The capital of Germany is Berlin"
)

```

The result is:

```

Next-token log-probabilities: [-9.6875, -0.76953125, -4.09375,
-0.30078125, -1.78125]
Joint log-probability: tensor(-16.6250, dtype=torch.bfloat16)

```

And now the second sequence:

```
calc_next_token_logprobas(
    model, tokenizer, device=device,
    prompt="The capital of Germany is Bridge"
)
```

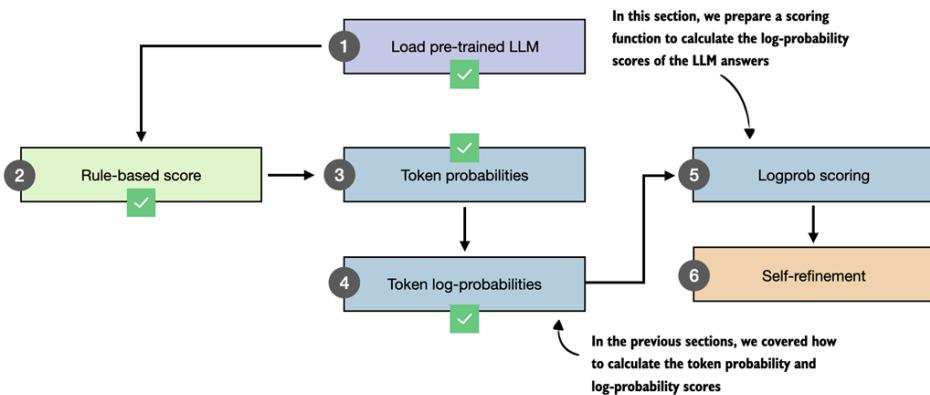
This returns:

```
Next-token log-probabilities: [-9.6875, -0.76953125, -4.09375,
-0.30078125, -15.0]
Joint log-probability: tensor(-29.8750, dtype=torch.bfloat16)
```

As we can see, the difference between the "Berlin" and the "Bridge" sequence is now much more pronounced (-16.6250 and -29.8750), with the former scoring much higher (better).

## 5.6 Scoring model confidence with log-probabilities

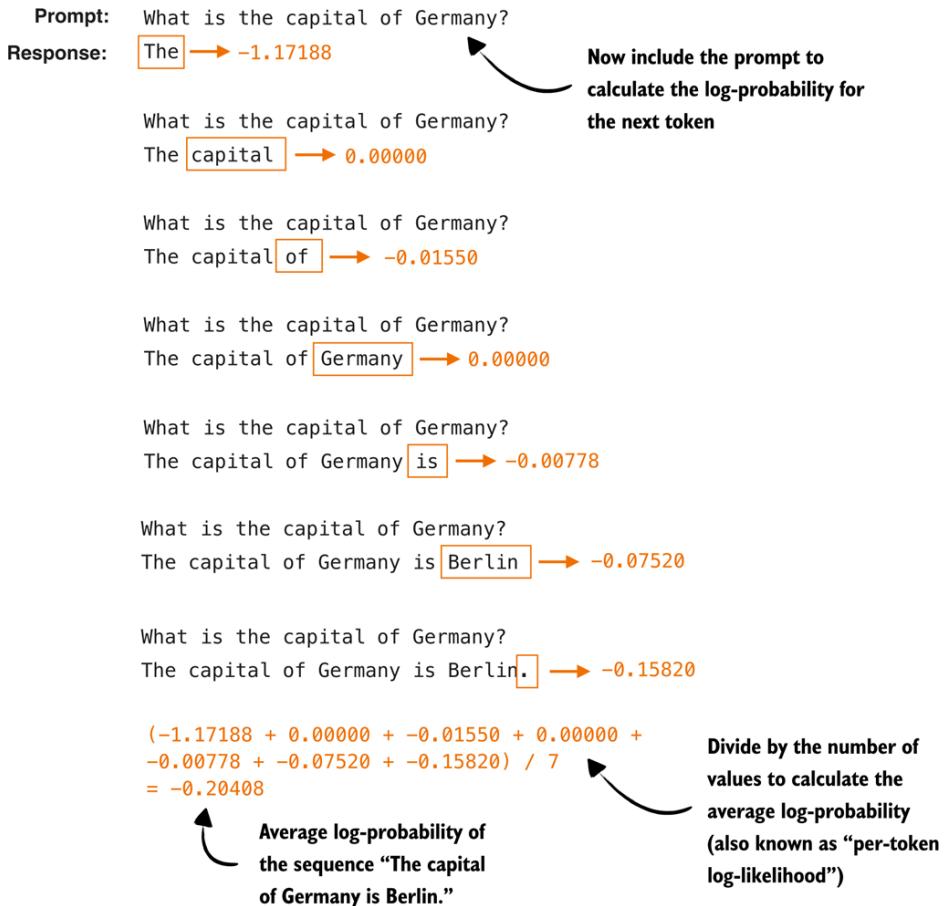
The previous two sections explained the concepts of token probabilities and log-probabilities in great detail. In this section, we add some slight modifications to the log-probability computation to develop a log-probability-based scorer function analogous to the heuristic scorer we developed at the beginning of this chapter, as illustrated in figure 5.17.



**Figure 5.17** This section implements a logprob scorer, based on token log-probabilities, which we will use in the self-refinement method later in this chapter.

The logprob scoring method (*logprob* is a common term in the AI literature that is short for log-probability) we develop is very similar to the procedure we covered in the previous section. However, we make two main modifications.

First, we exclude the prompt from the score calculation and only calculate the score for the answer tokens. Second, we average (instead of summing) over the token log-probabilities so that it is fairer to compare two different sequences of different lengths. The updated calculation with these two modifications is shown in figure 5.18.



**Figure 5.18 Illustration of the modified logprob scoring procedure. The prompt tokens are excluded from the calculation, and only the log-probabilities of the answer tokens are collected. These values are then averaged to obtain a length-normalized score, which allows us to compare answers of different lengths fairly.**

For illustration purposes, we can implement the two modifications shown in figure 5.18 using the `calc_next_token_logprobas` function from the previous section. For example, suppose we have the following example prompt and answer:

```

example_prompt = "What is the capital of Germany?"
example_answer = " The capital of Germany is Berlin."

calc_next_token_logprobas(
    model, tokenizer, device=device,
    prompt=example_prompt+example_answer
)

```

This prints the following output:

```

Next-token log-probabilities: [-0.451171875, -0.341796875, -8.3125,
-0.390625, -3.8125, -3.046875, -1.171875, 0.0,
-0.0155029296875, 0.0, -0.007781982421875,
-0.0751953125, -0.158203125]
Joint log-probability: tensor(-17.7500, dtype=torch.bfloat16)

```

We can then calculate the number of answer tokens via the following code, which yields 7:

```
print(len(tokenizer.encode(example_answer)))
```

Then, to calculate the average log-probability over the answer tokens, as shown in figure 5.18, we average over those 7 answer tokens (these are the last 7 tokens copied from the previous next-token log-probabilities output):

```

vals = torch.tensor([
    -1.171875,
    0.0,
    -0.0155029296875,
    0.0,
    -0.007781982421875,
    -0.0751953125,
    -0.158203125
])
print(torch.mean(vals))

```

This results in the value `-0.2041`, similar to what's shown in figure 5.19.

We can combine the `calc_next_token_logprobas` code with this calculation more conveniently into a new function, `avg_logprob_answer`, which is shown below:

**Listing 5.8 Average log-probability scoring for answer tokens**

```

@torch.inference_mode()
def avg_logprob_answer(model, tokenizer, prompt, answer, device="cpu"):

    prompt_ids = tokenizer.encode(prompt) #A
    answer_ids = tokenizer.encode(answer) #A
    full_ids = torch.tensor(prompt_ids + answer_ids, device=device)

    logits = model(full_ids.unsqueeze(0)).squeeze(0) #B
    logprobs = torch.log_softmax(logits, dim=-1) #B

    start = len(prompt_ids) - 1 #C
    end = full_ids.shape[0] - 1 #C

    #D
    t_idx = torch.arange(start, end, device=device)
    next_tokens = full_ids[start + 1 : end + 1]
    next_token_logps = logprobs[t_idx, next_tokens]

    #E
    return torch.mean(next_token_logps).item()

#A Encode prompt and answer tokens separately to get the prompt length later
#B Same as in calc_next_token_logprobas before
#C Index range for positions corresponding to answer tokens
#D Same as before, except for using start and end
#E Average over the answer token scores

```

The `avg_logprob_answer` function is overall similar to the `calc_next_token_logprobas` function from the previous section, except that we only calculate the logprobs for the answer tokens, instead of the whole sequence including the prompt, and average over the calculated logprob values instead of summing them.

Let's apply this new function to the prompt and answer from earlier in this section:

```

score_1 = avg_logprob_answer(
    model, tokenizer,
    prompt="What is the capital of Germany?",
    answer=" The capital of Germany is Berlin.",
    device=device
)
print(score_1)

```

This returns `-0.2041` similar to before, which indicates that we implemented the function correctly. We can now also apply this function to the nonsensical "Bridge" answer:

```
score_2 = avg_logprob_answer(
    model, tokenizer,
    prompt="What is the capital of Germany?",
    answer=" The capital of Germany is Bridge.",
    device=device
)
print(score_2)
```

The resulting score here is `-3.890625`, much lower than the "Berlin" score, as expected.

With the average logprob scoring function in place, we could now also calculate the scores for the MATH-500 prompt (`prompt_cot`) we defined at the beginning of this chapter and the respective responses we stored as `response_1` and `response_2`, which is left as an exercise for the reader.

We spend a lot of time in this chapter to go through the concepts of next-token probability and log-probability scoring. One reason for this is that we can use it as a scorer in the upcoming section on self-refinement. A second reason is that the concept of log-probabilities will also be relevant when we implement the *reinforcement learning with verifiable rewards* training in the upcoming chapter.

#### **EXERCISE 5.3: USING THE LOGPROB SCORER AS A TIE-BREAKER IN SELF-CONSISTENCY**

Using the logprob scorer instead of the heuristic scorer in exercise 5.1, extend the self-consistency implementation (`self_consistency_vote`) in the previous chapter so that it can handle ties among the candidate answers. Then run the two implementations on (a subset of) the MATH-500 dataset to see which tie-breaking method performs better.

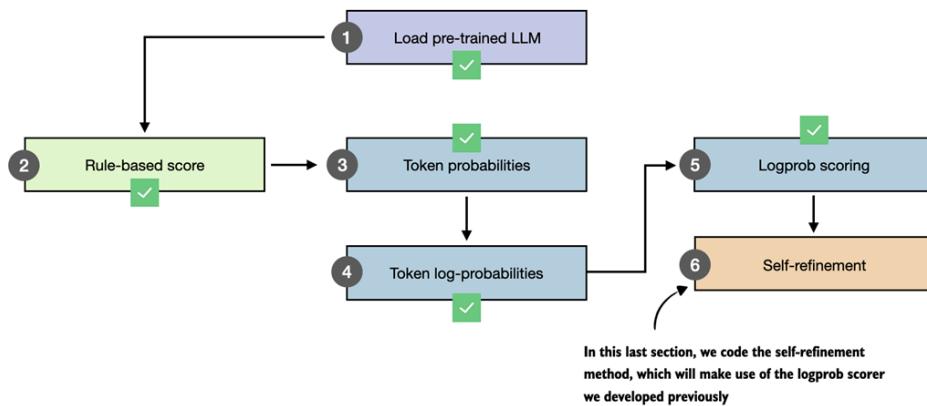
#### EXERCISE 5.4: USING THE LOGPROB SCORER IN A BEST-OF-N SETUP

Extend the self-consistency implementation so that the final answer is chosen using the logprob scorer (avg\_logprob\_answer) rather than the heuristic scorer or majority voting (similar to exercise 5.2). Then run the different implementations on (a subset of) the MATH-500 dataset to see which tie-breaking method performs better.

Tips: You don't need to modify the `self_consistency_vote` function itself to apply the tie-breaking, but you can apply the tie-breaking to the `self_consistency_vote` function's returned results dictionary, which is used inside the `evaluate_math500_stream` function from chapter 3.

## 5.7 Self-refinement through iterative feedback

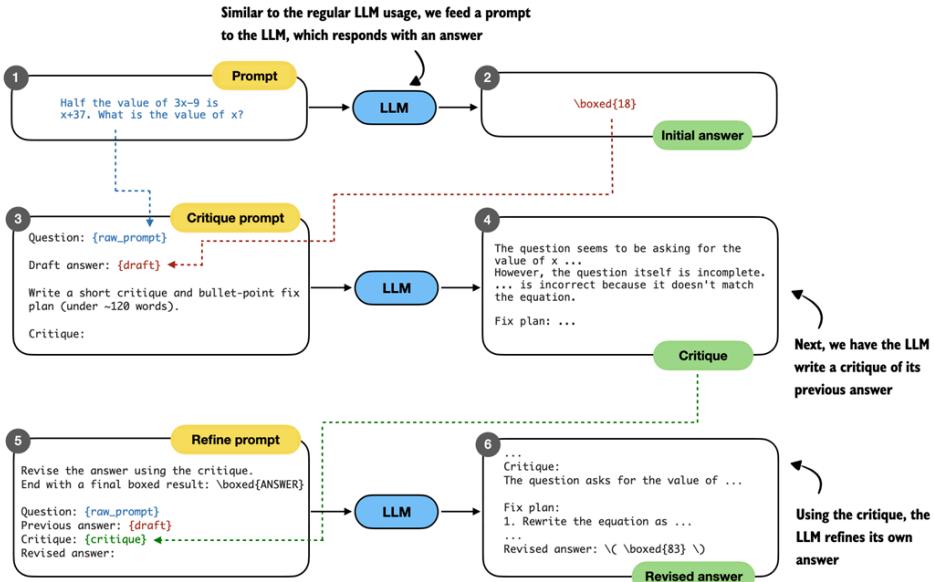
Having introduced several methods for scoring LLM answers, we now get to the core inference-scaling technique of this chapter, *self-refinement* (figure 5.19).



**Figure 5.19** Overview of the final step in our workflow, where the logprob scorer developed earlier is used inside the self-refinement method.

This section introduces self-refinement and walks through the process manually for illustration. The next section, as shown in figure 5.19, then automates the self-refinement loop and adds support for the scoring methods developed earlier in the chapter (the heuristic scorer and the average logprob scorer).

Self-refinement is a technique in which the LLM analyzes and refines its own answers. As shown in figure 5.20, the LLM starts with an initial answer to the prompt as in regular LLM usage. Then, it critiques the answer and refines it.



**Figure 5.20 Illustration of the self-refinement process. The LLM first produces an initial answer to the prompt, then receives a critique prompt that asks it to analyze its own response and produce a short critique with a plan to refine the answer. In the final step, the model is given a refine prompt that contains the original question, its draft answer, and the critique, and it generates a revised answer that incorporates the suggested improvements.**

The self-refinement procedure in figure 5.20 may look complicated at first, but it's essentially just a sequential application of the text generation function on different prompts and inputs. To make it more clear, let's walk through it step by step with a concrete code example.

---

**NOTE** For this code example, we don't use chain-of-thought prompting for illustration purposes. However, in practice, we can combine self-refinement with chain-of-thought prompting if we work with a base model.

---

We start with the base prompt and answer (steps 1 and 2 in figure 5.20), which is based on the code from section 5.2 at the beginning of this chapter, *Loading a pre-trained model*:

**Listing 5.9 Base prompt and answer**

```
raw_prompt = (
    "Half the value of $3x-9$ is $x+37$."
    "What is the value of $x$?"
)
prompt = render_prompt(raw_prompt)

torch.manual_seed(123)
initial_response = generate_text_stream_concat_flex(
    model, tokenizer, prompt, device,
    max_new_tokens=2048, verbose=True,
    generate_func=generate_text_top_p_stream_cache,
    temperature=0.7,
    top_p=0.9,
)
```

The LLM responds with "`\boxed{18}`", which is an incorrect answer (the correct answer is 83).

Next, we have the LLM critique the answer. To do this we write a critique prompt, which includes the original question (`raw_prompt`) and answer (`draft`), as shown in steps 3 and 4 in figure 5.20. In code, this looks like as follows:

**Listing 5.10 Critique prompt and refinement plan**

```
def make_critique_prompt(raw_prompt, draft):
    return (
        "You are a meticulous reviewer. Identify logical errors, missing "
        "steps, or arithmetic mistakes. If the answer seems correct, "
        "say so briefly. Then propose a concise plan to fix issues.\n\n"
        f"Question:\n{raw_prompt}\n\n"
        f"Draft answer:\n{draft}\n\n"
        "Write a short critique and bullet-point fix plan "
        "(under ~120 words).\n"
        "Critique:"
    )

critique_prompt = make_critique_prompt(raw_prompt, initial_response)
torch.manual_seed(123)
critique = generate_text_stream_concat_flex(
    model, tokenizer, critique_prompt, device,
    max_new_tokens=2048, verbose=True,
    generate_func=generate_text_top_p_stream_cache,
    temperature=0.7,
    top_p=0.9,
)
```

The critique prompt above lets the LLM write an elaborate critique, which even contains the correct answer itself:

The question seems to be asking for the value of  $\lfloor x \rfloor$  when half the value of  $\lfloor 3x - 9 \rfloor$  equals  $\lfloor x + 37 \rfloor$ . However, the question itself is incomplete. It doesn't provide enough information to solve for  $\lfloor x \rfloor$ . The answer provided,  $\boxed{18}$ , is incorrect because it doesn't match the equation.

Fix plan:

1. Rewrite the equation as  $\frac{1}{2}(3x - 9) = x + 37$ .
2. Multiply both sides by 2 to eliminate the fraction:  $3x - 9 = 2(x + 37)$ .
3. Distribute the 2 on the right side:  $3x - 9 = 2x + 74$ .
4. Subtract  $2x$  from both sides:  $x - 9 = 74$ .
5. Add 9 to both sides:  $x = 83$ .

Correct answer:  $x = 83$ .

Note that the critique itself can contain factual errors, too, though. For instance, it says "The question itself is incomplete", which is not true, as it then proceeds with a correct fix plan. However, ultimately what matters is that the fix plan is correct.

Finally, we use this refinement prompt to revise the original answer (steps 5 and 6 in figure 5.20):

#### **Listing 5.11 Answer refinement**

```
def make_refine_prompt(raw_prompt, draft, critique):
    return (
        "Revise the answer using the critique. Keep it concise and "
        "end with a final boxed result: \boxed{ANSWER}\n\n"
        f"Question:\n{raw_prompt}\n\n"
        f"Previous answer:\n{draft}\n\n"
        f"Critique:\n{critique}\n\n"
        "Revised answer:"
    )

refine_prompt = make_refine_prompt(raw_prompt, initial_response, critique)
torch.manual_seed(123)
revised_answer = generate_text_stream_concat_flex(
    model, tokenizer, refine_prompt, device,
    max_new_tokens=2048, verbose=True,
    generate_func=generate_text_top_p_stream_cache,
    temperature=0.7,
    top_p=0.9,
)
```

While the base model is not the best instruction-follower, the refinement prompt has the LLM generate the correct answer:

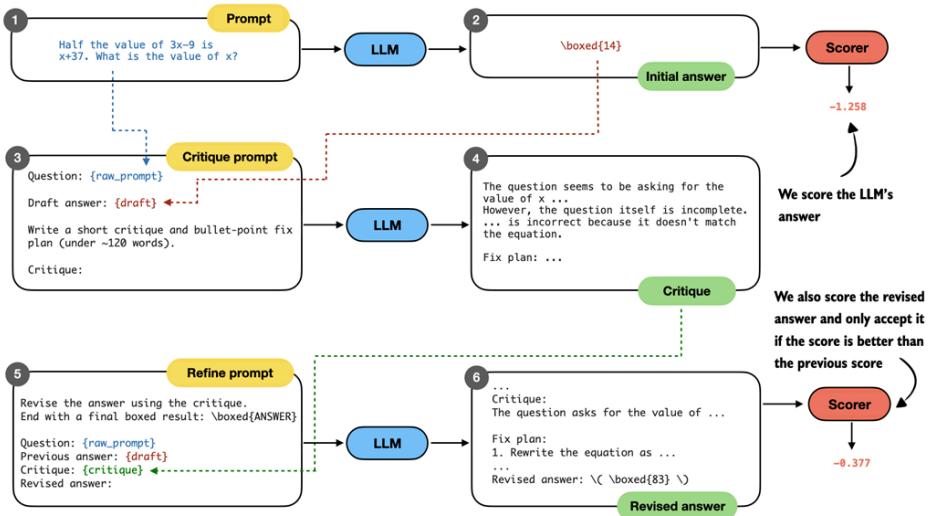
```
...
5. Add 9 to both sides:  $(x = 83)$ .
Correct answer:  $(x = 83)$ .
Revised answer:  $(\boxed{83})$ 
```

The procedure in this section illustrates a single refinement loop. In practice, it is not uncommon to repeat this loop for multiple iterations. In the next section, we will code a function to automate this process.

## 5.8 Coding the self-refinement loop

This final section packages the manual self-refinement steps from the previous section into a convenient function that simplifies using self-refinement and allows the process to be repeated for a fixed number of iterations.

In addition, the function supports plugging in the scoring methods developed earlier in the chapter (heuristic scoring and logprob scoring) to compute a score for each answer (figure 5.21), which can be used to decide whether a refined answer should be accepted.



**Figure 5.21** Overview of the self-refinement loop with optional scoring. The model first produces an initial answer, then critiques it, and generates a revised answer based on the critique. Both answers can be evaluated with the scoring functions from this chapter (for example, logprob scoring), and the revised answer is only accepted if its score improves on the previous one.

In figure 5.21, the revised answer receives a higher logprob score ( $-0.377$ ) than the initial answer ( $-1.258$ ), which makes it reasonable to accept the revision. In practice, however, self-refinement can also produce worse answers, especially when run for multiple iterations. The scorer therefore provides a way to decide whether a revised answer represents an improvement.

---

**NOTE** Scoring does not always improve the results. Whether and which scorer to use in self-refinement depends on the LLM and needs to be figured out via experimentation on a benchmark dataset like MATH-500.

---

The complete code, which combines the steps from the previous section and adds an `iterations` parameter and a scoring function (`score_fn`) option, is shown in listing 5.12 below:

**Listing 5.12 Self-refinement with multiple iterations and scoring support**

```

def self_refinement_loop(
    model,
    tokenizer,
    raw_prompt,
    device,
    iterations=2,
    max_response_tokens=2048,
    max_critique_tokens=256,
    score_fn=None,
    prompt_renderer=render_prompt,
    prompt_suffix="",
    verbose=False,
    temperature=0.7,
    top_p=0.9,
):
    steps = []

    #A
    prompt = prompt_renderer(raw_prompt) + prompt_suffix
    current_full = generate_text_stream_concat_flex(
        model=model,
        tokenizer=tokenizer,
        prompt=prompt,
        device=device,
        max_new_tokens=max_response_tokens,
        verbose=False,
        generate_func=generate_text_top_p_stream_cache,
        temperature=temperature,
        top_p=top_p,
    )

    current_extracted = extract_final_candidate(
        current_full, fallback="number_then_full"
    )
    if score_fn:
        current_score = score_fn(answer=current_full, prompt=prompt)
    else:
        current_score = 0.0

    #B
    for it in range(iterations):
        draft_before_full = current_full

```

```

draft_before_extracted = current_extracted
score_before = current_score

#C
critique_prompt = make_critique_prompt(
    raw_prompt, draft_before_full
)
critique_full = generate_text_stream_concat_flex(
    model=model,
    tokenizer=tokenizer,
    prompt=critique_prompt,
    device=device,
    max_new_tokens=max_critique_tokens,
    verbose=False,
    generate_func=generate_text_top_p_stream_cache,
    temperature=temperature,
    top_p=top_p,
)

#D
refine_prompt = make_refine_prompt(
    raw_prompt, draft_before_full, critique_full
)
revised_full = generate_text_stream_concat_flex(
    model=model,
    tokenizer=tokenizer,
    prompt=refine_prompt,
    device=device,
    max_new_tokens=max_response_tokens,
    verbose=False,
    generate_func=generate_text_top_p_stream_cache,
    temperature=temperature,
    top_p=top_p,
)

revised_extracted = extract_final_candidate(
    revised_full, fallback="number_then_full"
)
if score_fn:
    revised_score = score_fn(
        answer=revised_full, prompt=prompt
    )
else:
    revised_score = 0.0

```

```

#E
step = {
    "iteration": it + 1,
    "draft_full": draft_before_full,
    "draft_extracted": draft_before_extracted,
    "critique": critique_full,
    "revised_full": revised_full,
    "revised_extracted": revised_extracted,
    "score_before": score_before,
    "score_after": revised_score,
}
steps.append(step)

if verbose:
    print(
        f"[Refinement {it+1}/{iterations}]"
        f"\nCurrent: {draft_before_extracted}"
        f"\nRevised: {revised_extracted}"
        f"\nScore before: {score_before:.3f}"
        f"\nScore after: {revised_score:.3f}"
        f"\n{'=' * 25}\n"
    )

#F
if revised_score >= current_score:
    current_full = revised_full
    current_extracted = revised_extracted
    current_score = revised_score

return {
    "final_full": current_full,
    "final_extracted": current_extracted,
    "steps": steps,
}

#A Initial response (draft)
#B Run for one or more iterations
#C Critique the response
#D Refine the response
#E Log the results
#F Accept revised response if it's not worse

```

The `self_refinement_loop` function runs the self-refinement procedure from the previous section for one or more iterations (`for it in range(iterations)`). At the end of each iteration, it compares the revised score to the current score (`revised_score >= current_score`) and keeps the revised answer only if the score is equal or higher.

Using a scoring function is optional. When `score_fn=None` (the default), the score is always set to `0.0`. Since `0.0 >= 0.0` evaluates to `True`, the most recent answer is always accepted when running multiple refinement iterations.

Next, we run the self-refinement loop using the average logprob scorer, `avg_logprob_answer`. The `avg_logprob_answer` function from listing 5.8 (section 5.6) takes several arguments (`model`, `tokenizer`, `prompt`, `answer`, and `device`). However, as shown in the previous listing, the scoring function is called with only two arguments:

```
# ...
    if score_fn:
        revised_score = score_fn(
            answer=revised_full, prompt=prompt
        )
# ...
```

To make `avg_logprob_answer` compatible with this `score_fn` call, we use the `partial` function from Python's built-in `functools` module to pre-specify the remaining arguments:

#### **Listing 5.13 Creating an average log-probability scorer**

```
from functools import partial

avg_logprob_score = partial(
    avg_logprob_answer,
    model=model,
    tokenizer=tokenizer,
    device=device
)
```

Now, we can use the `avg_logprob_score` function in the self-refinement loop:

```

torch.manual_seed(0)

results_logprob = self_refinement_loop(
    model=model,
    tokenizer=tokenizer,
    raw_prompt=raw_prompt,
    device=device,
    iterations=2,
    max_response_tokens=2048,
    max_critique_tokens=256,
    score_fn=avg_logprob_score,
    verbose=True,
    temperature=0.7,
    top_p=0.9,
)

```

The output is shown below:

```

[Refinement 1/2]
Current: 28
Revised: 83
Score before: -0.855
Score after: -0.129
=====
[Refinement 2/2]
Current: 83
Revised: 83
Score before: -0.129
Score after: -1.320
=====
```

Looking at the results above, we see that the model corrected the initially incorrect answer (28 to 83) in the first iteration, and the score improved from -0.855 to -0.129. In the second iteration, the scores become worse, but that's okay since we already have the correct answer.

We can also access the extracted number of the best-scoring answer via `results_logprob["final_extracted"]`, which in this case returns 83. If you are interested in the full answer, use `results_logprob["final_full"]`. To read through the critiques and longer answers, you can print the `results_logprob` dictionary, which contains the detailed results for all iterations.

**EXERCISE 5.5: USING THE HEURISTIC SCORE FOR SELF-REFINEMENT**

Run the `self_refinement_loop` using the `heuristic_score` function, which we defined in section 5.3

The previous example showed that self-refinement can help the model with generating the correct answer. To get a better idea of how useful this self-refinement method really is, I ran this method on the MATH-500 from chapter 3 and summarized the results in table 5.1.

**Table 5.1 MATH-500 task accuracy for different self-refinement methods**

	<b>Method</b>	<b>Scoring</b>	<b>Iterations</b>	<b>Model</b>	<b>Accuracy</b>	<b>Time</b>
1	Baseline (chapter 3)	-	-	Base	15.2%	10.1 min
2	Self-refinement	None	1	Base	25.0%	84.8 min
3	Self-refinement	None	2	Base	22.0%	165.4 min
4	Self-refinement	Heuristic	1	Base	21.6%	84.7 min
5	Self-refinement	Heuristic	2	Base	20.8%	151.4 min
6	Self-refinement	Avg. logprob	1	Base	21.4%	85.3 min
7	Self-refinement	Avg. logprob	2	Base	22.0%	165.3 min
8	Baseline (chapter 3)	-	-	Reasoning	48.2%	182.1 min
9	Self-refinement	None	1	Reasoning	56.6%	498.8 min
10	Self-refinement	Heuristic	1	Reasoning	57.8%	498.6 min
11	Self-refinement	Avg. logprob	1	Reasoning	48.4%	499.7 min

The accuracy values shown in table 5.1 were computed on all 500 samples in the MATH-500 test set using a "cuda" GPU (DGX Spark).

As we can see in table 5.1, self-refinement improves the performance over the base model (rows 1-7). However, the improvement is only very moderate, with the best accuracy being achieved when no scoring is used in self-refinement. This means that both the heuristic score and average logprob score can sometimes lead to incorrect answers being accepted over the initial correct answer.

Note that when adding "Explain step by step." chain-of-thought for both the base model and the self-refinement, the model failed to improve over the base model (results not shown).

Looking at the reasoning model results (rows 8-11), we can see that the combination of self-refinement and heuristic scoring improved the answer accuracy by almost 10%. (As explained in chapter 3, the "reasoning" model can be used by changing `which_model="base"` to `which_model="reasoning"` in the `load_model_and_tokenizer` in section 5.2.)

In both cases, it seems that the average logprob scoring results in worse performance than no scorer or the heuristic scorer. This is likely because the logprob score favors answers that the model finds easy to produce rather than answers that are actually correct. Or, in other words, the logprob criterion can unintentionally select confident mistakes, whereas the heuristic score is more focused on the format and structure of the answer.

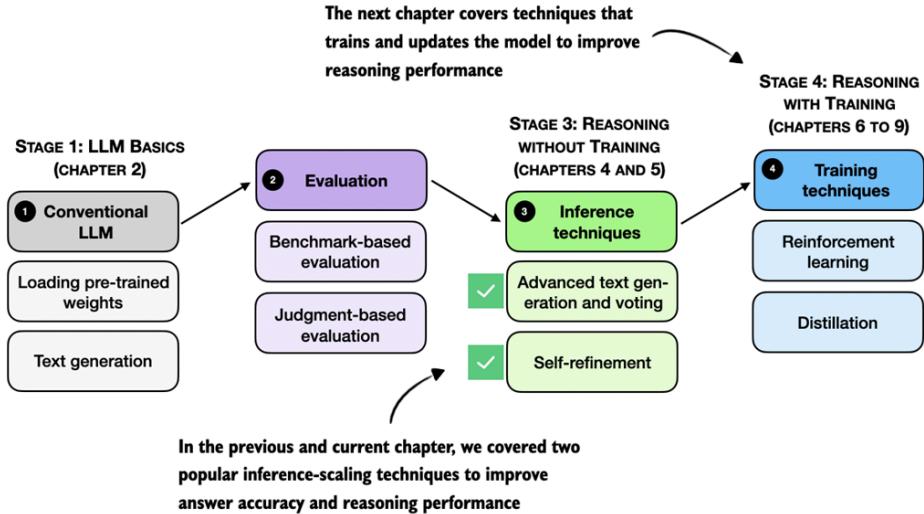
At this point, it seems we spent a lot of time discussing the average logprob scoring concept. However, logprob scoring is a fundamental concept when working with LLMs, and it will come handy in the upcoming chapters when we implement the reinforcement learning training procedure.

Comparing these results in table 5.1 with those from the previous chapter (table 4.1), we also observe that self-refinement is less effective than self-consistency for this model on this math task. Self-consistency remains widely used because it works well in practice, despite its simplicity.

Another method we have not explored in this chapter is self-refinement with an external model in an LLM-as-a-judge setup similar to what's described in appendix F in section F.5. For instance, instead of using the heuristic score or average logprob, we use a second LLM to compute a score and to write the critique.

In November 2025, with DeepSeekMath-V2, the DeepSeek team demonstrated that self-refinement with a second LLM can be very successful, leading to gold-level performance in several math competitions. Specifically, the DeepSeek team proposed a new method in which they trained a second LLM to be a good critique model and further trained their base model to become a better math problem solver in the context of self-refinement (in contrast, in this chapter, we applied self-refinement without any additional training).

Speaking of inference and training, this chapter concludes the inference scaling without additional training (figure 5.22). In the next chapter, we begin covering training techniques that update the model weights to become better at reasoning tasks.



**Figure 5.22** Overview of the book's progression from basic LLM usage to inference-time reasoning methods and finally to training-based techniques. This chapter concludes the inference-scaling methods without additional training, and the next chapters introduce approaches that update model weights to further improve reasoning performance.

## 5.9 Summary

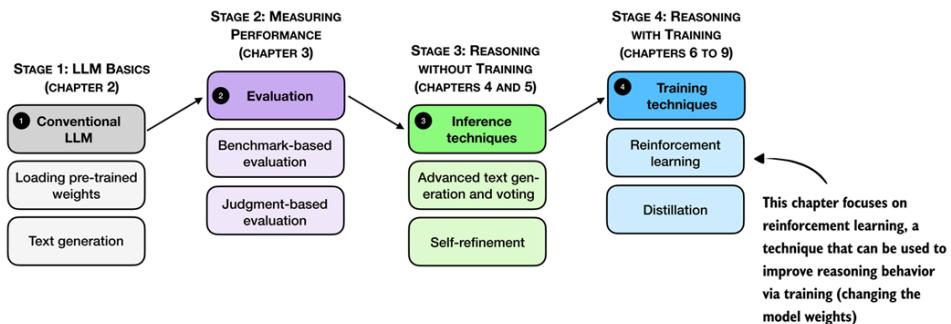
- Self-refinement extends the inference-time scaling ideas from the previous chapter by iteratively critiquing and improving a single answer instead of relying on multiple independent samples as in self-consistency.
- A simple rule-based scoring function ranks model outputs by rewarding extractable final answers and shorter, more economical completions.
- Next-token scoring quantifies model confidence by converting logits into normalized probabilities and combining these into a sequence-level likelihood.
- Log-probabilities replace raw probabilities to avoid numerical underflow and to turn products over many tokens into stable sums and averages.
- These scoring functions are more generally useful beyond self-refinement, for example, for breaking ties in self-consistency or implementing Best-of-N selection strategies.
- The self-refinement procedure consists of three stages: generating an initial draft, producing a short critique and fix plan, and generating a revised answer.
- A reusable refinement function automates the self-refinement workflow with multiple iterations (refinement rounds), and it can use a score-based acceptance to keep only revisions that do not degrade a computed answer score using one of the scoring functions we developed earlier.

## 6 *Training reasoning models with reinforcement learning*

### This chapter covers

- Training reasoning LLMs as a reinforcement learning problem with task-correctness rewards
- Sampling multiple responses per prompt to compute group-relative learning signals
- Updating the LLM weights using group-based policy optimization for improved reasoning

Reasoning performance and answer accuracy can be improved by both increasing the inference compute budget and by specific model training methods. This chapter, as shown in figure 6.1, focuses on *reinforcement learning*, which is the most commonly used training method for reasoning models.

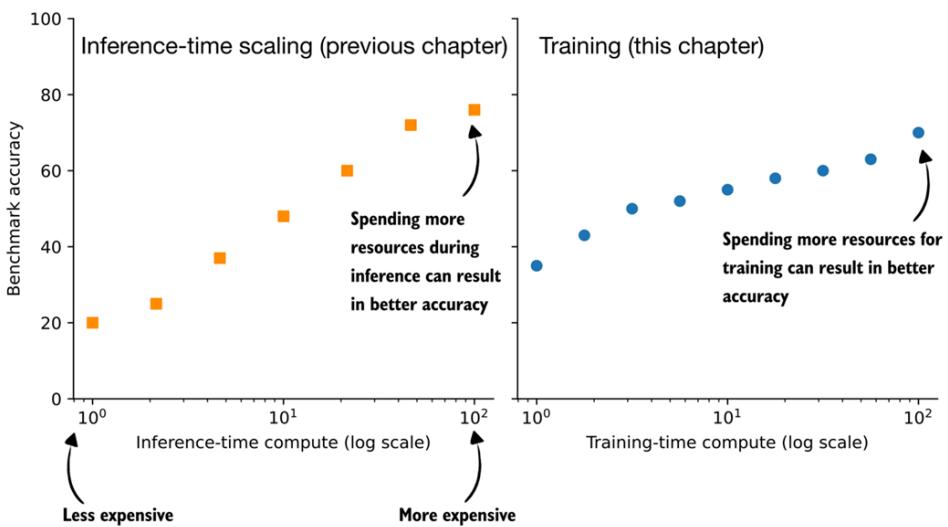


**Figure 6.1 A mental model of the topics covered in this book. This chapter focuses on techniques that improve reasoning with additional training (stage 4). Specifically, this chapter covers reinforcement learning.**

The next section provides a general introduction to reinforcement learning in the context of LLMs before discussing the two common reinforcement learning approaches used for LLMs.

## 6.1 Introduction to reinforcement learning for LLMs

*Inference-time scaling* and *training-time scaling* are two distinct approaches for improving the reasoning performance of large language models, as illustrated in figure 6.2. Inference-time scaling increases accuracy by spending more computation per generated answer, whereas training-time scaling improves accuracy by investing additional computation during training. This chapter focuses on training-time scaling.



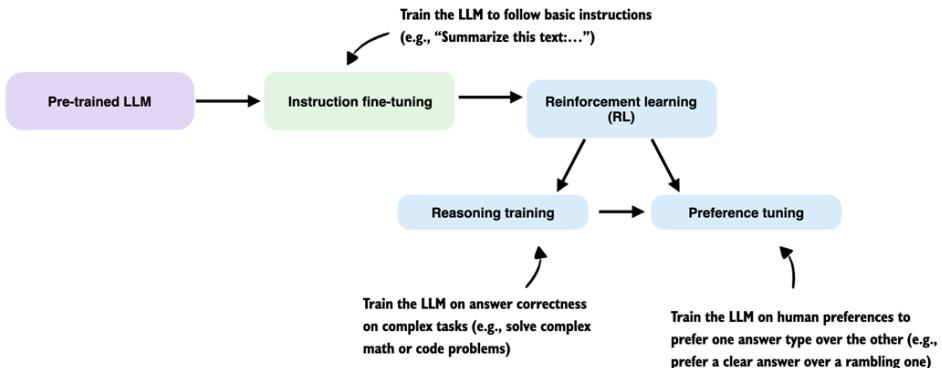
**Figure 6.2 Conceptual comparison of inference-time scaling and training-time scaling. Increasing compute at inference improves accuracy by spending more resources per answer generation, while increasing compute during training improves accuracy by investing more resources upfront.**

While figure 6.2 presents inference-time scaling and training-time scaling as separate concepts, in practice, they can be combined. For example, after improving a model's reasoning capabilities through *reinforcement learning* (RL), which is the core focus of this chapter, inference-time scaling techniques such as those introduced in chapters 4 and 5 can be applied to further boost performance.

RL is focused on how models learn from sequences of actions and their outcomes, with classic examples such as agents trained to play Chess, Go, or video games.

While RL for LLMs builds on this literature, it differs in important ways and often resembles familiar LLM training techniques. Since this book focuses on language models, we do not cover RL in general but instead focus on how RL is applied in LLM training.

So what does RL mean in practice for LLMs? At a high level, RL is typically applied as a post-training stage on top of a pre-trained language model, sometimes after instruction fine-tuning. This setup is illustrated in figure 6.3.



**Figure 6.3 Common training stages for LLMs.** The ordering of the reasoning training and preference tuning stages can vary, and some pipelines interleave reasoning and preference tuning rather than strictly sequencing them.

There are two common RL stages for LLMs, *reasoning training* and *preference tuning*. As shown in figure 6.3, RL is usually applied after *instruction fine-tuning*, and preference tuning often follows reasoning training. As demonstrated by the DeepSeek team in their DeepSeek-R1 work, though, reasoning-focused RL can also be applied directly to the pre-trained base model, skipping both instruction fine-tuning and preference tuning.

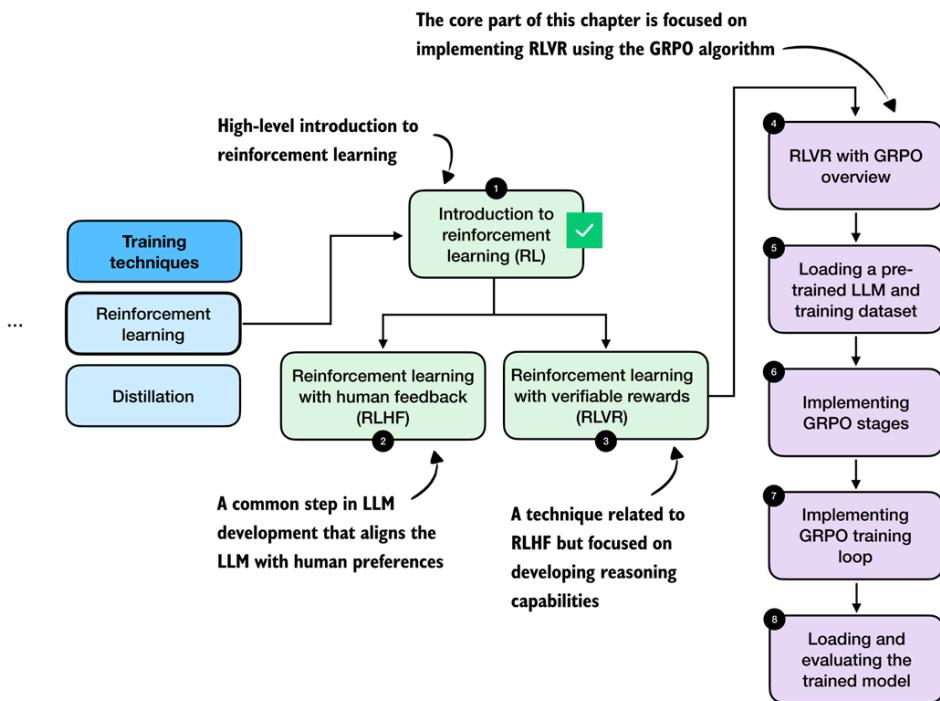
The resulting model is generally weaker in terms of reasoning accuracy than one that goes through the full training pipeline, but it still exhibits clear and robust reasoning behavior. More importantly, applying reasoning training directly to the base model avoids mixing the effects of multiple training stages, which makes it easier to attribute any observed improvements specifically to the reasoning training itself.

Before getting further into the details of how RL is implemented for LLMs, it is useful to briefly compare RL with pre-training at a conceptual level. During pre-training, an LLM is trained to predict the next token in large amounts of text, and many of the model's core abilities and emergent behaviors (for example, answering knowledge-based questions and following simple instructions) already develop at this stage.

Note that pre-training is the main focus of my earlier book, *Build a Large Language Model (From Scratch)*, but familiarity with that material is not required to follow this chapter or the rest of this book.

Applying RL on top of a pre-trained model is attractive because it lets us optimize whole outputs, such as answer correctness or preferences, rather than individual tokens and next-token prediction. In this sense, pre-training mainly builds knowledge, while RL shapes how the model uses that knowledge, including its reasoning behavior.

As shown in Figure 6.4, the next two subsections give a high-level overview of how RL is used in LLM training and set the context for the method used in this chapter.



**Figure 6.4** Roadmap of this chapter. After a brief introduction to reinforcement learning (RL) for LLMs in this section, we discuss the difference between two RL stages, RLHF and RLVR, in the next section. Then, we focus on implementing RLVR using the GRPO algorithm in the remainder of this chapter.

### 6.1.1 The original reinforcement learning pipeline with human feedback (RLHF)

In 2022, the InstructGPT paper (<https://arxiv.org/abs/2203.02155>) introduced *reinforcement learning with human feedback* (RLHF), a training approach that uses human preference labels to modify model behavior.

In fact, RLHF was a key ingredient in transforming GPT-3 into the original model used in ChatGPT, which was arguably what made LLMs broadly popular in 2022.

At a high level, RLHF is the most popular method to implement the preference tuning stage discussed in the previous section. Before RLHF, LLMs were primarily trained through pre-training and, in some cases, supervised fine-tuning, which are both based on next-token prediction objectives.

RLHF moved beyond this token-level optimization and optimizes models on the whole outputs (in this case, RLHF is optimizing for how humans rank and evaluate LLM outputs).

To make this more concrete, consider the following example. Suppose the prompt is: "I am looking to buy a laptop for programming and everyday use. What should I consider?"

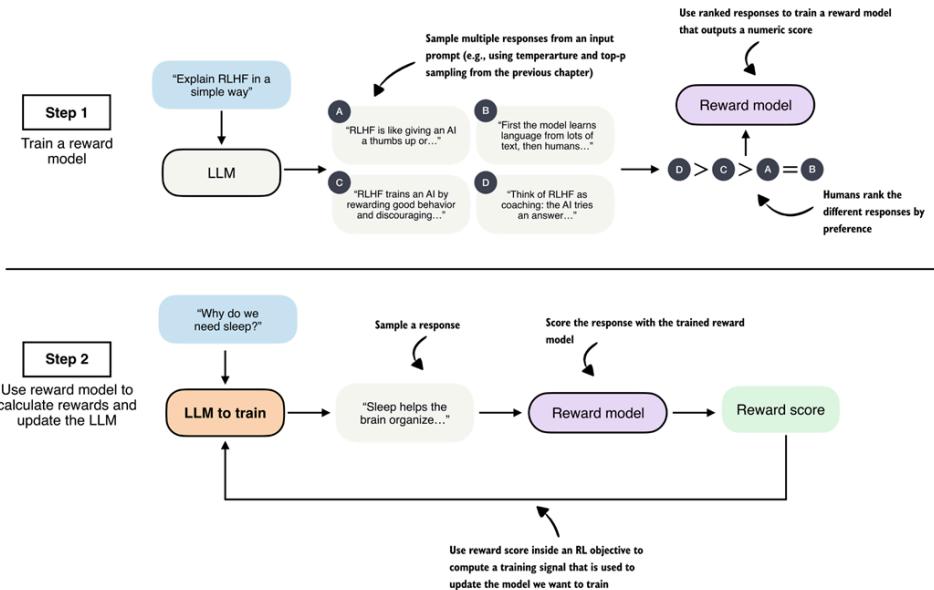
The model might generate two candidate responses:

- Response A: "You should consider the CPU, RAM, storage, GPU, screen resolution, battery life, keyboard quality, port selection, thermal design, and price."
- Response B: "For programming and everyday use, focus on a fast CPU, at least 16 GB of RAM, and an SSD with at least 256 GB storage. A comfortable keyboard and good battery life also matter. A GPU may only matter if you plan to train small LLMs locally."

Both responses mention relevant points, but a human data annotator might prefer response B because it is more concrete and specific to the use case stated in the prompt.

In RLHF, such pairwise preferences are collected across many prompts and used to train the model to favor responses that humans consistently rank higher.

The two-step process of RLHF, including reward model and LLM training, is illustrated in figure 6.5.



**Figure 6.5 Two-stage overview of reinforcement learning with human feedback (RLHF).** First, a reward model is trained on human-ranked responses to assign a preference score to each. Second, the LLM is updated using these reward scores within an RL objective to encourage preferred responses and discourage less desirable ones.

The first step in RLHF is training a *reward model*, as illustrated in the top subpanel of figure 6.5. Here, we have the LLM generate multiple responses per prompt (for example, using the temperature scaling and top-p filtering approach explained in chapter 4), and ask human annotators to rank the answers from best to worst based on their human preference.

These human preferences are converted into training targets for the reward model using a statistical preference model that maps pairwise comparisons into relative quality scores. The reward model, which is usually a separate model initialized from a pretrained LLM, is trained to output a single-number (scalar) reward score to each response that reflects its perceived quality.

The motivation is that the reward model can automatically score new model outputs, which eliminates the need for human annotation at every training step.

The second step in RLHF trains the LLM on the reward scores the reward model assigns to each of the LLMs' answers (for example, a bad answer may receive a -2, and a good answer may receive a +2).

We discuss RLHF here because it can be viewed as a precursor to reasoning-focused RL methods such as *reinforcement learning with verifiable rewards* (RLVR).

While the source of the training signal differs, human preferences versus automatically verifiable rewards, the underlying structure of the pipeline (generating responses, scoring them, and updating the model via RL) is closely related.

### 6.1.2 From human feedback to verifiable rewards (RLVR)

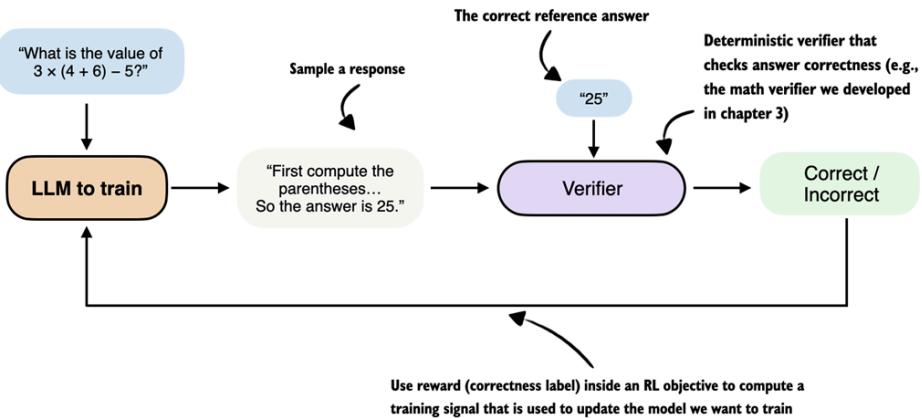
RLHF can be fairly involved because it requires training an additional reward model, which is often a large LLM that is (also) expensive to train.

*Reinforcement Learning with Verifiable Rewards* (RLVR) simplifies this setup by replacing the LLM reward model with verifiable rewards. Verifiable rewards are computed deterministically and without human annotation. For example, the math verifier introduced in chapter 3 is a verifiable reward generator for math problems.

For example, given a math problem (and a correct solution), a math verifier automatically checks whether a generated solution's final answer matches the ground truth and assigns a corresponding reward (1 for correct, and 0 for incorrect responses).

As a result, the two-step RLHF pipeline collapses into a single training loop in RLVR, which resembles step 2 in RLHF: the model generates responses, the verifier assigns rewards, and these rewards are used directly to update the model.

This simplified RLVR training procedure is outlined in figure 6.6.



**Figure 6.6 Overview of reinforcement learning with verifiable rewards (RLVR).** The LLM generates a response that is evaluated by a deterministic verifier, for example the math verifier from chapter 3, which assigns a correctness label used as a reward signal within an RL objective to update the model.

The popularity of RLVR can largely be traced to the success of DeepSeek-R1 in 2025 (<https://arxiv.org/abs/2501.12948>), which demonstrated that strong reasoning performance can be achieved without relying on human preference data or a learned reward model.

DeepSeek-R1 trained reasoning behavior using automatically verifiable rewards, including correctness checks for math problems (similar to our verifier in chapter 3) and code compilation and execution for code-related tasks.

Code execution is outside the scope of this book, as this would require training the LLM in a secure environment, which further complicates things compared to solving math problems. The idea behind verifying math tasks (checking whether an answer is correct, as a binary label) and code compilation (checking whether code compiles, also as a binary label) is similar, and the RLVR training algorithm would be identical.

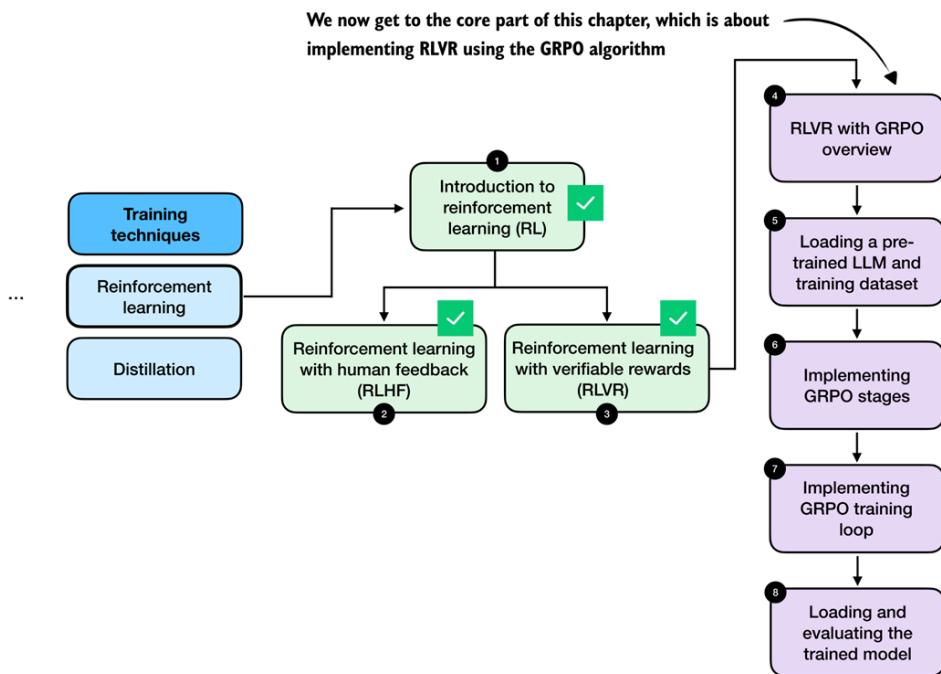
To summarize, RLVR offers several practical advantages over RLHF. It removes the need to train and maintain a separate reward model, which is often comparable in size and cost to the base LLM.

Verifiable rewards are also deterministic and reproducible, meaning they produce the same result every time for the same output, which avoids the noise and inconsistency that arise in human preference annotations.

Finally, RLVR scales naturally to large training sets, since, given a reference solution, rewards can be computed automatically for any number of model-generated answers without additional human labeling effort.

## 6.2 Reinforcement learning with verifiable rewards walkthrough using GRPO

Now that we have introduced the big picture and seen how RL fits within the overall development cycle of LLMs, we get to the concrete implementation. Specifically, we implement RLVR to train a reasoning model, as illustrated in the chapter roadmap in figure 6.7. This method is similar to DeepSeek-R1-Zero, but we apply it at a much smaller scale, on a smaller model with less data, since a comparable full training run would otherwise require several hundred thousand dollars in GPU compute.



**Figure 6.7 Chapter roadmap.** After introducing the two main reinforcement learning approaches for LLMs, RLHF and RLVR, the remaining sections focus on implementing RLVR using the GRPO algorithm, from dataset loading to implementing the full training loop.

Note that RLVR defines the overall training setup, namely, using automatically verifiable rewards. In addition, we need a concrete *policy optimization* algorithm that can be used within this RLVR framework to update the model weights and train the model. The term "policy" is an RL-specific jargon and refers, in this specific context, to the LLM we want to train.

Specifically, we will use the *group relative policy optimization* (GRPO) algorithm for policy optimization.

In short, RLVR determines what learning signal is available, and GRPO determines how that signal is used to update the model weights.

### POLICY GRADIENT ALGORITHMS

A widely used policy gradient algorithm for LLMs is *proximal policy optimization* (PPO), which was popularized in RLHF. In principle, the same algorithm could also be applied in an RLVR setting.

When training the DeepSeek-R1 reasoning models, the DeepSeek team opted for a simpler alternative, GRPO, which was originally introduced in their DeepSeekMath paper (<https://arxiv.org/abs/2402.03300>). GRPO is more resource-friendly than PPO because it does not require a separate value model to estimate a value function. Instead, GRPO derives its learning signal from relative comparisons within a group of sampled responses, which substantially reduces computational overhead.

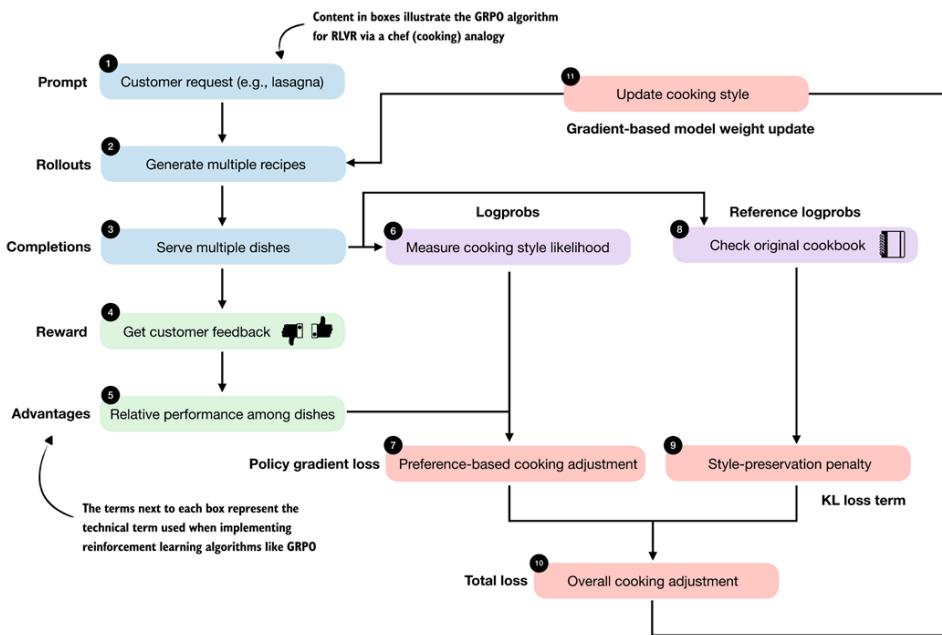
Interested readers can find a more detailed side-by-side comparison of PPO and GRPO in my article *The State of Reinforcement Learning for LLM Reasoning* (<https://magazine.sebastianraschka.com/p/the-state-of-lm-reasoning-model-training>). In this chapter, we focus on implementing RLVR using GRPO, which is similar to what DeepSeek-R1 and other popular reasoning models use.

In the following chapter, we build on this foundation and introduce several practical extensions to GRPO that further improve training stability and the resulting reasoning performance.

#### 6.2.1 High-level GRPO intuition via a chef analogy

The technical details of the GRPO algorithm for RLVR can be a bit overwhelming at first, since it has many moving parts and a lot of domain-specific RL terminology. So, before we start with a concrete technical overview and implementation, it may be beneficial to introduce the idea and mechanics behind GRPO with an example.

Figure 6.8 introduces the technical terms and approach of GRPO using a chef analogy preparing a meal.



**Figure 6.8 High-level overview of the GRPO algorithm for RLVR using a chef analogy. Multiple rollouts are generated and scored, relative advantages are computed, and a policy gradient objective with a KL-based regularization (loss) term is used to update the model parameters.**

So, walking through the flowchart in figure 6.8, imagine we are a chef running a small food delivery service:

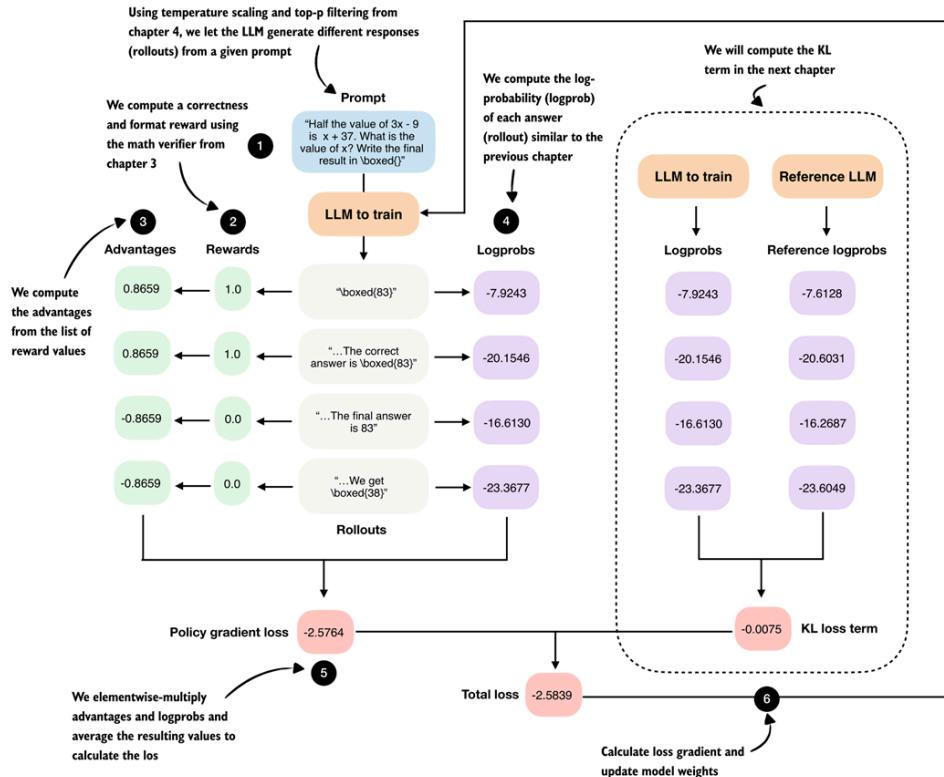
1. Each day, we receive a single customer request (the prompt) asking us to prepare a certain dish (for example, lasagna).
2. For that request, we cook our famous lasagna dish, but while doing so, we try out several different recipe variations (the *rollouts*).
3. After preparing the recipes, we create multiple dishes (*completions*). Note in the LLM setting, a rollout refers to the entire generation process for a prompt, while the completion is the resulting output text. In practice, the two terms are often used interchangeably.
4. Customers only give us feedback (the *reward*) after tasting the entire dish, not while it is being prepared. This means we judge the final result as a whole rather than individual cooking steps.
5. We compare how the dishes performed relative to each other based on the customer feedback from stage 4. This helps us identify which recipe variations received higher rewards and which received lower rewards for this particular request.

6. For each completed dish, we also keep track of how typical it was of our current cooking style, that is, how closely it followed our usual techniques and ingredient choices (*logprobs*).
7. Using relative feedback from stage 5 and how characteristic each dish was of our current cooking style from stage 6, we determine how to tweak the cooking style (*policy gradient loss*). Here, we want to reinforce choices that led to better dishes and reduce those that led to worse ones.
8. At the same time, we consult our original cookbook from the past.
9. We then apply a style-preservation penalty that encourages trying new recipe variations, but prevents overly drastic changes to our cooking style that could scare away existing customers.
10. The preference-based adjustment and the cooking style-preservation penalty are combined into a single overall measure or decision (the overall loss) about how the cooking style should change. The goal here is to balance customer satisfaction with consistency.
11. Finally, we update the cooking style (gradient-based model weight update) so that future dishes are more likely to satisfy customer preferences while still remaining close to the original cookbook.

Even though this is a relatively intuitive analogy, we can see that there are lots of components and moving parts in GRPO.

### **6.2.2 The high-level GRPO procedure**

Following the chef analogy from the previous section, the flowchart in figure 6.9 provides a technical overview of the GRPO algorithm for RLVR using concrete values and numbers that we will compute when implementing GRPO step by step.



**Figure 6.9 Step-by-step GRPO update for RLVR.** (1) A prompt is sampled and multiple rollouts are generated. (2) Each rollout is scored using a verifiable reward. (3) Group-relative advantages are computed from these rewards. (4) The log-probability of each rollout under the current model is calculated. (5) Advantages and log-probabilities are combined to form the policy gradient loss. (6) A KL regularization term against a reference model is added, and the resulting total loss is used to update the model parameters.

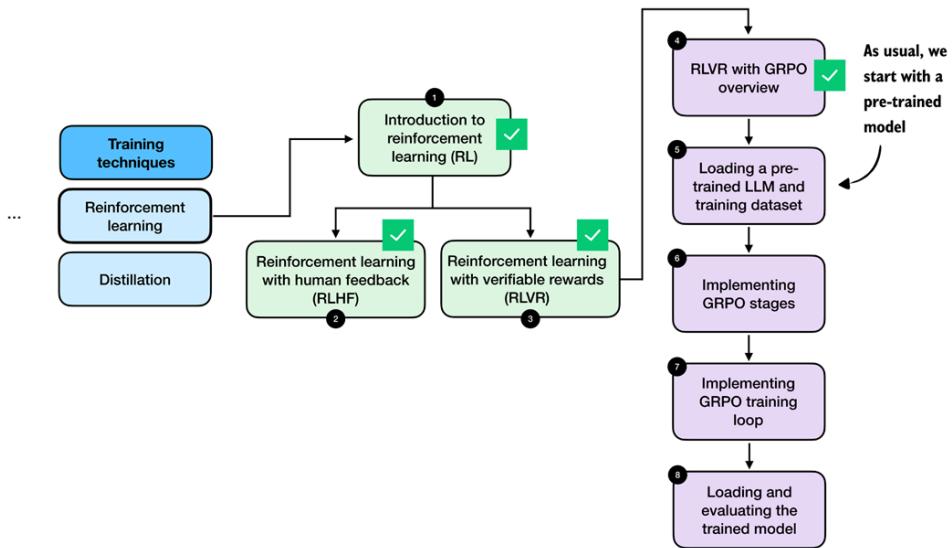
The GRPO outline in figure 6.9 contains many technical components, and at first glance, it may look overwhelming. We will start by implementing a simplified version of GRPO that omits the *KL loss* term shown on the right side of the figure.

In chapter 7, we will add this missing *KL loss* term for completeness. While the *KL loss* is part of the original GRPO formulation, it is not strictly essential. In fact, many LLM developers omit it in practice, as doing so both simplifies implementation and can sometimes improve modeling performance.

If this overview, shown in figure 6.9, feels complicated on first reading, don't worry. We will build up the algorithm step by step. For now, it is best to treat figure 6.9 as a high-level roadmap that helps orient us as we work through the individual pieces.

### 6.3 Loading a pre-trained model

Similar to previous chapters, we begin by loading the pre-trained model (see stage 5 in figure 6.10), which we will then use to generate the rollouts (answers to a given prompt) we need for GRPO.



**Figure 6.10** In stage 5, we load the pre-trained model (this section) and dataset (next section) that we will use for the model training.

Listing 6.1 is the same code we've used previously for loading the tokenizer and base model.

**Listing 6.1 Loading tokenizer and base model**

```
import torch
from reasoning_from_scratch.ch02 import get_device
from reasoning_from_scratch.ch03 import (
    load_model_and_tokenizer
)

device = get_device()
device = torch.device("cpu") #A

model, tokenizer = load_model_and_tokenizer(
    which_model="base",
    device=device,
    use_compile=False
)
```

#A Delete this line to run the code on a GPU (if supported by your machine)

Note that the code in listing 6.1 runs on the CPU by default to ensure you get results that are consistent with what's shown in this chapter. Once you have done a full pass through this chapter, I recommend deleting the line `device = torch.device("cpu")` and running the chapter code on a GPU.

Next, to ensure that the model is loaded correctly, let's use it together with the *temperature* and *top-p* sampler code from chapter 4 on a simple math prompt:

**Listing 6.2 Generating text with temperature scaling and top-p sampling**

```

from reasoning_from_scratch.ch03 import render_prompt
from reasoning_from_scratch.ch04 import (
    generate_text_stream_concat_flex,
    generate_text_top_p_stream_cache
)

raw_prompt = (
    "Half the value of $3x-9$ is $x+37$."
    "What is the value of $x$?"
)
prompt = render_prompt(raw_prompt)

torch.manual_seed(0)
response = generate_text_stream_concat_flex(
    model, tokenizer, prompt, device,
    max_new_tokens=2048, verbose=True,
    generate_func=generate_text_top_p_stream_cache,
    temperature=0.9,
    top_p=0.9
)
print(response)

```

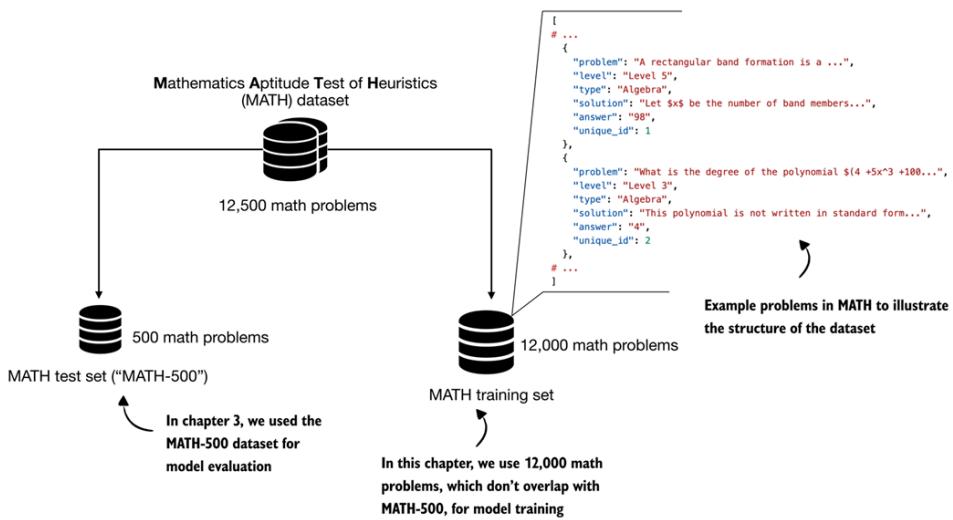
The model prints "`\boxed{58}`", which is an incorrect answer (83 is correct), but that's okay since the goal here was merely to make sure that the code runs without issue.

## 6.4 Loading a MATH training subset

Next, we load the dataset that we will use when training the model with GRPO.

We use a non-overlapping training subset derived from the original MATH dataset, which means that all MATH-500 evaluation problems are explicitly excluded from the training data. This prevents data leakage and ensures a clean separation between training and evaluation, as illustrated in figure 6.11.

**NOTE** For details on how this dataset was constructed, see [https://github.com/rasbt/math\\_full\\_minus\\_math500](https://github.com/rasbt/math_full_minus_math500).



**Figure 6.11 Structure and split of the MATH dataset.** The full dataset contains about 12,500 problems that are divided into a 500-problem test set (MATH-500), which we used for model evaluation in chapter 3. A non-overlapping set of 12,000 problems is used for training in this chapter.

To load the 12,000 math problems from the MATH training set depicted in figure 6.11, we define the following `load_math_train` function in listing 6.3, which is similar to the `load_math500_test` function in chapter 3, except that we specify a different file path.

**Listing 6.3 Loading the MATH training set**

```

import json
from pathlib import Path
import requests

def load_math_train(local_path="math_train.json", save_copy=True):
    local_path = Path(local_path)
    url = (
        "https://raw.githubusercontent.com/rasbt/"
        "math_full_minus_math500/refs/heads/main/"
        "math_full_minus_math500.json"
    )

    if local_path.exists():
        with local_path.open("r", encoding="utf-8") as f:
            data = json.load(f)
    else:
        r = requests.get(url, timeout=30)
        r.raise_for_status()
        data = r.json()

        if save_copy: # Saves a local copy
            with local_path.open("w", encoding="utf-8") as f:
                json.dump(data, f, indent=2)

    return data

math_train = load_math_train()
print("Dataset size:", len(math_train))

```

The above code should print "Dataset size: 12000" to indicate that the dataset was loaded correctly.

Next, we can also print one of the 12,000 entries to get an idea of the dataset structure. For this, we use the `pprint` function from the `pprint` standard library, as it provides nicer formatting for dictionary entries compared to the regular `print` function:

```

from pprint import pprint
pprint(math_train[4])

```

The resulting entry (index 4 corresponds to the fifth entry in the dataset) is shown below:

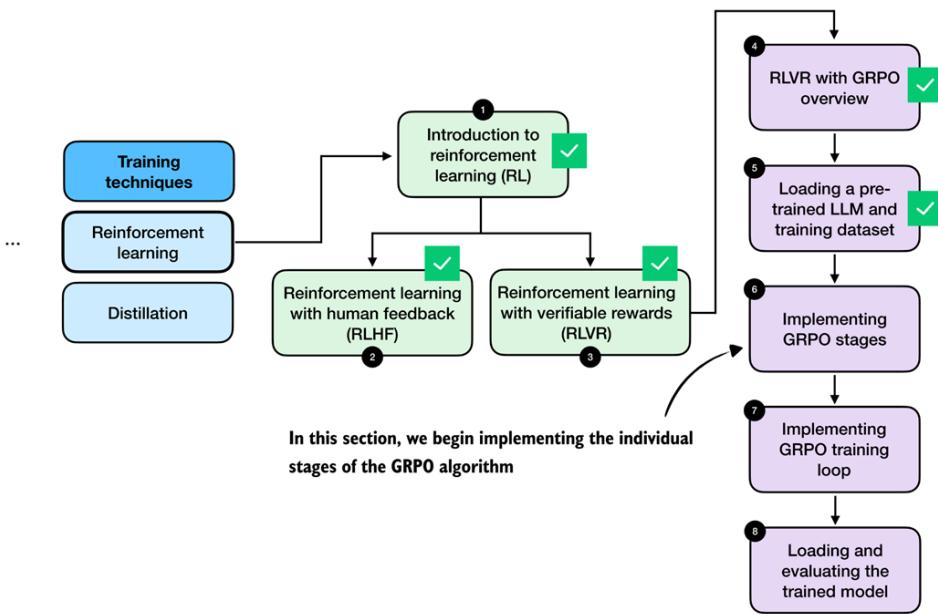
```
{
  "answer": "6",
  "level": "Level 3",
  "problem": (
    "Sam is hired for a 20-day period. On days that he "
    "works, he earns $\\$60. For each day that he does "
    "not work, $\\$30 is subtracted from his earnings. "
    "At the end of the 20-day period, he received "
    "$\\$660. How many days did he not work?"
  ),
  "solution": (
    "Call  $x$  the number of days Sam works and  $y$  the "
    "number of days he does not... Thus, Sam did not "
    "work for  $\\boxed{6}$  days."
  ),
  "type": "Algebra",
  "unique_id": 4,
}
}
```

As we can see, each dataset example is stored as a dictionary with several fields. For training, the relevant fields are "problem", which serves as the prompt, and "answer", which is the target we verify the model's output against using the math verifier.

The dataset also includes a full worked solution in the "solution" field. While such step-by-step solutions could in principle be used to evaluate intermediate reasoning steps, doing so would unnecessarily constrain the model and risk overfitting to a specific solution and style. Instead, we want to allow the model to explore the solution space more freely.

## 6.5 Sampling rollouts

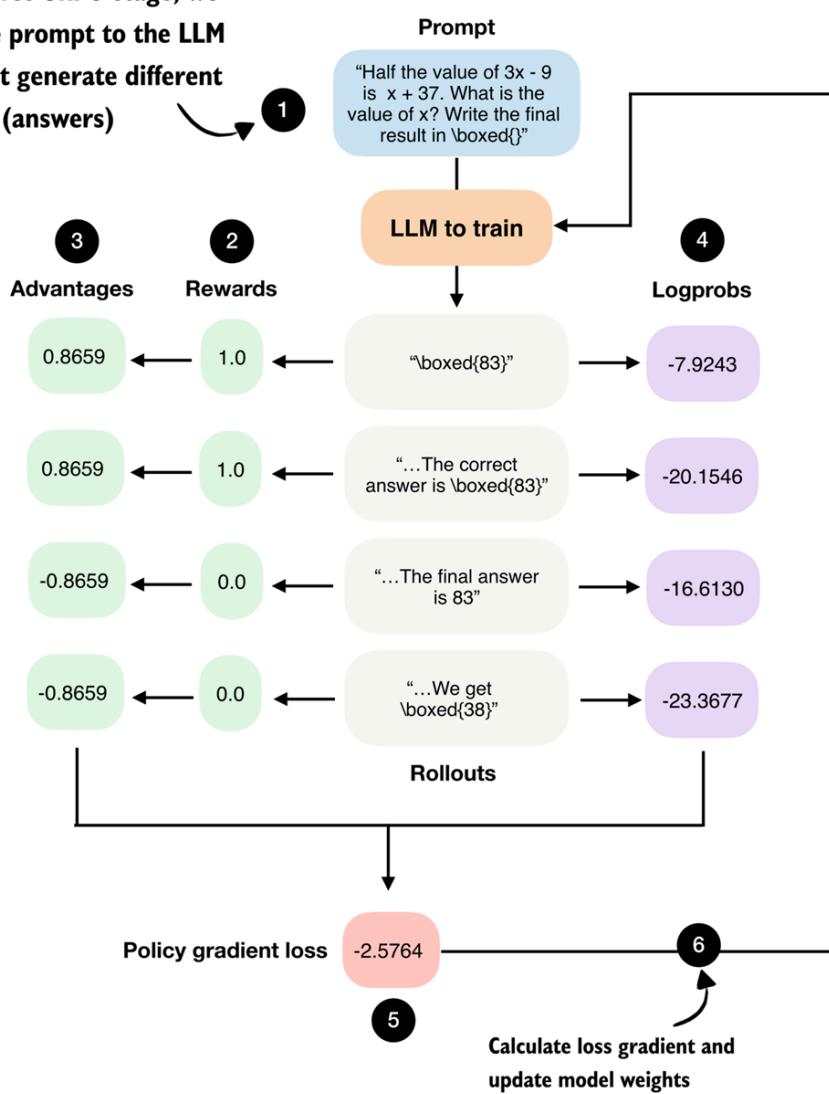
After setting up the pre-trained base model and loading the dataset we are now ready to implement the GRPO stages, as illustrated in figure 6.12.



**Figure 6.12** After outlining the RLVR method and GRPO algorithm, the following sections implement the individual GRPO stages that we need to train the LLM via verifiable rewards on the MATH dataset.

Before we begin, figure 6.13 revisits the GRPO overview introduced earlier, but in a simplified form that omits the KL loss term, which we will add in the next chapter. Referring back to this overview will help us orient ourselves as we step through the algorithm's components.

In this first GRPO stage, we pass the prompt to the LLM and let it generate different rollouts (answers)



**Figure 6.13 Step-by-step GRPO update for RLVR (without KL loss term).** We begin by prompting the LLM to generate the different rollouts.

As shown in figure 6.13, we first use the LLM to generate multiple *rollouts*. Here, "rollout" is a reinforcement learning term that simply refers to a complete answer generated by the model for a given prompt.

We could reuse the `generate_text_stream_concat_flex` function from listing 6.2 to generate the rollouts. When we defined that function earlier, though, we decorated it with `@inference_mode`, which disables several PyTorch features for efficiency. Since we will later perform a backward pass, this makes the function incompatible with our training setup. Instead, we need to use the `@torch.no_grad` decorator, which disables gradient tracking for the forward pass without switching PyTorch into inference-only mode.

Let's rewrite the function in a more compact form and call it `sample_response`. Importantly, this function generates text identical to `generate_text_stream_concat_flex(..., generate_func=generate_text_top_p_stream_cache)`. We can verify this by observing that, with the same `random_seed`, `temperature`, and `top_p` settings, both functions produce identical generated responses.

**Listing 6.4 Defining the rollout generation function**

```

from reasoning_from_scratch.qwen3 import KVCache
from reasoning_from_scratch.ch04 import top_p_filter


@torch.no_grad()
def sample_response(
    model,
    tokenizer,
    prompt,
    device,
    max_new_tokens=512,
    temperature=0.8,
    top_p=0.9,
):
    input_ids = torch.tensor(
        tokenizer.encode(prompt),
        device=device
    )

    cache = KVCache(n_layers=model.cfg["n_layers"]) #A
    model.reset_kv_cache()
    logits = model(input_ids.unsqueeze(0), cache=cache)[:, -1]

    generated = []
    for _ in range(max_new_tokens):
        if temperature and temperature != 1.0: #B
            logits = logits / temperature

        probas = torch.softmax(logits, dim=-1)
        probas = top_p_filter(probas, top_p) #C
        next_token = torch.multinomial(
            probas.cpu(), num_samples=1
        ).to(device)

        if (
            tokenizer.eos_token_id is not None
            and next_token.item() == tokenizer.eos_token_id
        ):
            break
        generated.append(next_token.item())
        logits = model(next_token, cache=cache)[:, -1]

```

```

full_token_ids = torch.cat(
    [input_ids,
     torch.tensor(generated, device=device, dtype=input_ids.dtype),]
)
return full_token_ids, input_ids.numel(), tokenizer.decode(generated)

#A Use KVCache similar to chapter 2
#B Apply temperature scaling from chapter 4
#C Apply top-p filter from chapter 4

```

Note that the function now also returns the token IDs of the prompt plus answer tokens and the number of tokens (`input_ids.numel()`) next to the answer text (`tokenizer.decode(generated)`). Returning these allows us to simplify several downstream functions later on.

Otherwise, there is nothing new here. The code is simply a leaner version of what we have been developing previously; it combines the `generate_text_basic_cache` function from chapter 2 with temperature and top-p sampling from chapter 4 directly.

Next, we call the function on an example prompt similar to listing 6.2.

#### **Listing 6.5 Generating rollouts with temperature scaling and top-p sampling**

```

torch.manual_seed(0)

raw_prompt = (
    "Half the value of $3x-9$ is $x+37$.\n"
    "What is the value of $x$?"
)
prompt = render_prompt(raw_prompt)

token_ids, prompt_len, answer_text = sample_response(
    model=model,
    tokenizer=tokenizer,
    prompt=prompt,
    device=device,
    max_new_tokens=512,
    temperature=0.9,
    top_p=0.9,
)
print(answer_text)

```

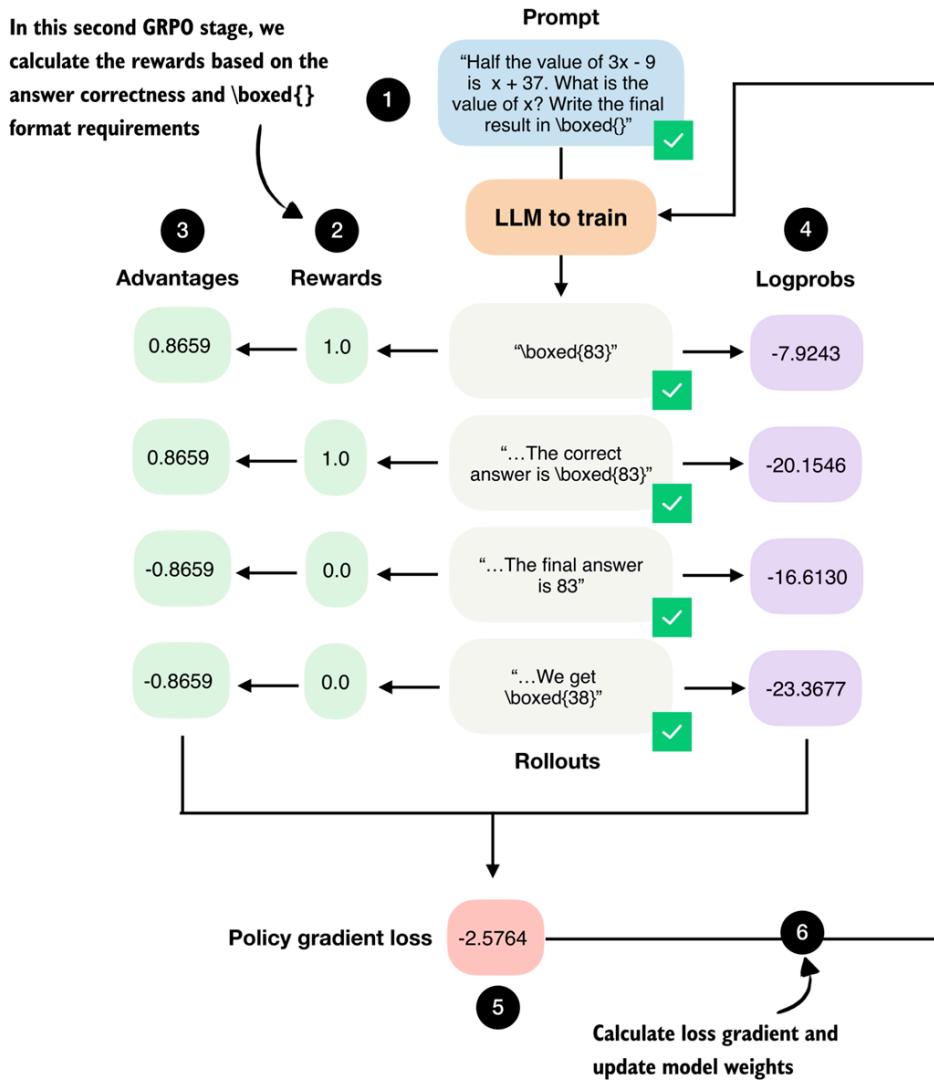
Similar to before, the function returns "`\boxed{58}`".

Next, we could call `sample_response` multiple times to generate different rollouts, which we will indeed do later when implementing the full training loop. For now, to keep the GRPO walkthrough example simple and to make it easier to follow the step-by-step GRPO outline from figure 6.13, we instead assume that the model produced the following four short responses:

```
rollouts = [
    r"\boxed{83}",
    r"The correct answer is \boxed{83}",
    r"The final answer is 83",
    r"We get \boxed{38}",
]
```

## 6.6 Calculating rewards

The second stage of the GRPO procedure involves computing rewards for each rollout, as shown in figure 6.14.



**Figure 6.14** The second stage in the GRPO pipeline computes the rewards for each rollout the LLM generated in the previous section.

The rewards are computed using the math verifier from chapter 3 and are primarily based on answer correctness. By setting `fallback=None` in the `reward_rlvr` function in listing 6.6, we also include an implicit format constraint. For instance, an answer only receives a reward of `1.0` if it is both correct and expressed in the required `\boxed{}` format.

**Listing 6.6 Implementing the reward function**

```
from reasoning_from_scratch.ch03 import (
    extract_final_candidate, grade_answer
)

def reward_rlvr(answer_text, ground_truth):
    extracted = extract_final_candidate(
        answer_text, fallback=None #A
    )
    if not extracted:
        return 0.0
    correct = grade_answer(extracted, ground_truth)
    return float(correct)

#A fallback=None requires \boxed{} format to return an extracted answer
```

Since the model receives a non-zero reward only if it answers correctly and writes the final answer in the `\boxed{}` format, we encourage it to learn to produce correct *and* properly formatted answers.

Let's give the `reward_rlvr` function a try and apply it to the rollouts.

**Listing 6.7 Applying the reward function to all rollouts**

```
rollout_rewards = []

for answer in rollouts:
    reward = reward_rlvr(answer_text=answer, ground_truth="83")
    print(f"Answer: {answer!r}")
    print(f"Reward: {reward}\n")
    rollout_rewards.append(reward)
```

The resulting outputs are as follows:

```
Answer: '\\boxed{83}'
```

Reward: 1.0

```
Answer: 'The correct answer is \\boxed{83}'
```

Reward: 1.0

```
Answer: 'The final answer is 83'
```

Reward: 0.0

```
Answer: 'We get \\boxed{38}'
```

Reward: 0.0

As shown above, the reward function works as intended and only provides a reward of 1.0 if the answer contains the correct result (83) and uses the `\boxed{}` format.

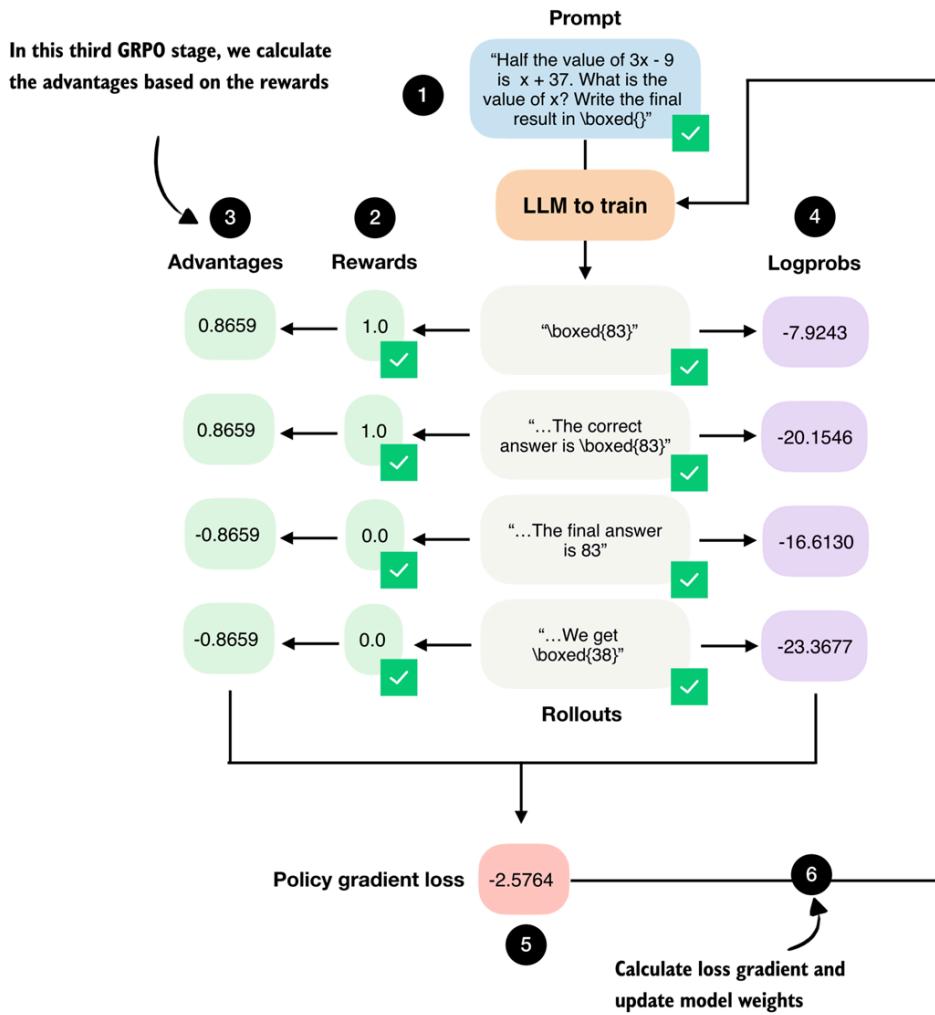
Note that the DeepSeek-R1 team also tried to use process reward models to score intermediate solution steps during training. These attempts were unsuccessful, and the researchers concluded that it is better to train only on final-answer correctness rewards, without intermediate rewards.

#### **EXERCISE 6.1: ADDING FORMAT-AWARE REWARD SHAPING**

Extend the `reward_rlvr` function from this chapter so that it assigns partial credit based on output format. Specifically, modify the reward function so that it returns 1.0 if the model produces the correct answer in the required `\boxed{}` format, 0.5 if the answer is correct but not boxed, and 0.0 otherwise.

## **6.7 Preparing learning signals from rollouts via advantages**

We now move on from rewards to the so-called *advantages*, as shown in figure 6.15. While rewards tell us how well each individual rollout performed, advantages capture how a rollout performed relative to other rollouts generated for the same prompt.



**Figure 6.15** The third GRPO stage computes the advantage values from the answer (rollout) correctness rewards.

The advantage values shown in figure 6.15 are computed by a simple formula:

$$\text{advantages}_i = (r_i - \mu_r) / (\sigma_r + \epsilon)$$

Where  $r_i$  denotes the reward of the  $i$ -th rollout,  $\mu_r$  is the mean reward across the group of rollouts,  $\sigma_r$  is the corresponding standard deviation, and  $\epsilon$  is a small constant added for numerical stability to avoid zero-division errors.

In code, we can implement the advantage calculation as follows:

#### **Listing 6.8 Calculating advantages**

```
rewards = torch.tensor(rollout_rewards, device=device)
advantages = (rewards - rewards.mean()) / (rewards.std() + 1e-4)
print(rewards)
print(advantages)
```

The rewards, which we computed in the previous section, are [1., 1., 0., 0.], and the corresponding advantages are [0.8659, 0.8659, -0.8659, -0.8659].

These advantage values capture which responses performed better than average and which performed worse than average within the group.

**NOTE** The "GR" (group relative) in GRPO refers to the fact that GRPO generates multiple answers (rollouts) per prompt, and compares them relative to each other to construct a learning signal.

You might still wonder what the point of converting rewards into advantages is. At a practical level, the advantage values directly scale the gradients during the policy update later.

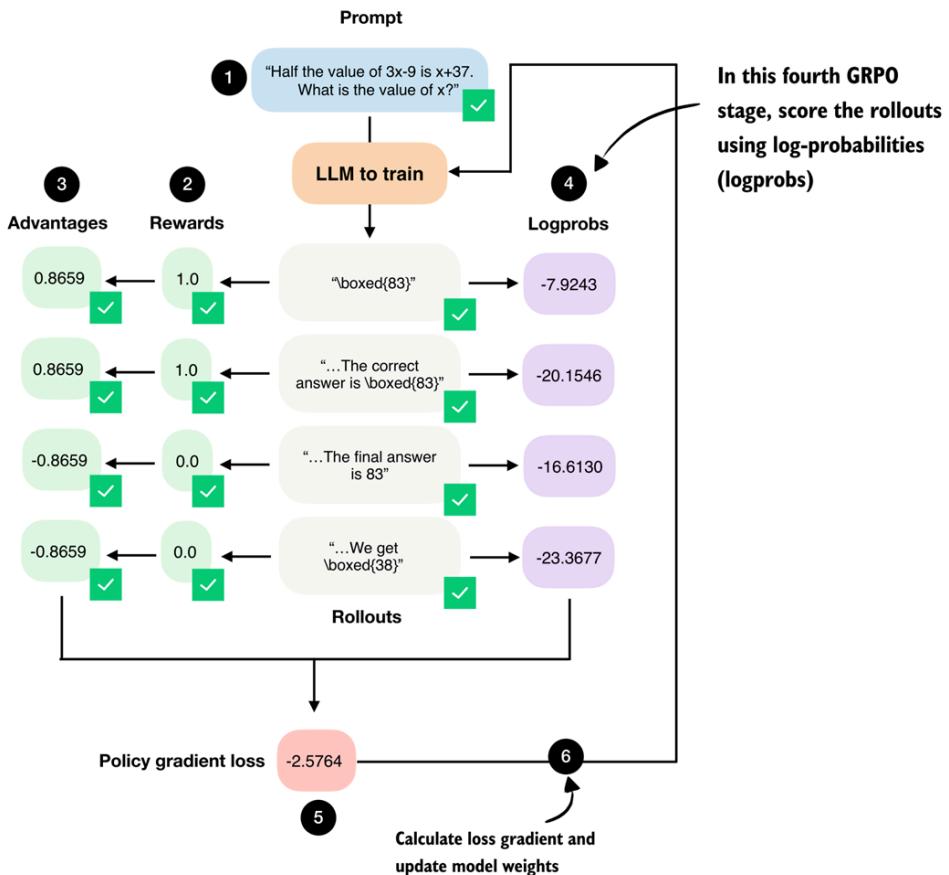
If the advantage is positive, the gradient increases the likelihood of the actions that produced that rollout. If it is negative, the gradient decreases their likelihood. Rollouts with advantage close to zero contribute very little.

#### **EXERCISE 6.2: ZERO-ADVANTAGE CASES**

Modify the rollout rewards so that all rollouts receive the same reward (for example, force all rewards to 0.0 or all to 1.0). Run the advantage computation and verify that the resulting GRPO loss is zero. Why is this behavior desirable in group-relative policy optimization?

## 6.8 Scoring rollouts with sequence log-probabilities

We calculated the rewards and advantages from the rewards. Now, we take a separate branch from the rollouts and compute their *log-probabilities* (*logprobs*), as shown in figure 6.16. Logprobs measure how likely the model considers each generated token under its current parameters, as discussed in the previous chapter.



**Figure 6.16** The fourth stage in the GRPO pipeline computes log-probabilities for each rollout, which is related to the logprob scorer we developed in the previous chapter.

As shown in figure 6.16, these logprobs, together with the advantages, form the core ingredients of the GRPO policy gradient loss, which we will implement later on.

Coming back to the logprobs, in the previous chapter we implemented an `avg_logprob_answer` function that computes the per-token logprobs for the model's answer tokens. These values, often referred to as token-level logprobs in the literature, are obtained by evaluating the likelihood that the model assigns to each generated token under its current parameters.

When averaged across the response, these values are often referred to as token-level logprobs and are commonly used for scoring LLM outputs. Averaging is preferred in this context because it provides length normalization, which makes the scores comparable across responses of different lengths.

### MATHEMATICAL DEFINITION OF TOKEN-LEVEL LOG-PROBABILITIES

Mathematically, the token-level logprobs we computed in the previous chapter can be written as

$$\frac{1}{T} \sum_{t=1}^T \log p_w(y_t | y_{<t}, x)$$

Here

- $y_1, y_2, \dots, y_T$  denote the tokens in the generated response of length  $T$
- $y_{<t}$  represents all previously generated tokens
- $x$  is the input prompt
- $W$  denotes the model's weight parameters.

This expression is mathematically identical to the one used in the previous chapter; we simply switch from  $x$  to  $y$  to distinguish the generated output tokens from the input prompt for clarity.

For reference, the `avg_logprob_answer` function from the previous chapter is copied below, which we use to compute the logprobs for the previous prompt and `answer_text` for illustration purposes.

**Listing 6.9 Computing token-level log-probabilities (similar to chapter 5)**

```

@torch.inference_mode()
def avg_logprob_answer(model, tokenizer, prompt, answer, device="cpu"):

    prompt_ids = tokenizer.encode(prompt)
    answer_ids = tokenizer.encode(answer)
    full_ids = torch.tensor(prompt_ids + answer_ids, device=device)

    logits = model(full_ids.unsqueeze(0)).squeeze(0)
    logprobs = torch.log_softmax(logits, dim=-1)

    start = len(prompt_ids) - 1
    end = full_ids.shape[0] - 1

    t_idx = torch.arange(start, end, device=device)
    next_tokens = full_ids[start + 1 : end + 1]
    next_token_logps = logprobs[t_idx, next_tokens]

    return torch.mean(next_token_logps).item()

avg_logprob_val = avg_logprob_answer(
    model, tokenizer,
    prompt=prompt,
    answer=answer_text,
    device=device)
print(avg_logprob_val)

```

This returns `-0.0517578125`.

In GRPO, however, we do not use averaged token-level logprobs. Instead, we work with sequence-level logprobs.

The reason is that GRPO assigns a single reward and advantage to each rollout, which applies to the entire generated response. Intuitively, logprob should reflect how likely the model was to generate the whole sequence, not an average per token.

We compute sequence-level logprobs by summing the logprobs of all generated tokens. (This sum corresponds to the log-likelihood of producing the full response under the current model parameters.)

In contrast, averaging token-level logprobs normalizes by sequence length. While this normalization is useful for scoring and comparing responses of different lengths, it would unintentionally rescale the learning signal in policy optimization and cause longer and shorter responses to contribute unevenly to the update. (While this is not the intention of the original GRPO formulation, we will revisit token-level logprobs when improving the algorithm in the next chapter.)

We can convert the token-level, length-normalized score into a sequence-level logprobs by dropping the averaging step and replacing `torch.mean(next_token_logps)` with `torch.sum(next_token_logps)` in the code in listing 6.9.

The same result can also be obtained by multiplying the averaged logprob by the number of answer tokens, as long as the token count matches exactly:

```
sequence_logprob_val = avg_logprob_val * (
    len(tokenizer.encode(answer_text)))
)
print(sequence_logprob_val)
```

This now returns `-11.490234375`.

These sequence-level logprobs scale linearly with the sequence length  $T$ , which means that longer responses generally receive more negative logprob values. As a result, for two equally good answers, the optimization implicitly favors the shorter one, since it is cheaper in terms of likelihood. Summed logprobs therefore encourage the model to stop earlier unless producing a longer response is justified by a higher reward.

So, as mentioned before, we can replace `torch.mean` with `torch.sum` in the function to obtain sequence-level logprobs. Since the function was run in inference mode in the previous chapter using the `@torch.inference_mode()` decorator, we need to redefine it (without the decorator) anyway so that PyTorch can track gradients.

In addition, because `sample_response` from section 6.5 already returns the `token_ids` and `prompt_len`, we can simplify the implementation by dropping the explicit tokenization step and the construction of `full_ids`.

**Listing 6.10 Computing sequence-level log-probabilities**

```

def sequence_logprob_draft(model, token_ids, prompt_len):
    logits = model(token_ids.unsqueeze(0)).squeeze(0).float()
    logprobs = torch.log_softmax(logits, dim=-1)

    start = prompt_len - 1          #A
    end = token_ids.shape[0] - 1    #A

    t_idx = torch.arange(start, end, device=token_ids.device)
    next_tokens = token_ids[start + 1 : end + 1]
    next_token_logps = logprobs[t_idx, next_tokens]

    return torch.sum(next_token_logps) #B

print(sequence_logprob_draft(model, token_ids, prompt_len))

```

#A Positions whose next-token probabilities we want to predict  
#B Sum log-probabilities over the answer tokens

This outputs:

```
tensor(-11.5178, grad_fn=<SumBackward0>)
```

First, the result is similar to the `-11.490234375` we got earlier (the difference is due to floating-point behavior and rounding).

PyTorch internally builds a computation graph that records each differentiable operation applied to a tensor. The `SumBackward0` entry shows that the summation of token log-probabilities is part of this graph, which means gradients can propagate back through the sequence-level log-probability to the model parameters.

This is exactly what we need for policy optimization later, when we implement the training loop to update the model weights via a backward pass. The backward pass applies the backpropagation algorithm, which is the standard method used in deep learning to compute gradients and update neural network weights.

## PYTORCH COMPUTATION GRAPHS, GRADIENTS, AND BACKPROPAGATION

As the model produces an output, PyTorch records each mathematical operation that leads from the model parameters to that output. This record is called the *computation graph*. When we later run the backward pass, PyTorch uses this graph to perform backpropagation, that is, computing gradients that describe how changes to the model parameters would affect the final result. If an operation is part of this graph, PyTorch can compute gradients through it during training.

It is not required to understand PyTorch's computation graphs for this chapter, but interested readers can read more about it in my tutorial at <https://sebastianraschka.com/teaching/pytorch-1h/#3-seeing-models-as-computation-graphs>.

While this draft implementation is a fully working code implementation, we can simplify it a bit and make it more efficient for GPUs. In particular, we can avoid explicitly constructing index ranges and instead use `torch.gather` to directly select the logprob corresponding to the generated tokens. Listing 6.11 shows a more compact, optimized version of the same computation that produces identical results while being easier to read and faster to execute.

### Listing 6.11 Optimized sequence-level log-probabilities code

```
def sequence_logprob(model, token_ids, prompt_len):
    logits = model(token_ids.unsqueeze(0)).squeeze(0).float()
    logprobs = torch.log_softmax(logits, dim=-1)
    selected = logprobs[:-1].gather(
        1, token_ids[1:].unsqueeze(-1)
    ).squeeze(-1)
    return torch.sum(selected[prompt_len - 1:])

print(sequence_logprob(model, token_ids, prompt_len))
```

Similar to the previous code, this returns `tensor(-11.5178, grad_fn=<SumBackward0>)`.

---

**TIP** We could compute these `logprobs` directly in the `sample_response` function to improve efficiency by avoiding calling `model(...)` twice (in both the `sample_response` and `sequence_logprob` functions).

---

Finally, with a robust and efficient sequence-level logprob function in place, we can calculate the respective logprobs of the four different rollouts.

**Listing 6.12 Computing sequence-level log-probabilities of all rollouts**

```
rollouts = [
    r"\boxed{83}",
    r"The correct answer is \boxed{83}",
    r"The final answer is 83",
    r"We get \boxed{38}",
]

rollout_logps = []

for text in rollouts:
    token_ids = tokenizer.encode(prompt + " " + text)
    logprob = sequence_logprob(
        model=model,
        token_ids=torch.tensor(token_ids, device=device),
        prompt_len=prompt_len,
    )

    print(f"Answer: {text}")
    print(f"Logprob: {logprob.item():.4f}\n")

    rollout_logps.append(logprob)
```

This prints the following:

```
Answer: \boxed{83}
Logprob: -7.9243

Answer: The correct answer is \boxed{83}
Logprob: -20.1546

Answer: The final answer is 83
Logprob: -16.6130

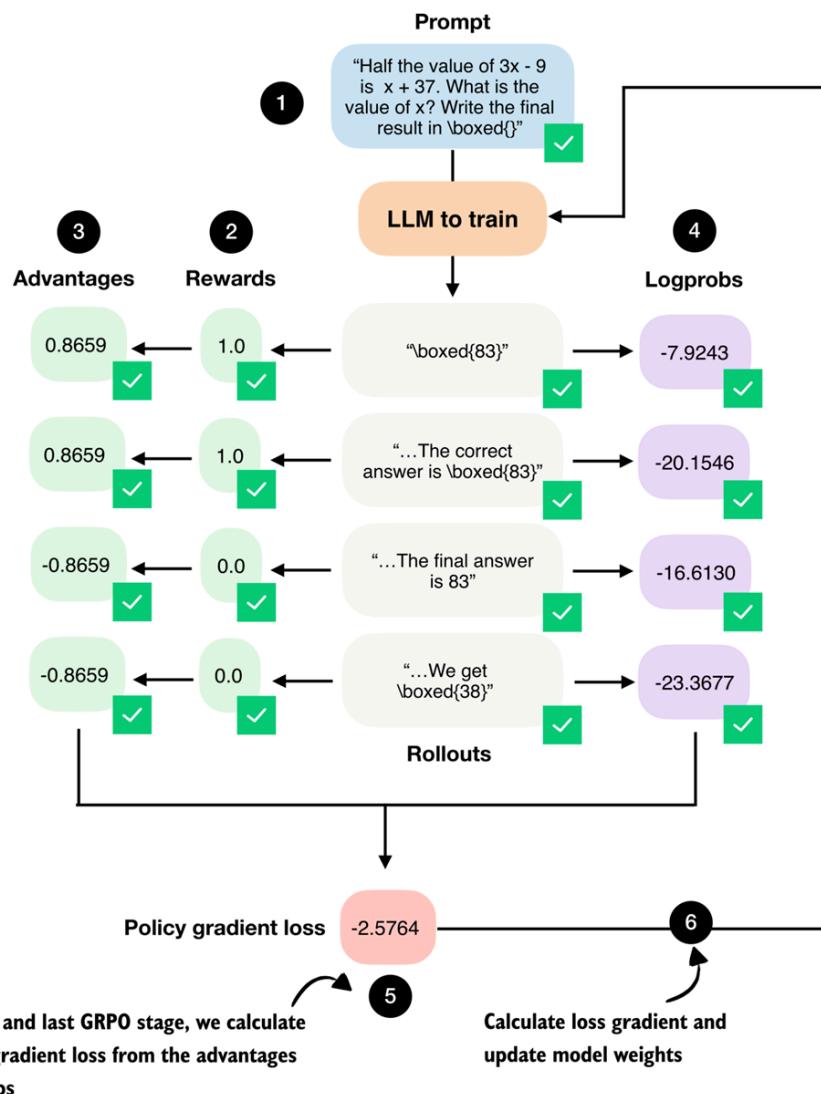
Answer: We get \boxed{38}
Logprob: -23.3677
```

As we can see, shorter and more concise answers receive higher (less negative) sequence-level logprobs than longer or more verbose responses. The exception is the last answer, which receives the lowest score. Note that this is the only answer that contains the incorrect numerical answer value (38 instead of 83), so this also makes intuitive sense.

Overall, this illustrates how summed logprobs naturally favor concise outputs, which is consistent with their role in GRPO when rewards and advantages are applied at the sequence level.

## 6.9 From advantages to policy updates via the GRPO loss

We now implement the fifth stage of the GRPO pipeline, where the previously computed advantages and logprobabs are combined into a policy gradient loss. We implement the subsequent weight update (stage six) later as part of the training loop, as shown in figure 6.17.



**Figure 6.17** The fifth stage in the GRPO pipeline computes the policy gradient loss that we use to update the model. Stage number 6, the model weight update, will be implemented as part of the training loop later.

First, we convert the list of the rollout logprobs from the previous section into a PyTorch tensor. Then, we compute the policy gradient loss by multiplying each rollout's sequence-level logprob by its corresponding advantage, taking the mean across rollouts, and applying a negative sign. Regarding the negative sign, because PyTorch optimizers are defined to *minimize* a loss, objectives that are naturally written as maximization problems must be sign-flipped.

**Listing 6.13 Computing the policy gradient loss**

```
logps = torch.stack(rollout_logps)
pg_loss = -(advantages.detach() * logps).mean()
print(logps)
print(pg_loss)
```

This prints

```
tensor([-7.9243, -20.1546, -16.6130, -23.3677],
grad_fn=<StackBackward0>)
```

for the logprobs, and the resulting policy gradient loss value is

```
tensor(-2.5764, grad_fn=<NegBackward0>)
```

Note that we use `.detach()` on the advantages because they are treated as fixed learning signals during the policy update. This prevents gradients from flowing back through the advantage computation. This ensures that only the model parameters influencing the logprobs are updated.

For instance, in policy gradient methods, we want to maximize the advantage-weighted logprob of the rollouts, since higher log-probabilities for high-advantage rollouts improve the policy.

So, by multiplying this objective by  $-1$ , we convert the maximization problem into an equivalent minimization problem. Note that minimizing the negative objective produces the exact same parameter updates as maximizing the original one, but this way, it remains compatible with PyTorch's optimizer implementations.

## MAXIMIZATION VERSUS MINIMIZATION

To illustrate the sign flipping further with a concrete example, suppose the objective we want to maximize is a simple scalar value  $f(x) = 3$ .

Under the maximization view, a larger value is better, so 3 is preferable to 2. However, PyTorch optimizers minimize, so they would try to *reduce* this value, which is the opposite of what we want.

Now, suppose we flip the sign and define the loss as  $L(x) = -f(x) = -3$ .

Minimizing  $L(x)$  is now equivalent to maximizing  $f(x)$ . For instance, if  $L(x)$  decreases from -2 to -3,  $f(x)$  increases from 2 to 3.

## MATHEMATICAL DEFINITION OF THE POLICY GRADIENT LOSS

For readers who find it easier to follow the computations via mathematical notation, the policy gradient loss used in GRPO can be written as

$$L_{PG} = -\frac{1}{N} \sum_{i=1}^N A_i \sum_{t=1}^{T_i} \log p_w(y_t^{(i)} | y_{<t}^{(i)}, x^{(i)})$$

Here,

- $N$  denotes the number of rollouts,
- $y_t^{(i)}, \dots, y_{T_i}^{(i)}$  are the tokens of the  $i$ -th generated response of length  $T_i$ ,
- $y_{<t}^{(i)}$  represents all previously generated tokens in that response,
- $x^{(i)}$  is the corresponding input prompt,
- And  $A_i$  is the advantage of the full rollout.

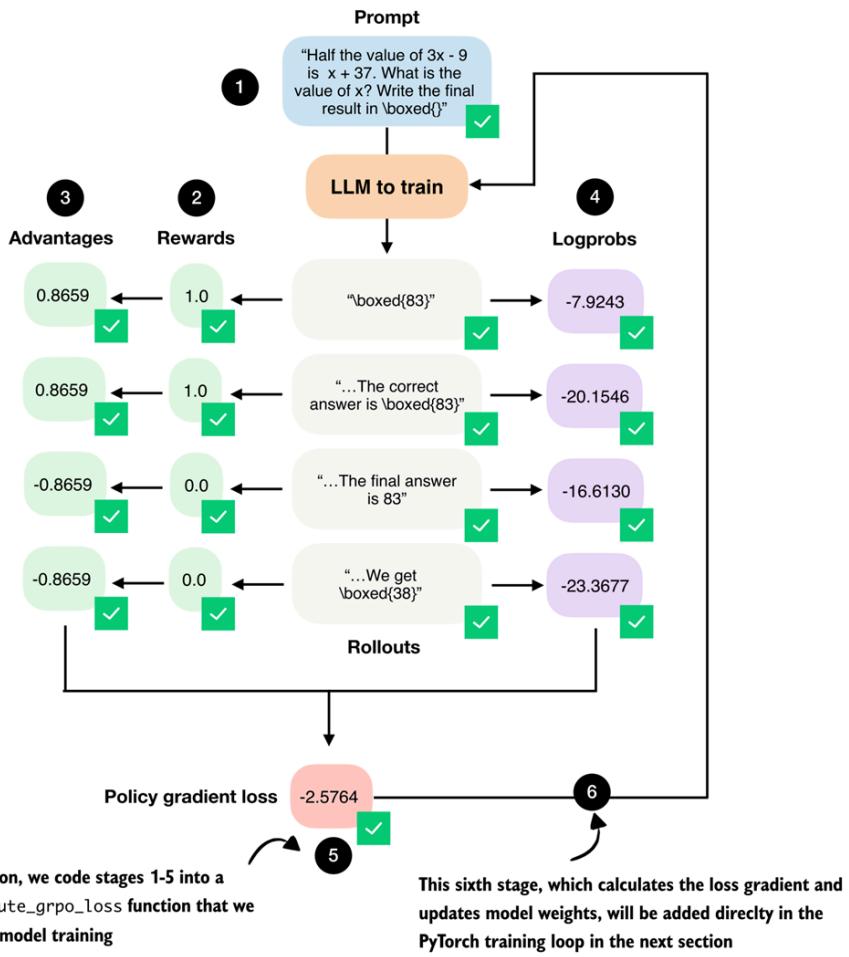
The inner sum computes the sequence-level log-probability of a rollout, while the outer average computes the advantage-weighted log-probabilities across rollouts.

## 6.10 Putting everything together in a single GRPO function

We have now completed the most challenging parts of this chapter and walked through each GRPO stage in isolation.

Next, we combine the five GRPO stages shown in figure 6.18 into a single `compute_grpo_loss` function, which we will later use as part of the full training loop.

(Note that here, the term stage refers to a conceptual component of the GRPO loss computation, based on how we walked through the GRPO algorithm step-by-step, not a training step in the outer optimization loop.)



**Figure 6.18** Overview of the complete GRPO workflow where (1) multiple rollouts are generated for a prompt, (2) assigned correctness rewards, (3) converted into group-relative advantages, and (4) combined with log probabilities to (5) compute the policy gradient loss. The loss gradients (6) will be computed and used to update the model in the next section.

The `compute_grpo_loss` function that combines all the stages from figure 6.18 that we discussed previously is shown in listing 6.14 below.

**Listing 6.14 Combining all GRPO stages**

```

def compute_grpo_loss(
    model,
    tokenizer,
    example,
    device,
    num_rollouts=2,
    max_new_tokens=256,
    temperature=0.8,
    top_p=0.9,
):
    assert num_rollouts >= 2
    roll_logps, roll_rewards, samples = [], [], []
    prompt = render_prompt(example["problem"])

    was_training = model.training
    model.eval()

    for _ in range(num_rollouts):
        #A
        token_ids, prompt_len, text = sample_response(
            model=model,
            tokenizer=tokenizer,
            prompt=prompt,
            device=device,
            max_new_tokens=max_new_tokens,
            temperature=temperature,
            top_p=top_p,
        )
        #B
        reward = reward_rlvr(text, example["answer"])
        #C
        logp = sequence_logprob(model, token_ids, prompt_len)

        roll_logps.append(logp)
        roll_rewards.append(reward)
        samples.append(
            {
                "text": text,
                "reward": reward,
                "gen_len": token_ids.numel() - prompt_len,
            }
        )

```

```

if was_training:
    model.train()
#D
rewards = torch.tensor(roll_rewards, device=device)
#E
advantages = (rewards - rewards.mean()) / (rewards.std() + 1e-4)
#F
logps = torch.stack(roll_logps)
#G
pg_loss = -(advantages.detach() * logps).mean()
loss = pg_loss # In the next chapter we add a KL term here

return {
    "loss": loss.item(),
    "pg_loss": pg_loss.item(),
    "rewards": roll_rewards,
    "advantages": advantages.detach().cpu().tolist(),
    "samples": samples,
    "loss_tensor": loss,
}

```

```

#A Stage 1: generate rollouts
#B Stage 2.1: compute rewards
#C Stage 4.1: compute logprobs
#D Stage 2.2: collect all rewards
#E Stage 3: compute advantages
#F Stage 4.2: collect all logprobs
#G Stage 5: compute policy gradient loss

```

The `compute_grobo_loss` function is relatively self-explanatory as it follows the exact stages we implemented manually in the previous sections. Note, though, that after stage 1, the code proceeds directly to stages 2 and 4, rather than stages 3 and 4. This is purely an implementation choice that simplifies the code structure by avoiding the need for multiple nested for-loops while preserving the same logical sequence of operations.

Also note that the `pg_loss` (policy gradient loss) is identical to the overall loss in our examples. This is because we intentionally omit the KL loss term here and add it later in chapter 7 for completeness. As discussed earlier, GRPO is often reported to perform better on math problems when the KL loss term is omitted, which is why we adopt this simplified objective at this stage.

Next, let's try the new `compute_grpo_loss` function on an example from the MATH training dataset to ensure that it works. Here, we sample only a small number of rollouts (`num_rollouts=2`) and generate a relatively small number of tokens (`max_new_tokens=256`) for testing purposes. It may still take several seconds until you see the function return the results.

**Listing 6.15 Computing GRPO stages on a MATH training example**

```
torch.manual_seed(123)

stats = compute_grpo_loss(
    model=model,
    tokenizer=tokenizer,
    example=math_train[4],
    device=device,
    num_rollouts=2,
    max_new_tokens=256,
    temperature=0.8,
    top_p=0.9
)

pprint(stats)
```

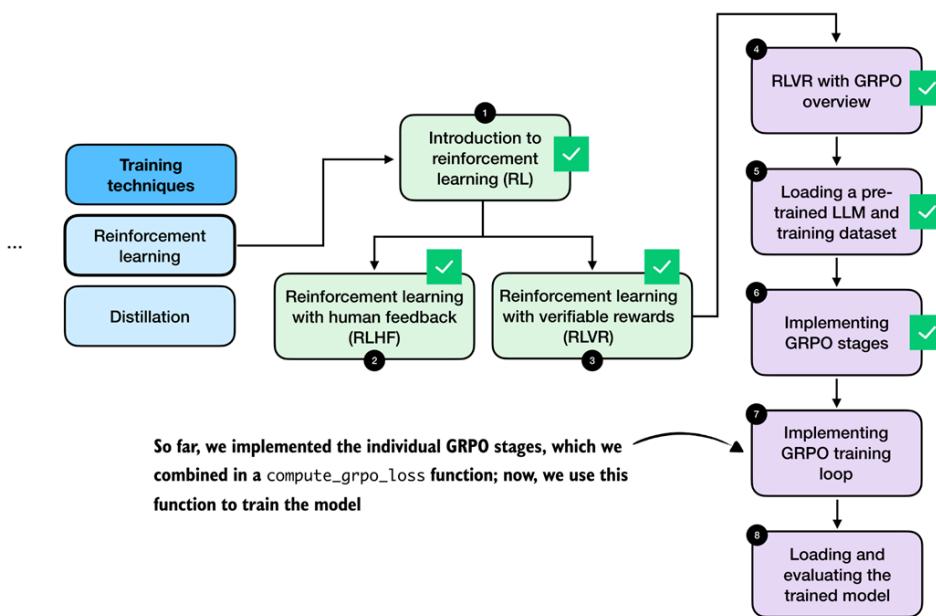
The output is as follows:

```
{'advantages': [0.0, 0.0],
 'loss': -0.0,
 'loss_tensor': tensor(-0., grad_fn=<NegBackward0>),
 'pg_loss': -0.0,
 'rewards': [0.0, 0.0],
 'samples': [{'gen_len': 3, 'reward': 0.0, 'text': ' 14'},
             {'gen_len': 256,
              'reward': 0.0,
              'text': ' 4\n'
                     'To solve this problem,
                     we can set up an equation based '
             ...
             '\\\\[ 60x - 600 + 30x = \\}]}}
```

As we can see from the 'rewards' and 'text' entries in the 'samples' field, the model answers incorrectly. As discussed earlier in section 6.4, a correct answer must include `\boxed{6}`. Since this condition is not met, the reward is zero, which in turn yields zero advantages and a zero loss. In this case, if we were training the model, the gradient would be zero and the model parameters would not be updated.

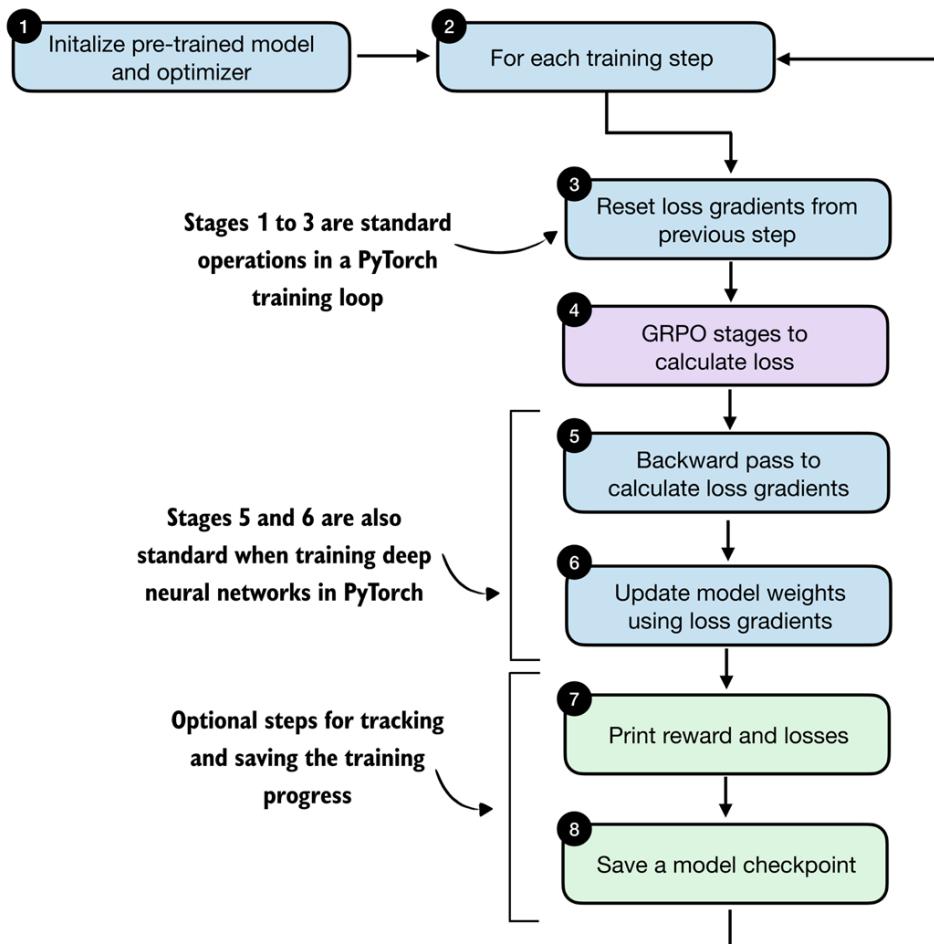
## 6.11 Implementing the GRPO training loop

We now have all the necessary components in place to implement the full GRPO training loop. In this final section of the chapter, we bring everything together to train the model via reinforcement learning with verifiable rewards using the GRPO algorithm, as illustrated in figure 6.19.



**Figure 6.19** After implementing the individual GRPO stages, we now implement the surrounding training loop to update the model weights.

The training loop consists of eight main stages, as shown in figure 6.20. Most of these stages are standard components of a typical PyTorch training loop used for deep neural networks, including LLMs. The only stage specific to reasoning models is stage 4, where we compute the GRPO loss rather than a standard classification loss used in many other types of deep neural networks and LLM pre-training.



**Figure 6.20 Outline of the training loop.** The overall structure follows a standard deep learning training loop.

The key difference lies in how the loss is computed: instead of a standard supervised objective, the loss is obtained via the **GRPO** stages (stage 4).

Before discussing the eight stages one by one, it is helpful to first see how they are implemented in code. Listing 6.16 shows the full training loop corresponding to the eight stages illustrated in figure 6.20.

**Listing 6.16 Implementing the RLVR training loop**

```

import time

def train_rlvr_grpo(
    model,
    tokenizer,
    math_data,
    device,
    steps=None,
    num_rollouts=2,
    max_new_tokens=256,
    temperature=0.8,
    top_p=0.9,
    lr=1e-5,
    checkpoint_every=50,
    checkpoint_dir=".",
    csv_log_path=None,
):

    if steps is None:
        steps = len(math_data)

    #A
    optimizer = torch.optim.AdamW(model.parameters(), lr=lr)
    current_step = 0

    if csv_log_path is None:
        timestamp = time.strftime("%Y%m%d_%H%M%S")
        csv_log_path = f"train_rlvr_grpo_metrics_{timestamp}.csv"
    csv_log_path = Path(csv_log_path)

    try:
        #B
        for step in range(steps):

            #C
            optimizer.zero_grad()

            current_step = step + 1
            example = math_data[step % len(math_data)]

            #D
            stats = compute_grpo_loss(

```

```

        model=model,
        tokenizer=tokenizer,
        example=example,
        device=device,
        num_rollouts=num_rollouts,
        max_new_tokens=max_new_tokens,
        temperature=temperature,
        top_p=top_p,
    )

#E
stats["loss_tensor"].backward()

#F
torch.nn.utils.clip_grad_norm_(model.parameters(), 1.0)

#G
optimizer.step()

#H
reward_avg = torch.tensor(stats["rewards"]).mean().item()
step_tokens = sum(
    sample["gen_len"] for sample in stats["samples"]
)
avg_response_len = (
    step_tokens / len(stats["samples"]) if stats["samples"] else 0.0
)
append_csv_metrics(
    csv_log_path, current_step, steps,
    stats["loss"], reward_avg, avg_response_len,
)
print(
    f"[Step {current_step}/{steps}] "
    f"loss={stats['loss']:.4f} "
    f"reward_avg={reward_avg:.3f} "
    f"avg_resp_len={avg_response_len:.1f}"
)

#I
if checkpoint_every and current_step % checkpoint_every == 0:
    ckpt_path = save_checkpoint(
        model=model,
        checkpoint_dir=checkpoint_dir,
        step=current_step,
    )

```

```

        )
        print(f"Saved checkpoint to {ckpt_path}")

#J
except KeyboardInterrupt:
    ckpt_path = save_checkpoint(
        model=model,
        checkpoint_dir=checkpoint_dir,
        step=max(1, current_step),
        suffix="interrupt",
    )
    print(f"\nKeyboardInterrupt. Saved checkpoint to {ckpt_path}")
    return model

return model


def save_checkpoint(model, checkpoint_dir, step, suffix=""):
    checkpoint_dir = Path(checkpoint_dir)
    checkpoint_dir.mkdir(parents=True, exist_ok=True)
    suffix = f"-{suffix}" if suffix else ""
    ckpt_path = (
        checkpoint_dir /
        f"qwen3-0.6B-rlvr-grpo-step{step:05d}{suffix}.pth"
    )
    torch.save(model.state_dict(), ckpt_path)
    return ckpt_path


#K
def append_csv_metrics(
    csv_log_path, step_idx, total_steps,
    loss, reward_avg, avg_response_len,
):
    if not csv_log_path.exists():
        csv_log_path.write_text(
            "step,total_steps,loss,reward_avg,avg_response_len\n",
            encoding="utf-8",
        )
    with csv_log_path.open("a", encoding="utf-8") as f:
        f.write(
            f"{step_idx},{total_steps},{loss:.6f},{reward_avg:.6f},"
            f"{avg_response_len:.6f}\n"
        )

```

```
#A Stage 1: initialize optimizer (model was already initialized outside the function)
#B Stage 2: Iterate over training steps
#C Stage 3: Reset loss gradient (best practice to do this at the beginning of each step)
#D Stage 4: calculate GRPO loss
#E Stage 5: Backward pass to calculate loss gradients
#F Clip large gradients to improve training stability
#G Stage 6: Update model weights using loss gradients
#H Stage 7: Collect rewards, average response lengths, and losses
#I Stage 8: Save model checkpoint
#J Save a model checkpoint if we interrupt the training early
#K Utility function to save the results to a CSV file
```

This function implements the full RLVR training loop using GRPO. As mentioned earlier, with the exception of stage 4 (the GRPO loss computation), the structure mirrors a conventional PyTorch training loop where we reset gradients, backpropagate a loss, optionally clip gradients, update parameters via an optimizer step, log metrics, and periodically save checkpoints.

**TIP** For a general introduction to training neural networks in PyTorch, see sections 3–8 of my *PyTorch in One Hour: From Tensors to Training Neural Networks on Multiple GPUs* article (<https://sebastianraschka.com/teaching/pytorch-1h/>).

The key difference here, compared to other standard training loops, is how the loss is constructed. Instead of a standard supervised objective (which is used for the majority of neural networks, for example classifiers, but also for pre-training LLMs), the training signal is computed via the GRPO algorithm as part of the RLVR procedure.

Note that, next to the logging steps that print results periodically to track progress, we also save the model checkpoint (a snapshot of the model weights) periodically so we can load, evaluate, and use it later, as discussed in the next section.

Let's now run the code and train the model. Earlier, we hardcoded the PyTorch device to "cpu" so that all intermediate results exactly match those shown in the book, since "mps" and "cuda" devices can introduce small floating-point differences. Training on a CPU is very slow, though.

For convenience, the listing below already includes two lines that automatically select and activate the appropriate device preceding the `train_rlvr_grpo()` function call. For instance, it will automatically switch to "cuda" or "mps" simply by running the code in the following listing.

**Listing 6.17 Training the model**

```

device = get_device()
model.to(device)

torch.manual_seed(1)

train_rlvr_grpo(
    model=model,
    tokenizer=tokenizer,
    math_data=math_train,
    device=device,
    steps=50,
    num_rollouts=4,
    max_new_tokens=512,
    temperature=0.8,
    top_p=0.9,
    lr=1e-5,
    checkpoint_every=5,
    checkpoint_dir=".",
)

```

Let's briefly talk about the settings before we inspect the results. The `temperature` and `top_p` settings were chosen in a common range that encourages some diversity in the answers but still results in coherent text for this model. Optionally, you can change these settings and see how they affect the results (reasonable and common ranges are 0.7-0.9).

The number of steps is relatively small at 50, even though the dataset contains 12,000 entries. This is purely due to achieving a reasonable runtime for this educational context, and can be increased. Now, though, it is already sufficient for good results.

The learning rate (`lr`) can be tweaked, but it is in a reasonable range and works well.

The number of rollouts (`num_rollouts=4`) and allowed tokens (`max_new_tokens=512`) are relatively small to reduce resource requirements. If you experience out-of-memory errors, you can further reduce these values, for example, by setting `num_rollouts=2` and `max_new_tokens=64` for testing purposes.

With the current `checkpoint_every=5` setting, the model is saved every 5 steps, and one checkpoint requires approximately 1.5 GB of disk space. In practice, in longer runs, I recommend setting this number to 50 or 100. In this context it is purposefully small for testing purposes. Also, note that an additional checkpoint should be created if the training run is manually interrupted, for instance, by interrupting the Jupyter notebook execution via the "Interrupt the kernel" button.

Let's now take a brief look at the run's output (where I interrupted the 50-step run at step 7):

```
Using Apple Silicon GPU (MPS)
[Step 1/50] loss=-0.0000 reward_avg=0.000 avg_resp_len=5.5
[Step 2/50] loss=-0.0000 reward_avg=0.000 avg_resp_len=6.8
[Step 3/50] loss=0.3592 reward_avg=0.250 avg_resp_len=7.8
[Step 4/50] loss=2.7401 reward_avg=0.250 avg_resp_len=56.5
[Step 5/50] loss=3.3214 reward_avg=0.500 avg_resp_len=251.2
Saved checkpoint to qwen3-0.6B-rlvr-grpo-step00005.pth
[Step 6/50] loss=-0.0000 reward_avg=0.000 avg_resp_len=14.2
KeyboardInterrupt. Saved checkpoint to qwen3-0.6B-rlvr-grpo-step00007-
interrupt.pth
```

We can see that both the loss and reward values fluctuate, which is normal in RL training.

Note that we print the average reward, which indicates the proportion of sampled responses that are correct. For example, with our `num_rollouts=4` setting, a `reward_avg` of 0.5 means that 2 of the 4 rollouts received a positive reward. Similarly, values of 0.25 and 0.0 indicate one or zero correct responses, respectively.

The loss values themselves should not be over-interpreted. Steps with `reward_avg=0.000` produce a near-zero loss because all rollouts receive the same reward, resulting in vanishing group-relative advantages and little to no gradient signal. Larger loss magnitudes, such as at step 3, simply reflect bigger relative differences between rollouts and are typical for GRPO-style objectives, especially at the beginning of training.

Ideally, we want to see two main trends over time:

1. The average reward should increase when averaged over many steps, since the model learns to produce more accurate responses.
2. The reasoning accuracy should improve (we evaluate this in the next section).

The supplementary materials contain a slightly more sophisticated script version that also contains an option to evaluate the model periodically on subsets of the MATH-500 test dataset to see if the model is improving on the target task: [https://github.com/rasbt/reasoning-from-scratch/tree/main/ch06/02\\_rlvr\\_grpo\\_scripts\\_intro](https://github.com/rasbt/reasoning-from-scratch/tree/main/ch06/02_rlvr_grpo_scripts_intro)

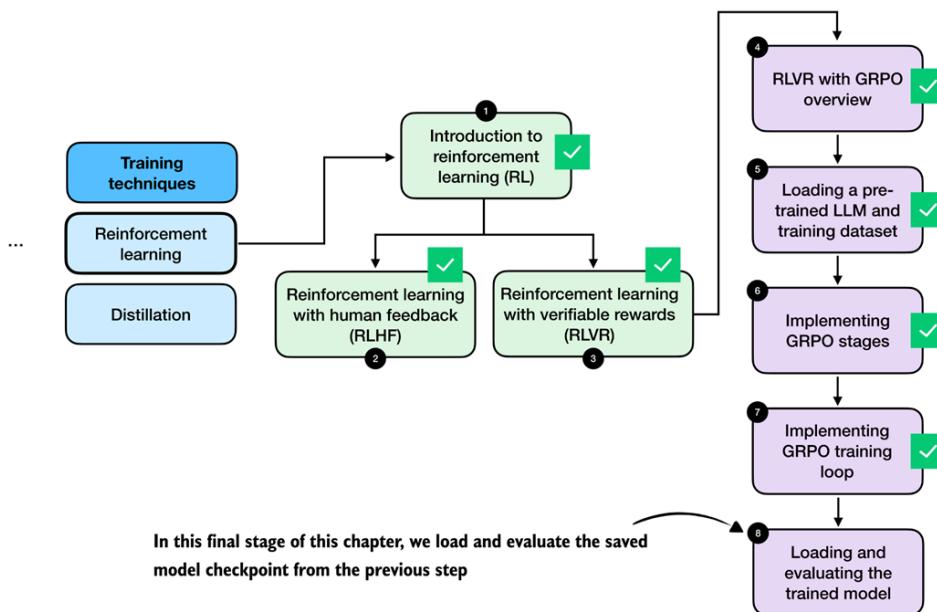
## BATCHED TRAINING

Reinforcement learning can be relatively resource-intensive due to the fact that we have to generate multiple (long) rollouts via the LLMs for each training step. For this reason, the implementation does not support batching.

If you have access to multiple GPUs you can use the optional version of this code with batch and multi-GPU support that can be found in the supplementary materials at [https://github.com/rasbt/reasoning-from-scratch/tree/main/ch06/02\\_rlrp\\_scripts\\_intro](https://github.com/rasbt/reasoning-from-scratch/tree/main/ch06/02_rlrp_scripts_intro), which trains the model faster.

## 6.12 Loading and evaluating saved model checkpoints

In this final section, as illustrated in figure 6.21, we discuss how to load the saved checkpoints from the RLVR training run of the previous section and how to evaluate the model on the MATH-500 dataset.



**Figure 6.21** The final step of this chapter discusses how we can load the saved model checkpoints and evaluate them.

The saved checkpoints can be loaded using the PyTorch state dict approach, as described in chapter 2, where `model_path` points to the corresponding .pth checkpoint file:

```
model.load_state_dict(torch.load("qwen3-0.6B-rlvr-grpo-step00050.pth"))
```

## DOWNLOADING RLVR CHECKPOINTS

If you prefer not to run the GRPO training loop locally due to its runtime cost, you can instead download pre-trained checkpoints that I have uploaded. These checkpoints can be fetched either manually or directly from Python using the provided helper function below, which downloads the selected checkpoint and makes it available for evaluation or further experimentation:

```
from reasoning_from_scratch.qwen3 import download_qwen3_grpo_checkpoints
download_qwen3_grpo_checkpoints(grpo_type="no_kl", step="00050")
```

Note that this checkpoint was created with similar settings as listing 6.17 except that `num_rollouts=4` was increased to `num_rollouts=8`.

These checkpoints are fully compatible with the model evaluation utilities introduced in chapter 3, which allows us to evaluate RL-trained models using the same MATH-500 verification pipeline.

For convenience, you can reuse the evaluation scripts provided in the chapter 3 bonus materials ([https://github.com/rasbt/reasoning-from-scratch/blob/main/ch03/02\\_math500-verifier-scripts/evaluate\\_math500.py](https://github.com/rasbt/reasoning-from-scratch/blob/main/ch03/02_math500-verifier-scripts/evaluate_math500.py)) to run evaluations on the MATH-500 dataset by specifying the desired checkpoint path.

For example, to evaluate the `qwen3-0.6B-rlvr-grpo-step00050.pth` checkpoint file you can run the aforementioned script as

```
uv run evaluate_math500.py \
--dataset_size 500 \
--which_model base \
--checkpoint_path "qwen3-0.6B-rlvr-grpo-step00050.pth"
```

(If you are not a `uv` user, replace `uv run` with `python`.)

Table 6.1 compares the original base and reasoning models (rows 1 and 2) to the base model we trained via GRPO in this chapter (rows 3 and 4).

**Table 6.1 MATH-500 task accuracy for different base and reasoning models**

	<b>Method</b>	<b>Step</b>	<b>Max tokens</b>	<b>Num rollouts</b>	<b>Accuracy</b>	<b>Average tokens</b>
1	Base model (chapter 3)	-	-	-	15.2%	78.85
2	Reasoning model (chapter 3)	-	-	-	48.2%	1369.79
3	GRPO (this chapter)	50	256	4	43.2%	560.22
4	GRPO (this chapter)	50	512	4	45.6%	579.81
5	GRPO (this chapter)	50	512	8	47.4%	586.11

The accuracy column in table 6.1 refers to the accuracy on the full 500-sample MATH-500 dataset we used in chapter 3. Note that the "Max tokens" column corresponds to the number of tokens that were allowed per rollout during training. If the number is 512, this encourages the LLM to provide the final boxed answer within this 512 token limit because otherwise it will not receive a reward during training.

The evaluation code was executed with a maximum token limit of 2048, allowing the LLM to generate longer responses during evaluation. The "Average tokens" column averages the response length during the evaluation on the MATH-500 dataset. What we can see is that, compared to the reference reasoning model (row 2), our GRPO models generate shorter responses on average, as expected due to the token-length restriction during training.

Note that the DeepSeek-R1 team observed that responses grow longer over the course of training as the model begins to write intermediate (chain-of-thought) explanations. So, to maximize accuracy, it makes sense not to restrict token length too aggressively during training. However, longer token lengths, especially when coupled with multiple rollouts, require more computational resources, which is why we capped them at a lower limit in this chapter.

As shown in table 6.1, training the reasoning model via GRPO results in 47.4% accuracy (row 5), which is close to the accuracy of the official Qwen3 0.6B reasoning model on MATH-500.

The model's accuracy could be further improved by increasing the response length, sampling more rollouts per prompt, and training for additional steps. The next chapter introduces practical tips and techniques for monitoring and improving GRPO training outcomes.

We trained the model for only 50 steps, which already appears sufficient to unlock reasoning behavior in the base model. In my experiments, training for longer does not necessarily improve performance and can even reduce accuracy, since the original GRPO formulation can be unstable over longer runs. We address this issue with common GRPO improvements in the next chapter.

Interested readers can find and download additional checkpoints in the range from 50 to 9000 from [https://huggingface.co/rasbt/qwen3-from-scratch-grpo-checkpoints/tree/main/grpo\\_original\\_no\\_kl](https://huggingface.co/rasbt/qwen3-from-scratch-grpo-checkpoints/tree/main/grpo_original_no_kl)

## 6.13 Summary

- Reinforcement learning (RL) can be used to train LLMs on human preference labels and verifiable rewards.
- RL is typically applied as post-training on top of a pre-trained base model, and it can be inserted at different stages of an LLM pipeline, including reasoning training and preference tuning.
- RL with human feedback (RLHF) optimizes for human preferences via a two-stage setup: train a reward model from ranked responses, then use reward scores to update the LLM.
- RL with verifiable rewards (RLVR) simplifies RLHF by replacing learned reward models with deterministic, automatically computed verifiers (for example, math answer checking)
- We focussed on RLVR for math reasoning.
- We used GRPO as the policy optimization algorithm that turns verifier rewards into parameter updates; because GRPO directly optimizes the model using sequence-level rewards without requiring a separate value model, it is particularly convenient.
- GRPO is a more resource-friendly alternative to other RL algorithms for LLMs because it avoids training a separate value model and instead derives learning signals from comparisons within a group of sampled rollouts.
- A "rollout" refers to a full model answer (completion) for a prompt; rewards, advantages, and log-probabilities are computed from the rollout in later steps.
- Rewards are computed from a verifier that only grants a reward if the final answer is both correct and extractable in a required format like "`\boxed{}`".
- Raw rewards are transformed into advantages by normalizing each rollout reward relative to the group mean and standard deviation.
- GRPO also relies on sequence-level log-probabilities, which are computed by summing token log-probabilities over the generated answer tokens.
- Sequence log-probabilities, together with the advantages, form the core policy-gradient objective in GRPO.

- The full GRPO loss computation is combined into a single function that performs rollout sampling, reward computation, advantage calculation, log-prob computation, and policy-gradient loss calculation.
- The surrounding training loop is a standard deep learning loop, with the key difference being that the loss comes from GRPO rather than conventional classification losses.
- Training is resource-intensive because each step requires generating multiple, potentially long rollouts, but even short GRPO runs can increase MATH-500 accuracy from 15% to 47%.

# *Appendix A. References and further reading*

## A.1 Chapter 1

### A.1.1 References

The announcement article of OpenAI's o1 model, which is regarded as the first LLM-based reasoning model:

- Introducing OpenAI o1-preview, <https://openai.com/index/introducing-openai-o1-preview/>

DeepSeek-R1 is the first open-source reasoning model that was accompanied by a comprehensive technical report, which was the first to show that reasoning emerges from reinforcement learning with verifiable rewards (a topic covered in more detail in chapter 5):

- DeepSeek-R1: Incentivizing Reasoning Capability in LLMs via Reinforcement Learning, <https://arxiv.org/abs/2501.12948>

OpenAI CEO's comment on the reasoning ("chain-of-thought") capabilities of future models:

- "[...] We will next ship GPT-4.5, the model we called Orion internally, as our last non-chain-of-thought model. [...]", <https://x.com/sama/status/1889755723078443244>

A research paper by AI researchers at Apple finding that reasoning models are sophisticated (but very capable) pattern matchers:

- The Illusion of Thinking: Understanding the Strengths and Limitations of Reasoning Models via the Lens of Problem Complexity,  
<https://machinelearning.apple.com/research/illusion-of-thinking>

An in-depth book and guide on implementing and training large language models step-by-step:

- Build a Large Language Model (From Scratch),<http://mng.bz/orYv>

### A.1.2 Further Reading

An introduction to how DeepSeek-R1 works, providing insights into the foundations of reasoning in LLMs:

- Understanding Reasoning LLMs, <https://magazine.sebastianraschka.com/p/understanding-reasoning-langs>

## A.2 Chapter 2

### A.2.1 References

Official installation page for the uv Python package and project manager:

- Installing uv, <https://docs.astral.sh/uv/getting-started/installation/>

Cloud compute platforms with GPU support:

- Lightning AI, <https://lightning.ai/>
- Google Colab, <https://colab.research.google.com/>

Qwen3 resources with additional benchmark performance and comparison to other models:

- Blog post, <https://qwenlm.github.io/blog/qwen3/>
- Technical report, <https://arxiv.org/abs/2505.09388>

### A.2.2 Further Reading

A PyTorch tutorial for readers who are new to PyTorch or would like a refresher:

- PyTorch in One Hour: From Tensors to Training Neural Networks on Multiple GPUs tutorial, <https://sebastianraschka.com/teaching/pytorch-1h>

Additional resources on tokenization:

- Build a Large Language Model (from Scratch) chapter 2, <https://mng.bz/M96o>
- Implementing A Byte Pair Encoding (BPE) Tokenizer From Scratch, <https://sebastianraschka.com/blog/2025/bpe-from-scratch.html>

## A.3 Chapter 3

### A.3.1 References

The MATH-500 dataset originated from the MATH dataset (with 12,500 problems across algebra, geometry, probability, number theory, and more) that was introduced in the following paper:

- Measuring Mathematical Problem Solving With the MATH Dataset,  
<https://arxiv.org/abs/2103.03874>

The MATH-500 split (created from the original MATH dataset) was proposed in the following paper:

- Let's Verify Step by Step, <https://arxiv.org/abs/2305.20050>

### A.3.2 Further Reading

Readers who are interested in learning more about SymPy (not required for this book) can learn about it in this official tutorial:

- SymPy introductory tutorial, <https://docs.sympy.org/latest/tutorials/intro-tutorial/index.html>

An example of a system (here, a fine-tuned LLM) to also evaluate intermediate reasoning steps:

- Evaluating Mathematical Reasoning Beyond Accuracy, <https://arxiv.org/pdf/2404.05692>

A large-scale dataset containing 800,000 step-level correctness labels for model-generated solutions to problems from the MATH dataset:

- Let's Verify Step by Step, <https://arxiv.org/abs/2305.20050>

An article describing the rising cost of LLM evaluation, finding that evaluating reasoning models such as o1 on (seven) popular benchmarks costs approximately \$1500:

- The rise of AI "reasoning" models is making benchmarking more expensive, <https://techcrunch.com/2025/04/10/the-rise-of-ai-reasoning-models-is-making-benchmarking-more-expensive/>

A comprehensive 2025 survey on LLM benchmarks:

- A Survey on Large Language Model Benchmarks, <https://arxiv.org/abs/2508.15361>

Instead of only relying on deterministic and symbolic verifiers, a recent research project highlighted that small reasoning models themselves can be used successfully as verifiers for other reasoning models:

- xVerify: Efficient Answer Verifier for Reasoning Model Evaluations,  
<https://arxiv.org/abs/2504.10481>

## A.4 Chapter 4

### A.4.1 References

The paper that formally described chain-of-thought prompting:

- Let's Verify Step by Step, <https://arxiv.org/abs/2305.20050>

A description of self-consistency sampling with additional comparison studies:

- Self-Consistency Improves Chain-of-Thought Reasoning in Language Models, <https://arxiv.org/abs/2203.11171>

### A.4.2 Further Reading

An overview and discussion of additional inference scaling methods:

- The State of LLM Reasoning Model Inference, <https://magazine.sebastianraschka.com/p/state-of-lm-reasoning-and-inference-scaling>

## A.5 Chapter F

### A.5.1 References

The paper that introduced the popular multiple-choice MMLU dataset:

- Measuring Massive Multitask Language Understanding, <https://arxiv.org/abs/2009.03300>

A detailed description of the Elo rating system:

- Elo rating system, [https://en.wikipedia.org/wiki/Elo\\_rating\\_system](https://en.wikipedia.org/wiki/Elo_rating_system)

The Chatbot Arena paper describing the original methodology behind the popular LLM leaderboard:

- Chatbot Arena: An Open Platform for Evaluating LLMs by Human Preference, <https://arxiv.org/abs/2403.04132>

### A.5.2 Further Reading

A paper discussing the problems with leaderboards such as LM Arena:

- The Leaderboard Illusion, <http://arxiv.org/abs/2504.20879>

An article by the author describing gpt-oss in more detail:

- From GPT-2 to gpt-oss: Analyzing the Architectural Advances,  
<https://magazine.sebastianraschka.com/p/from-gpt-2-to-gpt-oss-analyzing-the>

A survey of different LLM judge approaches:

- A Survey on LLM-as-a-Judge, <https://arxiv.org/abs/2411.15594>

Example of a small LLM fine-tuned to act as a judge:

- PHUDGE: Phi-3 as Scalable Judge, <https://arxiv.org/abs/2405.08029>

## *Appendix B. Exercise solutions*

The complete code examples for the exercise solutions can be found in the supplementary GitHub repository at <https://github.com/rasbt/reasoning-from-scratch>.

### **B.1 Chapter 2**

#### **EXERCISE 2.1**

We can use a prompt similar to "Hello, Ardwarklethyrx. Haus und Garten.", which contains a made-up word ("Ardwarklethyrx") and three words in a non-English language (German):

```
"Haus und Garten":  
prompt = "Hello, Ardwarklethyrx. Haus und Garten."  
input_token_ids_list = tokenizer.encode(prompt)  
for i in input_token_ids_list:  
    print(f"[{i}] --> {tokenizer.decode([i])}")
```

The output is:

```
[9707] --> Hello
[11] --> ,
[1644] --> Ar
[29406] --> dw
[838] --> ark
[273] --> le
[339] --> th
[10920] --> yr
[87] --> x
[13] --> .
[47375] --> Haus
[2030] --> und
[93912] --> Garten
[13] --> .
```

As we can see, unknown words are broken into smaller pieces of subwords or even single tokens; this allows the tokenizer and LLM to handle any input.

German words are not broken down into characters or even subwords here, suggesting that the tokenizer has seen German texts during training. This also suggests that the LLM was likely trained on German texts, too, and should be able to handle at least certain non-English languages well.

## **EXERCISE 2.2**

The updated `generate_text_basic` function, now called `generate_text_basic_stream`, looks like as follows:

```

@torch.inference_mode()
def generate_text_basic_stream(
    model,
    token_ids,
    max_new_tokens,
    eos_token_id=None
):

    # input_length = token_ids.shape[1] #A

    model.eval()
    for _ in range(max_new_tokens):
        out = model(token_ids)[:, -1]
        next_token = torch.argmax(out, dim=-1, keepdim=True)

        if (eos_token_id is not None
            and next_token.item() == eos_token_id):
            break

        yield next_token #B

    token_ids = torch.cat([token_ids, next_token], dim=1)
    # return token_ids[:, input_length:] #C


for token in generate_text_basic_stream(
    model=model,
    token_ids=input_token_ids_tensor,
    max_new_tokens=50,
    eos_token_id=tokenizer.eos_token_id
):
    token_id = token.squeeze(0).tolist()
    print(
        tokenizer.decode(token_id),
        end="",
        flush=True
    )

#A The input_length is no longer needed
#B We now yield each token as it's generated
#C Since we use yield, we no longer need the return statement

```

This prints the following text:

Large language models are artificial intelligence systems that can understand, generate, and process human language, enabling them to perform a wide range of tasks, from answering questions to writing articles, and even creating creative content.

Note that the text above is identical to the text generated by the `generate_text_basic` function when using the same prompt, as expected. However, if we execute the code on our computer, we should see each word being generated on the fly now.

Similarly, we can modify the KV cache variant (`generate_text_basic_cache`) as follows to add streaming support:

```
from reasoning_from_scratch.qwen3 import KVCache

@torch.inference_mode()
def generate_text_basic_stream_cache(
    model,
    token_ids,
    max_new_tokens,
    eos_token_id=None
):
    # input_length = token_ids.shape[1]  #A
    model.eval()
    cache = KVCache(n_layers=model.cfg["n_layers"])
    model.reset_kv_cache()

    out = model(token_ids, cache=cache)[:, -1]
    for _ in range(max_new_tokens):
        next_token = torch.argmax(out, dim=-1, keepdim=True)

        if (eos_token_id is not None
            and next_token.item() == eos_token_id):
            break

        yield next_token  #B
        # token_ids = torch.cat([token_ids, next_token], dim=1)
        out = model(next_token, cache=cache)[:, -1]

    # return token_ids[:, input_length:]  #C

#A The input_length is no longer needed
#B We now yield each token as it's generated
#C Since we use yield, we no longer need the return statement
```

## EXERCISE 2.3

We can simply delete the line `device = torch.device("cpu")` in section 2.5, and then rerun the rest of the code in chapter 2 as is. Reference numbers for the hardware I tried the code on are provided in table 2.1 at the end of chapter 2.

## B.2 Chapter 3

### EXERCISE 3.1

There is an endless number of different test cases we may add. Below is a selection of some interesting ones:

```
from reasoning_from_scratch.ch03 import (
    run_demos_table
)

more_tests = [
    ("check_16", "[1, 2]", "(1, 2)", True),  #A
    ("check_17", "1e-3", "0.001", True),  #B
    ("check_18", "(-3)^2", "9", True),  #C
    ("check_19", "-1", "-1", True),  #D
]

run_demos_table(more_tests)

#A Different bracket types
#B Scientific notation
#C Algebraic simplification with caret exponent
#D Unicode minus (U+2212) vs ASCII hyphen-minus
```

The output is:

Test	Expect	Got	Status
check_16	True	True	PASS
check_17	True	True	PASS
check_18	True	True	PASS
check_19	True	False	FAIL

As we can see, the tests pass in all cases except for `check_19`, which swaps the regular sign with a Unicode version of a minus sign that looks indistinguishable to the human eye (depending on which font or editor we use). We could fix this test case by adding one of the following lines anywhere to the `normalize_text` function:

```
text = text.replace("-", "-")
```

or

```
text = text.replace("\u2212", "-")
```

Another interesting test is the following one:

```
extra_tests_1 = [
    ("check_20", "Text around answer 3.", "3", True)
]
```

We can run it via the following code:

```
run_demos_table(extra_tests_1)
```

However, it fails the test:

Test	Expect	Got	Status
check_20	True	False	FAIL

Passed 0/1

While it may seem that our code cannot handle such text-containing cases, this is actually a poorly designed test. In practice, the `run_demos_table` function is intended specifically to test the `grade_answer` function; nothing more, nothing less.

The `grade_answer` function would never receive the entire answer in this text form, since the answer would have been extracted from the text before being passed to it. For instance, if we want to test text answers, we need to call the test as follows:

```
from reasoning_from_scratch.ch03 import (
    extract_final_candidate
)
extra_tests_2 = [
    ("check_20",
        extract_final_candidate("Text around answer 3."),
        "3", True)
]
```

As we can see based on the output, it now passes the test:

```
run_demos_table(extra_tests_2)
Test      | Expect | Got   | Status
check_20 | True   | True  | PASS

Passed 1/1
```

## EXERCISE 3.2

There are two options to calculate the average response length. The first option is to modify the `evaluate_math500_stream` function (listing 3.13 in chapter 3) by adding the following lines:

```
# ...
# below `num_correct = 0`
total_len = 0

# ...
# inside for i, row in enumerate(math_data, start=1):
# anywhere below `gen_text = ...`
total_len += len(tokenizer.encode(gen_text))

# ...
# anywhere at the bottom before the return statement
avg_len = total_len / num_examples
print(f"Average length: {avg_len:.2f} tokens")
```

Alternatively, the second option is to calculate the response lengths from the `.jsonl` files that were created when we ran the `evaluate_math500_stream` function in the main chapter. This way, we avoid having to rerun the evaluation.

First, we load the `.jsonl` file as follows:

```

import json
from pathlib import Path

WHICH_MODEL = "base"

local_path = Path(f"math500_{WHICH_MODEL}-mps.jsonl") #A
if not local_path.exists():
    raise FileNotFoundError
    f"{local_path} not found. Run ch03_main.ipynb to create it."
)

results = []
with open(local_path, "r") as f:
    for line in f:
        if line.strip():
            results.append(json.loads(line))

print("Number of entries:", len(results))

#A You may need to adjust this path

```

Let's print the dictionary keys to get a better idea of how the `results` dataset is structured:

```
print(results[0].keys())
```

This prints:

```
dict_keys(['index', 'problem', 'gtruth_answer', 'generated_text',
'extracted', 'correct'])
```

Note that each entry has multiple keys, however, we are only interested in the "generated\_text" key, which contains the model's full answer. Next, we need to load the tokenizer so that we can tokenize the answer text before we can calculate the number of tokens. This is similar to the code we used in listing 3.1 in chapter 3:

```

from reasoning_from_scratch.qwen3 import (
    download_qwen3_small,
    Qwen3Tokenizer
)

if WHICH_MODEL == "base":

    download_qwen3_small(
        kind="base", tokenizer_only=True, out_dir="qwen3"
    )
    tokenizer_path = Path("qwen3") / "tokenizer-base.json"
    tokenizer = Qwen3Tokenizer(tokenizer_file_path=tokenizer_path)

elif WHICH_MODEL == "reasoning":

    download_qwen3_small(
        kind="reasoning", tokenizer_only=True, out_dir="qwen3"
    )
    tokenizer_path = Path("qwen3") / "tokenizer-reasoning.json"
    tokenizer = Qwen3Tokenizer(
        tokenizer_file_path=tokenizer_path,
        apply_chat_template=True,
        add_generation_prompt=True,
        add_thinking=True,
    )

```

Then, we can calculate the average length as follows, which is similar to how we could have modified the `evaluate_math500_stream` function:

```

total_len = 0

for item in results:
    num_tokens = len(tokenizer.encode(item["generated_text"]))
    total_len += num_tokens

avg_len = total_len / len(results)
print(f"Average length: {avg_len:.2f} tokens")

```

The resulting average length is as follows:

```
Average length: 98.00 tokens
```

Table B.1 lists the average lengths for the different models and subsets.

**Table B.1 Average number of tokens on MATH-500**

Model	Device	Average length	MATH-500 size
Base	CPU	97.30	10
Base	CUDA	96.74	500
Reasoning	CPU	891.80	10
Reasoning	CUDA	1361.21	500

As we can see based on the results in table B.1, and as expected, the reasoning model generates much longer responses (in this case, approximately 10-times longer).

### EXERCISE 3.3

To evaluate the model on a larger dataset, we can simply change the `math_data[:10]` to a different slice or larger number (up to 500) in the following function call:

```
num_correct, num_examples, acc = evaluate_math500_stream(
    model, tokenizer, device,
    math_data=math_data[:10],
    max_new_tokens=2048,
    verbose=False
)
```

Table B.2 below shows the accuracy values for different dataset sizes. (Since the MATH-500 test set is already shuffled, no additional shuffling was applied.)

**Table B.2 Accuracies for different MATH-500 dataset sizes**

<b>Model</b>	<b>Device</b>	<b>Accuracy</b>	<b>MATH-500 size</b>
Base	CUDA	30.0%	10
Base	CUDA	34.0%	50
Base	CUDA	27.0%	100
Base	CUDA	15.3%	500
Reasoning	CUDA	90.0%	10
Reasoning	CUDA	58.0%	50
Reasoning	CUDA	56.0%	100
Reasoning	CUDA	48.2%	500

As we can see based on the results in table B.2, the first 10 examples are not very representative of the MATH-500 performance evaluated on the whole 500 examples.

In addition, we can create an entirely new dataset in a similar style to MATH-500. For example, a dataset in MATH-500 style is included in this repository; we can use it in the main chapter by changing the filename from `math500_test.json` to `math_new50_exercise.json` (this dataset is included in this book's GitHub repository at [https://github.com/rasbt/reasoning-from-scratch/tree/main/ch03/01\\_main-chapter-code](https://github.com/rasbt/reasoning-from-scratch/tree/main/ch03/01_main-chapter-code)).

The performance of the models is as follows:

- base: 36.0% (18/50)
- reasoning: 80.0% (40/50)

Accuracy is similar for the base model and higher for the reasoning model compared to the 50-example subset of the MATH-500 test set (table B.2). This indicates that, despite the possibility of overlap with Qwen3's training data, the model generalizes well to new math questions and does not show signs of extensive overfitting to the original MATH-500 data.

## EXERCISE 3.4

We could use the alternative prompt similar to the one suggested in the chapter, which modifies the prompt to use the word "problem" instead of "question":

```
def render_prompt(prompt):
    template = (
        "You are a helpful math assistant.\n"
        "Solve the problem and write the final "
        "result on a new line as:\n"
        "\\boxed{ANSWER}\n\n"
        f"Problem:{prompt}\n\nAnswer:"
    )
    return template
```

Using this prompt improves the performance of the base model, on the 500 examples, from 15.3% to 31.2%. Also, it improves the performance of the reasoning model from 48.2% to 50.0%

From these observations, we may conclude that the base model is much more sensitive to the prompt format (likely due to memorizing some prompt-formatted MATH-500 examples from the training set) than the reasoning model; the latter seems largely unaffected.

## B.3 Chapter 4

### EXERCISE 4.1

The modification only requires adding a prompt suffix such as "\n\nExplain step by step." after applying the prompt template. There is only a very small portion of code that needs to be updated in the MATH-500 evaluation function from chapter 3, as shown below:

```
def evaluate_math500_stream(...):
    # ...
    for i, row in enumerate(math_data, start=1):
        prompt = render_prompt(row["problem"])
        prompt += "\n\nExplain step by step." # NEW
        gen_text = generate_text_stream_concat(
            model, tokenizer, prompt, device,
            max_new_tokens=max_new_tokens,
            verbose=verbose,
        )

    # ...
```

The improvements are shown in row 3 in table 4.1, which can be found in section 4.6 in chapter 4.

## EXERCISE 4.2

Here, we replace the `generate_text_stream_concat` function with `generate_text_stream_concat_flex` and pass in `generate_text_top_p_stream_cache` as its generation function. The updated MATH-500 evaluation function from chapter 3 is shown below, and the changes are marked with comments labeled # NEW.

```
def evaluate_math500_stream(
    model,
    tokenizer,
    device,
    math_data,
    out_path=None,
    max_new_tokens=512,
    verbose=False,
    temperature=1.0,    # NEW
    top_p=1.0,          # NEW
):
    # ...

    with open(out_path, "w", encoding="utf-8") as f:
        for i, row in enumerate(math_data, start=1):
            prompt = render_prompt(row["problem"])
            gen_text = generate_text_stream_concat(
                model, tokenizer, prompt, device,
                max_new_tokens=max_new_tokens,
                verbose=verbose,
                generate_func=generate_text_stream_concat_flex,  # NEW
                temperature=temperature,                         # NEW
                top_p=top_p                                     # NEW
            )
    # ...
```

The difference between this modified function and the baseline from chapter 3 can be seen in rows 1 and 4 in table 4.1, which can be found in section 4.6 in chapter 4.

### EXERCISE 4.3

Starting from the evaluate\_math500\_stream function in chapter 3, the first modification is to replace the line `gen_text = generate_text_stream_concat(...)` with a call to `results = self_consistency_vote(...)` from chapter 4. The second modification adds a simple tie-breaking rule that selects the first occurrence of the most frequent answer. For instance, if the sampled results are 1, 3, 5, 3, 5, the function would return 3 because it is the earliest member of the most frequent group.

Since the most frequent answers are stored in `results["majority_winners"]`, one straightforward way to break ties is to take the first element of this list, that is, `results["majority_winners"][0]`.

Those changes are illustrated in the code excerpts below:

```

def evaluate_math500_stream(
    model,
    tokenizer,
    device,
    math_data,
    out_path=None,
    max_new_tokens=2048,
    verbose=False,
    prompt_suffix="",      # NEW
    temperature=1.0,       # NEW
    top_p=1.0,              # NEW
    seed=None,               # NEW
    num_samples=10,          # NEW
):
    if out_path is None:
        dev_name = str(device).replace(":", "-")
        out_path = Path(f"math500-{dev_name}.jsonl")

    num_examples = len(math_data)
    num_correct = 0
    start_time = time.time()

    with open(out_path, "w", encoding="utf-8") as f:
        for i, row in enumerate(math_data, start=1):
            prompt = render_prompt(row["problem"])

            ##### # NEW
            prompt += prompt_suffix
            results = self_consistency_vote(
                model=model,
                tokenizer=tokenizer,
                prompt=prompt,
                device=device,
                num_samples=num_samples,
                temperature=temperature,
                top_p=top_p,
                max_new_tokens=max_new_tokens,
                show_progress=False,
                show_long_answer=False,
                seed=seed,
            )

```

```

# resolve ties
if results["final_answer"] is None:
    extracted = results["majority_winners"][0]
else:
    extracted = results["final_answer"]

# extracted = extract_final_candidate(
#     gen_text
# )

# Optionally, get long answer
if extracted is not None:
    for idx, s in enumerate(results["short_answers"]):
        if s == extracted:
            long_answer = results["full_answers"][idx]
            break
gen_text = long_answer
#####
#####
```

is\_correct = grade\_answer(  
 extracted, row["answer"]  
>)  
 num\_correct += int(is\_correct)

# ...

The performance improvements when using self-consistency sampling are summarized and discussed in table 4.1 in chapter 4 (rows 5-7 and rows 9-12), which can be found in section 4.6 of chapter 4.

## **EXERCISE 4.4**

The early stopping check can be implemented by adding a few lines of code that check whether the given answer is already counted multiple times, or, more specifically, if the given answer count is greater than `num_samples / 2`:

```
if early_stop and counts[short] > num_samples / 2:  
    majority_winners = [short]  
    final_answer = short  
    break
```

The excerpt of the modified `self_consistency_vote` function below illustrates more specifically where to insert this code:

```

def self_consistency_vote(
    # ...
    early_stop=True,    # NEW
):
    # ...

    if show_progress:
        print(f"[Sample {i+1}/{num_samples}] → {short!r}")

    ##### NEW
    # Early stop if one answer already meets >= 50% majority
    if early_stop and counts[short] > num_samples / 2:
        majority_winners = [short]
        final_answer = short
        break
    #####
    ##### NEW

    if final_answer is None:
        mc = counts.most_common()
        if mc:
            top_freq = mc[0][1]
            majority_winners = [s for s, f in mc if f == top_freq]
            final_answer = mc[0][0] if len(majority_winners) == 1 else None

    return {
        "full_answers": full_answers,
        "short_answers": short_answers,
        "counts": dict(counts),
        "groups": groups,
        "majority_winners": majority_winners,
        "final_answer": final_answer,
    }
}

```

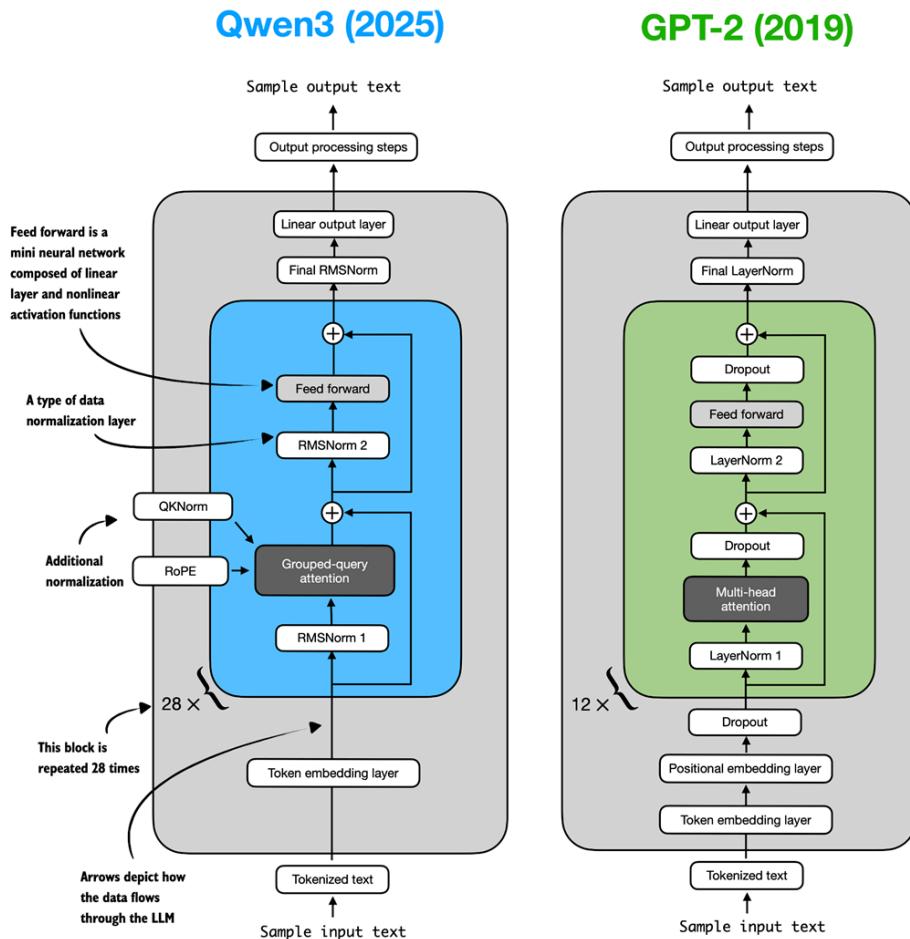
## Appendix C. Qwen3 LLM source code

While this is a *from scratch* book, as mentioned in the main chapters, the *from scratch* part refers to the reasoning techniques, not the LLM itself. Implementing an LLM entirely from scratch would require a separate book, which is the topic of my *Build A Large Language Model (From Scratch)* book (<http://mng.bz/orYv>).

However, for readers interested in seeing the Qwen3 implementation we use in this *Build A Reasoning Model (From Scratch)* book, this appendix lists the source code for the `Qwen3Model` model that I implemented in and that we import from the book's `reasoning_from_scratch` Python package:

```
from reasoning_from_scratch.qwen3 import Qwen3Model, Qwen3Tokenizer
```

As shown in figure C.1, the Qwen3 architecture is very similar to GPT-2, which is covered in my *Build A Large Language Model (From Scratch)* book. While familiarity with GPT-2 is not required for this book, this appendix mentions comparisons to GPT-2 for those who are familiar with it. In fact, I wrote the Qwen3 implementation by porting the GPT-2 model from my other book piece by piece into the Qwen3 architecture, such that it follows similar style conventions to improve readability.



**Figure C.1 Architectural comparison between Qwen3 and GPT-2. Both models process text through embedding layers and stacked transformer blocks, but they differ in certain design choices.**

As shown in figure C.1, both Qwen3 (released in 2025) and GPT-2 (released in 2019) are very similar overall in that they are both based on the decoder submodule of the original transformer architecture. However, some of the design choices have evolved since 2019. Note that most of these design choices found in Qwen3 are not unique to Qwen3 but are found in many other contemporary LLMs, which I discussed in my *The Big LLM Architecture Comparison* (<https://magazine.sebastianraschka.com/p/the-big-lm-architecture-comparison>) article.

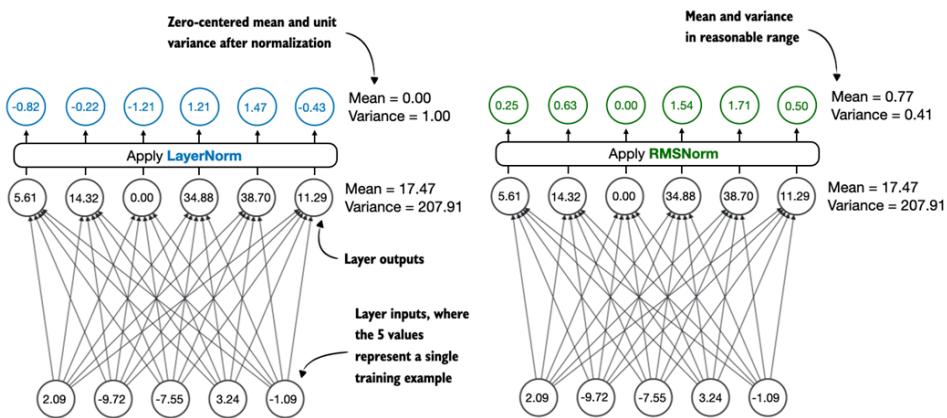
For readers new to LLMs who want to understand how these architectures are implemented, I recommend starting with GPT-2. Its design is simpler to implement, which makes it an easier entry point before exploring more modern variations.

Since this book does not focus on architecture implementations, the remainder of this appendix will cover only a brief overview of Qwen3's code.

## C.1 Root mean square layer normalization (RMSNorm)

In contrast to GPT-2, which used standard *LayerNorm*, the newer Qwen3 architecture replaces it with *root mean square layer normalization (RMSNorm)*. This is a trend that has become increasingly common in recent model architectures.

RMSNorm fulfills the same core function as LayerNorm: normalizing layer activations to stabilize and improve training. However, it simplifies the computation by removing the mean-centering step, as shown in figure C.2. This means that activations will still be normalized, but they are not centered at 0.



**Figure C.2 Comparison of LayerNorm and RMSNorm.** LayerNorm (left) normalizes activations so that their average value (mean) is exactly zero and their spread (variance) is exactly one. RMSNorm (right) instead scales activations based on their root mean square, which does not enforce zero mean or unit variance, but still keeps the mean and variance within a reasonable range for stable training.

As we can see in figure C.2, both LayerNorm and RMSNorm scale the layer outputs to be in a reasonable range.

LayerNorm subtracts the mean and divides by the standard deviation such that the layer outputs have a zero mean and unit variance (variance of one and standard deviation of one), which results in favorable properties, in terms of gradient values, for stable training.

RMSNorm divides the inputs by the root mean square. This scales activations to a comparable magnitude without enforcing zero mean or unit variance. In this particular example shown in figure C.2, the mean is 0.77 and the variance is 0.41.

Both LayerNorm and RMSNorm stabilize activation scales and improve optimization; however, RMSNorm is often preferred in large-scale LLMs because it is computationally cheaper. Unlike LayerNorm, RMSNorm does not use a bias (shift) term by default, which reduces the number of trainable parameters. Moreover, RMSNorm reduces the expensive mean and variance computations to a single root-mean-square operation. This reduces the number of cross-feature reductions from two to one, which lowers communication overhead on GPUs and slightly improves training efficiency.

Listing C.1 shows what RMSNorm looks like in code.

#### Listing C.1 RMSNorm

```
import torch.nn as nn

class RMSNorm(nn.Module):

    def __init__(self,
                 emb_dim,
                 eps=1e-6,
                 bias=False,
                 qwen3_compatible=True,
                 ):
        super().__init__()
        self.eps = eps
        self.qwen3_compatible = qwen3_compatible
        self.scale = nn.Parameter(torch.ones(emb_dim))
        self.shift = nn.Parameter(torch.zeros(emb_dim)) if bias else None

    def forward(self, x):
        input_dtype = x.dtype

        if self.qwen3_compatible:
            x = x.to(torch.float32)

        variance = x.pow(2).mean(dim=-1, keepdim=True)
        norm_x = x * torch.rsqrt(variance + self.eps)
        norm_x = norm_x * self.scale

        if self.shift is not None:
            norm_x = norm_x + self.shift

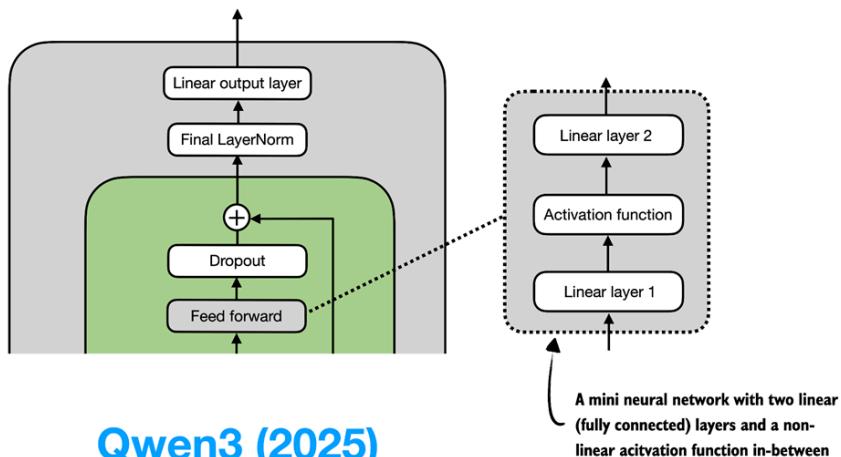
        return norm_x.to(input_dtype)
```

Note that, for brevity, this appendix does not provide detailed code walkthroughs for each LLM component. Instead, in section C.6, we will integrate all components into the Qwen3Model class, load the pre-trained weights into it, and then use this model to generate text in section C.9.

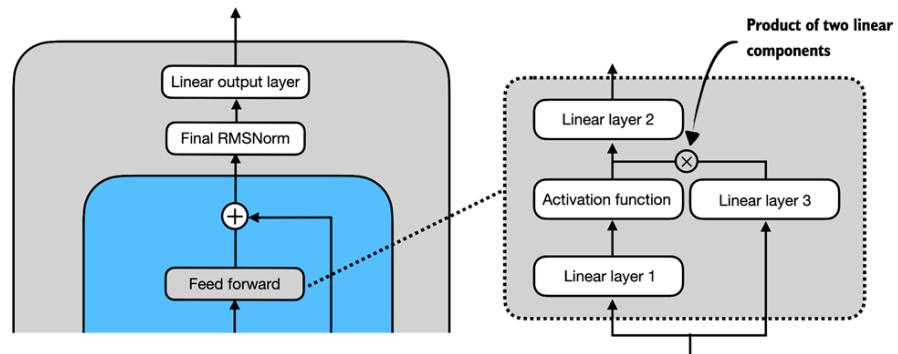
## C.2 Feed forward module

The *feed forward module* (a small multi-layer perceptron) is replaced with a *gated linear unit (GLU)* variant, introduced in a 2020 paper (<https://arxiv.org/abs/2002.05202>). In this design, the standard two fully connected layers are replaced by three, as shown in figure C.3.

## GPT-2 (2019)



## Qwen3 (2025)



**Figure C.3** In GPT-2 (top), the feed forward module consists of two fully connected (linear) layers separated by a non-linear activation function. In Qwen3 (bottom), this module is a gated linear unit (GLU) variant, which adds a third linear layer (linear layer 3) and multiplies the output of this linear layer 3 elementwise with the activated output of linear layer 1.

Qwen3's feed forward module (figure C.3) can be implemented as shown in listing C.2.

**Listing C.2 Qwen3 feed forward module**

```

class FeedForward(nn.Module):
    def __init__(self, cfg):
        super().__init__()
        self.fc1 = nn.Linear(
            cfg["emb_dim"], cfg["hidden_dim"], dtype=cfg["dtype"],
            bias=False
        )
        self.fc2 = nn.Linear(
            cfg["emb_dim"], cfg["hidden_dim"], dtype=cfg["dtype"],
            bias=False
        )
        self.fc3 = nn.Linear(
            cfg["hidden_dim"], cfg["emb_dim"], dtype=cfg["dtype"],
            bias=False
        )

    def forward(self, x):
        x_fc1 = self.fc1(x)
        x_fc2 = self.fc2(x)
        x = nn.functional.silu(x_fc1) * x_fc2 #A
        return self.fc3(x)

```

#A The non-linear activation function here is a SiLU function, which will be discussed later

At first glance, it might seem that the GLU feed forward variant used in Qwen3 should outperform the standard feed forward variant in GPT-2, simply because it adds an extra linear layer (three instead of two) and therefore appears to have more parameters.

However, this intuition is misleading. In practice, the `fc1` and `fc2` layers in the GLU variant are each half the width of the `fc1` layer in a standard feed forward module. In practice, the GLU variant has fewer parameters.

To illustrate this with a concrete example, suppose the input dimension to the "Linear layer 1" in figure C.3 is 1024. This corresponds to `cfg["emb_dim"]` in listing C.2. The output dimension of `fc1` is 3,072 (`cfg["hidden_dim"]`). Note that these are the actual numbers used in the Qwen3 0.6B variant. In this case, we have the following parameter counts for the GLU variant in listing C.2:

- `fc1`:  $1024 \times 3,072 = 3,145,728$
- `fc2`:  $1024 \times 3,072 = 3,145,728$
- `fc3`:  $1024 \times 3,072 = 3,145,728$
- Total:  $3 \times 3,145,728 = 9,437,184$  parameters

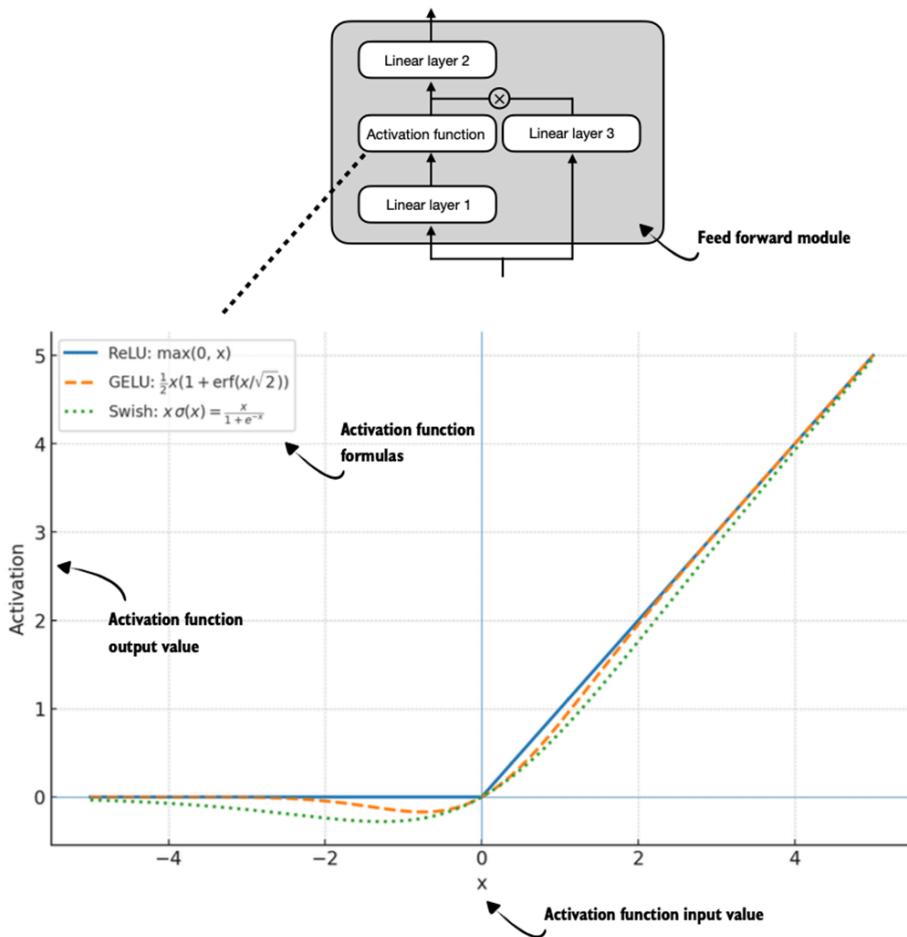
If we assume that `fc1` in this GLU variant has half the width as would be typically chosen for an `fc1` in a standard feed forward module, the parameter counts of the standard feed forward module would be as follows:

- `fc1`:  $1024 \times 2 \times 3,072 = 6,291,456$
- `fc2`:  $1024 \times 2 \times 3,072 = 6,291,456$
- Total:  $2 \times 6,291,456 = 12,582,912$  parameters

While GLU variants usually have fewer parameters than regular feed forward modules, they perform better. The improvement comes from the additional multiplicative interaction introduced by the gating mechanism, `activation(x_fc1) * x_fc2`, which increases the model's expressivity. This is similar to how deeper, slimmer networks can outperform shallower, wider ones, given proper training.

Before we proceed to the next section, there is one more thing to address. Note that the feed forward module shown in figure C.3 contains an element labeled as "Activation function," whereas we used a `nn.functional.silu` activation as a concrete example in listing C.2.

Historically, activation functions were a hot topic of debate until the deep learning community largely converged on the *rectified linear unit (ReLU)* more than a decade ago. ReLU is simple and computationally cheap, but it has a sharp kink at zero. This motivated researchers to explore smoother functions such as the *Gaussian error linear unit (GELU)* and the *sigmoid linear unit (SiLU)*, as shown in figure C.4.



**Figure C.4** Different activation functions that can be used in a feed forward module (neural network). GELU and SiLU (Swish) offer smooth alternatives to ReLU, which has a sharp kink at input zero.

GELU involves the Gaussian cumulative distribution function (CDF). Computing this CDF is slow because it uses piecewise logic and exponentials, which makes it hard to write fused, optimized GPU kernels (although a tanh approximation exists that uses cheaper operations and runs faster with near-identical results).

In short, while GELU produces smooth activation curves, it is overall computationally more expensive than simpler functions.

Newer models have largely replaced GELU with the SiLU (also known as *Swish*) function, which smoothly suppresses large negative inputs toward  $\sim 0$  and is approximately linear for large positive inputs, as shown in figure C.4.

SiLU has a similar smoothness, but it is slightly cheaper to compute than GELU and offers comparable modeling performance. In practice, SiLU is now used in most architectures, while GELU remains in use in only some models, such as Google's Gemma open-weight LLM. In the implementation of the feed forward module in listing C.2, this SiLU function is called via `nn.functional.silu`. The feed forward module in listing C.2 is also often called *SwiGLU*, an abbreviation that is derived from the terms Swish and GLU.

### C.3 Rotary position embeddings (RoPE)

In transformer-based LLMs, positional encoding is necessary because of the attention mechanism. By default, attention treats the input tokens as if they have no order. In the original GPT architecture, absolute positional embeddings addressed this by adding a learned embedding vector for each position in the sequence, which is then added to the token embeddings.

*RoPE* (short for *rotary position embeddings*) introduced a different approach: instead of adding position information as separate embeddings, it encodes position information by rotating the query and key vectors in the attention mechanism (section C.4) in a way that depends on each token's position. RoPE is an elegant idea, but also a long topic in itself. Interested readers can find more information in the original RoPE paper at <https://arxiv.org/abs/2104.09864>. (While first introduced in 2021, RoPE became widely adopted with the release of the original Llama model in 2023 and has since become a staple in modern LLMs, so it is not unique to Qwen3.)

RoPE can be implemented in two mathematically equivalent ways: the interleaved form, which pairs adjacent dimensions for rotation, or in a two-halves form, which splits the dimension into cosine and sine halves for convenience. Listing C.3 implements the two-halves variant, which can be easier to read.

**Listing C.3 RoPE functions**

```

import torch


def compute_rope_params(head_dim, theta_base=10_000, context_length=4096,
                       dtype=torch.float32):
    assert head_dim % 2 == 0, "Embedding dimension must be even"
    inv_freq = 1.0 / (theta_base ** (
        torch.arange(0, head_dim, 2, dtype=dtype)[: (head_dim // 2)].float() /
        head_dim
    ))
    positions = torch.arange(context_length, dtype=dtype)
    angles = positions.unsqueeze(1) * inv_freq.unsqueeze(0)
    angles = torch.cat([angles, angles], dim=1)

    cos = torch.cos(angles)
    sin = torch.sin(angles)

    return cos, sin


def apply_rope(x, cos, sin, offset=0):
    batch_size, num_heads, seq_len, head_dim = x.shape      #A
    assert head_dim % 2 == 0, "Head dimension must be even"

    x1 = x[..., : head_dim // 2]   # First half      #B
    x2 = x[..., head_dim // 2:]   # Second half     #B

    cos = cos[offset:offset + seq_len, :].unsqueeze(0).unsqueeze(0)
    sin = sin[offset:offset + seq_len, :].unsqueeze(0).unsqueeze(0)
    # Shape after: (1, 1, seq_len, head_dim)

    rotated = torch.cat((-x2, x1), dim=-1)
    x_rotated = (x * cos) + (rotated * sin)

    return x_rotated.to(dtype=x.dtype)      #C

#A The shape is (batch_size, num_heads, seq_len, head_dim)
#B Split x into first half and second half
#C It's ok to use lower-precision after applying cos and sin rotation

```

The RoPE code in listing C.3 will be used in the grouped query attention mechanism in section C.4.

### ROPE IMPLEMENTATION VARIANTS

Readers familiar with the original RoPE paper (<https://arxiv.org/abs/2104.09864>), you may be wondering about the particular implementation I have chosen.

There are two common styles to implement RoPE, which are mathematically equivalent. The implementations mainly differ in how the rotation matrix pairs dimensions. I chose the split-halves style as it is a bit easier to read and implement.

1) Split-halves style (this book, Hugging Face Transformers):

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{bmatrix}$$

$$\begin{array}{cccccccc} | & | & | & | & | & | & | & | \\ \downarrow & \downarrow \\ \cos & \cos & \cos & \cos & \sin & \sin & \sin & \sin \end{array}$$

Rotation matrix:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & \dots \\ \sin\theta & \cos\theta & 0 & 0 & \dots \\ 0 & 0 & \cos\theta & -\sin\theta & \dots \\ 0 & 0 & \sin\theta & \cos\theta & \dots \end{bmatrix}$$

Here, the embedding dimensions are split into two halves and then each one is rotated in blocks.

2) Interleaved (even/odd) style (original paper, Llama repo):

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{bmatrix}$$

$$\begin{array}{cccccccc} | & | & | & | & | & | & | & | \\ \downarrow & \downarrow \\ \cos & \sin & \cos & \sin & \cos & \sin & \cos & \sin \end{array}$$

Rotation matrix:

[ cosθ	-sinθ	0	0	...
[ sinθ	cosθ	0	0	...
[ 0	0	cosθ	-sinθ	...
[ 0	0	sinθ	cosθ	...

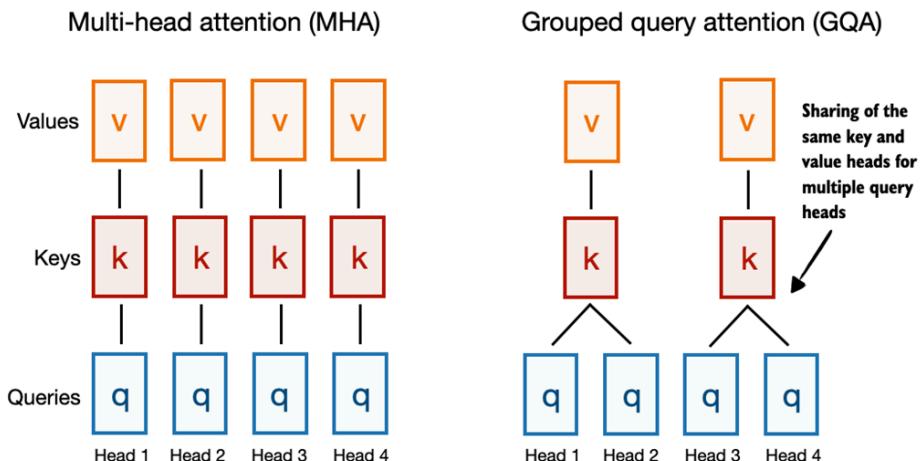
Here, embedding dims are interleaved as even/odd cosine/sine pairs.

Both layouts encode the same relative positions. The only difference is how dimensions are paired.

## C.4 Grouped query attention (GQA)

*Grouped query attention (GQA)* has become the standard, more compute- and parameter-efficient alternative to the original multi-head attention (MHA) mechanism.

Unlike MHA, where each head also has its own set of keys and values, to reduce memory usage, GQA groups multiple heads to share the same key and value projections, as shown in figure C.5.



**Figure C.5** A comparison between MHA and GQA. Here, the group size is 2, where a key and value pair is shared among 2 queries.

So, the core idea behind GQA, shown in figure C.5, is to reduce the number of key and value heads by sharing them across multiple query heads. This (1) lowers the model's parameter count and (2) reduces the memory bandwidth usage for key and value tensors during inference since fewer keys and values need to be stored and retrieved from the *KV cache* (section C.7).

While GQA is primarily a computational efficiency workaround for MHA, ablation studies (as presented in the original GQA paper, <https://arxiv.org/abs/2305.13245>) show that it performs comparably to standard MHA in terms of LLM modeling performance.

Listing C.4 implements the GQA mechanism with KV cache support.

**Listing C.4 Grouped query attention**

```

class GroupedQueryAttention(nn.Module):
    def __init__(self, d_in, num_heads, num_kv_groups, head_dim=None,
                 qk_norm=False, dtype=None):
        super().__init__()
        assert num_heads % num_kv_groups == 0

        self.num_heads = num_heads
        self.num_kv_groups = num_kv_groups
        self.group_size = num_heads // num_kv_groups

        if head_dim is None:
            assert d_in % num_heads == 0
            head_dim = d_in // num_heads

        self.head_dim = head_dim
        self.d_out = num_heads * head_dim

        self.W_query = nn.Linear(
            d_in, self.d_out, bias=False, dtype=dtype
        )
        self.W_key = nn.Linear(
            d_in, num_kv_groups * head_dim, bias=False, dtype=dtype
        )
        self.W_value = nn.Linear(
            d_in, num_kv_groups * head_dim, bias=False, dtype=dtype
        )

        self.out_proj = nn.Linear(self.d_out, d_in, bias=False, dtype=dtype)

        if qk_norm:
            self.q_norm = RMSNorm(head_dim, eps=1e-6)
            self.k_norm = RMSNorm(head_dim, eps=1e-6)
        else:
            self.q_norm = self.k_norm = None

    def forward(self, x, mask, cos, sin, start_pos=0, cache=None):
        b, num_tokens, _ = x.shape

        queries = self.W_query(x)                      #A
        keys = self.W_key(x)                          #B
        values = self.W_value(x)                      #B

```

```

        queries = queries.view(b, num_tokens, self.num_heads,
                               self.head_dim).transpose(1, 2)
        keys_new = keys.view(b, num_tokens, self.num_kv_groups,
                             self.head_dim).transpose(1, 2)
        values_new = values.view(b, num_tokens, self.num_kv_groups,
                                 self.head_dim).transpose(1, 2)

        if self.q_norm:
            queries = self.q_norm(queries)
        if self.k_norm:
            keys_new = self.k_norm(keys_new)

        queries = apply_rope(queries, cos, sin, offset=start_pos)
        keys_new = apply_rope(keys_new, cos, sin, offset=start_pos)

        if cache is not None:
            prev_k, prev_v = cache
            keys = torch.cat([prev_k, keys_new], dim=2)
            values = torch.cat([prev_v, values_new], dim=2)
        else:
            start_pos = 0                      #C
            keys, values = keys_new, values_new
            next_cache = (keys, values)

        keys = keys.repeat_interleave(          #D
                                     self.group_size, dim=1)           #D
        )                                    #D
        values = values.repeat_interleave(    #D
                                         self.group_size, dim=1)           #D
        )                                    #D

        attn_scores = queries @ keys.transpose(2, 3)
        attn_scores = attn_scores.masked_fill(mask, -torch.inf)
        attn_weights = torch.softmax(
            attn_scores / self.head_dim**0.5, dim=-1
        )

        context = (attn_weights @ values).transpose(1, 2)
        context = context.reshape(b, num_tokens, self.d_out)
        return self.out_proj(context), next_cache
    
```

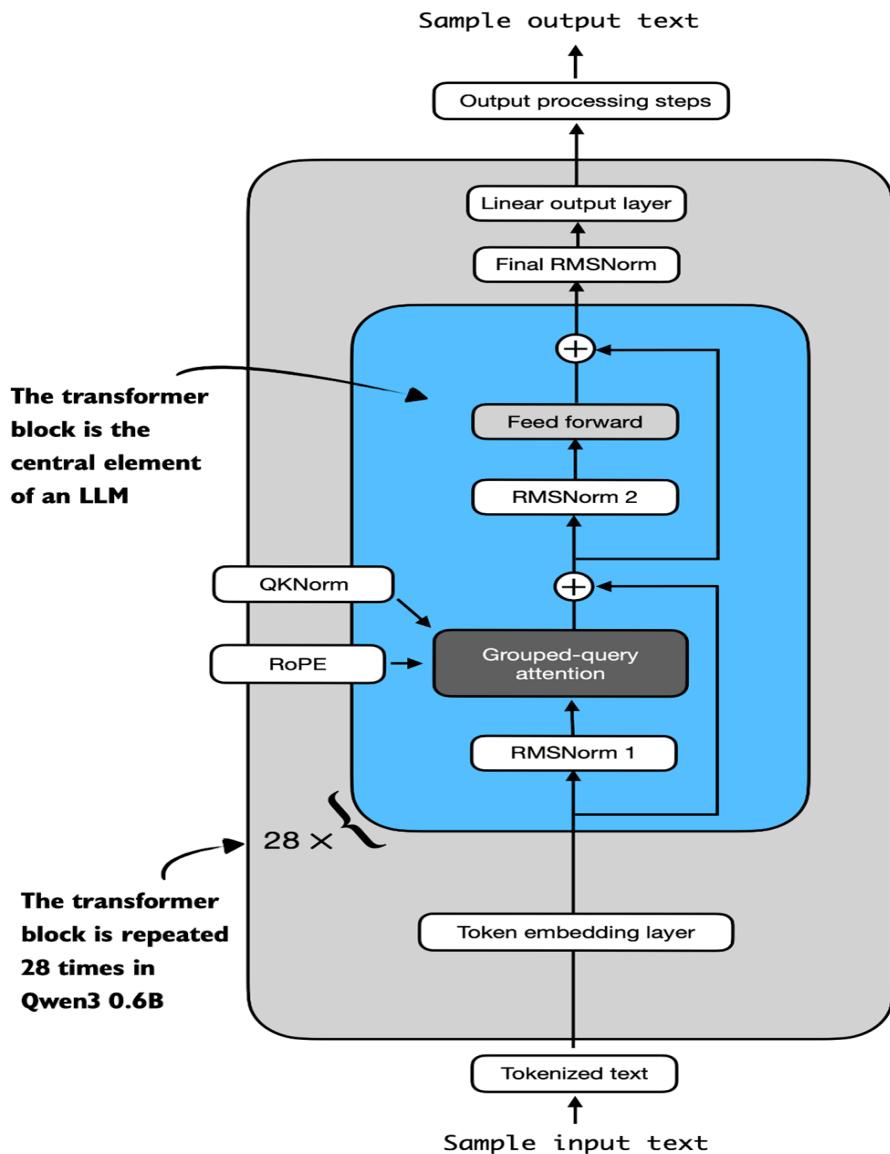
```
#A The shape is (b, num_tokens, num_heads * head_dim)
#B The shapes are (b, num_tokens, num_kv_groups * head_dim)
#C Reset RoPE
#D Expand K and V to match number of heads
```

You may have noticed that the GQA mechanism in listing C.4 also includes a `qk_norm` parameter. This is not part of the standard GQA design. When `qk_norm=True`, an additional Query/Key-RMSNorm-based normalization, called *QKNorm*, is applied to both the queries and keys, which is a technique used in Qwen3. As discussed earlier in the RMSNorm section (section C.1), QKNorm helps improve training stability.

## C.5 Transformer block

The *transformer block* is the central component of an LLM, which combines all the individual elements covered in this appendix so far. As shown in figure C.6, it is repeated multiple times; in the 0.6-billion-parameter version of Qwen3, it is repeated 28 times.

## Qwen3 0.6B



**Figure C.6 The Structure of the transformer block in Qwen3. Each block includes RMSNorm, RoPE, masked grouped-query attention, and a feed-forward module, and is repeated 28 times in the 0.6B-parameter model.**

Listing C.5 implements the transformer block shown in figure C.6.

**Listing C.5 Transformer block**

```

class TransformerBlock(nn.Module):
    def __init__(self, cfg):
        super().__init__()
        self.att = GroupedQueryAttention(
            d_in=cfg["emb_dim"],
            num_heads=cfg["n_heads"],
            head_dim=cfg["head_dim"],
            num_kv_groups=cfg["n_kv_groups"],
            qk_norm=cfg["qk_norm"],
            dtype=cfg["dtype"]
        )
        self.ff = FeedForward(cfg)
        self.norm1 = RMSNorm(cfg["emb_dim"], eps=1e-6)
        self.norm2 = RMSNorm(cfg["emb_dim"], eps=1e-6)

    def forward(self, x, mask, cos, sin, start_pos=0, cache=None):
        shortcut = x
        x = self.norm1(x)
        x, next_cache = self.att(
            x, mask, cos, sin, start_pos=start_pos, cache=cache
        ) #A
        x = x + shortcut

        shortcut = x
        x = self.norm2(x)
        x = self.ff(x)
        x = x + shortcut

    return x, next_cache

```

#A The shape is (batch\_size, num\_tokens, emb\_size)

As we can see, in listing C.5, the transformer block simply connects various elements we implemented in previous sections.

## C.6 Main model code

In this section, we will define the `Qwen3Model` class that we imported and used in chapter 2.

To implement the `Qwen3Model` class, the code in listing C.6 follows the architecture previously shown in figure C.6, where the transformer block sits at the heart of the LLM.

**Listing C.6 Main Qwen3Model code**

```

class Qwen3Model(nn.Module):
    def __init__(self, cfg):
        super().__init__()

        # Main model parameters
        self.tok_emb = nn.Embedding(cfg["vocab_size"], cfg["emb_dim"],
                                   dtype=cfg["dtype"])

        self.trf_blocks = nn.ModuleList(
            [TransformerBlock(cfg) for _ in range(cfg["n_layers"])]
        )
        self.final_norm = RMSNorm(cfg["emb_dim"])
        self.out_head = nn.Linear(
            cfg["emb_dim"], cfg["vocab_size"],
            bias=False, dtype=cfg["dtype"]
        )

        # Reusable utilities
        if cfg["head_dim"] is None:
            head_dim = cfg["emb_dim"] // cfg["n_heads"]
        else:
            head_dim = cfg["head_dim"]
        cos, sin = compute_rope_params(
            head_dim=head_dim,
            theta_base=cfg["rope_base"],
            context_length=cfg["context_length"]
        )
        self.register_buffer("cos", cos, persistent=False)
        self.register_buffer("sin", sin, persistent=False)
        self.cfg = cfg
        self.current_pos = 0 # Track current position in KV cache

    def forward(self, in_idx, cache=None):
        tok_embeds = self.tok_emb(in_idx)
        x = tok_embeds

        num_tokens = x.shape[1]
        if cache is not None:
            pos_start = self.current_pos
            pos_end = pos_start + num_tokens
            self.current_pos = pos_end
            mask = torch.triu(

```

```

        torch.ones(
            pos_end, pos_end, device=x.device, dtype=torch.bool
        ),
        diagonal=1
    )[pos_start:pos_end, :pos_end]
else:
    pos_start = 0 # Not strictly necessary but helps torch.compile
    mask = torch.triu(
        torch.ones(num_tokens, num_tokens, device=x.device,
                  dtype=torch.bool),
        diagonal=1
    )

mask = mask[None, None, :, :] #A

for i, block in enumerate(self.trf_blocks):
    blk_cache = cache.get(i) if cache else None
    x, new_blk_cache = block(x, mask, self.cos, self.sin,
                             start_pos=pos_start,
                             cache=blk_cache)
    if cache is not None:
        cache.update(i, new_blk_cache)

    x = self.final_norm(x)
    logits = self.out_head(x.to(self.cfg["dtype"]))
    return logits

def reset_kv_cache(self):
    self.current_pos = 0

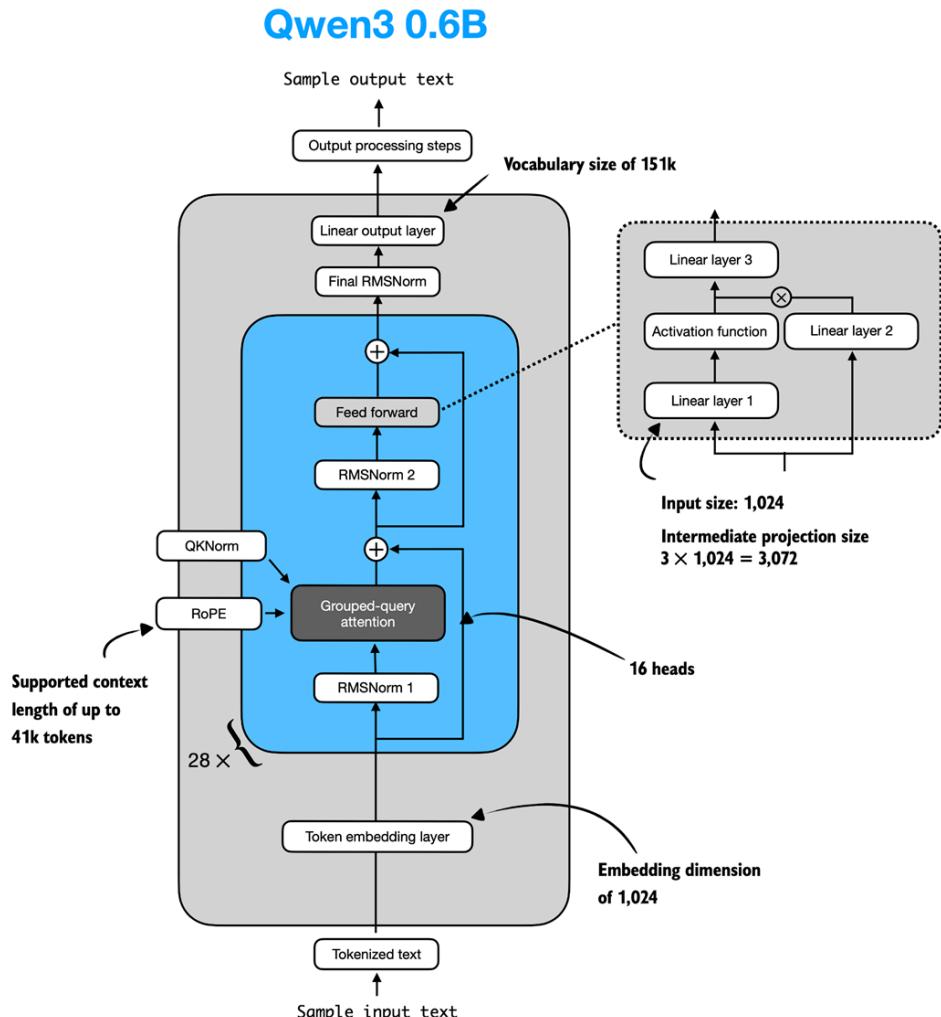
```

#A Prefill: Shape (1, 1, T, T) to broadcast across batch and heads. Cached: Shape (1, 1, T, K+T) where T=new tokens, K=cached keys.

Since we already have all the main ingredients, the `Qwen3Model` class in listing C.6 only adds a few more components around the transformer block, namely the embedding and output layers (including one more RMSNorm layer). However, the code may appear somewhat complicated, which is due to the KV cache option.

As discussed in chapter 2, the KV cache can speed up the text generation process, but it is a topic outside the scope of this book. Interested readers can find more information about KV caching in my *Understanding and Coding the KV Cache in LLMs from Scratch* article at <https://magazine.sebastianraschka.com/p/coding-the-kv-cache-in-langs.>

Note that the `Qwen3Model` class, as implemented in listing C.6, supports various model sizes (see appendix D for more information). In chapter 2, we use the 0.6-billion-parameter model as it is the least resource-intensive model in the Qwen3 model family. The specific configuration of this model is visualized in figure C.7.



**Figure C.7 Architecture of the Qwen3 0.6B model.** The model consists of a token embedding layer followed by 28 transformer blocks, each containing RMSNorm, RoPE, QKNorm, masked grouped-query attention with 16 heads, and a feed-forward module with an intermediate size of 3,072.

To use the 0.6B model shown in figure C.7 via the `Qwen3Model` class, we can define the following configuration in listing C.7 that we provide as input (`cfg=QWEN_CONFIG_06_B`) upon instantiating a new `Qwen3Model` instance.

#### **Listing C.7 Qwen3 0.6B configuration**

```
QWEN_CONFIG_06_B = {
    "vocab_size": 151_936,           # Vocabulary size
    "context_length": 40_960,         # Length originally used during training
    "emb_dim": 1024,                # Embedding dimension
    "n_heads": 16,                  # Number of attention heads
    "n_layers": 28,                 # Number of layers
    "hidden_dim": 3072,              # Size of intermediate dim in FeedForward
    "head_dim": 128,                # Size of the heads in GQA
    "qk_norm": True,                # Whether to normalize queries & keys in GQA
    "n_kv_groups": 8,               # Key-Value groups for GQA
    "rope_base": 1_000_000.0,        # The base in RoPE's "theta"
    "dtype": torch.bfloat16,          # Lower-precision dtype to reduce memory
}
```

We will use the `QWEN_CONFIG_06_B` configuration from listing C.7 to instantiate the Qwen3 0.6B model later in section C.9.

#### **MORE PRECISE CONTEXT LENGTH INFORMATION**

For Qwen3 0.6B, the default maximum supported context length is 40,960 tokens. However, the model itself was trained with only 32,768 tokens. The 40,960-token allocation reserves 32,768 tokens for model outputs (the generated text) and 8,192 tokens for typical prompts (the user's question or instruction).

To put this in perspective, 40 thousand tokens correspond to roughly one half of the first Harry Potter book.

Since this length is sufficient for most reasoning tasks, the developers note that context-extension methods (such as YaRN) are not recommended unless the average context regularly exceeds 32 thousand tokens, as enabling them in shorter contexts can slightly degrade performance.

## C.7 KV cache

The KV-cache-related heavy-lifting is mostly done in the `Qwen3Model` (listing C.6) and `GroupedQueryAttention` (listing C.4) code. The `KVCache`, shown in listing C.8, stores the key-value pairs themselves during text generation, which results in the speedup we experienced when enabling KV caching in chapter 2.

### Listing C.8 KV Cache

```
class KVCache:
    def __init__(self, n_layers):
        self.cache = [None] * n_layers

    def get(self, layer_idx):
        return self.cache[layer_idx]

    def update(self, layer_idx, value):
        self.cache[layer_idx] = value

    def get_all(self):
        return self.cache

    def reset(self):
        for i in range(len(self.cache)):
            self.cache[i] = None
```

The `KVCache` class in listing C.8 is used inside the `generate_text_basic_cache` function that we implemented in chapter 2.

## C.8 Tokenizer

The `tokenizer` code is somewhat complicated, as it supports a variety of special tokens, in addition to the base model and the so-called "Thinking" model variant of Qwen3, which is a reasoning model. The full reimplementation of the tokenizer is shown in listing C.9.

**Listing C.9 Tokenizer**

```

import re
from tokenizers import Tokenizer


class Qwen3Tokenizer:
    _SPECIALS = [
        "<|endoftext|>",
        "<|im_start|>", "<|im_end|>",
        "<|object_ref_start|>", "<|object_ref_end|>",
        "<|box_start|>", "<|box_end|>",
        "<|quad_start|>", "<|quad_end|>",
        "<|vision_start|>", "<|vision_end|>",
        "<|vision_pad|>", "<|image_pad|>", "<|video_pad|>",
    ]
    _SPLIT_RE = re.compile(r"(<\|[^\>]+\?\|>)")

    def __init__(self,
                 tokenizer_file_path="tokenizer-base.json",
                 apply_chat_template=False,
                 add_generation_prompt=False,
                 add_thinking=False):

        self.apply_chat_template = apply_chat_template
        self.add_generation_prompt = add_generation_prompt
        self.add_thinking = add_thinking

        tok_path = Path(tokenizer_file_path)
        if not tok_path.is_file():
            raise FileNotFoundError(
                f"Tokenizer file '{tok_path}' not found. "
            )

        self._tok = Tokenizer.from_file(str(tok_path))
        self._special_to_id = {t: self._tok.token_to_id(t)
                              for t in self._SPECIALS}

        self.pad_token = "<|endoftext|>"
        self.pad_token_id = self._special_to_id.get(self.pad_token)

        f = tok_path.name.lower()                      #A
        if "base" in f and "reasoning" not in f:       #A
            self.eos_token = "<|endoftext|>"          #A
        else:                                         #A

```

```

        self.eos_token = "<|im_end|>"                      #A
        self.eos_token_id = self._special_to_id.get(self.eos_token)

    def encode(self, prompt, chat_wrapped=None):
        if chat_wrapped is None:
            chat_wrapped = self.apply_chat_template

        stripped = prompt.strip()
        if stripped in self._special_to_id and "\n" not in stripped:
            return [self._special_to_id[stripped]]

        if chat_wrapped:
            prompt = self._wrap_chat(prompt)

        ids = []
        for part in filter(None, self._SPLIT_RE.split(prompt)):
            if part in self._special_to_id:
                ids.append(self._special_to_id[part])
            else:
                ids.extend(self._tok.encode(part).ids)
        return ids

    def decode(self, token_ids):
        return self._tok.decode(token_ids, skip_special_tokens=False)

    def _wrap_chat(self, user_msg):
        s = f"<|im_start|>user\n{user_msg}<|im_end|>\n"
        if self.add_generation_prompt:
            s += "<|im_start|>assistant"
        if self.add_thinking:
            s += "\n"           #B
        else:
            s += "\n<think>\n\n</think>\n\n"
        return s

```

#A Match HF behavior: chat model: <|im\_end|>, base model: <|endoftext|>  
#B insert no <think> tag, just a new line

Note that my Qwen3Tokenizer reimplementation in listing C.9 may appear somewhat complicated, as it aims to replicate the behavior of the official tokenizer released by the Qwen3 team in the Hugging Face Transformers library.

At first glance, it appears to have a few quirks. For example, when `add_thinking=True`, no "`\n<think>\n\n</think>\n\n`" tokens are inserted (where `\n` is a newline character), and when `add_thinking=False`, these tokens are added. This is intentional because the non-base Qwen3 0.6B model is a hybrid that supports both reasoning ("thinking") and standard modes.

## C.9 Using the model

Let's now instantiate and use the model to confirm that the code works by reusing the text generation approach from chapter 2.

First, we instantiate the model using the pre-trained model weights:

```
from pathlib import Path
import torch

from reasoning_from_scratch.ch02 import get_device
from reasoning_from_scratch.qwen3 import download_qwen3_small

# device = get_device()           #A
device = torch.device("cpu")

download_qwen3_small(kind="base", tokenizer_only=False, out_dir="qwen3")

tokenizer_file_path = Path("qwen3") / "tokenizer-base.json"
model_file = Path("qwen3") / "qwen3-0.6B-base.pth"

tokenizer = Qwen3Tokenizer(tokenizer_file_path=tokenizer_file_path)
model = Qwen3Model(QWEN_CONFIG_06_B)
model.load_state_dict(torch.load(model_file))

model.to(device)

#A Optional: Uncomment to use automatic device picker
```

The output shows the structure of the instantiated model, which should match the values we used in the configuration file in listing C.7:

```
✓ qwen3/qwen3-0.6B-base.pth already up-to-date
✓ qwen3/tokenizer-base.json already up-to-date
Qwen3Model(
    (tok_emb): Embedding(151936, 1024)
    (trf_blocks): ModuleList(
        (0-27): 28 x TransformerBlock(
            (att): GroupedQueryAttention(
                (W_query): Linear(in_features=1024, out_features=2048, bias=False)
                (W_key): Linear(in_features=1024, out_features=1024, bias=False)
                (W_value): Linear(in_features=1024, out_features=1024, bias=False)
                (out_proj): Linear(in_features=2048, out_features=1024, bias=False)
                (q_norm): RMSNorm()
                (k_norm): RMSNorm()
            )
            (ff): FeedForward(
                (fc1): Linear(in_features=1024, out_features=3072, bias=False)
                (fc2): Linear(in_features=1024, out_features=3072, bias=False)
                (fc3): Linear(in_features=3072, out_features=1024, bias=False)
            )
            (norm1): RMSNorm()
            (norm2): RMSNorm()
        )
    )
    (final_norm): RMSNorm()
    (out_head): Linear(in_features=1024, out_features=151936, bias=False)
)
```

Next, we re-use the text generation functions from chapter 2 to generate text:

```

import time

from reasoning_from_scratch.ch02 import (
    generate_stats,
    generate_text_basic_cache,
)

prompt = "Explain large language models in a single sentence."

input_token_ids_tensor = torch.tensor(
    tokenizer.encode(prompt),
    device=device
).unsqueeze(0)

start_time = time.time()

output_token_ids_tensor = generate_text_basic_cache(
    model=model,
    token_ids=input_token_ids_tensor,
    max_new_tokens=200,
    eos_token_id=tokenizer.eos_token_id,
)
end_time = time.time()

generate_stats(output_token_ids_tensor, tokenizer, start_time, end_time)

```

Since we used the same prompt as in chapter 2, the generated text matches the generated text from chapter 2 exactly:

Time: 1.46 sec  
28 tokens/sec

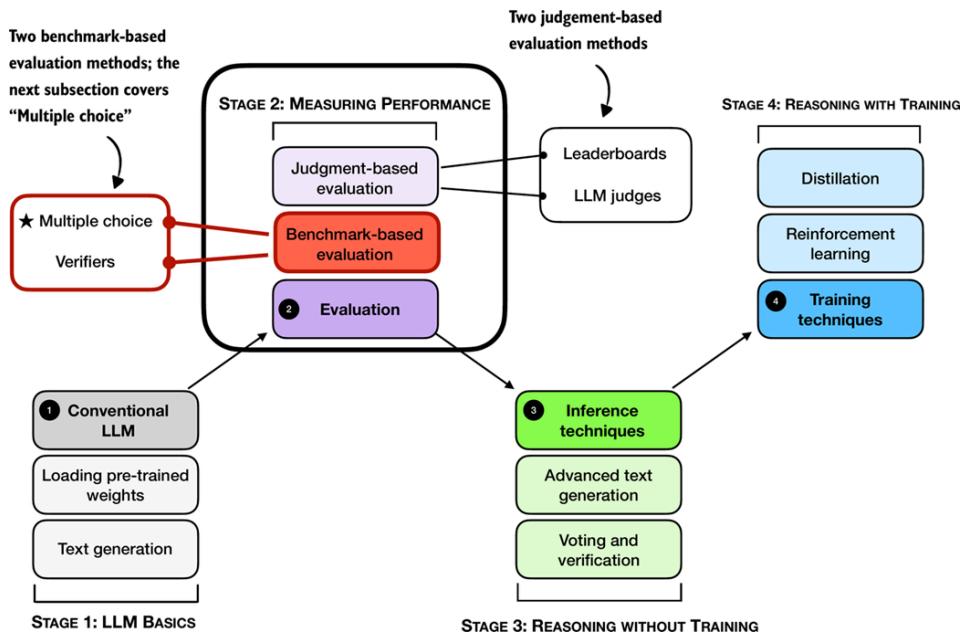
Large language models are artificial intelligence systems that can understand, generate, and process human language, enabling them to perform a wide range of tasks, from answering questions to writing articles, and even creating creative content.

While the main chapters use the 0.6-billion-parameter variant of Qwen3 to lower the resource requirements for this book, interested readers can find more information on how to use the larger models in appendix D.

## *Appendix F. Common approaches to model evaluation*

### **F.1 Understanding the main evaluation methods for LLMs**

There are four common ways of evaluating trained LLMs in practice: *multiple choice*, *verifiers*, *leaderboards*, and *LLM judges*, as shown in figure F.1. Research papers, marketing materials, technical reports, and model cards (a term for LLM-specific technical reports) often include results from two or more of these categories.



**Figure F.1** A mental model of the topics covered in this book with a focus on the two broad evaluation categories, *benchmark-based evaluation* and *judgment-based evaluation*, covered in this appendix.

Furthermore, as shown in figure F.1, the four categories introduced here fall into two groups: *benchmark-based evaluation* and *judgment-based evaluation*.

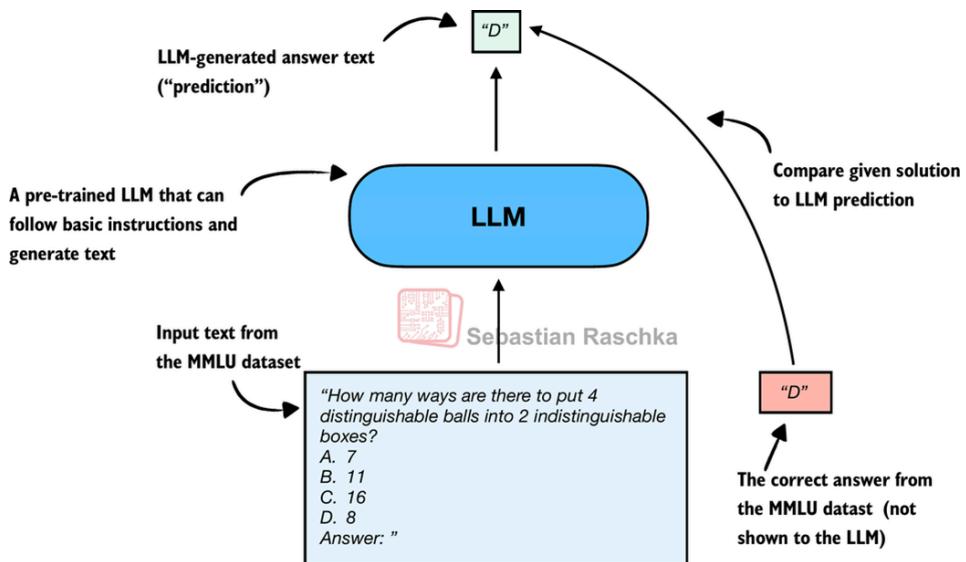
Other measures, such as *training loss*, *perplexity*, and *rewards*, are typically used internally during model development. (They are covered in the model training chapters.)

The following subsections provide brief overviews of each method.

## F.2 Evaluating answer-choice accuracy

We begin with a benchmark-based method: multiple-choice question answering.

Historically, one of the most widely used evaluation methods is multiple-choice benchmarks such as *MMLU* (short for Massive Multitask Language Understanding, <https://huggingface.co/datasets/cais/mmlu>). An example task from the MMLU dataset is shown in figure F.2.



**Figure F.2 Evaluating an LLM on MMLU by comparing its multiple-choice prediction with the correct answer from the dataset.**

Figure F.2 shows just a single example from the MMLU dataset. The complete MMLU dataset consists of 57 subjects (from high school math to biology) with about 16 thousand multiple-choice questions in total, and performance is measured in terms of accuracy (the fraction of correctly answered questions), for example 87.5% if 14,000 out of 16,000 questions are answered correctly.

Multiple-choice benchmarks, such as MMLU, test an LLM's knowledge recall in a straightforward, quantifiable way similar to standardized tests, many school exams, or theoretical driving tests.

Note that figure F.2 shows a simplified version of multiple-choice evaluation, where the model's predicted answer letter is compared directly to the correct one. Two other popular methods exist that involve *log-probability scoring* (log-probabilities are discussed in chapter 4 in more detail).

The following subsections illustrate how the MMLU scoring shown in figure F.2 can be implemented in code. End-to-end MMLU scripts, including the different scoring variants, will be provided as bonus materials in this book's code repository.

### F.2.1 Loading the model

First, before we can evaluate it on MMLU, we have to load the pre-trained model. The following code is identical to listing 3.1 in chapter 3.

**Listing F.1 Loading a pre-trained model**

```

from pathlib import Path
import torch
from reasoning_from_scratch.ch02 import get_device
from reasoning_from_scratch.qwen3 import (
    download_qwen3_small, Qwen3Tokenizer,
    Qwen3Model, QWEN_CONFIG_06_B
)

device = get_device()
torch.set_float32_matmul_precision("high")  #A

# device = "cpu"  #B

WHICH_MODEL = "base"  #C

if WHICH_MODEL == "base":
    download_qwen3_small(
        kind="base", tokenizer_only=False, out_dir="qwen3"
    )
    tokenizer_path = Path("qwen3") / "tokenizer-base.json"
    model_path = Path("qwen3") / "qwen3-0.6B-base.pth"
    tokenizer = Qwen3Tokenizer(tokenizer_file_path=tokenizer_path)

elif WHICH_MODEL == "reasoning":
    download_qwen3_small(
        kind="reasoning", tokenizer_only=False, out_dir="qwen3"
    )
    tokenizer_path = Path("qwen3") / "tokenizer-reasoning.json"
    model_path = Path("qwen3") / "qwen3-0.6B-reasoning.pth"
    tokenizer = Qwen3Tokenizer(
        tokenizer_file_path=tokenizer_path,
        apply_chat_template=True,
        add_generation_prompt=True,
        add_thinking=True,
    )

else:
    raise ValueError(f"Invalid choice: WHICH_MODEL={WHICH_MODEL}")

model = Qwen3Model(QWEN_CONFIG_06_B)
model.load_state_dict(torch.load(model_path))
model.to(device)

```

```

USE_COMPILE = False #D
if USE_COMPILE:
    torch._dynamo.config.allow_unspec_int_on_nn_module = True
    model = torch.compile(model)

#A Lower precision from "highest" to enable Tensor Cores if applicable
#B Uncomment this line if you have compatibility issues with your device
#C Uses the base model, similar to chapter 2, by default
#D Optionally set to true to enable model compilation

```

## F.2.2 Checking the generated answer letter

In this section, we implement the simplest and perhaps most intuitive MMLU scoring method, which relies on checking whether a generated multiple-choice answer letter matches the correct answer. This is similar to what was illustrated earlier in figure F.2.

For this, we will work with an example from the MMLU dataset:

```

example = {
    "question": (
        "How many ways are there to put 4 distinguishable"
        " balls into 2 indistinguishable boxes?"
    ),
    "choices": ["7", "11", "16", "8"],
    "answer": "D",
}

```

Next, we define a function to format the LLM prompts:

### Listing F.2 Loading a pre-trained model

```

def format_prompt(example):
    return (
        f"{example['question']}\n"
        f"A. {example['choices'][0]}\n"
        f"B. {example['choices'][1]}\n"
        f"C. {example['choices'][2]}\n"
        f"D. {example['choices'][3]}\n"
        "Answer: "
    )

```

Let's execute the function on the MMLU example to get an idea of what the formatted LLM input looks like:

```
prompt = format_prompt(example)
print(prompt)
```

The output is:

```
How many ways
are there to put 4 distinguishable balls into 2
indistinguishable boxes?
```

- A. 7
- B. 11
- C. 16
- D. 8

Answer:

The model prompt, as shown above, provides the model with a list of the different answer choices and ends with an "Answer: " text that encourages the model to generate the correct answer.

While it is not strictly necessary, it can sometimes also be helpful to provide additional questions along with the correct answers as input, so that the model can observe how it is expected to solve the task. (For example, cases where 5 examples are provided are also known as 5-shot MMLU.) However, for current generations of LLMs, where even the base models are quite capable, this is not required.

### LOADING DIFFERENT MMLU SAMPLES

You can load examples from the MMLU dataset directly via the datasets library (which can be installed via `pip install datasets` or `uv add datasets`):

```
from datasets import load_dataset
configs = get_dataset_config_names("cais/mmlu")
dataset = load_dataset("cais/mmlu", "high_school_mathematics")
# Inspect the first example from the test set:
example = dataset["test"][0]
print(example)
```

Above, we used the `"high_school_mathematics"` subset; to get a list of the other subsets, use the following code:

```
from datasets import get_dataset_config_names
subsets = get_dataset_config_names("cais/mmlu")
print(subsets)
```

Next, we tokenize the prompt and wrap it in a PyTorch tensor object as input to the LLM (similar to what we did in chapter 2):

```
prompt_ids = tokenizer.encode(prompt)
prompt_fmt = torch.tensor(prompt_ids, device=device).unsqueeze(0)
```

Then, we define the main scoring function in listing F.3, which generates a few tokens (here, 8 tokens by default) and extracts the first instance of letter A/B/C/D that the model prints.

**Listing F.3 Extracting the generated letter**

```

from reasoning_from_scratch.ch02_ex import (
    generate_text_basic_stream_cache
)

def predict_choice(
    model, tokenizer, prompt_fmt, max_new_tokens=8
):
    pred = None
    for t in generate_text_basic_stream_cache(
        model=model,
        token_ids=prompt_fmt,
        max_new_tokens=max_new_tokens,
        eos_token_id=tokenizer.eos_token_id,
    ):
        answer = tokenizer.decode(t.squeeze(0).tolist())
        for letter in answer:
            letter = letter.upper()
            if letter in "ABCD": #A
                pred = letter
                break
            if pred:
                break
    return pred

#A stop as soon as a letter appears

```

We can then check the generated letter using the function from listing F.3 as follows:

```

pred1 = predict_choice(model, tokenizer, prompt_fmt)
print(
    f"Generated letter: {pred1}\n"
    f"Correct? {pred1 == example['answer']}"
)

```

The result is:

```

Generated letter: C
Correct? False

```

As we can see, the generated answer is incorrect (`False`) in this case.

### MULTIPLE-CHOICE ANSWER FORMATS

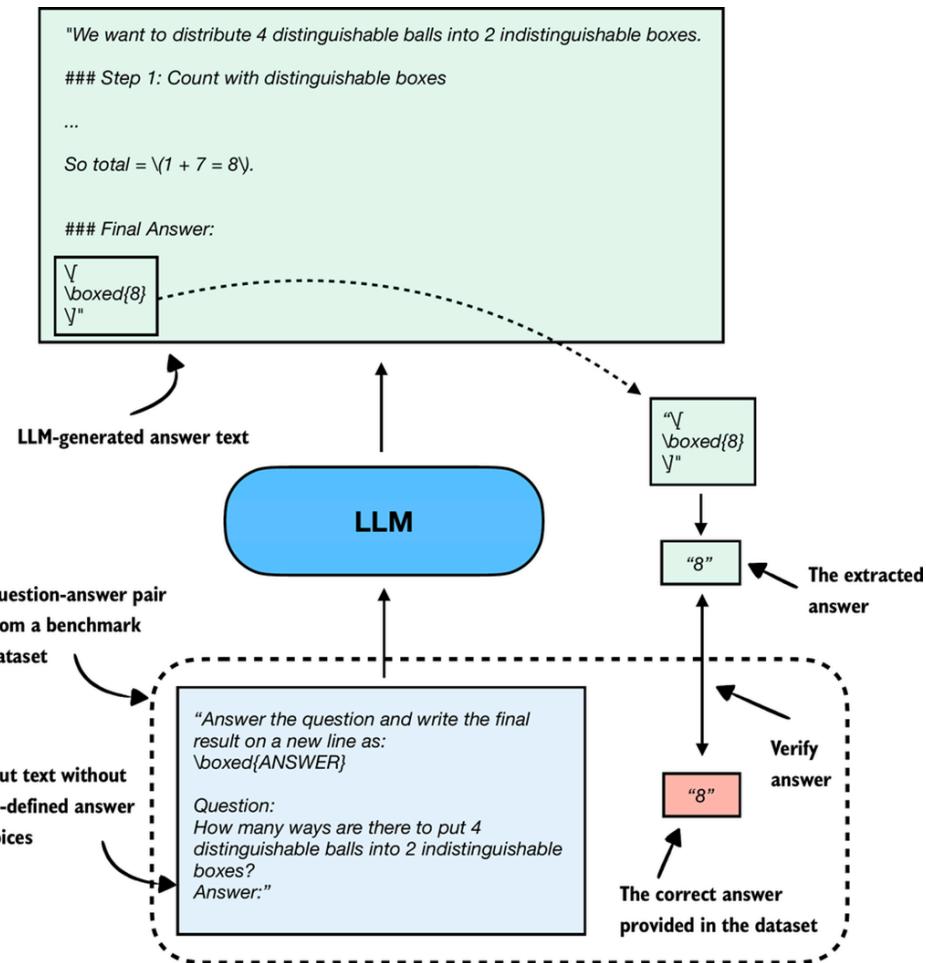
Note that this section implemented a simplified version of multiple-choice evaluation for illustration purposes, where the model's predicted answer letter is compared directly to the correct one. In practice, more widely used variations exist, such as log-probability scoring, where we measure how likely the model considers each candidate answer rather than just checking the final letter choice. (We discuss probability-based scoring in chapter 4.) For reasoning models, evaluation can also involve assessing the likelihood of generating the correct answer when it is provided as input.

Regardless of the variant, the evaluation still amounts to checking whether the model selects from the predefined answer options. Examples of these variations will be included in the code repository as optional bonus material.

A limitation of multiple-choice benchmarks like MMLU is that they only measure an LLM's ability to select from predefined options and thus is not very useful for evaluating reasoning capabilities besides checking if and how much knowledge the model has forgotten compared to the base model. It does not capture free-form writing ability or real-world utility. Still, it remains a simple and useful diagnostic: a high MMLU score doesn't necessarily mean the model is strong in practical use, but a low score can highlight potential knowledge gaps.

### F.3 Using verifiers to check answers

Related to multiple-choice question answering discussed in the previous section, verification-based approaches quantify the LLMs capabilities via an accuracy metric. However, in contrast to multiple-choice benchmarks, verification methods allow LLMs to provide a free-form answer. We then extract the relevant answer portion and use a so-called verifier to compare the answer portion to the correct answer provided in the dataset, as illustrated in figure F.3.



**Figure F.3 Evaluating an LLM with a verification-based method in free-form question answering. The model generates a free-form answer (which may include multiple steps) and a final boxed answer, which is extracted and compared against the correct answer from the dataset.**

When we compare the extracted answer with the provided answer, as shown in figure F.3, we can employ external tools, such as code interpreters or calculator software.

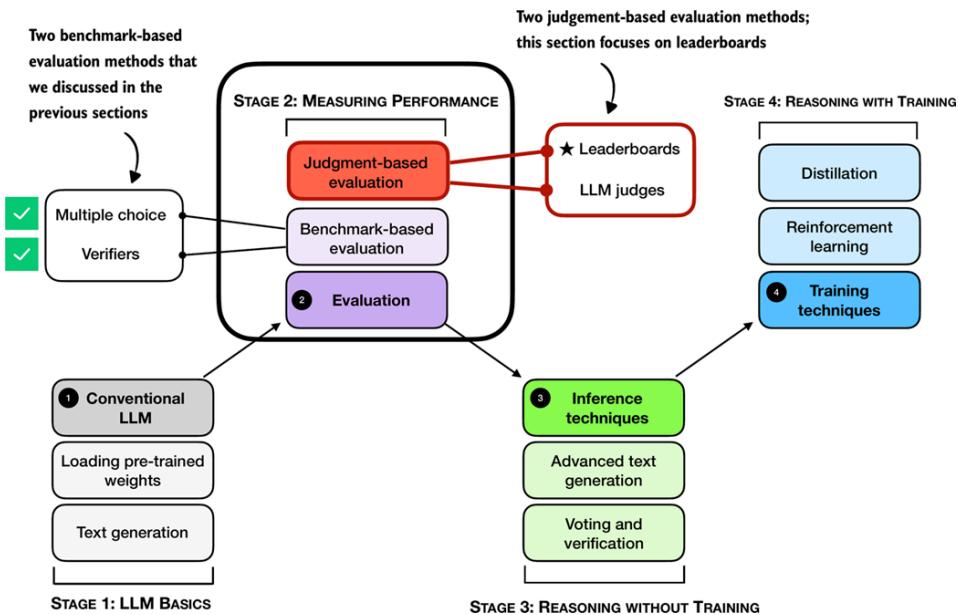
The downside is that this method can only be applied to domains that can be easily (and ideally deterministically) verified, such as math and code. Also, this approach can introduce additional complexity and dependencies, and it may shift part of the evaluation burden from the model itself to the external tool.

However, because it allows us to generate an unlimited number of math problem variations programmatically and benefits from step-by-step reasoning, it has become a cornerstone of reasoning model evaluation and development.

An extensive example of this method is provided in chapter 3, which is why we skip a code demonstration here.

## F.4 Comparing models using preferences and leaderboards

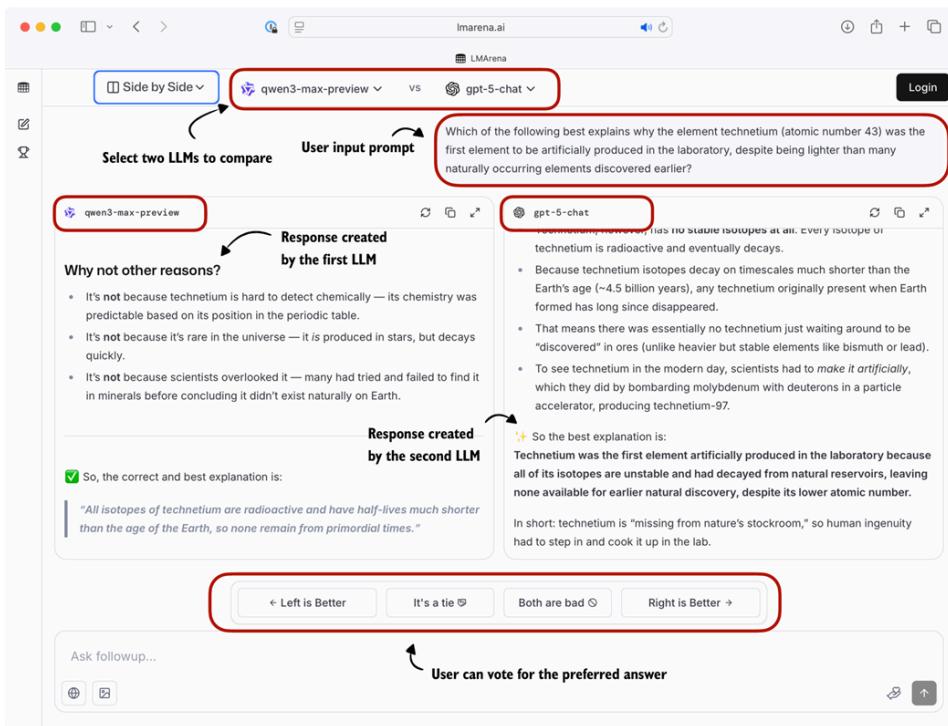
So far, we have covered two methods that offer easily quantifiable metrics such as model accuracy. However, none of the aforementioned methods evaluate LLMs in a more holistic way, including judging the style of the responses. In this section, as illustrated in figure F.4, we discuss a judgment-based method, namely, LLM leaderboards.



**Figure F.4** A mental model of the topics covered in this book with a focus on the judgment- and benchmark-based evaluation methods covered in this appendix. Having already covered benchmark-based approaches (multiple choice, verifiers) in the previous section, we now introduce judgment-based approaches to measure LLM performance, with this subsection focusing on leaderboards.

The leaderboard method mentioned in figure F.4 is a judgment-based approach where models are ranked not by accuracy values or other fixed benchmark scores but by user (or other LLM) preferences on their outputs.

A popular leaderboard is *LM Arena* (formerly Chatbot Arena, <https://lmarena.ai/>), where users compare responses from two user-selected or anonymous models and vote for the one they prefer, as shown in figure F.5.



**Figure F.5 Example of a judgment-based leaderboard interface (LM Arena). Two LLMs are given the same prompt, their responses are shown side by side, and users vote for the preferred answer.**

These preference votes, which are collected as shown in figure F.5, are then aggregated across all users into a leaderboard that ranks different models by user preference. In the remainder of this section, we will implement a simple example of a leaderboard.

To create a concrete example, consider users prompting different LLMs in a setup similar to figure F.5. The list below represents pairwise votes where the first model is the winner:

```
votes = [
    ("GPT-5", "Claude-3"),
    ("GPT-5", "Llama-4"),
    ("Claude-3", "Llama-3"),
    ("Llama-4", "Llama-3"),
    ("Claude-3", "Llama-3"),
    ("GPT-5", "Llama-3"),
]
```

In the list above, each tuple in the votes list represents a pairwise preference between two models, written as (winner, loser). So, ("GPT-5", "Claude-3") means that a user preferred GPT-5 over a Claude-3 model answer.

In the remainder of this section, we will turn the votes list into a leaderboard. For this, we will use the popular Elo rating system, which was originally developed for ranking chess players. Before we look at the concrete code implementation, in short, it works as follows. Each model starts with a baseline score. Then, after each comparison and the preference vote, the model's rating is updated. Specifically, if a user prefers a current model over a highly ranked model, the current model will get a relatively large ranking update and rank higher in the leaderboard. Vice versa, if the current model loses against a lowly ranked model, it increases the rating only a little. (And if the current model loses, it is updated in a similar fashion, but with ranking points getting subtracted instead of added.)

The code to turn these pairwise rankings into a leaderboard is shown in listing F.4.

#### **Listing F.4 Constructing a leaderboard**

```
def elo_ratings(vote_pairs, k_factor=32, initial_rating=1000):
    ratings = { #A
        model: initial_rating
        for pair in vote_pairs
        for model in pair
    }

    for winner, loser in vote_pairs: #B
        rating_winner, rating_loser = ratings[winner], ratings[loser]

        expected_winner = 1.0 / ( #C
            1.0 + 10 ** ((ratings[loser] - ratings[winner]) / 400.0)
        )

        ratings[winner] = ( #D
            ratings[winner] + k_factor * (1 - expected_winner)
        )
        ratings[loser] = (
            ratings[loser] + k_factor * (0 - (1 - expected_winner))
        )

    return ratings

#A Initialize all models with the same base rating
#B Update ratings after each match
#C Expected score for the current winner given the ratings
#D k_factor determines sensitivity of rating updates
```

The `elo_ratings` function in listing F.4 takes the `votes` as input and turns it into a leaderboard, as follows:

```
ratings = elo_ratings(votes, k_factor=32, initial_rating=1000)
for model in sorted(ratings, key=ratings.get, reverse=True):
    print(f"{model:8s} : {ratings[model]:.1f}")
```

This results in the following leaderboard ranking, where the higher the score, the better:

```
GPT-5      : 1043.7
Claude-3   : 1015.2
Llama-4    : 1000.7
Llama-3    : 940.4
```

So, how does this work? For each pair, we compute the expected score of the winner using the following formula:

```
expected_winner = 1 / (1 + 10 ** ((rating_loser - rating_winner) / 400))
```

This value `expected_winner` is the model's predicted chance to win in a no-draw setting based on the current ratings. It determines how large the rating update is.

First, each model starts at `initial_rating = 1000`. If the two ratings (winner and loser) are equal, we have `expected_winner = 0.5`, which indicates an even match. In this case, the updates are:

```
rating_winner + k_factor * (1 - 0.5) = rating_winner + 16
rating_loser   + k_factor * (0 - (1 - 0.5)) = rating_loser - 16
```

Now, if a heavy favorite (a model with a high rating) wins, we have `expected_winner ≈ 1`. The favorite gains only a small amount and the loser loses only a little:

```
rating_winner + 32 * (1 - 0.99) = rating_winner + 0.32
rating_loser   + 32 * (0 - (1 - 0.99)) = rating_loser - 0.32
```

However, if an underdog (a model with a low rating) wins, we have `expected_winner ≈ 0`, and the winner gets almost the full `k_factor` points while the loser loses about the same magnitude:

```
rating_winner + 32 * (1 - 0.01) = rating_winner + 31.68
rating_loser   + 32 * (0 - (1 - 0.01)) = rating_loser - 31.68
```

### ORDER MATTERS

The Elo approach updates ratings after each match (model comparisons), so later results build on ratings that have already been updated. This means the same set of outcomes, when presented in a different order, can end with slightly different final scores. This effect is usually mild, but it can happen especially when an upset happens early versus late.

To reduce this order effect, we can shuffle the `votes` pairs and run the `elo_ratings` function multiple times and average the ratings.

Leaderboard approaches such as the one described above provide a more dynamic view of model quality than static benchmark scores. However, the results can be influenced by user demographics, prompt selection, and voting biases. Benchmarks and leaderboards can also be gamed, and users may select responses based on style rather than correctness. Finally, compared to automated benchmark harnesses, leaderboards do not provide instant feedback on newly developed variants, which makes them harder to use during active model development.

### OTHER RANKING METHODS

The LM Arena originally used the Elo method described in this section but recently transitioned to a statistical approach based on the Bradley–Terry model. The main advantage of the Bradley–Terry model is that, being statistically grounded, it allows the construction of confidence intervals to express uncertainty in the rankings. Also, in contrast to the Elo ratings, the Bradley–Terry model estimates all ratings jointly using a statistical fit over the entire dataset, which makes it immune to order effects.

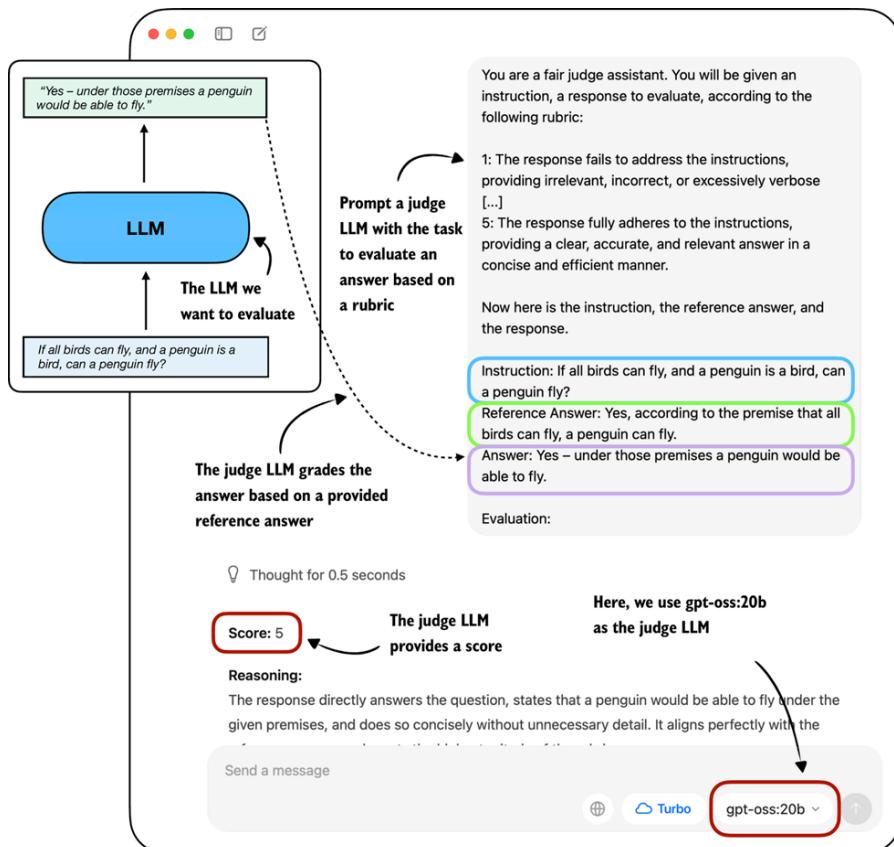
To keep the reported scores in a familiar range, the Bradley–Terry model is fitted to produce values comparable to Elo. Even though the leaderboard no longer officially uses Elo ratings, the term "Elo" remains widely used by LLM researchers and practitioners when comparing models. A code example showing the Elo rating is included in this book's bonus materials at [https://github.com/rasbt/reasoning-from-scratch/tree/main/chF/03\\_leaderboards](https://github.com/rasbt/reasoning-from-scratch/tree/main/chF/03_leaderboards).

## F.5 Judging responses with other LLMs

In the early days, LLMs were evaluated using statistical and heuristics-based methods, including a measure called *BLEU*, which is a crude measure of how well generated text matches reference text. The problem with such metrics is that they require exact word matches and don't account for synonyms, word changes, and so on.

One solution to this problem, if we want to judge the written answer text as a whole, is to use relative rankings and leaderboard-based approaches as discussed in the previous section. However, a downside of leaderboards is the subjective nature of the preference-based comparisons as it involves human feedback (as well as the challenges that are associated with collecting this feedback).

A related method is to use another LLM with a pre-defined grading *rubric* (i.e., an evaluation guide) to compare an LLM's response to a reference response and judge the response quality based on a pre-defined rubric, as illustrated in figure F.6.



**Figure F.6 Example of an LLM-judge evaluation. The model to be evaluated generates an answer, which is then scored by a separate judge LLM according to a rubric and a provided reference answer.**

In practice, the judge-based approach shown in figure F.6 works well when the judge LLM is strong. Common setups use leading proprietary LLMs via API, though specialized judge models also exist (see appendix A for references). One of the reasons why judges work so well is also that evaluating an answer is often easier than generating one.

To implement a judge-based model evaluation as shown in figure F.6 programmatically in Python, we could either load one of the Qwen3 models (appendix D) and prompt it with a grading rubric and the model answer we want to evaluate.

Alternatively, we can use other LLMs through an API, for example the ChatGPT or Ollama API. In the remainder of the section, we will implement the judge-based evaluation shown in figure F.6 using the Ollama API in Python.

Specifically, we will use the 20-billion parameter gpt-oss open-weight model by OpenAI as it offers a good balance between capabilities and efficiency. For more information about gpt-oss, please see my *From GPT-2 to gpt-oss: Analyzing the Architectural Advances* article at <https://magazine.sebastianraschka.com/p/from-gpt-2-to-gpt-oss-analyzing-the->.

### F.5.1 Implementing a LLM-as-a-judge approach in Ollama

Ollama (<https://ollama.com>) is an efficient open-source application for running LLMs on a laptop. It serves as a wrapper around the open-source llama.cpp library (<https://github.com/ggerganov/llama.cpp>), which implements LLMs in pure C/C++ to maximize efficiency. However, note that Ollama is only a tool for generating text using LLMs (inference) and does not support training or fine-tuning LLMs.

To execute the following code, please install Ollama by visiting <https://ollama.com> and follow the provided instructions for your operating system:

- For macOS and Windows users: Open the downloaded Ollama application. If prompted to install command-line usage, select "yes."
- For Linux users: Use the installation command available on the Ollama website.

Before implementing the model evaluation code, let's first download the gpt-oss model and verify that Ollama is functioning correctly by using it from the command line terminal.

Execute the following command on the command line (not in a Python session) to try out the 20 billion parameter gpt-oss model:

```
ollama run gpt-oss:20b
```

The first time you execute this command, the 20 billion parameter gpt-oss model, which takes up 14 GB of storage space, will be automatically downloaded. The output looks as follows:

```
$ ollama run gpt-oss:20b
pulling manifest
pulling b112e727c6f1: 100% [██████████] 13 GB
pulling fa6710a93d78: 100% [██████████] 7.2 KB
pulling f60356777647: 100% [██████████] 11 KB
pulling d8ba2f9a17b3: 100% [██████████] 18 B
pulling 55c108d8e936: 100% [██████████] 489 B
verifying sha256 digest
writing manifest
removing unused layers
success
```

### ALTERNATIVE OLLAMA MODELS

Note that the `gpt-oss:20b` in the `ollama run gpt-oss:20b` command refers to the 20 billion parameter gpt-oss model. Using Ollama with the `gpt-oss:20b` model requires approximately 13 GB of RAM. If your machine does not have sufficient RAM, you can try using a smaller model, such as the 4 billion parameter `qwen3:4b` model via `ollama run qwen3:4b`, which only requires around 4 GB of RAM.

For more powerful computers, you can also use the larger 120-billion parameter gpt-oss model by replacing `gpt-oss:20b` with `gpt-oss:120b`. However, keep in mind that this model requires significantly more computational resources.

Once the model download is complete, we are presented with a command-line interface that allows us to interact with the model. For example, try asking the model, "What is  $1+2$ ":

```
>>> What is 1+2?
Thinking...
User asks: "What is 1+2?" This is simple: answer 3. Provide explanation? Possibly
ask for simple
arithmetic. Provide answer: 3.
...done thinking.

1 + 2 = **3**
```

You can end this `ollama run gpt-oss:20b` session using the input `/bye`.

In the remainder of this section, we will use the ollama API. This approach requires that Ollama is running in the background. There are three different options to achieve this:

1. Run the `ollama serve` command in the terminal (recommended). This runs the Ollama backend as a server, usually on `http://localhost:11434`. Note that it doesn't load a model until it's called through the API (later in this section).

2. Run the `ollama run gpt-oss:20b` command similar to earlier, but keep it open and don't exit the session via `/bye`. As discussed earlier, this opens a minimal convenience wrapper around a local Ollama server. Behind the scenes, it uses the same server API as `ollama serve`.

3. Ollama desktop app. Opening the desktop app runs the same backend automatically and provides a graphical interface on top of it as shown in the earlier figure F.6.

#### **OLLAMA SERVER IP ADDRESS**

Ollama runs locally on our machine by starting a local server-like process. When running `ollama serve` in the terminal, as described above, you may encounter an error message saying `Error: listen tcp 127.0.0.1:11434: bind: address already in use`.

If that's the case, try use the command `OLLAMA_HOST=127.0.0.1:11435 ollama serve` (and if this address is also in use, try to increment the numbers by one until you find an address not in use.)

The following code verifies that the Ollama session is running properly before we use Ollama to evaluate the test set responses generated in the previous section:

**Listing F.5 Checking Ollama is running**

```
import psutil

def check_if_running(process_name):
    running = False
    for proc in psutil.process_iter([["name"]]):
        if process_name in proc.info["name"]:
            running = True
            break
    return running

ollama_running = check_if_running("ollama")

if not ollama_running:
    raise RuntimeError(
        "Ollama not running."
        "Launch ollama before proceeding."
    )
print("Ollama running:", check_if_running("ollama"))
```

Ensure that the output from executing the previous code displays `Ollama running: True`. If it shows `False`, please verify that the `ollama serve` command or the Ollama application is actively running.

In the remainder of this appendix, we will interact with the local gpt-oss model, running on our machine, through the Ollama REST API using Python. The following `query_model` function demonstrates how to use the API:

**Listing F.6 Querying a local Ollama model**

```

import json
import urllib.request

def query_model(
    prompt,
    model="gpt-oss:20b",
    url="http://localhost:11434/api/chat" #A
):
    data = { #B
        "model": model,
        "messages": [
            {"role": "user", "content": prompt}
        ],
        "options": { #C
            "seed": 123,
            "temperature": 0,
            "num_ctx": 2048
        }
    }

    payload = json.dumps(data).encode("utf-8") #D

    request = urllib.request.Request( #E
        url,
        data=payload,
        method="POST"
    )
    request.add_header("Content-Type", "application/json")

    response_data = ""
    with urllib.request.urlopen(request) as response: #F
        while True: #G
            line = response.readline().decode("utf-8")
            if not line:
                break
            response_json = json.loads(line) #H
            response_data += response_json["message"]["content"]

    return response_data

```

```
#A If you used OLLAMA_HOST=127.0.0.1:11435 ollama serve, update the address
#B Create the data payload as a dictionary
#C Settings required for deterministic responses
#D Convert the dictionary to JSON and encode it to bytes
#E Create a POST request and add headers
#F Send the request and capture the streaming response
#G Read and decode each line
#H Parse each line into JSON
```

Here's an example of how to use the `query_model` function from listing F.6 that we just implemented:

```
ollama_model = "gpt-oss:20b"
result = query_model("What is 1+2?", ollama_model)
print(result)
```

The resulting response is "3". (It differs from what we'd get if we ran Ollama run or the Ollama application due to different default settings.)

Using the `query_model` function, we can evaluate the responses generated by our model with a prompt that includes a grading rubric asking the gpt-oss model to rate our target model's responses on a scale from 1 to 5 based on a correct answer as a reference.

The prompt we use for this is shown in listing F7:

**Listing F.7 Setting up the prompt template including grading rubric**

```
def rubric_prompt(instruction, reference_answer, model_answer):
    rubric = (
        "You are a fair judge assistant. You will be given an instruction, "
        "a reference answer, and a candidate answer to evaluate, according "
        "to the following rubric:\n\n"
        "1: The response fails to address the instruction, providing "
        "irrelevant, incorrect, or excessively verbose content.\n"
        "2: The response partially addresses the instruction but contains "
        "major errors, omissions, or irrelevant details.\n"
        "3: The response addresses the instruction to some degree but is "
        "incomplete, partially correct, or unclear in places.\n"
        "4: The response mostly adheres to the instruction, with only minor "
        "errors, omissions, or lack of clarity.\n"
        "5: The response fully adheres to the instruction, providing a "
        "clear, accurate, and relevant answer in a concise and efficient "
        "manner.\n\n"
        "Now here is the instruction, the reference answer, and the "
        "response.\n"
    )

    prompt = (
        f"{rubric}\n"
        f"Instruction:{instruction}\n\n"
        f"Reference Answer:{reference_answer}\n\n"
        f"Answer:{model_answer}\n\n"
        f"Evaluation: "
    )
    return prompt
```

The `model_answer` in the `rubric_prompt` is intended to represent the response produced by our own model in practice. For illustration purposes, we hardcode a plausible model answer here rather than generating it dynamically. (However, feel free to use the Qwen3 model we loaded in section F.2.1 to generate a real `model_answer`).

Next, let's generate the rendered prompt for the Ollama model:

```
rendered_prompt = rubric_prompt(  
    instruction=  
        "If all birds can fly, and a penguin is a bird, "  
        "can a penguin fly?"  
,  
    reference_answer=  
        "Yes, according to the premise that all birds can fly, "  
        "a penguin can fly."  
,  
    model_answer=  
        "Yes - under those premises a penguin would be able to fly."  
)  
print(rendered_prompt)
```

The output is as follows:

You are a fair judge assistant. You will be given an instruction, a reference answer, and a candidate answer to evaluate, according to the following rubric:

- 1: The response fails to address the instruction, providing irrelevant, incorrect, or excessively verbose content.
- 2: The response partially addresses the instruction but contains major errors, omissions, or irrelevant details.
- 3: The response addresses the instruction to some degree but is incomplete, partially correct, or unclear in places.
- 4: The response mostly adheres to the instruction, with only minor errors, omissions, or lack of clarity.
- 5: The response fully adheres to the instruction, providing a clear, accurate, and relevant answer in a concise and efficient manner.

Now here is the instruction, the reference answer, and the response.

**Instruction:**

If all birds can fly, and a penguin is a bird, can a penguin fly?

**Reference Answer:**

Yes, according to the premise that all birds can fly, a penguin can fly.

**Answer:**

Yes - under those premises a penguin would be able to fly.

**Evaluation:**

Ending the prompt in "Evaluation: " incentivizes the model to generate the answer. Let's see how the gpt-oss:20b model judges the response:

```
result = query_model(rendered_prompt, ollama_model)
print(result)
```

The response is as follows:

**\*\*Score: 5\*\***

The candidate answer directly addresses the question, correctly applies the given premises, and concisely states that a penguin would be able to fly. It is accurate, relevant, and clear.

As we can see, the answer receives the highest score, which is reasonable, as it is indeed correct. While this was a simple example stepping through the process manually, we could take this idea further and implement a for-loop that iteratively queries the model (for example, the Qwen3 model from chapter 2 that we loaded in section F.2.1) with questions from an evaluation dataset and evaluate it via gpt-oss and calculate the average score. Then, doing this for two models (for example, the Qwen3 base and reasoning model), we can compare the models relative to each other.

### SCORING INTERMEDIATE REASONING STEPS WITH PROCESS REWARD MODELS

Related to symbolic verifiers and LLM judges, there is a class of learned models called *process reward models* (PRMs). Like judges, PRMs can evaluate reasoning traces beyond just the final answer, but unlike general judges, they focus specifically on the intermediate steps of reasoning. And unlike verifiers, which check correctness symbolically and usually only at the outcome level, PRMs provide step-by-step reward signals during training in reinforcement learning. We can categorize PRMs as "step-level judges," which are predominantly developed for training, not pure evaluation. (In practice, PRMs are difficult to train reliably at scale. For example, DeepSeek R1 did not adopt PRMs and instead combined verifiers for the reasoning training.)

Judge-based evaluations offer advantages over preference-based leaderboards, including scalability and consistency, as they do not rely on large pools of human voters. (Technically, it is possible to outsource the preference-based rating behind leaderboards to LLM judges as well). However, LLM judges also share similar weaknesses with human voters: results can be biased by model preferences, prompt design, and answer style. Also, there is a strong dependency on the choice of judge model and rubric, and they lack the reproducibility of fixed benchmarks.