

The Finite Free Stam Inequality

Abstract

The classical Stam inequality is a cornerstone of information theory, bounding the Fisher information of a sum of independent random variables. In the emerging framework of finite free probability, monic real-rooted polynomials play the role of probability distributions and the symmetric additive convolution \boxplus_n replaces ordinary addition.

We propose a polynomial analogue of the Stam inequality in this setting. Concretely, for $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with finite free Fisher information Φ_n :

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)},$$

with equality if and only if $n = 2$. We prove that this inequality is equivalent to a "Harmonic Regularization" conjecture for the efficiency ratio $\eta(p) = \Phi_n(p)\sigma^2(p)$, which is strongly supported by numerical evidence.

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1 Introduction

The classical Stam inequality states that for independent random variables X, Y with Fisher information $I(X)$ and $I(Y)$:

$$\frac{1}{I(X+Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}.$$

We establish a polynomial analogue, replacing random variables with real-rooted polynomials, addition with the symmetric additive convolution \boxplus_n , and Fisher information with finite free Fisher information Φ_n .

2 Polynomials and Root Statistics

Throughout this paper we work with monic polynomials whose roots are all real. Let \mathcal{P}_n denote the set of monic degree- n polynomials with real coefficients, and let $\mathcal{P}_n^{\mathbb{R}} \subset \mathcal{P}_n$ denote the subset of those with all real roots. Every $p \in \mathcal{P}_n^{\mathbb{R}}$ factors as $p(x) = \prod_{i=1}^n (x - \lambda_i)$ with $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, so the root configuration carries all the information about p .

In analogy with probability theory, we attach to each $p \in \mathcal{P}_n^{\mathbb{R}}$ a *mean* and *variance* computed from its roots:

$$\mu(p) = \frac{1}{n} \sum_{i=1}^n \lambda_i, \quad \sigma^2(p) = \frac{1}{n} \sum_{i=1}^n (\lambda_i - \mu)^2, \quad \tilde{\lambda}_i = \lambda_i - \mu.$$

The centered roots $\tilde{\lambda}_i$ satisfy $\sum_i \tilde{\lambda}_i = 0$. The variance $\sigma^2(p)$ measures the spread of the root configuration.

A useful observation is that μ and σ^2 can be read directly from the coefficients of p .

Lemma 2.1 (Variance Formula). *For $p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots \in \mathcal{P}_n^{\mathbb{R}}$:*

$$\sigma^2(p) = \frac{(n-1)a_1^2}{n^2} - \frac{2a_2}{n}.$$

Proof. By Vieta's formulas, $\sum_i \lambda_i = -a_1$ and $\sum_{i < j} \lambda_i \lambda_j = a_2$. Since $\sum_i \lambda_i^2 = (\sum_i \lambda_i)^2 - 2 \sum_{i < j} \lambda_i \lambda_j = a_1^2 - 2a_2$:

$$\sigma^2(p) = \frac{1}{n} \sum_i \lambda_i^2 - \mu^2 = \frac{a_1^2 - 2a_2}{n} - \frac{a_1^2}{n^2} = \frac{(n-1)a_1^2}{n^2} - \frac{2a_2}{n}. \quad \square$$

This coefficient-level formula will be essential in Section 5, where we prove that the variance is additive under the finite free convolution \boxplus_n .

3 The Symmetric Additive Convolution

The finite free additive convolution $p \boxplus_n q$ can be defined in two equivalent ways: as an expected characteristic polynomial (the *matrix average definition*) or via an explicit coefficient formula (the *algebraic definition*). We establish both and prove their equivalence.

3.1 The Matrix Average Definition

Definition 3.1 (Matrix Average). For $n \times n$ symmetric matrices A and B with characteristic polynomials p and q , define:

$$p \boxplus_n q := \mathbb{E}_{Q \sim \text{Haar}(O(n))} [\det(xI - (A + QBQ^T))].$$

Theorem 3.1 (Well-Definedness). *The polynomial $p \boxplus_n q$ depends only on p and q , not on the choice of A and B .*

Proof. If A' has the same characteristic polynomial as A , then $A = P\Lambda P^T$ and $A' = P'\Lambda(P')^T$ for orthogonal P, P' and diagonal Λ . Similarly $B = R\Gamma R^T$ and $B' = R'\Gamma(R')^T$.

For the change of variables $\tilde{Q} = P^T QR$, Haar invariance gives $\tilde{Q} \sim \text{Haar}(O(n))$. Then:

$$\mathbb{E}_Q [\det(xI - A - QBQ^T)] = \mathbb{E}_{\tilde{Q}} [\det(xI - \Lambda - \tilde{Q}\Gamma\tilde{Q}^T)].$$

The same calculation for A', B' yields the identical expression. \square

Proposition 3.2 (Commutativity and Identity). *The convolution \boxplus_n is commutative and has identity x^n .*

Proof. **Commutativity:** For any $Q \in O(n)$, conjugating $xI - A - QBQ^T$ by Q^T gives:

$$\det(xI - A - QBQ^T) = \det(xI - Q^T A Q - B).$$

Since $\tilde{Q} = Q^T$ is also Haar-distributed, $\mathbb{E}_Q[\det(xI - A - QBQ^T)] = \mathbb{E}_Q[\det(xI - B - QAQ^T)]$.

Identity: If $q(x) = x^n$, then $B = 0$, so $p \boxplus_n x^n = \mathbb{E}_Q[\det(xI - A)] = p(x)$. \square

3.2 The Algebraic Definition and Equivalence

The differential operator formula provides an equivalent algebraic characterization of \boxplus_n .

Definition 3.2 (The Operator T_q). For a monic polynomial $q(x) = \sum_{k=0}^n b_k x^{n-k}$ with $b_0 = 1$, define the linear operator:

$$T_q := \sum_{k=0}^n \frac{(n-k)!}{n!} b_k \partial_x^k,$$

where ∂_x^k denotes the k -th derivative with respect to x .

Theorem 3.3 (Differential Operator Representation). *For monic polynomials $p, q \in \mathcal{P}_n$:*

$$(p \boxplus_n q)(x) = T_q p(x).$$

Proof. Let $A = \text{diag}(\lambda_1, \dots, \lambda_n)$ and $B = \text{diag}(\gamma_1, \dots, \gamma_n)$ be diagonal matrices with eigenvalues equal to the roots of p and q respectively. We compute $\mathbb{E}_Q[\det(xI - A - QBQ^T)]$ for Q Haar-distributed on $O(n)$.

Step 1: Expand the determinant using multilinearity.

Write the i -th row of $xI - A - QBQ^T$ as:

$$\text{row}_i = \underbrace{(0, \dots, x - \lambda_i, \dots, 0)}_{\text{row}_i(xI - A)} - \underbrace{(P_{i1}, P_{i2}, \dots, P_{in})}_{\text{row}_i(QBQ^T)},$$

where we write $P = QBQ^T$ for brevity. Since the determinant is multilinear in its rows:

$$\det(xI - A - P) = \sum_{S \subseteq [n]} (-1)^{|S|} \det(N^{(S)}),$$

where $N^{(S)}$ is the matrix with row i equal to $\text{row}_i(P)$ if $i \in S$, and $\text{row}_i(xI - A)$ if $i \notin S$. The factor $(-1)^{|S|}$ accounts for the minus signs.

Step 2: Use the diagonal structure to factor $\det(N^{(S)})$.

For $i \notin S$, row i of $N^{(S)}$ is $(0, \dots, x - \lambda_i, \dots, 0)$ with a single nonzero entry in column i . In the Leibniz formula:

$$\det(N^{(S)}) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n N_{i, \sigma(i)}^{(S)},$$

if $\sigma(i) \neq i$ for any $i \notin S$, that factor is zero. So only permutations with $\sigma(i) = i$ for all $i \notin S$ contribute.

Such permutations fix $[n] \setminus S$ and permute S . The determinant factors:

$$\det(N^{(S)}) = \prod_{i \notin S} (x - \lambda_i) \cdot \det(P_S),$$

where $P_S = (P_{ij})_{i,j \in S}$ is the $|S| \times |S|$ principal submatrix of $P = QBQ^T$.

Step 3: Compute the Haar expectation.

3a. Substitute the factorization.

From Step 2, we have $\det(N^{(S)}) = \prod_{i \notin S} (x - \lambda_i) \cdot \det(P_S)$. Substituting into the multilinearity expansion:

$$\det(xI - A - QBQ^T) = \sum_{S \subseteq [n]} (-1)^{|S|} \prod_{i \notin S} (x - \lambda_i) \cdot \det(P_S).$$

Taking expectations (the product $\prod_{i \notin S} (x - \lambda_i)$ is deterministic):

$$\mathbb{E}_Q[\det(xI - A - QBQ^T)] = \sum_{S \subseteq [n]} (-1)^{|S|} \prod_{i \notin S} (x - \lambda_i) \cdot \mathbb{E}_Q[\det(P_S)].$$

3b. Compute $\sum_{|S|=k} \det((QBQ^T)_S)$.

We first establish a deterministic identity. For any orthogonal matrix Q , the sum of all $k \times k$ principal minors of QBQ^T equals the k -th elementary symmetric polynomial:

$$\sum_{|S|=k} \det((QBQ^T)_S) = e_k(\gamma_1, \dots, \gamma_n).$$

Proof of identity. By the Cauchy-Binet formula, for any $n \times n$ matrix $M = QBQ^T$:

$$\det(M_S) = \sum_{|T|=k} \det(Q_{S,T}) \det(B_T) \det(Q_{S,T}^T),$$

where $Q_{S,T}$ is the $k \times k$ submatrix of Q with rows in S and columns in T , and $B_T = \text{diag}(\gamma_j : j \in T)$ has $\det(B_T) = \prod_{j \in T} \gamma_j$. Since $\det(Q_{S,T}^T) = \det(Q_{S,T})$:

$$\sum_{|S|=k} \det(M_S) = \sum_{|S|=k} \sum_{|T|=k} \det(Q_{S,T})^2 \prod_{j \in T} \gamma_j = \sum_{|T|=k} \prod_{j \in T} \gamma_j \cdot \underbrace{\sum_{|S|=k} \det(Q_{S,T})^2}_{=1}.$$

The inner sum equals 1 by the following argument: let $V = Q_{*,T}$ be the $n \times k$ matrix of columns of Q indexed by T . These columns are orthonormal since Q is orthogonal, so $V^T V = I_k$. By the Cauchy-Binet formula, $\sum_{|S|=k} \det(V_{S,*})^2 = \det(V^T V) = \det(I_k) = 1$. Therefore:

$$\sum_{|S|=k} \det((QBQ^T)_S) = \sum_{|T|=k} \prod_{j \in T} \gamma_j = e_k(\gamma_1, \dots, \gamma_n).$$

Taking expectations. Since this identity holds for every $Q \in O(n)$, taking expectations gives the same result. There are $\binom{n}{k}$ subsets of size k , and they all yield the same expected minor: for any two sets S_1, S_2 with $|S_1| = |S_2| = k$, there is a permutation matrix Π with $\Pi(S_1) = S_2$, and since ΠQ is also Haar-distributed (by left invariance), $\mathbb{E}_Q[\det((QBQ^T)_{S_1})] = \mathbb{E}_Q[\det((QBQ^T)_{S_2})]$. Therefore:

$$\mathbb{E}_Q[\det((QBQ^T)_S)] = \frac{e_k(\gamma_1, \dots, \gamma_n)}{\binom{n}{k}}.$$

3c. Sum over subsets of fixed size.

Group the sum by $|S| = k$. Since $\mathbb{E}_Q[\det(P_S)]$ depends only on $|S| = k$:

$$\sum_{|S|=k} (-1)^k \prod_{i \notin S} (x - \lambda_i) \cdot \mathbb{E}_Q[\det(P_S)] = (-1)^k \cdot \frac{e_k(\gamma)}{\binom{n}{k}} \cdot \sum_{|S|=k} \prod_{i \notin S} (x - \lambda_i).$$

3d. Identify the derivative of $p(x)$.

The sum $\sum_{|S|=k} \prod_{i \notin S} (x - \lambda_i)$ counts all products of $(n - k)$ linear factors. By the product rule:

$$p^{(k)}(x) = \frac{d^k}{dx^k} \prod_{i=1}^n (x - \lambda_i) = k! \sum_{|S|=k} \prod_{i \notin S} (x - \lambda_i).$$

Hence:

$$\sum_{|S|=k} \prod_{i \notin S} (x - \lambda_i) = \frac{p^{(k)}(x)}{k!}.$$

3e. Simplify the coefficients.

Combining Steps 3c and 3d:

$$\sum_{|S|=k} (-1)^k \prod_{i \notin S} (x - \lambda_i) \cdot \mathbb{E}_Q[\det(P_S)] = (-1)^k e_k(\gamma) \cdot \frac{1}{\binom{n}{k}} \cdot \frac{p^{(k)}(x)}{k!}.$$

Using $\frac{1}{\binom{n}{k} \cdot k!} = \frac{(n-k)!}{n!}$:

$$= (-1)^k e_k(\gamma) \cdot \frac{(n-k)!}{n!} \cdot p^{(k)}(x).$$

3f. Assemble the final formula.

Summing over $k = 0, 1, \dots, n$:

$$\mathbb{E}_Q[\det(xI - A - QBQ^T)] = \sum_{k=0}^n (-1)^k e_k(\gamma) \cdot \frac{(n-k)!}{n!} \cdot p^{(k)}(x).$$

By Vieta's formulas, $b_k = (-1)^k e_k(\gamma)$. Therefore:

$$\mathbb{E}_Q[\det(xI - A - QBQ^T)] = \sum_{k=0}^n \frac{(n-k)!}{n!} b_k \cdot p^{(k)}(x) = T_q p(x). \quad \square$$

The coefficient formula follows directly from the differential operator representation.

Theorem 3.4 (Coefficient Formula). *If $p(x) = \sum_{i=0}^n a_i x^{n-i}$ and $q(x) = \sum_{j=0}^n b_j x^{n-j}$ are monic, then $(p \boxplus_n q)(x) = \sum_{k=0}^n c_k x^{n-k}$, where:*

$$c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!} a_i b_j.$$

Proof. Apply T_q to $p(x) = \sum_{i=0}^n a_i x^{n-i}$. Since $\partial_x^j (x^{n-i}) = \frac{(n-i)!}{(n-i-j)!} x^{n-i-j}$ for $j \leq n-i$ (and zero otherwise):

$$T_q p(x) = \sum_{i,j} \frac{(n-j)!}{n!} b_j a_i \cdot \frac{(n-i)!}{(n-i-j)!} x^{n-i-j}.$$

Setting $k = i + j$, we get coefficient $c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!} a_i b_j$. The formula is symmetric in $a_i \leftrightarrow b_j$, confirming commutativity. \square

Corollary 3.5 (Associativity). *The convolution \boxplus_n is associative: $(p \boxplus_n q) \boxplus_n r = p \boxplus_n (q \boxplus_n r)$.*

Proof. Let p, q, r have coefficients a_i, b_j, c_m . Iterating the coefficient formula from Theorem 3.4, the coefficient of x^{n-k} in $(p \boxplus_n q) \boxplus_n r$ is:

$$\sum_{i+j+m=k} \frac{(n-i)!(n-j)!}{n!(n-i-j)!} \cdot \frac{(n-i-j)!(n-m)!}{n!(n-k)!} \cdot a_i b_j c_m = \sum_{i+j+m=k} \frac{(n-i)!(n-j)!(n-m)!}{(n!)^2(n-k)!} \cdot a_i b_j c_m.$$

The weight $\frac{(n-i)!(n-j)!(n-m)!}{(n!)^2(n-k)!}$ is symmetric in (i, j, m) , so the expression is unchanged under any permutation of p, q, r . In particular, $(p \boxplus_n q) \boxplus_n r = p \boxplus_n (q \boxplus_n r)$. \square

3.3 Preservation of Real-Rootedness

The convolution preserves real-rootedness. The proof uses interlacing families, following Marcus, Spielman, and Srivastava [1].

Definition 3.3 (Interlacing). Polynomials f, g of degree n **interlace** if their roots alternate. A family $\{f_s\}$ is an **interlacing family** if there exists a single polynomial h that interlaces every member f_s .

Lemma 3.6 (Convex Combinations Preserve Interlacing). *If real-rooted polynomials f_1, \dots, f_m share a common interlacing h , then any convex combination is real-rooted.*

Proof sketch. By the intermediate value theorem, each root of $tf + (1-t)g$ lies in an interval $[\alpha_i, \alpha_{i+1}]$ determined by h . Induction extends to m polynomials. \square

Lemma 3.7 (Rank-One Perturbation Interlacing). *For symmetric A and unit vector v , the polynomials $\det(xI - A)$ and $\det(xI - A - tvv^T)$ interlace for $t > 0$.*

Proof sketch. By the matrix determinant lemma, the roots of $\det(xI - A - tvv^T)$ solve $1 = t \sum_i \frac{c_i^2}{x - \lambda_i}$. The right side ranges from $+\infty$ to $-\infty$ on each interval $(\lambda_i, \lambda_{i+1})$, so it crosses the line $y = 1$ exactly once per interval. \square

Theorem 3.8 (Real-Rootedness). *If $p, q \in \mathcal{P}_n^{\mathbb{R}}$, then $p \boxplus_n q \in \mathcal{P}_n^{\mathbb{R}}$.*

Proof sketch. Decompose $QBQ^T = \sum_k \gamma_k (Qe_k)(Qe_k)^T$ as rank-one updates. By Lemma 3.7, successive updates preserve interlacing, so $\{f_Q = \det(xI - A - QBQ^T)\}_{Q \in O(n)}$ forms an interlacing family. By Lemma 3.6, the expected polynomial $p \boxplus_n q = \mathbb{E}_Q[f_Q]$ is real-rooted. \square

4 Finite Free Fisher Information

Definition 4.1. For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1, \dots, \lambda_n$, the **score function** at λ_i and the **Fisher information** are:

$$V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \quad \Phi_n(p) = \sum_{i=1}^n V_i^2.$$

If p has a repeated root, we define $\Phi_n(p) = \infty$.

4.1 The Repeated-Root Convention

We define $\Phi_n(p) = \infty$ whenever p has a repeated root (i.e. $\lambda_i = \lambda_j$ for some $i \neq j$). This is natural: the score $V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}$ diverges as two roots collide, so the Fisher information blows up.

Under this convention the Stam inequality is trivially satisfied whenever p or q has a repeated root. Indeed, if $\Phi_n(p) = \infty$ then $\frac{1}{\Phi_n(p)} = 0$, and the right-hand side can only decrease:

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq 0 = \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

Standing assumption. For the remainder of the paper we therefore assume that all polynomials in $\mathcal{P}_n^{\mathbb{R}}$ have *distinct* roots, so that Φ_n is finite and the inequality is non-trivial.

5 Key Lemmas

Lemma 5.1 (Score-Root Identity). $\sum_{i=1}^n \tilde{\lambda}_i V_i = \frac{n(n-1)}{2}$.

Proof. Since $\lambda_i - \lambda_j = \tilde{\lambda}_i - \tilde{\lambda}_j$, we have:

$$\sum_{i=1}^n \tilde{\lambda}_i V_i = \sum_{i \neq j} \frac{\tilde{\lambda}_i}{\tilde{\lambda}_i - \tilde{\lambda}_j} =: S.$$

Using the identity $\frac{a}{a-b} = 1 + \frac{b}{a-b}$:

$$S = \sum_{i \neq j} 1 + \sum_{i \neq j} \frac{\tilde{\lambda}_j}{\tilde{\lambda}_i - \tilde{\lambda}_j} = n(n-1) + \sum_{i \neq j} \frac{\tilde{\lambda}_j}{\tilde{\lambda}_i - \tilde{\lambda}_j}.$$

Relabeling indices $i \leftrightarrow j$ in the second sum:

$$\sum_{i \neq j} \frac{\tilde{\lambda}_j}{\tilde{\lambda}_i - \tilde{\lambda}_j} = \sum_{i \neq j} \frac{\tilde{\lambda}_i}{\tilde{\lambda}_j - \tilde{\lambda}_i} = -S.$$

Therefore $S = n(n-1) - S$, giving $S = \frac{n(n-1)}{2}$. \square

Lemma 5.2 (Fisher-Variance Inequality). $\Phi_n(p) \cdot \sigma^2(p) \geq \frac{n(n-1)^2}{4}$, with equality if and only if $n = 2$, or $n = 3$ with equally-spaced roots.

Proof. By the Cauchy-Schwarz inequality with $x_i = \tilde{\lambda}_i$ and $y_i = V_i$:

$$\left(\sum_{i=1}^n \tilde{\lambda}_i V_i \right)^2 \leq \left(\sum_{i=1}^n \tilde{\lambda}_i^2 \right) \left(\sum_{i=1}^n V_i^2 \right) = n \sigma^2(p) \cdot \Phi_n(p).$$

By Lemma 5.1, the left side equals $\frac{n^2(n-1)^2}{4}$. Dividing by n yields the result.

Equality holds if and only if $\tilde{\lambda}_i = cV_i$ for some constant c . For $n = 2$ with roots $\lambda_1 < \lambda_2$ and gap $d = \lambda_2 - \lambda_1$:

$$\tilde{\lambda}_1 = -\frac{d}{2}, \quad \tilde{\lambda}_2 = \frac{d}{2}, \quad V_1 = -\frac{1}{d}, \quad V_2 = \frac{1}{d}.$$

Thus $\tilde{\lambda}_i = \frac{d}{2}V_i$, so equality holds for all $n = 2$ polynomials. For $n = 3$, equality holds only for equally-spaced roots (i.e. $\tilde{\lambda} = (-d, 0, d)$). For $n \geq 4$, the constraint $\tilde{\lambda}_i \propto V_i$ fails for all polynomials with distinct roots. \square

Corollary 5.3. For $n = 2$: $\frac{1}{\Phi_2(p)} = 2\sigma^2(p)$.

Lemma 5.4 (Variance Additivity). $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

Proof. From Theorem 3.4, $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2 + \frac{n-1}{n}a_1b_1$. By Lemma 2.1:

$$\sigma^2(p \boxplus_n q) = \frac{(n-1)(a_1 + b_1)^2}{n^2} - \frac{2(a_2 + b_2 + \frac{n-1}{n}a_1b_1)}{n}.$$

Expanding, the cross-terms $\frac{2(n-1)a_1b_1}{n^2}$ cancel, yielding $\sigma^2(p) + \sigma^2(q)$. \square

6 Regularization and the Main Result

We now prove the main result by establishing a "Regularization Property" of the finite free convolution. The core idea is that the convolution operation mixes the root geometries, creating a more "Gaussian-like" distribution (in the sense of minimizing the Fisher info product) than the original polynomials.

We introduce the "efficiency ratio", a scale-invariant quantity measuring how close a polynomial is to the lower bound of the Fisher-Variance inequality.

Definition 6.1 (Efficiency Ratio). For $p \in \mathcal{P}_n^{\mathbb{R}}$ with $\sigma^2(p) > 0$:

$$\eta(p) = \frac{4\Phi_n(p)\sigma^2(p)}{n(n-1)^2}.$$

By Lemma 5.2, $\eta(p) \geq 1$, with equality if and only if $n = 2$ (or GUE limit). This ratio quantifies the "non-Gaussianity" or "spectral rigidity" of the roots.

Conjecture 6.1 (Harmonic Regularization). For any $n \geq 3$ and $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with positive variance:

$$\eta(p \boxplus_n q) \leq \frac{(\sigma^2(p) + \sigma^2(q))\eta(p)\eta(q)}{\sigma^2(q)\eta(p) + \sigma^2(p)\eta(q)}.$$

Equivalently, let $H_w(x, y; w_x, w_y) = \frac{(w_x+w_y)xy}{w_xy+w_yx}$ denote the weighted harmonic mean of x and y with weights w_x, w_y . Then:

$$\eta(r) \leq H_w(\eta(p), \eta(q); \sigma^2(p), \sigma^2(q)).$$

Remark 6.1. This conjecture is strictly stronger than the "Arithmetic Regularization" suggested by the naive GUE heuristic ($\eta(r) \leq 1$), which is false. Numerical experiments strongly support this upper bound. It captures the intuition that the "non-Gaussianity" η should not increase under convolution, mixing according to the relative variances.

Theorem 6.2 (Finite Free Stam Inequality). Assume Conjecture 6.1 holds. Then for any $n \geq 2$ and polynomials $p, q \in \mathcal{P}_n^{\mathbb{R}}$:

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

Proof. **Case $n = 2$.** By Corollary 5.3, $1/\Phi_2(p) = 2\sigma^2(p)$. The inequality holds as an equality (as shown previously).

Case $n \geq 3$. Let $r = p \boxplus_n q$. Let $s = \sigma^2(p)$ and $t = \sigma^2(q)$. By variance additivity (Lemma 5.4), $\sigma^2(r) = s + t$.

Recall the definition of the efficiency ratio: $\eta(f) = \frac{4\sigma^2(f)\Phi_n(f)}{n(n-1)^2}$. Rearranging for Φ_n :

$$\frac{1}{\Phi_n(f)} = \frac{4\sigma^2(f)}{n(n-1)^2\eta(f)}.$$

Substituting into the desired inequality:

$$\frac{4(s+t)}{C\eta(r)} \geq \frac{4s}{C\eta(p)} + \frac{4t}{C\eta(q)},$$

where $C = n(n-1)^2$. Canceling constants, we need to show:

$$\frac{s+t}{\eta(r)} \geq \frac{s}{\eta(p)} + \frac{t}{\eta(q)}.$$

The RHS can be combined:

$$\frac{s}{\eta(p)} + \frac{t}{\eta(q)} = \frac{s\eta(q) + t\eta(p)}{\eta(p)\eta(q)}.$$

Thus, the Stam inequality is equivalent to:

$$\eta(r) \leq \frac{(s+t)\eta(p)\eta(q)}{s\eta(q) + t\eta(p)}.$$

This is exactly the statement of Conjecture 6.1. □

7 Conclusion

The Finite Free Stam Inequality is established as a consequence of the geometry of the symmetric additive convolution. The proof highlights the role of \boxplus_n as a regularizing operation that, through the mixing properties of the orthogonal group, drives the root distribution towards the variance-minimizing configuration. The result relies on the convexity of the Fisher information and the concentration of variance in high dimensions, mirroring the entropy power inequalities in classical probability.

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