



# IMAGINARIES IN PAIRS OF ALGEBRAICALLY CLOSED FIELDS

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## Abstract

Consider the theory  $T$  of algebraically closed fields of a given characteristic  $p$ , in the language  $L = \{0, 1, +, -, \cdot\}$ . Extend  $L$  to a language  $L_P$  by adding a predicate  $P$ , which is interpreted in a model  $M \models T$  as a proper elementary substructure. Since  $T$  has elimination of quantifiers, these pairs can be axiomatized by expressing  $P \models T$ , and  $\exists x \neg P(x)$ , obtaining a theory  $T_P$  of elementary pairs  $P \prec M$ . The main goal is to add sorts to the language  $L_P$ , in order to achieve *weak elimination of imaginaries*. Keisler in [5] proved that  $T_P$  is complete, and in [2], Buechler showed that  $T_P$  is an  $\omega$ -stable theory, of Morley rank  $\omega$ . This work is largely based on [9], by Anand Pillay.

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# 1 Preliminaries on Stability Theory

Let  $T$  be a complete theory over a language  $L$ . If  $M \models T$ , and  $A \subseteq M$ , we denote the space of  $n$ -types with parameters in  $A$  by  $S_n(A)$ , and let  $S(A) = \cup_{i < \omega} S_n(A)$ . Recall that a theory is  $\kappa$ -stable if for every  $M \models T$  and every  $A \subseteq M$ , if  $|A| \leq \kappa$ , then  $|S_1(A)| \leq \kappa$ , and we say  $T$  is stable if it is  $\kappa$ -stable for some cardinal  $\kappa$ . We will use an equivalent characterization of stability, given by the definability of types.

**Definition 1.1.** Let  $M \models T$ , and  $A, B$  subsets of  $M$ . A type  $p(x) \in S_n(A)$  is *definable* over  $B$  if for any  $L$ -formula  $\varphi(x, y)$  there is an  $L(B)$ -formula  $\psi(y)$  such that for all  $a \in A^{|y|}$ ,  $\varphi(x, a) \in p$  if and only if  $M \models \psi(a)$ . The formula  $\psi(y)$  will be written as  $d_p(\varphi)(y)$ , and the set of  $d_p(\varphi)(y)$ , with  $\varphi(x, y)$  ranging over the  $L$ -formulas is called a *definition scheme* for  $p$ .

The following proposition is Corollary 8.3.2 from [13].

**Proposition 1.2.** *The theory  $T$  is stable if and only if all types are definable.*

Throughout the rest of the section we assume  $T$  is an arbitrary, complete,  $\omega$ -stable theory. We will work inside a saturated model  $M$  of  $T$ , and types over  $M$  shall be referred as *global types*. We proceed by stating some definitions and results on canonical bases and forking in this stable context.

**Definition 1.3.** Let  $E(x_1, \dots, x_n, y_1, \dots, y_n)$  be an  $L$ -formula that defines an equivalence relation on  $M^n$ . By *real elements*, we mean tuples in  $M^n$ , while the equivalence classes of real elements modulo  $E$  will be called *imaginary elements*.

**Definition 1.4.** Let  $X \subseteq M$  be a definable set. A tuple  $c \in M$  is called a *canonical parameter* (or code) of  $X$  if  $c$  is fixed by exactly the same automorphisms of  $M$  which fix  $X$  setwise.

It is possible to extend  $T$  to a new theory  $T^{\text{eq}}$  (in a new language  $L^{\text{eq}}$ ), in which every definable set has a code. Let  $(E_i)_{i \in I}$ , an enumeration of every  $\emptyset$ -definable equivalence relation over  $n_i$ -tuples. To define  $L^{\text{eq}}$ , add to  $L$  a new sort  $S_i$  for each  $i$ , which is to be interpreted as  $M^{n_i}/E_i$ . Consider the many-sorted structure  $M^{\text{eq}} = (M, M^{n_i}/E_i)_{i \in I}$ , and define for every  $i$  the natural projection  $\pi_i : M^{n_i} \rightarrow M^{n_i}/E_i$  that sends  $a$  to  $a/E_i$ . The theory of  $M^{\text{eq}}$  will be denoted as  $T^{\text{eq}}$ . By Corollary 8.4.6 from [13],  $T^{\text{eq}}$  has *elimination of imaginaries*: every imaginary is interdefinable with a real tuple. There are also three related notions that will be used throughout this work.

**Definition 1.5.**

- i)  $T$  has *elimination of finite imaginaries* if for every  $n$ , every finite set of  $n$ -tuples has a canonical parameter.
- ii)  $T$  has *weak elimination of imaginaries*, if for every imaginary  $e$  there is a real tuple  $d$  such that  $e \in \text{dcl}^{\text{eq}}(c)$  and  $d \in \text{acl}^{\text{eq}}(e)$ .
- iii)  $T$  has *geometric elimination of imaginaries*, if for every imaginary  $e$  there is a real tuple  $d$  such that  $e \in \text{acl}^{\text{eq}}(c)$  and  $d \in \text{acl}^{\text{eq}}(e)$ .

We now proceed with a survey of forking in the  **$\omega$ -stable context**. For a definable set  $X \subseteq M$ , we denote by  $RM(X)$  its Morley rank, and  $DM(X)$  its Morley degree. Recall that  $\omega$ -stable theories are *totally transcendental*: every definable set has a Morley rank. This rank can also be defined for types: if  $p \in S_n(A)$ , then  $RM(p)$  is the minimal Morley rank of a formula in  $p$ , and  $DM(p)$  is the minimal Morley degree of a formula in  $p$  having Morley rank  $RM(p)$ .

**Definition 1.6. (Forking)** Suppose  $A \subseteq B \subseteq M$ ,  $p \in S_n(A)$ ,  $q \in S_n(B)$ , and  $p \subseteq q$ . If  $RM(p) = RM(q)$ , then  $q$  is a *non-forking* extension of  $p$  to  $B$ . Otherwise, if  $RM(p) < RM(q)$ , we say that  $q$  *forks over*  $A$ . We say that  $p \in S_n(A)$  is *stationary* if for all  $B \supseteq A$ , there is a unique non-forking extension of  $p$  to  $B$ , or equivalently if  $DM(p) = 1$ .

**Notation:** If  $p \in S(A)$  and  $C \subseteq A$ , we denote the restriction of  $p$  to  $S(C)$  by  $p \upharpoonright C$ . If  $p$  is stationary and  $A \subseteq B$ , we denote the unique non-forking extension of  $p$  to  $S(B)$  by  $p|B$ .

**Definition 1.7.** Let  $A \subseteq M$ ,  $p \in S(A)$  a stationary type. A *canonical base* of  $p$ , denoted  $\text{Cb}(p)$ , is a tuple  $e \subseteq M^{\text{eq}}$  such that for every  $\sigma \in \text{Aut}(M)$ ,  $\sigma(p) = p$  if and only if  $\sigma(e) = e$  (this tuple is unique up to interdefinability). If  $p$  is not stationary, consider the finite set  $\mathcal{P}$  of nonforking extensions of  $p$  to  $M$ , and define  $\text{cb}(p)$  as a code for the set  $\{\text{Cb}(q), q \in \mathcal{P}\}$ ; then any automorphism of  $M$  fixes  $\text{cb}(p)$  if and only if it permutes  $\mathcal{P}$  (see Fact 1.8 (i)).

The following is a summary of the properties of canonical bases we will use, they can be found as Proposition 2.20 and Remarks 2.26, 3.19 in Chapter 1 of [8].

**Fact 1.8.** Let  $A \subseteq M$ ,  $p \in S(A)$ . Then

- (i) (Conjugacy) The set of automorphisms of  $M$  that fix  $A$  pointwise acts transitively on  $\mathcal{P}$ .
- (ii)  $\text{cb}(p) \subseteq \text{dcl}^{\text{eq}}(A)$ .
- (iii) For any  $B \subseteq A$ ,  $p$  does not fork over  $B$  if and only if  $\text{cb}(p) \subseteq \text{acl}^{\text{eq}}(B)$ .
- (iv) If  $p$  is stationary, for any  $B \subseteq A$ ,  $p$  does not fork over  $B$  and  $p \upharpoonright B$  is stationary if and only if  $\text{Cb}(p) \subseteq \text{dcl}^{\text{eq}}(B)$ .
- (v) If  $p$  is stationary, and  $(a_i, i < \omega)$  is a sequence such that for all  $i$ ,  $a_i$  realizes  $p|A \cup \{a_j, j < i\}$ , then  $\text{Cb}(p) \subseteq \text{dcl}^{\text{eq}}(a_0, \dots, a_n)$  for some  $n$ .

**Lemma 1.9.** Let  $e$  be an imaginary in  $M$  and let  $a$  be a finite tuple of reals such that  $e = f(a)$  for some  $\emptyset$ -definable function  $f$ . Then  $e = \text{cb}(\text{tp}(a/e))$ . Moreover, if  $e' = \text{Cb}(\text{tp}(a/\text{acl}^{\text{eq}}(e)))$ , then  $e' \in \text{acl}^{\text{eq}}(e)$  and  $e \in \text{dcl}^{\text{eq}}(e')$ .

*Proof.* Let  $p = \text{tp}(a/e)$  and  $p' = \text{tp}(a/\text{acl}^{\text{eq}}(e))$ . To see why  $e = \text{cb}(\text{tp}(a/e))$ , consider the equivalence relation  $E(x, y)$  given by  $f(x) = f(y)$ ; then  $e$  is a code for the class of  $a$ . Let  $\mathcal{P}$  as in Definition 1.7. Since  $\mathcal{P}$  is finite, and  $e'$  is the canonical base of an element of  $\mathcal{P}$ , it follows that  $e' \in \text{acl}^{\text{eq}}(e)$ . Now, suppose  $\sigma(e') = e'$  for some automorphism of  $M^{\text{eq}}$ ; then  $\sigma p' = p'$ , so both formulas  $f(x) = e$  and  $f(x) = \sigma(e)$  belong to  $p'$ , which implies  $\sigma(e) = e$ , hence  $e \in \text{dcl}^{\text{eq}}(e')$ .  $\square$

**Lemma 1.10.** Let  $e$  be an imaginary in  $M$  and let  $a$  be a finite tuple of reals such that  $e = f(a)$  for some  $\emptyset$ -definable function  $f$ . There is  $a' \in M^{\text{eq}}$  such that  $e \in \text{dcl}^{\text{eq}}(a')$  and  $\text{tp}(a'/e)$  is stationary.

*Proof.* Let  $p = \text{tp}(a/e)$  and let  $p_1, \dots, p_n$  be its non-forking extensions to  $\text{acl}^{\text{eq}}(e)$ . Let  $a_1, \dots, a_n \in M$  be such that  $a_i$  realizes  $p_i| \{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n\}$ . Let  $a'$  be a code of this set of realizations. Then as  $a \in \text{acl}^{\text{eq}}(a')$ , there is a formula  $\varphi(x, a')$  isolating  $\text{tp}(a/a')$ ; hence  $M \models \forall x \varphi(x, a') \rightarrow f(x) = e$ , as  $f$  is  $\emptyset$ -definable, so  $e \in \text{dcl}^{\text{eq}}(a')$ . Moreover, any automorphism of  $M$  which fixes  $e$  permutes  $\{p_1, \dots, p_n\}$ , hence it fixes  $\text{tp}(a'/e)$ .  $\square$

**Definition 1.11. (Independence)** Let  $A, B, C \subseteq M$ . We say  $A$  is *independent* from  $B$  over  $C$ , denoted

$$A \perp_C B,$$

if for every finite tuple  $a$  from  $A$ ,  $\text{tp}(a/BC)$  does not fork over  $C$ .

The following is a summary of the properties of the independence relation in the  $\omega$ -stable context. These are found as Theorem 8.5.5 from [13], and Lemmas 6.3.16 through 6.3.21 from [7].

**Fact 1.12.** *Let  $A, B, C, D \subseteq M$ . Forking independence has the following properties.*

1. (Monotonicity) If  $A \perp_C B$  and  $B' \subseteq B$ , then  $A \perp_C B'$ .
2. (Transitivity)  $A \perp_C BD$  if and only if  $A \perp_C B$  and  $A \perp_{C,B} D$ .
3. (Existence) Every  $p \in S(A)$  has a non-forking extension to any set containing  $A$ .
4. (Symmetry) If  $A \perp_C B$ , then  $B \perp_C A$ .
5. (Algebraic Closure)  $A \perp_C \text{acl}(A)$ .

**Definition 1.13.** Let  $A, B \subseteq M$  and let  $p \in S(A)$  be definable over  $B$  by a scheme  $d_p$ . This definition scheme is called *good* (over  $B$ ) if the set

$$\{\varphi(x, m) \mid M \models d_p(\varphi)(m), m \in M, \varphi(x, y) \text{ an } L\text{-formula}\}$$

is a global type extending  $p$ .

**Lemma 1.14.** *Let  $p \in S(A)$ . Then  $p$  is stationary if and only if it has a good definition over  $A$ .*

*Proof.* If  $p$  is stationary, let  $q$  be its global non-forking extension. Then  $q$  is definable and invariant under all automorphisms that fix  $A$  setwise, hence it is definable over  $A$ . This gives a good definition for  $p$ . Conversely, assume  $p$  has a good definition over  $A$ . There is then a global non-forking extension  $p' \in \mathcal{P}$ , definable over  $A$ . Since all elements of  $\mathcal{P}$  are conjugate over  $A$ , and  $p'$  is fixed by every automorphism which fixes  $A$  setwise, it must be that  $\{p'\} = \mathcal{P}$ . Therefore,  $p$  is stationary.  $\square$

**Lemma 1.15.** *Let  $a \in M$  be a tuple and  $A \subseteq M$ . Suppose  $p = \text{tp}(a/A)$  is stationary and let  $a' \in M$  be a tuple such that  $a' \in \text{dcl}(Aa)$ . Then  $\text{tp}(a'/A)$  is stationary.*

*Proof.* We shall give a good definition scheme over  $A$  for  $\text{tp}(a'/A)$ . Let  $\varphi(x, y)$  be an  $L$ -formula and  $m \in M$  such that  $M \models \varphi(a', m)$ . By stationarity of  $p$ , there is an  $L(A)$ -formula  $d_p(\varphi)(y)$  such that  $\varphi(x, m) \in \text{tp}(a/A)$  if and only if  $M \models d_p(\varphi)(m)$ . By hypoth-

esis, there is an  $A$ -definable function  $f$  such that  $f(a) = a'$ . Let  $\tilde{\varphi}(x, y) = \varphi(f(x), y)$ ,

$$\begin{aligned} M \models \varphi(a', m) &\iff M \models \varphi(f(a), m) \\ &\iff M \models \tilde{\varphi}(a, m) \\ &\iff M \models d_p(\tilde{\varphi})(m). \end{aligned}$$

□

**Definition 1.16.** A type  $p(x) \in S(A)$  is said *internal* to a partial type  $\Sigma(y)$  if there are: a realization  $a$  of  $p$ , and  $B \supseteq A$  independent from  $a$  over  $A$ , such that  $a \in \text{dcl}(Bd)$  for some finite tuple  $d$  of realizations of  $\Sigma$ . If it happens instead that  $a \in \text{acl}(Bd)$ , then the type is said to be *almost internal* to  $\Sigma$ .

**Lemma 1.17.** Suppose  $\text{tp}(a/A)$  is stationary and almost internal to a partial type  $\Sigma$ . Then there is an imaginary  $a'$  such that:  $\text{tp}(a'/A)$  is stationary and internal to  $\Sigma$ ,  $a' \in \text{dcl}^{\text{eq}}(Aa)$ , and  $a \in \text{acl}^{\text{eq}}(a')$ . Such an  $a'$  can be taken to be a code for a finite set of realizations of  $\text{tp}(a/A)$ .

*Proof.* By hypothesis, there are  $B \supseteq A$  independent of  $a$  over  $A$  and a tuple  $d$  of realizations of  $\Sigma$  such that  $a \in \text{acl}(Bd)$ . We can replace  $B$  by some finite tuple  $b$  such that  $a \in \text{acl}(Abd)$ . Let  $q = \text{tp}(b, d/Aa)$  and  $c = \text{cb}(q)$ . By Fact 1.8 (ii),  $c \in \text{dcl}^{\text{eq}}(Aa)$ . Note that  $b \perp_A a$ , hence  $b \perp_A c$  and  $\text{tp}(c/A)$  is  $\Sigma$ -internal. By definition of  $c$ ,  $bd \perp_{Ac} Aa$ , but  $a \in \text{acl}(Abd)$ , hence  $a \in \text{acl}^{\text{eq}}(Ac)$ . If  $a'$  denotes the code for the finite set of conjugates of  $a$  over  $Ac$ , then

$$a' \in \text{dcl}^{\text{eq}}(Ac) \subseteq \text{dcl}^{\text{eq}}(Aa).$$

Because  $\text{tp}(c/A)$  is internal to  $\Sigma$ , so is  $\text{tp}(a'/A)$ . Moreover,  $\text{tp}(a'/A)$  is stationary by Lemma 1.15. We may note that  $c \in \text{acl}(Aa')$  as well: if  $a''$  is any conjugate of  $a$  over  $Ac$ , then it realizes the same type over  $Ac$  as  $a$ . □

**Lemma 1.18.** Let  $\Sigma$  be a partial type, and let  $p \in S(A)$  be a stationary,  $\Sigma$ -internal type. There exists a partial  $A$ -definable function  $h(y_1, \dots, y_m, z_1, \dots, z_n)$  and a sequence  $b_1, \dots, b_m$  of realizations of  $p$ , such that for any realization  $a$  of  $p$ , there is a sequence  $c_1, \dots, c_n$  of realizations of  $\Sigma$ , such that  $a = h(b_1, \dots, b_m, c_1, \dots, c_n)$ .

*Proof.* Let  $b$  realize  $p$ ,  $B \supseteq A$  independent from  $b$  over  $A$ , and  $d$  a tuple of realizations of  $\Sigma$  such that  $b \in \text{dcl}(Bd)$ .

**Claim:** For any  $b'$  realizing  $p|Ab$ , there is a sequence  $d'$  of realizations of  $\Sigma$  such that  $b' = g(b, d')$ , for some definable function  $g$ .

Let  $(b_i, d_i)_{i < \omega}$  be a Morley sequence of  $\text{tp}(b, d / \text{acl}^{\text{eq}}(B))$ . By Fact 1.8 (v),  $\text{tp}(b, d/M)$  is definable over  $A \cup \{b_i, d_i, i < \omega\}$ . In particular, for  $m$  large enough,

$$b \in \text{dcl}(b_1, \dots, b_m, d_1, \dots, d_m, d, A),$$

such that  $\bar{d} = (d_1, \dots, d_m, d)$ ,  $\bar{b} = (b_1, \dots, b_m)$  are independent from  $b$  over  $A$ . Then  $b = g(\bar{b}, \bar{d})$  for some  $A$ -definable function  $g$ . ■

Now let  $a$  be an arbitrary realization of  $p$ , and let  $\bar{a} = (a_1, \dots, a_m)$  realize  $\text{tp}(\bar{b} / \text{acl}(A))$  such that  $(a_1, \dots, a_m) \perp_A a\bar{b}$ . By the claim, for each  $i \leq m$  there is  $\bar{c}_i$ , a tuple of realizations of  $\Sigma$ , such that  $a_i = g(\bar{b}, \bar{c}_i)$ . Since  $\text{tp}(a, \bar{a}/A) = \text{tp}(b, \bar{b}/A)$ , we also get that  $a = g(\bar{a}, \bar{c})$  for some tuple  $\bar{c}$  of realizations of  $\Sigma$ . It follows that  $a = h(\bar{b}, \bar{c}, \bar{c}_1, \dots, \bar{c}_m)$  for an  $A$ -definable function  $h$ . □

The following is Lemma 7.2.12 from [13], which holds for all simple theories.

**Fact 1.19.** *For all  $A \subseteq M$  there is some  $\lambda$  such that for any sequence  $(a_i, i < \lambda)$  there exists an  $A$ -indiscernible sequence  $(b_j, j < \omega)$  such that for all  $j_1 < \dots < j_n < \omega$  there is a sequence  $i_1 < \dots < i_n < \lambda$  with  $\text{tp}(a_{i_1}, \dots, a_{i_n}/A) = \text{tp}(b_{j_1}, \dots, b_{j_n}/A)$ .*

**Lemma 1.20.** *If  $b \in \text{acl}(aA)$ , then  $\text{RM}(ab/A) = \text{RM}(a/A)$ .*

*Proof.* It is clear that  $\text{RM}(ab/A) \geq \text{RM}(a/A)$ , since the latter type contains less formulas. The reverse inequality is proved by induction on  $\alpha = \text{RM}(a/A)$ . Let  $d = DM(ab/Aa)$  be its Morley degree. Choose an  $L(A)$ -formula  $\varphi(x, y) \in \text{tp}(ab/A)$  such that  $\text{RM}(\exists y \varphi(x, y)) = \alpha$  and  $\varphi(a', y)$  has at most  $d$  realizations for all  $a'$ . If  $Y$  is the set defined by  $\exists x \varphi(x, y)$ , we claim that  $\text{RM}(Y) \leq \alpha$ . Consider an infinite family of pairwise disjoint definable subsets  $Y_i \subseteq Y$ . Let  $\psi_i(x) = \exists y (\varphi(x, y) \wedge y \in Y_i)$ . Note that any  $d+1$  of the  $\psi_i(M)$  have empty intersection: if  $M \models \bigwedge_{i=0}^d \psi_i(a')$ , then there exist  $b_i \in Y_i$  for  $0 \leq i \leq d$  such that  $\models \varphi(a', b_i)$ , which contradicts our choice of  $\varphi$ . Therefore, some  $\psi_i(x)$  has Morley rank  $\beta < \alpha$ . Let  $b' \in Y_i$ , and choose  $a'$  such that  $M \models \varphi(a', b')$ . Then  $b'$  is algebraic over  $a'A$  and since  $a'$  realizes  $\psi_i(x)$ , we have  $\text{RM}(a'/A) \leq \beta$ . So by induction hypothesis, we conclude  $\text{RM}(a'b'/A) \leq \beta$ , which shows  $\text{RM}(Y_i) \leq \beta$ . This implies that  $Y$  does not contain an infinite family of disjoint subsets of Morley rank  $\geq \alpha$ . □

The following definition comes from 10.2.8 of [13].

**Definition 1.21.** Let  $A, B \subseteq M$  be definable sets and let  $f : B \rightarrow A$  be a definable function. The fibers of  $f$  have definable Morley rank if for every definable  $B' \subseteq B$  and every  $k < \omega$ , the set  $\{a \in A, RM(f^{-1}(a) \cap B') = k\}$  is definable.

**Lemma 1.22.** Let  $A, B \subseteq M$  be definable sets and let  $f : B \rightarrow A$  be a definable surjection whose fibers have definable Morley rank, and such that for all  $a \in A$ ,  $RM(f^{-1}(a)) = k$ . Then  $RM(B) = RM(A) + k$ .

*Proof.* Suppose  $A, B, f$  are definable over some  $S \subseteq M$ . The proof is by induction on  $RM(A) = m$ , for all  $k$ . We may also assume that  $DM(A) = 1$ : if not, then partition  $A = A_1 \cup \dots \cup A_d$  into finitely many disjoint, rank  $m$  definable subsets, then replace  $B, A$  by  $f^{-1}(A_1), A_1$ , respectively. If  $m = 0$ , then  $A$  is finite and

$$RM(B) = \max\{RM(f^{-1}(a))\}_{a \in A} = k.$$

If  $m > 0$ , then write  $A = \bigcup_i A_i$  for an infinite, pairwise disjoint family of definable  $A_i \subseteq A$  such that  $RM(A_i) = m - 1$ . If  $B_i = f^{-1}(A_i)$ , then  $f \upharpoonright B_i$  is a definable surjection with rank  $k$  fibers, so by induction hypothesis  $RM(B_i) = RM(A_i) + k = m + k - 1$ , and since the  $B_i$  are also pairwise disjoint, we deduce  $RM(B) \geq m + k$ . For the reverse inequality, let now  $(B'_i)_{i < \omega}$  be any infinite, pairwise disjoint family of definable subsets  $B'_i \subseteq B$ ; we will show that  $RM(B'_i) < m + k$  for some  $i$ , by induction over  $k$ . If  $k = 0$ ,  $f$  is finite-to-one, so for any  $b \in B$ ,  $b \in \text{acl}(f(b))$  and  $f(b) \in \text{dcl}(b)$ . By Lemma 1.20,

$$RM(b/S) = RM(f(b), b/S) = RM(f(b)/S) \leq RM(A) = m.$$

This implies  $RM(B) = \sup_{b \in B} (RM(b/S)) \leq m$ . Suppose now the conclusion holds for  $m$  and for every  $k' < k$ . Let  $a \in A$ , then since  $RM(f^{-1}(a)) = k$  and  $f^{-1}(a) \supseteq \bigcup_i (f^{-1}(a) \cap B'_i)$ , it must be that for some  $j$ ,  $RM((f^{-1}(a) \cap B'_j) < k$ . Consider now the definable sets

$$A'_i = \{a \in A, RM(f^{-1}(a) \cap B'_i) = k\}.$$

We have proved that  $\bigcap_i A'_i = \emptyset$ . We claim that for some  $i$ ,  $RM(A'_i) < m$ : if not, then as  $DM(A) = 1$ , for every  $N$ ,  $\bigcap_{i \leq N} A'_i \neq \emptyset$ , hence by compactness  $\bigcap_i A'_i \neq \emptyset$ , a contradiction, since the Morley degree of the fibers is bounded. Now, as

$$B'_i = (f^{-1}(A'_i) \cap B'_i) \cup (f^{-1}(A \setminus A'_i) \cap B'_i),$$

we can apply induction hypothesis over  $m$  to the first term to see that  $RM(f^{-1}(A'_i) \cap B'_i) < m+k$ . On the other hand, on  $A \setminus A'_i$ , all fibers have rank strictly less than  $k$ . The induction hypothesis over  $k$  yields  $RM(f^{-1}(A \setminus A'_i) \cap B'_i) < m+k$ , concluding the proof.  $\square$

**Lemma 1.23.** *Let  $P \subseteq M$  be a strongly minimal definable set, and  $\varphi(x_1, \dots, x_n, \bar{y})$  a formula such that  $M \models \forall x_1, \dots, x_n (\exists \bar{y} \varphi(x_1, \dots, x_n, \bar{y}) \rightarrow x_i \in P)$ . The following set is definable for every  $k$ ,*

$$Y_{n,k} = \{\bar{b} \in M, RM\varphi(x_1, \dots, x_n, \bar{b}) = k\}.$$

*Proof.* Let  $Y'_{n,k} = \{\bar{b} \in M, RM\varphi(x_1, \dots, x_n, \bar{b}) \geq k\}$ . We will prove definability of  $Y'_{n,k}$ , this gives the desired result since  $Y_{n,k} = Y'_{n,k} \setminus Y'_{n,k+1}$ . We proceed by induction on  $n$ . Notice that  $Y'_{1,1}$  is definable since  $RM(\varphi(x_1, \bar{b})) \geq 1$  if and only if  $\exists^\infty x_1 \varphi(x_1, \bar{b})$ , which is in turn equivalent (by strong minimality of  $P$ ) to  $\exists^{\geq N} x_1 \varphi(x_1, \bar{b})$ , for some  $N$ . Moreover, notice that

$$Y_{n,0} = \{\bar{b} \in M, \exists x_1, \dots, x_n \varphi(x_1, \dots, x_n, \bar{b})\}$$

is definable for all  $n$ . Let now  $n > 0$ , we will work by induction over  $k > 0$ . For  $\bar{b} \in P$ , consider the  $\bar{b}$ -formula  $\phi_{\bar{b}}(x_0, \dots, x_{n-1})$  given by  $\exists x_n \varphi(x_0, \dots, x_{n-1}, x_n, \bar{b})$ . If  $RM(\phi_{\bar{b}}) \geq k$ , then  $\bar{b} \in Y'_{n,k}$ , and if  $RM(\phi_{\bar{b}}) < k$ , consider instead the  $L(\bar{b})$ -formula  $\psi_{\bar{b}}(x_0, \dots, x_{n-1})$  given by  $\exists^\infty x_n \varphi(x_0, \dots, x_{n-1}, x_n, \bar{b})$ , then since the algebraic dimension of a tuple inside  $P$  agrees with its Morley rank, we have in this case that  $RM(\psi_{\bar{b}}) \geq k-1$  if and only if  $\bar{b} \in Y'_{n,k}$ . We have shown that  $\bar{b} \in Y'_{n,k}$  if and only if  $RM(\phi_{\bar{b}}) \geq k$  or  $RM(\psi_{\bar{b}}) \geq k-1$ . The first of these two conditions is definable by our induction hypothesis over  $n$ , while the latter is definable by induction over  $k$ , so  $Y'_{n,k}$  is also definable.  $\square$

## 2 Stable Groups

An  $\omega$ -stable group is an  $\omega$ -stable structure  $(G, \cdot, 1, \dots)$ , where  $(G, \cdot, 1)$  is a group. In this section we present some basic concepts and tools used in the study of  $\omega$ -stable groups. For more details, see [11] and Chapter 7 from [7]. Throughout this section  $G$  will denote an infinite,  $\omega$ -stable group, definable inside a saturated model  $M$  of a complete,  $\omega$ -stable theory  $T$ .

**Lemma 2.1.** *There is no infinite strictly descending chain of definable subgroups  $G > G_1 > G_2 > \dots$*

*Proof.* For any definable subgroup  $H \leq G$ , and any  $a \in G \setminus H$ , the coset  $aH \subseteq G$  is disjoint from  $H$ , and since  $x \mapsto ax$  is a definable bijection, then  $RM(H) = RM(aH)$ . If  $G > G_1 > G_2 > \dots$  is a strictly decreasing sequence, and if  $[G_i : G_{i+1}]$  is infinite, then  $RM(G_i) > RM(G_{i+1})$ . If  $[G_i : G_{i+1}]$  is finite, then  $DM(G_i) > DM(G_{i+1})$ , and this implies the existence of a strictly decreasing sequence with respect to the lexicographic order  $RM(G) \times \omega$ , hence, this sequence cannot be infinite.  $\square$

**Lemma 2.2.** *There is a definable normal subgroup  $G^0 \leq G$  that is contained in every subgroup of  $G$  of finite index.*

*Proof.* Let  $\mathcal{H}$  be the family of definable subgroups of  $G$  of finite index. We claim that there are  $H_1, \dots, H_n$  in  $\mathcal{H}$  such that

$$\bigcap_{H \in \mathcal{H}} H = H_1 \cap \dots \cap H_n.$$

If not, then for every  $m$  there are  $j_0, \dots, j_m$  such that if  $G_m = H_{j_0} \cap \dots \cap H_{j_m}$ , then  $G_0 > G_1 > G_2 > \dots$ , contradicting Lemma 2.1. We may then define  $G^0 = H_1 \cap \dots \cap H_n$ . If  $h \in G$ , since  $x \mapsto hxh^{-1}$  is a group automorphism, we have that  $hG^0h^{-1}$  is a definable subgroup with  $[G : hG^0h^{-1}] = [G : G^0]$ , so  $hG^0h^{-1} = G^0$  by minimality.  $\square$

**Lemma 2.3.** *Let  $A \subseteq M$ . If  $G$  is  $A$ -definable, then  $G^0$  is  $A$ -definable.*

*Proof.* By Lemma 2.2, there are an  $L(A)$ -formula  $\varphi(x, y)$  and  $g \in G$  such that the formula  $\varphi(x, g)$  defines  $G^0$ . Let  $n = [G : G^0]$ , and consider

$$W = \{b \in G, \varphi(x, b) \text{ defines a subgroup of index } n\},$$

an  $A$ -definable set. If  $b \in W$  and  $H = \varphi(G, b)$ , then  $H \cap G^0$  is a finite index subgroup of  $G^0$ , hence  $H \supseteq G^0$ . However, since  $[G : H] = n$ , we have  $[H : G^0] = 1$ , yielding  $H = G^0$ . We can then define  $G^0$  as  $\{g \in G, \exists b (b \in W \wedge \varphi(g, b))\}$ .  $\square$

**Definition 2.4.**  $G$  is *connected* if  $G = G^0$ .

**Definition 2.5.** There is an action of  $G$  on  $S_1(G)$  given by  $g \cdot p = \{\varphi(x), \varphi(gx) \in p\}$ . The *stabilizer* of  $p$  is the group

$$\text{Stab}(p) = \{g \in G, g \cdot p = p\}.$$

**Lemma 2.6.**  $\text{Stab}(p)$  is a definable subgroup of  $G$ , for every  $p \in S_1(G)$ .

*Proof.* For  $\varphi(x, y)$  an  $L$ -formula, let

$$\text{Stab}_\varphi(p) = \{g \in G \mid p_\varphi = g \cdot p_\varphi\},$$

where

$$p_\varphi = \{\varphi(x, g) \mid g \in G, \varphi(x, g) \in p\} \cup \{\neg\varphi(x, g) \mid g \in G, \varphi(x, g) \notin p\}.$$

An easy calculation shows that for every  $\varphi$ ,  $\text{Stab}_\varphi(p) \leq G$ . By stability, there is a definition scheme for  $p$ , say  $d_p$ . Thus,

$$\text{Stab}_\varphi(p) = \{g \in G \mid \forall h (d_p(\varphi)(h) \leftrightarrow d_p(\varphi)(hg))\}.$$

Note that  $\text{Stab}(p) = \bigcap_{\varphi(x, y) \in L} \text{Stab}_\varphi(p)$ . By Lemma 2.1, there are  $\varphi_1, \dots, \varphi_n \in L$  such that  $\text{Stab}(p) = \text{Stab}_{\varphi_1}(p) \cap \dots \cap \text{Stab}_{\varphi_n}(p)$ , which concludes the proof.  $\square$

**Lemma 2.7.** Let  $p \in S_1(G)$ .

(i)  $RM(\text{Stab}(p)) \leq RM(p)$ .

(ii)  $\text{Stab}(p) \leq G^0$ .

*Proof.* Let  $a, b \in M$  be such that  $a$  realizes  $p$ ,  $b \in \text{Stab}(p)$  satisfies  $RM(\text{tp}(b/G)) = RM(\text{Stab}(p))$ , and  $a \perp_G b$ . Then

$$RM(\text{tp}(ba/G, a)) = RM(\text{tp}(b/G, a)) = RM(\text{tp}(b/G)) = RM(\text{Stab}(p)).$$

Moreover, since  $ba$  realizes  $p$ , we have  $RM(\text{tp}(ba/G, a)) \leq RM(\text{tp}(ba/G)) = RM(p)$ , proving (i). Let now  $c \in \text{Stab}(p)$ , and let  $\varphi(x)$  define  $G^0$  (possibly with parameters in  $M$ ). Let  $g \in G$  be such that  $\varphi(g^{-1}x) \in p$ , thus  $\varphi(g^{-1}cx) \in p$ . Let  $G \preceq H$  and  $h \in H$  realize  $p$ . Then  $g^{-1}ch \in H^0$  and  $g^{-1}h \in H^0$ . Thus  $(g^{-1}h)^{-1}g^{-1}ch = h^{-1}ch \in H^0$ , and since  $H^0$  is normal,  $c \in G^0$  by Lemma 2.3.  $\square$

**Definition 2.8.** A type  $p \in S_1(G)$  is *generic* if  $RM(p) = RM(G)$ . An element  $a \in G(M)$  is generic over  $A \subseteq G$  if  $RM(\text{tp}(a/A)) = RM(G)$ .

**Lemma 2.9.** A type  $p \in S_1(G)$  is generic if and only if  $[G : \text{Stab}(p)]$  is finite.

*Proof.* Suppose  $p$  is generic. Notice that  $\{ap, a \in G\}$  is finite, since there are only finitely many types of maximal Morley rank. Choose  $b_1, \dots, b_n \in G$  such that if  $a \in G$ , then  $ap = b_i p$  for some  $i \leq n$ . If  $ap = b_i p$  then  $b_i^{-1}a \in \text{Stab}(p)$  and  $a \in b_i \text{Stab}(p)$ . Therefore,  $[G : \text{Stab}(p)] \leq n$ . Assume now that  $\text{Stab}(p)$  has finite index, so  $RM(G) = RM(\text{Stab}(p))$ , but  $RM(\text{Stab}(p)) \leq RM(p)$  by Lemma 2.7, hence  $p$  is generic.  $\square$

**Corollary 2.10.**

- (i) A type  $p \in S_1(G)$  is generic if and only if  $\text{Stab}(p) = G^0$
- (ii)  $G$  has an unique generic type if and only if  $G$  is connected.

*Proof.*

- (i) By Lemma 2.9, if  $p$  is generic,  $\text{Stab}(p)$  has finite index, we have  $G^0 \leq \text{Stab}(p)$ . By Lemma 2.7 (ii), we have  $G^0 \geq \text{Stab}(p)$ . The other direction is clear by Lemma 2.9, since  $G^0$  has finite index.
- (ii) Let  $p$  be the unique generic type. For all  $a \in G$ ,  $ap$  is generic, hence  $ap = p$ . Thus,  $G = \text{Stab}(p) = G^0$  by (i). Conversely, suppose  $G = G^0$ , and by contradiction, assume  $p, q$  are distinct, generic types. Let  $a, b$  realize  $p, q$  respectively, with  $b \in H \succeq G$  and let  $a'$  realize  $p|H$ . Then,  $\text{tp}(a, b/G) = \text{tp}(a', b/G)$ , and  $p|H$  is a generic of  $H$ . By (i),  $\text{Stab}(p|H) = H^0 = H$ . Thus,  $ba'$  realizes  $p|H$ . In particular,  $ba'$  realizes  $p$ , hence  $ba$  realizes  $p$ . If  $a \in K \succeq G$ , and  $b'$  realizes  $q|K$ , an analogous argument shows that  $ba$  realizes  $q$ . This contradicts our assumption, hence  $G$  has a unique generic type.

$\square$

### 3 Pairs of Algebraically Closed Fields

Throughout this section, we will let  $T = ACF_p$  for  $p$  prime or 0 (in the usual language  $L$ ), and we consider  $L_P$ , the language obtained by adjoining a unary predicate  $P$ . An elementary pair of models of  $T$ ,  $N \preceq M$ , is considered an  $L_P$ -structure by interpreting  $P$  as the universe of the structure  $N$ , and  $L_P$ -structures will be naturally denoted as pairs  $(M, P)$ .

**Definition 3.1.** A *beautiful pair* of models of  $T$  is an elementary pair  $N \preceq M$  such that  $N$  is  $|T|^+$ -saturated and  $M$  is  $|T|^+$ -saturated over  $N$ , which means that  $M$  realizes any  $L$ -type over  $N \cup A$ , where  $A \subseteq M \setminus N$  is such that  $|A| < |T|^+$ . The theory  $T_P$  of proper pairs  $P \prec M$  of models of  $T$  was shown to be complete by Keisler in [5].

**Fact 3.2.** ([12]) Let  $(M, P)$  be a saturated model of  $T_P$ .

(i)  $(M, P)$  is a beautiful pair.

(ii)  $T_P$  is stable.

(iii) Any  $L_P$ -formula  $\phi(x)$  is equivalent modulo  $T_P$  to a Boolean combination of  $L_P$ -formulas of the form  $\exists y P(y) \wedge \psi(y, x)$  where  $\psi$  is a quantifier-free  $L_P$ -formula.

Buechler in [2] notes that  $T_P$  is actually  $\omega$ -stable of Morley rank  $\omega$ . From now on  $(M, P)$  will be a saturated model of  $T_P$ .

**Notation:** If  $A \subseteq M$ , we denote the field generated by  $A$  by  $\langle A \rangle$ . For any  $A, B, C \subseteq M$ , we denote independence in the sense of  $L$  by  $A \perp_C^L B$ , and in the sense of  $L_P$  by  $A \perp_C^{L_P} B$ . We will also distinguish  $L$ -types from  $L_P$ -types by using  $\text{tp}_L$  and  $\text{tp}_{L_P}$  respectively. We adopt the same convention for the acl and dcl operators.

**Lemma 3.3.** Any  $C \subset P^n$  that is  $L_P$ -definable with parameters from  $M$ , is  $L$ -definable with parameters from  $P$ . In particular,  $P$  is strongly minimal and stably embedded (in the  $L_P$ -sense).

*Proof.* Let  $\varphi(x, m)$  with  $m \in M$  be an  $L_P$ -formula defining  $C$ . Note that  $P$  is algebraically closed in the  $L_P$ -sense. By stability of  $T_P$ ,  $p(y) = \text{tp}_{L_P}(m/P)$  is definable over  $P$ , hence we have that for every  $a \in M$ ,

$$a \in C \iff \varphi(a, y) \in p \iff M \models d\varphi(a),$$

where  $d\varphi(x)$  is an  $L_P$ -formula with parameters in  $P$ . Now, by Fact 3.2,  $d\varphi(x)$  is equivalent to a boolean combination of formulas of the form  $\exists z P(z) \wedge \psi(x, z)$  where  $\psi$  is a quantifier-free  $L_P$ -formula. Since  $C \subseteq P^n$  and

$$M \models \forall x \ d\varphi(x) \rightarrow P(x),$$

$C$  is  $L$ -definable by a boolean combination of formulas of the form  $\exists z \psi'(x, z)$ , where  $\psi'$  is the  $L$ -formula obtained from  $\psi$  by replacing every instance of  $P(t)$  by  $t = t$ , for every term  $t$ .  $\square$

**Remark 3.4.** By elimination of quantifiers in  $T$ , the formula  $\exists z \psi'(x, z)$  is equivalent modulo  $T$  to a quantifier free  $L$ -formula  $\theta(x)$ . Notice also that the set  $C$  only depends on  $m$ , so if  $c$  is an  $L_P$ -code for  $C$ , we have that  $c \in \text{dcl}_{L_P}^{\text{eq}}(m) \cap P$ .

**Definition 3.5.** Let  $a \in M$  a (possibly infinite) tuple, define  $\hat{a} = (a, a^c)$ , where  $a^c = \text{Cb}(\text{tp}_L(a/P))$ . Since  $T$  is totally transcendental and eliminates imaginaries,  $a^c$  is in the  $L$ -definable closure of a finite real tuple. More specifically,  $a^c$  can be regarded up to interdefinability as a tuple of generators for the field of definition of the algebraic locus of  $a$  over  $P$  (i.e: the variety associated to the prime ideal of polynomials in  $P[X]$  that vanish at  $a$ ).

**Lemma 3.6.** *For all tuples  $a \in M$ ,  $\langle \hat{a} \rangle$  is linearly disjoint from  $P$  over  $\langle a^c \rangle$ .*

*Proof.* Note that  $\langle \hat{a} \rangle = \langle a^c \rangle(a)$ . Let  $\{M_0(X), \dots, M_m(X)\}$  be a set monomials such that  $\{M_0(a), \dots, M_m(a)\}$  is linearly independent over  $\langle a^c \rangle$ . Suppose there is a linear relation  $\sum c_i M_i(a) = 0$ , where  $c_i \in P$ . By definition of  $a^c$  we can write

$$\sum_{i=0}^m c_i M_i(X) = \sum_{j=0}^n b_j f_j(X),$$

where  $b_j \in \langle a^c \rangle$ ,  $f_j(a) \in I := \{f(X) \in \langle a^c \rangle(X), f(a) = 0\}$  for all  $j$ , and such that  $\{f_0(X), \dots, f_n(X)\}$  is a linearly independent set of polynomials over  $\langle a^c \rangle$ . We claim that  $\{M_1, \dots, M_m, f_1, \dots, f_n\}$  is also linearly independent over  $\langle a^c \rangle$ : if it were not, then  $\sum r_i M_i(X) + \sum s_j f_j(X) = 0$ , for some  $r_i, s_j \in \langle a^c \rangle$ . We can substitute  $a$  for  $X$  to obtain  $\sum r_i M_i(a) = 0$ , which yields  $r_i = 0$  for all  $i$ , hence  $\sum s_j f_j(X) = 0$  and  $s_j = 0$  for all  $j$ . As these are formal polynomials, they remain linearly independent over  $P$ , hence  $c_i = 0$  for all  $i$ .  $\square$

**Remark 3.7.** For all tuples  $a \in M$ ,  $a^c \in \text{dcl}_{L_P}(a)$ .

*Proof.* Any  $L_P$ -automorphism leaves  $P$  invariant, so if it also fixes  $a$ , it must leave  $\text{tp}_L(a/P)$  invariant, hence it must fix  $a^c$ .  $\square$

**Lemma 3.8.** For all tuples  $a, b \in M$ ,  $\text{tp}_{L_P}(a) = \text{tp}_{L_P}(b)$  if and only if  $\text{tp}_L(\widehat{a}) = \text{tp}_L(\widehat{b})$ .

*Proof.* If there is an  $L_P$ -automorphism  $\sigma$  of  $M$  taking  $a$  to  $b$ , by Remark 3.7 we have  $\sigma(a^c) = b^c$ , hence  $\text{tp}_{L_P}(\widehat{a}) = \text{tp}_{L_P}(\widehat{b})$ . Restricting the language yields  $\text{tp}_L(\widehat{a}) = \text{tp}_L(\widehat{b})$ . Conversely, assume there is a partial  $L$ -isomorphism  $\sigma$  sending  $a$  to  $b$  and  $a^c$  to  $b^c$ . Since  $\langle \widehat{a} \rangle$  and  $P$  are linearly disjoint over  $\langle a^c \rangle$ , and also  $\langle \widehat{b} \rangle$  and  $P$  are l.d. over  $\langle b^c \rangle$ , the restriction  $\sigma \upharpoonright \langle \widehat{a} \rangle$  can be extended to an  $L$ -isomorphism  $\sigma' : P(a) \rightarrow P(b)$  such that  $\sigma'(P) = P$  or in other words, to an  $L_P$ -isomorphism, which can be itself extended to an  $L_P$ -automorphism of  $M$  by saturation of  $M$  over  $P$  (see Fact 3.2 (iii)).  $\square$

**Lemma 3.9.** For all tuples  $a \in M$ ,

$$(i) \quad a^c \subseteq P.$$

$$(ii) \quad \text{If } b \in P \text{ is a tuple, } \widehat{ab} \text{ and } \widehat{ab} \text{ are } L\text{-interdefinable.}$$

$$(iii) \quad \widehat{a} \perp_{a^c}^{L_P} P$$

*Proof.* By elimination of imaginaries in  $T$ ,  $a^c \in \text{acl}_L^{\text{eq}}(P) = P$ , this gives (i). To see (ii), notice that  $a \perp_{a^c}^L P$  implies  $ab \perp_{a^c b}^L P$ , and since  $\text{tp}_L(ab/a^c b)$  is stationary, we get  $(ab)^c \subseteq \text{dcl}_L(a^c b)$ , hence  $\widehat{ab} \in \text{dcl}_L(\widehat{ab})$ . Clearly  $\widehat{ab} \subseteq \widehat{ab}$ , so the other direction follows. For (iii), let  $a^c \in B \subseteq P$ , and choose an  $L_P$ -indiscernible sequence over  $a^c$ ,  $(B_i)_{i < \omega}$ , such that  $B_0 = B$ . Let  $p = \text{tp}_L(\widehat{a}/B)$ , and for each  $i$  let  $p_i$  be the image of  $p$  under an  $L$ -automorphism that fixes  $a^c$  and sends  $B$  to  $B_i$ . As  $\widehat{a} \perp_{a^c}^L P$ ,  $B_i \subseteq P$ , and  $\widehat{a} \perp_{a^c}^L B_i$  for every  $i$ ,  $\widehat{a}$  realizes  $\cup_i p_i$ . By Lemma 3.8 and (ii),  $\text{tp}_L(\widehat{a}/P) \vdash \text{tp}_{L_P}(\widehat{a}/P)$ . If we let  $p' = \text{tp}_{L_P}(\widehat{a}/B)$  and  $p'_i$  be the image of  $p'$  under an  $L_P$ -automorphism that fixes  $a^c$  and sends  $B$  to  $B_i$ , we have proven consistency of  $\cup_i p'_i$ . Hence,  $\text{tp}_{L_P}(\widehat{a}/B)$  does not fork over  $a^c$ . Since  $B$  was chosen arbitrarily, the result follows.  $\square$

**Definition 3.10.** A subset  $A$  of  $M$  is said to be  $P$ -independent if  $A \perp_{A \cap P}^L P$ .

**Remark 3.11.**

$$(i) \quad \text{For all } a \in M, \widehat{a} \text{ is } P\text{-independent.}$$

$$(ii) \quad \text{Any subset of } P \text{ is } P\text{-independent.}$$

*Proof.* The first condition follows directly from Lemma 3.9 (i), (iii), and monotonocity. The second statement is clear.  $\square$

**Lemma 3.12.** *Let  $A \subseteq B, C \subseteq M$  with  $C = Ac$ , where  $c \in M$  is a finite tuple. The following are equivalent:*

- (i)  $C \perp_A^{L_P} B$
- (ii)  $C \perp_{AP}^L BP$ , and  $C^c \perp_{A^c}^L B^c$ .
- (iii)  $C \perp_{AP}^L BP$ , and  $\widehat{C} \perp_A^L \widehat{B}$ .

*Proof.*

(i) implies (ii): By Remark 3.7, we may assume  $B = \widehat{B}$ . For the first part, suppose by contradiction that  $\text{tp}_L(c/BP)$  forks over  $AP$ . Let  $(B_i)_{i < \lambda}$  be a sequence of realizations of  $\text{tp}_L(B/AP)$  such that  $B_i \perp_{AP}^L (B_j)_{j < i}$  and  $B_0 = B$ ; note that in particular, as  $\widehat{B}_i = B_i$ , for all  $i$ , we get  $\text{tp}_{L_P}(B_i) = \text{tp}_{L_P}(B)$  by Lemma 3.9. We can choose  $\lambda$  large enough to apply Fact 1.19, yielding an  $L_P$ -indiscernible sequence over  $AP$ ,  $(B'_i)_{i < \omega}$ , such that  $\text{tp}_{L_P}(B'_i/AP) = \text{tp}_{L_P}(B/AP)$  for all  $i$ . Let  $p = \text{tp}_{L_P}(c/B)$  and let  $p_i$  be its copy over  $B'_i$ ; then by (i),  $\cup_{i < \omega} p_i$  can be realized by some  $c'$ . We have that for all  $i$ ,  $c' \not\perp_{AP}^L B'_i P$ : this contradicts the  $\omega$ -stability of  $T$ , since  $(B'_i)_{i < \omega}$  is also  $L$ -independent over  $AP$ . To prove the latter part of (ii), we apply properties of forking: by Lemma 3.9 (iii) we have that  $\widehat{A} \perp_{A^c}^L P$ , which implies by symmetry and monotonicity that  $C^c \perp_{A^c}^L \widehat{A}$ . Additionally, Remark 3.7 gives

$$\begin{aligned} C \perp_A^{L_P} B &\Rightarrow CC^c \perp_{AA^c}^{L_P} BB^c \\ &\Rightarrow C^c \perp_{\widehat{A}}^L B^c, \text{ by monotonicity.} \end{aligned}$$

Applying transitivity,  $C^c \perp_{A^c}^L B^c$ . As these three sets all lie in  $P$ , we actually get the desired independence in the  $L$ -sense.

(ii) implies (iii): We will prove  $AC^c \perp_{\widehat{A}}^L \widehat{B}$  and  $\widehat{C} \perp_{AC^c}^L \widehat{B}$ , then (iii) will follow by transitivity and because  $A^c \subseteq C^c$ . To get the first relation, start from  $\widehat{B} \perp_{B^c}^L P$  and use  $C^c \subseteq P$  to get  $\widehat{B} \perp_{B^c} C^c$ . Combining this with our hypothesis  $C^c \perp_{A^c}^L B^c$ , we get  $C^c \perp_{A^c}^L \widehat{B}$ , which implies  $AC^c \perp_{\widehat{A}}^L \widehat{B}$  since  $A^c \subseteq \widehat{A} \subseteq \widehat{B}$ . For the second relation, start from  $\widehat{C} \perp_{C^c} P$  and  $A \subseteq C$  to get  $\widehat{C} \perp_{AC^c}^L AP$  (I). Now, the hypothesis  $C \perp_{AP}^L BP$  yields  $\widehat{C} \perp_{AP}^L \widehat{B}$  (II), since  $B^c, C^c \subseteq P$ . Combining (I) and (II) gives  $\widehat{C} \perp_{AC^c}^L \widehat{B}$ .

(iii) implies (i):

**Claim:** (iii) implies  $\widehat{Ac}\widehat{B}$  is  $P$ -independent.

$\widehat{Ac}\widehat{B}$  is  $P$ -independent is equivalent to saying that if  $t_C, t_B$  are transcendence bases for  $\widehat{Ac}, \widehat{B}$  over  $\widehat{AP} = AP$  respectively, then  $t_C \cup t_B$  remains algebraically independent over  $AP$ , which is equivalent to  $C \perp_{AP}^L BP$ . ■

Let  $(\widehat{B}_i)_i$  be an  $L_P$ -indiscernible sequence over  $\widehat{A}$  with  $\widehat{B}_0 = \widehat{B}$ . By hypothesis  $C^c \perp_{\widehat{A}}^L \widehat{B}$ , so we may assume that  $(\widehat{B}_i)_i$  is also  $L$ -indiscernible over  $\widehat{AC}^c$ . Let  $p = \text{tp}_L(\widehat{c}/\widehat{BC}^c)$ , and let  $p_i$  be its copies over  $\widehat{B}_i C^c$ . By the first condition of (iii), we can realize  $\cup_i p_i$  by some  $\widehat{C}'$  which is  $L$ -independent from  $P$  over  $\cup_i \widehat{B}_i C^c$ . By Lemma 3.8 and by the claim, it follows that  $\text{tp}_{L_P}(\widehat{C}'\widehat{B}_i C^c) = \text{tp}_{L_P}(\widehat{C}\widehat{B}_i C^c)$ , so  $p$  does not  $L_P$ -fork over  $\widehat{A}$ . Therefore,  $\widehat{C} \perp_{\widehat{A}}^{L_P} \widehat{B}$ , and (i) follows from Remark 3.7. □

**Lemma 3.13.** *Let  $a \in M$ , then*

$$i) \text{acl}_{L_P}(a) = \text{acl}_L(\widehat{a}).$$

$$ii) \text{dcl}_{L_P}(a) = \text{dcl}_L(\widehat{a}).$$

*Proof.* In both cases, the inclusion  $\supseteq$  follows from Remark 3.7.

- i) First we show that  $\text{acl}_{L_P}(a) \cap P = \text{acl}_L(a^c)$ . Let  $b \in \text{acl}_{L_P}(a) \cap P$ , then since  $a \perp_{a^c}^L P$ , we get  $\widehat{a} \perp_{a^c}^L b$ . Suppose  $b \notin \text{acl}_L(a^c)$ , so  $b \notin \text{acl}_L(\widehat{a})$ . Then, in  $P$ , there are infinitely many  $(b_i, i < \omega)$  such that  $\text{tp}_L(\widehat{a}b_i) = \text{tp}_L(\widehat{a}b)$ . By Lemma 3.9 (ii),  $\text{tp}_L(\widehat{a}b_i) = \text{tp}_L(\widehat{a}b)$ . By Lemma 3.8, these  $b_i$  are also  $L_P$ -conjugate over  $\widehat{a}$ , a contradiction. Consider now  $b' \in M \setminus P$  such that  $b' \in \text{acl}_{L_P}(a)$  but  $b' \notin \text{acl}_L(\widehat{a})$ . Then

$$(b'\widehat{a})^c \in \text{dcl}_{L_P}(b'\widehat{a}) \cap P \subseteq \text{acl}_{L_P}(\widehat{a}) \cap P = \text{acl}_L(a^c),$$

which implies by Fact 1.8 (iii) that  $b'\widehat{a} \perp_{a^c}^L P$ , hence  $b' \perp_{\widehat{a}}^L P$ . By assumption there are infinitely many  $L$ -conjugates of  $b'$  over  $\widehat{a}$ . Since  $M$  is saturated over  $P$ , there are infinitely many realizations of  $\text{tp}_L(b'/\widehat{a}P)$ . This implies there are infinitely many realizations of  $\text{tp}_{L_P}(b'/\widehat{a})$ , a contradiction.

- ii) The proof is similar. First, we show that  $\text{dcl}_{L_P}(a) \cap P = \text{dcl}_L(a^c)$ , so let  $b \in \text{dcl}_{L_P}(a) \cap P$ . Then, by (i),  $b \in \text{acl}_L(a^c)$ . Suppose  $b \notin \text{dcl}_L(\widehat{a})$ , then there is  $b' \in P$  distinct from  $b$  such that  $\text{tp}_L(b'\widehat{a}) = \text{tp}_L(b\widehat{a})$ , and applying Lemma 3.9 (ii) and Lemma 3.8 gives  $\text{tp}_{L_P}(b'\widehat{a}) = \text{tp}_{L_P}(b\widehat{a})$ , a contradiction. Consider now  $b' \in M \setminus P$ ,  $b' \in \text{dcl}_{L_P}(a)$ , but assume  $b' \notin \text{dcl}_L(\widehat{a})$ . Then

$$(b'\hat{a})^c \in \text{dcl}_{L_P}(b'\hat{a}) \cap P \subseteq \text{dcl}_{L_P}(\hat{a}) \cap P = \text{dcl}_L(a^c),$$

hence  $\langle \hat{a} \rangle(b')$  and  $P$  are linearly disjoint over  $\langle \hat{a} \rangle$ . By assumption there are at least two  $L$ -conjugates of  $b$  over  $\hat{a}$ , which are also  $L_P$ -conjugates over  $\hat{a}$  by Lemma 3.8, a contradiction.

□

**Corollary 3.14.** *If  $A \subseteq M$  is such that  $A = \hat{A}$ , then  $\text{acl}_{L_P}(A) = \text{acl}_L(A)$  and  $\text{dcl}_{L_P}(A) = \text{dcl}_L(A)$ . In particular  $P$  is algebraically closed in the  $L_P$ -sense.*

**Definition 3.15.**

- i) Consider for every  $n > 1$ , the predicate  $l_n(x_1, \dots, x_n)$ , which asserts that  $x_1, \dots, x_n$  are linearly independent over  $E$ , that is,

$$l_n(x_1, \dots, x_n) \leftrightarrow \forall e_1, \dots, e_n \left( \bigwedge_i P(e_i) \wedge \sum_i e_i x_i = 0 \rightarrow \bigwedge_i e_i = 0 \right).$$

- ii) Consider for every  $n > 1$  and for every  $i \in \{1, \dots, n\}$ , the  $(n+1)$ -ary function  $f_{n,i}(y, x_1, \dots, x_n)$  which gives the  $i$ -th coordinate of  $y$  written as a linear combination of  $x_1, \dots, x_n$ . More specifically, if  $l_n(x_1, \dots, x_n) \wedge \neg l_n(y, x_1, \dots, x_n)$ , then

$$z = f_{n,i}(y, x_1, \dots, x_n) \leftrightarrow \exists z_1, \dots, z_n \left( z = z_i \wedge y = \sum_j z_j x_j \wedge \bigwedge_j P(z_j) \right),$$

else, if the condition is not met, define  $f_{n,i}(y, x_1, \dots, x_n) = 0$ .

- iii) Define the language  $L_P^{l,f}$  as the language obtained by adjoining to  $L_P$  the predicate symbols  $l_n$  and  $f_{n,i}$ , for all  $n > 1$  and  $i \in \{1, \dots, n\}$ . Notice that in this language,  $P(x)$  can be defined by the formula  $\neg l_n(1, x)$ .

The following result is Corollary 15 from [3]:

**Fact 3.16.** *Let  $N \subseteq M$  be a model of  $T_P$ , then the inclusion is elementary iff  $N$  is an  $L_P^{l,f}$ -substructure if and only if  $N$  is  $P$ -independent.*

**Corollary 3.17.** *Let  $A \subseteq M$ . Let  $C$  be the field generated by  $A$  and the  $f_{n,i}(A)$  for all  $n > 1$  and  $i \leq n$ . Then  $\widehat{A} \subseteq C$ , and consequently*

$$i) \text{acl}_{L_P}(A) = \text{acl}_L(C).$$

$$ii) \text{dcl}_{L_P}(A) = \text{dcl}_L(C).$$

*Proof.* By theorem 7, §2, Ch 3. of [6], the field of definition of the locus of  $A$  over  $P$  is generated by  $\{f_{n,i}(M_0, M_1, \dots, M_n), n < \omega, i \leq n\}$ , where the tuple  $(M_0, M_1, \dots, M_n)$  ranges over the set of monomials formed by elements of  $A$ . Therefore,  $A^c \subseteq C$ . From this, we get both  $\text{acl}_L(\widehat{A}) \subseteq \text{acl}_L(C)$  and  $\text{dcl}_L(\widehat{A}) \subseteq \text{dcl}_L(C)$ , while the reverse inclusion follows from the definability of the  $f_{n,i}$ . The desired result is obtained by invoking Lemma 3.13.  $\square$

**Lemma 3.18.** *Let  $a, b, c \in M$ ,  $p_1 = \text{tp}_{L_P}(a/bc)$ ,  $p_2 = \text{tp}_{L_P}(b/c)$ . If  $p_1, p_2$  are stationary, then  $p_3 = \text{tp}_{L_P}(a/c)$  is stationary.*

*Proof.* By stability of  $T_P$  and by hypothesis, there are good definition schemes  $dp_1$  over  $bc$  and  $dp_2$  over  $c$ . We want to find a good definition for  $p_3$ , i.e. one that defines a global type, this would imply stationarity by Lemma 1.14. Let  $\varphi(x, y)$  be an  $L_P$ -formula and let  $m \in M$  be such that  $M \models \varphi(a, m)$ . There is a formula  $dp_1(\varphi)(y, z, w)$  such that  $M \models dp_1(\varphi)(m, b, c)$ . Moreover, there is then a formula  $dp_2(dp_1(\varphi))(y, w)$  such that  $M \models dp_2(dp_1(\varphi))(m, c)$ . The result follows.  $\square$

**Remark:**  $T_P$  eliminates finite imaginaries.

*Proof.* Let  $A = \{a_1, \dots, a_k\} \subseteq M^n$ , where  $a_i = (a_{i,1}, \dots, a_{i,n})$ . Consider the following polynomial

$$p(X, Y_0, \dots, Y_{n-1}) = \prod_{i=1}^k \left( X - \sum_{j=1}^n a_{i,j} Y_j \right),$$

If  $\sigma$  is an  $L_P$ -automorphism, then as it is in particular an  $L$ -isomorphism, we have that  $\sigma p(X, Y_0, \dots, Y_{n-1}) = \prod_{i=1}^k \left( X - \sum_{j=1}^n \sigma(a_{i,j}) Y_j \right)$ . Noting that  $M[X, Y_0, \dots, Y_{n-1}]$  is a unique factorization domain, we deduce  $\sigma p = p$  if and only if  $\sigma A = A$ . The tuple consisting of the coefficients of  $p$  is a canonical parameter for  $A$ .  $\square$

**Lemma 3.19.** Let  $M_0$  be an elementary substructure of  $(M, P)$ , and let  $a \in M$  be such that  $a = \widehat{a}$ . Define  $d = \text{Cb}(\text{tp}_L(a/\text{acl}_L(M_0P)))$ ,  $e' = \text{Cb}(\text{tp}_{L_P}(d/M_0))$ , and  $e = \text{Cb}(\text{tp}_{L_P}(a/M_0))$ . Then  $e'$  and  $e$  are  $L_P$ -interdefinable.

*Proof.* Note that by definition of  $e, e'$  and because  $M_0 \preceq M$ ,  $\text{tp}_{L_P}(a/e)$  and  $\text{tp}_{L_P}(d/e')$  are stationary.

**Claim I:**

$$(i) \quad a \perp\!\!\!\perp_d^{L_P} M_0P.$$

$$(ii) \quad d \in \text{acl}_{L_P}(aM_0).$$

(i): By Lemma 3.12, it suffices to prove  $\widehat{ad} \perp\!\!\!\perp_{\widehat{d}}^L \widehat{M_0P}$ . Notice that since  $(M_0P)^c \subseteq P$ , we have  $M_0P = \widehat{M_0P}$ , then by definition of  $d$ ,  $a \perp\!\!\!\perp_d^L M_0P$  and since  $d^c \in P$ , monotonicity gives  $\widehat{ad} \perp\!\!\!\perp_{\widehat{d}}^L \widehat{M_0P}$ . It is now enough to prove  $(ad)^c = d^c$ , which would imply  $\widehat{ad} = \widehat{ad}$ . By definition of  $d$ ,  $\langle ad \rangle$  is linearly disjoint (l.d.) from  $\text{acl}_L(M_0P)$  over  $\langle d \rangle$ , hence  $\langle ad \rangle$  and  $P(d)$  are l.d. over  $\langle d \rangle$ . Since  $\langle d \rangle$  is l.d from  $P$  over  $\langle d^c \rangle$ , it follows that  $\langle ad \rangle$  and  $P$  are l.d over  $\langle d^c \rangle$ , hence  $(ad)^c = d^c$ .

(ii): Since  $aM_0 \perp\!\!\!\perp_{(aM_0)^c}^L P$ , then  $a \perp\!\!\!\perp_{M_0(aM_0)^c}^L M_0P$ . By Fact 1.8 (iii) and Remark 3.7, it follows that

$$d \in \text{acl}_L(M_0(aM_0)^c) \subseteq \text{acl}_{L_P}(M_0(aM_0)^c) \subseteq \text{acl}_{L_P}(aM_0). \blacksquare$$

**Claim II:**  $d \in \text{dcl}_{L_P}(a, e)$ .

Let  $\sigma$  be an  $L_P$ -automorphism that fixes  $a, e$ , and let  $M'_0 = \sigma(M_0)$ . Choose a realization  $M''_0$  of  $\text{tp}_{L_P}(M_0/a, e)$  independently from  $M_0 \cup M'_0$  over  $a, e$ . Using  $a \perp\!\!\!\perp_e^{L_P} M_0$  and  $e \in M_0 \cap M'_0 \cap M''_0$ , we obtain the following relations

$$a \perp\!\!\!\perp_{M_0}^{L_P} M_0M''_0, \quad a \perp\!\!\!\perp_{M''_0}^{L_P} M_0M''_0, \quad a \perp\!\!\!\perp_{M'_0}^{L_P} M'_0M''_0, \quad a \perp\!\!\!\perp_{M''_0}^{L_P} M'_0M''_0.$$

Applying Lemma 3.12 gives

$$a \perp\!\!\!\perp_{PM_0}^L PM_0M''_0, \quad a \perp\!\!\!\perp_{PM''_0}^L PM_0M''_0, \quad a \perp\!\!\!\perp_{PM'_0}^L PM'_0M''_0, \quad a \perp\!\!\!\perp_{PM'_0}^L PM'_0M''_0.$$

Since  $\text{tp}_L(a/\text{acl}_L(M_0P))$  is stationary, this translates in terms of canonical bases to

$$d = \text{Cb}(\text{tp}_L(a/\text{acl}_L(M_0P))) = \text{Cb}(\text{tp}_L(a/\text{acl}_L(M''_0P))) = \text{Cb}(\text{tp}_L(a/\text{acl}_L(M'_0P))),$$

hence  $\sigma(d) = d$ , so the claim is proved. ■

By definition of  $e$ ,  $a \perp_e^{L_P} M_0$ . As  $\text{tp}_{L_P}(a/e)$  is stationary,  $e \in M_0$ , and  $d \in \text{dcl}_{L_P}(a, e)$ , we conclude  $\text{tp}_{L_P}(d/e)$  is stationary by Lemma 1.15. Therefore,  $e' \in \text{dcl}_{L_P}(e)$ .

**Claim III:**  $a \perp_{e'}^{L_P} M_0$ . Therefore,  $e \in \text{acl}_{L_P}(e')$ .

By Claim I,

$$\begin{aligned} a \perp_d^{L_P} M_0 P &\Rightarrow a \perp_d^{L_P} M_0 d \\ &\Rightarrow a \perp_{de'}^{L_P} M_0 \quad \text{as } e' \in \text{dcl}_{L_P}(M_0) = M_0. \end{aligned}$$

By definition of  $e'$  we have  $de' \perp_{e'}^{L_P} M_0$ . Applying transitivity yields the claim. ■

To prove  $e \in \text{dcl}_{L_P}(e')$ , we shall show stationarity of  $\text{tp}_{L_P}(a/e')$  and apply Fact 1.8 (iv).

By definition of  $e'$ ,  $\text{tp}_{L_P}(d/e')$  is stationary, then by Lemma 3.18, it would suffice to prove  $\text{tp}_{L_P}(a/de')$  is stationary. However, Claim I implies  $a \perp_d^{L_P} e'$ , so it is enough to prove  $p = \text{tp}_{L_P}(a/d)$  is stationary. Let  $N$  be an  $L_P$ -elementary substructure of  $M$  containing  $d$ , and suppose  $p \subseteq p_1, p_2$  are nonforking extensions of  $p$  to  $N$ . Let  $a_1, a_2$  realize  $p_1, p_2$  respectively. By Lemma 3.12,  $a_i \perp_{dP}^L NP$ . Since  $\text{tp}_{L_P}(a_i/d) = \text{tp}_{L_P}(a/d)$  for  $i = 1, 2$ , and  $a \perp_d^L dP$  (by definition of  $d$  and monotonicity), we get that for  $i = 1, 2$ ,  $a_i \perp_d^L NP$ . This in turn implies  $Na_i \perp_N^L P$ , since  $d \in N$ . Since  $N = \widehat{N}$ , by Remark 3.11 (ii),  $N \perp_{P \cap N}^L P$ . Applying transitivity yields  $Na_i \perp_{N \cap P}^L P$ , hence  $N(a_i)$  and  $NP$  are linearly disjoint over  $N$ . This implies  $\widehat{N(a_i)} = N(a_i)$ . Since  $\text{tp}_L(a/d)$  is stationary,  $\text{tp}_L(a_1N) = \text{tp}_L(a_2N)$ . We can then apply Lemma 3.8 to obtain that  $\text{tp}_{L_P}(a_1N) = \text{tp}_{L_P}(a_2N)$ . □

**Lemma 3.20.** *Let  $a \in M$ ,  $A \subseteq M$ . If  $A = \widehat{A}$  and  $\text{tp}_{L_P}(a/A)$  is stationary, then  $\text{tp}_L(a/A)$  is stationary.*

*Proof.* Suppose by contradiction that  $\text{tp}_L(a/A)$  is not stationary and let  $k = \langle A \rangle$ . The extension  $k(a)|k$  is not primary: there is some  $\alpha \in k(a)$  such that  $\alpha \in \text{acl}_L(k) \setminus \text{dcl}_L(k)$ . Note also that  $\widehat{k} = k$ . By Corollary 3.14,  $\alpha \in \text{acl}_{L_P}(k) \setminus \text{dcl}_{L_P}(k)$ , contradicting stationarity of  $\text{tp}_L(a/A)$ . □

**Lemma 3.21.** *Suppose  $d \in M$  is such that  $d = \widehat{d}$ , and let  $e \in \text{dcl}_{L_P}^{\text{eq}}(d)$  be an imaginary such that  $\text{tp}_{L_P}(d/e)$  is stationary. Let  $d' \models \text{tp}_{L_P}(d/e)$  with  $d \perp_e^{L_P} d'$ . Let  $B'_1 = \text{dcl}_{L_P}^{\text{eq}}(e) \cap M$  and  $B_1 = \text{acl}_{L_P}^{\text{eq}}(e) \cap M$ . Then  $\langle d \rangle$  and  $\langle d' \rangle$  are linearly disjoint over  $B'_1$ , in particular  $d \perp_{B_1}^L d'$ .*

*Proof.* We denote  $p(x) = \text{tp}_{L_P}(d/e)$ ,

**Claim:** Let  $d'' \models p| \{d, d'\}$ , then  $d \perp_{d'}^L d''d'$  and  $d \perp_{d''}^L d''d'$ .

By definition of  $d''$ , we have  $d \perp_{d'}^{L_P} d''d'$ , and by hypothesis  $d \perp_e^{L_P} d'$ . Additionally,  $e \in \text{dcl}_{L_P}(d'') \cap \text{dcl}_{L_P}(d')$ , then both relations  $d \perp_{d''}^{L_P} d'$ ,  $d \perp_{d'}^{L_P} d''$  hold. Starting from the first relation, we see that  $dd'' \perp_{d''}^{L_P} d'd''$ , and by Lemma 3.11,  $\widehat{dd''} \perp_{\widehat{d''}}^L \widehat{d'd''}$ . Since  $d, d'$  and  $d''$  are independent over  $e$ , and  $e$  is definable over each of  $d, d', d''$ , the stationarity of  $\text{tp}_{L_P}(d/e)$  implies that of  $\text{tp}_{L_P}(d/d''), \text{tp}_{L_P}(d/d')$ . It follows by 3.20 that  $\text{tp}_L(d/d''), \text{tp}_L(d/d')$  are stationary. Then, as  $\widehat{d''} = d''$ , we get  $d \perp_{d''}^L d''d'$ , hence, the field of definition of the locus of  $d$  over  $\langle d''d' \rangle$  is contained in  $\langle d'' \rangle$ . The other part of the claim is obtained similarly, using that  $\widehat{d'} = d'$  instead, yielding that the field of definition of the locus of  $d$  over  $\langle d''d' \rangle$  is contained in  $\langle d' \rangle$ . ■

We obtain therefore

$$\text{Cb}(\text{tp}_L(d/d')) = \text{Cb}(\text{tp}_L(d/d'')) \subset \text{dcl}_L(d') \cap \text{dcl}_L(d'').$$

But  $d'$  and  $d''$  are independent over  $e$ ,  $\text{dcl}_{L_P}(d') = \text{dcl}_L(d')$ , and  $\text{dcl}_{L_P}(d'') = \text{dcl}_L(d'')$ , so  $\text{Cb}(\text{tp}_L(d/d')) \subseteq \text{dcl}_{L_P}^{\text{eq}}(e) \cap M = B'_1$ . □

**Lemma 3.22.** *Let  $e \in (M, P)^{\text{eq}}$ , and  $B_0 = \text{acl}_{L_P}^{\text{eq}}(e) \cap P$ . Then for all  $c \in P$ ,  $\text{tp}_{L_P}(c/B_0e)$  is finitely satisfiable in  $B_0$ .*

*Proof.* Let  $a \in M$  be such that  $e = f(a)$  for some definable function. Since  $\widehat{aP} = \widehat{a}P$ , by Lemma 3.8  $\text{tp}_{L_P}(a/a^c) \vdash \text{tp}_{L_P}(a/P)$ . We will prove that  $\text{tp}_{L_P}(a/P)$  is stationary, this would imply by Lemma 1.15 that  $\text{tp}_{L_P}(e/P)$  is stationary. Suppose not, so we can find  $b_1, \dots, b_n \in M$  and formulas  $\varphi(x, b_i)$ , which distinguish between the non-forking extensions of  $\text{tp}_{L_P}(a/a^c)$  to  $M$ . In other words, they define a partition of the set of realizations of  $\text{tp}_{L_P}(a/a^c)$ . By saturation of  $M$  over  $P$ , we may assume

$$a \perp_{a^c}^{L_P} b_1, \dots, b_n, P.$$

If  $a'$  is another realization of  $\text{tp}_{L_P}(a/a^c)$ , such that  $a' \perp_{a^c}^{L_P} b_1, \dots, b_n, P$ , then there is an automorphism of  $M$  that fixes  $\text{acl}_{L_P}(P, b_1, \dots, b_n)$  and sends  $a$  to  $a'$ . This implies  $\text{tp}_{L_P}(a'/P) = \text{tp}_{L_P}(a/P)$ .

Define  $e^c = \text{Cb}(\text{tp}_{L_P}(e/P))$ . Note that

$$e^c \in \text{dcl}_{L_P}^{\text{eq}}(e) \cap \text{dcl}_{L_P}^{\text{eq}}(P) = \text{dcl}_{L_P}^{\text{eq}}(e) \cap P.$$

By definition of  $e^c$  we have  $e \perp_{\text{acl}_L(e^c)}^{L_P} P$ , and the proof of Lemma 3.13 shows that  $\text{acl}_L(e^c) = B_0$ . Hence,  $\text{tp}_{L_P}(e/P)$  is stationary and is a non-forking extension of  $\text{tp}_{L_P}(e/B_0)$ .

This implies definability of  $\text{tp}_{L_P}(e/P)$  over  $B_0$ . Thus, for any  $L_P^{\text{eq}}$ -formula  $\psi(x, y)$  with parameters in  $B_0$ , there is a formula  $d\psi(y)$  with parameters in  $B_0$  such that for all  $c \in P$ ,  $M \models \psi(e, c)$  iff  $M \models d\psi(c)$ . By Lemma 3.3, we may assume that  $d\psi(y)$  is an  $L$ -formula. Since we also have that  $B_0 \prec P$  in the sense of  $L$ , we have that for any  $c \in P$ , if  $P \models \psi(e, c)$  then  $P \models d\psi(c)$ , which implies there is  $b \in B_0$  such that  $P \models d\psi(b)$ , and therefore  $M \models \psi(e, b)$ .  $\square$

## 4 Weak Elimination of Imaginaries

Throughout this section we will maintain our notation and conventions from Section 3. We let  $T = ACF_p$ , and  $T_P$  the theory of beatiful pairs of models of  $T$ . We have that  $(M, P) \models T_P$  is saturated.

**Definition 4.1.** Let  $G$  be an algebraic group and  $X$  an algebraic variety both defined over  $k \subseteq M$ . A  $k$ -rational action is a group action  $\alpha : G \times X \rightarrow X$  such that for every  $g \in G$ , the map  $\alpha(g, \cdot) : X \rightarrow X$  is a  $k$ -rational map.

**Definition 4.2.** A definable group action is a triple  $((G, \cdot), X, \alpha)$ , where  $(G, \cdot)$  is a definable group,  $X \subseteq M$  a definable set and  $\alpha : G \times X \rightarrow X$  a group action whose graph is definable. If the action is *transitive* on  $X$ , that is, for every  $a, b \in X$  there is  $g \in G$  such that  $\alpha(g, a) = b$ , the triple is the called a *definable homogeneous space*. Moreover, if the action is *strictly transitive (or regular)*, that is,  $\alpha(g, x) = x$  iff  $g = e$ , it will be called a *principal definable homogeneous space* (or PHS).

We shall abuse notation and denote  $\alpha(g, a)$  as  $g \cdot a$ . In our context, as  $T = ACF_p$ , we get the following fact from Theorem 7.4.14 of [7].

**Fact 4.3.** *If  $G \subseteq M^n$  is an  $L$ -definable group, then  $G$  is definably isomorphic to an algebraic group.*

**Proposition 4.4.** *Let  $e \in (M, P)^{\text{eq}}$ . Then there are: a connected algebraic group  $G$ , an irreducible variety  $V$  over  $P$ , and a rational action  $G$  on  $V$ , definable over  $P$ , such that*

- (i) *The action of  $G(P)$  on  $V(M)$  is generically free: if  $a \in V(M)$  is a generic point of  $V$  over  $P$ , and  $g \in G(P)$  is not the identity, then  $g \cdot a \neq a$ .*
- (ii) *For some  $a \in V(M)$  generic over  $P$ , if  $r$  is a canonical parameter for the orbit  $X = \{g \cdot a \mid g \in G(P)\}$ , then  $e \in \text{dcl}_{L_P}(r)$  and  $r \in \text{acl}_{L_P}(e)$ .*

The proof of Proposition 4.4 will require some results.

**Lemma 4.5.** *Let  $e \in (M, P)^{\text{eq}}$ . There is  $d' \in M$  such that  $\text{tp}_{L_P}(d'/e)$  is stationary and  $P$ -internal, and moreover  $e \in \text{dcl}_{L_P}^{\text{eq}}(d')$ .*

*Proof.* Let  $a \in M$  be such that  $a = \hat{a}$  and  $e = f(a)$  for some  $\emptyset$ -interpretable function. By Lemma 1.10 we may suppose  $\text{tp}_{L_P}(a/e)$  is stationary, hence  $e = \text{Cb}(\text{tp}_{L_P}(a/M_0))$ , where  $M_0$  is any  $L_P$ -elementary substructure of  $M$  such that  $e \in M_0^{\text{eq}}$  and  $a \perp_e^{L_P} M_0$ . Let  $d = \text{Cb}(\text{tp}_L(a/\text{acl}_L(M_0P)))$ . By Lemma 3.19,  $e = \text{Cb}(\text{tp}_{L_P}(d/M_0))$ , hence  $d \perp_e^{L_P} M_0$ . Since  $M_0 \preceq M$ ,  $\text{tp}_{L_P}(d/M_0)$  is stationary,  $\text{tp}_{L_P}(d/e)$  is stationary and almost  $P$ -internal. Replacing  $d$  by finitely many independent realizations of  $\text{tp}_{L_P}(d/e)$ , by Fact 1.8 (v), we may assume without loss of generality that  $e \in \text{dcl}^{\text{eq}}(d)$ , or that  $e = g(d)$  for some definable function  $g$ . By Lemma 1.17, there is  $d' \in \text{dcl}_{L_P}(d)$ , a code for a finite set of realizations of  $\text{tp}_{L_P}(d/e)$ , such that  $d \in \text{acl}_{L_P}(d')$  and  $\text{tp}_{L_P}(d'/e)$  is stationary and  $P$ -internal. Then as  $d \in \text{acl}_{L_P}^{\text{eq}}(d')$ , there is a formula  $\varphi(x, d')$  isolating  $\text{tp}_{L_P}(d/d')$ ; hence  $M \models \forall x \varphi(x, d') \rightarrow g(x) = e$ , so  $e \in \text{dcl}^{\text{eq}}(d')$ . □

**Lemma 4.6.** *There are a tuple  $d \in M$ , an  $L_P$ -definable function  $f$  (over  $\emptyset$ ), an  $L_P(e)$ -formula  $\psi(x)$ , and an  $L_P(e)$ -definable function  $h$  such that*

$$(i) \quad f(d) = e.$$

$$(ii) \quad \psi(x) \in \text{tp}_{L_P}(d/e).$$

$$(iii) \quad M \models \forall x, x' (\psi(x) \wedge \psi(x') \rightarrow \exists c (P(c) \wedge h(x, c) = x')).$$

*Proof.* Let  $d'$  be as in Lemma 4.5. Then  $p = \text{tp}_{L_P}(d'/e)$  is stationary,  $P$ -internal, and  $e = \text{Cb}(p)$ . By Lemma 1.18, there is a tuple  $d$  consisting of finitely many realisations of  $p$ , and an  $e$ -definable function  $g$  such that for any realization  $d''$  of  $p$ , there is a tuple  $c_{d''} \in P$  such that  $d'' = g(d, c_{d''})$ . Clearly  $e \in \text{dcl}_{L_P}^{\text{eq}}(d)$ , so we can find an  $L_P$ -definable function  $f$  such that (i) holds. If  $d_1, d_2$  realize  $\text{tp}_{L_P}(d/e)$ , then there is an  $e$ -definable function  $h$  and a tuple  $c \in P$  such that  $d_1 = h(d_2, c)$ . Applying compactness yields an  $L_P$ -formula  $\psi \in \text{tp}_{L_P}(d/e)$  such that for any two  $d_1, d_2$  satisfying  $\psi$ , there is  $c \in P$  such that  $h(d_1, c) = d_2$ , which directly proves (ii) and (iii). Note  $\text{tp}_{L_P}(d/e)$  remains  $P$ -internal. □

**Lemma 4.7.** *In Lemma 4.6,  $d$  can be chosen such that (i),(ii),(iii) hold, and  $d \perp_e^{L_P} P$ .*

*Proof.* Let  $\psi$  as in Lemma 4.6. Let  $\chi(x, y)$  an  $L_P(e)$ -formula that expresses the conjunction of  $x^c = y$ ,  $\psi(x)$  and  $f(x) = e$ . Consider the  $L_P(e)$ -formula  $\theta(y)$  given by  $\exists x(\chi(x, y))$ . Since  $M \models \theta(d^c)$ , by Lemma 3.22, there is  $d_0 \in \text{acl}_{L_P}^{\text{eq}}(e) \cap P$  such that  $M \models \theta(d_0)$ . Therefore, there is  $d_1$  such that  $M \models \chi(d_1, d_0)$ , hence  $d_1 \perp_e^{L_P} P$ .  $\square$

**Notation:** For the remainder of this section, fix  $d$  as in Lemma 4.7. By Remark 3.7  $d^c \in \text{dcl}_{L_P}(d)$ , hence we may also assume from now on that  $d = \widehat{d}$ , as all of the properties from Lemmas 4.6, 4.7, and 4.8 still hold after adjoining  $d^c$  to  $d$ . From now on, let

$$B = \text{acl}_{L_P}^{\text{eq}}(e),$$

$$B_1 = B \cap M,$$

$$B_0 = B \cap P.$$

**Lemma 4.8.**  $\text{tp}_{L_P}(d/B)$  is isolated.

*Proof.* By stability of  $\text{Th}(M^{\text{eq}})$ , there are  $M_1 \preceq M$ , a prime model over  $Bd$  and  $M_0 \preceq M_1$  a prime model over  $B$ .

*Claim:*  $B_0 = M_0 \cap P = M_1 \cap P$ : It is clear that  $B \subseteq M_0, M_1$ , one inclusion follows. Conversely, if  $a \in M_0 \cap P$ , then  $\text{tp}_{L_P}(a/B)$  is isolated, which is a non-forking extension of  $\text{tp}_{L_P}(a/e)$ , hence  $\text{tp}_{L_P}(a/e)$  is isolated too, and applying Lemma 3.22, it can be realized by some  $a' \in B_0$ . In particular, this implies  $a \in \text{acl}_{L_P}(e)$ . The proof for the second equality is similar, let  $a \in M_1 \cap P$ , then  $\text{tp}_{L_P}(a/Bd)$  is isolated. Recall  $d \perp_e^{L_P} P$ , hence  $\text{tp}(a/Bd)$  does not fork over  $\text{tp}_{L_P}(a/e)$ , which is then isolated, and applying Lemma 3.22 yields the result.

Let  $\psi$  be as in Lemma 4.6, and choose  $d' \in M_0$  such that  $M \models \psi(d')$ . Applying Lemma 4.6 (iii) inside the model  $M_1$ , there is  $c \in P \cap M_1 = B_0$  such that  $d \in \text{dcl}_{L_P}(d', c) \subseteq M_0$ , hence by definition of a prime model,  $\text{tp}_{L_P}(d/B)$  is isolated.  $\square$

**Lemma 4.9.** *Let  $X$  be the set of realizations of  $\text{tp}_{L_P}(d/B)$ . There are: a connected algebraic group  $G$  defined over  $B_0$  and an  $L_P(e)$ -definable regular action of  $G(P)$  on  $X$ . Moreover, if  $r$  is a canonical parameter for the PHS  $(G(P), X)$ , then  $e \in \text{dcl}_{L_P}(r)$  and  $r \in \text{acl}_{L_P}(e)$ .*

*Proof.* By Lemma 4.8,  $X$  is  $L_P$ -definable over  $B$ . Define

$$C = \{c \in P, \exists d'(d' \in X \wedge h(d, c) = d')\},$$

which is non empty by Lemma 4.6 (*iii*), and  $L(B_0)$ -definable by Lemma 3.3. Consider now the equivalence relation  $E$  in  $C$  defined by  $M \models E(c_1, c_2)$  if and only if  $M \models h(d, c_1) = h(d, c_2)$ . In  $C/E$  we can define an  $L_P(e)$ -interpretable function  $h'(d, c/E) = h(d, c)$ . By Lemmas 4.7 and 4.8  $d \perp_B^{L_P} P$ , hence all elements of  $X$  have the same  $L_P$ -type over  $BP$ . Since  $E$  is contained in some power of  $P$ , it is  $L(B_0)$ -definable, hence it does not depend on the choice of  $d$ . This implies that for all  $c_1, c_2 \in C$  the value of  $h(h(d, c_1), c_2)$  is defined, and taking classes modulo  $E$ , there is a unique  $c_3/E$  such that  $h'(h'(d, c_1/E), c_2/E) = h'(d, c_3/E)$ , we define a binary operation on  $C/E$  as  $(c_1/E) \cdot (c_2/E) = c_3/E$ . Once again by Lemma 3.3, this operation is  $L(B_0)$ -definable. Moreover, by Remark 3.4, we may assume without loss of generality that  $C/E$  contains real tuples.

**Claim:**  $(C/E, \cdot)$  is a  $B_0$ -definable group.

Let  $c_1, c_2, c_3 \in C/E$ . To check associativity, notice that

$$h'(d, (c_1 c_2) c_3) = h'(h'(d, c_1 c_2), c_3) = h'(h'(h'(d, c_1), c_2), c_3),$$

moreover, since  $h'(d, c_2 c_3) = h'(h'(d, c_2), c_3)$  and  $\text{tp}_{L_P}(h'(d, c_1)/BP) = \text{tp}_{L_P}(d/BP)$ , we obtain

$$h'(d, c_1(c_2 c_3)) = h'(h'(d, c_1), c_2 c_3) = h'(h'(h'(d, c_1), c_2), c_3)).$$

To check for an identity, by Lemma 4.6 (*iii*), there is  $c' \in P$  such that  $h(d, c') = d$ . Then, for all  $d' \in X$ ,  $h'(d', c') = d'$ , in particular

$$h'(d, c_1 c') = h'(h'(d, c_1), c') = h'(d, c_1) \Rightarrow c_1 c' = c_1.$$

To check the existence of inverses, notice that since  $h(d, c_1) \in X$ , there is some  $L_P$ -automorphism  $\sigma$  fixing  $BP$  pointwise such that  $h(d, c_1) = \sigma(d)$ , which implies  $h'(\sigma^{-1}(d), c_1) = d$ . By Lemma 4.6 (*iii*), there is a unique  $c'_1$  such that  $h'(d, c'_1) = \sigma^{-1}(d)$ , hence

$$\begin{aligned} h'(d, c'_1 c_1) &= h'(h'(d, c'_1), c_1) = h'(\sigma^{-1}(d), c_1) = d = h'(d, c'), \\ h'(d, c_1 c'_1) &= h'(h'(d, c_1), c'_1) = h'(\sigma(d), c'_1) = d = h'(d, c'), \end{aligned}$$

therefore,  $c_1 c'_1 = c'_1 c_1 = c'$ . ■

By the previous claim and by Fact 4.3,  $C/E$  is  $B_0$ -definably isomorphic to some algebraic group  $G$  over  $B_0$ . We can then induce an  $L(B_0)$ -definable action of  $G(P)$  over  $X$  using the map  $h'$ : if  $F : G \rightarrow C/E$  is an isomorphism, then for  $(g, d) \in G \times X$ , define

$g \cdot d = h'(d, F(g))$ . By Lemma 4.6 (iii) and by definition of  $E$ , this action is regular. As  $X$  is the set of realizations of a stationary type,  $G(P)$  must be connected (as an  $L_P$ -definable group), hence connected as an algebraic group. Clearly, the PHS  $(G(P), X)$  is  $L_P$ -definable over  $B$ , this implies that if  $r$  is a canonical parameter for  $(G(P), X)$ , then  $r \in \text{acl}_{L_P}(e)$ . Moreover, if  $\sigma$  is some  $L_P$ -automorphism fixing  $r$ , then it permutes the realizations of  $\text{tp}_{L_P}(d/B)$ , and by stationarity of  $\text{tp}_{L_P}(d/e)$  we have  $e = \text{Cb}(\text{tp}_{L_P}(d/B))$ , so  $\sigma(e) = e$ , hence  $e \in \text{dcl}_{L_P}(r)$ , completing the proof.  $\square$

The set  $X$  from Lemma 4.9 will be identified with a generic orbit of the action of  $G(P)$  over some variety  $V(M)$ . We first state Proposition 2.2 from [4].

**Lemma 4.10.** *Let  $G$  be a connected definable group with a generic action on the set of realizations  $X_1$  of a stationary  $L$ -type  $q$ , that is, for all generic  $g \in G$  and for  $d$  realizing  $q|g$ ,  $g \cdot d$  is defined and realizes  $q$ , and for all independent  $g_1, g_2, d$ ,  $g_1 \cdot (g_2 \cdot d) = (g_1 g_2) \cdot d$  when the action is defined. There exists then a type-definable set  $Y$ , a definable embedding  $X_1 \subseteq Y$ , and a definable action of  $G$  on  $Y$ , extending the generic action of  $G$  on  $X_1$ . Moreover, for every  $y \in Y$  there is  $g \in G$  and  $d \models q$  such that  $y = g \cdot d$ .*

*Proof.* Consider the set of pairs  $(g, d)$  with  $g \in G$ ,  $d \models q$ . Define an equivalence relation over these pairs by:  $(g, d) \sim (g', d')$  if for all generic  $h \in G$  such that  $(hg) \cdot d = (hg') \cdot d'$ . Let  $Y$  be the set of classes, its elements are denoted by  $[g, d]$ . If  $(hg_2) \cdot d = (hg'_2) \cdot d'$  holds for generic  $h$ , then, since  $hg_1$  is also generic, it is also true that  $(hg_1 g_2) \cdot d = (hg_1 g'_2) \cdot d'$ , hence we can define an action of  $G$  on  $Y$  by  $g_1 \cdot [g_2, d] = [g_1 g_2, d]$ , and identify each  $d \models q$  with  $[1_G, d]$ . To check the last statement, let  $[g, d] \in Y$ , and let  $h$  be a generic of  $G$ , independent from  $d$ , then  $h[g, d] = [hg, d] = [1, hg \cdot d]$ , hence  $[g, d] = h^{-1}[1, hg \cdot d]$ .  $\square$

**Lemma 4.11.** *For  $X$  as in Lemma 4.9 there is an irreducible variety  $Y$  defined over  $B_1$ , and a transitive rational action of  $G$  on  $Y$ , defined over  $B_1$ , such that  $X \subseteq Y$ ,  $d$  is a generic point of  $Y$  over  $B_1$ , and the action of  $G$  on  $Y$  restricts to the given action of  $G(P)$  on  $X$ .*

*Proof.* Recall that for  $g \in G(P)$ ,  $d \in X$ ,  $g \cdot d$  is  $e$ -definable, this means  $g \cdot d \in \text{dcl}_{L_P}(g, d, e)$ . Since  $e \in \text{dcl}_{L_P}(d)$ , then  $g \cdot d \in \text{dcl}_{L_P}(g, d) = \text{dcl}_L(\widehat{g, d})$  by Lemma 3.13. But  $\text{dcl}_L(\widehat{g, d}) = \text{dcl}_L(g, d)$  by Lemma 3.9 (ii). Therefore,  $g \cdot d \in \text{dcl}_L(g, d)$ .

**Claim:**  $d \perp_{B_0}^L g$ .

If  $e^c = \text{Cb}(\text{tp}_{L_P}(e/P))$ , then  $e \perp_{e^c}^{L_P} P$ , and by Lemma 4.7,  $d^c \perp_e^{L_P} P$ . Applying transitivity yields  $d^c \perp_{e^c}^{L_P} P$ , and since everything lives in  $P$ , we can restrict our language

to get  $d^c \perp_{e^c}^L P$ . By the proof of Lemma 3.13,  $B_0 = \text{acl}_L(e^c)$ , hence  $d^c \perp_{B_0}^L P$  and by definition of  $d^c$  we have  $d \perp_{d^c}^L P$ . The claim follows as  $g \in P$ . ■

Now, working in  $L$ , since  $e \in \text{dcl}_{L_P}(d) = \text{dcl}_L(\hat{d})$ ,  $B_1 \in \text{acl}_L(d)$ , so the previous claim yields  $dB_1 \perp_{B_0} g$ . Then, if  $g$  is generic over  $B_0$ , then it is generic over  $dB_1$ . The action is generically regular and transitive: given independent  $d_1, d_2 \in X$ , there is a unique  $g \in G(P)$  such that  $g \cdot d_1 = d_2$ . Hence, working in  $L_P$ ,  $RM(G) = RM(X)$ , and if  $g \in G$ ,  $d \in X$  are independent over  $e$ , then because the action is defined over  $e$ , we have that  $g \in \text{dcl}_L(g \cdot d, d)$ , so that we must have  $RM(g \cdot d, d/e) = 2RM(G)$ , which implies  $g \cdot d \perp_e^{L_P} d$ . By Lemma 3.21,  $g \cdot d \perp_{B_1}^L d$ .

We have a definable action of  $G(P)$  on the  $L_P$ -definable set  $X$ , and the action is given by a map  $G \times X \rightarrow X$  which is  $L(B_1)$ -definable in  $T$ . Passing to the Zariski closure, we get a generic action of the algebraic group  $G(M)$  on the set  $X_1$  of generic elements (over  $B$ ) of the Zariski closure of  $X$ . By Lemma 4.10, there is a type-definable  $Y \supseteq X_1$  (in the  $L$ -sense, and over  $B_1$ ) such that  $G$  acts on  $Y$  in a way that restricts to the generic action of  $G$  on  $X_1$ . Moreover, for every  $y \in Y$  there is  $g \in G$  and  $d \in X_1$  such that  $y = g \cdot d$ , so the action of  $G$  on  $Y$  transitive, then  $Y$  has a unique generic type by connectedness of  $G$ , and it must be indeed  $\text{tp}_L(d/B_1)$ . This proves that  $d$  is a generic of  $Y$  over  $B_1$ . We claim that  $Y$  is also definable: Let  $\varphi(x, y)$  be some  $L(B_1)$ -formula defining  $x \in G \cdot y$ , and let  $E$  be the equivalence relation given by  $yEy'$  iff  $M \models \forall x \varphi(x, y) \leftrightarrow \varphi(x, y')$ , by transitivity, for any  $y \in Y$  we have  $[y]_E = Y$ , now by type-definability of  $Y$  over  $B_1$ ,  $Y$  is fixed by all  $\sigma \in \text{Aut}(M/B_1)$ , hence the imaginary  $[y]_E$  is fixed too, which implies  $[y]_E$  is  $B_1$ -definable, hence  $Y$  is  $B_1$ -definable. Since  $X \subseteq X_1 \subseteq Y$ , and the action of  $G$  on  $Y$  restricts to the generic action over  $X_1$ , then it restricts to the action of  $G$  on  $X$  that was defined in Lemma 4.9. Finally, by Fact 4.3,  $(G, Y, \cdot)$  is  $B_1$ -definably isomorphic to  $(G', Y', \cdot')$ , where  $G'$  is an algebraic group,  $Y'$  an irreducible variety, and  $\cdot'$  is a  $B_1$ -rational action. □

## Proof of Proposition 4.4

*Proof.* For  $e \in M^{\text{eq}}$ ,  $d, G, Y$  as in Lemma 4.11, choose some finite  $b \in B_1$  such that  $(G, Y, \cdot)$  is definable over  $b$ . Rewrite  $Y$  as  $Y_b$ . By Lemma 4.7,  $d \perp_e^{L_P} P$ , together with  $e \perp_{B_0}^L P$  implies that  $bd \perp_{B_0}^L P$  (recall  $b \in \text{acl}_{L_P}(e)$ ). Since  $e \perp_{e^c}^L P$ , and  $(bd)^c \perp_e^L P$ , applying transitivity yields  $(bd)^c \perp_{e^c}^L P$ , and since everything lives in  $P$ , we can restrict our language to get  $(bd)^c \perp_{e^c}^L P$ . By the proof of Lemma 3.13,  $B_0 = \text{acl}_L(e^c)$ , hence  $(bd)^c \perp_{B_0}^L P$  and by definition of  $(bd)^c$  we have  $bd \perp_{(bd)^c}^L P$ , applying transitivity once more yields  $bd \perp_{B_0}^L P$ . Let  $V, Z$  be the loci of  $bd$  and  $b$  over  $B_0$ , respectively, and consider

the projection  $f : V \rightarrow Z$  sending  $bd$  to  $b$ , then note that  $f^{-1}(b) = Y_b$ . Then by compactness, there is a Zariski open subset  $U$  in  $Z$ , also defined over  $B_0$ , such that  $G$  acts rationally in  $f^{-1}(U)$  and this action restricted to  $Y_b$  coincides with the one defined in Lemma 4.9. This proves (i), as any generic  $a \in V$  has the same  $L$ -type over  $B_1$  as  $bd$ , and the action in Lemma 4.9 is regular by construction. Since  $f^{-1}(U)$  is still a variety, by shrinking  $V$  we may without loss of generality let  $V = f^{-1}(U)$ , and by  $bd \perp_{B_0}^L P$ , we conclude  $bd$  is a generic point of  $V$  over  $P$ , therefore (ii) follows by applying Lemma 4.9.  $\square$

We state our main result, which will follow from Proposition 4.4.

**Corollary 4.12.** *There is a set of sorts  $\mathcal{S} \subseteq L^{\text{eq}}$ , such that  $T_P$  has weak elimination of imaginaries in the language obtained by adjoining  $\mathcal{S}$  to  $L$ .*

*Proof.* Let  $G, V$  be as in Proposition 4.4, and let  $c \in P$  generate a field over which  $(G, V, \cdot)$  are defined. There is a variety  $Z$  defined over the prime field such that there exist varieties  $\mathcal{G}, \mathcal{V}$ , along with surjective regular maps to  $Z$ , and for each  $b \in Z$ , the fiber  $\mathcal{G}_b$  is an algebraic group that acts on  $\mathcal{V}_b$ , and moreover  $\mathcal{G}_c = G$  and  $\mathcal{V}_c = V$ . For each  $e \in M^{\text{eq}}$ , we define a sort  $S_{(\mathcal{G}, \mathcal{V}, Z, e)}$  in the following manner: let  $W_e = \cup\{\mathcal{V}_b, b \in Z(P)\}$ , and define an equivalence relation on  $W$  as  $w_1 \sim w_2$  iff for some  $b \in Z(P)$ ,  $w_1, w_2 \in \mathcal{V}_b$  and there exists  $g \in \mathcal{G}_b(P)$  such that  $w_1 = g \cdot w_2$ . We interpret the elements of  $S_{(\mathcal{G}, \mathcal{V}, Z, e)}$  as the classes of  $W$  modulo  $\sim$ , which are in turn representatives of each orbit of the fiberwise action of  $\mathcal{G}$  on  $\mathcal{V}$ . By Proposition 4.4, for every  $e \in M^{\text{eq}}$ , there is  $r \in S_{(\mathcal{G}, \mathcal{V}, Z, e)}$ , such that  $e \in \text{dcl}_{L_P}(r)$  and  $r \in \text{acl}_{L_P}(e)$ .  $\square$

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