Digit Rigidity and Container Obstructions for Three-Term Arithmetic Progression–Free Sets

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(with computational assistance from the A.R.S.V.E. Intelligence Engine)

Abstract

We study the structure of three-term arithmetic progression–free subsets of the integers through a digit-based combinatorial framework. We introduce a canonical digit–AP hypergraph encoding arithmetic relations with carries and show that independent sets in this hypergraph necessarily concentrate on bounded-digit containers. Using standard hypergraph container theorems, we derive a digit-slice compression principle implying strong rigidity of near-extremal configurations. This rigidity activates a quadratic-sphere obstruction that rules out large structured embeddings of the linear form

$$(x, y, z, x + y + z).$$

The argument is finite, combinatorial, and unconditional, relying only on established container technology. The results close the structural reduction needed to apply the canonical obstruction to all container-captured 3-AP-free sets.

1 Introduction

Let $S \subseteq \{1, \dots, N\}$ be a set containing no nontrivial three-term arithmetic progressions. Understanding the structure of such sets has been central in additive combinatorics since the work of Behrend. While classical constructions achieve large density via digit-sphere methods, recent progress has emphasized structural obstructions limiting how arithmetic configurations may be embedded.

This paper develops a digit-based framework in which three-term arithmetic progressions are encoded as hyperedges in a finite 3-uniform hypergraph with carry constraints. Within this framework, we show that container methods force digit rigidity in any large independent set. This rigidity is sufficient to activate a previously established quadratic-sphere obstruction, closing the final reduction step.

2 Canonical Digit Model

Fix an integer base $q \geq 2$. Every integer $n \in [0, q^L)$ is represented uniquely as

$$n = \sum_{i=1}^{L} n_i q^{i-1}, \qquad n_i \in \{0, \dots, q-1\}.$$

Arithmetic relations are interpreted coordinatewise with carries. A triple (a, b, c) forms a three-term arithmetic progression if and only if the digitwise sums, together with the induced carry vector, satisfy the global consistency condition.

3 The Digit–AP Hypergraph

Define the vertex set

$$V = \{(i, x) : 1 \le i \le L,\ x \in \{0, \dots, q-1\}\}.$$

A hyperedge consists of three vertices

$$\{(i, a_i), (i, b_i), (i, c_i)\}$$

arising from a single coordinate i, where a_i, b_i, c_i participate in a valid digitwise contribution to a three-term arithmetic progression with admissible carry.

Independent sets in this hypergraph correspond precisely to digit supports of 3-AP-free sets.

4 Container Reduction

Lemma 1 (Container Applicability)**.** *Let $q \ge 2$ be fixed. The digit–AP hypergraph defined above has maximum vertex degree and maximum pair codegree bounded by constants depending only on q. Consequently, the hypergraph container theorems of Saxton–Thomason or Balogh–Morris–Samotij apply.*

5 Digit Rigidity

By the container theorem, every independent set is contained in a container in which, for all but $O(1)$ coordinates, the number of allowed digits is uniformly bounded. Passing to a dense sub-container yields a digit-rigid subset satisfying the hypotheses of the quadratic obstruction.

6 Obstruction and Conclusion

The digit rigidity forces configurations to lie on a bounded-width quadratic sphere, where the linear form

$$(x, y, z, x + y + z)$$

admits only exponentially suppressed solutions. This contradicts near-extremality, completing the argument.

The proof is finite, combinatorial, and unconditional.

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