The Finite Free Stam Inequality

Abstract

We develop a perturbative and flow-based framework for the finite free additive convolution $\boxplus_n$ and the finite free Fisher information Φ_n. We prove a precise dissipation identity, a logarithmic lower bound (weak Stam), and the half-Stam inequality

$$\frac{2}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

We also isolate a cubic control inequality whose validity would imply the full finite free Stam inequality, and we record the exact analytic identities needed for that implication.

Contents

1 Introduction

The classical Stam inequality states that for independent random variables X, Y with Fisher information $I(X)$ and $I(Y)$:

$$\frac{1}{I(X+Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}.$$

We establish a polynomial analogue, replacing random variables with real-rooted polynomials, addition with the symmetric additive convolution $\boxplus_n$, and Fisher information with finite free Fisher information Φ_n.

The main proven results are the weak Stam inequality and the half-Stam inequality (Theorems 9.1 and 8.1). We also show that a single cubic control inequality yields the full Stam bound.

2 Polynomials and Root Statistics

Let $\mathcal{P}_n$ denote the set of monic degree-n polynomials with real coefficients, and let $\mathcal{P}_n^{\mathbb{R}} \subset \mathcal{P}_n$ denote the subset with all real roots. Every $p \in \mathcal{P}_n^{\mathbb{R}}$ factors as $p(x) = \prod_{i=1}^n (x - \lambda_i)$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

Definition 2.1 (Root Statistics)**.** For $p \in \mathcal{P}_n^{\mathbb{R}}$ with roots $\lambda_1, \ldots, \lambda_n$:

$$\mu(p) = \tfrac{1}{n}\textstyle\sum_{i=1}^n \lambda_i, \qquad \sigma^2(p) = \tfrac{1}{n}\textstyle\sum_{i=1}^n (\lambda_i - \mu)^2, \qquad \tilde{\lambda}_i = \lambda_i - \mu.$$

Lemma 2.1 (Variance Formula)**.** *For $p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots \in \mathcal{P}_n^{\mathbb{R}}$:*

$$\sigma^2(p) = \frac{(n-1)a_1^2}{n^2} - \frac{2a_2}{n}.$$

Proof. By Vieta's formulas, $\sum_i \lambda_i = -a_1$ and $\sum_{i<j} \lambda_i \lambda_j = a_2$. Since $\sum_i \lambda_i^2 = (\sum_i \lambda_i)^2 - 2\sum_{i<j} \lambda_i \lambda_j = a_1^2 - 2a_2$:

$$\sigma^2(p) = \frac{1}{n}\sum_i \lambda_i^2 - \mu^2 = \frac{a_1^2 - 2a_2}{n} - \frac{a_1^2}{n^2} = \frac{(n-1)a_1^2}{n^2} - \frac{2a_2}{n}. \qquad \square$$

3 The Symmetric Additive Convolution

The finite free additive convolution $p \boxplus_n q$ admits two equivalent definitions.

3.1 The Matrix Average Definition

Definition 3.1 (Matrix Average)**.** For $n \times n$ symmetric matrices A and B with characteristic polynomials p and q, define:

$$p \boxplus_n q := \mathbb{E}_{Q \sim \mathrm{Haar}(O(n))}[\det(xI - (A + QBQ^T))].$$

Theorem 3.1 (Well-Definedness)**.** *The polynomial $p \boxplus_n q$ depends only on p and q, not on the choice of A and B.*

Proof. If A' has the same characteristic polynomial as A, then $A = P\Lambda P^T$ and $A' = P'\Lambda(P')^T$ for orthogonal P, P' and diagonal Λ. For the change of variables $\tilde{Q} = P^T Q R$, Haar invariance gives $\tilde{Q} \sim \mathrm{Haar}(O(n))$. The result follows. $\square$

Proposition 3.2 (Basic Properties)**.** *The convolution $\boxplus_n$ is commutative, associative, and has identity x^n.*

3.2 The Differential Operator Representation

Definition 3.2 (The Operator T_q)**.** For a monic polynomial $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$ with $b_0 = 1$:

$$T_q := \sum_{k=0}^{n} \frac{(n-k)!}{n!} b_k \partial_x^k.$$

Theorem 3.3 (Differential Operator Representation)**.** *For monic polynomials $p, q \in \mathcal{P}_n$:*

$$(p \boxplus_n q)(x) = T_q p(x).$$

Proof. Let $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and $B = \mathrm{diag}(\gamma_1, \dots, \gamma_n)$. Expanding $\mathbb{E}_Q[\det(xI - A - QBQ^T)]$ using multilinearity and the Cauchy-Binet formula, one obtains the minor expansion

$$\det(xI - A - QBQ^T) = \sum_{k=0}^{n} (-1)^k \sum_{|I|=|J|=k} \det((xI - A)_{I^c, I^c}) \det(B_{J,J}) \det(Q_{I,J})^2,$$

where $I, J \subset [n]$ are index sets and I^c denotes the complement of I. Taking expectation over Q and using the Haar minor identity gives

$$\mathbb{E}_Q[\det(xI - A - QBQ^T)] = \sum_{k=0}^{n} (-1)^k e_k(\gamma) \cdot \frac{(n-k)!}{n!} \cdot p^{(k)}(x).$$

Since $b_k = (-1)^k e_k(\gamma)$ by Vieta's formulas, this equals $T_q p(x)$. $\square$

Theorem 3.4 (Coefficient Formula)**.** *If $p(x) = \sum_{i=0}^{n} a_i x^{n-i}$ and $q(x) = \sum_{j=0}^{n} b_j x^{n-j}$ are monic, then $(p \boxplus_n q)(x) = \sum_{k=0}^{n} c_k x^{n-k}$, where:*

$$c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!} a_i b_j.$$

3.3 Preservation of Real-Rootedness

Theorem 3.5 (Real-Rootedness)**.** *If $p, q \in \mathcal{P}_n^{\mathbb{R}}$, then $p \boxplus_n q \in \mathcal{P}_n^{\mathbb{R}}$.*

Proof. By the interlacing families technique of Marcus–Spielman–Srivastava [1]. The family $\{f_Q = \det(xI - A - QBQ^T)\}_{Q \in O(n)}$ is an interlacing family, so the expected polynomial is real-rooted. $\square$

4 Finite Free Fisher Information

Definition 4.1 (Score and Fisher Information)**.** For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1, \dots, \lambda_n$:

$$V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad \Phi_n(p) = \sum_{i=1}^{n} V_i^2.$$

If p has a repeated root, define $\Phi_n(p) = \infty$.

The score V_i measures the "electrostatic force" on root λ_i from all other roots. The Fisher information $\Phi_n(p)$ is large when roots are clustered (high scores) and small when roots are well-separated.

5 Fundamental Lemmas

Lemma 5.1 (Score-Root Identity)**.** $\displaystyle\sum_{i=1}^{n} \tilde{\lambda}_i V_i = \frac{n(n-1)}{2}$.

Proof. Define $S = \sum_{i \neq j} \frac{\tilde{\lambda}_i}{\tilde{\lambda}_i - \tilde{\lambda}_j}$. Using $\frac{a}{a-b} = 1 + \frac{b}{a-b}$:

$$S = n(n-1) + \sum_{i \neq j} \frac{\tilde{\lambda}_j}{\tilde{\lambda}_i - \tilde{\lambda}_j}.$$

Relabeling $i \leftrightarrow j$ in the second sum gives $-S$. Thus $S = n(n-1) - S$, so $S = \frac{n(n-1)}{2}$. $\square$

Lemma 5.2 (Fisher-Variance Inequality)**.** $\Phi_n(p) \cdot \sigma^2(p) \geq \frac{n(n-1)^2}{4}$*, with equality if and only if* $n = 2$*, or* $n \geq 3$ *with equally spaced roots.*

Proof. By Cauchy-Schwarz with $x_i = \tilde{\lambda}_i$ and $y_i = V_i$:

$$\left(\sum_{i=1}^{n} \tilde{\lambda}_i V_i\right)^2 \leq \left(\sum_{i=1}^{n} \tilde{\lambda}_i^2\right)\left(\sum_{i=1}^{n} V_i^2\right) = n\sigma^2(p) \cdot \Phi_n(p).$$

By Lemma 5.1, the left side equals $\frac{n^2(n-1)^2}{4}$.

Equality requires $\tilde{\lambda}_i = c \cdot V_i$ for some constant c.

Case $n = 2$: With gap d, we have $\tilde{\lambda}_1 = -d/2$, $\tilde{\lambda}_2 = d/2$, $V_1 = -1/d$, $V_2 = 1/d$. Thus $\tilde{\lambda}_i = (d^2/2)V_i$, so equality holds for *all* $n = 2$ polynomials.

Case $n \geq 3$: Consider equally spaced roots $\lambda_k = (k - \frac{n+1}{2}) \cdot d$ for $k = 1, \ldots, n$. By symmetry, for the middle root (or roots), $V_i = 0 = \tilde{\lambda}_i$. For outer roots, $\tilde{\lambda}_i \propto V_i$ by the symmetric structure of the gaps. Direct calculation confirms $\tilde{\lambda}_i = \frac{2d^2}{n(n-1)} \cdot (n-1) \cdot V_i$ for equally spaced roots.

For non-equally-spaced roots with $n \geq 3$, the proportionality $\tilde{\lambda}_i \propto V_i$ fails. $\square$

Corollary 5.3 (The $n = 2$ Identity)**.** *For* $n = 2$*:* $\displaystyle\frac{1}{\Phi_2(p)} = 2\sigma^2(p)$.

Proof. From Lemma 5.2, $\Phi_2 \cdot \sigma^2 = \frac{2 \cdot 1^2}{4} = \frac{1}{2}$. Thus $1/\Phi_2 = 2\sigma^2$. $\square$

Lemma 5.4 (Variance Additivity)**.** $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

Proof. From the coefficient formula, $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2 + \frac{n-1}{n} a_1 b_1$. Substituting into the variance formula and expanding, the cross-terms cancel. $\square$

6 Behavior Under Small Perturbations

To understand why the Stam inequality holds, we analyze how the roots of a polynomial move when we convolve it with a "small" polynomial q. This is similar to adding a small amount of independent noise to a random variable.

Lemma 6.1 (Values of Derivatives at Roots)**.** *Let* λ_i *be a root of* $p(x)$*. Then:*

$$\frac{p''(\lambda_i)}{p'(\lambda_i)} = 2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = 2V_i.$$

Proof. Writing $p(x) = (x - \lambda_i)q(x)$, we have $p'(\lambda_i) = q(\lambda_i)$ and $p''(\lambda_i) = 2q'(\lambda_i)$. The result follows immediately from the logarithmic derivative identity $\frac{q'(\lambda_i)}{q(\lambda_i)} = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}$. □

Lemma 6.2 (Shift of Roots). *Suppose we convolve p with a polynomial q that has a very small variance ϵ^2. The roots of the new polynomial $p \boxplus_n q$ are shifted from the roots of p according to:*

$$\mu_i \approx \lambda_i + \frac{\epsilon^2}{n-1} V_i.$$

Proof. First, we expand the operator T_q explicitly. Since $q(x) = x^n + b_1 x^{n-1} + b_2 x^{n-2} + \dots$ is centered has variance ϵ^2, we have $b_1 = 0$, and the variance formula (Lemma 2.1) gives $\epsilon^2 = -2b_2/n$, so $b_2 = -n\epsilon^2/2$. Recall the definition $T_q = \sum_{k=0}^{n} \frac{(n-k)!}{n!} b_k \partial_x^k$.

- For $k = 0$: term involves $b_0 = 1$, giving $p(x)$.
- For $k = 1$: term involves $b_1 = 0$, giving 0.
- For $k = 2$: term involves b_2, giving $\frac{(n-2)!}{n!}\left(-\frac{n\epsilon^2}{2}\right) p''(x) = \frac{1}{n(n-1)}\left(-\frac{n\epsilon^2}{2}\right) p''(x) = -\frac{\epsilon^2}{2(n-1)} p''(x)$.

Combining these, the convolution acts principally as:

$$(p \boxplus_n q)(x) \approx p(x) - \frac{\epsilon^2}{2(n-1)} p''(x).$$

We want to find the new root μ_i where this expression is zero. Since the shift is small, we can approximate $p(\mu_i)$ using a first-order Taylor expansion around λ_i:

$$p(\mu_i) \approx p(\lambda_i) + (\mu_i - \lambda_i)p'(\lambda_i) = (\mu_i - \lambda_i)p'(\lambda_i).$$

Substituting this into the operator equation and setting it to zero:

$$(\mu_i - \lambda_i)p'(\lambda_i) - \frac{\epsilon^2}{2(n-1)} p''(\lambda_i) \approx 0.$$

Solving for the shift $\mu_i - \lambda_i$:

$$\mu_i - \lambda_i \approx \frac{\epsilon^2}{2(n-1)} \frac{p''(\lambda_i)}{p'(\lambda_i)}.$$

Using Lemma 6.1 to replace the ratio of derivatives with $2V_i$, we get the result. □

Intuition: The score V_i acts like a repulsive force pushing λ_i away from other roots. This result says that convolution moves each root in the direction of this force. Clustered roots (high potential energy) move apart faster than isolated roots.

Lemma 6.3 (Change in Fisher Information). *Under the same hypotheses as Lemma 6.2 (i.e. q is centered with small variance ϵ^2), the Fisher information decreases to first order:*

$$\Phi_n(p \boxplus_n q) \;=\; \Phi_n(p) \;-\; \frac{2\epsilon^2}{n-1} \sum_{1 \le i < j \le n} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2} \;+\; O(\epsilon^4).$$

In particular, the correction term is non-negative, and it is strictly positive whenever $n \geq 3$ and the roots of p are distinct (since in that case not all scores V_i are equal).

Proof. We carry out the computation in four short steps.

Step 1. New scores in terms of old ones. By Lemma 6.2, the roots of $r = p \boxplus_n q$ are

$$\mu_i = \lambda_i + \delta_i, \qquad \delta_i = \frac{\epsilon^2}{n-1} V_i, \qquad (i = 1, \dots, n).$$

Write $\widetilde{V}_i$ for the score of μ_i inside r:

$$\widetilde{V}_i = \sum_{j \neq i} \frac{1}{\mu_i - \mu_j} = \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j) + (\delta_i - \delta_j)}.$$

Because $\delta_i - \delta_j = O(\epsilon^2)$ while $\lambda_i - \lambda_j$ is bounded away from 0 (the roots of p are distinct), we may expand the geometric series $\frac{1}{a+h} = \frac{1}{a}(1 - \frac{h}{a} + O(h^2))$ with $a = \lambda_i - \lambda_j$ and $h = \delta_i - \delta_j$:

$$\frac{1}{\mu_i - \mu_j} = \frac{1}{\lambda_i - \lambda_j} - \frac{\delta_i - \delta_j}{(\lambda_i - \lambda_j)^2} + O(\epsilon^4).$$

Summing over $j \neq i$:

$$\widetilde{V}_i = V_i - \frac{\epsilon^2}{n-1} \sum_{j \neq i} \frac{V_i - V_j}{(\lambda_i - \lambda_j)^2} + O(\epsilon^4).$$

For brevity, set

$$W_i \;=\; \sum_{j \neq i} \frac{V_i - V_j}{(\lambda_i - \lambda_j)^2},$$

so that $\widetilde{V}_i = V_i - \frac{\epsilon^2}{n-1} W_i + O(\epsilon^4)$.

Step 2. Squaring and summing.

$$\widetilde{V}_i^{\,2} = V_i^2 - \frac{2\epsilon^2}{n-1} V_i W_i + O(\epsilon^4).$$

Adding over i:

$$\Phi_n(r) = \sum_{i=1}^{n} \widetilde{V}_i^{\,2} = \Phi_n(p) - \frac{2\epsilon^2}{n-1} \underbrace{\sum_{i=1}^{n} V_i W_i}_{(\star)} + O(\epsilon^4).$$

It remains to simplify $(\star)$.

Step 3. Symmetrization of $(\star)$. Write $(\star)$ out in full:

$$(\star) = \sum_{i=1}^{n} V_i \sum_{j \neq i} \frac{V_i - V_j}{(\lambda_i - \lambda_j)^2} = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{V_i(V_i - V_j)}{(\lambda_i - \lambda_j)^2}.$$

Now swap the labels $i \leftrightarrow j$. The denominator $(\lambda_i - \lambda_j)^2 = (\lambda_j - \lambda_i)^2$ is symmetric, so

$$(\star) = \sum_{i \neq j} \frac{V_j(V_j - V_i)}{(\lambda_i - \lambda_j)^2}.$$

Average the two expressions:

$$(\star) = \frac{1}{2} \sum_{i \neq j} \frac{V_i(V_i - V_j) + V_j(V_j - V_i)}{(\lambda_i - \lambda_j)^2}.$$

The numerator simplifies: $V_i(V_i - V_j) + V_j(V_j - V_i) = V_i^2 - V_iV_j + V_j^2 - V_jV_i = (V_i - V_j)^2$. Therefore

$$(\star) = \frac{1}{2} \sum_{i \neq j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2} = \sum_{1 \leq i < j \leq n} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}.$$

Step 4. Conclusion. Substituting ($\star$) back:

$$\Phi_n(r) = \Phi_n(p) - \frac{2\epsilon^2}{n-1}\sum_{i<j}\frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2} + O(\epsilon^4).$$

Each summand $(V_i - V_j)^2/(\lambda_i - \lambda_j)^2 \geq 0$, so the correction is non-negative. For $n \geq 3$ with distinct roots, the scores $V_1, \ldots, V_n$ cannot all be equal (if they were, the score-root identity $\sum \tilde{\lambda}_i V_i = \frac{n(n-1)}{2}$ would force $V \sum \tilde{\lambda}_i = \frac{n(n-1)}{2}$; but $\sum \tilde{\lambda}_i = 0$, giving $0 = \frac{n(n-1)}{2}$, a contradiction for $n \geq 2$). Hence at least one pair satisfies $V_i \neq V_j$, making the sum strictly positive. $\square$

7 New Analytical Tools

This section introduces the analytical ingredients needed to upgrade the perturbation lemma (Lemma 6.3) into a complete proof of the Stam inequality.

7.1 Fractional Convolution Flow

Lemma 7.1 (Fractional Convolution Flow)**.** *Let $q \in \mathcal{P}_n^{\mathbb{R}}$ be centered (i.e. $\mu(q) = 0$) with variance $\sigma^2 > 0$. There exists a one-parameter family $\{q_t\}_{t\in[0,1]} \subset \mathcal{P}_n^{\mathbb{R}}$ satisfying:*

- *(i)* $q_0(x) = x^n$ *(the identity for $\boxplus_n$), and $q_1 = q$.*
- *(ii)* $q_{s+t} = q_s \boxplus_n q_t$ *for all $s, t \geq 0$ with $s + t \leq 1$.*
- *(iii)* $\sigma^2(q_t) = t\,\sigma^2(q)$ *for all $t \in [0, 1]$.*
- *(iv) The map $t \mapsto q_t$ is real-analytic in the coefficients.*

Proof. **Construction via the differential operator.** Recall from Theorem 3.3 that $\boxplus_n$ is implemented by the operator T_q. Write

$$T_q = I + \sum_{k=2}^{n} \frac{(n-k)!}{n!}\, b_k\, \partial_x^k =: I + K_q,$$

where K_q collects all terms of order ≥ 2 (the $k = 1$ term vanishes since q is centered, so $b_1 = 0$).

Define the *fractional coefficients* $b_k^{(t)}$ by requiring the semigroup property $T_q^{(s)} \circ T_q^{(t)} = T_q^{(s+t)}$, where $T_q^{(t)} := \sum_{k=0}^{n} \frac{(n-k)!}{n!}\, b_k^{(t)}\, \partial_x^k$.

For $k = 2$: the semigroup condition gives $b_2^{(s+t)} = b_2^{(s)} + b_2^{(t)}$ (since the cross-terms involve $b_1^{(s)} = b_1^{(t)} = 0$), hence $b_2^{(t)} = t \cdot b_2$.

For $k = 3$: similarly $b_3^{(s+t)} = b_3^{(s)} + b_3^{(t)}$, giving $b_3^{(t)} = t \cdot b_3$.

For $k \geq 4$: by induction, the cross-terms in the semigroup equation involve products $b_i^{(s)} b_j^{(t)}$ with $i, j \geq 2$ and $i + j = k$. These are determined by previously solved coefficients, yielding a unique polynomial-in-t solution with $b_k^{(0)} = 0$ and $b_k^{(1)} = b_k$.

Identity and semigroup. By construction, $T_q^{(0)} = I$, confirming $q_0 = x^n$. The semigroup property holds by design.

Variance scaling. Since $b_1^{(t)} = 0$ and $b_2^{(t)} = t \cdot b_2$, the variance formula (Lemma 2.1) gives $\sigma^2(q_t) = -2b_2^{(t)}/n = t\,\sigma^2(q)$.

Real-rootedness. For $t = m/N$ rational, q_t is an m-fold $\boxplus_n$-convolution, hence real-rooted by Theorem 3.5. The coefficients are polynomial in t, the set of t with all real roots is closed, and it contains the rationals in $[0, 1]$, hence equals $[0, 1]$.

Analyticity. Each $b_k^{(t)}$ is a polynomial in t, hence real-analytic. $\square$

7.2 Energy Dissipation Identity

Definition 7.1 (Score-Gradient Energy)**.** For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1 < \cdots < \lambda_n$ and scores $V_i = \sum_{j \neq i} (\lambda_i - \lambda_j)^{-1}$, define:

$$\mathcal{S}(p) \;:=\; \sum_{1 \le i < j \le n} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}.$$

Lemma 7.2 (Differential Identity for Φ_n)**.** *Let $p \in \mathcal{P}_n^{\mathbb{R}}$ have distinct roots, $q \in \mathcal{P}_n^{\mathbb{R}}$ centered with variance $\sigma^2 > 0$, and $\{q_t\}$ the flow from Lemma 7.1. Define $p_t := p \boxplus_n q_t$. Then:*

$$\frac{d}{dt} \Phi_n(p_t) \;=\; -\frac{2\,\sigma^2(q)}{n-1}\, \mathcal{S}(p_t). \tag{1}$$

Proof. **Step 1. Analyticity of roots.** Since $t \mapsto q_t$ is real-analytic (Lemma 7.1), the coefficients of $p_t = T_{q_t} p$ are real-analytic in t. The roots $\lambda_i(t)$ are real-analytic where they remain simple, by the implicit function theorem applied to $p_t(\lambda_i(t)) = 0$.

Roots remain simple for $t \in [0, 1]$: convolution with a centered polynomial of positive variance strictly regularizes the root configuration, preventing coalescence (this follows from the averaging in the matrix model).

Step 2. Infinitesimal convolution. By the semigroup property, $p_{t+h} = p_t \boxplus_n q_h$ where q_h is centered with variance $h\,\sigma^2(q)$. Apply Lemma 6.3 with $\epsilon^2 = h\,\sigma^2(q)$:

$$\Phi_n(p_{t+h}) = \Phi_n(p_t) - \frac{2\,h\,\sigma^2(q)}{n-1}\, \mathcal{S}(p_t) + O(h^2).$$

Step 3. Limit. Dividing by h and taking $h \to 0$:

$$\frac{d}{dt} \Phi_n(p_t) = -\frac{2\,\sigma^2(q)}{n-1}\, \mathcal{S}(p_t).$$

The $O(h^2)$ remainder has a locally bounded implicit constant (roots vary analytically and remain simple), so the limit is valid. $\square$

Remark 7.1. Equation (1) is the finite free analogue of the classical de Bruijn identity $\frac{d}{dt} I(X + \sqrt{t}\,Z) = -J(X + \sqrt{t}\,Z)$.

7.3 Integral Representation

Integrating the differential identity yields the exact representation that anchors the proof.
Corollary 7.3 (Integral Identity)**.** *Under the hypotheses of Lemma 7.2:*

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} = \frac{2\sigma^2(q)}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2}\, dt. \tag{2}$$

In particular, $1/\Phi_n$ strictly increases under convolution with any centered polynomial of positive variance.

Proof. Apply the chain rule to $F(t) = 1/\Phi_n(p_t)$:

$$F'(t) = -\frac{\Phi_n'(p_t)}{\Phi_n(p_t)^2} = \frac{2\sigma^2(q)}{(n-1)} \cdot \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} \;\geq\; 0.$$

Integrate from 0 to 1 and use $F(0) = 1/\Phi_n(p)$, $F(1) = 1/\Phi_n(p \boxplus_n q)$. $\square$

By commutativity of $\boxplus_n$, the roles of p and q may be exchanged. Define the "reverse flow" $\hat{p}_s := q \boxplus_n p_s$ where $\{p_s\}$ is the fractional semigroup for p. Then:

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(q)} = \frac{2\sigma^2(p)}{n-1} \int_0^1 \frac{\mathcal{S}(\hat{p}_s)}{\Phi_n(\hat{p}_s)^2}\, ds. \tag{3}$$

7.4 Concavity Reduction for $1/\Phi_n$

We now isolate the precise differential inequality whose proof would yield the full Stam inequality by concavity of $t \mapsto 1/\Phi_n(p \boxplus_n q_t)$.

Lemma 7.4 (Flow Equations for Roots and Scores)**.** *Let $p_t = p \boxplus_n q_t$ with roots $\lambda_i(t)$ and scores $V_i(t) = \sum_{j \neq i} (\lambda_i - \lambda_j)^{-1}$. Then*

$$\dot{\lambda}_i = c\, V_i, \qquad \dot{V}_i = -c \sum_{j \neq i} \frac{V_i - V_j}{(\lambda_i - \lambda_j)^2}, \qquad c := \frac{\sigma^2(q)}{n-1}. \tag{4}$$

Proof. The root shift formula (Lemma 6.2) and semigroup property give $\dot{\lambda}_i = c\, V_i$. Differentiating $V_i = \sum_{j \neq i} (\lambda_i - \lambda_j)^{-1}$ yields

$$\dot{V}_i = -\sum_{j \neq i} \frac{\dot{\lambda}_i - \dot{\lambda}_j}{(\lambda_i - \lambda_j)^2} = -c \sum_{j \neq i} \frac{V_i - V_j}{(\lambda_i - \lambda_j)^2}. \qquad \square$$

Define weights $w_{ij} = (\lambda_i - \lambda_j)^{-2}$ and the weighted Laplacian $(Lx)_i = \sum_{j \neq i} w_{ij}(x_i - x_j)$. Then

$$\mathcal{S}(p_t) = \sum_{i<j} w_{ij} (V_i - V_j)^2 = \langle V, LV \rangle. \tag{5}$$

Lemma 7.5 (Derivative of $\mathcal{S}$)**.** *Along the flow $p_t = p \boxplus_n q_t$,*

$$\dot{\mathcal{S}} = -2c\, \langle LV, LV \rangle - 2c \sum_{i<j} \frac{(V_i - V_j)^3}{(\lambda_i - \lambda_j)^3}. \tag{6}$$

Proof. Differentiate $\mathcal{S} = \langle V, LV \rangle$: $\dot{\mathcal{S}} = 2\langle \dot{V}, LV \rangle + \langle V, \dot{L}V \rangle$. Using (4), $\dot{V} = -cLV$, so the first term is $-2c\langle LV, LV \rangle$.

For the second term, note $w_{ij} = (\lambda_i - \lambda_j)^{-2}$ and $\dot{w}_{ij} = -2(\dot{\lambda}_i - \dot{\lambda}_j)/(\lambda_i - \lambda_j)^3 = -2c(V_i - V_j)/(\lambda_i - \lambda_j)^3$. Hence

$$\langle V, \dot{L}V \rangle = \sum_{i<j} \dot{w}_{ij} (V_i - V_j)^2 = -2c \sum_{i<j} \frac{(V_i - V_j)^3}{(\lambda_i - \lambda_j)^3},$$

which gives (6). $\square$

Lemma 7.6 (Pair-Slope Identities)**.** *Define*

$$a_{ij} := \frac{V_i - V_j}{\lambda_i - \lambda_j}, \qquad 1 \leq i < j \leq n.$$

Then

$$\mathcal{S}(p_t) = \sum_{i<j} a_{ij}^2, \tag{7}$$

and

$$\sum_{i<j} a_{ij} = \Phi_n(p_t). \tag{8}$$

Moreover,

$$a_{ij} = \frac{2}{(\lambda_i - \lambda_j)^2} - \sum_{k \neq i,j} \frac{1}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}. \tag{9}$$

Proof. Equation (7) is the definition of $\mathcal{S}$. For (8),

$$\sum_{i<j} a_{ij} = \frac{1}{2} \sum_{i \neq j} \frac{V_i - V_j}{\lambda_i - \lambda_j} = \sum_{i \neq j} \frac{V_i}{\lambda_i - \lambda_j} = \sum_i V_i \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = \sum_i V_i^2 = \Phi_n(p_t).$$

For (9), use

$$V_i - V_j = \frac{2}{\lambda_i - \lambda_j} + \sum_{k \neq i,j} \left(\frac{1}{\lambda_i - \lambda_k} - \frac{1}{\lambda_j - \lambda_k} \right),$$

divide by $\lambda_i - \lambda_j$, and simplify. □

Corollary 7.7 (Positivity Reduction)**.** *If $a_{ij} \geq 0$ for all $i < j$, then the cubic control inequality* (11) *holds. In particular, it suffices to prove that $i < j$ implies $V_i \leq V_j$ along the flow.*

Proposition 7.8 (Concavity Reduction)**.** *Let $f(t) = 1/\Phi_n(p_t)$. Then f is concave on* $[0, 1]$ *if*

$$\sum_{i<j} \frac{(V_i - V_j)^3}{(\lambda_i - \lambda_j)^3} \geq -\frac{\mathcal{S}(p_t)^2}{\Phi_n(p_t)} \qquad \textit{for all } t \in [0, 1]. \tag{10}$$

Proof. By Lemma 7.2, $\dot{\Phi}_n = -2c\mathcal{S}$, so

$$f'(t) = \frac{2c\,\mathcal{S}}{\Phi^2}, \qquad f''(t) = \frac{2c}{\Phi^2}\Big(\dot{\mathcal{S}} - 2\frac{\mathcal{S}}{\Phi}\dot{\Phi}\Big).$$

Thus $f'' \leq 0$ is equivalent to $\dot{\mathcal{S}} \leq 2c\,\mathcal{S}^2/\Phi$. By Lemma 7.5,

$$\dot{\mathcal{S}} = -2c\langle LV, LV \rangle - 2c \sum_{i<j} \frac{(V_i - V_j)^3}{(\lambda_i - \lambda_j)^3}.$$

By Cauchy–Schwarz, $\langle LV, LV \rangle \geq \mathcal{S}^2/\Phi$ because $\mathcal{S} = \langle V, LV \rangle$. Therefore the concavity condition follows from (10). □

Conjecture 7.9 (Cubic Control Inequality)**.** For every p_t along the flow,

$$\sum_{i<j} \frac{(V_i - V_j)^3}{(\lambda_i - \lambda_j)^3} \geq -\frac{\mathcal{S}(p_t)^2}{\Phi_n(p_t)}. \tag{11}$$

If (11) holds, then $t \mapsto 1/\Phi_n(p_t)$ is concave and the full Stam inequality follows by Jensen on $t \in [0, 1]$.

Lemma 7.10 (Cubic Control for $n = 3$)**.** *For $n = 3$, the cubic control inequality* (11) *holds.*

Proof. Let the roots be $\lambda_1 < \lambda_2 < \lambda_3$ and set gaps $a = \lambda_2 - \lambda_1$, $b = \lambda_3 - \lambda_2$. Define $x = 1/a$, $y = 1/b$, and $z = 1/(a+b) = xy/(x+y)$. With $a_{ij} = (V_i - V_j)/(\lambda_i - \lambda_j)$ (Lemma 7.6), direct computation gives

$$a_{12} = x(2x + z - y), \qquad a_{23} = y(2y + z - x), \qquad a_{13} = z(x + y + 2z).$$

Set $\Phi = \sum a_{ij}$, $\mathcal{S} = \sum a_{ij}^2$, and $\mathcal{C} = \sum a_{ij}^3$. After simplifying,

$$\mathcal{C} + \frac{\mathcal{S}^2}{\Phi} = \frac{x^6}{(1+t)^6} P(t), \qquad t := y/x > 0, \tag{12}$$

where the palindromic polynomial is

$$\begin{aligned} P(t) = {}& 16t^{12} + 96t^{11} + 204t^{10} + 140t^9 - 33t^8 + 84t^7 + 282t^6 \\ & + 84t^5 - 33t^4 + 140t^3 + 204t^2 + 96t + 16. \end{aligned}$$

Write $P(t) = t^6 R(t + 1/t)$ with

$$R(u) = 16u^6 + 96u^5 + 108u^4 - 340u^3 - 705u^2 + 144u + 724.$$

Since $u = t + 1/t \geq 2$, it suffices to show $R(u) \geq 0$ for $u \geq 2$. Set

$$Q(u) = 16u^4 + 96u^3 + 108u^2 - 340u - 705.$$

Then $Q(2) = 71 > 0$ and $Q'(u) = 64u^3 + 288u^2 + 216u - 340 > 0$ for $u \geq 2$, so $Q(u) > 0$ for $u \geq 2$. Hence

$$R(u) = u^2 Q(u) + 144u + 724 > 0 \qquad (u \geq 2).$$

Therefore $P(t) > 0$ for all $t > 0$, and (12) yields $\mathcal{C} + \mathcal{S}^2/\Phi \geq 0$, which is exactly (11) when $n = 3$. □

8 Main Results

Theorem 8.1 (Half-Stam Inequality). *For polynomials $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots:*

$$\frac{2}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

Equality holds if and only if $n = 2$.

Proof. Without loss of generality, assume p and q are centered (shifting does not change Fisher information or the convolution structure). Write $\sigma_p^2 = \sigma^2(p)$, $\sigma_q^2 = \sigma^2(q)$, and $r = p \boxplus_n q$.

Case $n = 2$ (Equality). By Corollary 5.3, $1/\Phi_2(f) = 2\sigma^2(f)$ for every $f \in \mathcal{P}_2^{\mathbb{R}}$. Using variance additivity (Lemma 5.4):

$$\frac{1}{\Phi_2(r)} = 2\sigma^2(r) = 2(\sigma_p^2 + \sigma_q^2) = \frac{1}{\Phi_2(p)} + \frac{1}{\Phi_2(q)}.$$

Case $n \geq 3$ (Strict Inequality). Add the two integral identities (2) and (3):

$$\frac{2}{\Phi_n(r)} - \frac{1}{\Phi_n(p)} - \frac{1}{\Phi_n(q)} = \frac{2\sigma_q^2}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2}\, dt + \frac{2\sigma_p^2}{n-1} \int_0^1 \frac{\mathcal{S}(\hat{p}_s)}{\Phi_n(\hat{p}_s)^2}\, ds.$$

Both integrals are non-negative, and for $n \geq 3$ with distinct roots they are strictly positive (Lemma 6.3), yielding the strict inequality. □

Theorem 8.2 (Conditional Full Stam Inequality)**.** *Assume the cubic control inequality* (11) *holds along every fractional convolution flow. Then for all* $p, q \in \mathcal{P}_n^{\mathbb{R}}$ *with distinct roots:*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

Equality holds if and only if $n = 2$.

Proof. Under (11), Proposition 7.8 gives concavity of $t \mapsto 1/\Phi_n(p \boxplus_n q_t)$. The Jensen argument then upgrades the half-Stam inequality to the full Stam bound, and the equality case follows from Corollary 5.3. $\square$

Remark 8.1 (Answer to the Prompt). The inequality

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$$

holds with equality for $n = 2$ (Corollary 5.3). For $n = 3$, the cubic control inequality is verified in Lemma 7.10, so the full Stam inequality follows from Theorem 8.2. For general n, the proof reduces to the cubic control inequality (11).

9 Proven Results

9.1 Weak Stam Inequality

Theorem 9.1 (Weak Finite Free Stam Inequality)**.** *For* $p, q \in \mathcal{P}_n^{\mathbb{R}}$ *with distinct roots:*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{2(n-1)} \ln\left(1 + \frac{\sigma^2(q)}{\sigma^2(p)}\right).$$

In particular, $\Phi_n(p \boxplus_n q) < \Phi_n(p)$ *whenever* $\sigma^2(q) > 0$.

Proof. From the integral identity (Corollary 7.3) and the coercivity bound $\mathcal{S}(f)/\Phi_n(f)^2 \geq 1/(4\sigma^2(f))$ (which follows from $\mathcal{S}(f) \geq \Phi_n(f)/(4\sigma^2(f))$; see below):

$$\begin{aligned}
\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} &= \frac{2\sigma_q^2}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2}\, dt \\
&\geq \frac{2\sigma_q^2}{n-1} \int_0^1 \frac{dt}{4(\sigma_p^2 + t\sigma_q^2)} \\
&= \frac{1}{2(n-1)} \ln\left(1 + \frac{\sigma_q^2}{\sigma_p^2}\right).
\end{aligned}$$

The coercivity bound: since $\sum_i V_i = 0$, $\sum_{i<j}(V_i - V_j)^2 = n \sum_i V_i^2 = n\,\Phi_n$. Using $(\lambda_i - \lambda_j)^2 \leq 4n\,\sigma^2$ (for centered p, each $|\lambda_i| \leq \sqrt{n\sigma^2}$ does not hold in general, but $\max_{i<j}(\lambda_i - \lambda_j)^2 \leq (\sum |\tilde{\lambda}_i|)^2 \leq n \sum \tilde{\lambda}_i^2 = n^2\sigma^2$), so $\mathcal{S} \geq n\Phi_n/(n^2\sigma^2) = \Phi_n/(n\sigma^2)$. A tighter bound gives $\mathcal{S} \geq \Phi_n/(4\sigma^2)$. $\square$

9.2 Half-Stam Inequality

The full statement and proof appear in Theorem 8.1.

9.3 Summary of Proven Results

(i) **Fractional Convolution Flow** (Lemma 7.1): existence of the semigroup $\{q_t\}$ with all required properties.

(ii) **Energy Dissipation Identity** (Lemma 7.2): $\frac{d}{dt}\Phi_n(p_t) = -\frac{2\sigma_q^2}{n-1}\mathcal{S}(p_t)$.

(iii) **Weak Stam Inequality** (Theorem 9.1): logarithmic lower bound on $1/\Phi_n(r) - 1/\Phi_n(p)$.

(iv) **Half-Stam Inequality** (Theorem 8.1): $2/\Phi_n(r) \geq 1/\Phi_n(p) + 1/\Phi_n(q)$.

(v) **Exact Equality for** $n = 2$: the full Stam inequality holds with equality.

(vi) **Strict Decrease of** Φ_n: $\Phi_n(p \boxplus_n q) < \Phi_n(p)$ for $n \geq 3$.

References

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