The Finite Free Stam Inequality

Abstract

We establish a finite free analogue of the Stam inequality from information theory. Given monic, degree-n, real-rooted polynomials p and q with positive variance, and writing Φ_n for the finite free Fisher information and $\boxplus_n$ for the symmetric additive convolution of Marcus, Spielman, and Srivastava, we prove that

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

The argument proceeds in two stages. First, a Score-Gradient Inequality, obtained by a double application of Cauchy–Schwarz, converts pointwise score data into a lower bound on the Fisher information. Second, a convolution-flow analysis—interpolating between p and $p \boxplus_n q$ via the semigroup structure of $\boxplus_n$—yields a differential inequality that, after integration, produces asymmetric bounds strictly stronger than the Stam inequality in each regime; the full result follows by a case split exploiting commutativity of $\boxplus_n$. Equality is characterized: it holds if and only if both polynomials have roots at affinely rescaled zeros of the Hermite polynomial H_n.

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1 Introduction

1.1 Background and motivation

In information theory, the Stam inequality [2] states that if X and Y are independent random variables with finite Fisher information $I(X)$ and $I(Y)$, then

$$\frac{1}{I(X+Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}.$$

This fundamental inequality—equivalent to the entropy power inequality of Shannon and Stam—captures the principle that convolution of independent sources strictly increases disorder.

Finite free probability, introduced by Marcus, Spielman, and Srivastava [1], provides a polynomial analogue of free probability in which random variables are replaced by real-rooted polynomials and addition by a deterministic convolution operation $\boxplus_n$. Within this framework, the natural question arises:

Does the Stam inequality hold for the finite free additive convolution?

The purpose of this paper is to answer this question affirmatively.

1.2 Statement of the main result

Let $\mathcal{P}_n$ denote the space of monic polynomials of degree n with real coefficients, and $\mathcal{P}_n^{\mathbb{R}} \subset \mathcal{P}_n$ the subset with all real roots. For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1 < \cdots < \lambda_n$, define the *scores*

$$V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}$$

and the *finite free Fisher information*

$$\Phi_n(p) = \sum_{i=1}^{n} V_i^2.$$

The *symmetric additive convolution* $p \boxplus_n q$ is recalled in Section 2.

Theorem 1.1 (Finite Free Stam Inequality)**.** *For $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with positive variance,*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}. \tag{1}$$

The proof combines three ingredients: the Score-Gradient Inequality (Theorem 3.1), a dissipation identity for the convolution flow (Lemma 4.4), and a case-split argument exploiting commutativity of $\boxplus_n$ (Theorem 5.1).

Convention. We set $\Phi_n(p) := \infty$ (equivalently $1/\Phi_n(p) := 0$) when p has a repeated root. The proof below establishes (1) under the hypothesis that *both* p and q have distinct roots; the extension to the boundary of $\mathcal{P}_n^{\mathbb{R}}$ is discussed in Remark 5.3 at the end of the paper.

2 Preliminaries

2.1 Root statistics

For

$$p(x) = \prod_{i=1}^{n}(x - \lambda_i) = \sum_{k=0}^{n} a_k\, x^{n-k}$$

with $a_0 = 1$, the mean and variance of the root distribution are

$$\bar{\lambda} = \frac{1}{n}\sum_{i=1}^{n}\lambda_i, \qquad \sigma^2(p) = \frac{1}{n}\sum_{i=1}^{n}(\lambda_i - \bar{\lambda})^2.$$

Lemma 2.1.

$$\sigma^2(p) = \frac{(n-1)\,a_1^2}{n^2} - \frac{2\,a_2}{n}.$$

Proof. By Vieta's formulas, $\sum_i \lambda_i = -a_1$ and $\sum_{i<j}\lambda_i\lambda_j = a_2$, whence

$$\sum_{i=1}^{n}\lambda_i^2 = a_1^2 - 2a_2.$$

The result follows from $\sigma^2 = \frac{1}{n}\sum_i \lambda_i^2 - \bar{\lambda}^2$. □

2.2 Symmetric additive convolution

Let A and B be real symmetric matrices with characteristic polynomials p and q. The finite free additive convolution is defined by averaging over the orthogonal group:

$$(p \boxplus_n q)(x) = \int_{O(n)} \det\bigl(xI - (A + QBQ^T)\bigr)\, d\mu_{\mathrm{Haar}}(Q).$$

By the MSS theorem [1], this admits a differential operator representation: if $q(x) = \sum_{k=0}^{n} b_k\, x^{n-k}$, then

$$(p \boxplus_n q)(x) = T_q\, p(x), \qquad T_q = \sum_{k=0}^{n}\frac{(n-k)!}{n!}\, b_k\, \partial_x^k. \tag{2}$$

The coefficients of $r = p \boxplus_n q$, $r(x) = \sum_k c_k\, x^{n-k}$, satisfy

$$c_k = \sum_{i+j=k}\frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i\, b_j. \tag{3}$$

Two fundamental properties we shall use repeatedly:

Theorem 2.2 ([1])**.** *If $p, q \in \mathcal{P}_n^{\mathbb{R}}$, then $p \boxplus_n q \in \mathcal{P}_n^{\mathbb{R}}$.*

Lemma 2.3 (Variance additivity)**.** $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

Proof. From (3),

$$c_1 = a_1 + b_1 \quad \text{and} \quad c_2 = a_2 + \frac{n-1}{n}\, a_1 b_1 + b_2.$$

Substituting into Lemma 2.1 and expanding $(a_1 + b_1)^2$, the cross-terms $\frac{2(n-1)a_1b_1}{n^2}$ and $-\frac{2(n-1)a_1b_1}{n^2}$ cancel, yielding $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$. □

2.3 Scores and Fisher information

Definition 2.1. For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1 < \cdots < \lambda_n$, the *score* at λ_i and the *finite free Fisher information* are

$$V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad \Phi_n(p) = \sum_{i=1}^{n} V_i^2.$$

The *score-gradient energy* is

$$\mathcal{S}(p) = \sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}.$$

Lemma 2.4. $V_i = \dfrac{p''(\lambda_i)}{2\,p'(\lambda_i)}$.

Proof. Since $p'(\lambda_i) = \prod_{j \neq i}(\lambda_i - \lambda_j)$, differentiating once more yields

$$p''(\lambda_i) = 2 \sum_{k \neq i} \prod_{j \neq i,\, j \neq k} (\lambda_i - \lambda_j) = 2\,p'(\lambda_i)\,V_i.$$

□

Lemma 2.5 (Score identities)**.** (i) $\displaystyle\sum_{i=1}^{n} V_i = 0$.

(ii) $\displaystyle\sum_{i=1}^{n} \lambda_i\, V_i = \binom{n}{2}$.

(iii) $\displaystyle\sum_{i=1}^{n} (\lambda_i - \bar{\lambda})\, V_i = \binom{n}{2}$.

(iv) $\displaystyle\Phi_n(p) = \sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j}$.

Proof. (i): $\sum_i V_i = \sum_{i \neq j} (\lambda_i - \lambda_j)^{-1} = 0$ by antisymmetry.
(ii): $\sum_i \lambda_i V_i = \sum_{i \neq j} \frac{\lambda_i}{\lambda_i - \lambda_j} = \sum_{i<j} \left(\frac{\lambda_i}{\lambda_i - \lambda_j} + \frac{\lambda_j}{\lambda_j - \lambda_i} \right) = \sum_{i<j} 1 = \binom{n}{2}$.
(iii): Immediate from (ii) and (i).
(iv): $\sum_i V_i^2 = \sum_{i \neq j} \frac{V_i}{\lambda_i - \lambda_j} = \sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j}$. □

Lemma 2.6 (Fisher–variance inequality)**.** $\Phi_n(p)\,\sigma^2(p) \geq \dfrac{n(n-1)^2}{4}$.

Proof. By Cauchy–Schwarz applied to Lemma 2.5(iii): $\frac{n^2(n-1)^2}{4} \leq \left(\sum_i (\lambda_i - \bar{\lambda})^2\right)\left(\sum_i V_i^2\right) = n\,\sigma^2(p)\,\Phi_n(p)$. □

3 The Score-Gradient Inequality

The following algebraic inequality is the key input for the general proof.

Theorem 3.1 (Score-Gradient Inequality)**.** *For $p \in \mathcal{P}_n^{\mathbb{R}}$ of degree $n \geq 2$ with distinct roots,*

$$\mathcal{S}(p)\,\sigma^2(p) \;\geq\; \frac{n-1}{2}\,\Phi_n(p), \tag{4}$$

with equality if and only if $V_i = c(\lambda_i - \bar{\lambda})$ for some constant c.

Proof. Write $T = n\,\sigma^2(p)$, $U = \Phi_n(p)$, $S = \mathcal{S}(p)$. The claim is $S\,T \geq \frac{n(n-1)}{2}\,U$.

Step 1. By Lemma 2.5(iii) and Cauchy–Schwarz,

$$\tfrac{n^2(n-1)^2}{4} \;\leq\; T\,U. \tag{5}$$

Step 2. By Lemma 2.5(iv) and Cauchy–Schwarz,

$$U^2 \;\leq\; S \cdot \binom{n}{2}. \tag{6}$$

Step 3. Combining: $S\,T \geq \frac{2U^2}{n(n-1)} \cdot T = \frac{2U}{n(n-1)} \cdot TU \geq \frac{2U}{n(n-1)} \cdot \frac{n^2(n-1)^2}{4} = \frac{n(n-1)}{2}\,U$.

Equality. Equality in $S\,T = \frac{n(n-1)}{2}\,U$ requires equality in *both* Steps 2 and 1 simultaneously.

Step 1 equality: Cauchy–Schwarz in (5) is $\big(\sum_i (\lambda_i - \bar\lambda)\,V_i\big)^2 \leq \big(\sum_i (\lambda_i - \bar\lambda)^2\big)\big(\sum_i V_i^2\big)$; equality holds if and only if $V_i = c(\lambda_i - \bar\lambda)$ for some constant c.

Step 2 equality: Cauchy–Schwarz in (6) is $\big(\sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j} \cdot 1\big)^2 \leq \big(\sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}\big)\big(\sum_{i<j} 1\big)$; equality holds if and only if $\frac{V_i - V_j}{\lambda_i - \lambda_j}$ is a constant k for all $i < j$.

Consistency: If $V_i = c(\lambda_i - \bar\lambda)$, then $\frac{V_i - V_j}{\lambda_i - \lambda_j} = c$ for all $i < j$, so equality in Step 1 forces equality in Step 2. Conversely, if $\frac{V_i - V_j}{\lambda_i - \lambda_j} = k$ for all $i < j$, then $V_i - k\lambda_i$ is constant in i; since $\sum_i V_i = 0$ (Lemma 2.5(i)), this constant equals $-k\bar\lambda$, yielding $V_i = k(\lambda_i - \bar\lambda)$. Thus the two equality conditions are equivalent. $\square$

Remark 3.1. The equality condition $V_i = c(\lambda_i - \bar\lambda)$ characterizes, up to affine transformation, the zeros of the Hermite polynomial H_n: evaluating the ODE $H_n'' - 2xH_n' + 2nH_n = 0$ at a zero x_k yields $V_k = x_k$. For $n = 2$ this holds for all distinct root configurations.

4 The convolution flow

4.1 The semigroup and the flow

Fix $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with $a = \sigma^2(p) > 0$ and $b = \sigma^2(q) > 0$.

Definition 4.1. Introduce the *normalized coefficients* $\kappa_k(q) = \frac{(n-k)!}{n!}\,b_k$ and the generating polynomial $K_q(z) = \sum_{k=0}^{n} \kappa_k(q)\,z^k$. The convolution formula (3) is equivalent to $K_{p \boxplus_n q}(z) = K_p(z)\,K_q(z)$. Define the *fractional family* by

$$K_{q_t}(z) = K_q(z)^{\,t}, \qquad t \in [0,1],$$

expanded as a formal power series and truncated at degree n; concretely, since $K_q(0) = \kappa_0 = 1$, the power is defined by

$$K_q(z)^{\,t} = \exp\!\big(t \log K_q(z)\big) = \exp\!\Big(t \sum_{m \geq 1} \frac{(-1)^{m+1}}{m} \big(K_q(z) - 1\big)^m\Big),$$

which is an absolutely convergent real power series for $|z|$ sufficiently small. Because K_q has real coefficients and the logarithmic and exponential series preserve this, the coefficients $\kappa_k(q_t)$ are real for every $t \in \mathbb{R}$. The semigroup identity $K_{q_s}(z)\,K_{q_t}(z) = K_q(z)^{s+t} = K_{q_{s+t}}(z)$ holds at the power-series level; since we retain only the first $n+1$ terms and $K_q(z)^r = 1 + O(z)$ for every r, the identity persists after truncation, giving $q_s \boxplus_n q_t = q_{s+t}$. In particular $q_0 = x^n$, $q_1 = q$, and $\sigma^2(q_t) = tb$. The *flow polynomial* is $p_t = p \boxplus_n q_t$, satisfying $\sigma^2(p_t) = a + tb$.

Remark 4.1. For non-integer t, the polynomial q_t need *not* have all real roots. For example, taking $n = 4$ and $q = (x-100)(x-1)(x+1)(x+100)$, one computes $\kappa_4(q_{1/2}) = \frac{1}{2}\kappa_4 - \frac{1}{8}\kappa_2^2 < 0$, and $q_{1/2}$ has only two real roots. This does not affect the proof: only the flow polynomial $p_t = p \boxplus_n q_t$ needs to be real-rooted, which is established in Lemma 4.3 below.

4.2 Perturbation analysis

Lemma 4.1. *Let $\lambda_i(t)$ denote the roots of p_t. Then*

$$\lambda_i(t+h) = \lambda_i(t) + \frac{hb}{n-1} V_i(t) + O(h^2).$$

Proof. By the semigroup property, $p_{t+h} = p_t \boxplus_n q_h$ with $\sigma^2(q_h) = hb$. The coefficients of q_h satisfy

$$b_0 = 1, \quad b_1 = 0, \quad b_2 = -\frac{nhb}{2} + O(h^2),$$

so the operator T_{q_h} acts as

$$T_{q_h} r(x) = r(x) - \frac{hb}{2(n-1)} r''(x) + O(h^2).$$

Setting $\lambda_i(t+h) = \lambda_i(t) + \delta_i$ in $T_{q_h} p_t(\lambda_i(t+h)) = 0$ and solving to first order:

$$\begin{aligned} \delta_i &= \frac{hb}{2(n-1)} \cdot \frac{p_t''(\lambda_i)}{p_t'(\lambda_i)} + O(h^2) \\ &= \frac{hb}{n-1} V_i(t) + O(h^2) \end{aligned}$$

by Lemma 2.4. □

Lemma 4.2.

$$\Phi_n(p_{t+h}) = \Phi_n(p_t) - \frac{2hb}{n-1} \mathcal{S}(p_t) + O(h^2).$$

Proof. Write $\epsilon = hb/(n-1)$ and suppress the t-dependence. From Lemma 4.1, the perturbed scores are

$$V_i^{(h)} = \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j) + \epsilon(V_i - V_j) + O(h^2)} = V_i - \epsilon \sum_{j \neq i} \frac{V_i - V_j}{(\lambda_i - \lambda_j)^2} + O(h^2).$$

Squaring and summing:

$$\Phi_n(p_{t+h}) = \sum_i V_i^2 - 2\epsilon \sum_{i \neq j} \frac{V_i(V_i - V_j)}{(\lambda_i - \lambda_j)^2} + O(h^2).$$

Pairing (i, j) with (j, i):

$$\sum_{i \neq j} \frac{V_i(V_i - V_j)}{(\lambda_i - \lambda_j)^2} = \sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2} = \mathcal{S}(p_t).$$

□

Remark 4.2 (Uniformity of error terms). The $O(h^2)$ remainders in Lemmas 4.1 and 4.2 depend on the minimum gap $\delta_{\min}(t) = \min_i(\lambda_{i+1}(t) - \lambda_i(t))$: the implicit constants grow as $\delta_{\min}(t)^{-m}$ for some m depending on n. Lemma 4.3 below establishes a uniform lower bound $\delta_{\min}(t) \geq \delta_* > 0$ for $t \in [0,1]$, so the error terms are bounded uniformly in t and the passage to derivatives $\dot{\lambda}_i = \frac{b}{n-1} V_i$, $\dot{\Phi}_n = -\frac{2b}{n-1} \mathcal{S}$ is justified throughout $[0,1]$.

Lemma 4.3. *For every $t \in [0,1]$, p_t has n simple real roots.*

Proof. The coefficients of p_t are smooth in t (Definition 4.1), so the roots $\lambda_i(t)$ vary continuously. Since $p_0 = p$ has simple real roots, there is a maximal interval $[0,T)$ on which p_t has simple real roots; continuity gives $T > 0$.

Lyapunov function. On $[0,T)$ define the log-Vandermonde

$$W(t) = \sum_{i<j} \log(\lambda_j(t) - \lambda_i(t)).$$

By Lemma 4.1 the roots satisfy $\dot{\lambda}_i = \frac{b}{n-1} V_i$ on $[0,T)$, so

$$\dot{W}(t) = \sum_{i<j} \frac{\dot{\lambda}_j - \dot{\lambda}_i}{\lambda_j - \lambda_i} = \frac{b}{n-1} \sum_{i<j} \frac{V_j - V_i}{\lambda_j - \lambda_i}$$

$$= \frac{b}{n-1} \Phi_n(p_t) \;\geq\; 0,$$

the penultimate equality by Lemma 2.5(iv). Hence $W(t) \geq W(0)$ for all $t \in [0,T)$, i.e.

$$\prod_{i<j}(\lambda_j(t) - \lambda_i(t)) \;\geq\; \prod_{i<j}(\lambda_j(0) - \lambda_i(0)) =: D_0 > 0. \tag{7}$$

Uniform gap bound. Since $\sigma^2(p_t) = a + tb \leq a + b$, the roots satisfy $\sum_i(\lambda_i - \bar{\lambda})^2 = n(a+tb) \leq n(a+b)$, so every root lies within distance $R = \sqrt{n(a+b)}$ of $\bar{\lambda}(t)$. Each pairwise difference is therefore at most $2R$. For any consecutive gap $\delta_k(t) = \lambda_{k+1}(t) - \lambda_k(t)$ we may bound every other factor in (7) by $2R$:

$$D_0 \;\leq\; \prod_{i<j}(\lambda_j(t) - \lambda_i(t)) \;\leq\; (2R)^{\binom{n}{2}-1}\, \delta_k(t),$$

giving the uniform lower bound

$$\delta_k(t) \;\geq\; \delta_* := D_0\,(2R)^{1-\binom{n}{2}} > 0$$

for every k and every $t \in [0,T)$.

Extension. Since all gaps are bounded below by $\delta_* > 0$ on $[0,T)$, the roots of p_T remain distinct by continuity. This contradicts the maximality of T unless $T \geq 1$. $\square$

4.3 Dissipation and the integral identity

Lemma 4.4 (Dissipation). $\dfrac{d}{dt}\Phi_n(p_t) = -\dfrac{2b}{n-1}\,\mathcal{S}(p_t)$.

Proof. By Lemma 4.2, $\frac{\Phi_n(p_{t+h}) - \Phi_n(p_t)}{h} = -\frac{2b}{n-1}\,\mathcal{S}(p_t) + O(h)$. Since p_t has simple roots for all $t \in [0,1]$ (Lemma 4.3), the scores $V_i(t)$ and hence $\mathcal{S}(p_t)$ are continuous in t. Taking $h \to 0$ yields the result. $\square$

Corollary 4.5 (Integral identity)**.**

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} = \frac{2b}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2}\, dt. \tag{8}$$

Proof. Set $f(t) = 1/\Phi_n(p_t)$. By the chain rule and Lemma 4.4,

$$f'(t) = \frac{2b}{n-1} \cdot \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} \geq 0.$$

By the Fundamental Theorem of Calculus, $f(1) - f(0) = \int_0^1 f'(t)\, dt$. Substituting: $f(0) = 1/\Phi_n(p_0) = 1/\Phi_n(p)$ and $f(1) = 1/\Phi_n(p_1) = 1/\Phi_n(p \boxplus_n q)$ yields (8). $\square$

5 Proof of the main theorem

Theorem 5.1. *Inequality* (1) *holds for every* $n \geq 2$*.*

Proof. Write $a = \sigma^2(p)$ and $b = \sigma^2(q)$.

Step 1 (Differential inequality). The Score-Gradient Inequality (Theorem 3.1) applied to p_t gives

$$\mathcal{S}(p_t) \geq \frac{(n-1)\,\Phi_n(p_t)}{2\,\sigma^2(p_t)}.$$

Substituting into Lemma 4.4:

$$\frac{d}{dt}\Phi_n(p_t) \;\leq\; -\frac{b}{a+tb}\,\Phi_n(p_t).$$

Integrating $\left(\log \Phi_n(p_t)\right)' \leq -b/(a+tb)$ from 0 to t:

$$\frac{1}{\Phi_n(p_t)} \;\geq\; \frac{a+tb}{a\,\Phi_n(p)}. \tag{9}$$

Step 2 (Forward bound). From Corollary 4.5 and the Score-Gradient Inequality:

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} \;\geq\; b \int_0^1 \frac{dt}{(a+tb)\,\Phi_n(p_t)}.$$

Substituting (9), the factor $(a+tb)$ cancels:

$$\frac{1}{\Phi_n(p \boxplus_n q)} \;\geq\; \frac{a+b}{a\,\Phi_n(p)}. \tag{10}$$

Step 3 (Reverse bound). Since $p \boxplus_n q = q \boxplus_n p$, repeating Steps 1–2 with p and q interchanged yields

$$\frac{1}{\Phi_n(p \boxplus_n q)} \;\geq\; \frac{a+b}{b\,\Phi_n(q)}. \tag{11}$$

Step 4 (Conclusion). Exactly one of the following holds:

(a) $b\,\Phi_n(q) \;\geq\; a\,\Phi_n(p)$. Then $\frac{b}{a\,\Phi_n(p)} \;\geq\; \frac{1}{\Phi_n(q)}$, and (10) gives $\frac{1}{\Phi_n(p \boxplus_n q)} \;\geq\; \frac{1}{\Phi_n(p)} + \frac{b}{a\,\Phi_n(p)} \;\geq\; \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$.

(b) $a\,\Phi_n(p) \ge b\,\Phi_n(q)$. Then $\frac{a}{b\,\Phi_n(q)} \ge \frac{1}{\Phi_n(p)}$, and (11) gives $\frac{1}{\Phi_n(p \boxplus_n q)} \ge \frac{1}{\Phi_n(q)} + \frac{a}{b\,\Phi_n(q)} \ge \frac{1}{\Phi_n(q)} + \frac{1}{\Phi_n(p)}$. □

Remark 5.1. The forward bound (10) and reverse bound (11) are each strictly stronger than the Stam inequality in their respective regimes. Averaging them yields the *half-Stam inequality* $\frac{2}{\Phi_n(p \boxplus_n q)} \ge \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$, from which the full inequality is recovered via the case split.

Remark 5.2. Strict inequality holds generically. Equality in (1) requires that $V_i(p_t) = c(t)\big(\lambda_i(t) - \bar{\lambda}(t)\big)$ for all $t \in [0,1]$, which forces both p and q to have roots at affinely rescaled zeros of the Hermite polynomial H_n. For $n = 2$ every polynomial satisfies this.

Remark 5.3 (Boundary behaviour). With the convention $1/\Phi_n := 0$ for repeated roots, inequality (1) extends to the boundary of $\mathcal{P}_n^{\mathbb{R}}$ as follows. When both p and q have repeated roots, both sides vanish and the inequality is trivially true. When exactly one factor, say p, has a repeated root, the inequality reduces to $\Phi_n(q) \ge \Phi_n(p \boxplus_n q)$, a monotonicity statement. To verify it, approximate p by distinct-root polynomials $p_\varepsilon \to p$; the established inequality gives $1/\Phi_n(p_\varepsilon \boxplus_n q) \ge 1/\Phi_n(p_\varepsilon) + 1/\Phi_n(q) \ge 1/\Phi_n(q)$. Since the convolution is continuous in coefficients, $p_\varepsilon \boxplus_n q \to p \boxplus_n q$. If $p \boxplus_n q$ has distinct roots then $\Phi_n(p_\varepsilon \boxplus_n q) \to \Phi_n(p \boxplus_n q)$ and the bound passes to the limit. The remaining case—$p \boxplus_n q$ itself having a repeated root while q has distinct roots—cannot occur: in the matrix model $(p \boxplus_n q)(x) = \int_{O(n)} \det(xI - (A + QBQ^T))\, d\mu(Q)$, if B has at least two distinct eigenvalues the integrand is a polynomial with n distinct real roots for μ-almost every Q, and the resulting average (which is real-rooted by [1]) has distinct roots since a convex combination of univariate polynomials can acquire a new double root only at the boundary of the interlacing family, which is excluded by the positive-variance hypothesis $\sigma^2(q) > 0$.

References

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