

Théorie des modèles TD1

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Exercise 0.1 Let \mathcal{M} be an \mathcal{L} -structure, $m, n \in \mathbb{N}$ and $A \subseteq M^{n+m}$ be definable in \mathcal{M} . For $\bar{b} \in M^m$, let $A_{\bar{b}} = \{\bar{a} \in M^n, (\bar{a}, \bar{b}) \in A\}$ the fiber of A over b . Let $k \in \mathbb{N}$. Show that the set $\{\bar{b} \in M^m, |A_{\bar{b}}| < k\}$ is definable. (*) Is the set $\{\bar{b} \in M^m, |A_{\bar{b}}| < \infty\}$ definable?

Solution 0.1 If $A \subseteq M^{n+m}$ is definable, then there is some $\bar{s} \in M$, and some formula $\phi(\bar{x}, \bar{y}, \bar{z})$ such that $A = \{(\bar{x}, \bar{y}) \in M^{n+m}, \mathcal{M} \models \phi(\bar{x}, \bar{y}, \bar{s})\}$. The following formula states that the $A_{\bar{b}}$ -fiber has less than k elements.

$$\phi_k(y_1, \dots, y_m) = \forall \bar{x}_1 \dots \bar{x}_k \left(\bigwedge_{i=1}^k \phi(\bar{x}_i, \bar{y}) \Rightarrow \bigvee_{1 \leq i \neq j \leq k} \bar{x}_i = \bar{x}_j \right).$$

We see that $\mathcal{M} \models \phi_k(\bar{b})$ if and only if $A_{\bar{b}}$ has less than k elements. (*) Consider $\mathcal{N} = (\mathbb{N}, <)$, and let \mathcal{U} a non-principal ultrafilter over \mathbb{N} . Let $\mathcal{M} = \mathcal{N}^{\mathcal{U}}$. We can identify each $n \in \mathbb{N}$ with $[n, n, \dots]_{\mathcal{U}} \in \mathcal{M}$. Notice that for every $n \in \mathbb{N}$,

$$\omega = [0, 1, 2, \dots, n, n+1, \dots]_{\mathcal{U}} > n$$

We call elements bigger than every n , *infinite*. Suppose now the set $\{\bar{b} \in M^m, |A_{\bar{b}}| < \infty\}$ is definable by some formula $\phi(x, \bar{m})$ with parameters $\bar{m} \in \mathcal{M}$. In other words, $\mathcal{M} \models \phi(x, \bar{m})$ if and only if there is a finite number of elements below x (in the usual finite sense), we show this implies that x is necessarily finite: suppose not, then for every n , \mathcal{U} -almost everywhere, $x_i \neq n$. We want to prove that actually $x_i \geq n$: if it were not the case, then again, \mathcal{U} -almost everywhere $x_i < n \Rightarrow x_i \in \{0, 1, \dots, n-1\} = \bigcup_{j=0}^{n-1} \{j\}$. In other words, this means that

$$\bigcup_{j=0}^{n-1} \{i, x_i = j\} \in \mathcal{U}.$$

By ultrafilter properties, (if $A \cup B \in \mathcal{U}$ then either $A \in \mathcal{U}$ or $B \in \mathcal{U}$) we conclude that for some k , $[x] = [k]$, which is a contradiction since x is infinite. We consider now $\Sigma(x, \bar{m}) = \{\neg\phi_k(x, \bar{m})\}_{k \in \mathbb{N}} \cup \{\phi(x, \bar{m})\}$. It is finitely consistent, since if $\Sigma_N(x, \bar{m}) = \{\neg\phi_k(x, \bar{m})\}_{k < N} \cup \{\phi(x, \bar{m})\}$ is a finite part of Σ , then $\mathcal{M} \models \Sigma_N(N, \bar{m})$ (N has at least k elements below it for every $k > N$, and also has a finite number of elements below it since it is finite). By compactness, there is $N' \in \mathcal{M}$ such that $\mathcal{M} \models \Sigma_N(N', \bar{m})$. We conclude that N' has at least k elements below it for every k , and that N is finite by the above. This is a contradiction, so $\{\bar{b} \in M^n, |A_{\bar{b}}| < \infty\}$ is not definable.

Exercise 2. Let M be a set and $\mathcal{D} = \bigcup_n D_n$ be a collection of subsets of $\bigcup_n M^n$ containing \emptyset , M^n for every n , the diagonals, and closed under permutation of the coordinates, cartesian products, the boolean set operations and linear projections. Show that $\mathcal{D} = \text{Def}(\mathcal{M}, \emptyset)$, for some language \mathcal{L} and some \mathcal{L} -structure \mathcal{M} .

Solution 0.2 Take $\mathcal{C} = \emptyset$, and $\mathcal{R} = \bigcup_{n \in \mathbb{N}} \{(x_1, \dots, x_n), (x_1, \dots, x_n) \in D_n\}$. In other words, take no constants and set each of the D_n 's to be a predicate. For each n , if for every $(x_1, \dots, x_n) \in M^n$, there exists $y \in M$ such that $(x_1, \dots, x_n, y) \in D_{n+1}$, we set $f : M^n \rightarrow M$ which sends (x_1, \dots, x_n) to y . We may have to do this (possibly infinitely) many times since such y may not be unique.

Exercise 0.3 Let \mathcal{M} be an expansion of a total order equipped with the *order topology*. Let $A \subseteq M^n$ and $f : A \rightarrow M$ both definable.

- (1) Show that A°, \bar{A} and $\text{bd}(A)$ are all definable.
- (2) Show that the set of discontinuity points of f is definable.
- (3) Show that the following properties are definable: A is discrete, A is bounded.
- (4) What about A is compact and connected?

Solution 0.3 We use the following abbreviations: $\bar{x} < \bar{y}$ for $x_i < y_i$ for each i and if ϕ is a formula then $Qx \in A(\phi)$ (where Q is a quantifier) for $Qx(x \in A \Rightarrow \phi)$.

- (1) $\bar{x} \in A^\circ$ if and only if $\exists \bar{y}, \bar{z} \in A (\bar{z} < \bar{x} < \bar{y})$.
 $\bar{x} \in \bar{A}$ if and only if $\forall \bar{y}, \bar{z} ((\bar{z} < \bar{x} < \bar{y}) \Rightarrow \exists \bar{w} \in A (\bar{z} < \bar{w} < \bar{y}))$
 $\bar{x} \in \text{bd}(A)$ if and only if $\bar{x} \notin A^\circ \wedge \bar{x} \in \bar{A}$.

(2) \bar{x} is a discontinuity point of f if and only if

$$\exists r \exists s ((r < f(\bar{x}) < s) \wedge \forall \bar{y}, \bar{z} \in A ((\bar{z} < \bar{x} < \bar{y}) \Rightarrow \exists \bar{w} ((\bar{z} < \bar{w} < \bar{y}) \wedge (f(\bar{w}) < r \vee f(\bar{w}) > s)))$$

(3) We say A is discrete if $\forall \bar{x} \in A \exists \bar{y} \in A (\bar{x} < \bar{y} \Rightarrow \nexists \bar{z} \in A (\bar{x} < \bar{z} < \bar{y}))$. We say A is bounded if $\forall \bar{x} \in A \exists \bar{y}, \bar{z} (\bar{x} < \bar{y} \wedge \bar{x} > \bar{z})$.

(4) Consider \mathcal{U} a non-principal ultrafilter on \mathcal{N} and let $\mathbb{R}^* = \mathbb{R}^{\mathcal{U}}$. Consider

$$\varepsilon = [1, 1/2, 1/3, \dots, 1/n, \dots]_{\mathcal{U}}.$$

Notice that for $i \geq n$, $\varepsilon_i < 1/n$, and since $\{n, n+1, \dots\} \in \mathcal{U}$ (it is cofinite), we conclude that for every n , $\varepsilon < 1/n$. This proves that \mathbb{R}^* is not archimedean, and in particular this also proves that archimedeanity for a field is not axiomatizable, since if it were by, say, some theory T , we would have $\mathbb{R} \models T$ and $\mathbb{R}^* \not\models T$, contradicting Łos' theorem. We will show that connectedness and compactness are not 1st order expressible: consider E the set of *infinitesimal* elements in \mathbb{R}^* , i.e the set of elements smaller than every $1/n$.

E is closed: Let $\varepsilon \in \bar{E}$, then for every $x, y \in \mathbb{R}^*$ such that $x < \varepsilon < y$ there is $\epsilon \in E$ such that $x < \epsilon < y$. If $\varepsilon \geq 1/n$ for some n , then we can find some infinitesimal $1/n < \epsilon < 1$, a contradiction.

E is open: For any $\varepsilon \in E$ we have $\varepsilon/2 < \varepsilon < 2\varepsilon$. We have to show these are infinitesimal: for $\varepsilon/2$ is trivial since it is below an infinitesimal. Now if $2\varepsilon \notin E$, we can find n such that $1/n < 2\varepsilon \Rightarrow 1/2n < \varepsilon$, which is impossible.

E has no supremum: Let $r = \sup E$. We have that, $r \notin E$ (otherwise $r < 2r \in E$), now let $\epsilon \in E$, and we claim $r - \epsilon$ bounds E : suppose not, so there is $\varepsilon \in E$ such that

$$r - \epsilon < \varepsilon < r \Rightarrow r < \varepsilon + \epsilon < r + \epsilon.$$

Notice also that $\epsilon + \varepsilon \in E$ because, for any $n \in \mathbb{N}$ since $\epsilon, \varepsilon < 1/2n$, then $\epsilon + \varepsilon < 1/n$. We have that $r \leq$ some infinitesimal, a contradiction. E cannot have a supremum.

E is not compact: the sequence

$$\varepsilon < 2\varepsilon < \dots < n\varepsilon < \dots$$

is contained in E (by the above argument), it is strictly increasing and is bounded above by 1. The above argument can be used to show it has no limit point. This proves E is not compact. If compactness of some set A was given by some sentence ϕ_A , then we would have $\mathbb{R} \models \phi_{[0,1]}$ but $\mathbb{R}^* \not\models \phi_{[0,1]}$, contradicting Los' theorem.

Finally, since E is clopen and it is neither \emptyset nor $[0, 1]$, we conclude that $[0, 1]$ is not connected in \mathbb{R}^* , and we can infer that connectedness is also not 1st order expressible.

Exercise 1. Let $\mathcal{L} = \emptyset$ and \mathcal{M} be an \mathcal{L} -structure. Show that $A \subseteq M$ is definable in \mathcal{M} if and only if A is either finite or cofinite.

Solution 1.

Lemma: If A is S -definable, then every automorphism σ that fixes S pointwise fixes A pointwise.

Proof: Let $\psi(x, \bar{s})$ be a formula defining A , since automorphisms preserve formula, then we have

$$\mathcal{M} \models \psi(x, \bar{s}) \iff \mathcal{M} \models \psi(\sigma(x), \sigma(\bar{s})) \iff \mathcal{M} \models \psi(\sigma(x), s)$$

so that $\sigma(X) = X$.

If there was some definable A which is neither finite nor cofinite, then we can choose infinite sets

$$\{a_0, a_1, \dots, a_n, \dots\} \in A \setminus S$$

$$\{b_0, b_1, \dots, b_n, \dots\} \in (M \setminus A) \setminus S$$

Then the bijection which sends a_i to b_i and fixes everything else (in particular S) is an automorphism that doesn't fix A , a contradiction. Conversely, if $A = \{a_0, \dots, a_n\}$ is finite, the formula

$$\phi(x, \bar{a}) = \bigvee_{i=0}^n x = a_i$$

defines A . If A is cofinite repeat this argument for $M \setminus A$.

Exercise 2. Let \mathcal{M} be an \mathcal{L} -structure, let $m, n \in \mathbb{N}$. A collection $\mathcal{A} = \{A_{\bar{b}}\}_{\bar{b} \in M^m}$ of subsets of M^n is a *definable family* if there exists $S \subseteq M$ and a formula $\phi \in \mathcal{F}_{n+m}(\mathcal{L}_S)$ such that $A_{\bar{b}} = \{\bar{a} \in M^n, \mathcal{M}_S \models \phi(\bar{a}, \bar{b})\}$. Let $D \subseteq M$ be a finite set. Given a D -definable family $\mathcal{A} = \{A_{\bar{b}}\}_{\bar{b} \in M^m}$, let $A = \cup \mathcal{A}$ and let $f : A \rightarrow M$ a function.

(1) Show that f is D -definable if and only if all restrictions $f \upharpoonright A_{\bar{b}}$ are D -definable.

(2) What can we say if D is infinite?

Solution 2. Suppose there is $\phi \in \mathcal{F}_{n+1}(\mathcal{L}_D)$ such that $f(a_1, \dots, a_n) = y$ if and only if $\mathcal{M}_D \models \phi(\bar{a}, y)$. Then $f \upharpoonright_{A_{\bar{b}}}(\bar{a}) = y$ if and only if $\mathcal{M}_D \models \varphi(\bar{a}, \bar{b}) \wedge \phi(\bar{a}, y)$, where φ is the formula which defines the family $\{A_{\bar{b}}\}_{\bar{b} \in D^m}$. Conversely if there is $\phi_{\bar{b}}$ which defines each $f \upharpoonright_{A_{\bar{b}}}$, we have that $f(\bar{a}) = y$ if and only if

$$\mathcal{M}_D \models \bigvee_{\bar{b} \in D^m} \varphi(\bar{a}, \bar{b}) \wedge \phi_{\bar{b}}(\bar{a}, y).$$

In case D is infinite, the first direction holds, but the converse may not, for instance take $\mathcal{M} = \langle \mathbb{C}, +, -, \times, 0, 1 \rangle$ and $D = \mathbb{R}$. Take $A_b = \{a \in \mathbb{R}, a = b\} = \{b\}$, and $f : A_b \rightarrow \mathbb{C}$ as the identity. Since $A = \mathbb{R}$, and the inclusion $\mathbb{R} \subseteq \mathbb{C}$ is not definable, even though its restrictions are.

Exercise 3. Let $\bar{\mathbb{R}} = \langle \mathbb{R}, 0, 1, -, +, \cdot, < \rangle$ be the real ordered field. Let f be a unary symbol and $\mathcal{L} = \mathcal{L}_{OR} \cup \{f\}$.

- (1) Show that the \mathcal{L} -structures $\langle \bar{\mathbb{R}}, \sin \left(\frac{1}{1+x^2} \right) \rangle$ and $\langle \bar{\mathbb{R}}, \arctan x \rangle$
- (2) Show that $\bar{\mathbb{R}}$ is definable in the structure $\langle \mathbb{R}, +, \exp(x) \rangle$.
- (3) Let $\mathbb{R}_{\exp} = \langle \bar{\mathbb{R}}, \exp \rangle$ be the real ordered exponential field. An *exponential polynomial* is a function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ such that there exists a polynomial $P \in \mathbb{R}[X_1, \dots, X_n, Y_1, \dots, Y_m]$ such that $F(x_1, \dots, x_n) = P(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n})$. Show that every set $A \in \mathbb{R}^m$ existentially definable in \mathbb{R}_{\exp} is a linear projection of the zero-set of some exponential polynomial F_A .

Solution 3.

- (1) We first define $\sin(x) \upharpoonright_{[0,1]}$ from $\sin \left(\frac{1}{1+x^2} \right)$:

$$\text{graph } \sin(x) \upharpoonright_{[0,1]} = \left\{ (x, y), (x = 0 \wedge y = 0) \vee \exists z (x(1 + x^2) = 1 \wedge y = \sin \left(\frac{1}{1 + z^2} \right)) \right\}.$$

We can now define

$$x = \frac{\pi}{2} \iff 2 \sin^2 \upharpoonright_{[0,1]}(x/2) = 2$$

And define for $0 < x < \pi/2$,

$$\tan(x) = \frac{2 \sin \upharpoonright_{[0,1]}(x/2) \sqrt{1 - \sin^2 \upharpoonright_{[0,1]}(x/2)}}{1 - 2 \sin^2 \upharpoonright_{[0,1]}(x/2)}.$$

And for $-\pi/2 < x < 0$

$$\tan(x) = -\tan(-x).$$

Then finally set

$$y = \arctan(x) \iff \tan(x) = y.$$

For the other direction we can define $\tan x$ from $\arctan x$ and set

$$\sin \upharpoonright_{[0,1]} x = \frac{\tan x}{\sqrt{1 + \tan^2 x}}$$

define $\sin(1/(1+x^2))$ as above.

$$(2) \quad (a) \quad x = 0 \iff x + x = x$$

$$(b) \quad 1 = e^0$$

$$(c) \quad y = -x \iff x + y = 0$$

$$(d) \quad x > 0 \iff \exists y e^y = x$$

$$(e) \quad xy = \exp(\log x + \log y)$$

- (3) Let $\varphi(\bar{x}, \bar{c})$ and existential formula in \mathcal{L}_{exp} with parameters $\bar{c} \in \mathbb{R}$. We can assume φ has the form

$$\varphi(\bar{x}, \bar{c}) = \exists z_1, \dots, \exists z_n \bigvee_{i=1}^l \bigwedge_{j=1}^s \theta_{ij}(\bar{x}, \bar{z}, \bar{c})$$

where θ_{ij} is atomic or \neg -atomic. We know that atomic formulas have the form $t_1 = t_2, t_1 < t_2, t_1 = 0$ or $t_1 < 0$ for t_1, t_2 terms with parameters \bar{c} . We can replace in θ_{ij} , $t \neq 0$ for $t < 0 \wedge -t < 0$ and $t < 0$ for $\exists y ty^2 + 1 = 0 \wedge t \neq 0$, and $t_1 = t_2$ for $t_1 - t_2 = 0$. In other words, we can assume θ_{ij} to be of the form $t = 0$. We now show by induction on $t(\bar{x}, \bar{c})$ that any term can be replaced by a conjunction of existential formulas containing only terms of the form $F(\bar{x}, \bar{y}, \bar{c})$, where F is an exponential polynomial and \bar{y} are new variables. In other words, $t(\bar{x}, \bar{c}) = 0$ becomes a system of exponential polynomial equations on variables \bar{y} .

If $t = c$ then $c = 0$ is already of the form we want.

If $t = t_1 + t_2$ then, since the sum of exponential polynomials is also an exponential polynomial, we can just add each of the rows of each system of equations to get one for $t = 0$.

The case $t_1 t_2$ is similar.

If $t(\bar{x}, \bar{c}) = e^{t_1(\bar{x}, \bar{c})}$, then we can replace

$$t(\bar{x}, \bar{c}) = 0 \iff \exists w e^w = 0 \wedge w = t_1(\bar{x}, \bar{c})$$

and then we can apply induction on t_1 , adding the variable w to our exponential polynomial. We can then suppose (renaming variables and reindexing) that θ_{ij} has the form $F_{ij}(\bar{x}, \bar{z}, \bar{c}) = 0$ for some exponential polynomial. So that

$$\varphi(\bar{x}, \bar{c}) = \exists z_1, \dots, \exists z_n \bigvee_{i=1}^l \bigwedge_{j=1}^s F_{ij}(\bar{x}, \bar{z}, \bar{c}) = 0.$$

It is clear that the set of zeros of

$$F = \sum_{i=1}^l \left(\prod_{j=1}^s F_{ij}(\bar{x}, \bar{c}) \right)^2$$

defines the same set as $\varphi(\bar{x}, \bar{c})$.