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UFR Mathématiques

Thèorie des modeles TD2

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Exercise 0.1 Consider the following classes. For each class, decide whether it is elementary. If yes, find a suitable language and axiomatization. Is the class finitely axiomatizable? Is the theory of the class complete?

(1) The class of totally ordered sets.
(2) The class of well-ordered sets.
(3) The class of finite abelian groups.
(4) The class of algebraically closed. fields.
(5) The class of non-oriented connected graphs without loops.

Solution 0.1

(1) The class of totally ordered sets. Take $\mathcal{L} = \{<\}$. The axioms are
- $\forall x \forall y \forall z (x < y \wedge y < z \Rightarrow x < z)$
- $\forall x x \not< x$.
- $\forall x \forall y (x = y \vee x < y \vee y < x)$. This theory is not complete, $(\mathbb{N}, <)$ and $(\mathbb{R}, <)$ are total orders but the second one satisfies density:

$$\forall x \forall y (x < y \Rightarrow \exists z \; x < z < y)$$

(2) The class of well-ordered sets. It is not axiomatizable: as in TD1, take $(\mathbb{N}, <)$ and $\mathbb{N}^{\mathcal{U}}$ for some $\mathbb{N}$-ultrafilter $\mathcal{U}$. The first structure is a well order while the latter isn't. Consider the set I of infinite elements in $\mathbb{N}^{\mathcal{U}}$, it is not empty but has no minimum, because otherwise one would have for $x = \min I$,

$$\underbrace{\mathrm{pred}(x)}_{finite} < \underbrace{x}_{infinite}$$

which would imply that $\mathbb{N}$ has a maximum element. pred(x) is defined by

$$y = \text{pred}(x) \iff y < x \wedge \forall z (z < x \implies z \leq y).$$

(3) The class of finite abelian groups. Abelian groups can be axiomatized by group axioms and $\forall x, y\ x + y = y + x$ in the languange $\mathcal{L} = \{0, +\}$. Finite abelian groups, however, are not. Suppose that some theory T axiomatizes them. Take

$$T' = T_{ab.groups} \cup T \cup \{\phi_k\}_{k \in \mathbb{N}}$$

Where ϕ_k is a formula stating "has at least k-elements". For any finite

$$\Delta \subseteq T_{ab.groups} \cup T \cup \{\phi_k\}_{k<N}$$

we can give a model: $(\mathbb{Z}/N\mathbb{Z}, +)$. By compactness, T' is satisfiable. A model of T' is a finite abelian group with more than k-elements for all k, and this is imposible.

(4) The class of algebraically closed fields. We can take $\mathcal{L}$ to be tha language of rings $\{0, 1, +, \cdot, -\}$ and T_{fields} to be the field axioms. We add to T, for each n, the formula

$$\varphi_n = \forall a_1, \forall a_2 \ldots \forall a_{n-1} \exists x\ x^n + a_{n-1}x^{n-1} + \cdots + a_1 x_1 + a_0 = 0.$$

This theory is not complete: $\mathbb{C}$ and $\bar{\mathbb{F}_p}^{alg}$ are both algebraically closed but have different characteristic. This theory is also not finitely axiomatizable: suppose it is, so there is a formula ϕ that axiomatizes it. By a lemma we saw in class, there is a finite subset $\Delta \subseteq T \cup \{\varphi_n\}_{n \in \mathbb{N}}$ such that $\Delta \models \phi$. Since Δ is finite, for some n , $\varphi_n \notin \Delta$. We'll show that $\Delta \cup \{\neg\varphi_n\}$ is satisfiable.

Consider $\mathbb{Q}$ and let $\mathbb{Q}_n$ be the field obtained by adjoining every root for every polynomial of degree strictly less than n. Let $p > n$ be a prime, so the polynomial $x^p = 1$ has no roots in $\mathbb{Q}_n$ (because for every $\alpha \in \mathbb{Q}_n$, the degree of its minimal polynomial over $\mathbb{Q}$ is at most n). We have that

$$\mathbb{Q}_n \models \Delta \cup \{\neg\varphi_n\} \models \phi$$

so that $\mathbb{Q}_n$ is algebraically closed, a contradiction.

(5) The class of non-oriented connected graphs without loops. Consider $\mathcal{L} = \{R\}$ where R is a binary predicate symbol. A graph is an $\mathcal{L}$ structure whose domain is regarded a set of

vertices and we regard xRy as x, y being connected by an edge. The axioms for non-oriented graphs without loops are

- $\forall x \; \neg xRx$
- $\forall x, y \; xRy \Rightarrow yRx$

Connectednes is not axiomatizable, consider the formula

$$\varphi_n(x, y) = \neg\exists z_1, \ldots, z_n \left(z_1 = x, z_n = y, \bigwedge_{k=1}^{n-1} z_k R z_{k+1} \right)$$

Which says "there is no path of length n connecting x and y".
Assume by contradiction that a theory T axiomatizes connected graphs. Then, consider for c_1, c_2 new constants the theory

$$T' = T \cup \{\phi_k(c_1, c_2\}_{k \in \mathbb{N}}.$$

For a finite

$$\Delta \subseteq T \cup \{\phi_k(c_1, c_2\}_{k<N},$$

we construct the graph with vertices labeled $\{0, 1, \ldots, N+1\}$, such that for $0 \leq k \leq N$, $kRk+1$. This graph satisfies $\phi_k(0, N+1)$ for every $k < N$ so it is a model of Δ. By compactness theorem, there is a connected graph with two vertices that are not connected by any finite path, this is impossible.

Exercise 0.2 Let $\mathcal{L}$ be a language and I be an infinite set. Let $\{\mathcal{M}\}_{i \in I}$, and $\{\mathcal{N}\}_{i \in I}$ be two families of $\mathcal{L}$-structures. Show that the following are equivalent:

(1) for every non-principal ultrafilter $\mathcal{U}$ on I, we have $\prod_{i \in I} \mathcal{M}_i/\mathcal{U} \equiv \prod_{i \in I} \mathcal{N}_i/\mathcal{U}$.

(2) For every $\mathcal{L}$-sentence φ, $\mathcal{M}_i \models \varphi \iff \mathcal{N}_i \models \varphi$ holds for all but finitely many $i \in I$.

Solution 0.2

First, assume (1). By Łos theorem,

$$\begin{aligned} \{i, \mathcal{M}_i \models \varphi\} \in \mathcal{U} &\iff \prod_{i \in I} \mathcal{M}_i/\mathcal{U} \models \varphi \\ &\iff \prod_{i \in I} \mathcal{N}_i/\mathcal{U} \models \varphi \text{ by hypothesis} \\ &\iff \{i, \mathcal{N}_i \models \varphi\} \in \mathcal{U} \end{aligned}$$

Notice that $C = \{i, \mathcal{M}_i \models \varphi, \mathcal{N}_i \not\models \varphi\}$ is contained in both $A = \{i, \mathcal{M}_i \models \varphi\}$ and $B = \{i, \mathcal{N}_i \not\models \varphi\}$, so if $A \in \mathcal{U}$ then $B \notin \mathcal{U}$ by the above, and $C \notin \mathcal{U}$, otherwise, if $A \notin \mathcal{U}$ then again $C \notin \mathcal{U}$. We can repeat this argument to show that $C' = \{i, \mathcal{M}_i \not\models \varphi, \mathcal{N}_i \models \varphi\} \notin \mathcal{U}$ and conclude that

$$C \cup C' = \{i, \neg(\mathcal{M}_i \models \varphi \iff \mathcal{N}_i \models \varphi)\} \notin \mathcal{U}.$$

This shows that for any ultrafilter $\mathcal{U}$, $\{i, \mathcal{M}_i \models \varphi \iff \mathcal{N}_i \models \varphi\} \in \mathcal{U}$. Since this holds for any choice of $\mathcal{U}$, it holds for every $i \in \cap_{\mathcal{U}\text{ultrafilter}} \mathcal{U} \subseteq Frechet$, so it holds for a cofinite set of indices.

We now assume (2). Let $\mathcal{U}$ be any ultrafilter over I and suppose by contradiction that $\{i, \mathcal{M}_i \models \varphi\} \in \mathcal{U}$ and $\{i, \mathcal{N}_i \models \varphi\} \notin \mathcal{U}$. Then we have that their intersection, $\{i, \mathcal{M}_i \models \varphi, \mathcal{N}_i \not\models \varphi\} \in \mathcal{U}$ is contained in

$$\{i, \neg(\mathcal{M}_i \models \varphi \iff \mathcal{N}_i \models \varphi)\},$$

but this set cannot be in $\mathcal{U}$ since it would contradict our assumption (if a condition holds for a cofinite set of indices then that set belongs to the Fréchet filter which is contained in $\mathcal{U}$). We conclude from this that $\{i, \mathcal{M}_i \models \varphi\} \in \mathcal{U}$ if and only if $\{i, \mathbb{N}_i \models \varphi\} \in \mathcal{U}$. Now we can apply Łos theorem:

$$\begin{aligned}
\prod_{i \in I} \mathcal{M}_i/\mathcal{U} \models \varphi &\iff \{i, \mathcal{M}_i \models \varphi\} \in \mathcal{U} \\
&\iff \{i, \mathcal{N}_i \models \varphi\} \in \mathcal{U} \text{ by hypothesis} \\
&\iff \prod_{i \in I} \mathcal{N}_i/\mathcal{U} \models \varphi
\end{aligned}$$

Exercise 1. Show that an abelian group is orderable if and only if all of its finitely generated subgroups are.

Solution 1. Let G be an abelian group. One direction is clear, if $(G, <)$ is orderable then for any subgroup H, the restriction of $<$ to H orders it an respects the group structure. Now, suppose that every finitely generated subgroup of G is orderrable. Let $\mathcal{L}$ be the language of groups and consider the language $\mathcal{L}(G)$. Define Ψ as the theory containing the following $\mathcal{L}(G)$-sentences for every $a, b, c \in G$:

(1) $a \not< a$.
(2) $a < b \vee a = b \vee a > b$.
(3) $a < b \wedge b < c \Rightarrow a < c$.
(4) $a < b \wedge a + c < b + c$.

Consider also the set

$$D(G) = \{\varphi \text{ an } \mathcal{L}(G)\text{-sentence}, G^* \models \varphi\}$$

where G^* is the expansion of G obtained by interpreting every constant symbol as itself in G. We now consider the $\mathcal{L}(G)$-theory

$$T' = \Psi \cup D(G).$$

This theory is finitely satisfiable: if $\Delta \subseteq T'$ is finite, then there are finite $a_1, \ldots, a_n \in G$ appearing as symbols in T', and since $H = \langle a_1, \ldots a_n \rangle$ (the group generated by the a_i's) is a substructure of G^*, it satisfies every quantifier free sentence that "mentions" them that is true in G^*, so they satisfy whatever finite part of $D(G)$ that may be in Δ. By our hypothesis, H can be ordered, which means $H \models \Psi$. By compactness theorem, there is G', a model of T'. We need to show that G is contained in G', so consider the map $f : G \to G'$, that sends $a \mapsto a^{G'}$ (sends and element of G to the interpretation of its symbol in G'). This is an embedding since clearly $f(0) = 0$ and if, for $a, b, c \in G$, $a + b = c$ then $G' \models a + b = c$ or equivalently $G' \models a^{G'} + b^{G'} = c^{G'}$ so that $f(c) = f(a) + f(b)$. Similarly one proves that $f(-a) = -f(a)$. Finally, one can conclude that G is isomorphic to a subgroup of $G' \restriction \mathcal{L}$, and since G' is orderable, then G is.

Exercice 2. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and let $\mathcal{N} \equiv \mathcal{M}$.

(1) Show that $|M| = |N|$.

(2) Let $k \in \mathbb{N}$. Show that there exist finite formulas $\varphi_1, \dots, \varphi_N$ with k free variables such that for every $\varphi \in \mathcal{F}(\mathcal{L})$, and every $\bar{b} \in M^k$, for some φ_i

$$\mathcal{M} \models \varphi(\bar{b}) \leftrightarrow \varphi_i(b).$$

(3) Let $k = |M|$ and let $\varphi_1, \dots, \varphi_N$ as in the previous question. Write $M = \{a_1, \dots, a_n\}$. Show that there exists a bijection $\sigma : M \to N$ such that for every i,

$$\mathcal{M} \models \varphi_i(a_1, \dots, a_n) \iff \mathcal{N} \models \varphi_i(\sigma(a_1), \dots, \sigma(a_n)).$$

(4) Show that $\mathcal{M} \simeq \mathcal{N}$.

Solution 2.

(1) For finite structures, we already know that their cardinality can be expressed by a 1st order sentence. Since $\mathcal{N} \equiv \mathcal{M}$, then they both satisfy this sentence.

(2) Let $M = \{a_1, \dots, a_n\}$. We will label all the free variables we are going to use as $x_1, \dots, x_n, \dots$. For each symbol in the language, we consider the set of formulas composed by

- If c is a constant symbol and $c^{\mathcal{M}} = a_i$, then add the formula $x_i = c$. If not, add its negation.
- If f is a p-ary function symbol, and if $f(a_{i_1}, \dots, a_{i_p}) = a_{i_j}$, then add the formula $f(x_{i_1}, \dots, x_{i_p}) = x_{i_j}$. If not, add its negation.
- If R is a p-ary predicate symbol, and if $R(a_{i_1}, \dots, a_{i_p})$, then add the formula $R(x_{i_1}, \dots, x_{i_p})$. If not, add its negation.

Define F to be the set formed by formulas of the form

$$\phi(x_1, \dots, x_n) = \bigvee_{S_1} \bigwedge_{S_2} \varphi(x_1, \dots, x_n)$$

and their negations, where φ is one of the above and M_1, M_2 are any finite subsets of $\{1, 2, \dots, n\}$. We will show by induction that any other formula $\phi(x_1, \dots, x_k)$ is equivalent to one of these (perhaps adding at most finitely many more formulas). Notice that our finite set of formulas all have their free variables among $x_1, \dots, x_n$, this doesn't matter much because we compare two formulas with different number of free variables, we can add to one something of the form $x_{n+1} = x_{n+1}$ to make it depend on more variables. We will then assume $k = n$.

Let $t(\bar{x})$ be a term, then if $t^{\mathcal{M}}(\bar{a}) = a_i$ for some i, there is a formula $\varphi(\bar{x})$ in F such that $t^{\mathcal{M}}(\bar{a}) = a_i$ iff $\mathcal{M} \models \varphi(\bar{a})$, we show this by induction: if t is a constant c, then $t^{\mathcal{M}}(\bar{a}) = a_i$ iff $c = a_i$, and we have a formula for expressing this. If t is x_j then $t^{\mathcal{M}}(\bar{a}) = a_i$ iff $a_j = a_i$, so we can add $x_i = x_j$ to F . And if $t(\bar{x}) = f(u_1(\bar{x}), \ldots, u_p(\bar{x}))$, then $t^{\mathcal{M}}(\bar{a}) = a_i$ iff for $i = 1, 2, \ldots p$ and for some $a_{j_1}, \ldots a_{j_p} \in M$, $u_i(\bar{a}) = a_{j_i}$ and $t(a_{j_1}, \ldots a_{j_p}) = a_i$, but by induction hypothesis we have a formula in F for each u_i and also have the formula $t(x_{j_1}, \ldots x_{j_p}) = x_i$.

We can now apply induction over formulas. If $\phi(\bar{x})$ has the form $t_1(\bar{x}) = t_2(\bar{x})$, then $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a}) = a$ if and only if $\bigvee_{a \in M} t_1^{\mathcal{M}}(\bar{a}) = a \wedge t_2^{\mathcal{M}}(\bar{a}) = a$, we can then find a formula in F equivalent to each of $t_1^{\mathcal{M}}(\bar{a}) = a, t_2^{\mathcal{M}}(\bar{a}) = a$ connect them by $\wedge$ and distribute to get one in F. If $\phi(\bar{x})$ has the form $R(t_1(\bar{x}), \ldots, t_p(\bar{x}))$, then $R^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_p^{\mathcal{M}}(\bar{a}))$ if and only if for each $1 \leq i \leq p$ there is b_i in M such that $t_i(\bar{a}) = b_i$ and $R^{\mathcal{M}}(b_1, \ldots, b_p)$, and we have a formula in F for each of these.

For the boolean case, if $\phi(\bar{x}) = \psi(\bar{x}) \wedge \theta(\bar{x})$, then we can find one formula in F for each ψ and θ and consider their conjunction, then distribute to get one in F. For the negation case, it is a similar proof (recall by construction, F is closed under $\neg$).

If $\phi(\bar{x}) = \exists y \psi(y, \bar{x})$ then we find a formula in F equivalent to $\psi(y, \bar{x})$ and notice that since M is finite, $\phi(\bar{b})$ is equivalent to $\bigvee_{a \in M} \psi(a, \bar{b})$.

(3) Let $\bar{a} = \{a_1, \ldots, a_n\}$, for each $i = 1, \ldots, N$, either $\mathcal{M} \models \varphi_i(\bar{a})$ or not, so define $\varphi'(\bar{x})$ as $\varphi(\bar{x})$ if it does, and as $\neg\varphi(\bar{x})$ if it does not. Consider

$$\Phi = \exists x_1, \ldots x_n \bigwedge_{i=1}^{N} \varphi_i'(\bar{x})$$

By construction $\mathcal{M} \models \Phi$, so $\mathcal{N} \models \Phi$ by elementary equivalence. Let $b_1, \ldots, b_n$ be the tuple of n that satisfies this formula. Then we can take $a_i \mapsto b_i$, this map verifies what we want.

(4) We just found a bijection that respects every φ_i. Since these formulas (by construction) actually express what the constants, predicates and functions are, this bijection is an embedding. Therefore $\mathcal{M} \simeq \mathcal{N}$.

Exercise 3. Let $\mathcal{K}$ be a class of finite $\mathcal{L}$-structures and let $T = \text{Th}(\mathcal{K})$.

(1) Give a necessary and sufficient condition for T to have an infinite model.

(2) Suppose that T has infinite models. Find an axiomatization T^∞ fo the class of all infinite models of T.

(3) Show that for all sentences φ, φ is a consequence of T^∞ if and only if there exists $n \in \mathbb{N}$ such that φ is true in every $\mathcal{M} \in \mathcal{K}$ such that $|M| > n$.

Solution 3.

(1) T has an infinite model iff if has arbitrarily large finite models.

Proof: Suppose T has an arbitrarily large models first, and consider the theory

$$T' = T \cup \{\phi_k\}_{k \in \mathbb{N}}$$

Where ϕ_k is the statement " has more than k elements ". T' is finitely satisfiable by hypothesis, so by the compactness theorem there is a model of T', this model is inifite. Suppose that T has no infinite model, so the theory T' is not satisfiable, then by compactness there is a finite fragment of T' that is not satisfiable, so for some $N \in \mathbb{N}$,

$$T \cup \{\phi_k\}_{k \leq N}$$

has no model. This means there is no member of $\mathcal{K}$ with N or more elements.

(2) Take $T^\infty = T'$ as above.

(3) Suppose $T^\infty \models \varphi$. Then by compactness, for some finite $\Delta \subseteq T^\infty$, $\Delta \models \varphi$. Then for some $N \in \mathbb{N}$, $T \cup \{\phi_k\}_{k \leq N} \models \phi$, but every $\mathcal{M} \models T \cup \{\phi_k\}_{k \leq N}$ is a member of $\mathcal{K}$ with $|M| > n$. Conversely, if $T^\infty \not\models \varphi$, this means that $T^\infty \cup \{\neg\varphi\}$ is satisfiable, so this means there are infinite models of T that do not satisfy φ, and by (1), there are arbitarily large members of $\mathcal{K}$ where φ is false.

Exercise 4. Let G be a group which has elements of arbitrarily large finite order and let $T = \mathrm{Th}(G)$. Give two proofs of the following (one using ultraproducts and the other using compactness).

(1) There is $H \models T$ such that H has infinitely many elements of infinite order.

(2) There is no $\mathcal{L}_G$-formula that defines the set of elements of finite order.

Solution 4.

(1) *Ultraproducts:* Let $\mathcal{U}$ a non-principal ultrafilter over $\mathbb{N}$ and consider the ultrapower $G^{\mathcal{U}}$. Let $\{g_1, g_2, \dots\} \subseteq G$ where the order of g_i is i. Consider $x = [g_i]_{\mathcal{U}}$, this element has infinite order since for all k, $\{i, g_k^i \neq 1\}$ is finite, hence $x^k \neq 1$. Therefore we can consider the set of infinite order elements $\{x, x^2, x^3, \dots\}$

Compactness: Let c be a new constant symbol and consider the theory

$$T' = T \cup \{c^n \neq 1\}_{n\in\mathbb{N}}.$$

For any finite fragment $T_0 \subseteq T$, there is N such that

$$T_0 \subseteq T \cup \{c^n \neq 1\}_{n<N}.$$

So G can model T_0 by interpreting c as g_{N+1}. By compactness, we can find $\mathcal{H} \models T'$, where $x = c^{\mathcal{H}}$ is an element of infinite order. Therefore we can consider the set of infinite order elements $\{x, x^2, x^3, \dots\}$.

(2) Asssume such a formula exists, call it $\phi(y)$.

Ultraproducts: In the same construction as above, we have that $G^{\mathcal{U}} \models \phi(g_i)$ for all i, but $G^{\mathcal{U}} \not\models \phi(x)$, which is a contradiction.

Compactness: Consider the theory

$$T'' = T' \cup \{\phi(c)\}.$$

It is satisfiable by the same argument above and a model of T'' has an element c of finite order larger than any natural, which is impossible.

Exercise 5. An infinite $\mathcal{L}$-structure $\mathcal{M}$ is called *minimal* if every definable $A \subseteq M$ is either finite or co-finite. Suppose $\mathcal{L}$ contains an unary predicate P. Show that the class of minimal $\mathcal{L}$-structures is not elementary. What about an arbitrary language?

Solution: We will show that this class is not closed under ultraproducts. Consider the family of $\mathcal{L}$-structures indexed by $\mathbb{N}$ given by $\mathcal{N}_i = \langle \mathbb{N}, P \rangle$, where $P^{\mathcal{N}_i} = \{0, \ldots, i-1\}$. Notice that $\mathcal{N}_i$ is minimal for every i, because being definable in $\mathcal{N}_i$ amounts to being definable in $\mathbb{N}$ with the empty language [1], as P can be defined as

$$P(x) \iff \bigvee_{k<n} k = x.$$

Consider a non-principal ultrafiter $\mathcal{U}$ over $\mathbb{N}$, and define the ultraproduct

$$\mathcal{N} = (\prod_{i \in \mathbb{N}} \mathcal{N}_i)/\mathcal{U}.$$

$P^{\mathcal{N}}$ is infinite and co-infinite: observe that for all n, $[n, n, \ldots] \in P^{\mathcal{N}}$ and $[n, n+1, n+2, \ldots] \notin P^{\mathcal{N}}$. We conclude that $\mathcal{N}$ is not minimal.

For arbitrary languages, we can consider cases:

- If P is k-ary then we can define an unary predicate $P'(x) \iff P(x, \ldots, x)$, and repeat the above.
- If f is a k-ary function symbol, we can define a $k+1$-predicate $\mathrm{Graph}(f)$ and repeat the above.
- Constants do not affect definability since we can use parameters.

[1] We showed in TD1 that in the empty language every structure is minimal