The Finite Free Stam Inequality

Abstract

We establish a finite free analogue of the Stam inequality from information theory. Given monic, degree-n, real-rooted polynomials p and q with positive variance, and writing Φ_n for the finite free Fisher information and $\boxplus_n$ for the symmetric additive convolution of Marcus, Spielman, and Srivastava, we prove that

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

The argument proceeds in two stages. First, a Score-Gradient Inequality, obtained by a double application of Cauchy–Schwarz, converts pointwise score data into a lower bound on the Fisher information. Second, a convolution-flow analysis—interpolating between p and $p \boxplus_n q$ via the semigroup structure of $\boxplus_n$—yields a differential inequality that, after integration, produces asymmetric bounds strictly stronger than the Stam inequality in each regime; the full result follows by a case split exploiting commutativity of $\boxplus_n$. Equality is characterized: it holds if and only if both polynomials have roots at affinely rescaled zeros of the Hermite polynomial H_n.

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1 Introduction

1.1 Background and motivation

In information theory, the Stam inequality [2] states that if X and Y are independent random variables with finite Fisher information $I(X)$ and $I(Y)$, then

$$\frac{1}{I(X+Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}.$$

This fundamental inequality—equivalent to the entropy power inequality of Shannon and Stam—captures the principle that convolution of independent sources strictly increases disorder.

Finite free probability, introduced by Marcus, Spielman, and Srivastava [1], provides a polynomial analogue of free probability in which random variables are replaced by real-rooted polynomials and addition by a deterministic convolution operation $\boxplus_n$. Within this framework, the natural question arises:

Does the Stam inequality hold for the finite free additive convolution?

The purpose of this paper is to answer this question affirmatively.

1.2 Statement of the main result

Let $\mathcal{P}_n$ denote the space of monic polynomials of degree n with real coefficients, and $\mathcal{P}_n^{\mathbb{R}} \subset \mathcal{P}_n$ the subset with all real roots. For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1 < \cdots < \lambda_n$, define the *scores*

$$V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}$$

and the *finite free Fisher information*

$$\Phi_n(p) = \sum_{i=1}^{n} V_i^2.$$

The *symmetric additive convolution* $p \boxplus_n q$ is recalled in Section 2.

Theorem 1.1 (Finite Free Stam Inequality). *For $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with positive variance,*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}. \tag{1}$$

The proof combines three ingredients: the Score-Gradient Inequality (Theorem 3.1), a dissipation identity for the convolution flow (Lemma 4.10), and a case-split argument exploiting commutativity of $\boxplus_n$ (Theorem 5.1).

Convention. We set $\Phi_n(p) := \infty$ (equivalently $1/\Phi_n(p) := 0$) when p has a repeated root. The proof below establishes (1) under the hypothesis that *both* p and q have distinct roots; the extension to the boundary of $\mathcal{P}_n^{\mathbb{R}}$ is discussed in Remark 5.3 at the end of the paper.

2 Preliminaries

2.1 Root statistics

For

$$p(x) = \prod_{i=1}^{n}(x - \lambda_i) = \sum_{k=0}^{n} a_k \, x^{n-k}$$

with $a_0 = 1$, the mean and variance of the root distribution are

$$\bar{\lambda} = \frac{1}{n}\sum_{i=1}^{n}\lambda_i, \qquad \sigma^2(p) = \frac{1}{n}\sum_{i=1}^{n}(\lambda_i - \bar{\lambda})^2.$$

Lemma 2.1.

$$\sigma^2(p) = \frac{(n-1)\,a_1^2}{n^2} - \frac{2\,a_2}{n}.$$

Proof. By Vieta's formulas, $\sum_i \lambda_i = -a_1$ and $\sum_{i<j} \lambda_i\lambda_j = a_2$, whence

$$\sum_{i=1}^{n}\lambda_i^2 = a_1^2 - 2a_2.$$

The result follows from $\sigma^2 = \frac{1}{n}\sum_i \lambda_i^2 - \bar{\lambda}^2$. $\qquad\square$

2.2 Symmetric additive convolution

Let A and B be real symmetric matrices with characteristic polynomials p and q. The finite free additive convolution is defined by averaging over the orthogonal group:

$$(p \boxplus_n q)(x) = \int_{O(n)} \det(xI - (A + QBQ^T)) \, d\mu_{\text{Haar}}(Q).$$

By the MSS theorem [1], this admits a differential operator representation: if $q(x) = \sum_{k=0}^{n} b_k \, x^{n-k}$, then

$$(p \boxplus_n q)(x) = T_q \, p(x), \qquad T_q = \sum_{k=0}^{n} \frac{(n-k)!}{n!} \, b_k \, \partial_x^k. \tag{2}$$

The coefficients of $r = p \boxplus_n q$, $r(x) = \sum_k c_k \, x^{n-k}$, satisfy

$$c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!} \, a_i \, b_j. \tag{3}$$

Two fundamental properties we shall use repeatedly:

Theorem 2.2 ([1]). *If $p, q \in \mathcal{P}_n^{\mathbb{R}}$, then $p \boxplus_n q \in \mathcal{P}_n^{\mathbb{R}}$.*

Lemma 2.3 (Variance additivity). $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

Proof. From (3),

$$c_1 = a_1 + b_1 \quad \text{and} \quad c_2 = a_2 + \frac{n-1}{n}\,a_1 b_1 + b_2.$$

Substituting into Lemma 2.1 and expanding $(a_1 + b_1)^2$, the cross-terms $\frac{2(n-1)a_1 b_1}{n^2}$ and $-\frac{2(n-1)a_1 b_1}{n^2}$ cancel, yielding $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$. $\qquad\square$

2.3 Scores and Fisher information

Definition 2.1. For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1 < \cdots < \lambda_n$, the *score* at λ_i and the *finite free Fisher information* are

$$V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad \Phi_n(p) = \sum_{i=1}^n V_i^2.$$

The *score-gradient energy* is

$$\mathcal{S}(p) = \sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}.$$

Lemma 2.4. $V_i = \dfrac{p''(\lambda_i)}{2\,p'(\lambda_i)}.$

Proof. Since $p'(\lambda_i) = \prod_{j \neq i}(\lambda_i - \lambda_j)$, differentiating once more yields

$$p''(\lambda_i) = 2 \sum_{k \neq i} \prod_{j \neq i,\, j \neq k} (\lambda_i - \lambda_j) = 2\,p'(\lambda_i)\,V_i.$$

$\square$

Lemma 2.5 (Score identities). (i) $\displaystyle\sum_{i=1}^n V_i = 0.$

(ii) $\displaystyle\sum_{i=1}^n \lambda_i V_i = \binom{n}{2}.$

(iii) $\displaystyle\sum_{i=1}^n (\lambda_i - \bar{\lambda}) V_i = \binom{n}{2}.$

(iv) $\displaystyle\Phi_n(p) = \sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j}.$

Proof. (i): $\sum_i V_i = \sum_{i \neq j}(\lambda_i - \lambda_j)^{-1} = 0$ by antisymmetry.
   (ii): $\sum_i \lambda_i V_i = \sum_{i \neq j} \frac{\lambda_i}{\lambda_i - \lambda_j} = \sum_{i<j}\big(\frac{\lambda_i}{\lambda_i - \lambda_j} + \frac{\lambda_j}{\lambda_j - \lambda_i}\big) = \sum_{i<j} 1 = \binom{n}{2}.$
   (iii): Immediate from (ii) and (i).
   (iv): $\sum_i V_i^2 = \sum_{i \neq j} \frac{V_i}{\lambda_i - \lambda_j} = \sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j}.$ $\square$

Lemma 2.6 (Fisher–variance inequality). $\Phi_n(p)\,\sigma^2(p) \geq \dfrac{n(n-1)^2}{4}.$

Proof. By Cauchy–Schwarz applied to Lemma 2.5(iii):

$$\frac{n^2(n-1)^2}{4} \;\leq\; \Big(\sum_i (\lambda_i - \bar{\lambda})^2\Big)\Big(\sum_i V_i^2\Big) = n\,\sigma^2(p)\,\Phi_n(p). \qquad \square$$

3 The Score-Gradient Inequality

The following algebraic inequality is the key input for the general proof.

Theorem 3.1 (Score-Gradient Inequality). *For $p \in \mathcal{P}_n^{\mathbb{R}}$ of degree $n \geq 2$ with distinct roots,*

$$\mathcal{S}(p)\,\sigma^2(p) \;\geq\; \frac{n-1}{2}\,\Phi_n(p), \tag{4}$$

with equality if and only if $V_i = c(\lambda_i - \bar{\lambda})$ for some constant c.

Proof. Write $T = n\,\sigma^2(p)$, $U = \Phi_n(p)$, $S = \mathcal{S}(p)$. The claim is $ST \geq \frac{n(n-1)}{2}\,U$.

Step 1. By Lemma 2.5(iii) and Cauchy–Schwarz,

$$\frac{n^2(n-1)^2}{4} \;\leq\; TU. \tag{5}$$

Step 2. By Lemma 2.5(iv) and Cauchy–Schwarz,

$$U^2 \;\leq\; S \cdot \binom{n}{2}. \tag{6}$$

Step 3. Combining: $ST \geq \frac{2U^2}{n(n-1)} \cdot T = \frac{2U}{n(n-1)} \cdot TU \geq \frac{2U}{n(n-1)} \cdot \frac{n^2(n-1)^2}{4} = \frac{n(n-1)}{2}\,U$.

Equality. Equality in $ST = \frac{n(n-1)}{2}\,U$ requires equality in *both* Steps 2 and 1 simultaneously.

Step 1 equality: Cauchy–Schwarz in (5) is $\left(\sum_i (\lambda_i - \bar\lambda)\,V_i\right)^2 \leq \left(\sum_i (\lambda_i - \bar\lambda)^2\right)\left(\sum_i V_i^2\right)$; equality holds if and only if $V_i = c(\lambda_i - \bar\lambda)$ for some constant c.

Step 2 equality: Cauchy–Schwarz in (6) is $\left(\sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j} \cdot 1\right)^2 \leq \left(\sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}\right)\left(\sum_{i<j} 1\right)$; equality holds if and only if $\frac{V_i - V_j}{\lambda_i - \lambda_j}$ is a constant k for all $i < j$.

Consistency: If $V_i = c(\lambda_i - \bar\lambda)$, then $\frac{V_i - V_j}{\lambda_i - \lambda_j} = c$ for all $i < j$, so equality in Step 1 forces equality in Step 2. Conversely, if $\frac{V_i - V_j}{\lambda_i - \lambda_j} = k$ for all $i < j$, then $V_i - k\lambda_i$ is constant in i; since $\sum_i V_i = 0$ (Lemma 2.5(i)), this constant equals $-k\bar\lambda$, yielding $V_i = k(\lambda_i - \bar\lambda)$. Thus the two equality conditions are equivalent. $\qquad\square$

Remark 3.1. The equality condition $V_i = c(\lambda_i - \bar\lambda)$ characterizes, up to affine transformation, the zeros of the Hermite polynomial H_n: evaluating the ODE $H_n'' - 2xH_n' + 2nH_n = 0$ at a zero x_k yields $V_k = x_k$. For $n = 2$ this holds for all distinct root configurations.

4 The convolution flow

4.1 The semigroup and the flow

Fix $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with $a = \sigma^2(p) > 0$ and $b = \sigma^2(q) > 0$.

Definition 4.1. Write $q(x) = \sum_{k=0}^n b_k\, x^{n-k}$ with $b_0 = 1$. The *normalized coefficients* of q are

$$\kappa_k(q) = \frac{(n-k)!}{n!}\, b_k, \qquad k = 0, 1, \dots, n,$$

and the *generating polynomial* is $K_q(z) = \sum_{k=0}^n \kappa_k(q)\, z^k$.

Lemma 4.1 (Multiplicativity)**.** *The convolution formula* (3) *is equivalent to*

$$K_{p \boxplus_n q}(z) = K_p(z)\, K_q(z) \pmod{z^{n+1}},$$

i.e. the two sides agree in all coefficients of $z^0, z^1, \dots, z^n$.

Proof. By (3) the coefficient of z^k in $K_{p \boxplus_n q}(z)$ is

$$\kappa_k(p \boxplus_n q) = \frac{(n-k)!}{n!}\, c_k = \frac{(n-k)!}{n!} \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i\, b_j = \sum_{i+j=k} \kappa_i(p)\, \kappa_j(q),$$

which is exactly the coefficient of z^k in $K_p(z)\, K_q(z)$. $\qquad\square$

We now construct the *fractional family* $(q_t)_{t\in\mathbb{R}}$ interpolating between $q_0 = x^n$ and $q_1 = q$.

Definition 4.2 (Fractional family). Since $K_q(0) = \kappa_0 = 1$, the formal power series $\log K_q(z)$ is well-defined by

$$\log K_q(z) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (K_q(z) - 1)^m. \tag{7}$$

For any $t \in \mathbb{R}$, define

$$K_{q_t}(z) = K_q(z)^t := \exp(t \log K_q(z)) \pmod{z^{n+1}}, \tag{8}$$

where the right-hand side is first expanded as a formal power series and then truncated to degree n. The polynomial $q_t \in \mathcal{P}_n$ is the degree-n polynomial whose normalized coefficients are $(\kappa_k(q_t))_{k=0}^n$, i.e. $q_t(x) = \sum_{k=0}^n b_k(t)\,x^{n-k}$ with $b_k(t) = \frac{n!}{(n-k)!}\,\kappa_k(q_t)$. The *flow polynomial* is

$$p_t = p \boxplus_n q_t.$$

We now verify the properties of this construction.

Lemma 4.2 (Convergence and reality of coefficients). *The series* (7) *converges absolutely for* $|z| < R$ *with* $R = 1/\max_{1\leq k \leq n} |\kappa_k(q)|^{1/k}$. *For every* $t \in \mathbb{R}$, *the power series* $K_q(z)^t$ *has real coefficients; in particular the normalized coefficients* $\kappa_k(q_t)$ *are real.*

Proof. Write $K_q(z) = 1 + f(z)$ with $f(z) = \sum_{k=1}^n \kappa_k(q)\,z^k$. For $|z| < R$ we have $|f(z)| < 1$, so the standard series $\log(1 + w) = \sum_{m\geq 1} \frac{(-1)^{m+1}}{m} w^m$ converges absolutely at $w = f(z)$, and hence so does $K_q(z)^t = \exp(t \log K_q(z))$. Since $\kappa_k(q) \in \mathbb{R}$ and the log and exp series have real coefficients, every coefficient of $K_q(z)^t$ is real. $\qquad\square$

Lemma 4.3 (Semigroup property). *For all* $s, t \in \mathbb{R}$,

$$q_s \boxplus_n q_t = q_{s+t}.$$

In particular the family $(q_t)_{t\geq 0}$ *forms a semigroup under* $\boxplus_n$.

Proof. By multiplicativity (Lemma 4.1),

$$K_{q_s \boxplus_n q_t}(z) = K_{q_s}(z)\,K_{q_t}(z) \pmod{z^{n+1}}.$$

On the other hand, by definition (8),

$$K_{q_s}(z)\,K_{q_t}(z) = K_q(z)^s\,K_q(z)^t = \exp(s \log K_q(z))\,\exp(t \log K_q(z)).$$

We claim the right-hand side equals $K_q(z)^{s+t} = K_{q_{s+t}}(z)$ modulo z^{n+1}. Indeed, the identity $\exp(A)\exp(B) = \exp(A + B)$ holds for *commuting* formal power series (here $A = s \log K_q(z)$ and $B = t \log K_q(z)$, which trivially commute since both are power series in the same variable), so

$$K_{q_s}(z)\,K_{q_t}(z) = \exp((s + t) \log K_q(z)) = K_q(z)^{s+t} \pmod{z^{n+1}}.$$

Note that the truncation to degree n does not break the argument: the coefficient of z^k (for $0 \leq k \leq n$) in a product of two power series depends only on the first $k + 1$ coefficients of each factor. Since equality holds at the power-series level, it holds *a fortiori* after truncation. Thus $K_{q_s \boxplus_n q_t}(z) = K_{q_{s+t}}(z) \pmod{z^{n+1}}$, and since both sides are polynomials of degree at most n, they are identical. $\qquad\square$

Lemma 4.4 (Boundary values). $q_0 = x^n$ and $q_1 = q$.

Proof. For $t = 0$: $K_{q_0}(z) = K_q(z)^0 = \exp(0) = 1$, so $\kappa_0(q_0) = 1$ and $\kappa_k(q_0) = 0$ for $k \geq 1$. This gives $q_0(x) = x^n$.

For $t = 1$: $K_{q_1}(z) = K_q(z)^1 = K_q(z)$, so $\kappa_k(q_1) = \kappa_k(q)$ for all k. This gives $q_1 = q$. $\qquad\square$

Lemma 4.5 (Variance of the fractional family). *For every $t \in \mathbb{R}$,*

$$\sigma^2(q_t) = t\,b = t\,\sigma^2(q).$$

Proof. By Lemma 2.1, the variance depends only on $b_1(t)$ and $b_2(t)$, or equivalently on $\kappa_1(q_t)$ and $\kappa_2(q_t)$. We compute these from $K_{q_t}(z) = \exp(t \log K_q(z))$.

Write $K_q(z) = 1 + \kappa_1 z + \kappa_2 z^2 + \cdots$. Then

$$\log K_q(z) = \kappa_1 z + \left(\kappa_2 - \tfrac{\kappa_1^2}{2}\right)z^2 + O(z^3).$$

Exponentiating:

$$K_{q_t}(z) = \exp\!\left(t\kappa_1 z + t\left(\kappa_2 - \tfrac{\kappa_1^2}{2}\right)z^2 + O(z^3)\right) = 1 + t\kappa_1 z + \left(t\kappa_2 - \tfrac{t\kappa_1^2}{2} + \tfrac{t^2\kappa_1^2}{2}\right)z^2 + O(z^3).$$

Therefore $\kappa_1(q_t) = t\kappa_1$ and $\kappa_2(q_t) = t\kappa_2 + \frac{t(t-1)}{2}\kappa_1^2$. Translating back:

$$b_1(t) = \frac{n!}{(n-1)!}\,\kappa_1(q_t) = n\,t\kappa_1 = t\,b_1,$$

and, writing $B_2 = \frac{n!}{(n-2)!}\,\kappa_2(q_t) = n(n-1)\big[t\kappa_2 + \frac{t(t-1)}{2}\kappa_1^2\big]$:

$$\sigma^2(q_t) = \frac{(n-1)\,b_1(t)^2}{n^2} - \frac{2\,B_2}{n}$$

$$= \frac{(n-1)\,t^2\,b_1^2}{n^2} - 2(n-1)\Big[t\kappa_2 + \tfrac{t(t-1)}{2}\kappa_1^2\Big]$$

$$= \frac{(n-1)\,t^2\,b_1^2}{n^2} - \frac{2t\,b_2}{n} - \frac{(n-1)t(t-1)b_1^2}{n^2}$$

$$= \frac{(n-1)\,t\,b_1^2}{n^2} - \frac{2t\,b_2}{n} \;=\; t\,\sigma^2(q),$$

where the last step uses Lemma 2.1 for q. $\qquad\square$

Lemma 4.6 (Variance of the flow). $\sigma^2(p_t) = a + t\,b$ *for every* $t \in [0,1]$.

Proof. $p_t = p \boxplus_n q_t$, so by variance additivity (Lemma 2.3) and Lemma 4.5,

$$\sigma^2(p_t) = \sigma^2(p) + \sigma^2(q_t) = a + t\,b. \qquad\square$$

Remark 4.1. For non-integer t, the polynomial q_t need *not* have all real roots. For example, taking $n = 4$ and $q = (x-100)(x-1)(x+1)(x+100)$, one computes $\kappa_4(q_{1/2}) = \frac{1}{2}\kappa_4 - \frac{1}{8}\kappa_2^2 < 0$, and $q_{1/2}$ has only two real roots. This does not affect the proof: only the flow polynomial $p_t = p \boxplus_n q_t$ needs to be real-rooted, which is established in Lemma 4.9 below.

4.2 Perturbation analysis

Lemma 4.7. *Let $\lambda_i(t)$ denote the roots of p_t. Then*

$$\lambda_i(t+h) = \lambda_i(t) + \frac{hb}{n-1} V_i(t) + O(h^2).$$

Proof. By the semigroup property, $p_{t+h} = p_t \boxplus_n q_h$ with $\sigma^2(q_h) = hb$. The coefficients of q_h satisfy

$$b_0 = 1, \quad b_1 = 0, \quad b_2 = -\frac{nhb}{2} + O(h^2),$$

so the operator T_{q_h} acts as

$$T_{q_h} r(x) = r(x) - \frac{hb}{2(n-1)} r''(x) + O(h^2).$$

Setting $\lambda_i(t+h) = \lambda_i(t) + \delta_i$ in $T_{q_h} p_t(\lambda_i(t+h)) = 0$ and solving to first order:

$$\delta_i = \frac{hb}{2(n-1)} \cdot \frac{p_t''(\lambda_i)}{p_t'(\lambda_i)} + O(h^2)$$

$$= \frac{hb}{n-1} V_i(t) + O(h^2)$$

by Lemma 2.4. $\qquad\square$

Lemma 4.8.

$$\Phi_n(p_{t+h}) = \Phi_n(p_t) - \frac{2hb}{n-1} \mathcal{S}(p_t) + O(h^2).$$

Proof. Write $\epsilon = hb/(n-1)$ and suppress the t-dependence. From Lemma 4.7, the perturbed scores are

$$V_i^{(h)} = \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j) + \epsilon(V_i - V_j) + O(h^2)} = V_i - \epsilon \sum_{j \neq i} \frac{V_i - V_j}{(\lambda_i - \lambda_j)^2} + O(h^2).$$

Squaring and summing:

$$\Phi_n(p_{t+h}) = \sum_i V_i^2 - 2\epsilon \sum_{i \neq j} \frac{V_i(V_i - V_j)}{(\lambda_i - \lambda_j)^2} + O(h^2).$$

Pairing (i,j) with (j,i):

$$\sum_{i \neq j} \frac{V_i(V_i - V_j)}{(\lambda_i - \lambda_j)^2} = \sum_{i < j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2} = \mathcal{S}(p_t).$$

$\qquad\square$

Remark 4.2 (Uniformity of error terms). The $O(h^2)$ remainders in Lemmas 4.7 and 4.8 depend on the minimum gap $\delta_{\min}(t) = \min_i(\lambda_{i+1}(t) - \lambda_i(t))$: the implicit constants grow as $\delta_{\min}(t)^{-m}$ for some m depending on n. Lemma 4.9 below establishes a uniform lower bound $\delta_{\min}(t) \geq \delta_* > 0$ for $t \in [0,1]$, so the error terms are bounded uniformly in t and the passage to derivatives $\dot{\lambda}_i = \frac{b}{n-1} V_i$, $\dot{\Phi}_n = -\frac{2b}{n-1} \mathcal{S}$ is justified throughout $[0,1]$.

Lemma 4.9. *For every $t \in [0,1]$, p_t has n simple real roots.*

Proof. The coefficients of p_t are smooth in t (Definition 4.2), so the roots $\lambda_i(t)$ vary continuously. Since $p_0 = p$ has simple real roots, there is a maximal interval $[0, T)$ on which p_t has simple real roots; continuity gives $T > 0$.

Lyapunov function. On $[0, T)$ define the log-Vandermonde

$$W(t) = \sum_{i<j} \log(\lambda_j(t) - \lambda_i(t)).$$

By Lemma 4.7 the roots satisfy $\dot{\lambda}_i = \frac{b}{n-1} V_i$ on $[0, T)$, so

$$\dot{W}(t) = \sum_{i<j} \frac{\dot{\lambda}_j - \dot{\lambda}_i}{\lambda_j - \lambda_i} = \frac{b}{n-1} \sum_{i<j} \frac{V_j - V_i}{\lambda_j - \lambda_i}$$

$$= \frac{b}{n-1} \Phi_n(p_t) \ge 0,$$

the penultimate equality by Lemma 2.5(iv). Hence $W(t) \ge W(0)$ for all $t \in [0, T)$, i.e.

$$\prod_{i<j}(\lambda_j(t) - \lambda_i(t)) \ge \prod_{i<j}(\lambda_j(0) - \lambda_i(0)) =: D_0 > 0. \tag{9}$$

Uniform gap bound. Since $\sigma^2(p_t) = a + tb \le a + b$, the roots satisfy $\sum_i (\lambda_i - \bar{\lambda})^2 = n(a + tb) \le n(a+b)$, so every root lies within distance $R = \sqrt{n(a+b)}$ of $\bar{\lambda}(t)$. Each pairwise difference is therefore at most $2R$. For any consecutive gap $\delta_k(t) = \lambda_{k+1}(t) - \lambda_k(t)$ we may bound every other factor in (9) by $2R$:

$$D_0 \le \prod_{i<j}(\lambda_j(t) - \lambda_i(t)) \le (2R)^{\binom{n}{2}-1} \delta_k(t),$$

giving the uniform lower bound

$$\delta_k(t) \ge \delta_* := D_0 (2R)^{1-\binom{n}{2}} > 0$$

for every k and every $t \in [0, T)$.

Extension. Since all gaps are bounded below by $\delta_* > 0$ on $[0, T)$, the roots of p_T remain distinct by continuity. This contradicts the maximality of T unless $T \ge 1$. $\qquad \square$

4.3 Dissipation and the integral identity

Lemma 4.10 (Dissipation). $\dfrac{d}{dt}\Phi_n(p_t) = -\dfrac{2b}{n-1} \mathcal{S}(p_t).$

Proof. By Lemma 4.8, $\frac{\Phi_n(p_{t+h}) - \Phi_n(p_t)}{h} = -\frac{2b}{n-1} \mathcal{S}(p_t) + O(h)$. Since p_t has simple roots for all $t \in [0, 1]$ (Lemma 4.9), the scores $V_i(t)$ and hence $\mathcal{S}(p_t)$ are continuous in t. Taking $h \to 0$ yields the result. $\qquad \square$

Corollary 4.11 (Integral identity).

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} = \frac{2b}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} \, dt. \tag{10}$$

Proof. Set $f(t) = 1/\Phi_n(p_t)$. By the chain rule and Lemma 4.10,

$$f'(t) = \frac{2b}{n-1} \cdot \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} \ge 0.$$

By the Fundamental Theorem of Calculus, $f(1) - f(0) = \int_0^1 f'(t)\,dt$. Substituting: $f(0) = 1/\Phi_n(p_0) = 1/\Phi_n(p)$ and $f(1) = 1/\Phi_n(p_1) = 1/\Phi_n(p \boxplus_n q)$ yields (10). $\qquad \square$

5 Proof of the main theorem

Theorem 5.1. *Inequality* (1) *holds for every* $n \geq 2$.

Proof. Write $a = \sigma^2(p)$ and $b = \sigma^2(q)$.

Step 1 (Differential inequality). The Score-Gradient Inequality (Theorem 3.1) applied to p_t gives
$$\mathcal{S}(p_t) \geq \frac{(n-1)\,\Phi_n(p_t)}{2\,\sigma^2(p_t)}.$$

Substituting into Lemma 4.10:
$$\frac{d}{dt}\Phi_n(p_t) \;\leq\; -\frac{b}{a+tb}\,\Phi_n(p_t).$$

Integrating $\big(\log \Phi_n(p_t)\big)' \leq -b/(a+tb)$ from 0 to t:
$$\frac{1}{\Phi_n(p_t)} \;\geq\; \frac{a+tb}{a\,\Phi_n(p)}. \tag{11}$$

Step 2 (Forward bound). From Corollary 4.11 and the Score-Gradient Inequality:
$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} \;\geq\; b\int_0^1 \frac{dt}{(a+tb)\,\Phi_n(p_t)}.$$

Substituting (11), the factor $(a+tb)$ cancels:
$$\frac{1}{\Phi_n(p \boxplus_n q)} \;\geq\; \frac{a+b}{a\,\Phi_n(p)}. \tag{12}$$

Step 3 (Reverse bound). Since $p \boxplus_n q = q \boxplus_n p$, repeating Steps 1–2 with p and q interchanged yields
$$\frac{1}{\Phi_n(p \boxplus_n q)} \;\geq\; \frac{a+b}{b\,\Phi_n(q)}. \tag{13}$$

Step 4 (Conclusion). Exactly one of the following holds:

  (a) $b\,\Phi_n(q) \geq a\,\Phi_n(p)$. Then $\frac{b}{a\,\Phi_n(p)} \geq \frac{1}{\Phi_n(q)}$, and (12) gives $\frac{1}{\Phi_n(p\boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{b}{a\,\Phi_n(p)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$.

  (b) $a\,\Phi_n(p) \geq b\,\Phi_n(q)$. Then $\frac{a}{b\,\Phi_n(q)} \geq \frac{1}{\Phi_n(p)}$, and (13) gives $\frac{1}{\Phi_n(p\boxplus_n q)} \geq \frac{1}{\Phi_n(q)} + \frac{a}{b\,\Phi_n(q)} \geq \frac{1}{\Phi_n(q)} + \frac{1}{\Phi_n(p)}$. $\qquad\square$

Remark 5.1. The forward bound (12) and reverse bound (13) are each strictly stronger than the Stam inequality in their respective regimes. Averaging them yields the *half-Stam inequality* $\frac{2}{\Phi_n(p\boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$, from which the full inequality is recovered via the case split.

Remark 5.2. Strict inequality holds generically. Equality in (1) requires that $V_i(p_t) = c(t)\big(\lambda_i(t) - \bar{\lambda}(t)\big)$ for all $t \in [0,1]$, which forces both p and q to have roots at affinely rescaled zeros of the Hermite polynomial H_n. For $n = 2$ every polynomial satisfies this.

Remark 5.3 (Boundary behaviour). With the convention $1/\Phi_n := 0$ for repeated roots, inequality (1) extends to the boundary of $\mathcal{P}_n^{\mathbb{R}}$ as follows. When both p and q have repeated roots, both sides vanish and the inequality is trivially true. When exactly one factor, say p, has a repeated root, the inequality reduces to $\Phi_n(q) \geq \Phi_n(p \boxplus_n q)$, a monotonicity statement. To verify it, approximate p by distinct-root polynomials $p_\varepsilon \to p$; the established inequality gives $1/\Phi_n(p_\varepsilon \boxplus_n q) \geq 1/\Phi_n(p_\varepsilon) + 1/\Phi_n(q) \geq 1/\Phi_n(q)$. Since the convolution is continuous in coefficients, $p_\varepsilon \boxplus_n q \to p \boxplus_n q$. If $p \boxplus_n q$ has distinct roots then $\Phi_n(p_\varepsilon \boxplus_n q) \to \Phi_n(p \boxplus_n q)$ and the bound passes to the limit. The remaining case— $p \boxplus_n q$ itself having a repeated root while q has distinct roots—cannot occur: in the matrix model $(p \boxplus_n q)(x) = \int_{O(n)} \det(xI - (A + QBQ^T))\, d\mu(Q)$, if B has at least two distinct eigenvalues the integrand is a polynomial with n distinct real roots for μ-almost every Q, and the resulting average (which is real-rooted by [1]) has distinct roots since a convex combination of univariate polynomials can acquire a new double root only at the boundary of the interlacing family, which is excluded by the positive-variance hypothesis $\sigma^2(q) > 0$.

6 Alternative route via critical-point identity

We outline a different program that does not rely on a flow. The critical-point identity

$$\Phi_n(p) = -\frac{1}{4} \sum_{p'(\zeta)=0} \frac{p''(\zeta)}{p(\zeta)}$$

suggests controlling $\Phi_n(T_q p)$ by comparing $(T_q p)''/(T_q p)$ to p''/p at the critical points of $T_q p$.

6.1 Critical-point identity

Lemma 6.1. *Let $p \in \mathcal{P}_n^{\mathbb{R}}$ have distinct roots, and let $\zeta_1, \ldots, \zeta_{n-1}$ be the simple zeros of p'. Then*

$$\Phi_n(p) = -\frac{1}{4} \sum_{j=1}^{n-1} \frac{p''(\zeta_j)}{p(\zeta_j)}. \tag{14}$$

Proof. By Lemma 2.4, $\Phi_n = \frac{1}{4} \sum_{i=1}^n \frac{p''(\lambda_i)^2}{p'(\lambda_i)^2}$. Consider the meromorphic function

$$F(x) = \frac{p''(x)^2}{p'(x)\, p(x)}.$$

Poles at the roots. Since p has simple roots λ_i and $p'(\lambda_i) \neq 0$, $\operatorname{Res}_{x=\lambda_i} F = p''(\lambda_i)^2/p'(\lambda_i)^2$. Summing gives $\sum_i \operatorname{Res}_{\lambda_i} F = 4\Phi_n$.
Poles at critical points. At a simple zero ζ_j of p', one has $p(\zeta_j) \neq 0$ by interlacing, so $\operatorname{Res}_{x=\zeta_j} F = p''(\zeta_j)/p(\zeta_j)$.
Pole at infinity. As $x \to \infty$, $F(x) = n(n-1)^2/x^3 + O(x^{-4})$, so $\operatorname{Res}_\infty F = 0$.
The sum of residues on $\mathbb{P}^1$ vanishes, yielding $4\Phi_n + \sum_j p''(\zeta_j)/p(\zeta_j) = 0$. $\qquad\square$

6.2 Comparison program

Let $r = T_q p$. By Lemma 6.1,

$$\Phi_n(r) = -\frac{1}{4} \sum_{r'(\xi)=0} \frac{r''(\xi)}{r(\xi)}.$$

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Thus any pointwise comparison of $Y(\xi) = r''(\xi)/r(\xi)$ to $X(\xi) = p''(\xi)/p(\xi)$ at critical points of r can be converted into a global bound on $\Phi_n(r)$. Initial hypotheses suggested a linear bound of the form $Y \leq \alpha + \beta X$ (for positive quantities), but analysis of the $n = 2$ cases reveals a reciprocal relationship is exact.

Remark 6.1 (Case $n = 2$). For $n = 2$, with variances $u = \sigma^2(p)$ and $v = \sigma^2(q)$, the unique critical point ξ yielded $X = -2/u$ and $Y = -2/(u + v)$. These satisfy the exact reciprocal relation

$$\frac{1}{Y} = \frac{1}{X} - \frac{v}{2} = \frac{1}{X} - \frac{1}{4\Phi_2(q)}.$$

Summing over the (single) critical point gives the Stam inequality equality:

$$\frac{1}{-4\Phi_2(r)} = \frac{1}{-4\Phi_2(p)} - \frac{1}{4\Phi_2(q)} \implies \frac{1}{\Phi_2(r)} = \frac{1}{\Phi_2(p)} + \frac{1}{\Phi_2(q)}.$$

Remark 6.2 (Case $n = 3$). For $n = 3$, the gap is not constant. Explicit calculation shows

$$\frac{1}{Y(\xi)} = \frac{1}{X(\xi)} + \Delta(q, \xi),$$

where $\Delta(q, \xi)$ depends on the specific location of the critical point ξ. This suggests that for $n \geq 3$, the residue-sum comparison requires bounding the sum of these gaps rather than a uniform pointwise bound.

References

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