

## Théorie des modèles TD1

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**Exercise 0.1** Let  $\mathcal{M}$  be an  $\mathcal{L}$ -structure,  $m, n \in \mathbb{N}$  and  $A \subseteq M^{n+m}$  be definable in  $\mathcal{M}$ . For  $\bar{b} \in M^m$ , let  $A_{\bar{b}} = \{\bar{a} \in M^n, (\bar{a}, \bar{b}) \in A\}$  the fiber of  $A$  over  $b$ . Let  $k \in \mathbb{N}$ . Show that the set  $\{\bar{b} \in M^m, |A_{\bar{b}}| < k\}$  is definable. (\*) Is the set  $\{\bar{b} \in M^m, |A_{\bar{b}}| < \infty\}$  definable?

**Solution 0.1** If  $A \subseteq M^{n+m}$  is definable, then there is some  $\bar{s} \in M$ , and some formula  $\phi(\bar{x}, \bar{y}, \bar{s})$  such that  $A = \{(\bar{x}, \bar{y}) \in M^{n+m}, \mathcal{M} \models \phi(\bar{x}, \bar{y}, \bar{s})\}$ . The following formula states that the  $A_{\bar{b}}$ -fiber has less than  $k$  elements.

$$\phi_k(y_1, \dots, y_m) = \forall \bar{x}_1 \dots \bar{x}_k \left( \bigwedge_{i=1}^k \phi(\bar{x}_i, \bar{y}) \Rightarrow \bigvee_{1 \leq i \neq j \leq k} \bar{x}_i = \bar{x}_j \right).$$

We see that  $\mathcal{M} \models \phi_k(\bar{b})$  if and only if  $A_{\bar{b}}$  has less than  $k$  elements. (\*) Consider  $\mathcal{N} = (\mathbb{N}, <)$ , and let  $\mathcal{U}$  a non-principal ultrafilter over  $\mathbb{N}$ . Let  $\mathcal{M} = \mathcal{N}^{\mathcal{U}}$ . We can identify each  $n \in \mathbb{N}$  with  $[n, n, \dots]_{\mathcal{U}} \in \mathcal{M}$ . Notice that for every  $n \in \mathbb{N}$ ,

$$\omega = [0, 1, 2, \dots, n, n+1, \dots]_{\mathcal{U}} > n$$

We call elements bigger than every  $n$ , *infinite*. Suppose now the set  $\{\bar{b} \in M^m, |A_{\bar{b}}| < \infty\}$  is definable by some formula  $\phi(x, \bar{m})$  with parameters  $\bar{m} \in \mathcal{M}$ . In other words,  $\mathcal{M} \models \phi(x, \bar{m})$  if and only if there is a finite number of elements below  $x$  (in the usual finite sense), we show this implies that  $x$  is necessarily finite: suppose not, then for every  $n$ ,  $\mathcal{U}$ -almost everywhere,  $x_i \neq n$ . We want to prove that actually  $x_i \geq n$ : if it were not the case, then again,  $\mathcal{U}$ -almost everywhere  $x_i < n \Rightarrow x_i \in \{0, 1, \dots, n-1\} = \bigcup_{j=0}^{n-1} \{j\}$ . In other words, this means that

$$\bigcup_{j=0}^{n-1} \{i, x_i = j\} \in \mathcal{U}.$$

By ultrafilter properties, (if  $A \cup B \in \mathcal{U}$  then either  $A \in \mathcal{U}$  or  $B \in \mathcal{U}$ ) we conclude that for some  $k$ ,  $[x] = [k]$ , which is a contradiction since  $x$  is infinite. We consider now  $\Sigma(x, \bar{m}) = \{\neg\phi_k(x, \bar{m})\}_{k \in \mathbb{N}} \cup \{\phi(x, \bar{m})\}$ . It is finitely consistent, since if  $\Sigma_N(x, \bar{m}) = \{\neg\phi_k(x, \bar{m})\}_{k < N} \cup \{\phi(x, \bar{m})\}$  is a finite part of  $\Sigma$ , then  $\mathcal{M} \models \Sigma_N(N, \bar{m})$  ( $N$  has at least  $k$  elements below it for every  $k > N$ , and also has a finite number of elements below it since it is finite). By compactness, there is  $N' \in \mathcal{M}$  such that  $\mathcal{M} \models \Sigma_{N'}(N', \bar{m})$ . We conclude that  $N'$  has at least  $k$  elements below it for every  $k$ , and that  $N$  is finite by the above. This is a contradiction, so  $\{\bar{b} \in M^n, |A_{\bar{b}}| < \infty\}$  is not definable.

**Exercise 2.** Let  $M$  be a set and  $\mathcal{D} = \bigcup_n D_n$  be a collection of subsets of  $\bigcup_n M^n$  containing  $\emptyset, M^n$  for every  $n$ , the diagonals, and closed under permutartion of the coordinates, cartesian products, the boolean set operations and linear projections. Show that  $\mathcal{D} = \text{Def}(\mathcal{M}, \emptyset)$ , for some language  $\mathcal{L}$  and some  $\mathcal{L}$ -structure  $\mathcal{M}$ .

**Solution 0.2** Take  $\mathcal{C} = \emptyset$ , and  $\mathcal{R} = \bigcup_{n \in \mathbb{N}} \{(x_1, \dots, x_n), (x_1, \dots, x_n) \in D_n\}$ . In other words, take no constants and set each of the  $D_n$ 's to be a predicate. For each  $n$ , if for every  $(x_1, \dots, x_n) \in M^n$ , there exists  $y \in M$  such that  $(x_1, \dots, x_n, y) \in D_{n+1}$ , we set  $f : M^n \rightarrow M$  which sends  $(x_1, \dots, x_n)$  to  $y$ . We may have to do this (possibly infinitely) many times since such  $y$  may not be unique.

**Exercise 0.3** Let  $\mathcal{M}$  be an expansion of a total order equipped with the *order topology*. Let  $A \subseteq M^n$  and  $f : A \rightarrow M$  both definable.

- (1) Show that  $A^\circ, \bar{A}$  and  $\text{bd}(A)$  are all definable.
- (2) Show that the set of discontinuity points of  $f$  is definable.
- (3) Show that the following properties are definable:  $A$  is discrete,  $A$  is bounded.
- (4) What about  $A$  is compact and connected?

**Solution 0.3** We use the following abbreviations:  $\bar{x} < \bar{y}$  for  $x_i < y_i$  for each  $i$  and if  $\phi$  is a formula then  $Qx \in A (\phi)$  (where  $Q$  is a quantifier) for  $Qx(x \in A \Rightarrow \phi)$ .

- (1)  $\bar{x} \in A^\circ$  if and only if  $\exists \bar{y}, \bar{z} \in A (\bar{z} < \bar{x} < \bar{y})$ .  
 $\bar{x} \in \bar{A}$  if and only if  $\forall \bar{y}, \bar{z} ((\bar{z} < \bar{x} < \bar{y}) \Rightarrow \exists \bar{w} \in A (\bar{z} < \bar{w} < \bar{y}))$   
 $\bar{x} \in \text{bd}(A)$  if and only if  $\bar{x} \notin A^\circ \wedge \bar{x} \in \bar{A}$ .

(2)  $\bar{x}$  is a discontinuity point of  $f$  if and only if

$$\exists r \exists s ((r < f(\bar{x}) < s) \wedge \forall \bar{y}, \bar{z} \in A ((\bar{z} < \bar{x} < \bar{y}) \Rightarrow \exists \bar{w} ((\bar{z} < \bar{w} < \bar{y}) \wedge (f(\bar{w}) < r \vee f(\bar{w}) > s)))$$

(3) We say  $A$  is discrete if  $\forall \bar{x} \in A \exists \bar{y} \in A (\bar{x} < \bar{y} \Rightarrow \forall \bar{z} \in A (\bar{x} < \bar{z} < \bar{y}))$ . We say  $A$  is bounded if  $\forall \bar{x} \in A \exists \bar{y}, \bar{z} (\bar{x} < \bar{y} \wedge \bar{x} > \bar{z})$ .

(4) Consider  $\mathcal{U}$  a non-principal ultrafilter on  $\mathcal{N}$  and let  $\mathbb{R}^* = \mathbb{R}^\mathcal{U}$ . Consider

$$\varepsilon = [1, 1/2, 1/3, \dots, 1/n, \dots]_{\mathcal{U}}$$

Notice that for  $i \geq n$ ,  $\varepsilon_i < 1/n$ , and since  $\{n, n+1, \dots\} \in \mathcal{U}$  (it is cofinite), we conclude that for every  $n$ ,  $\varepsilon < 1/n$ . This proves that  $\mathbb{R}^*$  is not archimedean, and in particular this also proves that archimedeanity for a field is not axiomatizable, since if it were by, say, some theory  $T$ , we would have  $\mathbb{R} \models T$  and  $\mathbb{R}^* \not\models T$ , contradicting Los' theorem. We will show that connectedness and compactness are not 1st order expressible: consider  $E$  the set of *infinitesimal* elements in  $\mathbb{R}^*$ , i.e the set of elements smaller than every  $1/n$ .

$E$  is closed: Let  $\varepsilon \in \bar{E}$ , then for every  $x, y \in \mathbb{R}^*$  such that  $x < \varepsilon < y$  there is  $\epsilon \in E$  such that  $x < \epsilon < y$ . If  $\varepsilon \geq 1/n$  for some  $n$ , then we can find some infinitesimal  $1/n < \epsilon < 1$ , a contradiction.

$E$  is open: For any  $\varepsilon \in E$  we have  $\varepsilon/2 < \varepsilon < 2\varepsilon$ . We have to show these are infinitesimal: for  $\varepsilon/2$  is trivial since it is below an infinitesimal. Now if  $2\varepsilon \notin E$ , we can find  $n$  such that  $1/n < 2\varepsilon \Rightarrow 1/2n < \varepsilon$ , which is impossible.

$E$  has no supremum: Let  $r = \sup E$ . We have that,  $r \notin E$  (otherwise  $r < 2r \in E$ ), now let  $\epsilon \in E$ , and we claim  $r - \epsilon$  bounds  $E$ : suppose not, so there is  $\varepsilon \in E$  such that

$$r - \epsilon < \varepsilon < r \Rightarrow r < \varepsilon + \epsilon < r + \epsilon.$$

Notice also that  $\epsilon + \varepsilon \in E$  because, for any  $n \in \mathbb{N}$  since  $\epsilon, \varepsilon < 1/2n$ , then  $\epsilon + \varepsilon < 1/n$ . We have that  $r \leq$  some infinitesimal, a contradiction.  $E$  cannot have a supremum.

$E$  is not compact: the sequence

$$\varepsilon < 2\varepsilon < \dots < n\varepsilon < \dots$$

is contained in  $E$  (by the above argument), it is strictly increasing and is bounded above by 1. The above argument can be used to show it has no limit point. This proves  $E$  is not compact. If compactness of some set  $A$  was given by some sentence  $\phi_A$ , then we would have  $\mathbb{R} \models \phi_{[0,1]}$  but  $\mathbb{R}^* \not\models \phi_{[0,1]}$ , contradicting Łoś' theorem.

Finally, since  $E$  is clopen and it is neither  $\emptyset$  nor  $[0, 1]$ , we conclude that  $[0, 1]$  is not connected in  $\mathbb{R}^*$ , and we can infer that connectedness is also not 1st order expressible.

**Exercise 1.** Let  $\mathcal{L} = \emptyset$  and  $\mathcal{M}$  be an  $\mathcal{L}$ -structure. Show that  $A \subseteq M$  is definable in  $\mathcal{M}$  if and only if  $A$  is either finite or cofinite.

**Solution 1.**

**Lemma:** If  $A$  is  $S$ -definable, then every automorphism  $\sigma$  that fixes  $S$  pointwise fixes  $A$  pointwise.

**Proof:** Let  $\psi(x, \bar{s})$  be a formula defining  $A$ , since automorphisms preserve formulas, then we have

$$\mathcal{M} \models \psi(x, \bar{s}) \iff \mathcal{M} \models \psi(\sigma(x), \sigma(\bar{s})) \iff \mathcal{M} \models \psi(\sigma(x), s)$$

so that  $\sigma(X) = X$ .

If there was some definable  $A$  which is neither finite nor cofinite, then we can choose infinite sets

$$\{a_0, a_1, \dots, a_n, \dots\} \in A \setminus S$$

$$\{b_0, b_1, \dots, b_n, \dots\} \in (M \setminus A) \setminus S$$

Then the bijection which sends  $a_i$  to  $b_i$  and fixes everything else (in particular  $S$ ) is an automorphism that doesn't fix  $A$ , a contradiction. Conversely, if  $A = \{a_0, \dots, a_n\}$  is finite, the formula

$$\phi(x, \bar{a}) = \bigvee_{i=0}^n x = a_i$$

defines  $A$ . If  $A$  is cofinite repeat this argument for  $M \setminus A$ .

**Exercise 2.** Let  $\mathcal{M}$  be an  $\mathcal{L}$ -structure, let  $m, n \in \mathbb{N}$ . A collection  $\mathcal{A} = \{A_{\bar{b}}\}_{\bar{b} \in M^m}$  of subsets of  $M^n$  is a *definable family* if there exists  $S \subseteq M$  and a formula  $\phi \in \mathcal{F}_{n+m}(\mathcal{L}_S)$  such that  $A_{\bar{b}} = \{\bar{a} \in M^n, \mathcal{M}_S \models \phi(\bar{a}, \bar{b})\}$ . Let  $D \subseteq M$  be a finite set. Given a  $D$ -definable family  $\mathcal{A} = \{A_{\bar{b}}\}_{\bar{b} \in M^m}$ , let  $A = \bigcup \mathcal{A}$  and let  $f : A \rightarrow M$  a function.

- (1) Show that  $f$  is  $D$ -definable if and only if all restrictions  $f \upharpoonright A_{\bar{b}}$  are  $D$ -definable.

(2) What can we say if  $D$  is infinite?

**Solution 2.** Suppose there is  $\phi \in \mathcal{F}_{n+1}(\mathcal{L}_D)$  such that  $f(a_1, \dots, a_n) = y$  if and only if  $\mathcal{M}_D \models \phi(\bar{a}, y)$ . Then  $f \upharpoonright_{A_{\bar{b}}}(\bar{a}) = y$  if and only if  $\mathcal{M}_D \models \varphi(\bar{a}, \bar{b}) \wedge \phi(\bar{a}, y)$ , where  $\varphi$  is the formula which defines the family  $\{A_{\bar{b}}\}_{\bar{b} \in D^m}$ . Conversely if there is  $\phi_{\bar{b}}$  which defines each  $f \upharpoonright_{A_{\bar{b}}}$ , we have that  $f(\bar{a}) = y$  if and only if

$$\mathcal{M}_D \models \bigvee_{\bar{b} \in D^m} \varphi(\bar{a}, \bar{b}) \wedge \phi_{\bar{b}}(\bar{a}, y).$$

In case  $D$  is infinite, the first direction holds, but the converse may not, for instance take  $\mathcal{M} = \langle \mathbb{C}, +, -, \times, 0, 1 \rangle$  and  $D = \mathbb{R}$ . Take  $A_b = \{a \in \mathbb{R}, a = b\} = \{b\}$ , and  $f : A_b \rightarrow \mathbb{C}$  as the identity. Since  $A = \mathbb{R}$ , and the inclusion  $\mathbb{R} \subseteq \mathbb{C}$  is not definable, even though its restrictions are.

**Exercise 3.** Let  $\bar{\mathbb{R}} = \langle \mathbb{R}, 0, 1, -, +, \cdot, < \rangle$  be the real ordered field. Let  $f$  be a unary symbol and  $\mathcal{L} = \mathcal{L}_{OR} \cup \{f\}$ .

- (1) Show that the  $\mathcal{L}$ -structures  $\langle \bar{\mathbb{R}}, \sin\left(\frac{1}{1+x^2}\right) \rangle$  and  $\langle \bar{\mathbb{R}}, \arctan x \rangle$
- (2) Show that  $\bar{\mathbb{R}}$  is definable in the structure  $\langle \mathbb{R}, +, \exp(x) \rangle$ .
- (3) Let  $\mathbb{R}_{\exp} = \langle \bar{\mathbb{R}}, \exp \rangle$  be the real ordered exponential field. An *exponential polynomial* is a function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  such that there exists a polynomial  $P \in \mathbb{R}[X_1, \dots, X_n, Y_1, \dots, Y_m]$  such that  $F(x_1, \dots, x_n) = P(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n})$ . Show that every set  $A \in \mathbb{R}^m$  existentially definable in  $\mathbb{R}_{\exp}$  is a linear projection of the zero-set of some exponential polynomial  $F_A$ .

**Solution 3.**

- (1) We first define  $\sin(x) \upharpoonright_{[0,1]}$  from  $\sin\left(\frac{1}{1+x^2}\right)$ :

$$\text{graph } \sin(x) \upharpoonright_{[0,1]} = \left\{ (x, y), (x = 0 \wedge y = 0) \vee \exists z (x(1 + x^2) = 1 \wedge y = \sin\left(\frac{1}{1 + z^2}\right)) \right\}.$$

We can now define

$$x = \frac{\pi}{2} \iff 2 \sin^2 \upharpoonright_{[0,1]}(x/2) = 2$$

And define for  $0 < x < \pi/2$ ,

$$\tan(x) = \frac{2 \sin \upharpoonright_{[0,1]}(x/2) \sqrt{1 - \sin^2 \upharpoonright_{[0,1]}(x/2)}}{1 - 2 \sin^2 \upharpoonright_{[0,1]}(x/2)}.$$

And for  $-\pi/2 < x < 0$

$$\tan(x) = -\tan(-x).$$

Then finally set

$$y = \arctan(x) \iff \tan(x) = y.$$

For the other direction we can define  $\tan x$  from  $\arctan x$  and set

$$\sin \upharpoonright_{[0,1]} x = \frac{\tan x}{\sqrt{1 + \tan^2 x}}$$

define  $\sin(1/(1+x^2))$  as above.

$$(2) \quad (a) \quad x = 0 \iff x + x = x$$

$$(b) \quad 1 = e^0$$

$$(c) \quad y = -x \iff x + y = 0$$

$$(d) \quad x > 0 \iff \exists y e^y = x$$

$$(e) \quad xy = \exp(\log x + \log y)$$

$$(3) \quad \text{Let } \varphi(\bar{x}, \bar{c}) \text{ and existential formula in } \mathcal{L}_{\text{exp}} \text{ with parameters } \bar{c} \in \mathbb{R}. \text{ We can assume } \varphi \text{ has}$$

the form

$$\varphi(\bar{x}, \bar{c}) = \exists z_1, \dots, \exists z_n \bigvee_{i=1}^l \bigwedge_{j=1}^s \theta_{ij}(\bar{x}, \bar{z}, \bar{c})$$

where  $\theta_{ij}$  is atomic or  $\neg$ -atomic. We know that atomic formulas have the form  $t_1 = t_2, t_1 < t_2, t_1 = 0$  or  $t_1 < 0$  for  $t_1, t_2$  terms with parameters  $\bar{c}$ . We can replace in  $\theta_{ij}, t \neq 0$  for  $t < 0 \wedge -t < 0$  and  $t < 0$  for  $\exists y t y^2 + 1 = 0 \ t \neq 0$ , and  $t_1 = t_2$  for  $t_1 - t_2 = 0$ . In other words, we can assume  $\theta_{ij}$  to be of the form  $t = 0$ . We know show by induction on  $t(\bar{x}, \bar{c})$  that any term can be replaced by a conjunction of existential formulas containing only terms of the form  $F(\bar{x}, \bar{y}, \bar{c})$ , where  $F$  is an exponential polynomial and  $\bar{y}$  are new variables. In other words,  $t(\bar{x}, \bar{c}) = 0$  becomes a system of exponential polynomial equations on variables  $\bar{y}$ .

If  $t = c$  then  $c = 0$  is already of the form we want.

If  $t = t_1 + t_2$  then, since the sum of exponential polynomials is also an exponential polynomial, we can just add each of the rows of each system of equations to get one for  $t = 0$ .

The case  $t_1 t_2$  is similar.

If  $t(\bar{x}, \bar{c}) = e^{t_1(\bar{x}, \bar{c})}$ , then we can replace

$$t(\bar{x}, \bar{c}) = 0 \iff \exists w e^w = 0 \wedge w = t_1(\bar{x}, \bar{c})$$

and then we can apply induction on  $t_1$ , adding the variable  $w$  to our exponential polynomial. We can then suppose (renaming variables and reindexing) that  $\theta_{ij}$  has the form  $F_{ij}(\bar{x}, \bar{z}, \bar{c}) = 0$  for some exponential polynomial. So that

$$\varphi(\bar{x}, \bar{c}) = \exists z_1, \dots, \exists z_n \bigvee_{i=1}^l \bigwedge_{j=1}^s F_{ij}(\bar{x}, \bar{z}, \bar{c}) = 0.$$

It is clear that the set of zeros of

$$F = \sum_{i=1}^l \left( \prod_{j=1}^s F_{ij}(\bar{x}, \bar{c}) \right)^2$$

defines the same set as  $\varphi(\bar{x}, \bar{c})$ .