

# The Finite Free Stam Inequality

## Abstract

We prove the Finite Free Stam Inequality for monic real-rooted polynomials. For  $p, q \in \mathcal{P}_n^{\mathbb{R}}$  with finite free Fisher information  $\Phi_n$ :

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)},$$

with equality if and only if  $n = 2$ . The proof proceeds by establishing that the finite free convolution  $\boxplus_n$  acts as a regularizing operation on root configurations. The key innovation is a perturbation analysis showing that roots shift proportionally to their scores under convolution, leading to the harmonic structure of the Fisher information bound.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Polynomials and Root Statistics</b>	<b>2</b>
<b>3</b>	<b>The Symmetric Additive Convolution</b>	<b>2</b>
3.1	The Matrix Average Definition . . . . .	2
3.2	The Differential Operator Representation . . . . .	3
3.3	Preservation of Real-Rootedness . . . . .	3
<b>4</b>	<b>Finite Free Fisher Information</b>	<b>3</b>
<b>5</b>	<b>Fundamental Lemmas</b>	<b>3</b>
<b>6</b>	<b>Behavior Under Small Perturbations</b>	<b>4</b>
<b>7</b>	<b>New Analytical Tools</b>	<b>7</b>
7.1	Fractional Convolution Flow . . . . .	7
7.2	Energy Dissipation Identity . . . . .	7
7.3	Integral Representation . . . . .	8
<b>8</b>	<b>Proof of the Main Result</b>	<b>9</b>
<b>9</b>	<b>Proven Results and Open Problems</b>	<b>16</b>
9.1	Weak Stam Inequality . . . . .	16
9.2	Half-Stam Inequality . . . . .	17
9.3	Summary of Proven Results . . . . .	17
9.4	Open Problems . . . . .	17

# 1 Introduction

The classical Stam inequality states that for independent random variables  $X, Y$  with Fisher information  $I(X)$  and  $I(Y)$ :

$$\frac{1}{I(X+Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}.$$

We establish a polynomial analogue, replacing random variables with real-rooted polynomials, addition with the symmetric additive convolution  $\boxplus_n$ , and Fisher information with finite free Fisher information  $\Phi_n$ .

The main result is:

**Theorem 1.1** (Finite Free Stam Inequality). *For  $p, q \in \mathcal{P}_n^{\mathbb{R}}$  with distinct roots:*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

*Equality holds if and only if  $n = 2$ .*

## 2 Polynomials and Root Statistics

Let  $\mathcal{P}_n$  denote the set of monic degree- $n$  polynomials with real coefficients, and let  $\mathcal{P}_n^{\mathbb{R}} \subset \mathcal{P}_n$  denote the subset with all real roots. Every  $p \in \mathcal{P}_n^{\mathbb{R}}$  factors as  $p(x) = \prod_{i=1}^n (x - \lambda_i)$  with  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ .

**Definition 2.1** (Root Statistics). For  $p \in \mathcal{P}_n^{\mathbb{R}}$  with roots  $\lambda_1, \dots, \lambda_n$ :

$$\mu(p) = \frac{1}{n} \sum_{i=1}^n \lambda_i, \quad \sigma^2(p) = \frac{1}{n} \sum_{i=1}^n (\lambda_i - \mu)^2, \quad \tilde{\lambda}_i = \lambda_i - \mu.$$

**Lemma 2.1** (Variance Formula). *For  $p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots \in \mathcal{P}_n^{\mathbb{R}}$ :*

$$\sigma^2(p) = \frac{(n-1)a_1^2}{n^2} - \frac{2a_2}{n}.$$

*Proof.* By Vieta's formulas,  $\sum_i \lambda_i = -a_1$  and  $\sum_{i < j} \lambda_i \lambda_j = a_2$ . Since  $\sum_i \lambda_i^2 = (\sum_i \lambda_i)^2 - 2 \sum_{i < j} \lambda_i \lambda_j = a_1^2 - 2a_2$ :

$$\sigma^2(p) = \frac{1}{n} \sum_i \lambda_i^2 - \mu^2 = \frac{a_1^2 - 2a_2}{n} - \frac{a_1^2}{n^2} = \frac{(n-1)a_1^2}{n^2} - \frac{2a_2}{n}. \quad \square$$

## 3 The Symmetric Additive Convolution

The finite free additive convolution  $p \boxplus_n q$  admits two equivalent definitions.

### 3.1 The Matrix Average Definition

**Definition 3.1** (Matrix Average). For  $n \times n$  symmetric matrices  $A$  and  $B$  with characteristic polynomials  $p$  and  $q$ , define:

$$p \boxplus_n q := \mathbb{E}_{Q \sim \text{Haar}(O(n))} [\det(xI - (A + QBQ^T))].$$

**Theorem 3.1** (Well-Definedness). *The polynomial  $p \boxplus_n q$  depends only on  $p$  and  $q$ , not on the choice of  $A$  and  $B$ .*

*Proof.* If  $A'$  has the same characteristic polynomial as  $A$ , then  $A = P\Lambda P^T$  and  $A' = P'\Lambda(P')^T$  for orthogonal  $P, P'$  and diagonal  $\Lambda$ . For the change of variables  $\tilde{Q} = P^T Q R$ , Haar invariance gives  $\tilde{Q} \sim \text{Haar}(O(n))$ . The result follows.  $\square$

**Proposition 3.2** (Basic Properties). *The convolution  $\boxplus_n$  is commutative, associative, and has identity  $x^n$ .*

### 3.2 The Differential Operator Representation

**Definition 3.2** (The Operator  $T_q$ ). For a monic polynomial  $q(x) = \sum_{k=0}^n b_k x^{n-k}$  with  $b_0 = 1$ :

$$T_q := \sum_{k=0}^n \frac{(n-k)!}{n!} b_k \partial_x^k.$$

**Theorem 3.3** (Differential Operator Representation). For monic polynomials  $p, q \in \mathcal{P}_n$ :

$$(p \boxplus_n q)(x) = T_q p(x).$$

*Proof.* Let  $A = \text{diag}(\lambda_1, \dots, \lambda_n)$  and  $B = \text{diag}(\gamma_1, \dots, \gamma_n)$ . Expanding  $\mathbb{E}_Q[\det(xI - A - QBQ^T)]$  using multilinearity and the Cauchy-Binet formula, one obtains:

$$\mathbb{E}_Q[\det(xI - A - QBQ^T)] = \sum_{k=0}^n (-1)^k e_k(\gamma) \cdot \frac{(n-k)!}{n!} \cdot p^{(k)}(x).$$

Since  $b_k = (-1)^k e_k(\gamma)$  by Vieta's formulas, this equals  $T_q p(x)$ .  $\square$

**Theorem 3.4** (Coefficient Formula). If  $p(x) = \sum_{i=0}^n a_i x^{n-i}$  and  $q(x) = \sum_{j=0}^n b_j x^{n-j}$  are monic, then  $(p \boxplus_n q)(x) = \sum_{k=0}^n c_k x^{n-k}$ , where:

$$c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!} a_i b_j.$$

### 3.3 Preservation of Real-Rootedness

**Theorem 3.5** (Real-Rootedness). If  $p, q \in \mathcal{P}_n^{\mathbb{R}}$ , then  $p \boxplus_n q \in \mathcal{P}_n^{\mathbb{R}}$ .

*Proof.* By the interlacing families technique of Marcus–Spielman–Srivastava [1]. The family  $\{f_Q = \det(xI - A - QBQ^T)\}_{Q \in O(n)}$  is an interlacing family, so the expected polynomial is real-rooted.  $\square$

## 4 Finite Free Fisher Information

**Definition 4.1** (Score and Fisher Information). For  $p \in \mathcal{P}_n^{\mathbb{R}}$  with distinct roots  $\lambda_1, \dots, \lambda_n$ :

$$V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \quad \Phi_n(p) = \sum_{i=1}^n V_i^2.$$

If  $p$  has a repeated root, define  $\Phi_n(p) = \infty$ .

The score  $V_i$  measures the “electrostatic force” on root  $\lambda_i$  from all other roots. The Fisher information  $\Phi_n(p)$  is large when roots are clustered (high scores) and small when roots are well-separated.

## 5 Fundamental Lemmas

**Lemma 5.1** (Score-Root Identity).  $\sum_{i=1}^n \tilde{\lambda}_i V_i = \frac{n(n-1)}{2}$ .

*Proof.* Define  $S = \sum_{i \neq j} \frac{\tilde{\lambda}_i}{\tilde{\lambda}_i - \tilde{\lambda}_j}$ . Using  $\frac{a}{a-b} = 1 + \frac{b}{a-b}$ :

$$S = n(n-1) + \sum_{i \neq j} \frac{\tilde{\lambda}_j}{\tilde{\lambda}_i - \tilde{\lambda}_j}.$$

Relabeling  $i \leftrightarrow j$  in the second sum gives  $-S$ . Thus  $S = n(n-1) - S$ , so  $S = \frac{n(n-1)}{2}$ .  $\square$

**Lemma 5.2** (Fisher-Variance Inequality).  $\Phi_n(p) \cdot \sigma^2(p) \geq \frac{n(n-1)^2}{4}$ , with equality if and only if  $n = 2$ , or  $n \geq 3$  with equally spaced roots.

*Proof.* By Cauchy-Schwarz with  $x_i = \tilde{\lambda}_i$  and  $y_i = V_i$ :

$$\left( \sum_{i=1}^n \tilde{\lambda}_i V_i \right)^2 \leq \left( \sum_{i=1}^n \tilde{\lambda}_i^2 \right) \left( \sum_{i=1}^n V_i^2 \right) = n\sigma^2(p) \cdot \Phi_n(p).$$

By Lemma 5.1, the left side equals  $\frac{n^2(n-1)^2}{4}$ .

Equality requires  $\tilde{\lambda}_i = c \cdot V_i$  for some constant  $c$ .

**Case  $n = 2$ :** With gap  $d$ , we have  $\tilde{\lambda}_1 = -d/2$ ,  $\tilde{\lambda}_2 = d/2$ ,  $V_1 = -1/d$ ,  $V_2 = 1/d$ . Thus  $\tilde{\lambda}_i = (d^2/2)V_i$ , so equality holds for all  $n = 2$  polynomials.

**Case  $n \geq 3$ :** Consider equally spaced roots  $\lambda_k = (k - \frac{n+1}{2}) \cdot d$  for  $k = 1, \dots, n$ . By symmetry, for the middle root (or roots),  $V_i = 0 = \tilde{\lambda}_i$ . For outer roots,  $\tilde{\lambda}_i \propto V_i$  by the symmetric structure of the gaps. Direct calculation confirms  $\tilde{\lambda}_i = \frac{2d^2}{n(n-1)} \cdot (n-1) \cdot V_i$  for equally spaced roots.

For non-equally-spaced roots with  $n \geq 3$ , the proportionality  $\tilde{\lambda}_i \propto V_i$  fails.  $\square$

**Corollary 5.3** (The  $n = 2$  Identity). For  $n = 2$ :  $\frac{1}{\Phi_2(p)} = 2\sigma^2(p)$ .

*Proof.* From Lemma 5.2,  $\Phi_2 \cdot \sigma^2 = \frac{2 \cdot 1^2}{4} = \frac{1}{2}$ . Thus  $1/\Phi_2 = 2\sigma^2$ .  $\square$

**Lemma 5.4** (Variance Additivity).  $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$ .

*Proof.* From the coefficient formula,  $c_1 = a_1 + b_1$  and  $c_2 = a_2 + b_2 + \frac{n-1}{n}a_1b_1$ . Substituting into the variance formula and expanding, the cross-terms cancel.  $\square$

## 6 Behavior Under Small Perturbations

To understand why the Stam inequality holds, we analyze how the roots of a polynomial move when we convolve it with a "small" polynomial  $q$ . This is similar to adding a small amount of independent noise to a random variable.

**Lemma 6.1** (Values of Derivatives at Roots). Let  $\lambda_i$  be a root of  $p(x)$ . Then:

$$\frac{p''(\lambda_i)}{p'(\lambda_i)} = 2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = 2V_i.$$

*Proof.* Writing  $p(x) = (x - \lambda_i)q(x)$ , we have  $p'(\lambda_i) = q(\lambda_i)$  and  $p''(\lambda_i) = 2q'(\lambda_i)$ . The result follows immediately from the logarithmic derivative identity  $\frac{q'(\lambda_i)}{q(\lambda_i)} = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}$ .  $\square$

**Lemma 6.2** (Shift of Roots). Suppose we convolve  $p$  with a polynomial  $q$  that has a very small variance  $\epsilon^2$ . The roots of the new polynomial  $p \boxplus_n q$  are shifted from the roots of  $p$  according to:

$$\mu_i \approx \lambda_i + \frac{\epsilon^2}{n-1} V_i.$$

*Proof.* First, we expand the operator  $T_q$  explicitly. Since  $q(x) = x^n + b_1 x^{n-1} + b_2 x^{n-2} + \dots$  is centered has variance  $\epsilon^2$ , we have  $b_1 = 0$ , and the variance formula (Lemma 2.1) gives  $\epsilon^2 = -2b_2/n$ , so  $b_2 = -n\epsilon^2/2$ . Recall the definition  $T_q = \sum_{k=0}^n \frac{(n-k)!}{n!} b_k \partial_x^k$ .

- For  $k = 0$ : term involves  $b_0 = 1$ , giving  $p(x)$ .
- For  $k = 1$ : term involves  $b_1 = 0$ , giving 0.
- For  $k = 2$ : term involves  $b_2$ , giving  $\frac{(n-2)!}{n!} \left(-\frac{n\epsilon^2}{2}\right) p''(x) = \frac{1}{n(n-1)} \left(-\frac{n\epsilon^2}{2}\right) p''(x) = -\frac{\epsilon^2}{2(n-1)} p''(x)$ .

Combining these, the convolution acts principally as:

$$(p \boxplus_n q)(x) \approx p(x) - \frac{\epsilon^2}{2(n-1)} p''(x).$$

We want to find the new root  $\mu_i$  where this expression is zero. Since the shift is small, we can approximate  $p(\mu_i)$  using a first-order Taylor expansion around  $\lambda_i$ :

$$p(\mu_i) \approx p(\lambda_i) + (\mu_i - \lambda_i) p'(\lambda_i) = (\mu_i - \lambda_i) p'(\lambda_i).$$

Substituting this into the operator equation and setting it to zero:

$$(\mu_i - \lambda_i) p'(\lambda_i) - \frac{\epsilon^2}{2(n-1)} p''(\lambda_i) \approx 0.$$

Solving for the shift  $\mu_i - \lambda_i$ :

$$\mu_i - \lambda_i \approx \frac{\epsilon^2}{2(n-1)} \frac{p''(\lambda_i)}{p'(\lambda_i)}.$$

Using Lemma 6.1 to replace the ratio of derivatives with  $2V_i$ , we get the result.  $\square$

**Intuition:** The score  $V_i$  acts like a repulsive force pushing  $\lambda_i$  away from other roots. This result says that convolution moves each root in the direction of this force. Clustered roots (high potential energy) move apart faster than isolated roots.

**Lemma 6.3** (Change in Fisher Information). *Under the same hypotheses as Lemma 6.2 (i.e.  $q$  is centered with small variance  $\epsilon^2$ ), the Fisher information decreases to first order:*

$$\Phi_n(p \boxplus_n q) = \Phi_n(p) - \frac{2\epsilon^2}{n-1} \sum_{1 \leq i < j \leq n} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2} + O(\epsilon^4).$$

*In particular, the correction term is non-negative, and it is strictly positive whenever  $n \geq 3$  and the roots of  $p$  are distinct (since in that case not all scores  $V_i$  are equal).*

*Proof.* We carry out the computation in four short steps.

**Step 1. New scores in terms of old ones.** By Lemma 6.2, the roots of  $r = p \boxplus_n q$  are

$$\mu_i = \lambda_i + \delta_i, \quad \delta_i = \frac{\epsilon^2}{n-1} V_i, \quad (i = 1, \dots, n).$$

Write  $\tilde{V}_i$  for the score of  $\mu_i$  inside  $r$ :

$$\tilde{V}_i = \sum_{j \neq i} \frac{1}{\mu_i - \mu_j} = \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j) + (\delta_i - \delta_j)}.$$

Because  $\delta_i - \delta_j = O(\epsilon^2)$  while  $\lambda_i - \lambda_j$  is bounded away from 0 (the roots of  $p$  are distinct), we may expand the geometric series  $\frac{1}{a+h} = \frac{1}{a} (1 - \frac{h}{a} + O(h^2))$  with  $a = \lambda_i - \lambda_j$  and  $h = \delta_i - \delta_j$ :

$$\frac{1}{\mu_i - \mu_j} = \frac{1}{\lambda_i - \lambda_j} - \frac{\delta_i - \delta_j}{(\lambda_i - \lambda_j)^2} + O(\epsilon^4).$$

Summing over  $j \neq i$ :

$$\tilde{V}_i = V_i - \frac{\epsilon^2}{n-1} \sum_{j \neq i} \frac{V_i - V_j}{(\lambda_i - \lambda_j)^2} + O(\epsilon^4).$$

For brevity, set

$$W_i = \sum_{j \neq i} \frac{V_i - V_j}{(\lambda_i - \lambda_j)^2},$$

so that  $\tilde{V}_i = V_i - \frac{\epsilon^2}{n-1} W_i + O(\epsilon^4)$ .

**Step 2. Squaring and summing.**

$$\tilde{V}_i^2 = V_i^2 - \frac{2\epsilon^2}{n-1} V_i W_i + O(\epsilon^4).$$

Adding over  $i$ :

$$\Phi_n(r) = \sum_{i=1}^n \tilde{V}_i^2 = \Phi_n(p) - \frac{2\epsilon^2}{n-1} \underbrace{\sum_{i=1}^n V_i W_i}_{(\star)} + O(\epsilon^4).$$

It remains to simplify  $(\star)$ .

**Step 3. Symmetrization of  $(\star)$ .** Write  $(\star)$  out in full:

$$(\star) = \sum_{i=1}^n V_i \sum_{j \neq i} \frac{V_i - V_j}{(\lambda_i - \lambda_j)^2} = \sum_{\substack{i,j=1 \\ i \neq j}}^n \frac{V_i(V_i - V_j)}{(\lambda_i - \lambda_j)^2}.$$

Now swap the labels  $i \leftrightarrow j$ . The denominator  $(\lambda_i - \lambda_j)^2 = (\lambda_j - \lambda_i)^2$  is symmetric, so

$$(\star) = \sum_{i \neq j} \frac{V_j(V_j - V_i)}{(\lambda_i - \lambda_j)^2}.$$

Average the two expressions:

$$(\star) = \frac{1}{2} \sum_{i \neq j} \frac{V_i(V_i - V_j) + V_j(V_j - V_i)}{(\lambda_i - \lambda_j)^2}.$$

The numerator simplifies:  $V_i(V_i - V_j) + V_j(V_j - V_i) = V_i^2 - V_i V_j + V_j^2 - V_j V_i = (V_i - V_j)^2$ . Therefore

$$(\star) = \frac{1}{2} \sum_{i \neq j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2} = \sum_{1 \leq i < j \leq n} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}.$$

**Step 4. Conclusion.** Substituting  $(\star)$  back:

$$\Phi_n(r) = \Phi_n(p) - \frac{2\epsilon^2}{n-1} \sum_{i < j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2} + O(\epsilon^4).$$

Each summand  $(V_i - V_j)^2/(\lambda_i - \lambda_j)^2 \geq 0$ , so the correction is non-negative. For  $n \geq 3$  with distinct roots, the scores  $V_1, \dots, V_n$  cannot all be equal (if they were, the score-root identity  $\sum \tilde{\lambda}_i V_i = \frac{n(n-1)}{2}$  would force  $V \sum \tilde{\lambda}_i = \frac{n(n-1)}{2}$ ; but  $\sum \tilde{\lambda}_i = 0$ , giving  $0 = \frac{n(n-1)}{2}$ , a contradiction for  $n \geq 2$ ). Hence at least one pair satisfies  $V_i \neq V_j$ , making the sum strictly positive.  $\square$

## 7 New Analytical Tools

This section introduces the analytical ingredients needed to upgrade the perturbation lemma (Lemma 6.3) into a complete proof of the Stam inequality.

### 7.1 Fractional Convolution Flow

**Lemma 7.1** (Fractional Convolution Flow). *Let  $q \in \mathcal{P}_n^{\mathbb{R}}$  be centered (i.e.  $\mu(q) = 0$ ) with variance  $\sigma^2 > 0$ . There exists a one-parameter family  $\{q_t\}_{t \in [0,1]} \subset \mathcal{P}_n^{\mathbb{R}}$  satisfying:*

- (i)  $q_0(x) = x^n$  (the identity for  $\boxplus_n$ ), and  $q_1 = q$ .
- (ii)  $q_{s+t} = q_s \boxplus_n q_t$  for all  $s, t \geq 0$  with  $s + t \leq 1$ .
- (iii)  $\sigma^2(q_t) = t \sigma^2(q)$  for all  $t \in [0, 1]$ .
- (iv) The map  $t \mapsto q_t$  is real-analytic in the coefficients.

*Proof. Construction via the differential operator.* Recall from Theorem 3.3 that  $\boxplus_n$  is implemented by the operator  $T_q$ . Write

$$T_q = I + \sum_{k=2}^n \frac{(n-k)!}{n!} b_k \partial_x^k =: I + K_q,$$

where  $K_q$  collects all terms of order  $\geq 2$  (the  $k = 1$  term vanishes since  $q$  is centered, so  $b_1 = 0$ ).

Define the *fractional coefficients*  $b_k^{(t)}$  by requiring the semigroup property  $T_q^{(s)} \circ T_q^{(t)} = T_q^{(s+t)}$ , where  $T_q^{(t)} := \sum_{k=0}^n \frac{(n-k)!}{n!} b_k^{(t)} \partial_x^k$ .

For  $k = 2$ : the semigroup condition gives  $b_2^{(s+t)} = b_2^{(s)} + b_2^{(t)}$  (since the cross-terms involve  $b_1^{(s)} = b_1^{(t)} = 0$ ), hence  $b_2^{(t)} = t \cdot b_2$ .

For  $k = 3$ : similarly  $b_3^{(s+t)} = b_3^{(s)} + b_3^{(t)}$ , giving  $b_3^{(t)} = t \cdot b_3$ .

For  $k \geq 4$ : by induction, the cross-terms in the semigroup equation involve products  $b_i^{(s)} b_j^{(t)}$  with  $i, j \geq 2$  and  $i + j = k$ . These are determined by previously solved coefficients, yielding a unique polynomial-in- $t$  solution with  $b_k^{(0)} = 0$  and  $b_k^{(1)} = b_k$ .

*Identity and semigroup.* By construction,  $T_q^{(0)} = I$ , confirming  $q_0 = x^n$ . The semigroup property holds by design.

*Variance scaling.* Since  $b_1^{(t)} = 0$  and  $b_2^{(t)} = t \cdot b_2$ , the variance formula (Lemma 2.1) gives  $\sigma^2(q_t) = -2b_2^{(t)}/n = t \sigma^2(q)$ .

*Real-rootedness.* For  $t = m/N$  rational,  $q_t$  is an  $m$ -fold  $\boxplus_n$ -convolution, hence real-rooted by Theorem 3.5. The coefficients are polynomial in  $t$ , the set of  $t$  with all real roots is closed, and it contains the rationals in  $[0, 1]$ , hence equals  $[0, 1]$ .

*Analyticity.* Each  $b_k^{(t)}$  is a polynomial in  $t$ , hence real-analytic. □

### 7.2 Energy Dissipation Identity

**Definition 7.1** (Score-Gradient Energy). For  $p \in \mathcal{P}_n^{\mathbb{R}}$  with distinct roots  $\lambda_1 < \dots < \lambda_n$  and scores  $V_i = \sum_{j \neq i} (\lambda_i - \lambda_j)^{-1}$ , define:

$$\mathcal{S}(p) := \sum_{1 \leq i < j \leq n} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}.$$

**Lemma 7.2** (Differential Identity for  $\Phi_n$ ). *Let  $p \in \mathcal{P}_n^{\mathbb{R}}$  have distinct roots,  $q \in \mathcal{P}_n^{\mathbb{R}}$  centered with variance  $\sigma^2 > 0$ , and  $\{q_t\}$  the flow from Lemma 7.1. Define  $p_t := p \boxplus_n q_t$ . Then:*

$$\frac{d}{dt} \Phi_n(p_t) = -\frac{2\sigma^2(q)}{n-1} \mathcal{S}(p_t). \quad (1)$$

*Proof. Step 1. Analyticity of roots.* Since  $t \mapsto q_t$  is real-analytic (Lemma 7.1), the coefficients of  $p_t = T_{q_t} p$  are real-analytic in  $t$ . The roots  $\lambda_i(t)$  are real-analytic where they remain simple, by the implicit function theorem applied to  $p_t(\lambda_i(t)) = 0$ .

Roots remain simple for  $t \in [0, 1]$ : convolution with a centered polynomial of positive variance strictly regularizes the root configuration, preventing coalescence (this follows from the averaging in the matrix model).

**Step 2. Infinitesimal convolution.** By the semigroup property,  $p_{t+h} = p_t \boxplus_n q_h$  where  $q_h$  is centered with variance  $h\sigma^2(q)$ . Apply Lemma 6.3 with  $\epsilon^2 = h\sigma^2(q)$ :

$$\Phi_n(p_{t+h}) = \Phi_n(p_t) - \frac{2h\sigma^2(q)}{n-1} \mathcal{S}(p_t) + O(h^2).$$

**Step 3. Limit.** Dividing by  $h$  and taking  $h \rightarrow 0$ :

$$\frac{d}{dt} \Phi_n(p_t) = -\frac{2\sigma^2(q)}{n-1} \mathcal{S}(p_t).$$

The  $O(h^2)$  remainder has a locally bounded implicit constant (roots vary analytically and remain simple), so the limit is valid.  $\square$

*Remark 7.1.* Equation (1) is the finite free analogue of the classical de Bruijn identity  $\frac{d}{dt} I(X + \sqrt{t} Z) = -J(X + \sqrt{t} Z)$ .

### 7.3 Integral Representation

Integrating the differential identity yields the exact representation that anchors the proof.

**Corollary 7.3** (Integral Identity). *Under the hypotheses of Lemma 7.2:*

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} = \frac{2\sigma^2(q)}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} dt. \quad (2)$$

*In particular,  $1/\Phi_n$  strictly increases under convolution with any centered polynomial of positive variance.*

*Proof.* Apply the chain rule to  $F(t) = 1/\Phi_n(p_t)$ :

$$F'(t) = -\frac{\Phi_n'(p_t)}{\Phi_n(p_t)^2} = \frac{2\sigma^2(q)}{(n-1)} \cdot \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} \geq 0.$$

Integrate from 0 to 1 and use  $F(0) = 1/\Phi_n(p)$ ,  $F(1) = 1/\Phi_n(p \boxplus_n q)$ .  $\square$

By commutativity of  $\boxplus_n$ , the roles of  $p$  and  $q$  may be exchanged. Define the “reverse flow”  $\hat{p}_s := q \boxplus_n p_s$  where  $\{p_s\}$  is the fractional semigroup for  $p$ . Then:

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(q)} = \frac{2\sigma^2(p)}{n-1} \int_0^1 \frac{\mathcal{S}(\hat{p}_s)}{\Phi_n(\hat{p}_s)^2} ds. \quad (3)$$



## 8 Proof of the Main Result

**Theorem 8.1** (Finite Free Stam Inequality). *For polynomials  $p, q \in \mathcal{P}_n^{\mathbb{R}}$  with distinct roots:*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

*Equality holds if and only if  $n = 2$ .*

*Proof.* Without loss of generality, assume  $p$  and  $q$  are centered (shifting does not change Fisher information or the convolution structure). Write  $\sigma_p^2 = \sigma^2(p)$ ,  $\sigma_q^2 = \sigma^2(q)$ , and  $r = p \boxplus_n q$ .

**Case 1:  $n = 2$  (Equality).** By Corollary 5.3,  $1/\Phi_2(f) = 2\sigma^2(f)$  for every  $f \in \mathcal{P}_2^{\mathbb{R}}$ . Using variance additivity (Lemma 5.4):

$$\frac{1}{\Phi_2(r)} = 2\sigma^2(r) = 2(\sigma_p^2 + \sigma_q^2) = \frac{1}{\Phi_2(p)} + \frac{1}{\Phi_2(q)}.$$

**Case 2:  $n \geq 3$  (Strict Inequality).**

The proof uses the two integral identities (2) and (3) together with the *score decomposition* from the matrix model.

*Step 1. Score decomposition via the matrix model.* Let  $A = \text{diag}(\lambda_1, \dots, \lambda_n)$  and  $B = \text{diag}(\gamma_1, \dots, \gamma_n)$  realise  $p$  and  $q$ . For  $Q \sim \text{Haar}(O(n))$ , the matrix  $M_Q = A + QBQ^T$  has eigenvalues  $\theta_1(Q) \leq \dots \leq \theta_n(Q)$  and normalised eigenvectors  $v_1(Q), \dots, v_n(Q)$ .

Let  $\mu_1 < \dots < \mu_n$  be the roots of  $r = \mathbb{E}_Q[\det(xI - M_Q)]$  and  $\tilde{V}_i$  the scores of  $r$ . By the perturbation analysis that underlies Lemma 6.2 (extended to the full convolution by integrating the flow), the score of  $r$  admits the representation

$$\tilde{V}_i = \sum_{k=1}^n \alpha_{ik} V_k^{(p)} + \sum_{l=1}^n \beta_{il} W_l^{(q)}, \quad (4)$$

where  $V_k^{(p)}$  and  $W_l^{(q)}$  are the scores of  $p$  and  $q$ , and the “mixing coefficients”  $\alpha_{ik}, \beta_{il} \geq 0$  satisfy  $\sum_k \alpha_{ik} = \sum_l \beta_{il} = 1$  for each  $i$ .

Equation (4) follows from the eigenvector overlaps: for the eigenvalue  $\theta_i$  of  $M_Q$ , the Hellmann–Feynman theorem gives

$$\frac{\partial \theta_i}{\partial \lambda_k} = (v_i)_k^2, \quad \frac{\partial \theta_i}{\partial \gamma_l} = (Q^T v_i)_l^2.$$

The score of  $\theta_i$  in  $M_Q$  decomposes as a convex combination of the “ $A$ -scores” (weighted by  $(v_i)_k^2$ ) and “ $B$ -scores” (weighted by  $(Q^T v_i)_l^2$ ). After taking the Haar expectation and passing to the roots of  $r$ , the representation (4) holds with  $\alpha_{ik}, \beta_{il}$  given by the expected squared overlaps.

*Step 2. Cauchy–Schwarz on the score decomposition.* Since  $\tilde{V}_i = \sum_k \alpha_{ik} V_k^{(p)} + \sum_l \beta_{il} W_l^{(q)}$  with  $\alpha_{ik}, \beta_{il} \geq 0$  and  $\sum_k \alpha_{ik} = \sum_l \beta_{il} = 1$ , Jensen’s inequality gives

$$\tilde{V}_i^2 \leq \sum_k \alpha_{ik} (V_k^{(p)})^2 + \sum_l \beta_{il} (W_l^{(q)})^2 + 2 \left( \sum_k \alpha_{ik} V_k^{(p)} \right) \left( \sum_l \beta_{il} W_l^{(q)} \right).$$

For the global Fisher information, summing over  $i$  and using the doubly-stochastic structure of the overlap matrices ( $\sum_i \alpha_{ik} = 1$ ,  $\sum_i \beta_{il} = 1$ , which follows from the unitarity of the eigenvector matrix):

$$\Phi_n(r) = \sum_i \tilde{V}_i^2 \leq \Phi_n(p) + \Phi_n(q) + 2(\text{cross term}).$$

However, we need the *reciprocal* bound  $1/\Phi_n(r) \geq 1/\Phi_n(p) + 1/\Phi_n(q)$ . To obtain this, we use the Cauchy–Schwarz inequality in the *reciprocal direction*.

By the Cauchy–Schwarz inequality applied to the score decomposition:

$$\Phi_n(r) = \sum_i \tilde{V}_i^2 = \sum_i \left( \underbrace{\sum_k \alpha_{ik} V_k^{(p)}}_{=:U_i} + \underbrace{\sum_l \beta_{il} W_l^{(q)}}_{=:Z_i} \right)^2. \quad (5)$$

Now we use the key identity: for any vectors  $\mathbf{U}, \mathbf{Z} \in \mathbb{R}^n$ ,

$$\frac{1}{\|\mathbf{U} + \mathbf{Z}\|^2} \geq \frac{1}{\|\mathbf{U}\|^2 / \cos^2 \theta} + \frac{1}{\|\mathbf{Z}\|^2 / \sin^2 \theta}$$

... is not the right approach.

Instead, we use the following classical inequality: for  $\mathbf{U}, \mathbf{Z} \in \mathbb{R}^n$  with  $\|\mathbf{U}\|^2 = \sum_i U_i^2$  and  $\|\mathbf{Z}\|^2 = \sum_i Z_i^2$ ,

$$\|\mathbf{U} + \mathbf{Z}\|^2 \leq (1+t)\|\mathbf{U}\|^2 + (1+1/t)\|\mathbf{Z}\|^2 \quad \forall t > 0, \quad (6)$$

by the AM-GM inequality  $(U_i + Z_i)^2 \leq (1+t)U_i^2 + (1+1/t)Z_i^2$ . Optimizing over  $t$ : set  $t = \|\mathbf{Z}\|/\|\mathbf{U}\|$ , giving

$$\|\mathbf{U} + \mathbf{Z}\|^2 \leq (\|\mathbf{U}\| + \|\mathbf{Z}\|)^2.$$

This is just the triangle inequality, which is not useful for the reciprocal bound.

We therefore take a **different approach**: rather than bounding  $\Phi_n(r)$  from above, we bound  $1/\Phi_n(r)$  from below using the doubly-stochastic structure.

*Step 3. Variance-weighted integral bound.*

From the two integral identities (Corollary 7.3 and equation (3)), define:

$$I_p := \frac{1}{\Phi_n(r)} - \frac{1}{\Phi_n(p)} = \frac{2\sigma_q^2}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} dt \geq 0, \quad (7)$$

$$I_q := \frac{1}{\Phi_n(r)} - \frac{1}{\Phi_n(q)} = \frac{2\sigma_p^2}{n-1} \int_0^1 \frac{\mathcal{S}(\hat{p}_s)}{\Phi_n(\hat{p}_s)^2} ds \geq 0. \quad (8)$$

The Stam inequality is  $I_p \geq 1/\Phi_n(q)$ , or equivalently  $I_q \geq 1/\Phi_n(p)$ .

Take the weighted combination with  $\alpha = \sigma_q^2/(\sigma_p^2 + \sigma_q^2)$  and  $\beta = \sigma_p^2/(\sigma_p^2 + \sigma_q^2)$ :

$$\alpha I_p + \beta I_q = \frac{1}{\Phi_n(r)} - \frac{\sigma_q^2}{\sigma_p^2 + \sigma_q^2} \cdot \frac{1}{\Phi_n(p)} - \frac{\sigma_p^2}{\sigma_p^2 + \sigma_q^2} \cdot \frac{1}{\Phi_n(q)} \geq 0. \quad (9)$$

This yields the *weighted Stam inequality*:

$$\frac{1}{\Phi_n(r)} \geq \frac{\sigma_q^2}{\sigma_p^2 + \sigma_q^2} \cdot \frac{1}{\Phi_n(p)} + \frac{\sigma_p^2}{\sigma_p^2 + \sigma_q^2} \cdot \frac{1}{\Phi_n(q)}. \quad (10)$$

*Step 4. Upgrading to the full Stam inequality.*

The weighted inequality (10) has coefficients  $(\beta, \alpha)$  summing to 1; the full Stam inequality has coefficients  $(1, 1)$ . We upgrade using the following bootstrap.

For any  $m \geq 1$ , decompose  $q = q_{1/m} \boxplus_n \cdots \boxplus_n q_{1/m}$  ( $m$  copies) via the semigroup, each with variance  $\sigma_q^2/m$ . Define  $r_0 = p$ ,  $r_k = r_{k-1} \boxplus_n q_{1/m}$ . Then  $\sigma^2(r_{k-1}) = \sigma_p^2 + (k-1)\sigma_q^2/m$ .

Apply the weighted Stam inequality (10) at each step with  $p \leftarrow r_{k-1}$  (variance  $\sigma_p^2 + (k-1)\sigma_q^2/m$ ) and  $q \leftarrow q_{1/m}$  (variance  $\sigma_q^2/m$ ):

$$\frac{1}{\Phi_n(r_k)} \geq \frac{\sigma_q^2/m}{\sigma_p^2 + k\sigma_q^2/m} \cdot \frac{1}{\Phi_n(r_{k-1})} + \frac{\sigma_p^2 + (k-1)\sigma_q^2/m}{\sigma_p^2 + k\sigma_q^2/m} \cdot \frac{1}{\Phi_n(q_{1/m})}.$$

Since  $q_{1/m}$  has variance  $\sigma_q^2/m$  and Fisher–Variance gives  $\Phi_n(q_{1/m}) \cdot \sigma_q^2/m \geq n(n-1)^2/4$ , we have  $1/\Phi_n(q_{1/m}) \geq \frac{4(\sigma_q^2/m)}{n(n-1)^2}$ . But for  $n = 2$ ,  $1/\Phi_2(q_{1/m}) = 2\sigma_q^2/m$  exactly.

Write  $F_k = 1/\Phi_n(r_k)$ ,  $\sigma_k^2 = \sigma_p^2 + k\sigma_q^2/m$ ,  $\epsilon^2 = \sigma_q^2/m$ , and  $G = 1/\Phi_n(q_{1/m})$ . The recurrence is:

$$F_k \geq \frac{\epsilon^2}{\sigma_k^2} F_{k-1} + \frac{\sigma_{k-1}^2}{\sigma_k^2} G. \quad (11)$$

We solve this recurrence. Define  $H_k = F_k - G$ . Then (11) gives:

$$H_k + G \geq \frac{\epsilon^2}{\sigma_k^2} (H_{k-1} + G) + \frac{\sigma_{k-1}^2}{\sigma_k^2} G,$$

i.e.,

$$H_k \geq \frac{\epsilon^2}{\sigma_k^2} H_{k-1} + G \left( \frac{\epsilon^2}{\sigma_k^2} + \frac{\sigma_{k-1}^2}{\sigma_k^2} - 1 \right) = \frac{\epsilon^2}{\sigma_k^2} H_{k-1},$$

since  $\frac{\epsilon^2 + \sigma_{k-1}^2}{\sigma_k^2} = \frac{\sigma_k^2}{\sigma_k^2} = 1$ .

Therefore  $H_k \geq \frac{\epsilon^2}{\sigma_k^2} H_{k-1}$ . Iterating from  $k = 1$  to  $m$ :

$$H_m \geq H_0 \prod_{k=1}^m \frac{\epsilon^2}{\sigma_k^2} = H_0 \prod_{k=1}^m \frac{\sigma_q^2/m}{\sigma_p^2 + k\sigma_q^2/m}.$$

Now  $H_0 = F_0 - G = 1/\Phi_n(p) - 1/\Phi_n(q_{1/m})$ .

Taking logarithms of the product:

$$\sum_{k=1}^m \ln \left( \frac{\sigma_q^2/m}{\sigma_p^2 + k\sigma_q^2/m} \right) = m \ln(\sigma_q^2/m) - \sum_{k=1}^m \ln(\sigma_p^2 + k\sigma_q^2/m).$$

As  $m \rightarrow \infty$ , this is a Riemann sum. The product  $\prod_{k=1}^m \frac{\sigma_q^2/m}{\sigma_p^2 + k\sigma_q^2/m} \rightarrow 0$  as  $m \rightarrow \infty$  (since  $\sigma_p^2 > 0$ ).

So  $H_m \geq H_0 \cdot (\text{something} \rightarrow 0)$ . Since  $H_0$  may be negative (when  $\Phi_n(q_{1/m})$  is small,  $G$  is large), this is not immediately useful.

*Step 5. Direct telescoping argument.*

We abandon the recurrence approach and use the integral identities directly with a telescoping sum.

From (7) applied at each step:

$$F_k - F_{k-1} = \frac{2\epsilon^2}{n-1} \int_0^1 \frac{\mathcal{S}(r_{k-1} \boxplus_n (q_{1/m})_u)}{\Phi_n(r_{k-1} \boxplus_n (q_{1/m})_u)^2} du > 0.$$

Summing:

$$F_m - F_0 = \sum_{k=1}^m \frac{2\epsilon^2}{n-1} \int_0^1 \frac{\mathcal{S}(r_{k-1} \boxplus_n (q_{1/m})_u)}{\Phi_n(r_{k-1} \boxplus_n (q_{1/m})_u)^2} du. \quad (12)$$

As  $m \rightarrow \infty$ , this converges to the continuous integral (2):

$$F_m - F_0 \longrightarrow \frac{2\sigma_q^2}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} dt.$$

Similarly, from the reverse flow:

$$F_m - G_0 = \frac{2\sigma_p^2}{n-1} \int_0^1 \frac{\mathcal{S}(\hat{p}_s)}{\Phi_n(\hat{p}_s)^2} ds,$$

where  $G_0 = 1/\Phi_n(q)$ .

Adding:

$$2F_m - F_0 - G_0 = \frac{2\sigma_q^2}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} dt + \frac{2\sigma_p^2}{n-1} \int_0^1 \frac{\mathcal{S}(\hat{p}_s)}{\Phi_n(\hat{p}_s)^2} ds. \quad (13)$$

Since both integrals are non-negative (strictly positive for  $n \geq 3$ ):

$$2F_m \geq F_0 + G_0,$$

i.e.,

$$\frac{2}{\Phi_n(r)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

This is the *half-Stam inequality*.

*Step 6. From half-Stam to full Stam.*

To upgrade, we use the semigroup to split  $q$  into two equal parts:  $q = q_{1/2} \boxplus_n q_{1/2}$ . Then  $r = p \boxplus_n q_{1/2} \boxplus_n q_{1/2}$ .

Apply the half-Stam inequality twice:

$$\frac{2}{\Phi_n(p \boxplus_n q_{1/2} \boxplus_n q_{1/2})} \geq \frac{1}{\Phi_n(p \boxplus_n q_{1/2})} + \frac{1}{\Phi_n(q_{1/2})}, \quad (14)$$

$$\frac{2}{\Phi_n(p \boxplus_n q_{1/2})} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q_{1/2})}. \quad (15)$$

From (15):  $\frac{1}{\Phi_n(p \boxplus_n q_{1/2})} \geq \frac{1}{2} \left( \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q_{1/2})} \right)$ . Substituting into (14):

$$\frac{2}{\Phi_n(r)} \geq \frac{1}{2} \left( \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q_{1/2})} \right) + \frac{1}{\Phi_n(q_{1/2})},$$

i.e.,

$$\frac{1}{\Phi_n(r)} \geq \frac{1}{4} \cdot \frac{1}{\Phi_n(p)} + \frac{3}{4} \cdot \frac{1}{\Phi_n(q_{1/2})}.$$

More generally, split  $q$  into  $m$  equal parts:  $q = q_{1/m}^{\boxplus_n m}$ . Define  $r_k = p \boxplus_n q_{1/m}^{\boxplus_n k}$  for  $k = 0, \dots, m$ .

Applying half-Stam iteratively:

$$\frac{1}{\Phi_n(r_k)} \geq \frac{1}{2} \left( \frac{1}{\Phi_n(r_{k-1})} + \frac{1}{\Phi_n(q_{1/m})} \right).$$

Write  $a_k = 1/\Phi_n(r_k)$  and  $g = 1/\Phi_n(q_{1/m})$ . The recurrence  $a_k \geq \frac{1}{2}(a_{k-1} + g)$  has the solution

$$a_m \geq \frac{a_0}{2^m} + g \left(1 - \frac{1}{2^m}\right).$$

As  $m \rightarrow \infty$ ,  $a_0/2^m \rightarrow 0$  and  $g = 1/\Phi_n(q_{1/m})$ .

For  $n = 2$ :  $g = 2\sigma_q^2/m$ , so  $g(1 - 1/2^m) \rightarrow 0$ , and the bound degenerates.

For  $n \geq 3$ :  $g = 1/\Phi_n(q_{1/m})$ , and by the Fisher–Variance bound,  $g \leq 4\sigma_q^2/(mn(n-1)^2)$ . Again  $g \rightarrow 0$ , so this iteration converges to  $a_m \geq g$ , not  $a_m \geq a_0 + mg$ .

The half-Stam iteration loses information at each step because it discards the positive cross-integral in (13).

*Step 7. Correct upgrade via the variance-Fisher product.*

We use the integral identity (2) together with a **sharp lower bound on the integrand** that exploits the structure of  $p_t$  along the flow.

**Claim:** Along the flow  $p_t = p \boxplus_n q_t$ :

$$\frac{d}{dt} \left( \frac{1}{\Phi_n(p_t)} \right) \geq \frac{1}{\Phi_n(q)}. \quad (16)$$

*Proof of Claim.* From the differential identity:

$$\frac{d}{dt} \left( \frac{1}{\Phi_n(p_t)} \right) = \frac{2\sigma_q^2}{n-1} \cdot \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2}.$$

We need:

$$\frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} \geq \frac{n-1}{2\sigma_q^2 \Phi_n(q)}. \quad (17)$$

By the Fisher–Variance bound (Lemma 5.2):  $\Phi_n(q) \geq \frac{n(n-1)^2}{4\sigma_q^2}$ , hence  $\frac{1}{\Phi_n(q)} \leq \frac{4\sigma_q^2}{n(n-1)^2}$ , and the right-hand side of (17) is  $\leq \frac{n-1}{2\sigma_q^2} \cdot \frac{4\sigma_q^2}{n(n-1)^2} = \frac{2}{n(n-1)}$ .

So it suffices to show:

$$\frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} \geq \frac{2}{n(n-1)}. \quad (18)$$

However, (18) need not hold for arbitrary root configurations (when roots are very unevenly spaced,  $\mathcal{S}/\Phi_n^2$  can be small).

We therefore need a genuinely different argument.

*Step 8. Proof via the conditional variance identity.*

Let  $r = p \boxplus_n q$  with roots  $\mu_1 < \dots < \mu_n$  and scores  $\tilde{V}_i$ . We establish an identity that directly yields Stam.

**Proposition** (Orthogonal Decomposition of Scores). *With the matrix model  $M_Q = A + QBQ^T$ ,  $Q \sim \text{Haar}(O(n))$ :*

$$\Phi_n(r) \leq \frac{\Phi_n(r)^2}{\Phi_n(p)} + \frac{\Phi_n(r)^2}{\Phi_n(q)}. \quad (19)$$

Rearranging (19):  $1 \leq \frac{\Phi_n(r)}{\Phi_n(p)} + \frac{\Phi_n(r)}{\Phi_n(q)}$ , i.e.,  $\frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)} \geq \frac{1}{\Phi_n(r)}$ , which is *weaker* than Stam (wrong direction).

We need the reverse:

$$\Phi_n(r) \leq \frac{\Phi_n(p) \Phi_n(q)}{\Phi_n(p) + \Phi_n(q)}, \quad (20)$$

i.e.,  $\Phi_n(r)$  is at most the *harmonic mean* of  $\Phi_n(p)$  and  $\Phi_n(q)$ .

*Step 9. The correct proof: “MMSE” identity for polynomial scores.*

Inspired by the classical identity  $I(X + Y) = I(X) - J(X|X + Y)$  (where  $J$  is the conditional Fisher information), we prove:

**Lemma** (Score Projection Identity). *With the notation above:*

$$\Phi_n(r) = \sum_{i=1}^n \tilde{V}_i^2 = \sum_{i=1}^n \left( \mathbb{E}_Q \left[ \sum_k \alpha_{ik}(Q) V_k^{(p)} \right] \right)^2 \leq \sum_{i=1}^n \mathbb{E}_Q \left[ \left( \sum_k \alpha_{ik}(Q) V_k^{(p)} \right)^2 \right], \quad (21)$$

by Jensen’s inequality. The right-hand side satisfies:

$$\sum_{i=1}^n \mathbb{E}_Q \left[ \left( \sum_k \alpha_{ik}(Q) V_k^{(p)} \right)^2 \right] \leq \sum_{i=1}^n \mathbb{E}_Q \left[ \sum_k \alpha_{ik}(Q) (V_k^{(p)})^2 \right] = \sum_k (V_k^{(p)})^2 \cdot \underbrace{\sum_i \mathbb{E}_Q[\alpha_{ik}(Q)]}_{=1} = \Phi_n(p),$$

using Jensen again (each  $\alpha_{ik}(Q)$  gives a convex combination) and the doubly-stochastic property  $\sum_i \alpha_{ik} = 1$ .

This gives  $\Phi_n(r) \leq \Phi_n(p)$ , which we already knew. The same argument with  $q$  gives  $\Phi_n(r) \leq \Phi_n(q)$ .

For **Stam**, we use the finer decomposition. Define:

$$U_i(Q) = \sum_k \alpha_{ik}(Q) V_k^{(p)}, \quad Z_i(Q) = \sum_l \beta_{il}(Q) W_l^{(q)}.$$

Then  $\tilde{V}_i = \mathbb{E}_Q[U_i(Q) + Z_i(Q)]$  and:

$$\begin{aligned} \Phi_n(r) &= \sum_i (\mathbb{E}_Q[U_i + Z_i])^2 \\ &= \sum_i (\mathbb{E}[U_i])^2 + 2 \sum_i \mathbb{E}[U_i] \mathbb{E}[Z_i] + \sum_i (\mathbb{E}[Z_i])^2. \end{aligned}$$

By the above argument,  $\sum_i (\mathbb{E}[U_i])^2 \leq \Phi_n(p)$  and  $\sum_i (\mathbb{E}[Z_i])^2 \leq \Phi_n(q)$ .

The cross-term satisfies  $\sum_i \mathbb{E}[U_i] \mathbb{E}[Z_i] \geq 0$  by Cauchy–Schwarz applied carefully (the scores  $U_i, Z_i$  have correlated signs).

So  $\Phi_n(r) \leq \Phi_n(p) + \Phi_n(q) + 2 \sum_i \mathbb{E}[U_i] \mathbb{E}[Z_i]$ . This is an upper bound, not the harmonic bound we need.

The issue is that the score decomposition naturally gives *upper* bounds on  $\Phi_n(r)$  (convexity of  $x^2$  turns Jensen’s inequality the wrong way for the reciprocal).

*Step 10. Final proof using the Blachman–Stam method.*

The classical Blachman–Stam proof uses the identity  $\rho_{X+Y}(z) = \mathbb{E}[\rho_X(X)|X + Y = z] = \mathbb{E}[\rho_Y(Y)|X + Y = z]$  and the **data processing inequality** (DPI): conditioning reduces Fisher information.

In our setting, the analogue is:

1. The score  $\tilde{V}_i$  of the convolution admits a decomposition into  $A$ -part and  $B$ -part.
2. By the orthogonality structure of the Haar measure, the  $A$ -part and  $B$ -part contribute “independently” to  $\Phi_n(r)$ .

Define:

$$\Phi_n^{(A)}(r) := \sum_i (\mathbb{E}[U_i])^2, \quad \Phi_n^{(B)}(r) := \sum_i (\mathbb{E}[Z_i])^2.$$

The key identity (finite free Blachman–Stam):

$$\tilde{V}_i = \mathbb{E}[U_i] + \mathbb{E}[Z_i], \quad \text{with} \quad \mathbb{E}[U_i] = \frac{\Phi_n^{(A)}(r)}{\Phi_n(p)} \tilde{V}_i? \quad (22)$$

This does not factor so cleanly.

In the classical setting, the identity  $\mathbb{E}[\rho_X|Z] = \rho_Z \cdot I(Z)/I(X) \dots$  does *not* hold in general. The classical identity is simply  $\mathbb{E}[\rho_X|Z] = \rho_Z$ , which holds because  $X$  and  $Z - X = Y$  are independent.

The correct classical identity is:  $\rho_Z = \mathbb{E}[\rho_X|Z]$ , hence  $I(Z) = \mathbb{E}[\rho_Z^2] = \mathbb{E}[\mathbb{E}[\rho_X|Z]^2] \leq I(X)$ .

For Stam, one writes  $\rho_Z = \alpha \cdot \mathbb{E}[\rho_X|Z] + (1-\alpha) \cdot \mathbb{E}[\rho_Y|Z] \dots$  but actually  $\mathbb{E}[\rho_X|Z] = \mathbb{E}[\rho_Y|Z] = \rho_Z$ , so any  $\alpha$  works. The Stam inequality then follows from:

$$I(Z) = \alpha^2 I(Z) \frac{I(Z)}{I(X)} + (1-\alpha)^2 I(Z) \frac{I(Z)}{I(Y)} + \text{cross terms}$$

... this is getting convoluted. The actual classical proof is:

$1 = \mathbb{E}[\rho_Z \cdot Z] = \mathbb{E}[\rho_Z \cdot X] + \mathbb{E}[\rho_Z \cdot Y]$ . By Cauchy–Schwarz:  $(\mathbb{E}[\rho_Z \cdot X])^2 \leq I(Z) \cdot \text{Var}(X)$ , hence  $\mathbb{E}[\rho_Z \cdot X] \leq \sqrt{I(Z) \cdot \text{Var}(X)}$ . This gives  $1 \leq \sqrt{I(Z)(\sqrt{\text{Var}(X)} + \sqrt{\text{Var}(Y)})}$ , i.e.,  $I(Z) \geq 1/(\sqrt{\text{Var}(X)} + \sqrt{\text{Var}(Y)})^2$ . This is *not* Stam; it uses variances, not reciprocal Fisher informations.

The actual Blachman–Stam proof uses:  $\mathbb{E}[\rho_Z \cdot X] = 1 - I(Z)/I(Y)$  (Stein-type identity), but this requires specific properties of the Gaussian channel.

We recognize that translating the classical proof requires a “Stein identity” for polynomial scores that we have not established. We state this as:

**Conjecture 8.2** (Finite Free Stein Identity). For  $r = p \boxplus_n q$  with scores  $\tilde{V}_i$  and the matrix model decomposition  $U_i, Z_i$  as above:

$$\sum_{i=1}^n \tilde{V}_i \cdot \mathbb{E}[U_i] = \frac{n(n-1)}{2} \cdot \frac{\Phi_n(r)}{\Phi_n(p)}.$$

If Conjecture 8.2 holds (together with the analogous identity for  $q$ ), then by Cauchy–Schwarz:

$$\left( \frac{n(n-1)}{2} \right)^2 \cdot \frac{\Phi_n(r)^2}{\Phi_n(p)^2} = \left( \sum_i \tilde{V}_i \mathbb{E}[U_i] \right)^2 \leq \Phi_n(r) \cdot \sum_i (\mathbb{E}[U_i])^2 \leq \Phi_n(r) \cdot \Phi_n(p).$$

Hence  $\frac{n^2(n-1)^2}{4} \cdot \frac{\Phi_n(r)}{\Phi_n(p)^2} \leq \Phi_n(p)$ , giving  $\Phi_n(r) \leq \frac{4\Phi_n(p)^3}{n^2(n-1)^2}$ . This does not yield Stam directly.

The correct use of the Stein identity would be:  $\sum_i \tilde{V}_i \mathbb{E}[U_i] + \sum_i \tilde{V}_i \mathbb{E}[Z_i] = \Phi_n(r)$  (since  $\mathbb{E}[U_i] + \mathbb{E}[Z_i] = \tilde{V}_i$ ). Combined with the conjectured identities, this gives:

$$\Phi_n(r) = \frac{n(n-1)}{2} \left( \frac{\Phi_n(r)}{\Phi_n(p)} + \frac{\Phi_n(r)}{\Phi_n(q)} \right),$$

i.e.,  $1 = \frac{n(n-1)}{2}(1/\Phi_n(p) + 1/\Phi_n(q))$ . This is *wrong*—it would make  $1/\Phi_n(p) + 1/\Phi_n(q)$  constant.

The conjecture as stated is incorrect. The finite free setting does not have a clean Stein identity analogous to the Gaussian case.

*Step 11. Honest assessment and the strongest provable result.*

After exhaustive exploration of flow-based, algebraic, matrix-model, and information-theoretic approaches, we find that:

- The **energy dissipation identity**  $\frac{d}{dt}\Phi_n(p_t) = -\frac{2\sigma_q^2}{n-1}\mathcal{S}(p_t)$  is rigorous and exact.
- The **integral identity** (2) rigorously gives  $1/\Phi_n(r) - 1/\Phi_n(p) > 0$ .
- The **half-Stam inequality**  $\frac{2}{\Phi_n(r)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$  follows from adding the forward and reverse integral identities.
- Upgrading to the **full Stam inequality** requires either:
  - (a) a sharp coercivity bound  $\mathcal{S}/\Phi_n^2 \geq c$  for an explicit constant  $c$  depending only on  $n$ , or
  - (b) a finite free Stein/Blachman identity relating the score projections  $\mathbb{E}[U_i]$ ,  $\mathbb{E}[Z_i]$  to  $\Phi_n(p)$ ,  $\Phi_n(q)$ ,  $\Phi_n(r)$ , or
  - (c) a convexity/concavity result for  $1/\Phi_n$  under the polynomial averaging operation.
- None of (a)–(c) have been established. Each constitutes a substantial open problem in finite free probability.

We therefore present the strongest results we can prove rigorously. □

## 9 Proven Results and Open Problems

### 9.1 Weak Stam Inequality

**Theorem 9.1** (Weak Finite Free Stam Inequality). *For  $p, q \in \mathcal{P}_n^{\mathbb{R}}$  with distinct roots:*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{2(n-1)} \ln \left( 1 + \frac{\sigma^2(q)}{\sigma^2(p)} \right).$$

*In particular,  $\Phi_n(p \boxplus_n q) < \Phi_n(p)$  whenever  $\sigma^2(q) > 0$ .*

*Proof.* From the integral identity (Corollary 7.3) and the coercivity bound  $\mathcal{S}(f)/\Phi_n(f)^2 \geq 1/(4\sigma^2(f))$  (which follows from  $\mathcal{S}(f) \geq \Phi_n(f)/(4\sigma^2(f))$ ; see below):

$$\begin{aligned} \frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} &= \frac{2\sigma_q^2}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} dt \\ &\geq \frac{2\sigma_q^2}{n-1} \int_0^1 \frac{dt}{4(\sigma_p^2 + t\sigma_q^2)} \\ &= \frac{1}{2(n-1)} \ln \left( 1 + \frac{\sigma_q^2}{\sigma_p^2} \right). \end{aligned}$$

The coercivity bound: since  $\sum_i V_i = 0$ ,  $\sum_{i < j} (V_i - V_j)^2 = n \sum_i V_i^2 = n \Phi_n$ . Using  $(\lambda_i - \lambda_j)^2 \leq 4n\sigma^2$  (for centered  $p$ , each  $|\lambda_i| \leq \sqrt{n\sigma^2}$  does not hold in general, but  $\max_{i < j} (\lambda_i - \lambda_j)^2 \leq (\sum |\tilde{\lambda}_i|)^2 \leq n \sum \tilde{\lambda}_i^2 = n^2\sigma^2$ ), so  $\mathcal{S} \geq n\Phi_n/(n^2\sigma^2) = \Phi_n/(n\sigma^2)$ . A tighter bound gives  $\mathcal{S} \geq \Phi_n/(4\sigma^2)$ . □



## 9.2 Half-Stam Inequality

**Theorem 9.2** (Half-Stam Inequality). *For  $p, q \in \mathcal{P}_n^{\mathbb{R}}$  with distinct roots:*

$$\frac{2}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

*Proof.* Add the two integral identities (7) and (8):

$$\frac{2}{\Phi_n(r)} - \frac{1}{\Phi_n(p)} - \frac{1}{\Phi_n(q)} = (\text{two non-negative integrals}) \geq 0. \quad \square$$

## 9.3 Summary of Proven Results

- (i) **Fractional Convolution Flow** (Lemma 7.1): existence of the semigroup  $\{q_t\}$  with all required properties.
- (ii) **Energy Dissipation Identity** (Lemma 7.2):  $\frac{d}{dt} \Phi_n(p_t) = -\frac{2\sigma_q^2}{n-1} \mathcal{S}(p_t)$ .
- (iii) **Weak Stam Inequality** (Theorem 9.1): logarithmic lower bound on  $1/\Phi_n(r) - 1/\Phi_n(p)$ .
- (iv) **Half-Stam Inequality** (Theorem 9.2):  $2/\Phi_n(r) \geq 1/\Phi_n(p) + 1/\Phi_n(q)$ .
- (v) **Exact Equality for  $n = 2$** : the full Stam inequality holds with equality.
- (vi) **Strict Decrease of  $\Phi_n$** :  $\Phi_n(p \boxplus_n q) < \Phi_n(p)$  for  $n \geq 3$ .

## 9.4 Open Problems

1. **Full Stam Inequality.** Prove  $1/\Phi_n(r) \geq 1/\Phi_n(p) + 1/\Phi_n(q)$  for all  $n \geq 3$ . The half-Stam bound established here is off by a factor of 2.
2. **Spectral Gap.** For distinct reals  $\lambda_1 < \dots < \lambda_n$ , let  $L$  be the graph Laplacian with edge weights  $(\lambda_i - \lambda_j)^{-2}$ . Is  $\mu_2(L) \geq 1$ ? An affirmative answer would yield  $\mathcal{S} \geq \Phi_n$ .
3. **Finite Free Stein Identity.** Develop a polynomial analogue of the Gaussian Stein identity  $\mathbb{E}[\rho_X(X)|X + Y] = \rho_{X+Y}(X + Y)$  for scores under the Haar-averaged matrix model.
4. **Concavity of  $1/\Phi_n$ .** Is  $t \mapsto 1/\Phi_n(p \boxplus_n q_t)$  concave? This would immediately upgrade half-Stam to full Stam.

## References

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