FINITE FREE STAM INEQUALITY VIA SCORE-GRADIENT BOUNDS AND DILATION INTERPOLATION: RIGOROUS PARTIAL RESULTS, AND A COUNTEREXAMPLE TO A CONVEXITY HEURISTIC

ABSTRACT. We study the finite free analogue of Stam's inequality for the symmetric additive convolution $\boxplus_n$ of Marcus–Spielman–Srivastava. For monic degree-n real-rooted polynomials p, q with positive variance, the conjectured inequality is

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)},$$

where Φ_n is the finite free Fisher information (defined in terms of "scores" at the roots). This note is self-contained. We give complete proofs of: (i) the Score-Gradient Inequality (a double Cauchy–Schwarz estimate), (ii) a sharp Hermite semigroup bound, and (iii) the Stam inequality in low degrees $n = 2$ (equality) and $n = 3$. For general n we present a real-rooted interpolation (the *dilation path*). We record an explicit numerical example showing that a natural global convexity heuristic for $t \mapsto 1/\Phi_n(p \boxplus_n q_t)$ (and for the associated "dilation excess") fails. Thus any dilation-based proof of the full Stam inequality must use a different monotonicity/comparison principle.

CONTENTS

1. SETUP

1.1. Real-rooted polynomials and convolution.

Definition 1.1 (Real-rooted polynomials)**.** Fix $n \geq 2$. Let $\mathcal{P}_n^{\mathbb{R}}$ denote the set of monic degree-n polynomials with all roots real. For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1 < \cdots < \lambda_n$, write

$$p(x) = \prod_{i=1}^{n}(x - \lambda_i) = \sum_{k=0}^{n} a_k\, x^{n-k}.$$

Definition 1.2 (Symmetric additive convolution)**.** For $p(x) = \sum_{k=0}^n a_k x^{n-k}$ and $q(x) = \sum_{k=0}^n b_k x^{n-k}$ in $\mathcal{P}_n^{\mathbb{R}}$, set $r = p \boxplus_n q$ to be the monic degree-n polynomial with coefficients

$$r(x) = \sum_{k=0}^n c_k x^{n-k}, \qquad c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i b_j.$$

Equivalently (MSS), writing

$$T_q := \sum_{k=0}^n \frac{(n-k)!}{n!}\, b_k\, \partial_x^k,$$

one has $p \boxplus_n q = T_q p$.

Theorem 1.3 (Marcus–Spielman–Srivastava)**.** *If $p, q \in \mathcal{P}_n^{\mathbb{R}}$, then $p \boxplus_n q \in \mathcal{P}_n^{\mathbb{R}}$. Moreover, $\boxplus_n$ is commutative.*

1.2. **Scores, Fisher information, and variance.**

Definition 1.4 (Scores and Fisher information)**.** Let $p \in \mathcal{P}_n^{\mathbb{R}}$ have distinct roots $\lambda_1 < \cdots < \lambda_n$. Define the *score* at λ_i and the *finite free Fisher information* by

$$V_i := \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad \Phi_n(p) := \sum_{i=1}^n V_i^2.$$

If p has a repeated root, set $\Phi_n(p) := \infty$ (equivalently $1/\Phi_n(p) := 0$).

Definition 1.5 (Score-gradient energy)**.**

$$\mathcal{S}(p) := \sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}.$$

Definition 1.6 (Variance)**.** Let $\bar{\lambda} := \frac{1}{n}\sum_{i=1}^n \lambda_i$. Define

$$\sigma^2(p) := \frac{1}{n}\sum_{i=1}^n (\lambda_i - \bar{\lambda})^2.$$

Remark 1.7 (Affine invariances). Φ_n and $\mathcal{S}$ are translation-invariant. Under dilation $p(x) \mapsto p_t(x) = t^{-n} p(tx)$ (i.e. roots scale by t), the scores scale as $V_i \mapsto V_i/t$, hence $\Phi_n \mapsto \Phi_n/t^2$.

Lemma 1.8 (Translation covariance)**.** *For $c \in \mathbb{R}$ and a monic polynomial p, write $(\tau_c p)(x) := p(x-c)$. Then for all monic degree-n polynomials p, q,*

$$\tau_a p \boxplus_n \tau_b q = \tau_{a+b}(p \boxplus_n q).$$

In particular, since scores depend only on root differences, $\Phi_n(\tau_c p) = \Phi_n(p)$ and $\sigma^2(\tau_c p) = \sigma^2(p)$.

Proof. Let K_p denote the normalized generating function used in the MSS framework. Translation by c multiplies the generating function by e^{cz}: $K_{\tau_c p}(z) = e^{cz} K_p(z)$. Using $K_{p \boxplus_n q} = K_p K_q$ modulo z^{n+1} gives $K_{\tau_a p \boxplus_n \tau_b q} = e^{(a+b)z} K_p K_q = K_{\tau_{a+b}(p \boxplus_n q)}$. The invariance statements follow from the definitions. $\square$

2. Preliminary identities

Throughout this section, $p \in \mathcal{P}_n^{\mathbb{R}}$ has distinct roots $\lambda_1 < \cdots < \lambda_n$ and scores V_i.

Lemma 2.1 (Score–derivative relation)**.**

$$V_i = \frac{p''(\lambda_i)}{2\,p'(\lambda_i)}.$$

Proof. Since $p'(\lambda_i) = \prod_{j \neq i}(\lambda_i - \lambda_j)$, differentiating $p'(x) = \sum_{i=1}^n \prod_{j \neq i}(x - \lambda_j)$ and evaluating at $x = \lambda_i$ gives

$$p''(\lambda_i) = 2\,p'(\lambda_i) \sum_{k \neq i} \frac{1}{\lambda_i - \lambda_k} = 2\,p'(\lambda_i)\,V_i. \qquad \square$$

Lemma 2.2 (Score identities)**.** (i) $\sum_i V_i = 0$.
(ii) $\sum_i \lambda_i V_i = \binom{n}{2}$.
(iii) $\sum_i (\lambda_i - \bar{\lambda}) V_i = \binom{n}{2}$.
(iv) $\Phi_n(p) = \sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j}$.

Proof. (i) is antisymmetry of $(\lambda_i - \lambda_j)^{-1}$ in (i, j).
(ii) Pair (i, j) and (j, i): $\frac{\lambda_i}{\lambda_i - \lambda_j} + \frac{\lambda_j}{\lambda_j - \lambda_i} = 1$.
(iii) follows from (ii) and (i).
(iv) Expand $\sum_i V_i^2 = \sum_i V_i \sum_{j \neq i} (\lambda_i - \lambda_j)^{-1}$ and pair (i, j) and (j, i). $\square$

Lemma 2.3 (Variance via coefficients)**.** *If $p(x) = \sum_{k=0}^n a_k x^{n-k}$, then*

$$\sigma^2(p) = \frac{(n-1)a_1^2}{n^2} - \frac{2a_2}{n}.$$

Proof. By Vieta, $\sum_i \lambda_i = -a_1$ and $\sum_{i<j} \lambda_i \lambda_j = a_2$. Thus $\sum_i \lambda_i^2 = a_1^2 - 2a_2$, and $\sigma^2 = \frac{1}{n}\sum_i \lambda_i^2 - \bar{\lambda}^2$ with $\bar{\lambda} = -a_1/n$. $\square$

Lemma 2.4 (Variance additivity)**.** $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

Proof. From Definition 1.2, $c_1 = a_1 + b_1$ and $c_2 = a_2 + \frac{n-1}{n} a_1 b_1 + b_2$. Plugging into Lemma 2.3 and expanding $(a_1 + b_1)^2$ shows cross terms cancel. $\square$

3. Fisher–variance and the Score-Gradient Inequality

Lemma 3.1 (Fisher–variance inequality)**.**

$$\Phi_n(p)\,\sigma^2(p) \geq \frac{n(n-1)^2}{4}.$$

Equality holds iff $V_i = c(\lambda_i - \bar{\lambda})$ for some constant c.

Proof. By Lemma 2.2 ((iii)), $\sum_i (\lambda_i - \bar{\lambda}) V_i = \frac{n(n-1)}{2}$. Apply Cauchy–Schwarz:

$$\Big(\sum_i (\lambda_i - \bar{\lambda}) V_i\Big)^2 \leq \Big(\sum_i (\lambda_i - \bar{\lambda})^2\Big)\Big(\sum_i V_i^2\Big) = n\,\sigma^2(p)\,\Phi_n(p). \qquad \square$$

Theorem 3.2 (Score-Gradient Inequality)**.** *For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots,*

$$\mathcal{S}(p)\,\sigma^2(p) \geq \frac{n-1}{2}\,\Phi_n(p). \tag{1}$$

Equality holds iff $V_i = c(\lambda_i - \bar{\lambda})$ for some constant c.

Proof. Set $T := n\sigma^2(p)$, $U := \Phi_n(p)$, $S := \mathcal{S}(p)$. We show $ST \geq \frac{n(n-1)}{2} U$.

First, Lemma 2.2 ((iii)) and Cauchy–Schwarz give $\frac{n^2(n-1)^2}{4} \leq TU$. Second, Lemma 2.2 ((iv)) and Cauchy–Schwarz give $U^2 \leq S\binom{n}{2} = \frac{n(n-1)}{2} S$. Combine:

$$ST \geq \frac{2U^2}{n(n-1)} T = \frac{2U}{n(n-1)}\,(TU) \geq \frac{2U}{n(n-1)}\,\frac{n^2(n-1)^2}{4} = \frac{n(n-1)}{2} U. \qquad \square$$

The equality characterization is the standard "both Cauchy–Schwarz equalities" argument.

4. Low-degree Stam: $n = 2$ and $n = 3$

4.1. $n = 2$: equality and convexity along the dilation path.

Proposition 4.1 (Quadratic case)**.** *For $n = 2$, for all $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots,*

$$\frac{1}{\Phi_2(p \boxplus_2 q)} = \frac{1}{\Phi_2(p)} + \frac{1}{\Phi_2(q)}.$$

Moreover, along the dilation path $r_t = p \boxplus_2 q_t$, $F(t) := 1/\Phi_2(r_t)$ is a quadratic polynomial in t with $F''(t) > 0$.

Proof. If $p(x) = (x - \lambda_1)(x - \lambda_2)$ with $d = \lambda_2 - \lambda_1 > 0$, then $V_1 = -1/d$, $V_2 = 1/d$, hence $\Phi_2(p) = 2/d^2$ and $\sigma^2(p) = d^2/4$, so $1/\Phi_2(p) = 2\sigma^2(p)$. By variance additivity (Lemma 2.4), $1/\Phi_2(p \boxplus_2 q) = 2\sigma^2(p \boxplus_2 q) = 2\sigma^2(p) + 2\sigma^2(q)$.

For dilation: q_t scales the root gap by t, so $\sigma^2(q_t) = t^2\sigma^2(q)$, and the same identity gives $1/\Phi_2(r_t) = 2(\sigma^2(p) + t^2\sigma^2(q))$, with constant second derivative. □

4.2. $n = 3$: an explicit computation (centered cubics).

4.2.1. *A critical-value formula for Φ_n.*

Theorem 4.2 (Critical-value formula)**.** *Let $p \in \mathcal{P}_n^{\mathbb{R}}$ have distinct roots $\lambda_1 < \cdots < \lambda_n$, and let $\zeta_1, \ldots, \zeta_{n-1}$ be the simple zeros of p'. Then*

$$\Phi_n(p) = -\frac{1}{4}\sum_{j=1}^{n-1} \frac{p''(\zeta_j)}{p(\zeta_j)}. \tag{2}$$

Proof. By Lemma 2.1, $\Phi_n(p) = \frac{1}{4}\sum_i \frac{p''(\lambda_i)^2}{p'(\lambda_i)^2}$. Consider the meromorphic function on the Riemann sphere:

$$F(x) := \frac{p''(x)^2}{p'(x)\, p(x)}.$$

Residues at the roots. Since p has a simple zero at λ_i and $p'(\lambda_i) \neq 0$,

$$\operatorname{Res}_{x=\lambda_i} F = \frac{p''(\lambda_i)^2}{p'(\lambda_i)^2}.$$

Summing over i gives $\sum_i \operatorname{Res}_{\lambda_i} F = 4\Phi_n(p)$.

Residues at the critical points. At a simple zero ζ_j of p', interlacing implies $p(\zeta_j) \neq 0$. Thus

$$\operatorname{Res}_{x=\zeta_j} F = \frac{p''(\zeta_j)^2}{p''(\zeta_j)\, p(\zeta_j)} = \frac{p''(\zeta_j)}{p(\zeta_j)}.$$

Residue at infinity. As $x \to \infty$, $p(x) \sim x^n$, $p'(x) \sim nx^{n-1}$, and $p''(x) \sim n(n-1)x^{n-2}$, so $F(x) = \frac{n(n-1)^2}{x^3}(1 + O(x^{-1}))$. Hence $\operatorname{Res}_\infty F = 0$.

By the global residue theorem, the sum of all residues on the sphere is zero: $4\Phi_n(p) + \sum_{j=1}^{n-1} \frac{p''(\zeta_j)}{p(\zeta_j)} = 0$, proving (2). □

A centered monic cubic has the form $r(x) = x^3 - Sx + T$ with $S \geq 0$. It has three distinct real roots iff its discriminant $\Delta := 4S^3 - 27T^2$ is positive.

Proposition 4.3 (Closed form for Φ_3)**.** *For a centered cubic $r(x) = x^3 - Sx + T$ with $\Delta > 0$,*

$$\Phi_3(r) = \frac{18S^2}{\Delta}.$$

Proof. Apply Theorem 4.2. The critical points are $\zeta_\pm = \pm\alpha$ with $\alpha := \sqrt{S/3}$ and $r''(x) = 6x$. Thus

$$4\Phi_3(r) = -\frac{6\alpha}{r(\alpha)} + \frac{6\alpha}{r(-\alpha)} = 6\alpha\,\frac{r(\alpha) - r(-\alpha)}{r(\alpha)r(-\alpha)}.$$

Compute $r(\alpha) - r(-\alpha) = -\frac{4S\alpha}{3}$ and $r(\alpha)r(-\alpha) = T^2 - \frac{4S^3}{27} = -\frac{\Delta}{27}$. Substituting gives $4\Phi_3(r) = \frac{72S^2}{\Delta}$. □

Proposition 4.4 (Convolution preserves cubic shape (centered))**.** *If $p(x) = x^3 - S_1x + T_1$ and $q(x) = x^3 - S_2x + T_2$ are centered, then $(p \boxplus_3 q)(x) = x^3 - (S_1 + S_2)x + (T_1 + T_2)$.*

Proof. With $a_1 = b_1 = 0$ the only surviving coefficient contributions are additive for a_2, a_3. □

Theorem 4.5 (Stam for $n = 3$)**.** *The finite free Stam inequality holds for $n = 3$. Equality holds iff $T_1 = T_2 = 0$ in the centered parametrization.*

Proof. By Propositions 4.3 and 4.4, $1/\Phi_3 = \Delta/(18S^2) = 2S/9 - 3T^2/(2S^2)$. Cancelling the linear terms in S, the inequality reduces to

$$\frac{(T_1 + T_2)^2}{(S_1 + S_2)^2} \le \frac{T_1^2}{S_1^2} + \frac{T_2^2}{S_2^2},$$

which is Jensen/convexity for $t \mapsto t^2$. □

5. Hermite semigroup bound

5.1. **Hermite kernel.**

Definition 5.1 (Hermite kernel)**.** For $t \ge 0$, let $G_t \in \mathcal{P}_n^{\mathbb{R}}$ be the monic degree-n polynomial whose normalized generating function satisfies

$$K_{G_t}(z) = \exp\Big(-\frac{t}{2(n-1)}z^2\Big) \pmod{z^{n+1}}.$$

The *Hermite flow* is $p_t := p \boxplus_n G_t$.

Lemma 5.2 (Semigroup and variance)**.** *For $s, t \ge 0$:*

(i) $G_s \boxplus_n G_t = G_{s+t}$.
(ii) $\sigma^2(G_t) = t$.
(iii) $\sigma^2(p_t) = \sigma^2(p) + t$.

Proof. (i) follows from $K_{G_s}K_{G_t} = K_{G_{s+t}}$ modulo z^{n+1}. (ii) is read from the quadratic term. (iii) is Lemma 2.4. □

5.2. **Root ODE and dissipation.**

Lemma 5.3 (Hermite root ODE)**.** *Along the Hermite flow, if $\lambda_i(t)$ are the roots of p_t and $V_i(t)$ their scores, then*

$$\dot\lambda_i = \frac{1}{n-1}V_i(t).$$

Proof. Using $K_{G_h}(z) = 1 - \frac{h}{2(n-1)}z^2 + O(h^2)$, one has $T_{G_h}f = f - \frac{h}{2(n-1)}f'' + O(h^2)$. Differentiate $0 = T_{G_h}p_t(\lambda_i(t+h))$ to first order and use Lemma 2.1. □

Lemma 5.4 (Hermite dissipation)**.**

$$\frac{d}{dt}\Phi_n(p_t) = -\frac{2}{n-1}\,\mathcal{S}(p_t).$$

Proof. Differentiate $V_i(t) = \sum_{j\ne i}(\lambda_i - \lambda_j)^{-1}$ using the root ODE, then sum $\dot\Phi_n = 2\sum_i V_i\dot V_i$ and symmetrize. □

Theorem 5.5 (Hermite flow bound). *Let $a := \sigma^2(p) > 0$ and $b > 0$. Then*

$$\frac{1}{\Phi_n(p \boxplus_n G_b)} \geq \frac{a+b}{a\,\Phi_n(p)}.$$

Proof. Apply the Score-Gradient Inequality (Theorem 3.2) to p_t: $\mathcal{S}(p_t) \geq \frac{(n-1)\Phi_n(p_t)}{2\sigma^2(p_t)} = \frac{(n-1)\Phi_n(p_t)}{2(a+t)}$. With Lemma 5.4, $\dot{\Phi}_n(p_t) \leq -\Phi_n(p_t)/(a+t)$. Integrate $(\log \Phi_n)' \leq -(a+t)^{-1}$ from 0 to b. ☐

6. Dilation interpolation and a convexity heuristic

6.1. The dilation path.

Definition 6.1 (Dilation family). Let $q(x) = \prod_{i=1}^n (x - \mu_i) \in \mathcal{P}_n^{\mathbb{R}}$. For $t \in [0,1]$, define

$$q_t(x) := \prod_{i=1}^n (x - t\mu_i), \qquad r_t := p \boxplus_n q_t.$$

Lemma 6.2 (Basic properties). *Let $a := \sigma^2(p)$ and $b := \sigma^2(q)$. Then:*

(i) $r_0 = p$ *and* $r_1 = p \boxplus_n q$.
(ii) $\sigma^2(q_t) = t^2\sigma^2(q)$ *and* $\sigma^2(r_t) = a + t^2 b$.
(iii) $\Phi_n(q_t) = \Phi_n(q)/t^2$ *for* $t > 0$.
(iv) $r_t \in \mathcal{P}_n^{\mathbb{R}}$ *for all* $t \in [0,1]$.

Proof. (i) is immediate since $q_0 = x^n$ is the identity for $\boxplus_n$. (ii) follows from scaling of roots and variance additivity. (iii) is score scaling under dilation. (iv) is Theorem 1.3. ☐

6.2. The excess functional.

Definition 6.3 (Dilation excess). For the dilation path r_t, define

$$E(t) := \frac{1}{\Phi_n(r_t)} - \frac{1}{\Phi_n(p)} - \frac{t^2}{\Phi_n(q)}.$$

Lemma 6.4 (Endpoints). $E(0) = 0$, *and* $E(1) \geq 0$ *is equivalent to the finite free Stam inequality.*

Proof. Immediate from the definition and $r_0 = p$, $r_1 = p \boxplus_n q$. ☐

Conjecture 6.5 (Excess convexity (false in general)). Along the dilation path, E is convex on $(0,1)$, i.e. $E''(t) \geq 0$. Equivalently, $\frac{d^2}{dt^2}(1/\Phi_n(r_t)) \geq 2/\Phi_n(q)$.

Remark 6.6. Conjecture 6.5 is a clean sufficient condition for Stam via Theorem 6.8, but it is *not true* in full generality; see Appendix A.

It is worth stressing a simple diagnostic: if we write $F(t) := 1/\Phi_n(r_t)$, then $E(t) = F(t) - 1/\Phi_n(p) - t^2/\Phi_n(q)$ satisfies

$$E''(t) = F''(t) - \frac{2}{\Phi_n(q)}.$$

Thus even if one happens to observe $F''(t) \geq 0$ in a given example, the subtraction of $t^2/\Phi_n(q)$ shifts the curvature by a negative constant and can force $E''(t) < 0$.

Lemma 6.7 (Vanishing first derivative at $t = 0$). *Assume q is centered (i.e. the sum of its roots is zero, equivalently its x^{n-1} coefficient vanishes). Then $E'(0) = 0$.*

Proof. Write $q(x) = \sum_{k=0}^n b_k x^{n-k}$. Centering means $b_1 = 0$.

Along the dilation family, q_t has coefficients $b_k(t) = t^k b_k$. By Definition 1.2,

$$r_t(x) = \sum_{k=0}^n \frac{(n-k)!}{n!}\, b_k(t)\, p^{(k)}(x) = p(x) + \sum_{k=1}^n \frac{(n-k)!}{n!}\, t^k b_k\, p^{(k)}(x).$$

Since $b_1 = 0$, the first nonzero term is order t^2, hence $\partial_t r_t|_{t=0} = 0$ as a polynomial. In particular, the coefficient vector of r_t has no linear term in t. When p has distinct roots, the roots of r_t depend smoothly on the coefficients for t in a neighborhood of 0, so each root trajectory has zero first derivative at $t = 0$. Since Φ_n is a smooth function of the roots as long as they remain distinct, this implies $\frac{d}{dt}\frac{1}{\Phi_n(r_t)}\big|_{t=0} = 0$. Also $\frac{d}{dt}(t^2/\Phi_n(q))|_{t=0} = 0$. Thus $E'(0) = 0$. ☐

Theorem 6.8 (Convexity reduction)**.** *If Conjecture 6.5 holds for all $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with positive variance, then the finite free Stam inequality holds for all such p, q.*

Proof. By Lemma 1.8, we may replace q by its centered translate without changing either side of the Stam inequality; assume henceforth that q is centered.

Assuming Conjecture 6.5, the convex function E satisfies $E(t) \geq E(0) + tE'(0)$ for all $t \in [0, 1]$. By Lemma 6.4, $E(0) = 0$, and by Lemma 6.7, $E'(0) = 0$. Hence $E(1) \geq 0$, which is exactly the Stam inequality. ☐

Remark 6.9 (What remains for general n)*.* The counterexample in Appendix A shows that a global convexity strategy along the dilation path cannot be the final mechanism behind the Stam inequality. The open problem is to find a different comparison principle along a real-rooted interpolation (such as the dilation path or the constant-variance path) that implies $E(1) \geq 0$ without requiring pointwise convexity.

Appendix A. A numerical counterexample to dilation convexity

We record one explicit example (found by brute-force search) showing that neither $t \mapsto 1/\Phi_n(r_t)$ nor the dilation excess $E(t)$ need be convex.

For $n = 3$, take p with roots $(-2, -\frac{3}{2}, \frac{3}{2})$ and q with roots $(-5, 2, 3)$ (so q is centered). Along the dilation path $r_t = p \boxplus_3 q_t$, define $F(t) = 1/\Phi_3(r_t)$ and $E(t) = F(t) - 1/\Phi_3(p) - t^2/\Phi_3(q)$.

A finite-difference computation (step size $h = 10^{-5}$) that verifies all roots of r_t have imaginary parts below 10^{-8} for the sampled t values yields a negative second derivative:

$$F''(t^*) \approx -8.16 \quad \text{at } t^* \approx 0.435.$$

Since $2/\Phi_3(q) \approx 0.965$, this also forces $E''(t^*) \approx -9.12 < 0$. Nevertheless $E(1) \approx 2.18 > 0$, so the Stam inequality holds in this example.

Raw convexity $F''(t) \geq 0$ also fails in higher degrees. For $n = 4$, take p with roots $(-1.10743, -0.81774, -0.36839, 0$ and q with centered roots $(-1.57864, -1.22305, -0.93765, 3.73934)$. A finite-difference computation (step size $h = 2 \cdot 10^{-4}$) gives $F''(0.3) \approx -0.14$ (and already $F''(0.2) \approx -0.12$), so $t \mapsto 1/\Phi_4(r_t)$ need not be convex.

This appendix is included to prevent overfitting the analysis to a false convexity narrative.

Appendix B. Bibliographic notes

Remark B.1. This file is intended to be arXiv-style and self-contained. The repository `math-docs` contains additional related notes, including a critical-value formula via residues and further numerical experiments.

References

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