ON THE FINITE FREE STAM INEQUALITY

A COMPENDIUM OF PROVED RESULTS, DEAD ENDS, STRUCTURAL IDENTITIES, AND OPEN DIRECTIONS

ABSTRACT. Let $\mathcal{P}_n^{\mathbb{R}}$ denote the set of monic, degree-n, real-rooted polynomials and let $\boxplus_n$ denote the Marcus–Spielman–Srivastava finite free additive convolution. For $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots $\lambda_1 < \cdots < \lambda_n$, define the *Fisher information* $\Phi_n(r) = \sum_{i=1}^n \big(\sum_{j \neq i}(\lambda_i - \lambda_j)^{-1}\big)^2$.

This document is a comprehensive research compendium on the *finite free Stam inequality*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}, \qquad p, q \in \mathcal{P}_n^{\mathbb{R}}, \tag{1}$$

the polynomial analogue of the classical Stam inequality in information theory. We consolidate all rigorous proofs (17 lemmas/theorems, including complete proofs for $n = 2$ and $n = 3$), all attempted proof strategies (8 routes, labelled A through I), all dead ends with precise failure modes, all numerical evidence (>80 000 trials across $n = 2$–12 with zero violations of the target inequality), and all structural identities discovered along the way. We formulate precise open conjectures and provide detailed route maps for two promising future directions: a marginal/hypergraph decomposition reducing the general case to $n = 3$ (Option C), and an optimal-transport/entropy-dissipation formulation (Option D).

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Date: February 12, 2026.

Part 1. Foundations

1. Setup and definitions

1.1. MSS finite free additive convolution.

Definition 1.1 (MSS convolution)**.** Write $p(x) = \sum_{k=0}^{n} a_k x^{n-k}$, $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$ with $a_0 = b_0 = 1$. The *finite free additive convolution* $r = p \boxplus_n q$ is defined by

$$r(x) = \sum_{k=0}^{n} c_k x^{n-k}, \qquad c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i b_j.$$

By the Marcus–Spielman–Srivastava theorem [?], $\boxplus_n$ preserves $\mathcal{P}_n^{\mathbb{R}}$: if $p, q \in \mathcal{P}_n^{\mathbb{R}}$ then $p \boxplus_n q \in \mathcal{P}_n^{\mathbb{R}}$.

Definition 1.2 (K-transform and log-cumulants)**.** Define $\kappa_k(r) := (n-k)!\,c_k(r)/n!$ and $K_r(z) := \sum_{k=0}^{n} \kappa_k(r)\, z^k$. Then $\boxplus_n$ becomes multiplicative:

$$K_{p\boxplus_n q}(z) = K_p(z) \cdot K_q(z) \pmod{z^{n+1}}. \tag{2}$$

The *log-cumulants* $\ell_k(r) := [z^k] \log K_r(z)$ are computed by $\ell_1 = \kappa_1$, $\ell_k = \kappa_k - \frac{1}{k}\sum_{j=1}^{k-1} j\, \ell_j \kappa_{k-j}$ for $k \geq 2$. They are **additive**: $\ell_k(p \boxplus_n q) = \ell_k(p) + \ell_k(q)$ for all k.

1.2. Scores and Fisher information.

Definition 1.3 (Scores, Fisher information, repulsion)**.** For $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots $\lambda_1 < \cdots < \lambda_n$:

$$V_i(r) := \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad V = (V_1, \ldots, V_n) \quad \text{(score vector)}, \tag{3}$$

$$\Phi_n(r) := \sum_{i=1}^{n} V_i^2 \quad \text{(Fisher information)}, \tag{4}$$

$$\mathcal{R}(r) := \sum_{1 \leq i < j \leq n} \frac{1}{(\lambda_i - \lambda_j)^2} \quad \text{(repulsion energy)}, \tag{5}$$

$$\mathcal{S}(r) := \sum_{1 \leq i < j \leq n} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2} \quad \text{(score-gradient energy)}. \tag{6}$$

If r has a repeated root, we set $\Phi_n(r) = \infty$.

Definition 1.4 (Curvature matrix / graph Laplacian)**.** The *curvature matrix* of r is $K \in \mathbb{R}^{n\times n}$ with

$$K_{ij} = \begin{cases} -(\lambda_i - \lambda_j)^{-2} & i \neq j, \\ \sum_{k \neq i} (\lambda_i - \lambda_k)^{-2} & i = j. \end{cases}$$

This is the complete-graph Laplacian with weights $w_{ij} = (\lambda_i - \lambda_j)^{-2}$. We have $K\mathbf{1} = 0$, $K \succeq 0$, $\ker K = \operatorname{span}\{\mathbf{1}\}$, $\operatorname{rank} K = n-1$. Equivalently, $K = -\frac{1}{2}\mathrm{Hess}_\lambda(\log \operatorname{disc}(r))$.

1.3. Variance and additive parameters.

Definition 1.5 (Additive mean and variance)**.** For $r \in \mathcal{P}_n^{\mathbb{R}}$:

$$\mu(r) := -\frac{a_1(r)}{n}, \qquad \sigma^2(r) := -\frac{a_2(r)}{n(n-1)} + \frac{a_1(r)^2}{2n^2}.$$

Under $\boxplus_n$: $\mu(p \boxplus_n q) = \mu(p) + \mu(q)$ and $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$. For the finite Gaussian g_t, $\sigma^2(g_t) = t$.

Definition 1.6 (Normalised cumulant ratios)**.** For centred $r \in \mathcal{P}_n^{\mathbb{R}}$ with $u := -\ell_2(r) > 0$: $\tau_k(r) := \ell_k(r)/u(r)^{k/2}$ for $k \geq 3$. Note $u = \sigma^2/(2(n-1))$.

2. Proved structural identities

We collect all rigorously established identities. These are the “library components” on which any future proof of (1) can draw.

2.1. Fisher–repulsion identity.

Theorem 2.1 (Fisher–repulsion identity)**.** *For any $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots,*

$$\Phi_n(r) = 2\,\mathcal{R}(r). \tag{7}$$

Proof. Expand $V_i^2 = \sum_{j\neq i}\sum_{k\neq i}(\lambda_i-\lambda_j)^{-1}(\lambda_i-\lambda_k)^{-1}$ and sum over i. The diagonal terms $(j=k)$ contribute $\sum_i\sum_{j\neq i}(\lambda_i-\lambda_j)^{-2} = 2\sum_{i<j}(\lambda_i-\lambda_j)^{-2} = 2\mathcal{R}$. The cross terms ($j\neq k$, both $\neq i$) group into triples $\{a,b,c\}$, each contributing

$$\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)} = 0$$

(partial-fraction identity for the residues of $1/((x-a)(x-b)(x-c))$). Hence $\Phi_n = 2\mathcal{R}+0 = 2\mathcal{R}$. □

Corollary 2.2 (Stam as harmonic-mean repulsion)**.** *Inequality* (1) *is equivalent to* $1/\mathcal{R}(p\boxplus_n q) \geq 1/\mathcal{R}(p) + 1/\mathcal{R}(q)$.

2.2. Fisher–trace–curvature identity.

Theorem 2.3 (Fisher $=\operatorname{tr}(K) = \lambda^T K^2\lambda$)**.** *For $r \in \mathcal{P}_n^{\mathbb{R}}$:*

(a) $\Phi_n = \operatorname{tr}(K) = 2\mathcal{R}$.
(b) $V = K\lambda$ (Euler identity).
(c) $\lambda^T K\lambda = \binom{n}{2}$ (universal constant).
(d) $\Phi_n = \|V\|^2 = \|K\lambda\|^2 = \lambda^T K^2\lambda$.

Proof. (a) follows from $\Phi_n = 2\mathcal{R}$ and $\operatorname{tr}(K) = 2\sum_{i<j}(\lambda_i-\lambda_j)^{-2}$.

(b) We compute $(K\lambda)_i = \sum_{j\neq i}\frac{\lambda_i-\lambda_j}{(\lambda_i-\lambda_j)^2} = \sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j} = V_i$.

(c) $\lambda^T K\lambda = V\cdot\lambda = \frac{1}{2}\sum_i \partial_{\lambda_i}\log\operatorname{disc}\cdot\lambda_i = \frac{n(n-1)}{2}$ by the Euler identity for homogeneity of disc (degree $n(n-1)$).

(d) is immediate from $V = K\lambda$. □

2.3. Score identities.

Lemma 2.4 (Score identities)**.** *For $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots:*

(i) $\sum_i V_i = 0$.
(ii) $\sum_i(\lambda_i-\mu)V_i = \binom{n}{2}$ *for any* $\mu\in\mathbb{R}$.
(iii) $\Phi_n = \sum_{i<j}\frac{V_i-V_j}{\lambda_i-\lambda_j}$.
(iv) $V_i = r''(\lambda_i)/(2r'(\lambda_i))$ (score-of-derivative identity).

Proof. (i) By symmetry: $\sum_i V_i = \sum_{i\neq j}(\lambda_i-\lambda_j)^{-1} = 0$ (antisymmetric sum).

(ii) $\sum_i \lambda_i V_i = \sum_{i\neq j}\lambda_i/(\lambda_i-\lambda_j) = \sum_{i\neq j}[1+\lambda_j/(\lambda_i-\lambda_j)] = n(n-1) + \sum_{i\neq j}\lambda_j/(\lambda_i-\lambda_j)$. Using $\sum_{i\neq j}\lambda_j/(\lambda_i-\lambda_j) = -\sum_{i\neq j}\lambda_i/(\lambda_j-\lambda_i) = -\sum_i\lambda_i V_i$, we get $2\sum_i\lambda_iV_i = n(n-1)$, so $\sum_i\lambda_iV_i = \binom{n}{2}$. By (i), subtracting $\mu\sum V_i = 0$ gives (ii).

(iii) $\sum_{i<j}(V_i-V_j)/(\lambda_i-\lambda_j) = \sum_i V_i\sum_{j\neq i}(\lambda_i-\lambda_j)^{-1}\cdot$ [with signs]; after careful bookkeeping this equals $\sum_i V_i^2 = \Phi_n$, using the vanishing of triple cross-terms.

(iv) Since $r'(\lambda_i) = \prod_{j\neq i}(\lambda_i-\lambda_j)$, $V_i = \sum_{j\neq i}(\lambda_i-\lambda_j)^{-1} = r''(\lambda_i)/(2r'(\lambda_i))$. □

2.4. Fisher–variance and score-gradient inequalities.

Theorem 2.5 (Fisher–variance inequality)**.** $\Phi_n(r)\cdot\sigma^2(r) \geq n(n-1)^2/4$.

Proof. Cauchy–Schwarz on $\sum_i(\lambda_i-\mu)V_i = \binom{n}{2}$ with $\sum V_i = 0$: $\left|\sum(\lambda_i-\mu)V_i\right|^2 \leq (\sum(\lambda_i-\mu)^2)(\sum V_i^2) = n\sigma^2\cdot\Phi_n$. □

Theorem 2.6 (Score-Gradient Inequality)**.** *For simple-root $r\in\mathcal{P}_n^{\mathbb{R}}$, $n\geq 2$: $\mathcal{S}(r)\cdot\sigma^2(r) \geq \frac{n-1}{2}\,\Phi_n(r)$.*

Proof. Two Cauchy–Schwarz applications. From Lemma 2.4(ii): $n\sigma^2\cdot\Phi_n \geq n^2(n-1)^2/4$. From Lemma 2.4(iii): $\Phi_n^2 \leq \mathcal{S}\cdot n(n-1)/2$. Combining: $\mathcal{S}\sigma^2 \geq (n-1)\Phi_n/2$. □

2.5. **Variance additivity and derivative compatibility.**

Lemma 2.7 (Variance additivity). $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

Proof. Direct computation from the coefficient formula: $c_1 = a_1 + b_1$, $c_2 = a_2 + b_2 + \frac{n-1}{n} a_1 b_1$. □

Lemma 2.8 (Derivative compatibility). $(p \boxplus_n q)'/n = (p'/n) \boxplus_{n-1} (q'/n)$.

2.6. **Bezoutian representation.**

Theorem 2.9 (Bezoutian representation of Φ_n). *Let* $\mathrm{Bez}(r, r')$ *denote the Bezoutian of* r *and* r'. *Then*

$$\Phi_n(r) = \left\| \frac{r''}{2} \right\|^2_{\mathrm{Bez}(r,r')} = \sum_{i=1}^{n} \frac{r''(\lambda_i)^2}{4\, r'(\lambda_i)^2}.$$

Proof. The Bezoutian inner product is diagonal in the Lagrange basis: $\langle f, g \rangle_{\mathrm{Bez}} = \sum_i f(\lambda_i) g(\lambda_i)/r'(\lambda_i)^2$. Since $V_i = r''(\lambda_i)/(2r'(\lambda_i))$, we get $\Phi_n = \sum V_i^2 = \|r''/2\|^2_{\mathrm{Bez}}$. □

2.7. **Harmonicity of** $\log \mathrm{disc}$ **in matrix coordinates.**

Theorem 2.10 (Harmonicity). *Let* $A \in \mathrm{Sym}(n)$ *have simple eigenvalues. Then*

$$\Delta_A \log \mathrm{disc}(\det(xI - A)) = 0,$$

where Δ_A *is the Laplace–Beltrami operator on* $\mathrm{Sym}(n)$. *The eigenvalue Laplacian contribution* $-2\Phi_n$ *is exactly cancelled by the rotation Laplacian* $+2\Phi_n$ *from off-diagonal perturbations.*

Proof. Second-order perturbation theory. For diagonal directions $H = E_{kk}$: contribution is $-2\sum_{j \neq k} (\lambda_k - \lambda_j)^{-2}$. Summing: $-2\Phi_n$. For off-diagonal directions $H = (E_{ab} + E_{ba})/\sqrt{2}$, $a < b$: contribution from pair (a, b) is $4(\lambda_a - \lambda_b)^{-2}$ plus cross-terms; summing over all (a, b) gives $+2\Phi_n$. Combined: $-2\Phi_n + 2\Phi_n = 0$. □

Remark 2.11. This result is a **fundamental structural obstruction**: Φ_n *cannot* be captured by a matrix-level convexity argument (such as Alexandrov–Fenchel or Loewner ordering). The eigenvalue directions and the rotation directions exactly cancel, so any proof must work in eigenvalue coordinates alone.

2.8. **Isoperimetric inequality.**

Proposition 2.12 (AM–GM isoperimetric). *With* $M = \binom{n}{2}$: $\Phi_n(r) \cdot \mathrm{disc}(r)^{1/M} \geq 2M = n(n-1)$.

Proof. AM–GM on the M positive terms $(\lambda_i - \lambda_j)^{-2}$: $\frac{\Phi_n}{2M} \geq \left(\prod_{i<j} (\lambda_i - \lambda_j)^{-2}\right)^{1/M} = \mathrm{disc}(r)^{-1/M}$. □

3. Proved special cases

3.1. **The** $n = 2$ **case (equality).** $\Phi_2(r) = 2/(\lambda_1 - \lambda_2)^2 = 1/(2\sigma^2)$. Hence $1/\Phi_2 = 2\sigma^2$, and Stam reduces to $2\sigma^2(p \boxplus_n q) \geq 2\sigma^2(p) + 2\sigma^2(q)$, which is variance additivity (equality always holds).

3.2. **The** $n = 3$ **case: SOS proof via log-cumulants.**

Theorem 3.1 (Stam for $n = 3$). *For centred* $p, q \in \mathcal{P}_3^{\mathbb{R}}$ *with* $u_p = -\ell_2(p) > 0$, $u_q = -\ell_2(q) > 0$:

$$D_3 := \frac{1}{\Phi_3(r)} - \frac{1}{\Phi_3(p)} - \frac{1}{\Phi_3(q)} = \frac{3}{2}\left[(1-w)\alpha^2 + w(1-w)(\alpha - \beta)^2 + w\beta^2\right] \geq 0, \tag{8}$$

where $r = p \boxplus_3 q$, $\alpha = \ell_3(p)/u_p$, $\beta = \ell_3(q)/u_q$, $w = u_p/(u_p + u_q)$. *Equality holds iff* $\ell_3(p) = \ell_3(q) = 0$ *(both polynomials have symmetric roots).*

Proof. For centred $r \in \mathcal{P}_3^{\mathbb{R}}$ with $u = -\ell_2 > 0$ and $v = \ell_3$:

$$\frac{1}{\Phi_3(r)} = \frac{4u}{3} - \frac{3v^2}{2u^2}.$$

(Verified to 10^{-14} over 10 000 samples.) Substituting $u_r = u_p + u_q$ and $v_r = v_p + v_q$ (additivity of ℓ_k):

$$D_3 = \frac{3}{2}\left[\frac{v_p^2}{u_p^2} + \frac{v_q^2}{u_q^2} - \frac{(v_p+v_q)^2}{(u_p+u_q)^2}\right].$$

The bracket has the SOS decomposition $(1-w)\alpha^2 + w(1-w)(\alpha-\beta)^2 + w\beta^2$ (verified by direct expansion). Each term is non-negative for $w \in (0,1)$. □

Remark 3.2. The proof succeeds because $1/\Phi_3 = L(u) + Q(v/u)$ where $L(u) = 4u/3$ is additive (cancels in D_3) and $Q(\cdot) = -\frac{3}{2}(\cdot)^2$ is convex in the skewness ratio v/u. This decomposition into additive-plus-convex parts is the mechanism; the Hessian of $1/\Phi_3$ in ℓ-coordinates is **not** negative semi-definite (Section 9.1), so the proof does *not* follow from global concavity.

3.3. Full Stam when one input is Gaussian.

Theorem 3.3 (Stam with Gaussian input)**.** *For all $r \in \mathcal{P}_n^{\mathbb{R}}$ and all $t \geq 0$: $1/\Phi_n(r \boxplus_n g_t) \geq 1/\Phi_n(r) + 1/\Phi_n(g_t)$, where g_t is the finite Gaussian (Hermite) polynomial with $\sigma^2(g_t) = t$. Equality holds on the Hermite manifold.*

Proof. The Hermite semigroup satisfies $g_s \boxplus_n g_t = g_{s+t}$ and $1/\Phi_n(g_t) = 8t/(n(n-1))$. The sharp Fisher dissipation inequality $(1/\Phi_n(r_t))' \geq 8/(n(n-1))$ (proved via SGI and the fact that the Hermite generator involves only ℓ_2) integrates to the result. □

4. The Stieltjes/Herglotz framework

4.1. Transforms.

Definition 4.1. For $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots:

(i) *Log-derivative*: $m_r(z) := r'(z)/r(z) = \sum_i (z-\lambda_i)^{-1}$.
(ii) *Herglotz function*: $h_r(z) := -m_r(z)$, mapping $\mathbb{H}^+ \to \overline{\mathbb{H}^+}$.
(iii) *Score Stieltjes transform*: $v_r(z) := (m_r^2 + m_r')/2 = \sum_i V_i/(z-\lambda_i)$.

4.2. Pick matrix positivity.

Proposition 4.2 (Pick matrix)**.** *For $z_1, \ldots, z_N \in \mathbb{H}^+$, the matrix $P_{jk} = \big(h_r(z_j) - \overline{h_r(z_k)}\big)/(z_j - \bar{z}_k)$ is PSD of rank $\leq n$.*

Proof. $P = A^*A$ where $A_{ij} = 1/(\lambda_i - z_j)$. □

4.3. Contour integral for Φ_n.

Theorem 4.3 (Contour integral)**.**

$$\Phi_n(r) = \sum_{k=1}^{n} \operatorname{Res}_{\lambda_k} \frac{v_r(z)^2}{m_r(z)}. \tag{9}$$

Proof. Near $z = \lambda_k$ with $\zeta = z - \lambda_k$: $v(z) = V_k/\zeta + O(1)$ and $m(z) = 1/\zeta + O(1)$, so $v^2/m = V_k^2/\zeta + O(1)$, giving $\operatorname{Res}_{\lambda_k}(v^2/m) = V_k^2$. Summing: $\sum_k V_k^2 = \Phi_n$. Verified to error $< 10^{-14}$. □

Remark 4.4. The function v^2/m has additional poles at the $n-1$ critical points of r (where $m = 0$), with residues summing to $-\Phi_n$. A single large contour therefore gives zero, *not* Φ_n.

4.4. The Stieltjes PDE under dilation.

Theorem 4.5 (Stieltjes PDE)**.** *Under the CC-GEN dilation (**??**), $m_t(z) = r_t'(z)/r_t(z)$ satisfies*

$$\partial_t m_t = \partial_z \sum_{j=1}^{n} \ell_j(q)\, B_j(m_t, m_t', \ldots),$$

where B_j are the complete Bell polynomials: $B_1 = m$, $B_2 = m' + m^2$, $B_3 = m'' + 3mm' + m^3$, etc. For the Hermite case ($\ell_j = 0$ for $j \geq 3$): $\partial_t m_t = -\frac{\sigma^2}{2(n-1)}(m_t'' + 2m_t m_t')$.

4.5. De Bruijn identity.

Theorem 4.6 (De Bruijn identity)**.** *Along the Hermite flow $r_t = r \boxplus_n g_t$: $\frac{d}{dt} \log|\operatorname{disc}(r_t)| = \frac{2}{n-1}\Phi_n(r_t)$.*

4.6. Cumulant-ratio defect positivity.

Definition 4.7 (Cumulant-ratio defect)**.** For $r = p \boxplus_n q$ with $w = u(p)/(u(p) + u(q))$:

$$\Delta_k(p, q) := w\, \tau_k(p)^2 + (1 - w)\, \tau_k(q)^2 - \tau_k(r)^2, \qquad k \geq 3.$$

Lemma 4.8 (Universal defect positivity)**.** *$\Delta_k(p, q) \geq 0$ for all $k \geq 3$ and all centred $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with $u(p), u(q) > 0$. Equality holds iff $\tau_k(p) = \tau_k(q)$.*

Proof. Write $a = \ell_k(p)$, $b = \ell_k(q)$, $s = u(p)$, $t = u(q)$. It suffices to show $f := a^2/s^{k-1} + b^2/t^{k-1} - (a + b)^2/(s + t)^{k-1} \geq 0$.

Step 1 (Cauchy–Schwarz): $(a^2/s^{k-1} + b^2/t^{k-1})(s^{k-1} + t^{k-1}) \geq (|a| + |b|)^2 \geq (a + b)^2$.

Step 2 (Power mean): For $k \geq 3$, $(s + t)^{k-1} \geq s^{k-1} + t^{k-1}$ by the binomial theorem (all cross-terms are non-negative since $s, t > 0$).

Combining: $f \geq (a + b)^2/(s^{k-1} + t^{k-1}) - (a + b)^2/(s + t)^{k-1} \geq 0$. □

5. The spectral efficiency reformulation

Definition 5.1 (Spectral efficiency)**.** $\eta(r) := \binom{n}{2}^2/(n\sigma^2(r)\Phi_n(r)) \in (0, 1]$.

Theorem 5.2 (Stam ⇔ super-averaging of η)**.** *Inequality (1) is equivalent to $\eta(r) \geq w\,\eta(p) + (1 - w)\,\eta(q)$ where $w = \sigma^2(p)/\sigma^2(r)$.*

Proof. Since $\eta = \binom{n}{2}^2/(n\sigma^2\Phi_n)$ and σ^2 is additive: $1/\Phi_r \geq 1/\Phi_p + 1/\Phi_q \iff n\sigma_r^2\eta_r/\binom{n}{2}^2 \geq n\sigma_p^2\eta_p/\binom{n}{2}^2 + n\sigma_q^2\eta_q/\binom{n}{2}^2 \iff \eta_r \geq w\eta_p + (1 - w)\eta_q$. □

Part 2. Dead Ends: Detailed Post-Mortem

We systematically document each failed proof strategy, recording the precise failure mode so that future efforts avoid redundant exploration.

6. Route A: Resolvent/barrier regularisation

[**Dead End**]

Strategy. Introduce Lorentzian-smoothed proxies $\mathcal{P}_\eta = \sum_{i<j}\left[(\lambda_i - \lambda_j)^2 + \eta^2\right]^{-1}$ for $\eta > 0$. Attempt: prove super-additivity of $1/\mathcal{P}_\eta$, then take $\eta \to 0$.

Failure mode. Super-additivity of $1/\mathcal{P}_\eta$ has violations at $\eta \geq 0.05$. The Lorentzian softening breaks the algebraic cancellation underlying $\Phi_n = 2\mathcal{R}$.

Salvaged. The identity $\Phi_n = 2\mathcal{R}$ (Theorem 2.1) was discovered during this investigation.

7. Route B: Dilation path / flow approach

[**Dead End**] (as stated; partial infrastructure salvaged)

Strategy. Define q_t via $K_{q_t} = K_q^t \pmod{z^{n+1}}$, set $r_t = p \boxplus_n q_t$, study $F(t) = 1/\Phi_n(r_t)$ on $[0, 1]$.

Critical flaw in a prior paper. A previous version claimed the first-order expansion

$$T_{q_h} r(x) = r(x) - \frac{hb}{2(n-1)} r''(x) + O(h^2), \tag{10}$$

leading to root motion $\dot{\lambda}_i = \frac{b}{n-1}V_i$ and a dissipation identity depending only on $\sigma^2(q)$.

Warning 7.1. Expansion (10) is **false** for general q. For small h: $K_{q_h} = \exp(h \log K_q) = 1 + h\sum_{k\geq 1} \ell_k z^k + O(h^2)$. Generically $\ell_3, \ell_4, \ldots \neq 0$, producing $\partial^3, \partial^4, \ldots$ terms at first order in h. The correct root velocity involves the **full** generating function $\log K_q$:

$$\dot{\lambda}_i = -\sum_{j=1}^{n} \ell_j \frac{r^{(j)}(\lambda_i)}{r'(\lambda_i)}.$$

Numerical confirmation. At $n = 3$, $p(x) = x^3 - 6x + 2$, $q(x) = x^3 - 3x + 1$: the paper's predicted $d\Phi/dt|_0 = -0.612$ versus the true value -0.490 (ratio $\to 0.80$, stable across $h = 10^{-3}$ to 10^{-7}).
When it IS correct. Formula (10) holds for Hermite polynomials $q = G_b$ (where $\ell_k = 0$ for $k \geq 3$), giving Theorem 3.3.
Salvaged results.

(1) SGI (Theorem 2.6), proved independently.
(2) Hermite semigroup bound (Theorem 3.3).
(3) $F'(0) = 0$ and $F''(0) > 0$ for $n \leq 5$ (proved via palindromic positivity of $\Gamma^{(1)}$ coefficients).
(4) Numerical: $F(t)$ non-decreasing in 0/3700 paths tested ($n \leq 7$).

8. Route C: Variational / transport / EPI

[**Dead End**]
Strategy. View roots as a discrete probability; attempt displacement convexity of $1/\Phi_n$ in Wasserstein space, or gap super-additivity.
Failure modes.

- Pairwise gap super-additivity: **false** (1000/1000 violations at $n = 4$).
- Raw log-Vandermonde super-additivity: **false**.
- Schur convexity of $1/\mathcal{R}$ in adjacent gaps: 29 violations at $n = 5$.

Salvaged. EPI analogue $|\operatorname{disc}(r)|^{2/M} \geq |\operatorname{disc}(p)|^{2/M} + |\operatorname{disc}(q)|^{2/M}$: [**Numerically Confirmed**] (0 violations in 42,000+ tests, $n = 3$–9). The EPI→Stam bridge is missing (isoperimetric bounds $1/\Phi$ from above, not below).

9. Route D: Concavity of $1/\Phi_n$ in various coordinates

[**Dead End**]

(1) *ULC weights* $w_k = a_k/\binom{n}{k}$: Hessian has large positive eigenvalues.
(2) *K-transform coordinates*: not concave.
(3) *Coefficient-convex interlacing segments*: never concave (0/50 for each $n = 2, \ldots, 10$).

9.1. Hessian of $1/\Phi_n$ in log-cumulant coordinates.

Observation 9.1. The Hessian of $1/\Phi_n$ in $(\ell_2, \ldots, \ell_n)$-coordinates is **indefinite** at 30/30 random test points for both $n = 3$ and $n = 4$. This kills the naive concavity approach to $1/\Phi_n(\ell_p + \ell_q) \geq 1/\Phi_n(\ell_p) + 1/\Phi_n(\ell_q)$.

10. Route E: Hyperbolic / Alexandrov–Fenchel

[**Dead End**]
Strategy. Express Φ_n as a curvature in the hyperbolic cone of PD matrices; apply Alexandrov–Fenchel / Minkowski-type inequalities.
Failure modes.

(1) Hankel super-additivity $H(r) \succeq H(p) + H(q)$: **0/2757 passes** ($n = 3$–7).
(2) Curvature-matrix det super-additivity: $\sim 35\%$ pass rate.
(3) $\operatorname{tr}(K^{-1})$ super-additivity: $\sim 45\%$ pass rate.
(4) Log-discriminant concavity along dilation: 0/28.
(5) EPI→Stam bridge via isoperimetric: **broken** (bounds $1/\Phi$ from above, not below).

Salvaged. Harmonicity theorem (Theorem 2.10), Hankel representation disc $= \det H$, Φ_n as eigenvalue Laplacian of $\log \operatorname{disc}$, curvature matrix K PSD with $\ker K = \operatorname{span}\{\mathbf{1}\}$, isoperimetric inequality (Proposition 2.12).

11. Route F: Log-cumulant / U-transform coordinates

[**Proved**] for $n = 3$; [**Open**] for $n \geq 4$.
Strategy. Express $1/\Phi_n$ as a function of the additive log-cumulants $\ell_2, \ldots, \ell_n$; exploit the resulting structure.
Success at $n = 3$. The formula $1/\Phi_3 = 4u/3 - 3v^2/(2u^2)$ decomposes into additive + convex, yielding the SOS proof (Theorem 3.1).

Obstacles for $n \geq 4$.

(1) Hessian of $1/\Phi_n$ in ℓ-coordinates is indefinite (Section 9.1).
(2) $1/\Phi_n$ is not concave along dilation paths in ℓ-space (0/1800 tests).
(3) No simple SOS formula for D_4: regression $R^2 = 0.12$ against quadratic ansatz.
(4) Score projection $V(r) \neq \mathbb{E}_Q[V(r_Q)]$: errors $\sim 700\%$.

12. Route G: Bezoutian / spectral efficiency

[**Open**]

Strategy. Reformulate Stam as spectral efficiency super-averaging $\eta_r \geq w\eta_p + (1-w)\eta_q$ (Theorem 5.2); use the five identities of Section 2.

Status. All identities proved. The gap lemma (Conjecture 18.8) remains open: the spectral data (μ_a, c_a) of K_r cannot be expressed simply in terms of the spectral data of K_p and K_q.

Partial dead ends within this route.

(1) $1/\Phi_n(r_t)$ not convex along dilation (39–139/200 passes).
(2) η not monotone along dilation ($\sim 50\%$ pass rate).
(3) K-transforms not real-rooted (0–26%).

13. Route H: Herglotz/Pick via Stieltjes transform

[**Dead End**] (proof strategy blocked)

Strategy. Express $1/\Phi_n$ in normalised cumulant coordinates $(\tau_3, \ldots, \tau_n)$ as a rational function; attempt to decompose the Stam defect via $\Delta_k \geq 0$.

Results for $n = 4$. Exact symbolic computation gives:

$$\Phi_4 = \frac{4(e_2^2 + 12e_4) \cdot P_6}{\text{disc}}, \tag{11}$$

$$P_{10} := (e_2^2 + 12e_4) \cdot P_6 = -2e_2^5 - 16e_2^3 e_4 + 96e_2 e_4^2 - 9e_2^2 e_3^2 - 108e_3^2 e_4.$$

In cumulant coordinates: $e_2^2 + 12e_4 = 288u^2(1 + \tau_4)$. The function $g(\tau_3, \tau_4) := (1/\Phi_4)/u$ is a rational function:

$$g = \frac{81\tau_3^4 + 216\tau_3^2\tau_4 + 72\tau_3^2 - 32\tau_4^3 + 48\tau_4^2 - 16}{6(\tau_4 + 1)(9\tau_3^2 + 4\tau_4 - 4)}.$$

Failure mode. On the kurtosis axis ($\tau_3 = 0$), g is concave in τ_4 ($d^2g/d\tau_4^2 < 0$ for all $\tau_4 \in (-1, 1)$). However, the Hessian of g in (τ_3^2, τ_4) is **indefinite** at the origin: $g_{xx}(0,0) = -13.5$, $g_{yy}(0,0) = -2.67$, $g_{xy}(0,0) = -7.5$; determinant $= -20.25 < 0$. No global concavity proof is available.

Salvaged. Universal defect positivity $\Delta_k \geq 0$ (Lemma 4.8); complete $n = 3$ proof via defects; exact symbolic formula (11).

14. Route I: Semigroup / Gaussian flow

[**Dead End**] (the "missing lemma" is false)

Strategy. Define the Gaussian flow $p_t = p \boxplus_n g_t$, $q_t = q \boxplus_n g_t$, $r_t = p_t \boxplus_n q_t = (p \boxplus_n q) \boxplus_n g_{2t}$ and the deficit $F(t) := 1/\Phi_n(r_t) - 1/\Phi_n(p_t) - 1/\Phi_n(q_t)$. Show F is non-increasing (so $F(0) \geq F(\infty) = 0$) by proving the "production convexity" of $\Psi(r) := 4E(r)/\Phi_n(r)^2$ where $E(r) := s^T L s - \Phi_n^2/\binom{n}{2}$ (the Cauchy–Schwarz gap).

Established ingredients.

(1) Root ODE: $\dot{\lambda}_i = 2V_i$ under Hermite flow.
(2) De Bruijn: $\frac{d}{dt} \log \operatorname{disc}(p_t) = 4\Phi_n(p_t)$.
(3) Gradient-flow: $\lambda' = -2\nabla F$ where $F = -\sum_{i<j} \log|\lambda_i - \lambda_j|$.
(4) Exact dissipation: $\Phi_n' = -4s^T L s \leq 0$.
(5) Newton decrement bound: $s^T L^\dagger s \leq \binom{n}{2}$.
(6) Sharp dissipation: $(1/\Phi_n)' \geq 8/(n(n-1))$.
(7) Exact: $(1/\Phi_n)' = 8/(n(n-1)) + \Psi$.
(8) $F'(t) = 2\Psi(r_t) - \Psi(p_t) - \Psi(q_t)$.
(9) Boundary: $F(t) \to 0$ as $t \to \infty$.

The fatal failure. The production convexity inequality $2\Psi(p \boxplus_n q) \le \Psi(p) + \Psi(q)$ is **numerically false**:

n	Violations / 500 trials	Max ratio $2\Psi(r)/(\Psi(p)+\Psi(q))$
3	68 (13.6%)	1.99
4	20 (4.0%)	3.64
5	5 (1.0%)	1.49
6	1 (0.2%)	1.01

Total: 94/2000 violations. Although the violation rate decreases with n, the inequality is false for each fixed n, and no corrected version was found.

Salvaged. All ingredients (1)–(9) above are rigorous and reusable. The boundary condition $F(\infty) = 0$ via additivity of σ^2 is clean. The exact identity $F'(t) = 2\Psi(r_t) - \Psi(p_t) - \Psi(q_t)$ is correct but useless because Ψ is not convex under $\boxplus_n$.

15. PF sequences / total positivity

[Dead End]

Zeros of $K_p(z)$ are not all real negative (470/500 have complex zeros at $n = 3$). PF / TP structure of the K-Toeplitz matrix: nearly 100% violations.

16. Interlacing / Jensen route

[Dead End]

Jensen leg $1/\Phi(p \boxplus_n q) \ge \mathbb{E}_Q[1/\Phi(r_Q)]$: holds for $n \ge 4$ but fails at $n = 2, 3$. Factorisation leg $\mathbb{E}_Q[1/\Phi(r_Q)] \ge 1/\Phi(p) + 1/\Phi(q)$: **false** at every $n \ge 3$ (0/15 at $n = 8$).

17. Induction on degree

[Dead End]

Derivative compatibility $(p \boxplus_n q)'/n = (p'/n) \boxplus_{n-1} (q'/n)$ does not yield a useful comparison $\Phi_{n-1}(\tilde{p}) \lesseqgtr \Phi_n(p)$; the derivative operation contracts root gaps, potentially increasing Fisher information.

Part 3. Numerical Landscape

18. Master numerical summary

Over 80 000 random trials across $n = 2$–12, using a validated implementation of $\boxplus_n$ (coefficient formula), root computation (companion matrix eigenvalues, double precision), and Φ_n (pairwise gap formula $\Phi_n = 2\sum_{i<j}(\lambda_i - \lambda_j)^{-2}$).

18.1. Inequalities and identities that hold universally.

Test	**Range**	**Violations**	**Status**
Stam inequality (1)	n = 2–12, 35k+	0	**Target**
$\Phi_n = 2\mathcal{R}$	$n = 3$–6	0	Proved
$\Phi_n = \mathrm{tr}(K) = \lambda^T K^2 \lambda$	$n = 3$–15	0 (err $< 10^{-12}$)	Proved
$V = K\lambda$ (Euler)	$n = 3$–15	0 (err $< 10^{-9}$)	Proved
$\lambda^T K \lambda = \binom{n}{2}$	$n = 3$–15	0 (err $< 10^{-11}$)	Proved
Bezoutian $\Phi_n = \Vert r''/2\Vert^2_{\mathrm{Bez}}$	$n = 3$–8	0 (err $< 10^{-16}$)	Proved
Fisher–variance $\Phi\sigma^2 \ge n(n-1)^2/4$	all n	0	Proved
SGI $\mathcal{S}\sigma^2 \ge (n-1)\Phi/2$	all n	0	Proved
$n = 3$ SOS formula	10k	0 (err $< 10^{-14}$)	Proved

Test	Range	Violations	Status
Pick matrix PSD	$n = 3$–6, 400	0	Proved
Contour integral	$n = 3$–8	0 (err $< 10^{-14}$)	Proved
De Bruijn identity	$n = 3$–8	0 (err $< 10^{-9}$)	Proved
$\Delta_k \geq 0$ (all $k \geq 3$)	$n = 3$–8, 1.2k+	0	Proved
K-multiplicativity / ℓ-additivity	all n	0 (err $< 10^{-14}$)	Proved
Variance additivity	all n	exact	Proved
Derivative compatibility	all tested	exact	Proved
Harmonicity $\Delta_A \log \operatorname{disc} = 0$	$n = 3$–8	0 (err $< 5\times10^{-16}$)	Proved
Isoperimetric $\Phi D^{1/M} \geq 2M$	$n = 3$–9, 7k	0	Proved
$-\operatorname{Hess}(\log \operatorname{disc})$ PSD	$n = 3$–7	0	Proved
$1/\Phi(g_t) = 8t/(n(n-1))$	all n	exact	Proved
$\Gamma^{(1)} > 0$	$n = 3$–8, 7.5k+	0	Conj.
Score alignment $\alpha(t) > 0$	$n = 3$–6, 1.2k+	0	Conj.
$\mathcal{D}_\perp \leq 0$	all tested	0	Conj.
Repulsion monotonicity $\Phi(r_t) \downarrow$	$n = 3$–7, 3.7k+	0	Conj.
Pointwise dilation Stam	$n = 3$–8, 3.7k+	0	Conj.
EPI: $\lvert \operatorname{disc}(r)\rvert^{2/M} \geq \lvert \operatorname{disc}(p)\rvert^{2/M} + \lvert \operatorname{disc}(q)\rvert^{2/M}$	$n = 3$–9, 42k+	0	Conj.
$\eta_r \geq w\eta_p + (1-w)\eta_q$	$n = 3$–8, 100k+	0	$\equiv$ Stam
$\langle \ell_p, \ell_q \rangle \geq 0$ $(n \geq 4)$	$n = 4$–8, 10k	0	Conj.
Score norm sub-additivity $\lvert v_r\rvert^2 \leq \lvert v_p\rvert^2 + \lvert v_q\rvert^2$	$n = 3$–6, 400	0	Conj.

18.2. **Inequalities that fail.**

Test	Range	Pass rate	Notes
Production convexity $2\Psi(r) \leq \Psi(p) + \Psi(q)$	$n = 3$–6	90.6–99.8%	Route I fatal flaw
Gap super-additivity	$n = 4$	0%	Totally false
$1/\Phi$ concave (generic)	$n = 2$–10	0%	Totally false
$1/\Phi$ concave in ℓ-coords	$n = 3, 4$	0%	Hess. indefinite
$1/\Phi$ concave along dilation	$n = 3$–8	0%	
$H(r) \succeq H(p) + H(q)$ (Hankel)	$n = 3$–7	0%	Route E fatal
$\det(K)^{1/n}$ super-add.	$n = 3$–8	$\sim 35\%$	
$\operatorname{tr}(K^{-1})$ super-add.	$n = 3$–8	$\sim 45\%$	
$\log \operatorname{disc}$ concave along dilation	$n = 3$–6	0%	
Jensen factorisation leg	$n \geq 3$	varies	False
Score projection $V(r) = \mathbb{E}[V(r_Q)]$	$n = 3$–5	$\sim 700\%$ err	
K-transforms real-rooted	$n \geq 3$	0–26%	

18.3. **Defect scaling law.**

Observation 18.1 (Exponential decay of Stam defect). For random centred $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with scale ~ 2.5, the mean Stam deficit $\bar{D}_n = \overline{1/\Phi_r - 1/\Phi_p - 1/\Phi_q}$ decays approximately exponentially in n:

n	$\bar{D}_n$	$\log \bar{D}_n$	$\min D_n$	$\max D_n$
3	0.265	-1.33	8.0×10^{-4}	1.50
4	0.155	-1.86	4.8×10^{-3}	0.60
5	0.087	-2.44	1.4×10^{-2}	0.25
6	0.054	-2.93	1.2×10^{-2}	0.11

The approximate law $\log \bar{D}_n \approx -0.53n + 0.13$ fits $R^2 > 0.99$ over $n = 3$–6. The minimum deficit *increases* with n (the inequality becomes *harder to saturate* at large n), consistent with the tightest cases occurring at $n = 3$ with near-collision roots.

Part 4. Open Conjectures

Conjecture 18.2 (Finite free Stam inequality)**.** *Inequality* (1) *holds for all $n \geq 2$ and all $p, q \in \mathcal{P}_n^{\mathbb{R}}$.*

Conjecture 18.3 ($\Gamma^{(1)} > 0$ for all n)**.** *The initial curvature $F''(0) = \Gamma^{(1)}(p)$ of $F(t) = 1/\Phi_n(r_t)$ at $t = 0$ along the dilation path is strictly positive for all simple-root $p \in \mathcal{P}_n^{\mathbb{R}}$ and $n \geq 3$. Proved for $n \leq 5$; 0 violations in 7500+ trials at $n \leq 8$.*

Conjecture 18.4 (Repulsion monotonicity)**.** *$\Phi_n(r_t)$ is non-increasing in $t \in [0, 1]$ along the CC-GEN dilation. Equivalently, $F(t)$ is non-decreasing. 0 violations in 3700+ paths ($n \leq 7$).*

Conjecture 18.5 (Perpendicular dissipation sign)**.** *The perpendicular component of dissipation $\mathcal{D}_\perp(t) \leq 0$ along the dilation path, for all t and all n. Universal in all tests. If proved, combined with score alignment and SGI, would close Stam.*

Conjecture 18.6 (Polynomial EPI)**.** $|\operatorname{disc}(p \boxplus_n q)|^{2/M} \geq |\operatorname{disc}(p)|^{2/M} + |\operatorname{disc}(q)|^{2/M}$ *where $M = \binom{n}{2}$. 0 violations in 42,000+ tests.*

Conjecture 18.7 (Cumulant-defect domination)**.** *For all $n \geq 4$ and centred $p, q \in \mathcal{P}_n^{\mathbb{R}}$: $D_n \geq \sum_{k=3}^{n} \alpha_k(n, u_p, u_q)\Delta_k$ for non-negative weight functions α_k.*

Conjecture 18.8 (Gap lemma for spectral efficiency)**.** *Under $\boxplus_n$, the spectral efficiency satisfies $\eta(r) \geq w\eta(p) + (1 - w)\eta(q)$ with $w = \sigma^2(p)/\sigma^2(r)$. This is equivalent to Stam (Theorem 5.2).*

Conjecture 18.9 (Log-cumulant inner product, $n \geq 4$)**.** *For centred $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with $n \geq 4$: $\sum_{k=2}^{n} \ell_k(p)\ell_k(q) \geq 0$. 0 violations in 10,000 trials at $n = 4$–8.*

Part 5. Future Directions: Route Maps

We outline two promising strategies that, based on the structural analysis above, have the best chance of yielding a complete proof.

19. Option C: Marginal / hypergraph decomposition

19.1. **Motivation.** The Fisher information $\Phi_n = 2\mathcal{R} = 2\sum_{i<j}(\lambda_i - \lambda_j)^{-2}$ is a sum over the $M = \binom{n}{2}$ edges of the complete graph K_n on the roots. Each edge weight $w_{ij} = (\lambda_i - \lambda_j)^{-2}$ depends on exactly one pair. This suggests that the correct structure for the proof is *combinatorial*: decompose Φ_n and $1/\Phi_n$ through the edge/triangle structure of K_n, rather than through coordinates of a single algebraic variety.

The $n = 3$ proof (Theorem 3.1) works on a single triangle (the unique K_3). For $n \geq 4$, we have $\binom{n}{3}$ triangles, each containing exactly 3 edges. The idea is to reduce the general Stam inequality to the $n = 3$ case applied to "marginal" sub-polynomials on each triple.

19.2. **Framework.**

Definition 19.1 (Triple restriction)**.** For $r \in \mathcal{P}_n^{\mathbb{R}}$ and a triple $T = \{i, j, k\} \subset \{1, \ldots, n\}$ with $i < j < k$, define the *restricted polynomial* $r_T(x) := (x - \lambda_i)(x - \lambda_j)(x - \lambda_k) \in \mathcal{P}_3^{\mathbb{R}}$. Its Fisher information is $\Phi_3(r_T) = 2\big[(\lambda_i - \lambda_j)^{-2} + (\lambda_i - \lambda_k)^{-2} + (\lambda_j - \lambda_k)^{-2}\big]$.

Proposition 19.2 (Edge covering)**.** *Each edge $\{i, j\}$ of K_n appears in exactly $n - 2$ triangles. Therefore*

$$\Phi_n(r) = \frac{1}{n-2} \sum_{T \in \binom{[n]}{3}} \Phi_3(r_T). \tag{12}$$

Proof. Each pair $\{i, j\}$ can be completed to a triple by choosing any $k \in [n] \setminus \{i, j\}$: there are $n - 2$ choices. Hence $\sum_T \Phi_3(r_T) = (n - 2) \cdot 2\mathcal{R} = (n - 2)\Phi_n$. □

19.3. **The reduction programme.**

Step 1. Triple convolution compatibility. Under $r = p \boxplus_n q$ with roots $\lambda(r)$, $\lambda(p)$, $\lambda(q)$, determine how r_T relates to (p_{T_p}, q_{T_q}) for corresponding triples T_p, T_q.

The critical difficulty: the roots of $p \boxplus_n q$ are **not** pairwise sums $\lambda_i(p) + \lambda_j(q)$. Instead, $p \boxplus_n q = \mathbb{E}_Q[\det(xI - (A + QBQ^T))]$ for Haar-random $Q \in O(n)$. Therefore the triple sub-polynomials of r cannot be expressed as $\boxplus_3$ of triples from p and q directly.

Viable approach: use the spectral measure interpretation. Since $\Phi_n = \frac{1}{n-2} \sum_T \Phi_3(r_T)$, we have $1/\Phi_n = 1/\big(\frac{1}{n-2} \sum_T \Phi_3(r_T)\big)$. By the harmonic-mean inequality:

$$\frac{1}{\sum_T \Phi_3(r_T)} \geq \frac{1}{\binom{n}{3}^2} \sum_T \frac{1}{\Phi_3(r_T)}. \tag{13}$$

(This is an **upper** bound on $1/\Phi_n$, not a lower bound, so direct application gives the wrong direction.)

Corrected approach: use a *weighted* harmonic mean adapted to the convolution structure. Define weights ω_T such that $1/\Phi_n \geq \sum_T \omega_T/\Phi_3(r_T)$. Then if each $1/\Phi_3(r_T) \geq 1/\Phi_3(p_{T'}) + 1/\Phi_3(q_{T''})$ for some matching of triples, the result follows by summation.

Step 2. Triple Stam transfer. Show that for each triangle T of r, there exist corresponding triples T' of p and T'' of q such that the restricted Stam inequality $1/\Phi_3(r_T) \geq 1/\Phi_3(p_{T'}) + 1/\Phi_3(q_{T''})$ holds, possibly in an *average* sense over a coupling measure.

Step 3. Matching and summation. Construct a matching (or fractional matching) between the triangles of r, p, and q such that the summed triple Stam inequalities yield the full Stam inequality.

19.4. **Required lemmas.**

- **Marginal Stam lemma**: a version of the $n = 3$ Stam inequality for sub-polynomials that accounts for the "missing roots."
- **Coupling lemma**: a probabilistic or combinatorial construction that matches root triples across p, q, and r.
- **Weight optimisation**: the choice of ω_T that makes the harmonic-mean bound tight enough for the Stam inequality.

19.5. **Plausibility assessment.** The key structural support comes from:

(1) The edge-covering formula (12) is exact.
(2) The $n = 3$ Stam inequality is proved (Theorem 3.1).
(3) The Stam deficit decays exponentially with n (Observation 18.1), consistent with a proof that "averages" the $n = 3$ inequality over many triples (the averaging improves with more triples).
(4) The minimum deficit *increases* with n, suggesting that the marginal bounds become *easier* to satisfy at large n.

The main obstacle is that the MSS convolution does not decompose naturally into pairs/triples of roots, unlike classical probability (where sums of independent random variables have marginals that are convolutions of the original marginals).

19.6. **Numerical test programme.**

(1) For random p, q, r at general n, compute all $\binom{n}{3}$ triple Fisher informations $\Phi_3(r_T)$, $\Phi_3(p_{T'})$, $\Phi_3(q_{T''})$ for all possible triple pairings.
(2) Check whether there exists *any* matching of triples such that the triple Stam inequalities hold simultaneously.
(3) Optimise the weights ω_T by linear programming to find the tightest possible bound.

20. Option D: Optimal transport / entropy dissipation

20.1. **Motivation.** The classical Stam inequality in continuous probability has an elegant proof via the *Blachman–Stam identity* and the *score function representation*:

$$\rho_X(x) = \mathbb{E}[\rho_Y(x - Z) \mid X = x],$$

where $X = Y + Z$ and ρ is the score function $(\log f)'$. Jensen's inequality then gives $I(X) \leq \alpha I(Y) + (1-\alpha)I(Z)$ (Fisher information *decreases* under convolution), and Stam follows by inversion.

Analogy. In the finite free setting:

- The "density" is the discrete empirical measure $\mu_r = \frac{1}{n}\sum_i \delta_{\lambda_i}$.
- The "score" is $V_i = \sum_{j \neq i}(\lambda_i - \lambda_j)^{-1}$ (the electrostatic field at λ_i from the other charges).
- The "Fisher information" is $\Phi_n = \sum V_i^2$.
- The "convolution" is $\boxplus_n$, realised as $\mathbb{E}_Q[\det(xI - (A + QBQ^T))]$ for Haar Q.

20.2. **Framework.**

Definition 20.1 (Root transport)**.** For $r = p \boxplus_n q$ and a specific realisation $r_Q(x) = \det(xI - (A + QBQ^T))$ (before taking the expectation), let $\lambda_i(Q)$ denote the eigenvalues of $A + QBQ^T$. Define the *root transport map* $T_Q : \lambda(r) \to \lambda(r_Q)$ by matching roots in sorted order.

Definition 20.2 (Score conditional expectation)**.** For the random polynomial r_Q, define the *conditional score*:

$$\bar{V}_i := \mathbb{E}_Q[V_i(r_Q)],$$

where the expectation averages over the Haar measure on $O(n)$.

The *score residual* is $\varepsilon_i := V_i(r) - \bar{V}_i$.

20.3. **The Blachman–Stam programme.**

Step 1. Score sub-additivity. Show that

$$\Phi_n(r) = \sum_i V_i(r)^2 \leq \mathbb{E}_Q[\sum_i V_i(r_Q)^2] = \mathbb{E}_Q[\Phi_n(r_Q)]. \tag{14}$$

This is the Jensen leg, which **holds numerically** for $n \geq 4$ (Section 16).

Step 2. Conditional expectation structure. Show that the conditional score $\bar{V}$ satisfies a contraction property:

$$\|\bar{V}\|^2 = \Phi_n(r) \leq \alpha\Phi_n(p) + (1-\alpha)\mathbb{E}_Q[\Phi_n(B_Q)] \tag{15}$$

for some $\alpha \in (0, 1)$ depending on the variance ratio. Here $B_Q = QBQ^T$ and the second term captures the contribution from q.

Step 3. Inversion. Convert the Fisher-information upper bound into a reciprocal-Fisher lower bound by the standard inversion lemma: if $\Phi(r) \leq f(\Phi(p), \Phi(q))$ for a suitable sublinear function f, then $1/\Phi(r) \geq g(1/\Phi(p), 1/\Phi(q))$ with g superadditive.

20.4. **Required lemmas.**

- **Jensen for Φ_n over Haar measure**: the convexity $\Phi_n(\mathbb{E}[r_Q]) \leq \mathbb{E}[\Phi_n(r_Q)]$ is *not* automatic because Φ_n is not convex in polynomial coefficients. However, it holds numerically for $n \geq 4$ and might be provable via the second-order expansion of Φ_n in matrix perturbation (using the harmonicity theorem, Theorem 2.10).
- **Score decomposition lemma**: express $V_i(r)$ as a conditional expectation of $V_i(r_Q)$ plus a mean-zero remainder, in a way that enables Cauchy–Schwarz contraction.

- **Haar integration formula**: compute $\mathbb{E}_Q[\Phi_n(A+QBQ^T)]$ in terms of the spectra of A and B. This is a Harish-Chandra / Itzykson–Zuber integral question and has known asymptotics for large n; the finite-n formula is more delicate.

20.5. **Connection to existing results.** This programme connects to:

(1) **Score norm sub-additivity** (Section 4): the pointwise inequality $|v_r(z)|^2 \leq |v_p(z)|^2 + |v_q(z)|^2$, if proved, would give a function-level version of the score contraction.
(2) **Harmonicity theorem** (Theorem 2.10): the exact cancellation $\Delta_A \log \operatorname{disc} = 0$ implies that any Jensen inequality for Φ_n over the Haar measure must come from a *fourth-order* effect (the second-order term vanishes).
(3) **Classical Stam proof structure**: our framework follows exactly the Blachman–Stam architecture, adapted to the finite discrete setting.

20.6. **Plausibility assessment.** Supporting evidence:

- Jensen leg holds for $n \geq 4$ (100% pass rate).
- Score norm sub-additivity holds numerically (100%).
- The mean ratio $\Phi_n(r)/\mathbb{E}_Q[\Phi_n(r_Q)]$ lies in $(0,1)$ for $n \geq 4$ (typically ~ 0.2–0.5), suggesting strong contraction.
- Large-n asymptotics match the Voiculescu free entropy framework.

Obstacles:

- Jensen leg fails at $n = 2, 3$ (needs separate treatment; $n = 3$ is done).
- The factorisation leg is numerically false as a *sharp* inequality, so any approach must correct for the "gap" between $\mathbb{E}_Q[\Phi(r_Q)]$ and $\Phi(p) + \Phi(q)$.
- The Haar integration formula for $\mathbb{E}_Q[\Phi_n(A+QBQ^T)]$ is not known in closed form for finite n.

21. Additional plausible ideas

Based on the structural analysis, we record several additional directions that warrant exploration.

21.1. **Integrated dilation comparison (corrected Route B).** The dilation path $r_t = p \boxplus_n q_t$ with the **correct** generator $\log K_q$ provides:

- $F(0) = 1/\Phi_n(p)$, $F(1) = 1/\Phi_n(p \boxplus_n q)$.
- $F'(0) = 0$, $F''(0) > 0$ (proved for $n \leq 5$).
- $F(t)$ non-decreasing (numerically universal).

If one can show that $\int_0^1 F'(t)\,dt \geq 1/\Phi_n(q)$ (the "integrated Stam"), the inequality follows. The correct root velocity (**??**) involves all log-cumulants, but the dominant term $-2\ell_2 V_i$ gives the Hermite contribution, and the corrections from $\ell_3, \ell_4, \ldots$ are controlled by $\mathcal{D}_\perp \leq 0$ (Conjecture 18.5).

21.2. **Free cumulant / moment-cumulant duality.** The log-cumulant additivity $\ell_k(r) = \ell_k(p) + \ell_k(q)$ is the structural heart of $\boxplus_n$. For $n \geq 4$, $1/\Phi_n$ is a *nonlinear* function of $(\ell_2, \ldots, \ell_n)$. The key question is whether there exists a *dual* representation—a moment-cumulant-type formula—that expresses $1/\Phi_n$ as an integral or sum over a space where additivity of ℓ translates into a convexity or super-additivity property.

The $n = 3$ proof uses such a structure: $1/\Phi_3$ separates into additive + convex-in-ratio. For general n, one might seek a representation

$$\frac{1}{\Phi_n(\ell)} = \int_\Omega h(\ell;\omega)\,d\mu(\omega)$$

where $h(\ell_p + \ell_q;\omega) \geq h(\ell_p;\omega) + h(\ell_q;\omega)$ for μ-a.e. ω.

21.3. **Operator convexity via the K-transform.** Since $K_{p \boxplus_n q} = K_p \cdot K_q$ and Φ_n can be expressed through $r = K^{-1}$[product] (via the inverse K-transform = coefficient extraction), one might seek an operator-theoretic proof:

- View $K_r(z)$ as a "transfer function" of a linear system.
- Express $1/\Phi_n$ as a norm or spectral radius in this system.
- Use multiplicativity of K and sub-multiplicativity of norms to obtain super-additivity of $1/\Phi_n$.

This connects to the Schur–Hadamard product structure and the Oppenheim inequality for permanents/determinants.

21.4. **Stam for $n = 4$ via exact formula.** The exact rational expression (Section 13)

$$g(\tau_3, \tau_4) = \frac{81\tau_3^4 + 216\tau_3^2\tau_4 + 72\tau_3^2 - 32\tau_4^3 + 48\tau_4^2 - 16}{6(\tau_4 + 1)(9\tau_3^2 + 4\tau_4 - 4)}$$

with $1/\Phi_4 = u \cdot g(\tau_3, \tau_4)$ is fully explicit. Although the Hessian of g is indefinite (ruling out a direct concavity proof), one might:

(1) Decompose D_4 explicitly as a function of $(\tau_3^{(p)}, \tau_4^{(p)}, \tau_3^{(q)}, \tau_4^{(q)}, w)$.
(2) Use computer algebra (Sturm sequences, CAD, sum-of-squares relaxations) to certify positivity of D_4 over the feasible region (a semi-algebraic set defined by real-rootedness constraints on the τ_k).
(3) Such a certificate, while not "elegant," would establish the $n = 4$ case rigorously and might reveal the structure needed for general n.

Part 6. Conclusion

22. Summary of the mathematical landscape

The finite free Stam inequality (1) has been:

- **Proved** for $n = 2$ (equality via variance additivity) and $n = 3$ (SOS decomposition in log-cumulant coordinates, with two independent proofs).
- **Proved** when one input is a finite Gaussian (Hermite polynomial), for all n.
- **Verified numerically** with zero violations across $> 80{,}000$ trials at $n = 2$–12.
- **Supported** by 17 rigorously proved structural identities and inequalities.
- **Reformulated** equivalently as spectral efficiency super-averaging (Theorem 5.2), harmonic-mean repulsion (Corollary 2.2), cumulant-defect domination (Conjecture 18.7), and production convexity (disproved for Ψ).

Eight proof routes (A–I) have been explored in depth. Each has produced valuable structural results, but none has yielded a complete proof for $n \geq 4$. The precise failure modes are documented in Part 2.

23. The core obstruction

The fundamental difficulty is the **nonlinearity of $1/\Phi_n$ in the additive coordinates**. The log-cumulants ℓ_k are additive under $\boxplus_n$, but $1/\Phi_n$ is a complicated rational function of $(\ell_2, \ldots, \ell_n)$ whose Hessian is *indefinite*. The $n = 3$ proof circumvents this by a fortunate algebraic accident: $1/\Phi_3$ decomposes into additive + convex-in-ratio. For $n \geq 4$, no such simple decomposition exists ($R^2 = 0.12$ for the natural quadratic ansatz at $n = 4$).

The harmonicity theorem (Theorem 2.10) provides a deeper explanation: Φ_n captures only the "radial" (eigenvalue) part of the matrix Laplacian of log disc, while the "angular" (rotation) part cancels exactly. This means that any proof via matrix-level convexity is fundamentally mismatched; the proof must work in eigenvalue coordinates, where the algebraic structure is more opaque.

24. Recommended priorities

(1) **Pursue Option C** (marginal/hypergraph, Section 19): the combinatorial structure of Φ_n as an edge sum is natural, and the $n = 3$ proof is the elementary building block.
(2) **Pursue Option D** (optimal transport, Section 20): the random-matrix coupling $p \boxplus_n q = \mathbb{E}_Q[\det(xI - (A + QBQ^T))]$ provides a concrete integral representation amenable to Jensen-type arguments.
(3) **Prove the perpendicular dissipation conjecture** (Conjecture 18.5): if established, this would close the dilation-path approach (corrected Route B) and give the full inequality.

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