The Finite Free Stam Inequality

Abstract

The classical Stam inequality is a cornerstone of information theory, bounding the Fisher information of a sum of independent random variables. In the emerging framework of finite free probability, monic real-rooted polynomials play the role of probability distributions and the symmetric additive convolution $\boxplus_n$ replaces ordinary addition.

We propose and prove a polynomial analogue of the Stam inequality in this setting. Concretely, for $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with finite free Fisher information Φ_n:

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)},$$

with equality if and only if $n = 2$. Our proof relies on a "Regularization Property" of the convolution, which we establish using the convexity of the Fisher information functional and the concentration of measure for the matrix average definition of $\boxplus_n$.

Contents

1 Introduction

The classical Stam inequality states that for independent random variables X, Y with Fisher information $I(X)$ and $I(Y)$:

$$\frac{1}{I(X+Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}.$$

We establish a polynomial analogue, replacing random variables with real-rooted polynomials, addition with the symmetric additive convolution $\boxplus_n$, and Fisher information with finite free Fisher information Φ_n.

2 Polynomials and Root Statistics

Throughout this paper we work with monic polynomials whose roots are all real. Let $\mathcal{P}_n$ denote the set of monic degree-n polynomials with real coefficients, and let $\mathcal{P}_n^{\mathbb{R}} \subset \mathcal{P}_n$ denote the subset of those with all real roots. Every $p \in \mathcal{P}_n^{\mathbb{R}}$ factors as $p(x) = \prod_{i=1}^n (x - \lambda_i)$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, so the root configuration carries all the information about p.

In analogy with probability theory, we attach to each $p \in \mathcal{P}_n^{\mathbb{R}}$ a *mean* and *variance* computed from its roots:

$$\mu(p) = \tfrac{1}{n} \textstyle\sum_{i=1}^n \lambda_i, \qquad \sigma^2(p) = \tfrac{1}{n} \textstyle\sum_{i=1}^n (\lambda_i - \mu)^2, \qquad \tilde{\lambda}_i = \lambda_i - \mu.$$

The centered roots $\tilde{\lambda}_i$ satisfy $\sum_i \tilde{\lambda}_i = 0$. The variance $\sigma^2(p)$ measures the spread of the root configuration.

A useful observation is that μ and σ^2 can be read directly from the coefficients of p.

Lemma 2.1 (Variance Formula)**.** *For $p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots \in \mathcal{P}_n^{\mathbb{R}}$:*

$$\sigma^2(p) = \frac{(n-1)a_1^2}{n^2} - \frac{2a_2}{n}.$$

Proof. By Vieta's formulas, $\sum_i \lambda_i = -a_1$ and $\sum_{i<j} \lambda_i \lambda_j = a_2$. Since $\sum_i \lambda_i^2 = (\sum_i \lambda_i)^2 - 2\sum_{i<j} \lambda_i \lambda_j = a_1^2 - 2a_2$:

$$\sigma^2(p) = \frac{1}{n} \sum_i \lambda_i^2 - \mu^2 = \frac{a_1^2 - 2a_2}{n} - \frac{a_1^2}{n^2} = \frac{(n-1)a_1^2}{n^2} - \frac{2a_2}{n}. \qquad \square$$

This coefficient-level formula will be essential in Section 5, where we prove that the variance is additive under the finite free convolution $\boxplus_n$.

3 The Symmetric Additive Convolution

The finite free additive convolution $p \boxplus_n q$ can be defined in two equivalent ways: as an expected characteristic polynomial (the *matrix average definition*) or via an explicit coefficient formula (the *algebraic definition*). We establish both and prove their equivalence.

3.1 The Matrix Average Definition

Definition 3.1 (Matrix Average)**.** For $n \times n$ symmetric matrices A and B with characteristic polynomials p and q, define:

$$p \boxplus_n q := \mathbb{E}_{Q \sim \mathrm{Haar}(O(n))}[\det(xI - (A + QBQ^T))].$$

Theorem 3.1 (Well-Definedness)**.** *The polynomial $p \boxplus_n q$ depends only on p and q, not on the choice of A and B.*

Proof. If A' has the same characteristic polynomial as A, then $A = P\Lambda P^T$ and $A' = P'\Lambda(P')^T$ for orthogonal P, P' and diagonal Λ. Similarly $B = R\Gamma R^T$ and $B' = R'\Gamma(R')^T$.

For the change of variables $\tilde{Q} = P^T Q R$, Haar invariance gives $\tilde{Q} \sim \mathrm{Haar}(O(n))$. Then:

$$\mathbb{E}_Q[\det(xI - A - QBQ^T)] = \mathbb{E}_{\tilde{Q}}[\det(xI - \Lambda - \tilde{Q}\Gamma\tilde{Q}^T)].$$

The same calculation for A', B' yields the identical expression. $\square$

Proposition 3.2 (Commutativity and Identity)**.** *The convolution $\boxplus_n$ is commutative and has identity x^n.*

Proof. **Commutativity:** For any $Q \in O(n)$, conjugating $xI - A - QBQ^T$ by Q^T gives:

$$\det(xI - A - QBQ^T) = \det(xI - Q^T AQ - B).$$

Since $\tilde{Q} = Q^T$ is also Haar-distributed, $\mathbb{E}_Q[\det(xI - A - QBQ^T)] = \mathbb{E}_Q[\det(xI - B - QAQ^T)]$.

Identity: If $q(x) = x^n$, then $B = 0$, so $p \boxplus_n x^n = \mathbb{E}_Q[\det(xI - A)] = p(x)$. □

3.2 The Algebraic Definition and Equivalence

The differential operator formula provides an equivalent algebraic characterization of $\boxplus_n$.

Definition 3.2 (The Operator T_q)**.** For a monic polynomial $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$ with $b_0 = 1$, define the linear operator:

$$T_q := \sum_{k=0}^{n} \frac{(n-k)!}{n!} b_k \partial_x^k,$$

where ∂_x^k denotes the k-th derivative with respect to x.

Theorem 3.3 (Differential Operator Representation)**.** *For monic polynomials $p, q \in \mathcal{P}_n$:*

$$(p \boxplus_n q)(x) = T_q p(x).$$

Proof. Let $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and $B = \operatorname{diag}(\gamma_1, \dots, \gamma_n)$ be diagonal matrices with eigenvalues equal to the roots of p and q respectively. We compute $\mathbb{E}_Q[\det(xI - A - QBQ^T)]$ for Q Haar-distributed on $O(n)$.

Step 1: Expand the determinant using multilinearity.

Write the i-th row of $xI - A - QBQ^T$ as:

$$\mathrm{row}_i = \underbrace{(0, \dots, x - \lambda_i, \dots, 0)}_{\mathrm{row}_i(xI-A)} - \underbrace{(P_{i1}, P_{i2}, \dots, P_{in})}_{\mathrm{row}_i(QBQ^T)},$$

where we write $P = QBQ^T$ for brevity. Since the determinant is multilinear in its rows:

$$\det(xI - A - P) = \sum_{S \subseteq [n]} (-1)^{|S|} \det(N^{(S)}),$$

where $N^{(S)}$ is the matrix with row i equal to $\mathrm{row}_i(P)$ if $i \in S$, and $\mathrm{row}_i(xI - A)$ if $i \notin S$. The factor $(-1)^{|S|}$ accounts for the minus signs.

Step 2: Use the diagonal structure to factor $\det(N^{(S)})$*.*

For $i \notin S$, row i of $N^{(S)}$ is $(0, \dots, x - \lambda_i, \dots, 0)$ with a single nonzero entry in column i. In the Leibniz formula:

$$\det(N^{(S)}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} N^{(S)}_{i,\sigma(i)},$$

if $\sigma(i) \neq i$ for any $i \notin S$, that factor is zero. So only permutations with $\sigma(i) = i$ for all $i \notin S$ contribute.

Such permutations fix $[n] \setminus S$ and permute S. The determinant factors:

$$\det(N^{(S)}) = \prod_{i \notin S} (x - \lambda_i) \cdot \det(P_S),$$

where $P_S = (P_{ij})_{i,j \in S}$ is the $|S| \times |S|$ principal submatrix of $P = QBQ^T$.

Step 3: Compute the Haar expectation.

3a. **Substitute the factorization.**

From Step 2, we have $\det(N^{(S)}) = \prod_{i \notin S}(x - \lambda_i) \cdot \det(P_S)$. Substituting into the multilinearity expansion:

$$\det(xI - A - QBQ^T) = \sum_{S \subseteq [n]} (-1)^{|S|} \prod_{i \notin S} (x - \lambda_i) \cdot \det(P_S).$$

Taking expectations (the product $\prod_{i \notin S}(x - \lambda_i)$ is deterministic):

$$\mathbb{E}_Q[\det(xI - A - QBQ^T)] = \sum_{S \subseteq [n]} (-1)^{|S|} \prod_{i \notin S} (x - \lambda_i) \cdot \mathbb{E}_Q[\det(P_S)].$$

3b. **Compute $\sum_{|S|=k} \det((QBQ^T)_S)$.**

We first establish a deterministic identity. For any orthogonal matrix Q, the sum of all $k \times k$ principal minors of QBQ^T equals the k-th elementary symmetric polynomial:

$$\sum_{|S|=k} \det((QBQ^T)_S) = e_k(\gamma_1, \ldots, \gamma_n).$$

Proof of identity. By the Cauchy-Binet formula, for any $n \times n$ matrix $M = QBQ^T$:

$$\det(M_S) = \sum_{|T|=k} \det(Q_{S,T}) \det(B_T) \det(Q^T_{S,T}),$$

where $Q_{S,T}$ is the $k \times k$ submatrix of Q with rows in S and columns in T, and $B_T = \operatorname{diag}(\gamma_j : j \in T)$ has $\det(B_T) = \prod_{j \in T} \gamma_j$. Since $\det(Q^T_{S,T}) = \det(Q_{S,T})$:

$$\sum_{|S|=k} \det(M_S) = \sum_{|S|=k} \sum_{|T|=k} \det(Q_{S,T})^2 \prod_{j \in T} \gamma_j = \sum_{|T|=k} \prod_{j \in T} \gamma_j \cdot \underbrace{\sum_{|S|=k} \det(Q_{S,T})^2}_{=1}.$$

The inner sum equals 1 by the following argument: let $V = Q_{*,T}$ be the $n \times k$ matrix of columns of Q indexed by T. These columns are orthonormal since Q is orthogonal, so $V^T V = I_k$. By the Cauchy–Binet formula, $\sum_{|S|=k} \det(V_{S,*})^2 = \det(V^T V) = \det(I_k) = 1$. Therefore:

$$\sum_{|S|=k} \det((QBQ^T)_S) = \sum_{|T|=k} \prod_{j \in T} \gamma_j = e_k(\gamma_1, \ldots, \gamma_n).$$

Taking expectations. Since this identity holds for every $Q \in O(n)$, taking expectations gives the same result. There are $\binom{n}{k}$ subsets of size k, and they all yield the same expected minor: for any two sets S_1, S_2 with $|S_1| = |S_2| = k$, there is a permutation matrix Π with $\Pi(S_1) = S_2$, and since ΠQ is also Haar-distributed (by left invariance), $\mathbb{E}_Q[\det((QBQ^T)_{S_1})] = \mathbb{E}_Q[\det((QBQ^T)_{S_2})]$. Therefore:

$$\mathbb{E}_Q[\det((QBQ^T)_S)] = \frac{e_k(\gamma_1, \ldots, \gamma_n)}{\binom{n}{k}}.$$

3c. **Sum over subsets of fixed size.**

Group the sum by $|S| = k$. Since $\mathbb{E}_Q[\det(P_S)]$ depends only on $|S| = k$:

$$\sum_{|S|=k} (-1)^k \prod_{i \notin S} (x - \lambda_i) \cdot \mathbb{E}_Q[\det(P_S)] = (-1)^k \cdot \frac{e_k(\gamma)}{\binom{n}{k}} \cdot \sum_{|S|=k} \prod_{i \notin S} (x - \lambda_i).$$

3d. **Identify the derivative of $p(x)$.**

The sum $\sum_{|S|=k}\prod_{i\notin S}(x-\lambda_i)$ counts all products of $(n-k)$ linear factors. By the product rule:

$$p^{(k)}(x) = \frac{d^k}{dx^k}\prod_{i=1}^{n}(x-\lambda_i) = k!\sum_{|S|=k}\prod_{i\notin S}(x-\lambda_i).$$

Hence:

$$\sum_{|S|=k}\prod_{i\notin S}(x-\lambda_i) = \frac{p^{(k)}(x)}{k!}.$$

3e. **Simplify the coefficients.**

Combining Steps 3c and 3d:

$$\sum_{|S|=k}(-1)^k\prod_{i\notin S}(x-\lambda_i)\cdot\mathbb{E}_Q[\det(P_S)] = (-1)^k e_k(\gamma)\cdot\frac{1}{\binom{n}{k}}\cdot\frac{p^{(k)}(x)}{k!}.$$

Using $\frac{1}{\binom{n}{k}\cdot k!} = \frac{(n-k)!}{n!}$:

$$= (-1)^k e_k(\gamma)\cdot\frac{(n-k)!}{n!}\cdot p^{(k)}(x).$$

3f. **Assemble the final formula.**

Summing over $k=0,1,\ldots,n$:

$$\mathbb{E}_Q[\det(xI-A-QBQ^T)] = \sum_{k=0}^{n}(-1)^k e_k(\gamma)\cdot\frac{(n-k)!}{n!}\cdot p^{(k)}(x).$$

By Vieta's formulas, $b_k = (-1)^k e_k(\gamma)$. Therefore:

$$\mathbb{E}_Q[\det(xI-A-QBQ^T)] = \sum_{k=0}^{n}\frac{(n-k)!}{n!}b_k\cdot p^{(k)}(x) = T_q p(x). \qquad \square$$

The coefficient formula follows directly from the differential operator representation.

Theorem 3.4 (Coefficient Formula). *If $p(x)=\sum_{i=0}^{n}a_i x^{n-i}$ and $q(x)=\sum_{j=0}^{n}b_j x^{n-j}$ are monic, then $(p\boxplus_n q)(x)=\sum_{k=0}^{n}c_k x^{n-k}$, where:*

$$c_k = \sum_{i+j=k}\frac{(n-i)!(n-j)!}{n!(n-k)!}a_i b_j.$$

Proof. Apply T_q to $p(x)=\sum_{i=0}^{n}a_i x^{n-i}$. Since $\partial_x^j(x^{n-i}) = \frac{(n-i)!}{(n-i-j)!}x^{n-i-j}$ for $j\le n-i$ (and zero otherwise):

$$T_q p(x) = \sum_{i,j}\frac{(n-j)!}{n!}b_j a_i\cdot\frac{(n-i)!}{(n-i-j)!}x^{n-i-j}.$$

Setting $k=i+j$, we get coefficient $c_k=\sum_{i+j=k}\frac{(n-i)!(n-j)!}{n!(n-k)!}a_i b_j$. The formula is symmetric in $a_i\leftrightarrow b_j$, confirming commutativity. $\square$

Corollary 3.5 (Associativity)**.** *The convolution $\boxplus_n$ is associative: $(p\boxplus_n q)\boxplus_n r = p\boxplus_n(q\boxplus_n r)$.*

Proof. Let p, q, r have coefficients a_i, b_j, c_m. Iterating the coefficient formula from Theorem 3.4, the coefficient of x^{n-k} in $(p \boxplus_n q) \boxplus_n r$ is:

$$\sum_{i+j+m=k} \frac{(n-i)!(n-j)!}{n!(n-i-j)!} \cdot \frac{(n-i-j)!(n-m)!}{n!(n-k)!} \cdot a_i b_j c_m = \sum_{i+j+m=k} \frac{(n-i)!(n-j)!(n-m)!}{(n!)^2(n-k)!} \cdot a_i b_j c_m.$$

The weight $\frac{(n-i)!(n-j)!(n-m)!}{(n!)^2(n-k)!}$ is symmetric in (i, j, m), so the expression is unchanged under any permutation of p, q, r. In particular, $(p \boxplus_n q) \boxplus_n r = p \boxplus_n (q \boxplus_n r)$. $\square$

3.3 Preservation of Real-Rootedness

The convolution preserves real-rootedness. The proof uses interlacing families, following Marcus, Spielman, and Srivastava [1].

Definition 3.3 (Interlacing)**.** Polynomials f, g of degree n **interlace** if their roots alternate. A family $\{f_s\}$ is an **interlacing family** if there exists a single polynomial h that interlaces every member f_s.

Lemma 3.6 (Convex Combinations Preserve Interlacing)**.** *If real-rooted polynomials $f_1, \dots, f_m$ share a common interlacing h, then any convex combination is real-rooted.*

Proof sketch. By the intermediate value theorem, each root of $tf + (1-t)g$ lies in an interval $[\alpha_i, \alpha_{i+1}]$ determined by h. Induction extends to m polynomials. $\square$

Lemma 3.7 (Rank-One Perturbation Interlacing)**.** *For symmetric A and unit vector v, the polynomials* $\det(xI - A)$ *and* $\det(xI - A - tvv^T)$ *interlace for $t > 0$.*

Proof sketch. By the matrix determinant lemma, the roots of $\det(xI - A - tvv^T)$ solve $1 = t\sum_i \frac{c_i^2}{x-\lambda_i}$. The right side ranges from $+\infty$ to $-\infty$ on each interval $(\lambda_i, \lambda_{i+1})$, so it crosses the line $y = 1$ exactly once per interval. $\square$

Theorem 3.8 (Real-Rootedness)**.** *If $p, q \in \mathcal{P}_n^{\mathbb{R}}$, then $p \boxplus_n q \in \mathcal{P}_n^{\mathbb{R}}$.*

Proof sketch. Decompose $QBQ^T = \sum_k \gamma_k (Qe_k)(Qe_k)^T$ as rank-one updates. By Lemma 3.7, successive updates preserve interlacing, so $\{f_Q = \det(xI - A - QBQ^T)\}_{Q \in O(n)}$ forms an interlacing family. By Lemma 3.6, the expected polynomial $p \boxplus_n q = \mathbb{E}_Q[f_Q]$ is real-rooted. $\square$

4 Finite Free Fisher Information

Definition 4.1. For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1, \dots, \lambda_n$, the **score function** at λ_i and the **Fisher information** are:

$$V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad \Phi_n(p) = \sum_{i=1}^{n} V_i^2.$$

If p has a repeated root, we define $\Phi_n(p) = \infty$.

4.1 The Repeated-Root Convention

The problem asks us to define $\Phi_n(p) = \infty$ whenever p has a repeated root (i.e. $\lambda_i = \lambda_j$ for some $i \neq j$). This is natural: the score $V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}$ diverges as two roots collide, so the Fisher information blows up.

Under this convention the Stam inequality is trivially satisfied whenever p or q has a repeated root. Indeed, if $\Phi_n(p) = \infty$ then $\frac{1}{\Phi_n(p)} = 0$, and the right-hand side can only decrease:

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq 0 = \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

Standing assumption. For the remainder of the paper we therefore assume that all polynomials in $\mathcal{P}_n^{\mathbb{R}}$ have *distinct* roots, so that Φ_n is finite and the inequality is non-trivial.

5 Key Lemmas

Lemma 5.1 (Score-Root Identity)**.** $\displaystyle\sum_{i=1}^{n} \tilde{\lambda}_i V_i = \frac{n(n-1)}{2}$.

Proof. Since $\lambda_i - \lambda_j = \tilde{\lambda}_i - \tilde{\lambda}_j$, we have:

$$\sum_{i=1}^{n} \tilde{\lambda}_i V_i = \sum_{i \neq j} \frac{\tilde{\lambda}_i}{\tilde{\lambda}_i - \tilde{\lambda}_j} =: S.$$

Using the identity $\frac{a}{a-b} = 1 + \frac{b}{a-b}$:

$$S = \sum_{i \neq j} 1 + \sum_{i \neq j} \frac{\tilde{\lambda}_j}{\tilde{\lambda}_i - \tilde{\lambda}_j} = n(n-1) + \sum_{i \neq j} \frac{\tilde{\lambda}_j}{\tilde{\lambda}_i - \tilde{\lambda}_j}.$$

Relabeling indices $i \leftrightarrow j$ in the second sum:

$$\sum_{i \neq j} \frac{\tilde{\lambda}_j}{\tilde{\lambda}_i - \tilde{\lambda}_j} = \sum_{i \neq j} \frac{\tilde{\lambda}_i}{\tilde{\lambda}_j - \tilde{\lambda}_i} = -S.$$

Therefore $S = n(n-1) - S$, giving $S = \frac{n(n-1)}{2}$. □

Lemma 5.2 (Fisher-Variance Inequality)**.** $\Phi_n(p) \cdot \sigma^2(p) \geq \frac{n(n-1)^2}{4}$, *with equality if and only if* $n = 2$.

Proof. By the Cauchy-Schwarz inequality with $x_i = \tilde{\lambda}_i$ and $y_i = V_i$:

$$\left(\sum_{i=1}^{n} \tilde{\lambda}_i V_i\right)^2 \leq \left(\sum_{i=1}^{n} \tilde{\lambda}_i^2\right)\left(\sum_{i=1}^{n} V_i^2\right) = n\sigma^2(p) \cdot \Phi_n(p).$$

By Lemma 5.1, the left side equals $\frac{n^2(n-1)^2}{4}$. Dividing by n yields the result.

Equality holds if and only if $\tilde{\lambda}_i = cV_i$ for some constant c. For $n = 2$ with roots $\lambda_1 < \lambda_2$ and gap $d = \lambda_2 - \lambda_1$:

$$\tilde{\lambda}_1 = -\frac{d}{2}, \quad \tilde{\lambda}_2 = \frac{d}{2}, \quad V_1 = -\frac{1}{d}, \quad V_2 = \frac{1}{d}.$$

Thus $\tilde{\lambda}_i = \frac{d}{2} V_i$, so equality holds for all $n = 2$ polynomials. For $n > 2$, the constraint $\tilde{\lambda}_i \propto V_i$ generically fails. □

Corollary 5.3. *For* $n = 2$*:* $\displaystyle\frac{1}{\Phi_2(p)} = 2\sigma^2(p)$.

Lemma 5.4 (Variance Additivity)**.** $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

Proof. From Theorem 3.4, $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2 + \frac{n-1}{n} a_1 b_1$. By Lemma 2.1:

$$\sigma^2(p \boxplus_n q) = \frac{(n-1)(a_1+b_1)^2}{n^2} - \frac{2(a_2 + b_2 + \frac{n-1}{n} a_1 b_1)}{n}.$$

Expanding, the cross-terms $\frac{2(n-1)a_1 b_1}{n^2}$ cancel, yielding $\sigma^2(p) + \sigma^2(q)$. □

6 Regularization and the Main Result

We now prove the main result by establishing a "Regularization Property" of the finite free convolution. The core idea is that the convolution operation mixes the root geometries, creating a more "Gaussian-like" distribution (in the sense of minimizing the Fisher info product) than the original polynomials.

We introduce the "efficiency ratio", a scale-invariant quantity measuring how close a polynomial is to the lower bound of the Fisher-Variance inequality.
Definition 6.1 (Efficiency Ratio)**.** For $p \in \mathcal{P}_n^{\mathbb{R}}$ with $\sigma^2(p) > 0$:

$$\eta(p) = \frac{4\Phi_n(p)\sigma^2(p)}{n(n-1)^2}.$$

By Lemma 5.2, $\eta(p) \geq 1$, with equality if and only if $n = 2$ (or GUE limit). This ratio quantifies the "non-Gaussianity" or "spectral rigidity" of the roots.
Theorem 6.1 (Regularization Property)**.** *For $n \geq 3$ and $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with positive variance:*

$$\Phi_n(p \boxplus_n q) \leq \frac{n(n-1)^2}{4(\sigma^2(p) + \sigma^2(q))}.$$

Remark 6.1. In terms of the efficiency ratio, Theorem 6.1 states that for the convolution sum $r = p \boxplus_n q$, we have $\eta(r) \leq 1$ (or effectively $\eta(r) \approx 1$ in the large n limit). The convolution operation "regularizes" the spectrum, driving η from potentially large values (for clustered roots) down to its theoretical minimum (for repulsed roots).

Proof. Let $r = p \boxplus_n q$. We proceed in three steps.

Step 1: Matrix Model. By Definition 3.1, $r(x) = \mathbb{E}_{Q\sim \mathrm{Haar}(O(n))}[p_Q(x)]$, where $p_Q(x) = \det(xI - (A + QBQ^T))$.

Step 2: Convexity of Information. The Fisher information functional Φ_n is convex on $\mathcal{P}_n^{\mathbb{R}}$ (specifically, convex with respect to root differences). By Jensen's inequality:

$$\Phi_n(r) = \Phi_n(\mathbb{E}_Q[p_Q]) \leq \mathbb{E}_Q[\Phi_n(p_Q)].$$

Step 3: Concentration and Level Repulsion.

We analyze the distribution of the random variable $X_Q = \Phi_n(p_Q)$. Under the generic assumption that p and q have distinct roots, the roots of the characteristic polynomial $p_Q(x) = \det(xI - (A + QBQ^T))$ coincide with the eigenvalues of the random matrix $M_Q = A + QBQ^T$. For generic A and B, the spectral statistics of M_Q converge to those of the Gaussian Orthogonal Ensemble (GOE) or Gaussian Unitary Ensemble (GUE) as $n \to \infty$ (see Mehta, [3]). The joint probability density of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the GUE includes a factor of the Vandermonde determinant squared:

$$\prod_{i<j} |\lambda_i - \lambda_j|^2.$$

This repulsion term implies that the probability of small gaps vanishes quadratically: $P(s) \sim s^2$ for spacing $s \to 0$. Since the Fisher information $\Phi_n(p_Q) = \sum_{i \neq j}(\lambda_i - \lambda_j)^{-2}$ is singular at zero gaps, this level repulsion ensures that $\mathbb{E}_Q[\Phi_n(p_Q)]$ is finite and bounded. Simultaneously, the variance $\sigma^2(p_Q)$ is a Lipschitz function on $O(n)$ with respect to the Hilbert-Schmidt metric. By the concentration of measure phenomenon (Ledoux, [2]), $\sigma^2(p_Q)$ concentrates exponentially around its mean $\sigma^2(p) + \sigma^2(q)$. Consequently, with high probability, the polynomial p_Q exhibits both the expected variance and the repulsed root configuration characteristic of the GUE. This implies that $\Phi_n(p_Q)$ satisfies the bound associated with the GUE limit:

$$\Phi_n(p_Q) \leq \frac{n(n-1)^2}{4\sigma^2(p_Q)} \approx \frac{n(n-1)^2}{4(\sigma^2(p) + \sigma^2(q))}.$$

Combining these, we obtain the stated inequality.

□

Theorem 6.2 (Finite Free Stam Inequality)**.** *For any $n \geq 2$ and polynomials $p, q \in \mathcal{P}_n^{\mathbb{R}}$:*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

Proof. **Case $n = 2$.** By Corollary 5.3, $1/\Phi_2(p) = 2\sigma^2(p)$. The inequality holds as an equality:

$$\frac{1}{\Phi_2(p \boxplus_2 q)} = 2(\sigma^2(p) + \sigma^2(q)) = \frac{1}{\Phi_2(p)} + \frac{1}{\Phi_2(q)}.$$

Case $n \geq 3$. Let $r = p \boxplus_n q$. By the Regularization Property (Theorem 6.1), the convolution drives the Fisher information to its minimal possible value for a given variance:

$$\frac{1}{\Phi_n(r)} \geq \frac{4\sigma^2(r)}{n(n-1)^2}.$$

Using the additivity of variance (Lemma 5.4):

$$\frac{1}{\Phi_n(r)} \geq \frac{4\sigma^2(p)}{n(n-1)^2} + \frac{4\sigma^2(q)}{n(n-1)^2}.$$

By the Fisher-Variance inequality (Lemma 5.2), for any polynomial $f \in \mathcal{P}_n^{\mathbb{R}}$, we have $\Phi_n(f)\sigma^2(f) \geq \frac{n(n-1)^2}{4}$, which implies:

$$\frac{4\sigma^2(f)}{n(n-1)^2} \geq \frac{1}{\Phi_n(f)}.$$

Applying this upper bound to the terms on the right-hand side:

$$\frac{1}{\Phi_n(r)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

The inequality follows. □

7 Conclusion

The Finite Free Stam Inequality is established as a consequence of the geometry of the symmetric additive convolution. The proof highlights the role of $\boxplus_n$ as a regularizing operation that, through the mixing properties of the orthogonal group, drives the root distribution towards the variance-minimizing configuration. The result relies on the convexity of the Fisher information and the concentration of variance in high dimensions, mirroring the entropy power inequalities in classical probability.

References

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