TOWARDS A PROOF OF THE FINITE FREE STAM INEQUALITY: CORE IDENTITIES, VERIFIED CASES, AND REDUCTION PRINCIPLES

ABSTRACT. Let $\mathcal{P}_n^{\mathbb{R}}$ denote the set of monic, degree-n, real-rooted polynomials and let $\boxplus_n$ be the Marcus–Spielman–Srivastava finite free additive convolution. For $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots $\lambda_1 < \cdots < \lambda_n$, the *finite free Fisher information* is $\Phi_n(r) := \sum_{i=1}^n V_i^2$, where $V_i := \sum_{j \neq i}(\lambda_i - \lambda_j)^{-1}$. The *finite free Stam inequality* asserts

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}, \qquad p, q \in \mathcal{P}_n^{\mathbb{R}}.$$

We prove this inequality for all $n \leq 3$, giving two independent proofs at $n = 3$ (a sum-of-squares identity and a Cauchy–Schwarz mixing argument). We derive equivalent defect-based reformulations, establish a Cauchy–Schwarz mixing mechanism that yields a manifestly non-negative quadratic lower bound on the Stam defect, and present a degree-telescoping framework that reduces the full conjecture to controlling explicit correction terms $C_k = D_k - D_{k-1}$ for $k \geq 4$. The Gaussian-input Stam inequality at all n is proved conditionally on a root ODE. The general conjecture remains open for $n \geq 4$.

Proof-status conventions. [**Proved**] fully rigorous; [**Conditional**] depends on stated hypotheses; [**Computer-verified**] numerically verified; [**Proof sketch**] outline only.

1. Introduction

The classical Stam inequality [7] states that for independent continuous random variables X, Y with finite Fisher informations $J(X), J(Y)$:

$$\frac{1}{J(X+Y)} \geq \frac{1}{J(X)} + \frac{1}{J(Y)}. \tag{1}$$

This is a cornerstone of information theory, closely related to the entropy power inequality [2] and the Cramér–Rao bound (see [3] for a survey).

Marcus, Spielman, and Srivastava [6, 5] introduced the *finite free additive convolution* $\boxplus_n$ on monic real-rooted polynomials of degree n, a finite-dimensional analogue of free additive convolution in the sense of [8]. A natural question is whether the Stam inequality (1) has a polynomial analogue. Define the *finite free Fisher information* of $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots $\lambda_1 < \cdots < \lambda_n$ by $\Phi_n(r) := \sum_{i=1}^n (\sum_{j \neq i}(\lambda_i - \lambda_j)^{-1})^2$. The *finite free Stam inequality* is the conjecture that for all $p, q \in \mathcal{P}_n^{\mathbb{R}}$:

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}. \tag{2}$$

Preserving real-rootedness under $\boxplus_n$ is guaranteed by [6]; see [1] for the connection to linear operators preserving stability.

Contributions.

- (i) Structural identities (Section 3): $\Phi_n = 2\mathcal{R} = \mathrm{tr}(L)$, score identities, variance additivity, Bezoutian and Laplacian formulations.
- (ii) Full proofs for $n = 2$ (equality) and $n = 3$ (two independent proofs); Gaussian-input Stam at all n, conditional on a root ODE (Section 4).
- (iii) Equivalent reformulations: Stam is equivalent to sub-averaging of a spectral efficiency defect R_n (Section 5).
- (iv) A Cauchy–Schwarz mixing inequality yielding a manifestly non-negative quadratic lower bound on the Stam defect (Section 6).
- (v) A degree-induction framework via the Cauchy interlacing matrix: K-cumulant preservation, Score–Cauchy identity, Frobenius norm identity, and deficit telescoping (Section 7).
- (vi) Discussion of remaining obstructions and open problems (Section 8).

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2. Preliminaries

Definition 2.1 (MSS convolution [6])**.** For $p(x) = \sum_{k=0}^{n} a_k x^{n-k}$ and $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$ with $a_0 = b_0 = 1$, the *finite free additive convolution* $r = p \boxplus_n q$ is defined by $r(x) = \sum_{k=0}^{n} c_k x^{n-k}$ with

$$c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i b_j.$$

By [6], $\boxplus_n$ preserves $\mathcal{P}_n^{\mathbb{R}}$.

Definition 2.2 (K-transform and log-cumulants)**.** Define $\kappa_k(r) := (n-k)!\,c_k(r)/n!$ and $K_r(z) := \sum_{k=0}^{n} \kappa_k(r)\, z^k$. Then $K_{p\boxplus_n q}(z) = K_p(z)\cdot K_q(z) \pmod{z^{n+1}}$. The *log-cumulants* $\ell_k(r) := [z^k] \log K_r(z)$ are computed by $\ell_1 = \kappa_1,\ \ \ell_k = \kappa_k - \frac{1}{k}\sum_{j=1}^{k-1} j\,\ell_j \kappa_{k-j}$ for $k \ge 2$. They are **additive**: $\ell_k(p \boxplus_n q) = \ell_k(p) + \ell_k(q)$ for all k.

Definition 2.3 (Scores and Fisher information)**.** For $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots $\lambda_1 < \cdots < \lambda_n$, let $V_i(r) := \sum_{j\ne i} (\lambda_i - \lambda_j)^{-1}$ (the *score vector* $V = (V_1, \ldots, V_n)$), $\Phi_n(r) := \sum_i V_i^2$ (the *Fisher information*), $\mathcal{R}(r) := \sum_{i<j} (\lambda_i - \lambda_j)^{-2}$ (the *repulsion energy*), and $\mathcal{S}(r) := \sum_{i<j} (V_i - V_j)^2/(\lambda_i - \lambda_j)^2$ (the *score-gradient energy*). If r has a repeated root, set $\Phi_n(r) = \infty$.

Definition 2.4 (Graph Laplacian)**.** The graph Laplacian of r is $L \in \mathbb{R}^{n\times n}$ with $L_{ij} = -(\lambda_i - \lambda_j)^{-2}$ for $i \ne j$ and $L_{ii} = \sum_{k\ne i} (\lambda_i - \lambda_k)^{-2}$. We have $L\mathbf{1} = 0$, $L \succeq 0$, $\operatorname{rank} L = n-1$. Equivalently, $L = -\frac{1}{2}\mathrm{Hess}_\lambda(\log \mathrm{disc}(r))$.

Definition 2.5 (Variance and Gaussian polynomials)**.** For $r \in \mathcal{P}_n^{\mathbb{R}}$: $\mu(r) := n^{-1}\sum_i \lambda_i$, $\sigma^2(r) := n^{-1}\sum_i (\lambda_i - \mu)^2$. Both are additive under $\boxplus_n$. The *additive variance parameter* $u := \sigma^2/(2(n-1))$ satisfies $u(p \boxplus_n q) = u(p) + u(q)$. The *finite Gaussian* $g_t \in \mathcal{P}_n^{\mathbb{R}}$ has $\sigma^2(g_t) = t$ and $\ell_k(g_t) = 0$ for $k \ge 3$. The Hermite semigroup satisfies $g_s \boxplus_n g_t = g_{s+t}$.

Definition 2.6 (Normalised cumulant ratios)**.** For centred $r \in \mathcal{P}_n^{\mathbb{R}}$ with $u := -\ell_2(r) > 0$, define $\tau_k(r) := \ell_k(r)/u(r)^{k/2}$ for $k \ge 3$.

Lemma 2.7 (Normalisation identities)**.** *For centred $r \in \mathcal{P}_n^{\mathbb{R}}$ (i.e., $\mu(r) = 0$), the parameters κ_2, ℓ_2, u, and σ^2 are related by:*

$$\ell_2 \;=\; \kappa_2 \;=\; \frac{(n-2)!\,a_2}{n!} \;=\; \frac{a_2}{n(n-1)}, \qquad u \;:=\; -\ell_2 \;>\; 0, \qquad \sigma^2 \;=\; 2(n-1)\,u. \tag{3}$$

Here a_2 is the coefficient of x^{n-2} in r (so $a_2 < 0$ for centred real-rooted r with $n \ge 2$).

Proof. From Definition 2.2: $\kappa_2 = (n-2)!\,a_2/n!$. The log-cumulant recurrence (Definition 2.2) gives $\ell_2 = \kappa_2 - \frac{1}{2}\kappa_1^2 = \kappa_2$ when r is centred ($\kappa_1 = \ell_1 = 0$). From the variance formula with $a_1 = 0$: $\sigma^2 = -2a_2/n = -2n(n-1)\ell_2/n = 2(n-1)(-\ell_2) = 2(n-1)u$. All three parameters are additive under $\boxplus_n$ because ℓ_2 is additive (Definition 2.2). □

3. Structural identities

We collect the main identities connecting Φ_n to spectral quantities. Throughout this section, $r \in \mathcal{P}_n^{\mathbb{R}}$ has simple roots.

Theorem 3.1 (Fisher–repulsion identity)**.** $\Phi_n(r) = 2\,\mathcal{R}(r)$.

Proof. Expand $\Phi_n = \sum_i V_i^2 = \sum_i \sum_{j\ne i}\sum_{k\ne i} (\lambda_i - \lambda_j)^{-1}(\lambda_i - \lambda_k)^{-1}$. The diagonal terms ($j = k$) sum to $2\sum_{i<j}(\lambda_i - \lambda_j)^{-2} = 2\mathcal{R}$. The cross-terms ($j \ne k$, both $\ne i$) group into triples $\{a, b, c\}$, each contributing $(a-b)^{-1}(a-c)^{-1} + (b-a)^{-1}(b-c)^{-1} + (c-a)^{-1}(c-b)^{-1} = 0$ by the partial-fraction identity. □

Theorem 3.2 (Fisher–Laplacian identities)**.** (a) $\Phi_n = \mathrm{tr}(L)$.
(b) $V = L\lambda$ (Euler identity).
(c) $\lambda^T L \lambda = \binom{n}{2}$.
(d) $\Phi_n = \|L\lambda\|^2 = \lambda^T L^2 \lambda$.

Proof. (a) $\mathrm{tr}(L) = \sum_i \sum_{k\ne i}(\lambda_i - \lambda_k)^{-2} = 2\sum_{i<j}(\lambda_i - \lambda_j)^{-2} = 2\mathcal{R} = \Phi_n$.
(b) $(L\lambda)_i = \sum_{j\ne i}(\lambda_i - \lambda_j)/(\lambda_i - \lambda_j)^2 = \sum_{j\ne i}(\lambda_i - \lambda_j)^{-1} = V_i$.
(c) $\lambda^T L\lambda = V\cdot\lambda = \sum_i \lambda_i V_i = \binom{n}{2}$ by the Euler identity for disc (degree $n(n-1)$ homogeneous).
(d) Immediate from $V = L\lambda$. □

Lemma 3.3 (Score identities)**.** (i) $\sum_i V_i = 0$.
(ii) $\sum_i (\lambda_i - \mu)V_i = \binom{n}{2}$.
(iii) $\Phi_n = \sum_{i<j}(V_i - V_j)/(\lambda_i - \lambda_j)$.
(iv) $V_i = r''(\lambda_i)/(2r'(\lambda_i))$.

Proof. (i) $\sum_i V_i = \sum_{i\neq j}(\lambda_i-\lambda_j)^{-1} = 0$ (antisymmetric).

(ii) $\sum_i \lambda_i V_i = \sum_{i\neq j} \lambda_i/(\lambda_i-\lambda_j) = \sum_{i\neq j}\big[1+\lambda_j/(\lambda_i-\lambda_j)\big] = n(n-1)+\sum_{i\neq j}\lambda_j/(\lambda_i-\lambda_j)$. Using $\sum_{i\neq j}\lambda_j/(\lambda_i-\lambda_j) = -\sum_{i\neq j}\lambda_i/(\lambda_j-\lambda_i) = -\sum_i \lambda_i V_i$, we get $2\sum_i \lambda_i V_i = n(n-1)$, so $\sum_i \lambda_i V_i = \binom{n}{2}$. By (i), subtracting $\mu\sum V_i = 0$ gives (ii).

(iii) Expand the right-hand side:

$$\begin{aligned}\sum_{i<j}\frac{V_i-V_j}{\lambda_i-\lambda_j} &= \sum_{i<j}\frac{1}{\lambda_i-\lambda_j}\Big(\sum_{k\neq i}\frac{1}{\lambda_i-\lambda_k}-\sum_{k\neq j}\frac{1}{\lambda_j-\lambda_k}\Big)\\ &= \sum_{i<j}\sum_{k\neq i}\frac{1}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}-\sum_{i<j}\sum_{k\neq j}\frac{1}{(\lambda_i-\lambda_j)(\lambda_j-\lambda_k)}.\end{aligned}$$

Relabelling $i\leftrightarrow j$ in the second sum and combining yields $2\sum_{i<j}\sum_{k\neq i}1/((\lambda_i-\lambda_j)(\lambda_i-\lambda_k))$. Separating diagonal ($k=j$) from cross ($k\neq i,j$) terms: the diagonal gives $\sum_i\sum_{j\neq i}(\lambda_i-\lambda_j)^{-2} = \Phi_n$; the cross-terms group into triples $\{i,j,k\}$, each contributing $\sum_{\mathrm{cyc}} 1/((a-b)(a-c)) = 0$ by the same partial-fraction identity as in Theorem 3.1.

(iv) Since $r'(\lambda_i) = \prod_{j\neq i}(\lambda_i-\lambda_j)$, we have $V_i = r''(\lambda_i)/(2r'(\lambda_i))$. □

Theorem 3.4 (Fisher–variance inequality)**.** $\Phi_n(r)\cdot\sigma^2(r)\geq n(n-1)^2/4$.

Proof. Cauchy–Schwarz on $\sum_i(\lambda_i-\mu)V_i = \binom{n}{2}$ with $\sum V_i = 0$: $\left|\sum(\lambda_i-\mu)V_i\right|^2 \leq \left(\sum(\lambda_i-\mu)^2\right)\left(\sum V_i^2\right) = n\sigma^2\cdot\Phi_n$. Hence $n\sigma^2\cdot\Phi_n\geq\binom{n}{2}^2 = n^2(n-1)^2/4$. □

Theorem 3.5 (Score-gradient inequality)**.** $\mathcal{S}(r)\cdot\sigma^2(r)\geq(n-1)\Phi_n(r)/2$.

Proof. Write $\lambda_c := \lambda-\mu\mathbf{1}$ for the centred root vector. Since $L\mathbf{1}=0$, $V = L\lambda = L\lambda_c$. The Cauchy–Schwarz inequality for the positive semi-definite form $\langle u,v\rangle_L := u^TLv$ gives $(\lambda_c^TL^2\lambda_c)^2\leq(\lambda_c^TL\lambda_c)(\lambda_c^TL^3\lambda_c)$, i.e., $\Phi_n^2\leq\binom{n}{2}\cdot\mathcal{S}$. Combining with the Fisher–variance inequality (Theorem 3.4):

$$\mathcal{S}\,\sigma^2 \;\geq\; \frac{\Phi_n^2}{\binom{n}{2}}\,\sigma^2 \;=\; \frac{\Phi_n\,\sigma^2}{\binom{n}{2}}\cdot\Phi_n \;\geq\; \frac{n(n-1)^2/4}{n(n-1)/2}\cdot\Phi_n \;=\; \frac{(n-1)\,\Phi_n}{2}. \qquad \square$$

Theorem 3.6 (Bezoutian representation)**.** $\Phi_n(r) = \sum_{i=1}^n r''(\lambda_i)^2/(4\,r'(\lambda_i)^2) = \|r''/2\|^2_{\mathrm{Bez}(r,r')}$.

Proof. The Bezoutian matrix $\mathrm{Bez}(r,r')$ is the unique symmetric $B\in\mathbb{R}^{n\times n}$ satisfying $\sum_{i,j}B_{ij}x^{n-1-i}y^{n-1-j} = \big(r(x)r'(y)-r'(x)r(y)\big)/(x-y)$. The associated inner product is diagonal in the Lagrange basis $\{L_i(x) = \prod_{j\neq i}(x-\lambda_j)/\prod_{j\neq i}(\lambda_i-\lambda_j)\}$: $\langle f,g\rangle_{\mathrm{Bez}} = \sum_i f(\lambda_i)g(\lambda_i)/r'(\lambda_i)^2$ (see [4] for the diagonalisation). Since $V_i = r''(\lambda_i)/(2r'(\lambda_i))$ (Lemma 3.3(iv)), we get $\Phi_n = \sum V_i^2 = \|r''/2\|^2_{\mathrm{Bez}}$. □

Lemma 3.7 (Variance additivity)**.** $\sigma^2(p\boxplus_n q) = \sigma^2(p)+\sigma^2(q)$.

Proof. From the MSS coefficient formula (Definition 2.1): $c_1 = a_1+b_1$, $c_2 = a_2+b_2+\frac{n-1}{n}a_1b_1$. Using $\sigma^2 = \frac{(n-1)a_1^2-2na_2}{n^2}$:

$$\begin{aligned}\sigma^2(p\boxplus_n q) &= \frac{(n-1)(a_1+b_1)^2-2n(a_2+b_2+\frac{n-1}{n}a_1b_1)}{n^2}\\ &= \frac{(n-1)a_1^2-2na_2}{n^2}+\frac{(n-1)b_1^2-2nb_2}{n^2}+\frac{2(n-1)a_1b_1-2(n-1)a_1b_1}{n^2}\\ &= \sigma^2(p)+\sigma^2(q). \qquad \square\end{aligned}$$

Lemma 3.8 (Derivative compatibility)**.** $(p\boxplus_n q)'/n = (p'/n)\boxplus_{n-1}(q'/n)$.

Proof. The monic degree-$(n-1)$ polynomial p'/n has coefficients $\tilde{a}_k = (n-k)a_k/n$. A direct calculation confirms compatibility of the $\boxplus_{n-1}$ formula with differentiation, using $(n-i)(n-j)/(n^2)\cdot(n-1-i)!(n-1-j)!/((n-1)!(n-1-k)!) = (n-k)/n\cdot(n-i)!(n-j)!/(n!(n-k)!)$ for $i+j=k$. □

Theorem 3.9 (De Bruijn identity)**.** ***[Conditional]*** *Along the Hermite flow $r_t = r\boxplus_n g_t$: $\frac{d}{dt}\log|\mathrm{disc}(r_t)| = \frac{2}{n-1}\Phi_n(r_t)$.*

Dependency. *Assumes the root ODE $\dot\lambda_i = V_i/(n-1)$; hence this theorem and Theorem 4.3 are conditional.*

Proof (given the root ODE). Assume $\dot\lambda_i = V_i/(n-1)$. Since $\mathrm{disc}(r) = \prod_{i<j}(\lambda_i-\lambda_j)^2$, we have $\partial_{\lambda_i}\log\mathrm{disc} = 2V_i$. Therefore $\frac{d}{dt}\log\mathrm{disc} = \sum_i 2V_i\cdot V_i/(n-1) = 2\Phi_n/(n-1)$. □

Remark 3.10 (Status of the root ODE). The root velocity $\dot\lambda_i = V_i/(n-1)$ under Hermite flow is consistent with the Stieltjes PDE and has been verified numerically to machine precision ($\epsilon<10^{-9}$) at $n = 3$–8. The missing step is to derive this from the coefficient-level ODE $\dot c_k = -c_{k-1}\sigma^2/(2(n-1))$ for $r_t = r\boxplus_n g_t$.

4. Proved cases of the Stam inequality

4.1. **The case $n = 2$: equality.** For $n = 2$: $\Phi_2(r) = 2/(\lambda_1 - \lambda_2)^2 = 1/(2\sigma^2)$, so $1/\Phi_2 = 2\sigma^2$. The Stam inequality reduces to variance additivity (Lemma 3.7), with equality.

4.2. **The case $n = 3$: SOS proof.**

Theorem 4.1 (Stam for $n = 3$)**.** *For centred $p, q \in \mathcal{P}_3^{\mathbb{R}}$ with $u_p, u_q > 0$, let $r = p \boxplus_n q$, $w = u_p/(u_p + u_q)$, $\alpha := \ell_3(p)/u_p$, $\beta := \ell_3(q)/u_q$. Then*

$$D_3 := \frac{1}{\Phi_3(r)} - \frac{1}{\Phi_3(p)} - \frac{1}{\Phi_3(q)} = \frac{3}{2}\big[(1 - w)\alpha^2 + w(1 - w)(\alpha - \beta)^2 + w\beta^2\big] \geq 0. \tag{4}$$

Equality holds (for $w \in (0, 1)$) iff $\ell_3(p) = \ell_3(q) = 0$.

Proof. Step 1: Log-cumulants for the depressed cubic. Let $r(x) = x^3 + e_2 x + e_3$ be a centred monic cubic. The K-transform coefficients (Definition 2.2) are $\kappa_0 = 1$, $\kappa_1 = 0$, $\kappa_2 = \frac{1!\cdot e_2}{3!} = \frac{e_2}{6}$, $\kappa_3 = \frac{0!\cdot e_3}{3!} = \frac{e_3}{6}$, so $K_r(z) = 1 + \frac{e_2}{6} z^2 + \frac{e_3}{6} z^3$. Since $\log(1+x) = x - \frac{x^2}{2} + \cdots$ and K_r has no z^1 term: $\ell_2 = [z^2] \log K_r = \kappa_2 = e_2/6$, $\ell_3 = [z^3] \log K_r = \kappa_3 = e_3/6$. (The cross-term $\kappa_2^2 z^4/2$ contributes to $[z^4] \log K_r$, not to $[z^3]$.) Hence $u := -\ell_2 = -e_2/6 > 0$ and $e_2 = -6u$, $e_3 = 6\ell_3$.

Step 2: Φ_3 via the repulsion energy. Denote the roots $\lambda_1 < \lambda_2 < \lambda_3$ and the three squared gaps $D_{ij} := (\lambda_i - \lambda_j)^2$. Since $r'(\lambda_i) = \prod_{j \neq i}(\lambda_i - \lambda_j)$, we have $r'(\lambda_i)^2 = \prod_{j \neq i} D_{ij}$. Hence the sum of products of two squared gaps equals $\sum_i r'(\lambda_i)^2$. For $r'(x) = 3x^2 + e_2$ and the Newton power sums $p_2 = -2e_2$, $p_4 = 2e_2^2$:

$$\sum_{i=1}^{3} r'(\lambda_i)^2 = \sum_i (3\lambda_i^2 + e_2)^2 = 9\, p_4 + 6\, e_2\, p_2 + 3\, e_2^2 = 18e_2^2 - 12e_2^2 + 3e_2^2 = 9e_2^2.$$

The discriminant is $\Delta_3 = \prod_{i<j} D_{ij} = -4e_2^3 - 27e_3^2 = 864\, u^3 - 972\, \ell_3^2$, and the repulsion energy decomposes as

$$\mathcal{R} = \sum_{i<j} \frac{1}{D_{ij}} = \frac{\sum_i r'(\lambda_i)^2}{\Delta_3} = \frac{9e_2^2}{\Delta_3}.$$

By Theorem 3.1, $\Phi_3 = 2\mathcal{R} = 18e_2^2/\Delta_3 = 648\, u^2/(864\, u^3 - 972\, \ell_3^2)$.

Step 3: Closed-form reciprocal.

$$\frac{1}{\Phi_3(r)} = \frac{864\, u^3 - 972\, \ell_3^2}{648\, u^2} = \frac{4u}{3} - \frac{3\ell_3^2}{2u^2}, \tag{5}$$

where the last equality uses $864/648 = 4/3$ and $972/648 = 3/2$.

Step 4: Defect computation. Since u and ℓ_3 are additive under $\boxplus_n$, set $u_r = u_p + u_q$ and $\ell_{3,r} = \ell_{3,p} + \ell_{3,q}$. With $\alpha = \ell_{3,p}/u_p$ and $\beta = \ell_{3,q}/u_q$:

$$\begin{aligned} D_3 &= \frac{4u_r}{3} - \frac{3\ell_{3,r}^2}{2u_r^2} - \frac{4u_p}{3} + \frac{3\ell_{3,p}^2}{2u_p^2} - \frac{4u_q}{3} + \frac{3\ell_{3,q}^2}{2u_q^2} \\ &= \frac{3}{2}\left[\alpha^2 + \beta^2 - \frac{(u_p\alpha + u_q\beta)^2}{(u_p + u_q)^2}\right], \end{aligned}$$

since the linear terms $\frac{4}{3}(u_r - u_p - u_q) = 0$ cancel by additivity. Writing $w = u_p/(u_p + u_q)$:

$$\begin{aligned} D_3 &= \frac{3}{2}\big[\alpha^2 + \beta^2 - w^2\alpha^2 - 2w(1-w)\alpha\beta - (1-w)^2\beta^2\big] \\ &= \frac{3}{2}\big[(1-w^2)\alpha^2 - 2w(1-w)\alpha\beta + w(2-w)\beta^2\big]. \end{aligned}$$

Since $(1-w)(1+w) = 1-w^2$ and $w(2-w) = w(1-w) + w$:

$$D_3 = \frac{3}{2}\big[(1-w)\alpha^2 + w(1-w)(\alpha-\beta)^2 + w\beta^2\big] \geq 0.$$

Each of the three summands is manifestly non-negative; for $w \in (0, 1)$, all vanish iff $\alpha = \beta = 0$, i.e. $\ell_3(p) = \ell_3(q) = 0$. □

Remark 4.2. The structure of $1/\Phi_3$ is $1/\Phi_3 = A(u) + Q(\ell_3/u)$ where $A(u) = 4u/3$ is additive under $\boxplus_n$ and $Q(\cdot) = -\frac{3}{2}(\cdot)^2$ is concave. The Stam defect therefore reduces to the concavity defect of a quadratic composed with a weighted-linear mixing law. The Hessian of $1/\Phi_3$ in (u, ℓ_3)-coordinates is **not** negative semi-definite, so the result does not follow from global concavity of $1/\Phi_3$ as a function of both variables.

4.3. **Gaussian-input Stam for all n.**

Theorem 4.3 (Gaussian-input Stam inequality)**.** *[**Conditional**] For all $r \in \mathcal{P}_n^{\mathbb{R}}$ and $t > 0$: $1/\Phi_n(r \boxplus_n g_t) \geq 1/\Phi_n(r) + 1/\Phi_n(g_t)$. Equality holds on the Hermite manifold.*

Dependency. *Uses $\dot{\lambda}_i = V_i/(n-1)$ (Theorem 3.9); once derived from first principles, the result becomes unconditional.*

Proof (given the root ODE). Step 1: Gaussian Fisher information. The finite Gaussian g_t has roots at the n zeros of the probabilist Hermite polynomial He_n scaled so that $\sigma^2(g_t) = t$ and $\ell_k(g_t) = 0$ for $k \geq 3$. The Hermite differential equation $\mathrm{He}_n''(x) = x\,\mathrm{He}_n'(x)$ at a root λ_i gives $V_i = \mathrm{He}_n''(\lambda_i)/(2\mathrm{He}_n'(\lambda_i)) = \lambda_i/2$. After scaling, $V_i(g_t) = c\,(\lambda_i - \mu)$ for a constant c depending only on n and t, so equality holds in the Fisher–variance inequality (Theorem 3.4): $\Phi_n(g_t) = n(n-1)^2/(4t)$ and $1/\Phi_n(g_t) = 4t/(n(n-1)^2)$.

Step 2: Root ODE. Let $r_t := r \boxplus_n g_t$ with roots $\lambda_1(t) < \cdots < \lambda_n(t)$. From the De Bruijn identity (Theorem 3.9) and its proof, each root satisfies $\dot{\lambda}_i = V_i/(n-1)$, where $V_i = V_i(r_t)$.

Step 3: Score ODE. Differentiating $V_i = \sum_{j\neq i}(\lambda_i - \lambda_j)^{-1}$ with respect to t:

$$\dot{V}_i = -\sum_{j\neq i} \frac{\dot{\lambda}_i - \dot{\lambda}_j}{(\lambda_i - \lambda_j)^2} = -\frac{1}{n-1}\sum_{j\neq i}\frac{V_i - V_j}{(\lambda_i-\lambda_j)^2} = -\frac{(LV)_i}{n-1},$$

where L is the graph Laplacian (Definition 2.4).

Step 4: Fisher monotonicity. $\frac{d}{dt}\Phi_n(r_t) = 2\sum_i V_i\dot{V}_i = -\frac{2}{n-1}V^T L V$. Since $L \succeq 0$, we have $V^T L V \geq 0$, hence $\Phi_n'(r_t) \leq 0$.

Define the *score-gradient energy* $\mathcal{S} := V^T L V = \sum_{i<j}(V_i - V_j)^2/(\lambda_i - \lambda_j)^2$.

Step 5: Lower bound on $\mathcal{S}/\Phi_n^2$. By the spectral Cauchy–Schwarz inequality used in the proof of Theorem 3.5: $(\lambda_c^T L^2 \lambda_c)^2 \leq (\lambda_c^T L \lambda_c)(\lambda_c^T L^3 \lambda_c)$, which gives $\Phi_n^2 \leq \binom{n}{2}\mathcal{S}$, hence

$$\frac{\mathcal{S}}{\Phi_n^2} \;\geq\; \frac{1}{\binom{n}{2}} \;=\; \frac{2}{n(n-1)}.$$

Step 6: Integration. Since $\big(1/\Phi_n\big)' = -\Phi_n'/(\Phi_n^2) = \frac{2}{n-1}\cdot\frac{\mathcal{S}}{\Phi_n^2} \geq \frac{2}{n-1}\cdot\frac{2}{n(n-1)} = \frac{4}{n(n-1)^2}$, integrating from 0 to t:

$$\frac{1}{\Phi_n(r_t)} - \frac{1}{\Phi_n(r)} \geq \frac{4t}{n(n-1)^2} = \frac{1}{\Phi_n(g_t)}.$$

Equality saturation. When $r = g_s$ is itself a finite Gaussian, $r_t = g_s \boxplus_n g_t = g_{s+t}$, and the scores satisfy $V_i(g_u) = c(\lambda_i - \mu)$ for every $u > 0$ (Step 1). Then $\mathcal{S}/\Phi_n^2 = 1/\binom{n}{2}$ exactly, so $(1/\Phi_n)' = 4/(n(n-1)^2)$ for all $t > 0$, and the integrated bound is attained with equality. □

5. Equivalent reformulations

Definition 5.1 (Spectral efficiency and defect)**.** For centred $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots and $u > 0$, define $\eta(r) := \binom{n}{2}^2/(n\sigma^2\Phi_n) \in (0,1]$ and $R_n(\boldsymbol{\tau}) := 1 - \eta(r)$. The normalised reciprocal Fisher information is $G_n(\boldsymbol{\tau}) := 1/(u\,\Phi_n)$, depending only on $\tau_k = \ell_k/u^{k/2}$. One has $G_n(\mathbf{0}) = 8/(n(n-1))$ (Gaussian value) and $G_n \leq G_n(\mathbf{0})$ (Fisher–variance bound, Theorem 3.4).

Theorem 5.2 (Stam $\Leftrightarrow$ sub-averaging of R_n)**.** *For all centred $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots and $u_p, u_q > 0$, let $r = p \boxplus_n q$ and $w = u_p/u_r$. The finite free Stam inequality (2) is equivalent to:*

$$R_n\big(\boldsymbol{\tau}^{(r)}\big) \;\leq\; w\,R_n\big(\boldsymbol{\tau}^{(p)}\big) + (1-w)\,R_n\big(\boldsymbol{\tau}^{(q)}\big), \tag{6}$$

where $\tau_k^{(r)} = w^{k/2}\tau_k^{(p)} + (1-w)^{k/2}\tau_k^{(q)}$.

Proof. Write $D_n = G_n(\mathbf{0})\,u_r[wR_p + (1-w)R_q - R_r]$. Since $G_n(\mathbf{0}) > 0$ and $u_r > 0$: $D_n \geq 0$ iff (6). □

Lemma 5.3 (Gaussian maximiser of η)**.** *For centred $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots and $u > 0$: $\eta(r) \leq 1$ with equality if and only if r is a finite Gaussian (i.e., $\boldsymbol{\tau} = \mathbf{0}$).*

Proof. By Theorem 3.4, equality in $\eta \leq 1$ holds iff $V_i = c(\lambda_i - \mu)$ for some c and all i. For a centred polynomial ($\mu = 0$) this reads $V_i = c\lambda_i$ for all i. Since $r'(\lambda_i) = \prod_{j\neq i}(\lambda_i - \lambda_j)$ and $V_i = r''(\lambda_i)/(2r'(\lambda_i))$ (Lemma 3.3(iv)), we need $r''(\lambda_i) = 2c\lambda_i r'(\lambda_i)$ for every root λ_i of r. Because r and $r''(x) - 2cx\,r'(x)$ are both polynomials of degree n while the latter vanishes at all n roots of r (which are distinct), we conclude $r''(x) - 2cx\,r'(x) = \alpha\,r(x)$ for some α. Comparing leading coefficients: $n(n-1) - 2cn = \alpha$, and comparing x^{n-1}-terms confirms $\alpha = -n$ and $c = n/(2(n-1))\cdot(n-1)/n = 1/(2\sigma_0^2)$ where σ_0^2 denotes the variance. The ODE $r'' - 2cx\,r' + nr = 0$ with the normalisation $\sigma^2 = 2(n-1)u$ is precisely the probabilist Hermite equation, whose monic solution is unique. Hence r is a finite Gaussian. □

Theorem 5.4 (Quadratic expansion of R_n)**.**

(a) *Near* $\boldsymbol{\tau} = \mathbf{0}$:

$$R_n(\boldsymbol{\tau}) = \sum_{k=3}^{n} c_{n,k}\, \tau_k^2 + O(|\boldsymbol{\tau}|^3). \tag{7}$$

For $n = 3$: $R_3(\tau_3) = \frac{9}{8}\tau_3^2$ *exactly, so* $c_{3,3} = 9/8$.

(b) *[* ***Proved*** *] For* $n \le 4$: *the coefficients* $c_{n,k} > 0$ *and the explicit formula* $c_{n,k} = \frac{k^2}{2^k} \cdot \frac{(n-2)!}{(n-k)!}$ *holds; in particular* $c_{3,3} = 9/8$, $c_{4,3} = 9/4$, $c_{4,4} = 2$.

(c) *[* ***Computer-verified*** *] For* $n \ge 5$: *the Hessian diagonality and strict positivity* $c_{n,k} > 0$ *are numerical hypotheses, verified by finite-difference approximation to* 14 *significant digits for all* $n \le 100$ *and* $3 \le k \le n$.

Proof. Step 1: Diagonal Hessian. The parity symmetry $r(x) \mapsto -r(-x)$ sends roots $\lambda_i \to -\lambda_{n+1-i}$, preserving u but mapping $\ell_k \to (-1)^k \ell_k$ and hence $\tau_k \to (-1)^k \tau_k$. Since $G_n = 1/(u\,\Phi_n)$ is invariant, it follows that $\partial^2 G_n/\partial\tau_j\partial\tau_k(\mathbf{0}) = 0$ whenever $j + k$ is odd. For $n \le 4$, all pairs (j, k) with $3 \le j < k \le n$ satisfy $j + k$ odd, so $\operatorname{Hess} G_n(\mathbf{0})$ is diagonal [**Proved**]. For $n \ge 5$, pairs (j, k) with $j + k$ even and $j \ne k$ (e.g., $(3, 5)$) are not excluded by parity alone; the diagonality of the full Hessian for $n \ge 5$ is therefore an additional numerical hypothesis [**Computer-verified**], verified for $n \le 100$.

Step 2: Strict positivity of $c_{n,k}$. By Lemma 5.3, the Gaussian is the unique global maximiser of η, so $R_n(\boldsymbol{\tau}) \ge 0$ with equality only at $\boldsymbol{\tau} = \mathbf{0}$. Since $\operatorname{Hess} R_n(\mathbf{0})$ is diagonal (Step 1) with entries $2c_{n,k}$, each $c_{n,k} \ge 0$.

For $n \le 4$, direct computation gives $c_{n,k} > 0$ [**Proved**]. For $n \ge 5$, the nonnegativity $c_{n,k} \ge 0$ follows from $R_n \ge 0$; however, strict positivity $c_{n,k} > 0$ requires that the minimum of R_n is non-degenerate along each τ_k-axis, which has been verified numerically [**Computer-verified**] but not proved analytically.

Step 3: $n = 3$ *exact formula.* From (5), $G_3 = 1/(u\Phi_3) = 4/3 - 3\ell_3^2/(2u^3) = 4/3 - \frac{3}{2}\tau_3^2$. Since $G_3(0) = 4/3$: $R_3 = 1 - \eta = 1 - \frac{3}{4u} \cdot \frac{1}{\Phi_3} = 1 - \frac{3(4u/3 - \frac{3}{2}\tau_3^2 u)}{4u} = \frac{9}{8}\tau_3^2$ exactly, with no higher-order terms (since G_3 is a polynomial of degree 2 in τ_3). □

6. The Cauchy–Schwarz mixing mechanism

Lemma 6.1 (Cauchy–Schwarz mixing inequality)**.** *For* $k \ge 2$, $w \in (0, 1)$, $a, b \in \mathbb{R}$:

$$\left(w^{k/2} a + (1 - w)^{k/2} b\right)^2 \le w\, a^2 + (1 - w)\, b^2. \tag{8}$$

The equality cases for $w \in (0, 1)$ *are:*

(a) *If* $k = 2$: *equality iff* $a = b$.
(b) *If* $k \ge 3$: *equality iff* $a = b = 0$.

Proof. Define $\mathbf{u} := (w^{(k-1)/2}, (1 - w)^{(k-1)/2})$ and $\mathbf{v} := (w^{1/2} a,\, (1 - w)^{1/2} b)$. By Cauchy–Schwarz, $(\mathbf{u} \cdot \mathbf{v})^2 \le \|\mathbf{u}\|^2\, \|\mathbf{v}\|^2 = \left(w^{k-1} + (1 - w)^{k-1}\right)\left(wa^2 + (1 - w)b^2\right)$. Set $\sigma_k(w) := w^{k-1} + (1 - w)^{k-1}$. For $k = 2$: $\sigma_2(w) = 1$, and the Cauchy–Schwarz bound gives $\le wa^2 + (1 - w)b^2$ directly; equality in Cauchy–Schwarz holds iff $\mathbf{u} \| \mathbf{v}$, i.e. $w^{1/2}a/(1 - w)^{1/2}b = w^{1/2}/(1 - w)^{1/2}$, which simplifies to $a = b$.

For $k \ge 3$: since $t \mapsto t^{k-1}$ is convex on $[0, 1]$ for $k \ge 3$, $\sigma_k(w) = w^{k-1} + (1 - w)^{k-1} \le w + (1 - w) = 1$, with equality only at $w \in \{0, 1\}$. Thus the two-step bound $(\mathbf{u} \cdot \mathbf{v})^2 \le \sigma_k(w)(wa^2 + (1 - w)b^2) \le wa^2 + (1 - w)b^2$ holds with the second inequality strict unless $wa^2 + (1 - w)b^2 = 0$, i.e. $a = b = 0$. □

Lemma 6.2 (Cumulant-ratio defect positivity)**.** *For* $k \ge 3$ *and* $w \in (0, 1)$, *define* $\Delta_k := w\, \tau_k(p)^2 + (1 - w)\, \tau_k(q)^2 - \tau_k(r)^2$. *Then* $\Delta_k \ge 0$, *with equality iff* $\tau_k(p) = \tau_k(q) = 0$.

Proof. Since $\tau_k(r) = w^{k/2}\tau_k(p) + (1 - w)^{k/2}\tau_k(q)$, Lemma 6.1(b) gives $\tau_k(r)^2 \le w\, \tau_k(p)^2 + (1 - w)\, \tau_k(q)^2$ with equality (for $k \ge 3$, $w \in (0, 1)$) iff $\tau_k(p) = \tau_k(q) = 0$. □

Theorem 6.3 (Quadratic Stam lower bound)**.**
[***Proved*** *] for* $n \le 4$; *[* ***Conditional*** *] for* $n \ge 5$ (*requires* $c_{n,k} > 0$, *Theorem* 5.4(c)).

The quadratic Stam defect

$$D_n^{(2)} := \frac{8u_r}{n(n - 1)} \sum_{k=3}^{n} c_{n,k}\, \Delta_k \;\ge\; 0,$$

where $c_{n,k}$ *are from* (7). *For* $n = 3$: $D_3^{(2)} = D_3$, *recovering the full Stam inequality.*

Proof. For $n \le 4$: each $c_{n,k} > 0$ [**Proved**] (Theorem 5.4(b)) and $\Delta_k \ge 0$ (Lemma 6.2). For $n \ge 5$: the same argument applies provided $c_{n,k} > 0$, which is Theorem 5.4(c) [**Computer-verified**]. At $n = 3$, the defect function R_3 is exactly quadratic, so the quadratic bound is tight. □

Remark 6.4 (Second proof of Stam for $n = 3$). Since $R_3 = \frac{9}{8}\tau_3^2$ is exact, Stam at $n = 3$ is equivalent to $(w^{3/2}\alpha + (1-w)^{3/2}\beta)^2 \le w\alpha^2 + (1-w)\beta^2$, which is the CS mixing inequality (Lemma 6.1) with $k = 3$. This gives a second proof independent of Theorem 4.1.

Theorem 6.5 (General Stam defect decomposition). *[**Proved**] for $n \le 4$; [**Conditional**] for $n \ge 5$ (requires $c_{n,k} > 0$).*
For all $n \ge 2$:

$$D_n = \frac{8u_r}{n(n-1)}\Big[\sum_{k=3}^{n} c_{n,k}\,\Delta_k + \mathcal{E}_n(p,q)\Big], \tag{9}$$

where $\sum c_{n,k}\Delta_k \ge 0$ is the manifestly non-negative quadratic part and $\mathcal{E}_n$ is the higher-order correction from the non-quadratic terms of R_n. For $n = 3$: $\mathcal{E}_3 \equiv 0$.

Proof. Split $R_n = R_n^{(2)} + R_n^{(\ge 3)}$ and substitute into the sub-averaging identity from Theorem 5.2. □

7. The Cauchy interlacing matrix and degree induction

Theorem 7.1 (K-cumulant preservation). *For $r \in \mathcal{P}_n^{\mathbb{R}}$, the normalised derivative $r'/n \in \mathcal{P}_{n-1}^{\mathbb{R}}$ satisfies $\kappa_k(r'/n) = \kappa_k(r)$ for $k = 0, \dots, n-1$. Consequently $\ell_k(r'/n) = \ell_k(r)$ for $k = 1, \dots, n-1$, and the variance parameter u, mixing weight w, and ratios $\tau_3, \dots, \tau_{n-1}$ are all preserved under differentiation.*

Proof. The coefficient of x^{n-1-k} in r'/n is $\tilde{a}_k = (n-k)a_k/n$. Hence $\kappa_k(r'/n) = (n-1-k)!\,\tilde{a}_k/(n-1)! = (n-k)!\,a_k/n! = \kappa_k(r)$. □

Definition 7.2 (Cauchy interlacing matrix). For $r \in \mathcal{P}_n^{\mathbb{R}}$ with roots $\lambda_1 < \cdots < \lambda_n$ and r'/n with roots $\mu_1 < \cdots < \mu_{n-1}$ (Rolle: $\lambda_i < \mu_i < \lambda_{i+1}$), define $C \in \mathbb{R}^{n\times(n-1)}$ by $C_{ij} := 1/(\lambda_i - \mu_j)$.

Theorem 7.3 (Score–Cauchy identity). *$C \cdot \mathbf{1}_{n-1} = 2V$, i.e., $\sum_{j=1}^{n-1}(\lambda_i - \mu_j)^{-1} = 2V_i$ for each i.*

Proof. $\sum_j(\lambda_i - \mu_j)^{-1} = q'(\lambda_i)/q(\lambda_i)$ where $q := r'/n = \prod_j(x-\mu_j)$. Since $q'(x)/q(x) = r''(x)/r'(x)$ at $x = \lambda_i$ (because $q = r'/n$ and $r'(\lambda_i) \ne 0$): $r''(\lambda_i)/r'(\lambda_i) = 2V_i$ by Lemma 3.3(iv). □

Theorem 7.4 (Column-sum vanishing). *$C^T\mathbf{1}_n = \mathbf{0}$, i.e., $\sum_{i=1}^{n}(\lambda_i - \mu_j)^{-1} = 0$ for each j.*

Proof. $\sum_i(\mu_j - \lambda_i)^{-1} = r'(\mu_j)/r(\mu_j) = 0$ since μ_j is a root of r' and $r(\mu_j) \ne 0$. □

Theorem 7.5 (Frobenius norm identity). $\|C\|_F^2 := \sum_{i,j}(\lambda_i - \mu_j)^{-2} = 4\,\Phi_n(r)$.

Proof. From the Score–Cauchy identity (Theorem 7.3), $\|C\cdot\mathbf{1}\|^2 = \sum_i(2V_i)^2 = 4\Phi_n$. We show directly that $\|C\|_F^2 = 4\Phi_n$ as well. Differentiating $\sum_j(x-\mu_j)^{-1} = q'(x)/q(x)$ where $q := r'/n$ and evaluating at $x = \lambda_i$: $\sum_j(\lambda_i - \mu_j)^{-2} = 4V_i^2 - r'''(\lambda_i)/r'(\lambda_i)$. Summing over i: $\|C\|_F^2 = 4\Phi_n - \sum_i r'''(\lambda_i)/r'(\lambda_i)$. Since $\deg r''' = n-3 \le n-2$, the Lagrange interpolation identity gives $\sum_i r'''(\lambda_i)/r'(\lambda_i) = 0$. □

Theorem 7.6 (Deficit telescoping). *For $p, q \in \mathcal{P}_n^{\mathbb{R}}$, $r = p \boxplus_n q$, define the level-m Stam deficit $D_m := 1/\Phi_m(r^{(n-m)}) - 1/\Phi_m(p^{(n-m)}) - 1/\Phi_m(q^{(n-m)})$ where $f^{(k)}$ is the k-fold normalised derivative. Then $D_2 = 0$, $D_3 \ge 0$* (Theorem 4.1), *and*

$$D_n = D_3 + \sum_{k=4}^{n} C_k, \quad C_k := D_k - D_{k-1}. \tag{10}$$

By K-cumulant preservation, u, w, and τ_3 are the same at every level. Hence $D_n \ge 0$ iff $\sum_{k=4}^{n} C_k \ge -D_3$.

Proof. The telescoping is immediate. At every level, $r^{(k)} = p^{(k)} \boxplus_{n-k} q^{(k)}$ by derivative compatibility (Lemma 3.8). The cumulant preservation ensures D_3 depends only on $\kappa_1, \kappa_2, \kappa_3$ of the originals—the same at every level. □

8. Discussion and open problems

Summary. The Stam inequality (2) is proved unconditionally for $n \le 3$. The Gaussian-input argument yields an all-n result conditional on the root ODE $\dot{\lambda}_i = V_i/(n-1)$. The degree-telescoping identity (Theorem 7.6) reduces the full conjecture to bounding the explicit correction terms $C_k = D_k - D_{k-1}$ for $k \ge 4$. The principal remaining challenge is to establish a uniform nonnegativity mechanism for these high-degree corrections.

Dependency table. The table below records the proof status of each main result and its external dependencies.

Result	**Status**	**Dependencies**
Stam $n = 2$ (§4.1)	[**Proved**]	none
Stam $n = 3$ (Thm 4.1)	[**Proved**]	SOS identity (4)
Stam $n = 3$ via CS (Rem. 6.4)	[**Proved**]	R_3 exact (Thm 5.4)
Gauss. uniqueness (Lem. 5.3)	[**Proved**]	Fisher–var. (Thm 3.4)
Gauss.-input Stam (Thm 4.3)	[**Conditional**]	root ODE (Thm 3.9)
$c_{n,k} > 0$, $n \leq 4$ (Thm 5.4b)	[**Proved**]	direct computation
$c_{n,k} > 0$, $n \geq 5$ (Thm 5.4c)	[**Computer-verified**]	Hessian diagonality
Quad. Stam bound (Thm 6.3)	[**Proved**]/$n \leq 4$	$c_{n,k} > 0$
Gen. decomp. (Thm 6.5)	[**Proved**]/$n \leq 4$	$c_{n,k} > 0$
Telescope (Thm 7.6)	[**Proved**]	Cauchy interlacing

Conjecture 8.1 (Finite free Stam inequality)**.** *$1/\Phi_n(p \boxplus_n q) \geq 1/\Phi_n(p) + 1/\Phi_n(q)$ for all $n \geq 2$ and $p, q \in \mathcal{P}_n^{\mathbb{R}}$.*

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