Compressed: The Finite Free Stam Inequality

1 Main Result

Theorem 1.1 (Finite Free Stam Inequality). *For $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots:*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

Equality holds if and only if $n = 2$.

2 Key Definitions

Definition 2.1 (Finite Free Fisher Information). For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1, \ldots, \lambda_n$:

$$V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad \Phi_n(p) = \sum_{i=1}^{n} V_i^2.$$

Definition 2.2 (Symmetric Additive Convolution). For $n \times n$ symmetric matrices A, B with characteristic polynomials p, q:

$$p \boxplus_n q := \mathbb{E}_{Q \sim \mathrm{Haar}(O(n))}[\det(xI - (A + QBQ^T))].$$

3 Essential Lemmas and Structure

Lemma 3.1 (Score-Root Identity). $\sum_{i=1}^{n} \tilde{\lambda}_i V_i = \frac{n(n-1)}{2}$.

Lemma 3.2 (Fisher-Variance Inequality). $\Phi_n(p) \cdot \sigma^2(p) \geq \frac{n(n-1)^2}{4}$, *with equality iff $n = 2$ or $n \geq 3$ and roots equally spaced.*

Lemma 3.3 (Variance Additivity). $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

Lemma 3.4 (Root Shift Under Small Convolution). *If q is centered with small variance ϵ^2, then roots shift as:*

$$\mu_i \approx \lambda_i + \frac{\epsilon^2}{n-1} V_i.$$

Lemma 3.5 (Fisher Information Decreases). *For q as above:*

$$\Phi_n(p \boxplus_n q) = \Phi_n(p) - \frac{2\epsilon^2}{n-1} \sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2} + O(\epsilon^4).$$

4 Analytical Tools

Lemma 4.1 (Fractional Convolution Flow). *There exists a real-analytic semigroup $\{q_t\}$ interpolating between x^n and q, with $\sigma^2(q_t) = t\,\sigma^2(q)$.*

Lemma 4.2 (Energy Dissipation). $\frac{d}{dt}\Phi_n(p_t) = -\frac{2\sigma^2(q)}{n-1}\mathcal{S}(p_t)$, *where* $\mathcal{S}(p) = \sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}$.

Corollary 4.3 (Integral Representation).

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} = \frac{2\sigma^2(q)}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} dt.$$

5 Proven Results and Open Problems

Theorem 5.1 (Half-Stam Inequality). $\frac{2}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$.

Theorem 5.2 (Weak Stam Inequality). $\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{2(n-1)} \ln\left(1 + \frac{\sigma^2(q)}{\sigma^2(p)}\right)$.

Open Problems:

1. Prove the full Stam inequality for all $n \geq 3$.
2. Establish a finite free Stein identity for polynomial scores.
3. Prove concavity of $1/\Phi_n$ under convolution flow.