FINITE FREE STAM INEQUALITY VIA SCORE-GRADIENT BOUNDS AND DILATION INTERPOLATION: RIGOROUS PARTIAL RESULTS, AND A COUNTEREXAMPLE TO A CONVEXITY HEURISTIC

Abstract. We study the finite free analogue of Stam's inequality for the symmetric additive convolution $\boxplus_n$ of Marcus–Spielman–Srivastava. For monic degree-n real-rooted polynomials p, q with positive variance, the conjectured inequality is

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)},$$

where Φ_n is the finite free Fisher information (defined in terms of "scores" at the roots). This note is largely self-contained: all definitions are recalled from scratch, and the only external input is the Marcus–Spielman–Srivastava theorem on real-rootedness preservation under $\boxplus_n$ (stated without proof). We give complete proofs of: (i) the Score-Gradient Inequality (a double Cauchy–Schwarz estimate), (ii) a Hermite semigroup bound, and (iii) the Stam inequality in low degrees $n = 2$ (equality) and $n = 3$. For general n we present a real-rooted interpolation (the *dilation path*). We record an explicit numerical example showing that a natural global convexity heuristic for $t \mapsto 1/\Phi_n(p \boxplus_n q_t)$ (and for the associated "dilation excess") fails. Thus any dilation-based proof of the full Stam inequality must use a different monotonicity/comparison principle. From a variational/transport perspective, we prove a finite-free de Bruijn identity (linking the log-Vandermonde to Φ_n along the Hermite flow), an AM-GM isoperimetric inequality $\mathcal{R} \cdot \mathcal{N} \geq \binom{n}{2}$, and displacement convexity of the repulsion energy. We discover and conjecture a *discriminant power inequality* $\operatorname{disc}(p \boxplus_n q)^{1/N} \geq \operatorname{disc}(p)^{1/N} + \operatorname{disc}(q)^{1/N}$ (the finite-free analogue of Shannon's entropy power inequality), which survives 29,000+ random tests for $n = 2$ to 8 and is proved for $n \leq 3$.

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1. Setup

1.1. Real-rooted polynomials and convolution.

Definition 1.1 (Real-rooted polynomials)**.** Fix $n \geq 2$. Let $\mathcal{P}_n^{\mathbb{R}}$ denote the set of monic degree-n polynomials with all roots real. Every $p \in \mathcal{P}_n^{\mathbb{R}}$ can be written

$$p(x) = \prod_{i=1}^{n}(x - \lambda_i) = \sum_{k=0}^{n} a_k\, x^{n-k}$$

with $\lambda_1 \leq \cdots \leq \lambda_n$ (roots listed with multiplicity). We say p has *simple roots* if all inequalities are strict. In the sequel, we shall always state explicitly whenever simple roots are required.

Definition 1.2 (Symmetric additive convolution)**.** For $p(x) = \sum_{k=0}^{n} a_k x^{n-k}$ and $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$ in $\mathcal{P}_n^{\mathbb{R}}$, set $r = p \boxplus_n q$ to be the monic degree-n polynomial with coefficients

$$r(x) = \sum_{k=0}^{n} c_k x^{n-k}, \qquad c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i b_j.$$

Equivalently (MSS), writing

$$T_q := \sum_{k=0}^{n} \frac{(n-k)!}{n!}\, b_k\, \partial_x^k,$$

one has $p \boxplus_n q = T_q p$.

Theorem 1.3 (Marcus–Spielman–Srivastava)**.** *If $p, q \in \mathcal{P}_n^{\mathbb{R}}$, then $p \boxplus_n q \in \mathcal{P}_n^{\mathbb{R}}$. Moreover, $\boxplus_n$ is commutative.*

1.2. Scores, Fisher information, and variance.

Definition 1.4 (Scores and Fisher information)**.** Let $p \in \mathcal{P}_n^{\mathbb{R}}$ have distinct roots $\lambda_1 < \cdots < \lambda_n$. Define the *score* at λ_i and the *finite free Fisher information* by

$$V_i := \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad \Phi_n(p) := \sum_{i=1}^{n} V_i^2.$$

If p has a repeated root, set $\Phi_n(p) := \infty$ (equivalently $1/\Phi_n(p) := 0$).

Definition 1.5 (Score-gradient energy)**.**

$$\mathcal{S}(p) := \sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}.$$

Definition 1.6 (Variance)**.** Let $\bar{\lambda} := \frac{1}{n}\sum_{i=1}^n \lambda_i$. Define

$$\sigma^2(p) := \frac{1}{n}\sum_{i=1}^n (\lambda_i - \bar{\lambda})^2.$$

Remark 1.7 (Affine invariances). Φ_n and $\mathcal{S}$ are translation-invariant. Under dilation $p(x) \mapsto p_t(x) = t^{-n}p(tx)$ (i.e. roots scale by t), the scores scale as $V_i \mapsto V_i/t$, hence $\Phi_n \mapsto \Phi_n/t^2$.

Lemma 1.8 (Translation covariance)**.** *For $c \in \mathbb{R}$ and a monic polynomial p, write $(\tau_c p)(x) := p(x-c)$. Then for all monic degree-n polynomials p, q,*

$$\tau_a p \boxplus_n \tau_b q = \tau_{a+b}(p \boxplus_n q).$$

In particular, since scores depend only on root differences, $\Phi_n(\tau_c p) = \Phi_n(p)$ and $\sigma^2(\tau_c p) = \sigma^2(p)$.

Proof. For $p(x) = \sum_{k=0}^n a_k x^{n-k} \in \mathcal{P}_n^{\mathbb{R}}$, define the *normalized generating function*

$$K_p(z) := \sum_{k=0}^n \frac{a_k}{\binom{n}{k}} z^k,$$

so that $K_{p\boxplus_n q}(z) = K_p(z)\,K_q(z) \pmod{z^{n+1}}$ (see [1]). Translation by c replaces each root λ_i by $\lambda_i + c$. Since K_p is binomial-normalized, the shift identity $K_{\tau_c p}(z) = e^{cz}K_p(z) \pmod{z^{n+1}}$ is exactly the standard MSS binomial-convolution relation (see [1]). Therefore $K_{\tau_a p\boxplus_n \tau_b q} = e^{(a+b)z}K_pK_q = K_{\tau_{a+b}(p\boxplus_n q)} \pmod{z^{n+1}}$, and since K determines the monic polynomial, the identity follows. The invariance statements follow from the definitions. □

2. Preliminary identities

Standing hypothesis for Section 2. Throughout this section, $p \in \mathcal{P}_n^{\mathbb{R}}$ has simple (i.e. distinct) roots $\lambda_1 < \cdots < \lambda_n$ and scores V_i as in Definition 1.4.

Lemma 2.1 (Score–derivative relation)**.**

$$V_i = \frac{p''(\lambda_i)}{2\,p'(\lambda_i)}.$$

Proof. Since $p'(\lambda_i) = \prod_{j\neq i}(\lambda_i - \lambda_j)$, differentiating $p'(x) = \sum_{i=1}^n \prod_{j\neq i}(x - \lambda_j)$ and evaluating at $x = \lambda_i$ gives

$$p''(\lambda_i) = 2\,p'(\lambda_i)\sum_{k\neq i}\frac{1}{\lambda_i - \lambda_k} = 2\,p'(\lambda_i)\,V_i.$$ □

Lemma 2.2 (Score identities)**.**
(i) $\sum_i V_i = 0$.
(ii) $\sum_i \lambda_i V_i = \binom{n}{2}$.
(iii) $\sum_i (\lambda_i - \bar{\lambda})V_i = \binom{n}{2}$.
(iv) $\Phi_n(p) = \sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j}$.

Proof. (i) is antisymmetry of $(\lambda_i - \lambda_j)^{-1}$ in (i,j).
(ii) Pair (i,j) and (j,i): $\frac{\lambda_i}{\lambda_i - \lambda_j} + \frac{\lambda_j}{\lambda_j - \lambda_i} = 1$.
(iii) follows from (ii) and (i).
(iv) Write $\Phi_n(p) = \sum_i V_i^2 = \sum_i V_i \sum_{j\neq i}(\lambda_i - \lambda_j)^{-1}$. Pairing (i,j) and (j,i) gives $\frac{V_i - V_j}{\lambda_i - \lambda_j}$, and summing over $i<j$ yields the claim. □

Lemma 2.3 (Variance via coefficients)**.** *If* $p(x) = \sum_{k=0}^{n} a_k x^{n-k}$*, then*

$$\sigma^2(p) = \frac{(n-1)a_1^2}{n^2} - \frac{2a_2}{n}.$$

Proof. By Vieta, $\sum_i \lambda_i = -a_1$ and $\sum_{i<j} \lambda_i\lambda_j = a_2$. Thus $\sum_i \lambda_i^2 = a_1^2 - 2a_2$, and $\sigma^2 = \frac{1}{n}\sum_i \lambda_i^2 - \bar{\lambda}^2$ with $\bar{\lambda} = -a_1/n$. □

Lemma 2.4 (Variance additivity)**.** $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

Proof. From Definition 1.2, $c_1 = a_1 + b_1$ and $c_2 = a_2 + \frac{n-1}{n}a_1 b_1 + b_2$. Plugging into Lemma 2.3 and expanding $(a_1 + b_1)^2$ shows cross terms cancel. □

3. Fisher–variance and the Score-Gradient Inequality

Lemma 3.1 (Fisher–variance inequality)**.**

$$\Phi_n(p)\,\sigma^2(p) \geq \frac{n(n-1)^2}{4}.$$

Equality holds iff $V_i = c(\lambda_i - \bar{\lambda})$ *for some constant* c*.*

Proof. By Lemma 2.2 ((iii)), $\sum_i (\lambda_i - \bar{\lambda})V_i = \frac{n(n-1)}{2}$. Apply Cauchy–Schwarz:

$$\Big(\sum_i (\lambda_i - \bar{\lambda})V_i\Big)^2 \leq \Big(\sum_i (\lambda_i - \bar{\lambda})^2\Big)\Big(\sum_i V_i^2\Big) = n\,\sigma^2(p)\,\Phi_n(p).$$ □

Theorem 3.2 (Score-Gradient Inequality)**.** *For* $p \in \mathcal{P}_n^{\mathbb{R}}$ *with distinct roots,*

$$\mathcal{S}(p)\,\sigma^2(p) \geq \frac{n-1}{2}\,\Phi_n(p). \tag{1}$$

Equality holds iff $V_i = c(\lambda_i - \bar{\lambda})$ *for some constant* c*.*

Proof. Set $T := n\sigma^2(p)$, $U := \Phi_n(p)$, $S := \mathcal{S}(p)$. We show $ST \geq \frac{n(n-1)}{2}U$.

First, Lemma 2.2 ((iii)) and Cauchy–Schwarz give $\frac{n^2(n-1)^2}{4} \leq TU$. Second, Lemma 2.2 ((iv)) writes $U = \sum_{i<j} a_{ij}$ with $a_{ij} := (V_i - V_j)/(\lambda_i - \lambda_j)$. Applying Cauchy–Schwarz to the vector $(a_{ij})_{i<j}$ and the all-ones vector gives $U^2 \leq \binom{n}{2}\sum_{i<j} a_{ij}^2 = \frac{n(n-1)}{2}S$. Combine:

$$ST \geq \frac{2U^2}{n(n-1)}T = \frac{2U}{n(n-1)}(TU) \geq \frac{2U}{n(n-1)}\,\frac{n^2(n-1)^2}{4} = \frac{n(n-1)}{2}U.$$ □

Equality characterization. Equality in (1) requires simultaneous equality in both Cauchy–Schwarz applications. The first (Fisher–variance) gives $V_i = c(\lambda_i - \bar{\lambda})$ for a constant c. The second (score-gap) gives $\frac{V_i - V_j}{\lambda_i - \lambda_j} = \mu$ for every $i < j$ and a common value μ. Substituting $V_i = c(\lambda_i - \bar{\lambda})$ into the second condition yields $c = \mu$, which is consistent, so the equality case is exactly $V_i = c(\lambda_i - \bar{\lambda})$.

4. Low-degree Stam: $n = 2$ and $n = 3$

4.1. $n = 2$: equality and convexity along the dilation path.

Proposition 4.1 (Quadratic case)**.** *For* $n = 2$*, for all* $p, q \in \mathcal{P}_n^{\mathbb{R}}$ *with distinct roots,*

$$\frac{1}{\Phi_2(p \boxplus_2 q)} = \frac{1}{\Phi_2(p)} + \frac{1}{\Phi_2(q)}.$$

Moreover, along the dilation path $r_t = p \boxplus_2 q_t$*,* $F(t) := 1/\Phi_2(r_t)$ *is a quadratic polynomial in* t *with* $F''(t) > 0$*.*

Proof. If $p(x) = (x - \lambda_1)(x - \lambda_2)$ with $d = \lambda_2 - \lambda_1 > 0$, then $V_1 = -1/d$, $V_2 = 1/d$, hence $\Phi_2(p) = 2/d^2$ and $\sigma^2(p) = d^2/4$, so $1/\Phi_2(p) = 2\sigma^2(p)$. Variance additivity gives $1/\Phi_2(p \boxplus_2 q) = 2\sigma^2(p \boxplus_2 q) = 2\sigma^2(p) + 2\sigma^2(q)$. Along dilation, $\sigma^2(q_t) = t^2\sigma^2(q)$, hence $1/\Phi_2(r_t) = 2(\sigma^2(p) + t^2\sigma^2(q))$, a quadratic in t with positive constant second derivative. □

4.2. $n = 3$**: an explicit computation (centered cubics).** The following residue formula is valid for all degrees; we state it here because its only application in this paper is to the cubic case.

Theorem 4.2 (Critical-value formula)**.** *Let $p \in \mathcal{P}_n^{\mathbb{R}}$ have distinct roots $\lambda_1 < \cdots < \lambda_n$, and let $\zeta_1, \ldots, \zeta_{n-1}$ be the simple zeros of p'. Then*

$$\Phi_n(p) = -\frac{1}{4}\sum_{j=1}^{n-1} \frac{p''(\zeta_j)}{p(\zeta_j)}. \tag{2}$$

Proof. By Lemma 2.1, $\Phi_n(p) = \frac{1}{4}\sum_i \frac{p''(\lambda_i)^2}{p'(\lambda_i)^2}$. Consider the meromorphic function on the Riemann sphere:

$$F(x) := \frac{p''(x)^2}{p'(x)\,p(x)}.$$

Residues at the roots. Since p has a simple zero at λ_i and $p'(\lambda_i) \neq 0$,

$$\operatorname{Res}_{x=\lambda_i} F = \frac{p''(\lambda_i)^2}{p'(\lambda_i)^2}.$$

Summing over i gives $\sum_i \operatorname{Res}_{\lambda_i} F = 4\Phi_n(p)$.

Residues at the critical points. At a simple zero ζ_j of p', interlacing implies $p(\zeta_j) \neq 0$. Thus

$$\operatorname{Res}_{x=\zeta_j} F = \frac{p''(\zeta_j)^2}{p''(\zeta_j)\,p(\zeta_j)} = \frac{p''(\zeta_j)}{p(\zeta_j)}.$$

Residue at infinity. We need $\operatorname{Res}_\infty F = 0$. Equivalently, the coefficient of x^{-1} in the Laurent expansion of F at ∞ must vanish. Write $p(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots$, so that

$$\begin{aligned}
p(x) &= x^n(1 + a_1x^{-1} + a_2x^{-2} + \cdots),\\
p'(x) &= nx^{n-1}(1 + \tfrac{n-1}{n}a_1x^{-1} + \tfrac{n-2}{n}a_2x^{-2} + \cdots),\\
p''(x) &= n(n-1)x^{n-2}(1 + \tfrac{n-2}{n-1}a_1x^{-1} + \tfrac{(n-3)(n-2)}{n(n-1)}a_2x^{-2}\cdots).
\end{aligned}$$

Therefore

$$F(x) = \frac{p''(x)^2}{p'(x)\,p(x)} = \frac{n(n-1)^2}{x^3}\,\frac{\left(1 + \frac{n-2}{n-1}a_1x^{-1} + \cdots\right)^2}{\left(1 + \frac{n-1}{n}a_1x^{-1} + \cdots\right)\left(1 + a_1x^{-1} + \cdots\right)}.$$

Expanding numerator and denominator to order x^{-1}:

$$\begin{aligned}
\text{numerator:}\quad & 1 + \frac{2(n-2)}{n-1}\,a_1x^{-1} + O(x^{-2}),\\
\text{denominator:}\quad & 1 + \Big(\frac{n-1}{n} + 1\Big)a_1x^{-1} + O(x^{-2}) = 1 + \frac{2n-1}{n}\,a_1x^{-1} + O(x^{-2}).
\end{aligned}$$

Their ratio is $1 + (\frac{2(n-2)}{n-1} - \frac{2n-1}{n})a_1x^{-1} + O(x^{-2}) = 1 - \frac{3}{n(n-1)}a_1x^{-1} + O(x^{-2})$. Hence

$$F(x) = \frac{n(n-1)^2}{x^3} - \frac{3(n-1)a_1}{x^4} + O(x^{-5}).$$

Since the expansion starts at x^{-3}, no x^{-1} term is present, so $\operatorname{Res}_\infty F = 0$.

By the global residue theorem, the sum of all residues on the sphere is zero: $4\Phi_n(p) + \sum_{j=1}^{n-1} \frac{p''(\zeta_j)}{p(\zeta_j)} = 0$, proving (2). □

A centered monic cubic has the form $r(x) = x^3 - Sx + T$ with $S \geq 0$. It has three distinct real roots iff its discriminant $\Delta := 4S^3 - 27T^2$ is positive.

Proposition 4.3 (Closed form for Φ_3)**.** *For a centered cubic $r(x) = x^3 - Sx + T$ with $\Delta > 0$,*

$$\Phi_3(r) = \frac{18S^2}{\Delta}.$$

Proof. Apply Theorem 4.2. The critical points are $\zeta_\pm = \pm\alpha$ with $\alpha := \sqrt{S/3}$ and $r''(x) = 6x$. Thus

$$4\Phi_3(r) = -\frac{6\alpha}{r(\alpha)} + \frac{6\alpha}{r(-\alpha)} = 6\alpha\,\frac{r(\alpha) - r(-\alpha)}{r(\alpha)r(-\alpha)}.$$

Compute $r(\alpha) - r(-\alpha) = -\frac{4S\alpha}{3}$ and $r(\alpha)r(-\alpha) = T^2 - \frac{4S^3}{27} = -\frac{\Delta}{27}$. Substituting gives $4\Phi_3(r) = \frac{72S^2}{\Delta}$. □

Proposition 4.4 (Convolution preserves cubic shape (centered))**.** *If $p(x) = x^3 - S_1x + T_1$ and $q(x) = x^3 - S_2x + T_2$ are centered, then $(p \boxplus_3 q)(x) = x^3 - (S_1 + S_2)x + (T_1 + T_2)$.*

Proof. For $n = 3$ we have $a_0 = b_0 = 1$ (monic), $a_1 = b_1 = 0$ (centered), $a_2 = -S_1$, $a_3 = T_1$, and similarly for q. Using Definition 1.2 with $n = 3$, the convolution coefficients are

$$c_k = \sum_{i+j=k} \frac{(3-i)!\,(3-j)!}{3!\,(3-k)!}\,a_ib_j.$$

A direct substitution gives $c_0 = 1$, $c_1 = 0$, $c_2 = a_2 + b_2 = -(S_1 + S_2)$, and $c_3 = a_3 + b_3 = T_1 + T_2$. Therefore $(p \boxplus_3 q)(x) = x^3 - (S_1 + S_2)x + (T_1 + T_2)$. □

Theorem 4.5 (Stam for n=3)**.** *The finite free Stam inequality holds for $n = 3$. More precisely, after replacing p, q by their centered translates (which preserves both sides of the inequality by Lemma 1.8), equality holds if and only if $T_1 = T_2 = 0$, i.e. both polynomials are even.*

Proof. By Propositions 4.3 and 4.4, $1/\Phi_3 = \Delta/(18S^2) = 2S/9 - 3T^2/(2S^2)$. Cancelling the linear terms in S, the inequality reduces to

$$\frac{(T_1 + T_2)^2}{(S_1 + S_2)^2} \leq \frac{T_1^2}{S_1^2} + \frac{T_2^2}{S_2^2},$$

which follows from convexity of $x \mapsto x^2$.

To see the equality statement, note that three distinct real roots force $\Delta_i = 4S_i^3 - 27T_i^2 > 0$, hence $S_i > 0$; set $x_i := T_i/S_i$. Then

$$\frac{(T_1 + T_2)^2}{(S_1 + S_2)^2} = \Big(\frac{S_1}{S_1 + S_2}x_1 + \frac{S_2}{S_1 + S_2}x_2\Big)^2 \leq \frac{S_1}{S_1 + S_2}x_1^2 + \frac{S_2}{S_1 + S_2}x_2^2 \leq x_1^2 + x_2^2.$$

If equality holds at the endpoints, both inequalities are equalities. The first forces $x_1 = x_2$, and the second then forces $x_1 = x_2 = 0$. Hence $T_1 = T_2 = 0$. □

5. Hermite semigroup bound

5.1. **Hermite kernel.**

Definition 5.1 (Hermite kernel)**.** For $t \geq 0$, let $G_t \in \mathcal{P}_n^{\mathbb{R}}$ be the monic degree-n polynomial whose normalized generating function satisfies

$$K_{G_t}(z) = \exp\Big(-\frac{t}{2(n-1)}z^2\Big) \pmod{z^{n+1}}.$$

The *Hermite flow* is $p_t := p \boxplus_n G_t$.

Lemma 5.2 (Semigroup and variance)**.** *For $s, t \geq 0$:*

(i) $G_s \boxplus_n G_t = G_{s+t}$.
(ii) $\sigma^2(G_t) = t$.
(iii) $\sigma^2(p_t) = \sigma^2(p) + t$.

Proof. (i) follows from $K_{G_s} K_{G_t} = K_{G_{s+t}}$ modulo z^{n+1}. (ii) is read from the quadratic term. (iii) is Lemma 2.4. ☐

5.2. **Root ODE and dissipation.**

Lemma 5.3 (Hermite root ODE). *Suppose p_t has simple roots for all t in an open interval I. Then the root trajectories $\lambda_i(t)$ are C^∞ on I (by the implicit function theorem, since $p_t'(\lambda_i(t)) \neq 0$), and*

$$\dot{\lambda}_i = \frac{1}{n-1} V_i(t).$$

Proof. For small $h > 0$, expanding the exponential gives $K_{G_h}(z) = 1 - \frac{h}{2(n-1)} z^2 + O(h^2)$, so the MSS operator acts as $T_{G_h} f = f - \frac{h}{2(n-1)} f'' + O(h^2)$. Since $p_{t+h} = p_t \boxplus_n G_h = T_{G_h} p_t$, evaluating at $\lambda_i(t+h)$ gives $0 = p_{t+h}(\lambda_i(t+h)) = T_{G_h} p_t(\lambda_i(t+h))$. Expand to first order in h:

$$0 = p_t(\lambda_i + \dot{\lambda}_i h) - \frac{h}{2(n-1)} p_t''(\lambda_i) + O(h^2) = p_t'(\lambda_i)\, \dot{\lambda}_i\, h - \frac{h}{2(n-1)} p_t''(\lambda_i) + O(h^2),$$

where we used $p_t(\lambda_i) = 0$. Dividing by $h\, p_t'(\lambda_i)$ and applying Lemma 2.1 ($V_i = p_t''(\lambda_i)/(2p_t'(\lambda_i))$) yields $\dot{\lambda}_i = V_i/(n-1)$. ☐

Lemma 5.4 (Hermite dissipation). *Under the same simple-root hypothesis as Lemma 5.3,*

$$\frac{d}{dt} \Phi_n(p_t) = -\frac{2}{n-1}\, \mathcal{S}(p_t).$$

Proof. Differentiate $V_i(t) = \sum_{j \neq i} (\lambda_i - \lambda_j)^{-1}$ using the root ODE $\dot{\lambda}_i = V_i/(n-1)$:

$$\dot{V}_i = -\sum_{j \neq i} \frac{\dot{\lambda}_i - \dot{\lambda}_j}{(\lambda_i - \lambda_j)^2} = -\frac{1}{n-1} \sum_{j \neq i} \frac{V_i - V_j}{(\lambda_i - \lambda_j)^2}.$$

Then

$$\dot{\Phi}_n = 2 \sum_i V_i \dot{V}_i = -\frac{2}{n-1} \sum_i V_i \sum_{j \neq i} \frac{V_i - V_j}{(\lambda_i - \lambda_j)^2}.$$

Symmetrizing: for each unordered pair $\{i, j\}$, the contributions from the i-sum and the j-sum total $-\frac{2}{n-1} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}$. Summing over pairs gives $\dot{\Phi}_n = -\frac{2}{n-1} \mathcal{S}(p_t)$. ☐

Theorem 5.5 (Hermite flow bound). *Let $p \in \mathcal{P}_n^{\mathbb{R}}$ have simple roots, $a := \sigma^2(p) > 0$, and $b > 0$. Assume that $p_t := p \boxplus_n G_t$ has simple roots for all $t \in [0, b]$. (This holds, for instance, whenever p has simple roots and b is sufficiently small; by the semigroup property one may iterate to cover any finite interval.) Then*

$$\frac{1}{\Phi_n(p \boxplus_n G_b)} \geq \frac{a+b}{a\, \Phi_n(p)}.$$

Proof. Apply the Score-Gradient Inequality (Theorem 3.2) to p_t: $\mathcal{S}(p_t) \geq \frac{(n-1)\Phi_n(p_t)}{2\sigma^2(p_t)} = \frac{(n-1)\Phi_n(p_t)}{2(a+t)}$. With Lemma 5.4, $\dot{\Phi}_n(p_t) \leq -\Phi_n(p_t)/(a+t)$. Since roots remain simple on $[0, b]$, $\Phi_n(p_t) > 0$ throughout (every score is finite), so we may divide by $\Phi_n(p_t)$ to obtain $(\log \Phi_n)' \leq -(a+t)^{-1}$. Integrating from 0 to b gives the result. ☐

6. Dilation interpolation and a convexity heuristic

6.1. The dilation path.

Definition 6.1 (Dilation family)**.** Let $q(x) = \prod_{i=1}^n (x - \mu_i) \in \mathcal{P}_n^{\mathbb{R}}$. For $t \in [0,1]$, define

$$q_t(x) := \prod_{i=1}^n (x - t\mu_i), \qquad r_t := p \boxplus_n q_t.$$

Lemma 6.2 (Basic properties)**.** *Let $a := \sigma^2(p)$ and $b := \sigma^2(q)$. Then:*
(i) $r_0 = p$ *and* $r_1 = p \boxplus_n q$.
(ii) $\sigma^2(q_t) = t^2\sigma^2(q)$ *and* $\sigma^2(r_t) = a + t^2 b$.
(iii) $\Phi_n(q_t) = \Phi_n(q)/t^2$ *for* $t > 0$.
(iv) $r_t \in \mathcal{P}_n^{\mathbb{R}}$ *for all* $t \in [0,1]$.

Proof. (i) is immediate since $q_0 = x^n$ is the identity for $\boxplus_n$. (ii) follows from scaling of roots and variance additivity. (iii) is score scaling under dilation. (iv) is Theorem 1.3. □

6.2. The excess functional.

Definition 6.3 (Dilation excess)**.** For the dilation path r_t, define

$$E(t) := \frac{1}{\Phi_n(r_t)} - \frac{1}{\Phi_n(p)} - \frac{t^2}{\Phi_n(q)}.$$

Lemma 6.4 (Endpoints)**.** $E(0) = 0$, *and* $E(1) \geq 0$ *is equivalent to the finite free Stam inequality.*

Proof. Immediate from the definition and $r_0 = p$, $r_1 = p \boxplus_n q$. □

Conjecture 6.5 (Excess convexity (false in general))**.** Along the dilation path, E is convex on $(0,1)$, i.e. $E''(t) \geq 0$. Equivalently, $\frac{d^2}{dt^2}(1/\Phi_n(r_t)) \geq 2/\Phi_n(q)$.

Remark 6.6. Conjecture 6.5 is a clean sufficient condition for Stam via Theorem 6.8, but it is *not true* in full generality; see Appendix A.

It is worth stressing a simple diagnostic: if we write $F(t) := 1/\Phi_n(r_t)$, then $E(t) = F(t) - 1/\Phi_n(p) - t^2/\Phi_n(q)$ satisfies

$$E''(t) = F''(t) - \frac{2}{\Phi_n(q)}.$$

Thus even if one happens to observe $F''(t) \geq 0$ in a given example, the subtraction of $t^2/\Phi_n(q)$ shifts the curvature by a negative constant and can force $E''(t) < 0$.

Lemma 6.7 (Vanishing first derivative at t=0)**.** *Assume q is centered (i.e. the sum of its roots is zero, equivalently its x^{n-1} coefficient vanishes). Then $E'(0) = 0$.*

Proof. Write $q(x) = \sum_{k=0}^n b_k x^{n-k}$. Centering means $b_1 = 0$.

Along the dilation family, q_t has coefficients $b_k(t) = t^k b_k$. By Definition 1.2,

$$r_t(x) = \sum_{k=0}^n \frac{(n-k)!}{n!}\, b_k(t)\, p^{(k)}(x) = p(x) + \sum_{k=1}^n \frac{(n-k)!}{n!}\, t^k b_k\, p^{(k)}(x).$$

Since $b_1 = 0$, the first nonzero perturbation is order t^2, so $\partial_t r_t\big|_{t=0} = 0$ as a polynomial.

Now differentiate the identity $r_t(\lambda_i(t)) \equiv 0$ in t. Since $r_0 = p$ has simple roots, the implicit function theorem (applied to $r_t(\lambda_i(t)) = 0$ with $r_0'(\lambda_i) = p'(\lambda_i) \neq 0$) guarantees that each $\lambda_i(t)$ is C^∞ near $t = 0$. Differentiating gives

$$\dot{\lambda}_i(0) = -\frac{\partial_t r_t(\lambda_i)\big|_{t=0}}{r_0'(\lambda_i)} = -\frac{0}{p'(\lambda_i)} = 0,$$

so each root trajectory has zero first derivative at $t = 0$. Since Φ_n is a smooth function of the roots as long as they remain distinct, this implies $\frac{d}{dt}\frac{1}{\Phi_n(r_t)}\big|_{t=0} = 0$. Also $\frac{d}{dt}(t^2/\Phi_n(q))|_{t=0} = 0$. Thus $E'(0) = 0$. ☐

Theorem 6.8 (Convexity reduction). *If Conjecture 6.5 holds for all $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with positive variance, then the finite free Stam inequality holds for all such p, q.*

Proof. By Lemma 1.8, we may replace q by its centered translate without changing either side of the Stam inequality; assume henceforth that q is centered.

Assuming Conjecture 6.5, the convex function E satisfies $E(t) \geq E(0) + tE'(0)$ for all $t \in [0, 1]$. By Lemma 6.4, $E(0) = 0$, and by Lemma 6.7, $E'(0) = 0$. Hence $E(1) \geq 0$, which is exactly the Stam inequality. ☐

Remark 6.9 (What remains for general n). Appendix A rules out global convexity of the dilation excess as a general mechanism. The remaining task is to find a different one-sided comparison principle along a real-rooted interpolation (dilation path, constant-variance path, or similar) that still forces $E(1) \geq 0$.

7. Auxiliary toolkit for future attempts

The results in this section are *not* used elsewhere in the paper. They record operator identities and proxy functionals intended as scaffolding for future general-n attempts at the Stam inequality, and are included for the convenience of the reader.

7.1. **Operator paths in the normalized derivative basis.**

Definition 7.1 (Normalized derivative operators). For $0 \leq k \leq n$, set

$$D_k := \frac{(n-k)!}{n!}\,\partial_x^k.$$

Then for $q(x) = \sum_{k=0}^n b_k x^{n-k}$,

$$T_q = \sum_{k=0}^n b_k D_k,$$

and $p \boxplus_n q = T_q p$.

Lemma 7.2 (Dilation and Hermite as coefficient rays). *Let $q_t(x) = \prod_{i=1}^n (x - t\mu_i) = \sum_{k=0}^n t^k b_k x^{n-k}$. Then*

$$T_{q_t} = \sum_{k=0}^n t^k b_k D_k.$$

If G_t is the Hermite kernel (Definition 5.1), then

$$T_{G_t} = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{1}{m!}\Big(-\frac{t}{2(n-1)}\Big)^m \partial_x^{2m}.$$

In particular, both the dilation family and the Hermite flow are explicit paths in the commutative algebra generated by ∂_x.

Proof. The dilation identity is immediate from coefficient scaling $b_k(t) = t^k b_k$. For G_t, use $K_{G_t}(z) = \exp(-\frac{t}{2(n-1)}z^2) \pmod{z^{n+1}}$ and the MSS operator rule $T_q = \sum_k \frac{(n-k)!}{n!} b_k \partial_x^k$; only even powers occur, with the displayed coefficients. Commutativity follows because all operators are polynomials in ∂_x. ☐

Remark 7.3 (Why this matters). Lemma 7.2 identifies the operator coefficients as the control variables for each interpolation, which is the natural format for MSS-style one-sided estimates.

7.2. Fisher–repulsion identity and its consequences.

Definition 7.4 (Pairwise repulsion energy)**.** For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1 < \cdots < \lambda_n$, define

$$\mathcal{R}(p) := \sum_{1 \le i < j \le n} \frac{1}{(\lambda_i - \lambda_j)^2}.$$

Theorem 7.5 (Fisher–repulsion identity)**.** *For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots,*

$$\Phi_n(p) = 2\,\mathcal{R}(p). \tag{3}$$

In particular, $1/\Phi_n(p) = 1/(2\mathcal{R}(p))$.

Proof. Expand $\Phi_n = \sum_{i=1}^n V_i^2$ by writing each V_i as a sum:

$$\Phi_n = \sum_i \sum_{j \ne i} \sum_{k \ne i} \frac{1}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)} = \underbrace{\sum_i \sum_{j \ne i} \frac{1}{(\lambda_i - \lambda_j)^2}}_{=\,2\mathcal{R}} + \underbrace{\sum_i \sum_{\substack{j \ne i,\, k \ne i \\ j \ne k}} \frac{1}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)}}_{=:\,C}.$$

The first sum counts each unordered pair $\{i, j\}$ twice, giving $2\mathcal{R}$. To show $C = 0$, group by unordered triples $\{a, b, c\}$. Each triple contributes (with two orderings of the non-pivot pair)

$$2\Big[\frac{1}{(\lambda_a - \lambda_b)(\lambda_a - \lambda_c)} + \frac{1}{(\lambda_b - \lambda_a)(\lambda_b - \lambda_c)} + \frac{1}{(\lambda_c - \lambda_a)(\lambda_c - \lambda_b)}\Big].$$

Set $u = \lambda_a - \lambda_b$, $v = \lambda_a - \lambda_c$. The bracket becomes $\frac{1}{uv} - \frac{1}{u(v-u)} + \frac{1}{v(v-u)}$. Putting over the common denominator $uv(v-u)$ gives $\frac{v-u-v+u}{uv(v-u)} = 0$. Since each bracket vanishes, $C = 0$ and $\Phi_n = 2\mathcal{R}$. $\square$

Remark 7.6. This identity is purely algebraic: it holds for any list of distinct reals, with no real-rootedness hypothesis. It sharpens the Cauchy–Schwarz bound $\Phi_n \le 2(n-1)\mathcal{R}$ by a factor of $n - 1$.

Corollary 7.7 (Stam as a harmonic-mean bound on repulsion)**.** *The Stam inequality $1/\Phi_n(p \boxplus_n q) \ge 1/\Phi_n(p) + 1/\Phi_n(q)$ is equivalent to*

$$\frac{1}{\mathcal{R}(p \boxplus_n q)} \ge \frac{1}{\mathcal{R}(p)} + \frac{1}{\mathcal{R}(q)},$$

i.e. $\mathcal{R}(p \boxplus_n q)$ is bounded above by the harmonic mean of $\mathcal{R}(p)$ and $\mathcal{R}(q)$.

Proof. Divide (3) through by 2. $\square$

Corollary 7.8 (Improved Fisher–variance via repulsion)**.** *For every $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots and positive variance,*

$$\frac{1}{\Phi_n(p)} = \frac{1}{2\mathcal{R}(p)} \le \frac{4\,\sigma^2(p)}{n(n-1)^2}.$$

Equivalently, $\mathcal{R}(p)\,\sigma^2(p) \ge n(n-1)^2/8$.

Proof. Combine Theorem 7.5 with Lemma 3.1. $\square$

Remark 7.9 (Reformulation of Stam). The identity $\Phi_n = 2\mathcal{R}$ gives the cleanest formulation of the Stam inequality: $\mathcal{R}$ depends only on root gaps (no "scores" needed), and the target is the classical harmonic-mean inequality for an energy functional.

7.3. **Barrier-inspired resolvent quantities.**

Definition 7.10 (Cauchy transform at height η)**.** For $\eta > 0$ and real x, define

$$g_{p,\eta}(x) := \frac{1}{n}\frac{p'(x+i\eta)}{p(x+i\eta)} = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{x+i\eta-\lambda_j}.$$

Write $g_{p,\eta} = u_{p,\eta} + iv_{p,\eta}$ with $u, v \in \mathbb{R}$.

Lemma 7.11 (Uniform bounds away from the real axis)**.** *For every $\eta > 0$ and every real x,*

$$|u_{p,\eta}(x)| \le \frac{1}{\eta}, \qquad 0 < v_{p,\eta}(x) \le \frac{1}{\eta}.$$

Proof. Using Definition 7.10,

$$u_{p,\eta}(x) = \frac{1}{n}\sum_{j=1}^{n}\frac{x-\lambda_j}{(x-\lambda_j)^2+\eta^2}, \qquad v_{p,\eta}(x) = \frac{1}{n}\sum_{j=1}^{n}\frac{\eta}{(x-\lambda_j)^2+\eta^2}.$$

Each summand has absolute value at most $1/\eta$ in the real part and lies in $(0, 1/\eta]$ in the imaginary part, and averaging preserves these bounds. ☐

Remark 7.12 (Connection to MSS barriers)*.* In MSS-style arguments one tracks logarithmic derivatives off the real axis. Here $g_{p,\eta}(x)$ is precisely that quantity at height η, and unlike Φ_n it stays uniformly bounded and avoids collision singularities.

7.4. **Discussion and next steps.**

(L1) The dilation-path convexity heuristic for $E(t)$ is false globally, so the final proof mechanism is likely to be one-sided and robust rather than purely curvature-based.

(L2) The operator identities in Lemma 7.2 provide a clean decomposition of both interpolation families into differential-operator coordinates. This is the correct format for importing MSS-style barrier ideas.

(L3) The identity $\Phi_n = 2\mathcal{R}$ (Theorem 7.5) and the resulting harmonic-mean reformulation (Corollary 7.7) provide the cleanest formulation of the Stam conjecture.

(L4) Numerical experiments (see `route_a_experiments.py` and `test_repulsion_stam.py`) show that Lorentzian regularisation of the repulsion energy at any fixed $\eta > 0$ *breaks* the super-additivity; thus a direct resolvent-barrier proof at fixed height is unlikely to succeed without substantial modification. However, the identity $\Phi_n = 2\mathcal{R}$ was discovered via this investigation and is independently useful.

(L5) The most promising remaining strategies are: (a) a dilation-path ODE argument for $1/\mathcal{R}(r_t)$ combined with the SGI and the identity $\Phi_n = 2\mathcal{R}$ (see Section 9 for Route B analysis); (b) the constant-variance path combined with monotonicity of Φ_n under higher-cumulant perturbation; (c) a random-matrix / HCIZ representation of $\mathcal{R}$ as a Haar-averaged trace; (d) a variational/transport approach via the discriminant power inequality discovered in Section 8. Route B (Section 9) provides extensive evidence that $\Phi_n(r_t)$ is non-increasing along the dilation path and that the "pointwise dilation Stam" (Conjecture 9.18) holds; the key open sub-problem is controlling the sign of the perpendicular dissipation (Conjecture 9.15).

(L6) Route C (Section 8) introduces a transport/variational framework and discovers the *discriminant power inequality* (Conjecture 8.13): $\mathcal{N}(p \boxplus_n q) \ge \mathcal{N}(p) + \mathcal{N}(q)$, the finite-free analogue of Shannon's entropy-power inequality. The conjecture survives 29 000+ random tests for $n = 2$ to 8 with zero violations. For $n = 2$ it reduces to variance additivity (exact equality), and for $n = 3$ it is proved via the same convexity argument that establishes the Stam inequality. Several naïve transport proxies (pairwise gap super-additivity, dilation vs. displacement comparison, weighted gap coupling) are disproved by numerical counterexample.

8. Route C: Variational / transport approach

We now develop a variational perspective on the Stam inequality based on optimal-transport ideas and energy functionals of root configurations. The key outcome is a new conjectured inequality (the *discriminant power inequality*, Conjecture 8.13) that is the finite-free analogue of Shannon's entropy power inequality.

8.1. Root measures, log-Vandermonde, and discriminant power.

Definition 8.1 (Empirical root measure)**.** For $p \in \mathcal{P}_n^{\mathbb{R}}$ with roots $\lambda_1 \le \cdots \le \lambda_n$, define $\mu_p := \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i}$.

Definition 8.2 (Log-Vandermonde ("entropy"))**.** For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1 < \cdots < \lambda_n$, define

$$H(p) := \sum_{1\le i<j\le n} \log(\lambda_j - \lambda_i) = \log \prod_{i<j}(\lambda_j - \lambda_i) = \tfrac{1}{2}\log \operatorname{disc}(p),$$

where $\operatorname{disc}(p) = \prod_{i<j}(\lambda_j - \lambda_i)^2$ is the discriminant. If p has a repeated root, set $H(p) := -\infty$.

Definition 8.3 (Discriminant power ("entropy power"))**.** With $N := \binom{n}{2}$, define the *discriminant power*

$$\mathcal{N}(p) := \operatorname{disc}(p)^{1/N} = e^{2H(p)/N} = \Big(\prod_{i<j}(\lambda_j - \lambda_i)^2\Big)^{1/N},$$

i.e. the geometric mean of the squared gaps. Set $\mathcal{N}(p) := 0$ when p has a repeated root.

Remark 8.4. For $n = 2$, $N = 1$ and $\mathcal{N}(p) = (\lambda_2 - \lambda_1)^2 = 4\sigma^2(p)$. More generally, $\mathcal{N}(p)$ captures the "typical squared gap" of p.

8.2. Gradient structure of the log-Vandermonde.

Lemma 8.5 (Score as gradient of H)**.** *Let $p \in \mathcal{P}_n^{\mathbb{R}}$ have distinct roots. Viewing H as a function of the root vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ in the open Weyl chamber $W_n := \{\lambda_1 < \cdots < \lambda_n\} \subset \mathbb{R}^n$,*

$$\frac{\partial H}{\partial \lambda_k} = V_k \qquad (k = 1, \dots, n).$$

In particular $\Phi_n(p) = \|\nabla H(\boldsymbol{\lambda})\|^2$.

Proof. Differentiate $H = \sum_{i<j}\log(\lambda_j - \lambda_i)$. For fixed k, every pair (k, j) with $j > k$ contributes $\partial_{\lambda_k}\log(\lambda_j - \lambda_k) = -1/(\lambda_j - \lambda_k)$, and every pair (i, k) with $i < k$ contributes $\partial_{\lambda_k}\log(\lambda_k - \lambda_i) = 1/(\lambda_k - \lambda_i)$. Combining, $\partial_k H = \sum_{j\ne k} 1/(\lambda_k - \lambda_j) = V_k$. □

Lemma 8.6 (Hessian and trace identity)**.** *For distinct roots,*

$$\frac{\partial^2 H}{\partial\lambda_i \partial\lambda_j} = \begin{cases} \dfrac{1}{(\lambda_i - \lambda_j)^2} & i \ne j, \\ -\displaystyle\sum_{k\ne i} \frac{1}{(\lambda_i - \lambda_k)^2} & i = j. \end{cases}$$

In particular, $\operatorname{tr}\big(\operatorname{Hess}(H)\big) = -2\mathcal{R}(p) = -\Phi_n(p)$*. Moreover,* $\operatorname{Hess}(H)$ *is negative semidefinite with kernel* $\operatorname{span}(\mathbf{1})$ *(translation direction).*

Proof. The formula follows by direct differentiation. The trace sums the diagonal entries: $\sum_i(-\sum_{k\ne i}(\lambda_i - \lambda_k)^{-2}) = -\sum_i\sum_{k\ne i}(\lambda_i - \lambda_k)^{-2} = -2\mathcal{R} = -\Phi_n$.

For negative semidefiniteness, let $v \in \mathbb{R}^n$ with $\sum_i v_i = 0$. Then

$$v^\top \operatorname{Hess}(H)\, v = \sum_i v_i^2\Big(-\sum_{k\ne i}\frac{1}{d_{ik}^2}\Big) + \sum_{i\ne j}\frac{v_i v_j}{d_{ij}^2} = -\sum_{i<j}\frac{(v_i - v_j)^2}{d_{ij}^2} \le 0,$$

with equality iff all v_i are equal; combined with $\sum v_i = 0$, this gives $v = 0$. On the full space, $\mathbf{1}^\top \operatorname{Hess}(H)\mathbf{1} = \operatorname{tr} + 2\sum_{i<j} d_{ij}^{-2} = -2\mathcal{R} + 2\mathcal{R} = 0$, so $\mathbf{1}$ is in the kernel. $\square$

8.3. **De Bruijn identity and isoperimetric inequality.**

Theorem 8.7 (Finite free de Bruijn identity)**.** *Under the same simple-root hypothesis as Lemma 5.3, along the Hermite flow $p_t = p \boxplus_n G_t$,*

$$\frac{d}{dt} H(p_t) = \frac{\Phi_n(p_t)}{n-1}. \tag{4}$$

Proof. By the chain rule, Lemma 8.5, and the Hermite root ODE $\dot\lambda_i = V_i/(n-1)$:

$$\frac{d}{dt} H(p_t) = \sum_{k=1}^n \frac{\partial H}{\partial \lambda_k}\,\dot\lambda_k = \sum_{k=1}^n V_k \cdot \frac{V_k}{n-1} = \frac{1}{n-1}\sum_k V_k^2 = \frac{\Phi_n(p_t)}{n-1}. \qquad \square$$

Remark 8.8. Identity (4) is the exact analogue of the classical de Bruijn identity $\frac{d}{dt} h(X + \sqrt{t}\,Z) = \frac{1}{2} J(X + \sqrt{t}\,Z)$ for the differential entropy h and Fisher information J. Combined with the dissipation identity $\dot\Phi_n = -\frac{2}{n-1}\mathcal{S}$ (Lemma 5.4), it gives a closed system on $(H, \Phi_n, \mathcal{S})$ along the Hermite flow.

Lemma 8.9 (AM-GM isoperimetric inequality)**.** *For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots,*

$$\mathcal{R}(p) \cdot \mathcal{N}(p) \ge N = \binom{n}{2}. \tag{5}$$

Equivalently, $\Phi_n(p) \cdot \mathcal{N}(p) \ge 2\binom{n}{2} = n(n-1)$. Equality holds if and only if all $\binom{n}{2}$ gaps $|\lambda_i - \lambda_j|$ are equal (i.e. the roots are equally spaced).

Proof. By AM-GM applied to the N positive numbers $1/d_{ij}^2$ $(i<j)$:

$$\frac{1}{N}\sum_{i<j} \frac{1}{d_{ij}^2} \ge \Big(\prod_{i<j} \frac{1}{d_{ij}^2}\Big)^{1/N} = \frac{1}{\mathcal{N}(p)}.$$

Rearranging, $\mathcal{R}/N \ge 1/\mathcal{N}$. Equality in AM-GM holds iff all d_{ij}^{-2} coincide. $\square$

8.4. **Displacement convexity of the repulsion energy.**

Definition 8.10 (Displacement interpolation)**.** For two ordered root vectors $\boldsymbol{\lambda} = (\lambda_1 < \cdots < \lambda_n)$ and $\boldsymbol{\rho} = (\rho_1 < \cdots < \rho_n)$ in W_n, define $\boldsymbol{\gamma}(t) := (1-t)\boldsymbol{\lambda} + t\boldsymbol{\rho}$ for $t \in [0,1]$. This is the McCann displacement interpolation (monotone-coupling geodesic) between the empirical measures $\mu_{\boldsymbol{\lambda}}$ and $\mu_{\boldsymbol{\rho}}$.

Lemma 8.11 (Displacement convexity of $\mathcal{R}$)**.** *The repulsion energy $\mathcal{R}$ is convex along displacement interpolations in W_n. That is, for every $\boldsymbol{\lambda}, \boldsymbol{\rho} \in W_n$,*

$$t \mapsto \mathcal{R}(\boldsymbol{\gamma}(t)) \quad \textit{is convex on } [0,1].$$

Proof. Each gap along the displacement interpolation is linear: $\gamma_j(t) - \gamma_i(t) = (1-t)(\lambda_j - \lambda_i) + t(\rho_j - \rho_i)$, and is positive for all $t \in [0,1]$ because both endpoints are positive (since $i < j$ and both $\boldsymbol{\lambda}, \boldsymbol{\rho}$ are strictly ordered). Thus $\mathcal{R}(\boldsymbol{\gamma}(t)) = \sum_{i<j} \frac{1}{(\gamma_j(t) - \gamma_i(t))^2}$ is a sum of functions $f \circ \ell$ where $f(d) = 1/d^2$ is convex on $(0,\infty)$ (since $f''(d) = 6/d^4 > 0$) and $\ell(t)$ is a positive affine function. A convex function composed with an affine function is convex, and sums of convex functions are convex. $\square$

Remark 8.12. In optimal-transport terminology, this says the Riesz ($s{=}-2$) interaction energy is *displacement convex* on W_n. In contrast, the log-energy H is easily verified to be displacement *concave* (since $\log d$ is concave), and its exponential $\mathcal{N} = e^{2H/N}$ inherits the log-concavity.

8.5. The discriminant power inequality.

Conjecture 8.13 (Discriminant power inequality ("finite-free EPI")). For all $p, q \in \mathcal{P}_n^{\mathbb{R}}$,

$$\mathcal{N}(p \boxplus_n q) \geq \mathcal{N}(p) + \mathcal{N}(q). \tag{6}$$

Equivalently, $\operatorname{disc}(p \boxplus_n q)^{1/N} \geq \operatorname{disc}(p)^{1/N} + \operatorname{disc}(q)^{1/N}$.

Proposition 8.14 (EPI for $n = 2$: equality). *For $n = 2$, the discriminant power inequality is an exact equality and reduces to variance additivity: $\mathcal{N}(p \boxplus_2 q) = 4\sigma^2(p \boxplus_2 q) = 4\sigma^2(p) + 4\sigma^2(q) = \mathcal{N}(p) + \mathcal{N}(q)$.*

Proof. For $n = 2$, $N = 1$ and $\mathcal{N}(p) = (\lambda_2 - \lambda_1)^2 = 4\sigma^2(p)$. The result is Lemma 2.4. □

Theorem 8.15 (EPI for $n = 3$). *The discriminant power inequality* (6) *holds for $n = 3$. More precisely, after centering (which preserves $\mathcal{N}$ because all gaps are translation-invariant), write $p(x) = x^3 - S_1 x + T_1$ and $q(x) = x^3 - S_2 x + T_2$ with $S_i > 0$ and $\Delta_i = 4S_i^3 - 27T_i^2 > 0$. Then*

$$\Delta_r^{1/3} \geq \Delta_1^{1/3} + \Delta_2^{1/3},$$

where $\Delta_r = 4(S_1 + S_2)^3 - 27(T_1 + T_2)^2$.

Proof. Write $\delta_i := \Delta_i/(4S_i^3) = 1 - 27T_i^2/(4S_i^3) \in (0, 1]$, and $w_i := S_i/(S_1 + S_2) \in (0, 1)$, so $w_1 + w_2 = 1$. Then $\Delta_i^{1/3} = 4^{1/3} S_i \delta_i^{1/3}$, and the inequality becomes

$$\delta_r^{1/3} \geq w_1 \delta_1^{1/3} + w_2 \delta_2^{1/3}. \tag{7}$$

Step 1: reduce to a linear comparison. Since $t \mapsto t^{1/3}$ is concave on $(0, \infty)$, $\delta_r^{1/3} \geq (w_1\delta_1 + w_2\delta_2)^{1/3}$ implies (7) by Jensen. It thus suffices to prove the *stronger* inequality

$$\delta_r \geq w_1\delta_1 + w_2\delta_2. \tag{8}$$

Step 2: prove the linear comparison. Expanding and cancelling, (8) is equivalent to

$$\frac{(T_1 + T_2)^2}{(S_1 + S_2)^2} \leq \frac{T_1^2}{S_1^2} + \frac{T_2^2}{S_2^2},$$

which holds by the convexity of $f(x) = x^2$ applied to $x_i = T_i/S_i$ with weights w_i: $(w_1x_1 + w_2x_2)^2 \leq w_1x_1^2 + w_2x_2^2$ and a fortiori $\leq x_1^2 + x_2^2$. □

Remark 8.16 (Relationship between EPI and Stam for $n = 3$). The intermediate inequality (8) is *equivalent* to the Stam inequality at $n = 3$ (compare the proof of Theorem 4.5). Thus for $n = 3$, Stam ⇒ EPI (since Stam gives the stronger linear bound, from which the cube-root bound follows by concavity). For general n the two statements appear to be independent — neither obviously implies the other — although both hold in all numerical tests.

8.6. Numerical evidence.

We record the numerical evidence for Route C (code: `route_c_experiments.py`, `route_c_epi_deep.py`).

Discriminant power inequality (Conjecture 8.13). Over 29 000 random tests ($n = 2$ to 8, 3000 per degree, both centered and non-centered q), we observe zero violations. For $n = 2$ the minimum relative excess $(\mathcal{N}(r) - \mathcal{N}(p) - \mathcal{N}(q))/(\mathcal{N}(p) + \mathcal{N}(q))$ is $\sim 10^{-13}$ (machine-precision equality); for $n = 3$, $\sim 9 \times 10^{-6}$; and for $n \geq 4$ the minimum relative excess grows with n:

n	3	4	5	6	7	8
min relative excess	9×10^{-6}	5×10^{-4}	5×10^{-3}	3×10^{-2}	4×10^{-2}	6×10^{-2}

Near-equality occurs when one polynomial has very small variance (nearly degenerate roots).

Stam vs. EPI excess correlation. The Pearson correlation between the Stam excess $1/\Phi_n(r) - 1/\Phi_n(p) - 1/\Phi_n(q)$ and the EPI excess $\mathcal{N}(r) - \mathcal{N}(p) - \mathcal{N}(q)$ is strong (0.90 for $n = 3$, 0.83 for $n = 4$, 0.72 for $n = 6$), suggesting the two functionals measure related aspects of root spreading under $\boxplus_n$.

False proxies. Several naïve transport-inspired proxies do *not* hold:

(a) *Pairwise gap super-additivity* ($d_{ij}(r)^2 \geq d_{ij}(p)^2 + d_{ij}(q)^2$ for all pairs): fails in 100% of tests; worst ratio ≈ 0.32.
(b) *Dilation path $\leq$ displacement for $\mathcal{R}$* ($\mathcal{R}(r_t) \leq \mathcal{R}(\gamma(t))$): fails in $> 84\%$ of evaluations.
(c) *Raw log-Vandermonde super-additivity* ($H(r) \geq H(p) + H(q)$): fails in $\sim 19\%$ of tests.
(d) *Repulsion-weighted gap coupling*: fails generically.

These negative results constrain the class of viable transport arguments.

Displacement convexity and transport–repulsion bound. Lemma 8.11 (displacement convexity of $\mathcal{R}$) is confirmed with zero violations in 37 000 midpoint checks. Additionally, the ratio $(1/\mathcal{R}(r) - 1/\mathcal{R}(p))/W_2^2(\mu_p, \mu_r)$ is always positive (tested for $n = 3$ to 6, 500 trials each), suggesting that convolution-driven root spreading always reduces repulsion in proportion to the transport cost.

8.7. Discussion and next steps for Route C.

(C1) The discriminant power inequality (Conjecture 8.13) is numerically robust and has a clean algebraic statement: $\operatorname{disc}(p \boxplus_n q)^{1/N} \geq \operatorname{disc}(p)^{1/N} + \operatorname{disc}(q)^{1/N}$. A proof would constitute a finite-free analogue of Shannon's entropy power inequality.
(C2) The de Bruijn identity (Theorem 8.7) and the AM-GM isoperimetric (Lemma 8.9) provide the finite-free analogues of the two pillar identities in the classical proof of the entropic Stam inequality. The question is whether they can be combined to yield either Stam or EPI for general n.
(C3) For a proof of the EPI at general n, possible approaches include: (i) the MSS random-matrix representation $p \boxplus_n q(x) = \mathbb{E}_U \det(xI - A - UBU^*)$ combined with Minkowski-type determinantal inequalities; (ii) induction on n using interlacing and root-deletion relations for the discriminant; (iii) an operator-semigroup argument along the Hermite flow, exploiting the de Bruijn identity and the dissipation formula.
(C4) Even without a direct EPI⇒Stam implication, the EPI serves as an independent "litmus test" for proposed approaches: any strategy that can prove both Stam *and* EPI likely captures the essential structure of root spreading under $\boxplus_n$.

9. Route B: Operator-coefficient monotone path

We study the dilation path $r_t = p \boxplus_n q_t$ (Definition 6.1) from the perspective of root-velocity dissipation. The key outcomes are: (i) a general dissipation formula that decomposes $\dot{\Phi}_n$ into score-aligned and perpendicular components (Lemma 9.1), (ii) a proof that the initial root acceleration is proportional to the score (Proposition 9.5), (iii) a new functional $\Gamma^{(1)}$ whose positivity controls $F''(0)$, proved positive for $n = 3$ (Proposition 9.10), and (iv) several conjectures supported by extensive numerical experiments.

9.1. General dissipation along arbitrary root motions.

Lemma 9.1 (General Fisher dissipation)**.** *Let $\boldsymbol{\lambda}(t) = (\lambda_1(t), \dots, \lambda_n(t))$ be a smooth path of distinct ordered configurations in W_n, and set $V_i(t) := \sum_{j \neq i}(\lambda_i(t) - \lambda_j(t))^{-1}$. Then*

$$\frac{d}{dt}\Phi_n(\boldsymbol{\lambda}(t)) = -2\sum_{i<j}\frac{(V_i - V_j)(\dot{\lambda}_i - \dot{\lambda}_j)}{(\lambda_i - \lambda_j)^2}. \tag{9}$$

Proof. $\Phi_n = \sum_i V_i^2$. Differentiating $V_i = \sum_{j \neq i}(\lambda_i - \lambda_j)^{-1}$ gives $\dot{V}_i = -\sum_{j \neq i} \frac{\dot{\lambda}_i - \dot{\lambda}_j}{(\lambda_i - \lambda_j)^2}$. Thus $\dot{\Phi}_n = 2\sum_i V_i \dot{V}_i$. Substituting $\dot{V}_i$ and symmetrizing over unordered pairs $\{i, j\}$ (exactly as in the proof of Lemma 5.4) yields (9). ∎

Definition 9.2 (Dissipation pairing)**.** For a root configuration $\boldsymbol{\lambda} \in W_n$ with scores V_i and velocity vector $\dot{\boldsymbol{\lambda}}$, define

$$\mathcal{D}(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}) := \sum_{i<j} \frac{(V_i - V_j)(\dot{\lambda}_i - \dot{\lambda}_j)}{(\lambda_i - \lambda_j)^2}.$$

Thus $\dot{\Phi}_n = -2\mathcal{D}$ and $\dot{F} := \frac{d}{dt}\frac{1}{\Phi_n} = \frac{2\mathcal{D}}{\Phi_n^2}$.

Remark 9.3. Writing $s_{ij} := (V_i - V_j)/d_{ij}$ and $w_{ij} := (\dot{\lambda}_i - \dot{\lambda}_j)/d_{ij}$, we have $\mathcal{D} = \sum_{i<j} s_{ij}\, w_{ij}$ and $\mathcal{S} = \sum_{i<j} s_{ij}^2$. For the Hermite flow ($w_{ij} = s_{ij}/(n-1)$), $\mathcal{D} = \mathcal{S}/(n-1) > 0$ always. For a general path the sign of $\mathcal{D}$ depends on the alignment between the score-gradient and velocity-gradient vectors.

9.2. **Dilation root velocity and initial acceleration.**

Lemma 9.4 (Dilation root ODE)**.** *Along $r_t = p \boxplus_n q_t$ (with p, q_t having all distinct roots in a neighbourhood), the root trajectories satisfy*

$$\dot{\lambda}_i(t) = -\frac{(\partial_t r_t)(\lambda_i(t))}{r_t'(\lambda_i(t))}, \tag{10}$$

where $\partial_t r_t(x) = \displaystyle\sum_{k=1}^{n} k\, t^{k-1}\, b_k\, \frac{(n-k)!}{n!}\, p^{(k)}(x)$ *and b_k are the coefficients of* $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$.

Proof. Direct implicit differentiation of $r_t(\lambda_i(t)) \equiv 0$. The formula for $\partial_t r_t$ follows from the MSS operator representation $r_t = \sum_k t^k b_k D_k p$ (Lemma 7.2). ☐

Proposition 9.5 (Initial root acceleration for centered q)**.** *Let q be centered ($b_1 = 0$). Then $\dot{\lambda}_i(0) = 0$ and*

$$\ddot{\lambda}_i(0) = \frac{2\,\sigma^2(q)}{n-1}\, V_i(p), \tag{11}$$

where $V_i(p)$ are the scores of p.

Proof. Step 1: $\dot{\lambda}_i(0) = 0$. Since $b_1 = 0$, the term $k = 1$ in (10) vanishes at $t = 0$, and all terms $k \geq 2$ carry a factor $t^{k-1} \to 0$, so $(\partial_t r_t)|_{t=0} = 0$. By (10), $\dot{\lambda}_i(0) = 0$.

Step 2: second-order expansion. Differentiating $r_t(\lambda_i(t)) = 0$ twice at $t = 0$ (using $r_0 = p$, $\dot{\lambda}_i(0) = 0$):

$$p'(\lambda_i)\ddot{\lambda}_i(0) + (\partial_t^2 r_t)(\lambda_i)\big|_{t=0} = 0.$$

Now $\partial_t^2 r_t(x) = \sum_{k\geq 2} k(k-1)t^{k-2} b_k D_k p(x)$, and at $t = 0$ only $k = 2$ survives: $(\partial_t^2 r_t)(x)\big|_{t=0} = 2b_2\, D_2 p(x) = \frac{2b_2}{n(n-1)} p''(x)$. Thus:

$$\ddot{\lambda}_i(0) = -\frac{2b_2}{n(n-1)}\,\frac{p''(\lambda_i)}{p'(\lambda_i)} = -\frac{4b_2}{n(n-1)}\, V_i(p),$$

using Lemma 2.1. Since q is centered with $b_1 = 0$, the variance formula (Lemma 2.3) gives $\sigma^2(q) = -2b_2/n$, i.e. $b_2 = -n\sigma^2(q)/2$. Substituting: $\ddot{\lambda}_i(0) = \frac{4\cdot n\sigma^2(q)}{2n(n-1)}\, V_i(p) = \frac{2\sigma^2(q)}{n-1}\, V_i(p)$. ☐

Remark 9.6 (Score-aligned acceleration). Equation (11) shows that the initial root acceleration under dilation is proportional to the *score* V_i, with a positive coefficient $2\sigma^2(q)/(n-1)$. This is exactly the velocity field of the Hermite flow (Lemma 5.3), scaled by $2\sigma^2(q)$. In other words, the leading-order dilation perturbation “looks like” a Hermite perturbation at rate $2\sigma^2(q)$ per unit t^2.

9.3. The weighted score-gap functional $\Gamma^{(1)}$.

Definition 9.7 (Weighted score-gap functionals)**.** For integer $s \geq 0$ and $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots, define

$$\Gamma^{(s)}(p) := \sum_{i<j} \frac{V_j - V_i}{(\lambda_j - \lambda_i)^{1+2s}}.$$

Remark 9.8. $\Gamma^{(0)} = \Phi_n$ by Lemma 2.2 (iv).

Proposition 9.9 (Initial curvature of $1/\Phi_n$ along dilation)**.** *For centered q,*

$$F''(0) = \frac{d^2}{dt^2}\Big(\frac{1}{\Phi_n(r_t)}\Big)\bigg|_{t=0} = \frac{2\,\sigma^2(q)\,\Gamma^{(1)}(p)}{(n-1)\,\mathcal{R}(p)^2}. \tag{12}$$

In particular, $F''(0) > 0$ if and only if $\Gamma^{(1)}(p) > 0$.

Proof. From Proposition 9.5, $\dot\lambda_i(0) = 0$ and $\ddot\lambda_i(0) = \frac{2\sigma^2(q)}{n-1} V_i$. Using $\Phi_n = 2\mathcal{R}$ and $F = 1/\Phi_n = 1/(2\mathcal{R})$:

$$F'(t) = -\frac{\dot{\mathcal{R}}}{2\mathcal{R}^2}, \qquad \dot{\mathcal{R}} = -2\sum_{i<j} \frac{\dot\lambda_j - \dot\lambda_i}{(\lambda_j - \lambda_i)^3}.$$

Since all $\dot\lambda_i(0) = 0$, we get $\dot{\mathcal{R}}(0) = 0$ and $F'(0) = 0$. At second order: $\ddot{\mathcal{R}}(0) = -2\sum_{i<j} \frac{\ddot\lambda_j(0) - \ddot\lambda_i(0)}{(\lambda_j-\lambda_i)^3} = -\frac{4\sigma^2(q)}{n-1}\sum_{i<j}\frac{V_j - V_i}{(\lambda_j-\lambda_i)^3} = -\frac{4\sigma^2(q)}{n-1}\Gamma^{(1)}(p)$. Since $F'(0) = 0$: $F''(0) = -\frac{\ddot{\mathcal{R}}(0)}{2\mathcal{R}(p)^2} = \frac{2\sigma^2(q)\,\Gamma^{(1)}(p)}{(n-1)\mathcal{R}(p)^2}$. ☐

Proposition 9.10 ($\Gamma^{(1)} > 0$ for $n = 3$)**.** *For every $p \in \mathcal{P}_3^{\mathbb{R}}$ with distinct roots, $\Gamma^{(1)}(p) > 0$.*

Proof. Let the roots be $\lambda_1 < \lambda_2 < \lambda_3$ and set $d := \lambda_2 - \lambda_1 > 0$, $e := \lambda_3 - \lambda_2 > 0$. Computing the scores $V_1 = -1/d - 1/(d+e)$, $V_2 = 1/d - 1/e$, $V_3 = 1/(d+e) + 1/e$ and substituting into $\Gamma^{(1)} = \sum_{i<j}(V_j - V_i)/(\lambda_j - \lambda_i)^3$, a mechanical computation (verified with `sympy`) yields

$$\Gamma^{(1)} = \frac{(d^2 + de + e^2)\,f(d,e)}{d^4\,e^4\,(d+e)^4}, \tag{13}$$

where

$$f(d,e) := 2d^6 + 6d^5e + 3d^4e^2 - 4d^3e^3 + 3d^2e^4 + 6de^5 + 2e^6.$$

The denominator and the factor $d^2 + de + e^2$ are manifestly positive. For f, note that f is a palindromic polynomial in $t := e/d$: $f = d^6(2t^6 + 6t^5 + 3t^4 - 4t^3 + 3t^2 + 6t + 2) =: d^6\,\tilde f(t)$. Setting $s := t + 1/t \geq 2$ (by AM-GM), we find $\tilde f(t) = t^3\,g(s)$ with $g(s) = 2s^3 + 6s^2 - 3s - 16$. Now $g(2) = 18 > 0$ and $g'(s) = 6s^2 + 12s - 3 > 0$ for $s \geq 2$, so $g(s) \geq 18$ on $[2,\infty)$, hence $\tilde f(t) > 0$ for all $t > 0$. ☐

Conjecture 9.11 ($\Gamma^{(1)}$ positivity for general n)**.** For every $n \geq 2$ and every $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots, $\Gamma^{(1)}(p) > 0$.

Numerical evidence: 2500 random tests for $n = 3, 4, 5, 6, 7$ (500 per degree) with zero violations; additionally 300 near-collision stress tests per degree with zero violations.

9.4. Velocity decomposition and score alignment.

Definition 9.12 (Score alignment and orthogonal decomposition)**.** Given root velocities $\dot\lambda_i$ and scores V_i, define the *score-alignment coefficient*

$$\alpha := \frac{(n-1)\,\sum_i V_i\,\dot\lambda_i}{\sum_i V_i^2}$$

and decompose $\dot\lambda_i = \frac{\alpha}{n-1}\,V_i + \varepsilon_i$, where $\sum_i V_i\varepsilon_i = 0$ (so ε is orthogonal to the score vector in ℓ^2). The *aligned dissipation* is $\mathcal{D}_\parallel := \frac{\alpha}{n-1}\mathcal{S}$, and the *perpendicular dissipation* is $\mathcal{D}_\perp := \mathcal{D} - \mathcal{D}_\parallel$.

Remark 9.13. In the Hermite case, $\alpha \equiv 1$ and $\varepsilon \equiv 0$, so $\mathcal{D} = \mathcal{D}_{\|} = \mathcal{S}/(n-1)$.

The following conjectures are supported by extensive numerical experiments (code: `route_b_experiments.py` and `route_b_deep.py`).

Conjecture 9.14 (Score alignment)**.** Along the dilation path $r_t = p \boxplus_n q_t$ (with centered q, $t \in (0,1]$), the score-alignment coefficient satisfies $\alpha(t) > 0$.

Numerical evidence: 1200+ random trials for $n = 3, 4, 5, 6$, all with $\alpha(t) > 0$ along 100-point grids. Combined with Proposition 9.5, which shows $\alpha \to 2\sigma^2(q) \cdot t$ as $t \to 0^+$.

Conjecture 9.15 (Perpendicular dissipation sign)**.** Along the dilation path, $\mathcal{D}_\perp(t) \leq 0$ for all $t \in (0,1]$. In other words, the non-Hermite component of the velocity always *retards* the Fisher dissipation, but does not reverse it.

Numerical evidence: In all sampled examples ($n = 3, 4, 5$), $\mathcal{D}_\perp(t) < 0$ pointwise, with $|\mathcal{D}_\perp| < \mathcal{D}_{\|}$, so $\mathcal{D} = \mathcal{D}_{\|} + \mathcal{D}_\perp > 0$.

9.5. **Repulsion monotonicity and pointwise dilation Stam.**

Conjecture 9.16 (Repulsion monotonicity)**.** Along the dilation path $r_t = p \boxplus_n q_t$, the repulsion energy is non-increasing:

$$\frac{d}{dt}\mathcal{R}(r_t) \leq 0 \qquad \text{for all } t \in [0,1].$$

Equivalently, $\Phi_n(r_t)$ is non-increasing, and $F(t) := 1/\Phi_n(r_t)$ is non-decreasing.

Numerical evidence: In 800 random tests ($n = 3, 4, 5, 6$, 200-point grids), $\mathcal{R}(r_t)$ is non-increasing with zero violations. An additional 1200+ tests confirm $F'(t) \geq 0$ at all sampled grid points.

Remark 9.17. Conjecture 9.16 combined with Conjectures 9.14 and 9.15 would follow from $\mathcal{D}(t) = \mathcal{D}_{\|}(t) + \mathcal{D}_\perp(t) \geq 0$, which says the (positive) score-aligned dissipation always dominates the (negative) perpendicular correction.

Conjecture 9.18 (Pointwise dilation Stam)**.** For all $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots and positive variance,

$$\frac{1}{\Phi_n(r_t)} \geq \frac{1}{\Phi_n(p)} + \frac{t^2}{\Phi_n(q)} \qquad \text{for all } t \in [0,1]. \tag{14}$$

Evaluating at $t = 1$ recovers the Stam inequality.

Numerical evidence: 2500 random tests ($n = 3, 4, 5, 6, 7$, 200-point grids) plus 1200 near-collision stress tests, with zero violations. The global minimum of $E(t) := 1/\Phi_n(r_t) - 1/\Phi_n(p) - t^2/\Phi_n(q)$ is $\geq -5 \times 10^{-12}$ (machine precision).

Remark 9.19 (Hierarchy of conjectures). The implications are:

$$\text{Conj. 9.18 (pointwise)} \implies \text{Stam inequality} \implies E(1) \geq 0.$$

Also, Conjecture 9.16 (monotonicity $F' \geq 0$) alone gives only $F(1) \geq F(0) = 1/\Phi_n(p)$, which is weaker than Stam. However, monotonicity *combined with an initial curvature bound* (Proposition 9.9 and $\Gamma^{(1)} > 0$) could potentially yield Stam via an integrated comparison argument.

9.6. Lemma status and proof landscape.

Result	Status	Notes
General Fisher dissipation (Lemma 9.1)	Proven	Algebraic
Dilation root ODE (Lemma 9.4)	Proven	Implicit differentiation
Initial acceleration $\propto$ score (Prop. 9.5)	Proven	Second-order expansion
Initial curvature $F''(0)$ formula (Prop. 9.9)	Proven	From acceleration formula
$\Gamma^{(1)} > 0$ for $n = 3$ (Prop. 9.10)	Proven	Palindromic positivity
$\Gamma^{(1)} > 0$ for general n (Conj. 9.11)	Plausible	0 violations in 2500+ tests
Score alignment $\alpha > 0$ (Conj. 9.14)	Plausible	0 violations in 1200+ tests
$\mathcal{D}_\perp \leq 0$ (Conj. 9.15)	Plausible	Observed universally
Repulsion monotonicity (Conj. 9.16)	Plausible	0 violations in 800 tests
Pointwise dilation Stam (Conj. 9.18)	Plausible	0 violations in 3700+ tests
$F'(t) \geq 2t/\Phi_n(q)$	False	$\sim 3\%$ failures at $n = 3$
$F'(t)/(2t)$ non-decreasing	False	Fails almost always
SGI-based pointwise bound	False	$\sim 50\%$ failures
Dilation $\geq$ Hermite for $1/\Phi_n$	False	$\sim 10\%$ failures

9.7. Discussion and next steps for Route B.

(B1) The initial acceleration formula (Proposition 9.5) identifies a deep structural similarity between the dilation perturbation and the Hermite flow: both drive roots proportionally to the score V_i. The difference is that dilation introduces *higher cumulants* of q at higher orders in t, creating a perpendicular correction ε_i.

(B2) The fact that $\mathcal{D}_\perp \leq 0$ (Conjecture 9.15) means the perpendicular correction always *slows* the Fisher decay compared to pure Hermite flow; crucially, it does not reverse it. A proof of this sign condition would immediately yield Conjecture 9.16 (repulsion monotonicity).

(B3) The pointwise dilation Stam (Conjecture 9.18) is strictly stronger than the Stam inequality and provides a natural one-parameter strengthening. A proof would benefit from the convexity of the reference curve $t \mapsto t^2/\Phi_n(q)$ combined with the initial conditions $E(0) = E'(0) = 0$ and $E''(0) > 0$ (from $\Gamma^{(1)} > 0$).

(B4) For a minimal proof of Stam, one could try to show: (a) $F'(t) \geq 0$ (Conjecture 9.16), (b) $\int_0^1 F'(t)\,dt \geq 1/\Phi_n(q)$ using a *comparison principle* along the path rather than a pointwise bound (since $F'(t) \geq 2t/\Phi_n(q)$ fails). One possible comparison: exploit the fact that $F'(t)$ "front-loads" its mass near $t = 1$ (observed numerically), so the integral exceeds $1/\Phi_n(q)$ even when the pointwise bound fails at intermediate t.

(B5) The Hermite comparison fails ($1/\Phi_n(r_t)$ can be less than $1/\Phi_n(p \boxplus_n G_{t^2\sigma^2(q)})$), so a direct reduction to the Hermite bound (Theorem 5.5) is insufficient. However, the Hermite bound *does* give $1/\Phi_n(r_1) \geq (a+b)/(a\Phi_n(p))$ in the Gaussian case, and combining this with information about non-Gaussian corrections is a viable path.

Appendix A. A numerical counterexample to dilation convexity

We record an explicit brute-force example showing that neither $t \mapsto 1/\Phi_n(r_t)$ nor the dilation excess $E(t)$ must be convex.

Computational protocol. All computations were done in Python 3 with `numpy` (64-bit IEEE 754). We approximate $F''(t)$ by the centered stencil $\big(F(t+h) - 2F(t) + F(t-h)\big)/h^2$ on a uniform 200-point grid in $[0,1]$, compute $r_t = p \boxplus_n q_t$ from Definition 1.2, and extract roots with `numpy.roots`.

Real-rootedness is checked by $|\operatorname{Im}\lambda| < 10^{-8}$ (well above observed noise $\sim 10^{-15}$). Scripts are available in the cited repository.

Example 1 ($n = 3$). p with roots $(-2, -\frac{3}{2}, \frac{3}{2})$ and q with roots $(-5, 2, 3)$ (so q is centered). Along the dilation path $r_t = p \boxplus_3 q_t$, define $F(t) = 1/\Phi_3(r_t)$ and $E(t) = F(t) - 1/\Phi_3(p) - t^2/\Phi_3(q)$.

A finite-difference computation (step $h = 10^{-5}$, checked on the full 200-point grid) yields a negative second derivative:

$$F''(t^*) \approx -8.16 \quad \text{at } t^* \approx 0.435.$$

Since $2/\Phi_3(q) \approx 0.965$, this also forces $E''(t^*) \approx -9.12 < 0$. Nevertheless $E(1) \approx 2.18 > 0$, so the Stam inequality holds in this example.

Raw convexity $F''(t) \geq 0$ fails in higher degrees as well.

Example 2 ($n = 4$). Take p with roots $(-1.10743, -0.81774, -0.36839, 0.42118)$ and q with centered roots $(-1.57864, -1.22305, -0.93765, 3.73934)$. A finite-difference computation (step size $h = 2 \cdot 10^{-4}$) gives $F''(0.3) \approx -0.14$ (and already $F''(0.2) \approx -0.12$), so $t \mapsto 1/\Phi_4(r_t)$ need not be convex.

This appendix is included to prevent overfitting the analysis to a false convexity narrative.

Appendix B. Bibliographic notes

References

[1] A. Marcus, D. A. Spielman, and N. Srivastava, *Interlacing families II: Mixed characteristic polynomials and the Kadison–Singer problem*, Ann. of Math. **182** (2015), 327–350.

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Data availability. Supplementary Python scripts and further numerical experiments are available at `https://github.com/omegaestable/math-docs`. Scripts include: `route_a_core.py` (core library), `route_a_experiments.py` (Route A proxies), `test_repulsion_stam.py` (large-scale Stam tests), `route_b_experiments.py` (Route B dilation path experiments), `route_b_deep.py` (Route B targeted follow-ups and stress tests), `route_c_experiments.py` (Route C transport experiments), `route_c_epi_deep.py` (discriminant power inequality stress tests), `route_c_n3_epi_proof.py` (n=3 EPI proof verification).