Université de Paris **(Date: 08/11/2020)**

UFR Mathématiques

Théorie des modèles TD3

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Exercise 0.1. Let I be an infinite set, $I_0 \subseteq I$ and $\mathcal{U}$ an ultrafilter on I such that $I_0 \in \mathcal{U}$.

(1) Show that $\mathcal{U} \restriction_{I_0} = \{X \cap I_0, X \in \mathcal{U}\}$ is an ultrafilter over I_0.

(2) Show that $\prod_{i\in I} \mathcal{M}_i/\mathcal{U} \simeq \prod_{i\in I_0} \mathcal{M}_i/\mathcal{U} \restriction I_0$

Solution.

(1) Let $A \subseteq I_0$, then either $A \in \mathcal{U}$ or not. If yes, then since $A = A \cap I_0$ then $A \in \mathcal{U} \restriction_{I_0}$, if not, then $I \setminus A \in \mathcal{U}$ and $I_0 \setminus A = I_0 \cap (I \cap A) \in \mathcal{U}$.

(2) Consider the map that sends $[a_i]_{i\in I}$ to its restriction $[a_i]_{i\in I_0}$ (equivalence classes in $\mathcal{U}$ and $\mathcal{U} \restriction I_0$ respectively). It is well defined since if $[a_i]_{i\in I} = [b_i]_{i\in I}$, then $\{i, a_i = b_i\} \in \mathcal{U}$, so $\{i, a_i = b_i\} \cap I_0 \in \mathcal{U} \restriction I_0$, and hence $[a_i]_{i\in I_0} = [b_i]_{i\in I_0}$. It is also surjective: given $[a_i]_{i\in I_0}$ we can define $[b_i]_{i\in I_0}$ by setting $b_i = a_i$ for $i \in I_0$ and $b_i =$ anything for $i \notin I_0$, clearly $[a_i]_{i\in I_0}$ is a restriction of $[b_i]_{i\in I}$. Let $\varphi(\bar{x}) \in \mathcal{F}(\mathcal{L})$, then if $\bar{a} \in \prod_{i\in I} \mathcal{M}_i/\mathcal{U}$, we have that

$$\begin{aligned} \bar{a} \in \prod_{i\in I} \mathcal{M}_i/\mathcal{U} &\iff \{i, \mathcal{M}_i \models \varphi(\bar{a}_i)\} \in \mathcal{U} \\ &\iff \{i, \mathcal{M}_i \models \varphi(\bar{a}_i)\} \cap I_0 \in \mathcal{U} \restriction I_0 \\ &\iff \prod_{i\in I_0} \mathcal{M}_i/\mathcal{U} \restriction I_0 \models \varphi(\bar{a}) \end{aligned}$$

Exercise 0.2. Let φ be a sentence in the language of rings. Suppose that $\mathsf{ACF}_0 \models \varphi$. Show that there exists N such that $\mathsf{ACF}_n \models \varphi$ for all $n > N$.

Solution. We use the axiomatization for algebraically closed fields of char. 0 given by

$$T = T_{\text{fields}} \cup \{\underbrace{1 + 1 + \cdots + 1}_{n-\text{times}} \neq 0\}_{n\in\mathbb{N}}$$

Since $T \models \varphi$, there is some finite $\Delta \subseteq T$ such that $\Delta \models \varphi$. In particular, there is N such that

$$\Delta \subseteq T_{\text{fields}} \cup \{\underbrace{1+1+\cdots+1}_{n-\text{times}} \neq 0\}_{n<N},$$

so if F is a field of characteristic $n > N$ then $F \models \Delta$, hence $F \models \varphi$.

Exercise 1. Let I be an infinite set and $\{\mathcal{M}_i\}$ a collection of $\mathcal{L}$-structures. Let $\mathcal{U}$, $\mathcal{V}$ ultrafilters on I and consider the ultraproducts $\mathcal{M} = \prod_{i\in I} \mathcal{M}_i/\mathcal{U}$ and $\mathcal{N} = \prod_{i\in I} \mathcal{M}_i/\mathcal{V}$. Discuss whether $\mathcal{M} \simeq \mathcal{N}$ depending on the choice of $\mathcal{U}$, $\mathcal{V}$.

(1) Let $I = \mathbb{N}$ and $M_i = \overline{\mathbb{Q}[x_1,\dots,x_i]}^{\text{alg}}$.
(2) Let $I = \{p, p \text{ prime }\}$, and let $\mathcal{M}_p = \mathbb{F}_p$.

Solution.

(1) First, if $\mathcal{U}$ and $\mathcal{V}$ are both principal, then $\mathcal{M} \simeq \overline{\mathbb{Q}[x_1,\dots,x_i]}^{\text{alg}}$ and $\mathcal{N} \simeq \overline{\mathbb{Q}[x_1,\dots,x_j]}^{\text{alg}}$, so $\mathcal{M} \not\simeq \mathcal{N}$ unless $i = j$, as they would have different transcendence degree over $\mathbb{Q}$. Now, if both $\mathcal{U}$ and $\mathcal{V}$ are non-principal, then by a theorem of the lectures, as M_i is countably infinite for every i, we have that $|M| = |N| = 2^{\aleph_0}$, so that $\mathcal{M}, \mathcal{N}$ are algebraically closed fields of characteristic 0, and by an algebra fact these are both $\simeq \mathbb{C}$. If one is principal and the other isn't, they can't be isomorphic for cardinality reasons.
(2) First, if $\mathcal{U}$ and $\mathcal{V}$ are both principal, then $\mathcal{M} \simeq \mathbb{F}_i$ and $\mathcal{N} \simeq \mathbb{F}_j$, so $\mathcal{M} \not\simeq \mathcal{N}$ unless $i = j$. On the other hand, consider I_0 to be the set of primes congruent to 1 modulo 4. By a number-theoretic fact, I_0 is infinite and co-infinite, so we can find non-principal ultrafilters $\mathcal{U}$,$\mathcal{V}$ containing I_0 and $I \setminus I_0$ respectively. Consider the sentence $\exists x\ x^2 + 1 = 0$. By another number-theoretic fact, we know that $\mathbb{F}_p \models \phi$ if and only if $p \in I_0$, which allows us to conclude $\mathcal{M} \models \phi$ and $\mathcal{N} \not\models \phi$. Finally, if one is principal and the other isn't, they can't be isomorphic again for cardinality reasons.

Exercise 2. Let $\bar{\mathbb{Q}}$ be the ordered field of the rationals and let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Consider the ultrapower $\mathcal{K} = \bar{\mathbb{Q}}^{\mathcal{U}}$, and let i be the diagonal embedding of $\mathbb{Q}$ into $\mathcal{K}$.

(1) Show that $\mathcal{K}$ is an ordered field, and give at least two reasons why $\mathcal{K} \not\simeq \mathbb{R}$.
(2) Let

$$\mathcal{O} = \{a \in K, \exists q \in \mathbb{Q}^{>0}\ i(-q) < a < i(q)\}$$

and

$$\mathcal{M} = \{a \in K, \forall q \in \mathbb{Q}^{>0}\ i(-q) < a < i(q)\}$$

Show that $\mathcal{O}$ is a ring and that $\mathcal{M}$ is a maximal ideal in $\mathcal{O}$.

(3) Let $R = \mathcal{O}/\mathcal{M}$. Show that R can be equipped with a structure $\mathcal{R}$ of ordered field.

(4) Show that $\mathbb{Q}$ can be identified with a dense subset of $\mathcal{R}$.

(5) Show that $\mathcal{R} \simeq \mathbb{R}$ as ordered fields

Solution.

(1) $\mathcal{K}$ is an ultraproduct of structures belonging to an elementary class (ordered fields), therefore $\mathcal{K}$ belongs to the same class. $\mathcal{K}$ has infinitesimal elements: for example let $n \in \mathbb{N}$ and $\varepsilon = (1, 1/2, 1/3, \dots)_{\mathcal{U}}$ positive but less than $i(1/n)$ a.e with respect to $\mathcal{U}$, so $\mathcal{K} \models \varepsilon < i(1/n)$. Also $\mathcal{K}$ has infinitely large elements, for example $1/\varepsilon$.

(2) We have that $0 = i(0), 1 = i(1) \in \mathcal{O}$. It suffices to show that $\mathcal{O}$ is closed under addition, multiplication, and additive inverse. Let a,$b \in \mathcal{O}$ and pick $q, r \in \mathbb{Q}^{>0}$ such that $i(-q) < a < i(q)$ and $i(-r) < b < i(r)$. Recall that i is an elementary embedding, so we sum $i(-q)+i(-r) < a+b < i(q)+i(r)$ and get $i(-q-r) < a+b < i(q+r)$. Similarly one proves that $i(-qr) < ab < i(qr)$ (there are 3 cases to consider), it is clear also that $i(-q) < -a < i(q)$, so $\mathcal{O}$ is a ring. Now consider $a, b \in \mathcal{M}$, and let $q \in \mathbb{Q}$, then $i(-q/2) < a, b < i(q/2)$, so by adding the two inequalities we obtain $i(-q) < a + b < i(-q)$, and also it's easy to see that $\mathcal{M}$ is closed under $-$. To check that $\mathcal{M}$ is an ideal, let $a \in \mathcal{M}$, $b \in \mathcal{O}$, and $q \in \mathbb{Q}^{>0}$. Since $b \in \mathcal{O}$ there is some q' such that $i(-q') < b < i(q')$, and since $a \in \mathcal{M}$ we have that in particular $i(-q/q') < a < i(q/q')$, so by multiplying these inequalities we get $i(-q) < ab < i(q)$ for any q, hence $ab \in \mathcal{M}$. Finally, suppose there exists some ideal I such that $\mathcal{M} \subsetneq I \subseteq \mathcal{O}$, and let $j \in I \setminus \mathcal{M}$, so that there is some q such that $j \notin (i(-q), i(q))$. This implies $1/j \in (i(-q), i(q)) \to 1/j \in \mathcal{O}$ so that $j/j = 1 \in I$, hence $I = \mathcal{O}$. This proves that $\mathcal{M}$ is maximal.

(3) We denote $r + \mathcal{M}$ for the equivalence class of r modulo $\mathcal{M}$. We already know $\mathcal{O}/\mathcal{M}$ is a field, as $\mathcal{M}$ is a maximal ideal, so it suffices to define an ordering that preserves its field structure. Define $r + \mathcal{M} < s + \mathcal{M}$ iff $\mathcal{K} \models r < s$. This is well-defined, since given $r + \mathcal{M} < s + \mathcal{M}$, in particular $s - r \notin \mathcal{M}$ and also $\mathcal{K} \models 0 < s - r$. Let q such that $\mathcal{K} \models i(q) < s - r$ and notice that for all $\varepsilon_1, \varepsilon_2 \in \mathcal{M}$, in $\mathcal{K}$ it's true that:

$$\begin{aligned} r + \varepsilon_1 &< r + i(q/2) \\ &< s - i(q/2) \\ &< s + \varepsilon_2 \end{aligned}$$

so $r+\varepsilon_1+\mathcal{M}<s+\varepsilon_2+\mathcal{M}$. Let's check it respects the field structure: suppose $r+\mathcal{M}<s+\mathcal{M}$

- Let $a+\mathcal{M}\in\mathcal{O}/\mathcal{M}$, then we have $\mathcal{K}\models r+a<s+a$ since $\mathcal{K}$ is an ordered field, so $r+a\mathcal{M}<s+a\mathcal{M}$.
- Let $0<a+\mathcal{M}\in\mathcal{O}/\mathcal{M}$ then we have $\mathcal{K}\models ra<sa$ since $\mathcal{K}$ is an ordered field, so $ra+\mathcal{M}<sa+\mathcal{M}$.

Also $<_{\mathcal{R}}$ is a total order relation since $<_{\mathcal{K}}$ is.

(4) Notice that $\mathcal{R}$ is archimedean: suppose that $\mathcal{R}\models 0\leq\varepsilon<i(1/n)$ for all $n\in\mathbb{N}$, then $\varepsilon\leq i(q)$ for all $q\in\mathbb{Q}^{>0}$, therefore $\varepsilon\in\mathcal{M}$ which implies $\mathcal{R}\models\varepsilon=0$. Let $r,s\in\mathbb{R}$ such that $r<s$, pick n such that $i(1/n)<s-r$, then for some $m\in\mathbb{N}$ it must happen that $r<i(m/n)<s$, for otherwise there would be k such that $i(k/n)\leq r$ and $i((k+1)/n)\geq s$, which is a contradiction. Hence we can identify $\mathbb{Q}$ as a dense subfield of $\mathcal{R}$.

(5) We regard $\mathbb{R}$ as the set of equivalence classes of Cauchy sequences (denoted (a_i)) over $\mathbb{Q}$ under the relationship $(a_i)\sim(b_i)$ iff $\lim_{n\to\infty}|a_i-b_i|=0$. Also the field operations are defined point-wise and the order is defined as $(a_i)<(b_i)$ iff there is N such that $a_n<b_n$ for $n\geq N$.

We define the mapping from $\mathbb{R}$ to $\mathcal{R}$ that sends $(a_i)_\sim$ to $(a_i)_{\mathcal{U}}+\mathcal{M}$ (we will drop the $+\mathcal{M}$ for commodity). This map is well defined: if $(a_i)\sim(b_i)$, then for all n there is N such that for $n\geq N$, $|a_i-b_i|<1/n$, in particular the set $\{i,-1/n<a_i-b_i<1/n\}\in\mathcal{U}$ for all n, so $(a_i)_{\mathcal{U}}-(b_i)_{\mathcal{U}}\in\mathcal{M}$. It is easy to see that this map is a field embedding, since $(a+b)_i=(a_i)+(b_i)$, so that $(a_i)_\sim+(b_i)_\sim$ is sent to $(a_i)_{\mathcal{U}}+(b_i)_{\mathcal{U}}=(a_i+b_i)_{\mathcal{U}}$, and similarly for the product. This map also preserves order: if $(a_i)_\sim<(b_i)_\sim$, then there is N such that for $n\geq N$, $a_i<b_i$, so again $\{i,a_i<b_i\}\in\mathcal{U}$, and $(a_i)_{\mathcal{U}}<(b_i)_{\mathcal{U}}$. Finally we only need to check surjectivity: let $r\in\mathcal{R}$, and choose a sequence of rationals (q_i) such that for all $k>0$, there is N such that $q_n\in(r-1/k,r+1/k)$ for $n\geq N$ (in other words pick a convergent sequence), this is possible by the previous item. Notice that $\mathcal{R}\models(q_i)=r$, since otherwise, if for example $(q_i)<r$ then there is k such that the set $\{i,q_i<r_i-1/k\}\in\mathcal{U}$ so it is an infinite set, contradicting the fact that (q_i) converges to r. We claim that (q_i) is Cauchy: let $k>0$ and pick N such that for all $n\geq N$, $q_n\in(r-1/2k,r+1/2k)$, then if $m,n>N$, we have $q_n,q_m\in(r-1/2k,r+1/2k)$ so necessarily $-1/k<q_n-q_m<1/k$, so we can conclude that r is the image of (q_i), and therefore $\mathbb{R}\simeq\mathcal{R}$.