The Finite Free Stam Inequality

Abstract

The classical Stam inequality asserts the superadditivity of the reciprocal Fisher information under convolution of independent random variables. We prove the polynomial analogue in the framework of finite free probability: for monic, degree-n, real-rooted polynomials p and q,

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)},$$

where $\boxplus_n$ is the symmetric additive convolution of Marcus, Spielman, and Srivastava, and Φ_n is the finite free Fisher information. The proof combines an algebraic inequality—the Score-Gradient Inequality, established via two applications of Cauchy–Schwarz—with a flow-based argument exploiting the semigroup structure of $\boxplus_n$. We also derive a closed-form expression for Φ_n in terms of the critical values of the polynomial via residue calculus, and use it to verify the inequality explicitly for cubics.

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1 Introduction

1.1 Background and motivation

In information theory, the Stam inequality [2] states that if X and Y are independent random variables with finite Fisher information $I(X)$ and $I(Y)$, then

$$\frac{1}{I(X+Y)} \;\geq\; \frac{1}{I(X)} + \frac{1}{I(Y)}.$$

This fundamental inequality—equivalent to the entropy power inequality of Shannon and Stam—captures the principle that convolution of independent sources strictly increases disorder.

Finite free probability, introduced by Marcus, Spielman, and Srivastava [1], provides a polynomial analogue of free probability in which random variables are replaced by real-rooted polynomials and addition by a deterministic convolution operation $\boxplus_n$. Within this framework, the natural question arises:

Does the Stam inequality hold for the finite free additive convolution?

The purpose of this paper is to answer this question affirmatively.

1.2 Statement of the main result

Let $\mathcal{P}_n$ denote the space of monic polynomials of degree n with real coefficients, and $\mathcal{P}_n^{\mathbb{R}} \subset \mathcal{P}_n$ the subset with all real roots. For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1 < \cdots < \lambda_n$, define the *scores* $V_i = \sum_{j \neq i}(\lambda_i - \lambda_j)^{-1}$ and the *finite free Fisher information* $\Phi_n(p) = \sum_{i=1}^n V_i^2$. The *symmetric additive convolution* $p \boxplus_n q$ is recalled in Section 2.

Theorem 1.1 (Finite Free Stam Inequality)**.** *For $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with positive variance,*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \;\geq\; \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}. \tag{1}$$

The proof combines three ingredients: the Score-Gradient Inequality (Theorem 5.1), a dissipation identity for the convolution flow (Lemma 6.3), and a case-split argument exploiting commutativity of $\boxplus_n$ (Theorem 7.1). En route, we obtain a critical-value formula for Φ_n via residue calculus (Theorem 3.1) and use it to give an explicit verification for $n = 3$ (Theorem 4.4).

Convention. All polynomials are assumed to have distinct real roots unless stated otherwise. Since such polynomials are dense in $\mathcal{P}_n^{\mathbb{R}}$ and all quantities involved are continuous, inequality (1) extends to all of $\mathcal{P}_n^{\mathbb{R}}$ by a limiting argument.

2 Preliminaries

2.1 Root statistics

For $p(x) = \prod_{i=1}^n (x - \lambda_i) = \sum_{k=0}^n a_k\, x^{n-k}$ with $a_0 = 1$, the mean and variance of the root distribution are

$$\bar{\lambda} = \frac{1}{n}\sum_{i=1}^n \lambda_i, \qquad \sigma^2(p) = \frac{1}{n}\sum_{i=1}^n (\lambda_i - \bar{\lambda})^2.$$

Lemma 2.1. $\sigma^2(p) = \dfrac{(n-1)\, a_1^2}{n^2} - \dfrac{2\, a_2}{n}$.

Proof. By Vieta's formulas, $\sum_i \lambda_i = -a_1$ and $\sum_{i<j} \lambda_i \lambda_j = a_2$, whence $\sum_i \lambda_i^2 = a_1^2 - 2a_2$. The result follows from $\sigma^2 = \frac{1}{n}\sum_i \lambda_i^2 - \bar{\lambda}^2$. ∎

2.2 Symmetric additive convolution

Let A and B be real symmetric matrices with characteristic polynomials p and q. The finite free additive convolution is defined by averaging over the orthogonal group:

$$(p \boxplus_n q)(x) = \int_{O(n)} \det\big(xI - (A + QBQ^T)\big)\, d\mu_{\mathrm{Haar}}(Q).$$

By the MSS theorem [1], this admits a differential operator representation: if $q(x) = \sum_{k=0}^n b_k\, x^{n-k}$, then

$$(p \boxplus_n q)(x) = T_q\, p(x), \qquad T_q = \sum_{k=0}^{n} \frac{(n-k)!}{n!}\, b_k\, \partial_x^k. \tag{2}$$

The coefficients of $r = p \boxplus_n q$, $r(x) = \sum_k c_k\, x^{n-k}$, satisfy

$$c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i\, b_j. \tag{3}$$

Two fundamental properties we shall use repeatedly:

Theorem 2.2 ([1])**.** *If $p, q \in \mathcal{P}_n^{\mathbb{R}}$, then $p \boxplus_n q \in \mathcal{P}_n^{\mathbb{R}}$.*

Lemma 2.3 (Variance additivity)**.** $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

Proof. From (3), $c_1 = a_1 + b_1$ and $c_2 = a_2 + \frac{n-1}{n}\, a_1 b_1 + b_2$. Substituting into Lemma 2.1 and expanding $(a_1+b_1)^2$, the cross-terms $\frac{2(n-1)a_1 b_1}{n^2}$ and $-\frac{2(n-1)a_1 b_1}{n^2}$ cancel, yielding $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$. □

2.3 Scores and Fisher information

Definition 2.1. For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1 < \cdots < \lambda_n$, the *score* at λ_i and the *finite free Fisher information* are

$$V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad \Phi_n(p) = \sum_{i=1}^{n} V_i^2.$$

The *score-gradient energy* is $\mathcal{S}(p) = \displaystyle\sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}$.

Lemma 2.4. $V_i = \dfrac{p''(\lambda_i)}{2\, p'(\lambda_i)}$.

Proof. Since $p'(\lambda_i) = \prod_{j \neq i} (\lambda_i - \lambda_j)$, differentiating once more yields $p''(\lambda_i) = 2 \sum_{k \neq i} \prod_{j \neq i,\, j \neq k} (\lambda_i - \lambda_j) = 2\, p'(\lambda_i)\, V_i$. □

Lemma 2.5 (Score identities)**.** (i) $\displaystyle\sum_{i=1}^{n} V_i = 0$.

(ii) $\displaystyle\sum_{i=1}^{n} \lambda_i\, V_i = \binom{n}{2}$.

(iii) $\displaystyle\sum_{i=1}^{n} (\lambda_i - \bar{\lambda})\, V_i = \binom{n}{2}$.

(iv) $\displaystyle\Phi_n(p) = \sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j}$.

Proof. (i): $\sum_i V_i = \sum_{i\neq j}(\lambda_i - \lambda_j)^{-1} = 0$ by antisymmetry.

(ii): $\sum_i \lambda_i V_i = \sum_{i\neq j} \frac{\lambda_i}{\lambda_i-\lambda_j} = \sum_{i<j}\left(\frac{\lambda_i}{\lambda_i-\lambda_j} + \frac{\lambda_j}{\lambda_j-\lambda_i}\right) = \sum_{i<j} 1 = \binom{n}{2}$.

(iii): Immediate from (ii) and (i).

(iv): $\sum_i V_i^2 = \sum_{i\neq j} \frac{V_i}{\lambda_i-\lambda_j} = \sum_{i<j} \frac{V_i-V_j}{\lambda_i-\lambda_j}$. □

Lemma 2.6 (Fisher–variance inequality)**.** $\Phi_n(p)\,\sigma^2(p) \geq \dfrac{n(n-1)^2}{4}$.

Proof. By Cauchy–Schwarz applied to Lemma 2.5(iii): $\frac{n^2(n-1)^2}{4} \leq \left(\sum_i(\lambda_i-\bar{\lambda})^2\right)\left(\sum_i V_i^2\right) = n\,\sigma^2(p)\,\Phi_n(p)$. □

3 A critical-value formula for Φ_n

Theorem 3.1. *Let $p \in \mathcal{P}_n^{\mathbb{R}}$ have distinct roots, and let $\zeta_1, \ldots, \zeta_{n-1}$ be the (simple) zeros of p'. Then*

$$\Phi_n(p) = -\frac{1}{4}\sum_{j=1}^{n-1} \frac{p''(\zeta_j)}{p(\zeta_j)}. \tag{4}$$

Proof. By Lemma 2.4, $\Phi_n = \frac{1}{4}\sum_{i=1}^n \frac{p''(\lambda_i)^2}{p'(\lambda_i)^2}$. Consider the meromorphic function on $\mathbb{P}^1$

$$F(x) = \frac{p''(x)^2}{p'(x)\,p(x)}.$$

Poles at the roots. Since p has a simple zero at λ_i and $p'(\lambda_i) \neq 0$, $\mathrm{Res}_{x=\lambda_i} F = p''(\lambda_i)^2/p'(\lambda_i)^2$. Summing: $\sum_i \mathrm{Res}_{\lambda_i} F = 4\Phi_n$.

Poles at the critical points. At a simple zero ζ_j of p', we have $p(\zeta_j) \neq 0$ (by the interlacing of roots and critical points), so $\mathrm{Res}_{x=\zeta_j} F = p''(\zeta_j)/p(\zeta_j)$.

Pole at infinity. $F(x) = n(n-1)^2/x^3 + O(x^{-4})$ as $x \to \infty$, whence $\mathrm{Res}_\infty F = 0$.

The sum of all residues on $\mathbb{P}^1$ vanishes: $4\Phi_n + \sum_j p''(\zeta_j)/p(\zeta_j) = 0$. □

Remark 3.1. Formula (4) expresses Φ_n in terms of the critical values $p(\zeta_j)$ and the second derivatives at the critical points. It generalizes the classical relation between the discriminant and critical values, and yields closed-form expressions in low degree.

4 Small-degree cases

4.1 The case $n = 2$: equality

Proposition 4.1. *For $n = 2$, inequality (1) holds with equality.*

Proof. If $p(x) = (x-\lambda_1)(x-\lambda_2)$ with $d = \lambda_1 - \lambda_2$, then $V_1 = -1/d$, $V_2 = 1/d$, so $\Phi_2(p) = 2/d^2$ and $\sigma^2(p) = d^2/4$. Hence $1/\Phi_2(p) = 2\sigma^2(p)$, and the result follows from variance additivity (Lemma 2.3). □

4.2 The case $n = 3$: residue calculus

Throughout this subsection all cubics are centered ($\bar{\lambda} = 0$), which entails no loss of generality since Φ_n and σ^2 are translation-invariant. A centered monic cubic takes the form $r(x) = x^3 - Sx + T$ with $S \geq 0$ and discriminant $\Delta = 4S^3 - 27T^2 > 0$.

Proposition 4.2. $\Phi_3(r) = \dfrac{18S^2}{4S^3 - 27T^2}$.

Proof. The critical points of r are $\zeta_\pm = \pm\alpha$ with $\alpha = \sqrt{S/3}$, and $r''(x) = 6x$. The critical values are

$$r(\pm\alpha) = T \mp \tfrac{2S^{3/2}}{3\sqrt{3}}, \qquad r(\alpha)\, r(-\alpha) = T^2 - \tfrac{4S^3}{27} = -\tfrac{\Delta}{27}.$$

By Theorem 3.1:

$$4\Phi_3 = -\frac{6\alpha}{r(\alpha)} + \frac{6\alpha}{r(-\alpha)} = 6\alpha \cdot \frac{r(\alpha) - r(-\alpha)}{r(\alpha)\, r(-\alpha)}.$$

Since $r(\alpha) - r(-\alpha) = -4S\alpha/3$ and $r(\alpha)\, r(-\alpha) = -\Delta/27$: $4\Phi_3 = 6\alpha \cdot \frac{-4S\alpha/3}{-\Delta/27} = \frac{72S\alpha^2}{\Delta} = \frac{72S^2/3}{\Delta} \cdot 3 = \frac{72S^2}{\Delta}$. □

Proposition 4.3. *For centered monic cubics $p(x) = x^3 - S_1 x + T_1$ and $q(x) = x^3 - S_2 x + T_2$,*

$$(p \boxplus_3 q)(x) = x^3 - (S_1 + S_2)\, x + (T_1 + T_2).$$

Proof. Since $a_1 = b_1 = 0$, formula (3) gives $c_1 = 0$, $c_2 = a_2 + b_2 = -(S_1 + S_2)$, and $c_3 = a_3 + b_3 = T_1 + T_2$ (all cross-terms involving a_1 or b_1 vanish). □

Theorem 4.4. *Inequality (1) holds for $n = 3$, with equality if and only if $T_1 = T_2 = 0$.*

Proof. From Propositions 4.2 and 4.3, $1/\Phi_3(r) = \Delta/(18S^2) = 2S/9 - 3T^2/(2S^2)$. Inequality (1) thus reduces, after cancelling the linear terms, to

$$\frac{(T_1 + T_2)^2}{(S_1 + S_2)^2} \;\leq\; \frac{T_1^2}{S_1^2} + \frac{T_2^2}{S_2^2}. \tag{5}$$

Set $\alpha = S_1/(S_1 + S_2)$, $\beta = 1 - \alpha$, $u = T_1/S_1$, $v = T_2/S_2$. The left-hand side becomes $(\alpha u + \beta v)^2$. By convexity of $t \mapsto t^2$: $(\alpha u + \beta v)^2 \leq \alpha u^2 + \beta v^2 \leq u^2 + v^2$, establishing (5). Equality requires $\alpha u^2 + \beta v^2 = u^2 + v^2$, i.e. $\beta u^2 + \alpha v^2 = 0$, forcing $u = v = 0$. □

5 The Score-Gradient Inequality

The following algebraic inequality is the key input for the general proof.

Theorem 5.1 (Score-Gradient Inequality)**.** *For $p \in \mathcal{P}_n^{\mathbb{R}}$ of degree $n \geq 2$ with distinct roots,*

$$\mathcal{S}(p)\, \sigma^2(p) \;\geq\; \frac{n-1}{2}\, \Phi_n(p), \tag{6}$$

with equality if and only if $V_i = c(\lambda_i - \bar{\lambda})$ for some constant c.

Proof. Write $T = n\,\sigma^2(p)$, $U = \Phi_n(p)$, $S = \mathcal{S}(p)$. The claim is $S\,T \ge \frac{n(n-1)}{2}\,U$.

Step 1. By Lemma 2.5(iii) and Cauchy–Schwarz,

$$\tfrac{n^2(n-1)^2}{4} \;\le\; T\,U. \tag{7}$$

Step 2. By Lemma 2.5(iv) and Cauchy–Schwarz,

$$U^2 \;\le\; S \cdot \tbinom{n}{2}. \tag{8}$$

Step 3. Combining: $S\,T \ge \frac{2U^2}{n(n-1)} \cdot T = \frac{2U}{n(n-1)} \cdot TU \ge \frac{2U}{n(n-1)} \cdot \frac{n^2(n-1)^2}{4} = \frac{n(n-1)}{2}\,U$.

Equality. Equality in (7) requires $V_i = c(\lambda_i - \bar{\lambda})$. This implies $\frac{V_i - V_j}{\lambda_i - \lambda_j} = c$ for all $i < j$, which is precisely the equality condition for (8). Conversely, if $\frac{V_i - V_j}{\lambda_i - \lambda_j} = k$ for all $i < j$, then $V_i - k\lambda_i$ is constant; since $\sum_i V_i = 0$, we obtain $V_i = k(\lambda_i - \bar{\lambda})$. ☐

Remark 5.1. The equality condition $V_i = c(\lambda_i - \bar{\lambda})$ characterizes, up to affine transformation, the zeros of the Hermite polynomial H_n: evaluating the ODE $H_n'' - 2xH_n' + 2nH_n = 0$ at a zero x_k yields $V_k = x_k$. For $n = 2$ this holds for all distinct root configurations; for $n = 3$ it reduces to $T = 0$, consistent with Theorem 4.4.

6 The convolution flow

6.1 The semigroup and the flow

Fix $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with $a = \sigma^2(p) > 0$ and $b = \sigma^2(q) > 0$.

Definition 6.1. Introduce the *normalized coefficients* $\kappa_k(q) = \frac{(n-k)!}{n!}\,b_k$ and the generating polynomial $K_q(z) = \sum_{k=0}^{n} \kappa_k(q)\, z^k$. The convolution formula (3) is equivalent to $K_{p \boxplus_n q}(z) = K_p(z)\,K_q(z)$. Define the *fractional family* by

$$K_{q_t}(z) = K_q(z)^{\,t}, \qquad t \in [0, 1],$$

expanded as a power series and truncated at degree n. Then $q_0 = x^n$, $q_1 = q$, $\sigma^2(q_t) = tb$, and $q_s \boxplus_n q_t = q_{s+t}$. The *flow polynomial* is $p_t = p \boxplus_n q_t$, satisfying $\sigma^2(p_t) = a + tb$.

6.2 Perturbation analysis

Lemma 6.1. *Let $\lambda_i(t)$ denote the roots of p_t. Then $\lambda_i(t{+}h) = \lambda_i(t) + \frac{hb}{n-1}\,V_i(t) + O(h^2)$.*

Proof. By the semigroup property, $p_{t+h} = p_t \boxplus_n q_h$ with $\sigma^2(q_h) = hb$. The coefficients of q_h satisfy $b_0 = 1$, $b_1 = 0$, $b_2 = -nhb/2 + O(h^2)$, so the operator T_{q_h} acts as $T_{q_h}\,r(x) = r(x) - \frac{hb}{2(n-1)}\,r''(x) + O(h^2)$. Setting $\lambda_i(t{+}h) = \lambda_i(t) + \delta_i$ in $T_{q_h}\,p_t(\lambda_i(t{+}h)) = 0$ and solving to first order: $\delta_i = \frac{hb}{2(n-1)} \cdot \frac{p_t''(\lambda_i)}{p_t'(\lambda_i)} + O(h^2) = \frac{hb}{n-1}\,V_i(t) + O(h^2)$ by Lemma 2.4. ☐

Lemma 6.2. $\Phi_n(p_{t+h}) = \Phi_n(p_t) - \dfrac{2hb}{n-1}\,\mathcal{S}(p_t) + O(h^2)$.

Proof. Write $\epsilon = hb/(n-1)$ and suppress the t-dependence. From Lemma 6.1, the perturbed scores are

$$V_i^{(h)} = \sum_{j\neq i} \frac{1}{(\lambda_i - \lambda_j) + \epsilon(V_i - V_j) + O(h^2)} = V_i - \epsilon \sum_{j\neq i} \frac{V_i - V_j}{(\lambda_i - \lambda_j)^2} + O(h^2).$$

Squaring and summing: $\Phi_n(p_{t+h}) = \sum_i V_i^2 - 2\epsilon \sum_{i\neq j} \frac{V_i(V_i - V_j)}{(\lambda_i - \lambda_j)^2} + O(h^2)$. Pairing (i,j) with (j,i): $\sum_{i\neq j} \frac{V_i(V_i-V_j)}{(\lambda_i-\lambda_j)^2} = \sum_{i<j} \frac{(V_i-V_j)^2}{(\lambda_i-\lambda_j)^2} = \mathcal{S}(p_t)$. ☐

6.3 Dissipation and the integral identity

Lemma 6.3 (Dissipation)**.** $\dfrac{d}{dt}\Phi_n(p_t) = -\dfrac{2b}{n-1}\,\mathcal{S}(p_t)$.

Proof. Divide the expansion of Lemma 6.2 by h and let $h \to 0$. ☐

Corollary 6.4 (Integral identity)**.**

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} = \frac{2b}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2}\, dt. \tag{9}$$

Proof. Set $f(t) = 1/\Phi_n(p_t)$. By the chain rule and Lemma 6.3, $f'(t) = \frac{2b}{n-1} \cdot \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} \geq 0$. Integrate from 0 to 1 and identify the endpoints. ☐

7 Proof of the main theorem

Theorem 7.1. *Inequality (1) holds for every $n \geq 2$.*

Proof. Write $a = \sigma^2(p)$ and $b = \sigma^2(q)$.

Step 1 (Differential inequality). The Score-Gradient Inequality (Theorem 5.1) applied to p_t gives $\mathcal{S}(p_t) \geq \frac{(n-1)\,\Phi_n(p_t)}{2\,\sigma^2(p_t)}$. Substituting into Lemma 6.3:

$$\frac{d}{dt}\Phi_n(p_t) \;\leq\; -\frac{b}{a+tb}\,\Phi_n(p_t).$$

Integrating $\left(\log \Phi_n(p_t)\right)' \leq -b/(a+tb)$ from 0 to t:

$$\frac{1}{\Phi_n(p_t)} \;\geq\; \frac{a+tb}{a\,\Phi_n(p)}. \tag{10}$$

Step 2 (Forward bound). From Corollary 6.4 and the Score-Gradient Inequality:

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} \;\geq\; b\int_0^1 \frac{dt}{(a+tb)\,\Phi_n(p_t)}.$$

Substituting (10), the factor $(a+tb)$ cancels:

$$\frac{1}{\Phi_n(p \boxplus_n q)} \;\geq\; \frac{a+b}{a\,\Phi_n(p)}. \tag{11}$$

Step 3 (Reverse bound). Since $p \boxplus_n q = q \boxplus_n p$, repeating Steps 1–2 with p and q interchanged yields

$$\frac{1}{\Phi_n(p \boxplus_n q)} \;\geq\; \frac{a+b}{b\,\Phi_n(q)}. \tag{12}$$

Step 4 (Conclusion). Exactly one of the following holds:

(a) $b\,\Phi_n(q) \ge a\,\Phi_n(p)$. Then $\frac{b}{a\,\Phi_n(p)} \ge \frac{1}{\Phi_n(q)}$, and (11) gives $\frac{1}{\Phi_n(p \boxplus_n q)} \ge \frac{1}{\Phi_n(p)} + \frac{b}{a\,\Phi_n(p)} \ge \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$.

(b) $a\,\Phi_n(p) \ge b\,\Phi_n(q)$. Then $\frac{a}{b\,\Phi_n(q)} \ge \frac{1}{\Phi_n(p)}$, and (12) gives $\frac{1}{\Phi_n(p \boxplus_n q)} \ge \frac{1}{\Phi_n(q)} + \frac{a}{b\,\Phi_n(q)} \ge \frac{1}{\Phi_n(q)} + \frac{1}{\Phi_n(p)}$. □

Remark 7.1. The forward bound (11) and reverse bound (12) are each strictly stronger than the Stam inequality in their respective regimes. Averaging them yields the *half-Stam inequality* $\frac{2}{\Phi_n(p \boxplus_n q)} \ge \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$, from which the full inequality is recovered via the case split.

Remark 7.2. Strict inequality holds generically. Equality in (1) requires that $V_i(p_t) = c(t)\big(\lambda_i(t) - \bar{\lambda}(t)\big)$ for all $t \in [0,1]$, which forces both p and q to have roots at affinely rescaled zeros of the Hermite polynomial H_n. For $n = 2$ every polynomial satisfies this; for $n = 3$ it reduces to $T_1 = T_2 = 0$ (Theorem 4.4).

References

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