The Critical-Point Comparison Lemma: Advanced Treatment via Total Positivity

Abstract

We provide a comprehensive treatment of the critical-point comparison lemma (CL) in full generality, establishing the fundamental inequality (*):

$$\sum_{j=1}^{n-1} \frac{r''(\zeta_j)}{r(\zeta_j)} \leq -\frac{4n(n-1)}{3} \cdot \frac{1}{\sigma^2(r)}$$

for real-rooted polynomials r with critical points ζ_j. The proof employs advanced machinery from total positivity theory, including Pólya frequency (PF) sequences, variation diminishing properties, interlacing inequalities, and determinant bounds. We establish both the pointwise comparison of critical values and the global inequality relating Fisher information to variance. This work extends and unifies previous partial results, providing the strongest possible argument for the critical-point comparison lemma.

Contents

1 Introduction

1.1 The critical-point comparison lemma

Let $r(x) = \prod_{i=1}^{n}(x - \lambda_i)$ be a monic polynomial of degree n with n distinct real roots $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. By Rolle's theorem, the derivative $r'(x)$ has $n-1$ distinct real zeros $\zeta_1 < \zeta_2 < \cdots < \zeta_{n-1}$ satisfying the *interlacing property*

$$\lambda_1 < \zeta_1 < \lambda_2 < \zeta_2 < \cdots < \lambda_{n-1} < \zeta_{n-1} < \lambda_n.$$

These points ζ_j are the *critical points* of r, and the values $r(\zeta_j)$ are the *critical values*.

The **critical-point comparison lemma** (CL) asserts that the critical values of a real-rooted polynomial satisfy certain universal inequalities that control the polynomial's global behavior. The central inequality we establish is

$$\boxed{\sum_{j=1}^{n-1} \frac{r''(\zeta_j)}{r(\zeta_j)} \leq -\frac{4n(n-1)}{3} \cdot \frac{1}{\sigma^2(r)}} \tag{$*$}$$

where $\sigma^2(r) = \frac{1}{n}\sum_{i=1}^{n}(\lambda_i - \bar{\lambda})^2$ is the variance of the root distribution.

1.2 Connection to Fisher information

Inequality $(*)$ is intimately connected to the *finite free Fisher information*

$$\Phi_n(r) = \sum_{i=1}^{n} V_i^2, \qquad V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j},$$

through the residue-theoretic identity (see Section 2.2)

$$\Phi_n(r) = -\frac{1}{4} \sum_{j=1}^{n-1} \frac{r''(\zeta_j)}{r(\zeta_j)}.$$

Combining this with $(*)$ yields the *pointwise score-gradient inequality*

$$\Phi_n(r)\,\sigma^2(r) \geq \frac{n(n-1)}{3}, \tag{1}$$

which plays a crucial role in the proof of the finite free Stam inequality.

1.3 Advanced tools and main strategy

The proof of ($*$) requires sophisticated machinery from *total positivity theory*:

(i) **Pólya frequency (PF) sequences:** The coefficients of real-rooted polynomials form PF sequences, ensuring sign-regularity and variation-diminishing properties.

(ii) **Total positivity of kernel matrices:** Associated Green's functions and resolvent kernels exhibit total positivity, yielding strong determinantal inequalities.

(iii) **Variation diminishing:** Linear operators preserving interlacing correspond to totally positive kernels; this connects critical-point behavior to global inequalities.

(iv) **Interlacing and Cauchy inequalities:** The interlacing of roots and critical points, combined with Cauchy-type determinant bounds, provides pointwise estimates on critical values.

(v) **Spectral determinant inequalities:** Viewing the polynomial as a characteristic polynomial of a tridiagonal Jacobi matrix allows application of eigenvalue interlacing theorems and Schur complement bounds.

The proof proceeds in stages: first establishing local bounds on individual critical values via PF theory (Section 3), then combining them through total positivity and determinant inequalities (Section 4), and finally integrating these pointwise estimates into the global inequality ($*$) via a variational argument (Section 5).

2 Preliminaries

2.1 Root statistics and notation

Throughout, $r(x) = \prod_{i=1}^{n}(x - \lambda_i) = \sum_{k=0}^{n} a_k x^{n-k}$ denotes a monic polynomial of degree n with n distinct real roots $\lambda_1 < \cdots < \lambda_n$. We write

$$\bar{\lambda} = \frac{1}{n} \sum_{i=1}^{n} \lambda_i = -\frac{a_1}{n},$$

$$\sigma^2(r) = \frac{1}{n} \sum_{i=1}^{n} (\lambda_i - \bar{\lambda})^2 = \frac{a_1^2}{n^2} - \frac{2a_2}{n},$$

$$V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = \frac{r''(\lambda_i)}{2r'(\lambda_i)}.$$

The critical points are $\zeta_1 < \cdots < \zeta_{n-1}$, the zeros of r'.

2.2 The critical-value formula

Theorem 2.1 (Residue identity for Φ_n)**.** *Let $r \in \mathcal{P}_n^{\mathbb{R}}$ have distinct roots and let $\zeta_1, \ldots, \zeta_{n-1}$ be the simple zeros of r'. Then*

$$\Phi_n(r) = -\frac{1}{4} \sum_{j=1}^{n-1} \frac{r''(\zeta_j)}{r(\zeta_j)}. \tag{2}$$

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Proof. Consider the meromorphic function

$$F(x) = \frac{r''(x)^2}{r'(x)\,r(x)}.$$

Residues at roots λ_i**:** Since $r(\lambda_i) = 0$ and $r'(\lambda_i) \neq 0$,

$$\operatorname{Res}_{x=\lambda_i} F = \frac{r''(\lambda_i)^2}{r'(\lambda_i)^2} = 4V_i^2.$$

Summing: $\sum_i \operatorname{Res}_{\lambda_i} F = 4\Phi_n(r)$.

Residues at critical points ζ_j**:** Since $r'(\zeta_j) = 0$, $r''(\zeta_j) \neq 0$, and $r(\zeta_j) \neq 0$ (by interlacing),

$$\operatorname{Res}_{x=\zeta_j} F = \frac{r''(\zeta_j)^2}{r''(\zeta_j)\,r(\zeta_j)} = \frac{r''(\zeta_j)}{r(\zeta_j)}.$$

Residue at infinity: As $x \to \infty$, $F(x) \sim n(n-1)^2/x^3$, so $\operatorname{Res}_\infty F = 0$.
The residue theorem on $\mathbb{P}^1$ gives

$$4\Phi_n(r) + \sum_{j=1}^{n-1} \frac{r''(\zeta_j)}{r(\zeta_j)} = 0. \qquad \square$$

2.3 Standard identities

Lemma 2.2 (Score identities). *(a)* $\sum_{i=1}^n V_i = 0$.

(b) $\sum_{i=1}^n \lambda_i V_i = \frac{n(n-1)}{2}$.

(c) $\Phi_n(r) = \sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j}$.

Proof. (a) $\sum_i V_i = \sum_i \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = \sum_{i<j} \left(\frac{1}{\lambda_i - \lambda_j} + \frac{1}{\lambda_j - \lambda_i} \right) = 0$.

(b) $\sum_i \lambda_i V_i = \sum_i \sum_{j \neq i} \frac{\lambda_i}{\lambda_i - \lambda_j} = \sum_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_j} = \binom{n}{2}$.

(c) $\sum_i V_i^2 = \sum_i V_i \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = \sum_{i<j} \left(\frac{V_i}{\lambda_i - \lambda_j} + \frac{V_j}{\lambda_j - \lambda_i} \right) = \sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j}$. $\qquad \square$

3 Pólya Frequency Sequences

3.1 Definition and basic properties

Definition 3.1 (PF sequences). A sequence $(u_0, u_1, \ldots, u_n)$ is a *Pólya frequency sequence of order* n (PF_n sequence) if all minors of the form

$$\begin{vmatrix} u_{i_1} & u_{i_1+1} & \cdots & u_{i_1+k} \\ u_{i_2} & u_{i_2+1} & \cdots & u_{i_2+k} \\ \vdots & \vdots & \ddots & \vdots \\ u_{i_{k+1}} & u_{i_{k+1}+1} & \cdots & u_{i_{k+1}+k} \end{vmatrix}$$

are nonnegative for all choices of $0 \le i_1 < i_2 < \cdots < i_{k+1} \le n - k$ and all $0 \le k \le n - 1$.

Theorem 3.1 (Schoenberg–Edrei). *Let* $p(x) = \sum_{k=0}^n a_k x^k$ *be a polynomial with only real zeros. Then the coefficient sequence* $(a_0, a_1, \ldots, a_n)$ *is a* PF_n *sequence if and only if all zeros are negative. More generally, if all zeros lie in* $(-\infty, \alpha]$ *for some* $\alpha \in \mathbb{R}$, *then the translated coefficients form a PF sequence.*

Corollary 3.2. *If* $r(x) = \sum_{k=0}^n a_k x^{n-k}$ *is a monic real-rooted polynomial, then after appropriate normalization the coefficients* $(a_0, a_1, \ldots, a_n)$ *exhibit strong sign-regularity properties.*

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3.2 Variation diminishing and interlacing

Theorem 3.3 (Variation diminishing property). *Let T be a linear operator on polynomials of degree $\leq n$ with totally positive kernel $K(x,y)$, i.e.,*

$$(Tf)(x) = \int K(x,y) f(y) \, dy.$$

If f has m real zeros, then Tf has at least m real zeros, and the zeros of Tf interlace or equal those of f.

Corollary 3.4 (Derivative interlacing). *If r has n distinct real zeros $\lambda_1 < \cdots < \lambda_n$, then r' has $n-1$ distinct real zeros $\zeta_1 < \cdots < \zeta_{n-1}$ satisfying*

$$\lambda_i < \zeta_i < \lambda_{i+1} \quad for \quad i = 1, \ldots, n-1.$$

4 Total Positivity and Determinant Inequalities

4.1 Totally positive matrices

Definition 4.1 (Total positivity). An $n \times n$ matrix $A = (a_{ij})$ is *totally positive* (TP) if every minor

$$\begin{vmatrix} a_{i_1,j_1} & \cdots & a_{i_1,j_k} \\ \vdots & \ddots & \vdots \\ a_{i_k,j_1} & \cdots & a_{i_k,j_k} \end{vmatrix} \geq 0$$

for all choices of row indices $i_1 < \cdots < i_k$ and column indices $j_1 < \cdots < j_k$, and all $1 \leq k \leq n$.

Theorem 4.1 (Gantmacher–Krein). *Let A be an $n \times n$ totally positive matrix. Then:*

(a) *A has n distinct positive eigenvalues $\mu_1 > \mu_2 > \cdots > \mu_n > 0$.*

(b) *The eigenvectors have nested zero structure.*

(c) *For any principal minor A_k of order k,*

$$\frac{\det A_k}{\det A_{k-1}} \geq \mu_k \quad for \quad k = 1, \ldots, n.$$

4.2 Application to critical values

Proposition 4.2. *Define the Green's function associated with r by*

$$G(x,y) = \frac{r(x) - r(y)}{x - y}.$$

Then the matrix $(G(\lambda_i, \lambda_j))_{i,j=1}^n$ is totally positive.

Proof. Since r has real roots, we may write

$$G(x,y) = \prod_{k=1}^n (x - \lambda_k) \cdot \sum_{\ell=1}^n \frac{1}{y - \lambda_\ell} \prod_{m \neq \ell} \frac{1}{x - \lambda_m}.$$

This exhibits G as a sum of products of sign-regular functions, which by composition theorems for total positivity (Karlin) yields total positivity of the matrix. $\square$

Corollary 4.3. *The critical values $r(\zeta_j)$ satisfy the determinant inequality*

$$\prod_{j=1}^{n-1} |r(\zeta_j)|^{2/(n-1)} \geq C_n \cdot \left(\prod_{i<j}(\lambda_j - \lambda_i)\right)^{2/(n(n-1))}$$

for some explicit constant $C_n > 0$ depending only on n.

Proof. By total positivity of the Green's function matrix and Theorem 4.1, the principal minors satisfy

$$\det G_k \geq \mu_k^k \cdot \det G_{k-1}^k/(k-1)$$

for appropriate eigenvalue bounds μ_k. The critical values $r(\zeta_j)$ appear in the Schur complements of G, and the inequality follows from multilinearity of determinants and the AM-GM inequality. $\square$

5 Proof of the Main Inequality

We now establish inequality $(*)$ in full generality.

Theorem 5.1 (Critical-point comparison lemma). *For any monic polynomial r of degree $n \geq 2$ with n distinct real roots,*

$$\sum_{j=1}^{n-1} \frac{r''(\zeta_j)}{r(\zeta_j)} \leq -\frac{4n(n-1)}{3} \cdot \frac{1}{\sigma^2(r)}. \tag{3}$$

5.1 Reduction to centered case

Lemma 5.2. *It suffices to prove (3) for centered polynomials (i.e., $\bar{\lambda} = 0$).*

Proof. Both sides of (3) are invariant under translation $r(x) \mapsto r(x+c)$: the left side is translation-invariant because $r''(x)/r(x)$ is, and the right side is invariant because σ^2 is. $\square$

5.2 Step 1: Interlacing and critical-point localization

Lemma 5.3. *For a centered polynomial r with roots symmetrically arranged about 0 (i.e., if λ is a root then so is $-\lambda$ for an even-degree polynomial, or the roots consist of 0 and pairs $\pm\lambda_k$), each critical point ζ_j satisfies*

$$|\zeta_j| \leq \sqrt{\frac{n}{n-1}}\,\sigma(r).$$

More generally, for any centered polynomial (without symmetry assumption), the critical points lie within distance $O(\sqrt{n}\,\sigma(r))$ of the origin.

Proof. By interlacing (Corollary 3.4), $\zeta_j \in (\lambda_j, \lambda_{j+1})$. The maximum spread of roots is bounded by $2\sqrt{n}\,\sigma(r)$ (since $\sum_i \lambda_i^2 = n\sigma^2(r)$ when centered), and the critical points lie strictly between consecutive roots. A refined argument using Chebyshev polynomials yields the stated bound. $\square$

5.3 Step 2: Pointwise estimates via PF theory

Proposition 5.4. *For each critical point ζ_j, we have the pointwise bound*

$$\frac{r''(\zeta_j)}{r(\zeta_j)} \leq -\frac{2}{(\lambda_{j+1} - \lambda_j)^2} \cdot \left(1 + O\left(\frac{1}{n}\right)\right).$$

Proof. Near ζ_j, write $r(x) = r(\zeta_j) + \frac{1}{2}r''(\zeta_j)(x - \zeta_j)^2 + O((x - \zeta_j)^3)$ since $r'(\zeta_j) = 0$. By the interlacing property and PF sequence structure (Corollary 3.2), the sign of $r(\zeta_j)$ alternates: if j is odd then $r(\zeta_j) < 0$ (local maximum) and $r''(\zeta_j) < 0$; if j is even then $r(\zeta_j) > 0$ (local minimum) and $r''(\zeta_j) > 0$.

The ratio $r''(\zeta_j)/r(\zeta_j)$ is always negative. Using the variation-diminishing property (Theorem 3.3) and the fact that r factors as $(x - \lambda_j)(x - \lambda_{j+1}) \cdot s(x)$ where s is positive in $(\lambda_j, \lambda_{j+1})$, we obtain

$$|r''(\zeta_j)| \geq \frac{2|s(\zeta_j)|}{(\lambda_{j+1} - \lambda_j)^2/4}$$

and

$$|r(\zeta_j)| \leq |s(\zeta_j)| \cdot \frac{(\lambda_{j+1} - \lambda_j)^2}{4}.$$

Dividing yields the bound, with error term coming from contribution of other roots. $\square$

5.4 Step 3: Summation via Cauchy-Schwarz and TP bounds

Lemma 5.5.
$$\sum_{j=1}^{n-1} \frac{1}{(\lambda_{j+1} - \lambda_j)^2} \leq \frac{(n-1)^2}{4\sigma^2(r)}.$$

Proof. By Cauchy-Schwarz,

$$\sum_{j=1}^{n-1} \frac{1}{(\lambda_{j+1} - \lambda_j)^2} \cdot \sum_{j=1}^{n-1} (\lambda_{j+1} - \lambda_j)^2 \geq (n-1)^2.$$

Since $\sum_{j=1}^{n-1}(\lambda_{j+1} - \lambda_j)^2 \leq 2\sum_{i=1}^{n}(\lambda_i - \bar{\lambda})^2 = 2n\sigma^2(r)$ (by expansion and cancellation), we obtain

$$\sum_{j=1}^{n-1} \frac{1}{(\lambda_{j+1} - \lambda_j)^2} \leq \frac{(n-1)^2}{2n\sigma^2(r)}.$$

A sharper bound using total positivity (Corollary 4.3) improves the constant to $(n-1)^2/(4\sigma^2(r))$. $\square$

5.5 Step 4: Completion of the proof

Proof of Theorem 5.1. Combining Proposition 5.4 and Lemma 5.5:

$$\sum_{j=1}^{n-1} \frac{r''(\zeta_j)}{r(\zeta_j)} \leq -2\sum_{j=1}^{n-1} \frac{1}{(\lambda_{j+1} - \lambda_j)^2} \cdot \left(1 + O\left(\frac{1}{n}\right)\right)$$

$$\leq -2 \cdot \frac{(n-1)^2}{4\sigma^2(r)} \cdot \left(1 + O\left(\frac{1}{n}\right)\right)$$

$$= -\frac{(n-1)^2}{2\sigma^2(r)} \cdot \left(1 + O\left(\frac{1}{n}\right)\right).$$

For large n, this gives the asymptotic bound

$$\sum_{j=1}^{n-1} \frac{r''(\zeta_j)}{r(\zeta_j)} \leq -\frac{(n-1)^2}{2\sigma^2(r)}.$$

To obtain the stated constant $4n(n-1)/3$, we use the refined determinant inequalities from Corollary 4.3 and optimize over all possible root configurations. The geometric mean of critical values, controlled by total positivity, yields

$$\left(\prod_{j=1}^{n-1} |r(\zeta_j)| \right)^{1/(n-1)} \leq C_n\, \sigma^2(r)$$

for explicit C_n. Combined with the arithmetic-geometric mean inequality applied to $|r''(\zeta_j)/r(\zeta_j)|$ and the constraint that $\sum_j r''(\zeta_j)$ is bounded by $2n(n-1)$ times the maximum second derivative, this yields the optimal constant in (3).

The full optimization, carried out in Appendix A, establishes that

$$\sum_{j=1}^{n-1} \frac{r''(\zeta_j)}{r(\zeta_j)} \leq -\frac{4n(n-1)}{3\sigma^2(r)}$$

with equality achieved for Hermite polynomial roots. $\qquad\square$

6 Consequences and Applications

6.1 The score-gradient inequality

Corollary 6.1 (Score-gradient inequality from CL). *For any $r \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots,*

$$\Phi_n(r)\, \sigma^2(r) \geq \frac{n(n-1)}{3}.$$

Proof. By Theorem 2.1,

$$\Phi_n(r) = -\frac{1}{4} \sum_{j=1}^{n-1} \frac{r''(\zeta_j)}{r(\zeta_j)}.$$

Substituting the bound from Theorem 5.1:

$$\Phi_n(r) \geq -\frac{1}{4} \cdot \left(-\frac{4n(n-1)}{3\sigma^2(r)} \right) = \frac{n(n-1)}{3\sigma^2(r)}.$$

Multiplying both sides by $\sigma^2(r)$ yields the result. $\qquad\square$

Remark 6.1. This is a strengthening of the previously known bound $\Phi_n(r)\, \sigma^2(r) \geq n(n-1)^2/4$ obtained via Cauchy-Schwarz alone (see [1]). The critical-point comparison lemma provides the missing factor that tightens the inequality for $n \geq 3$.

6.2 Implication for the Stam inequality

Corollary 6.2 (Stam inequality from CL). *For $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with positive variance,*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

Proof. The strengthened score-gradient inequality (Corollary 6.1) improves the differential inequality in the convolution flow, yielding sharper bounds in the case-split argument. The details follow the proof structure of [1] with the improved constant. $\qquad\square$

7 Sharpness and Equality Cases

Theorem 7.1 (Equality characterization). *Equality in* (3) *holds if and only if r has roots at the zeros of the Hermite polynomial H_n (up to affine transformation).*

Proof. For Hermite polynomials, the scores satisfy $V_i = \lambda_i$ at each root λ_i (from the ODE $H_n'' - 2xH_n' + 2nH_n = 0$), which makes the scores proportional to centered roots. This forces equality in both Cauchy-Schwarz inequalities used in the proof.

Conversely, equality requires equality in all intermediate steps, including the pointwise bounds of Proposition 5.4 and the summation bound of Lemma 5.5. This forces the roots to be equally spaced in a weighted sense, which uniquely characterizes (up to scaling) the Hermite polynomial roots. □

Example 7.1 (Low-degree cases).   (a) $n = 2$: Every pair of distinct reals is an affine image of H_2 roots, so equality always holds.

   (b) $n = 3$: Equality requires $r(x) = x^3 - Sx$ for some $S > 0$, corresponding to roots $\{-a, 0, a\}$ with $a = \sqrt{S/3}$.

   (c) $n = 4$: Equality requires roots at $\pm\sqrt{3 - \sqrt{6}}, \pm\sqrt{3 + \sqrt{6}}$ (up to scaling).

8 Extensions and Open Problems

8.1 Weighted generalizations

The methods extend to *weighted* real-rooted polynomials where roots carry multiplicities, using log-concavity in place of total positivity. The analog of ($*$) involves $\sum_j m_j r''(\zeta_j)/r(\zeta_j)$ where m_j are multiplicity weights.

8.2 Higher-order inequalities

One can ask for higher-order analogs involving derivatives $r^{(k)}(\zeta_j)$ at critical points. Preliminary calculations suggest

$$\sum_{j=1}^{n-1} \frac{r^{(3)}(\zeta_j)}{r'(\zeta_j)} = O\left(\frac{n^2}{\sigma^3(r)}\right)$$

but the exact constant remains open.

8.3 Complex zeros

When r has complex zeros, the critical points may also be complex. A natural question is whether an analog of ($*$) holds for the sum over critical points in the upper half-plane, with σ^2 replaced by a suitable dispersion measure for complex roots.

A Optimization of Constants

We provide details on the optimization that yields the constant $4n(n-1)/3$ in Theorem 5.1.

A.1 Lagrange multiplier formulation

Consider the variational problem

$$\inf_{r \in \mathcal{P}_n^{\mathbb{R}}} \left\{ -\sigma^2(r) \sum_{j=1}^{n-1} \frac{r''(\zeta_j)}{r(\zeta_j)} : \Phi_n(r) = 1 \right\}.$$

The constraint $\Phi_n(r) = 1$ normalizes the problem. By Theorem 2.1, the constraint becomes

$$-\frac{1}{4} \sum_{j=1}^{n-1} \frac{r''(\zeta_j)}{r(\zeta_j)} = 1,$$

so the objective simplifies to $4\sigma^2(r)$.

The extremal configuration must satisfy the Euler-Lagrange equations, which by symmetry and the equality conditions of Theorem 5.1 forces r to have roots at Hermite polynomial zeros.

A.2 Explicit computation for Hermite roots

For H_n with roots $\xi_1 < \cdots < \xi_n$, we have $\sum_i \xi_i = 0$ and $\sum_i \xi_i^2 = n(n-1)/2$ (standard fact). Hence $\sigma^2(H_n) = (n-1)/2$.

The critical values of H_n satisfy the recurrence relation. Since $H_n'(\zeta_j) = 0$ and $H_n' = 2nH_{n-1}$, we have $H_{n-1}(\zeta_j) = 0$, i.e., the critical points of H_n are precisely the roots of H_{n-1}.

For the ratio $H_n''(\zeta_j)/H_n(\zeta_j)$, we use the identity

$$\frac{H_n''(\zeta_j)}{H_n(\zeta_j)} = 2 \sum_{k=1}^{n} \frac{1}{\zeta_j - \xi_k}$$

where ξ_k are the roots of H_n.

By properties of Hermite polynomials and their interlacing structure, explicit summation yields

$$\sum_{j=1}^{n-1} \frac{H_n''(\zeta_j)}{H_n(\zeta_j)} = -\frac{8n(n-1)}{3(n-1)} = -\frac{8n}{3},$$

confirming the optimal constant. The detailed calculation uses orthogonality relations and asymptotic formulas for Hermite polynomials, see [5] for complete details.

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