Université de Paris

IMAGINARIES IN PAIRS OF ALGEBRAICALLY CLOSED FIELDS

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Abstract

Consider the theory T of algebraically closed fields of a given characteristic p, in the language $L = \{0, 1, +, -, \cdot\}$. Extend L to a language L_P by adding a predicate P, which is interpreted in a model $M \models T$ as a proper elementary substructure. Since T has elimination of quantifiers, these pairs can be axiomatized by expressing $P \models T$, and $\exists x\, \neg P(x)$, obtaining a theory T_P of elementary pairs $P \prec M$. The main goal is to add sorts to the language L_P, in order to achieve *weak elimination of imaginaries*. Keisler in [5] proved that T_P is complete, and in [2], Buechler showed that T_P is an ω-stable theory, of Morley rank ω. This work is largely based on [9], by Anand Pillay.

Contents

1 Preliminaries on Stability Theory

Let T be a complete theory over a language L. If $M \models T$, and $A \subseteq M$, we denote the space of n-types with parameters in A by $S_n(A)$, and let $S(A) = \cup_{i<\omega} S_n(A)$. Recall that a theory is κ-stable if for every $M \models T$ and every $A \subseteq M$, if $|A| \leq \kappa$, then $|S_1(A)| \leq \kappa$, and we say T is stable if it is κ-stable for some cardinal κ. We will use an equivalent characterization of stability, given by the definability of types.

Definition 1.1. Let $M \models T$, and A, B subsets of M. A type $p(x) \in S_n(A)$ is *definable* over B if for any L-formula $\varphi(x, y)$ there is an $L(B)$-formula $\psi(y)$ such that for all $a \in A^{|y|}$, $\varphi(x, a) \in p$ if and only if $M \models \psi(a)$. The formula $\psi(y)$ will be writen as $d_p(\varphi)(y)$, and the set of $d_p(\varphi)(y)$, with $\varphi(x, y)$ ranging over the L-formulas is called a *definition scheme* for p.

The following proposition is Corollary 8.3.2 from [13].

Proposition 1.2. *The theory T is stable if and only if all types are definable.*

Throughout the rest of the section we assume T is an arbitrary, complete, ω-stable theory. We will work inside a saturated model M of T, and types over M shall be refered as *global types*. We proceed by stating some definitions and results on canonical bases and forking in this stable context.

Definition 1.3. Let $E(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be an L-formula that defines an equivalence relation on M^n. By *real elements*, we mean tuples in M^n, while the equivalence classes of real elements modulo E will be called *imaginary elements.*

Definition 1.4. Let $X \subseteq M$ be a definable set. A tuple $c \subset M$ is called a *canonical parameter* (or code) of X if c is fixed by exactly the same automorphisms of M which fix X setwise.

It is possible to extend T to a new theory T^{eq} (in a new language L^{eq}), in which every definable set has a code. Let $(E_i)_{i\in I}$, an enumeration of every $\varnothing$-definable equivalence relation over n_i-tuples. To define L^{eq}, add to L a new sort S_i for each i, which is to be interpreted as M^{n_i}/E_i. Consider the many-sorted structure $M^{\text{eq}} = (M, M^{n_i}/E_i)_{i\in I}$, and define for every i the natural projection $\pi_i : M^{n_i} \mapsto M^{n_i}/E_i$ that sends a to a/E_i. The theory of M^{eq} will be denoted as T^{eq}. By Corollary 8.4.6 from [13], T^{eq} has *elimination of imaginaries:* every imaginary is interdefinable with a real tuple. There are also three related notions that will be used throughout this work.

Definition 1.5.

i) T has *elimination of finite imaginaries* if for every n, every finite set of n-tuples has a canonical parameter.

ii) T has *weak elimination of imaginaries*, if for every imaginary e there is a real tuple d such that $e \in \mathrm{dcl}^{\mathrm{eq}}(c)$ and $d \in \mathrm{acl}^{\mathrm{eq}}(e)$.

iii) T has *geometric elimination of imaginaries*, if for every imaginary e there is a real tuple d such that $e \in \mathrm{acl}^{\mathrm{eq}}(c)$ and $d \in \mathrm{acl}^{\mathrm{eq}}(e)$.

We now proceed with a survey of forking in the **ω-stable context**. For a definable set $X \subseteq M$, we denote by $RM(X)$ its Morley rank, and $DM(X)$ its Morley degree. Recall that ω-stable theories are *totally transcendental*: every definable set has a Morley rank. This rank can also be defined for types: if $p \in S_n(A)$, then $RM(p)$ is the minimal Morley rank of a formula in p, and $DM(p)$ is the minimal Morley degree of a formula in p having Morley rank $RM(p)$.

Definition 1.6. (Forking) Suppose $A \subseteq B \subseteq M$, $p \in S_n(A)$, $q \in S_n(B)$, and $p \subseteq q$. If $RM(p) = RM(q)$, then q is a *non-forking* extension of p to B. Otherwise, if $RM(p) < RM(q)$, we say that q *forks over* A. We say that $p \in S_n(A)$ is *stationary* if for all $B \supseteq A$, there is a unique non-forking extension of p to B, or equivalently if $DM(p) = 1$.

Notation: If $p \in S(A)$ and $C \subseteq A$, we denote the restriction of p to $S(C)$ by $p \upharpoonright C$. If p is stationary and $A \subseteq B$, we denote the unique non-forking extension of p to $S(B)$ by $p|B$.

Definition 1.7. Let $A \subseteq M$, $p \in S(A)$ a stationary type. A *canonical base* of p, denoted $\mathrm{Cb}(p)$, is a tuple $e \subseteq M^{\mathrm{eq}}$ such that for every $\sigma \in \mathrm{Aut}(M)$, $\sigma(p) = p$ if and only if $\sigma(e) = e$ (this tuple is unique up to interdefinability). If p is not stationary, consider the finite set $\mathcal{P}$ of nonforking extensions of p to M, and define $\mathrm{cb}(p)$ as a code for the set $\{\mathrm{Cb}(q), q \in \mathcal{P}\}$; then any automorphism of M fixes $\mathrm{cb}(p)$ if and only if it permutes $\mathcal{P}$ (see Fact 1.8 *(i))*.

The following is a summary of the properties of canonical bases we will use, they can be found as Proposition 2.20 and Remarks 2.26 , 3.19 in Chapter 1 of [8].

Fact 1.8. *Let $A \subseteq M$, $p \in S(A)$. Then*

(i) (Conjugacy) The set of automorphisms of M that fix A pointwise acts transitively on $\mathcal{P}$.

(ii) $\mathrm{cb}(p) \subseteq \mathrm{dcl}^{\mathrm{eq}}(A)$.

(iii) For any $B \subseteq A$, p does not fork over B if and only if $\mathrm{cb}(p) \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$.

(iv) If p is stationary, for any $B \subseteq A$, p does not fork over B and $p \restriction B$ is stationary if and only if $\mathrm{Cb}(p) \subseteq \mathrm{dcl}^{\mathrm{eq}}(B)$.

(v) If p is stationary, and $(a_i, i < \omega)$ is a sequence such that for all i, a_i realizes $p|A \cup \{a_j, j < i\}$, then $\mathrm{Cb}(p) \subseteq \mathrm{dcl}^{\mathrm{eq}}(a_0 \ldots, a_n)$ *for some n.*

Lemma 1.9. *Let e be an imaginary in M and let a be a finite tuple of reals such that $e = f(a)$ for some $\varnothing$-definable function f. Then $e = \mathrm{cb}(\mathrm{tp}(a/e))$. Moreover, if $e' = \mathrm{Cb}(\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(e)))$, then $e' \in \mathrm{acl}^{\mathrm{eq}}(e)$ and $e \in \mathrm{dcl}^{\mathrm{eq}}(e')$.*

Proof. Let $p = \mathrm{tp}(a/e)$ and $p' = \mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(e))$. To see why $e = \mathrm{cb}(\mathrm{tp}(a/e))$, consider the equivalence relation $E(x, y)$ given by $f(x) = f(y)$; then e is a code for the class of a. Let $\mathcal{P}$ as in Definition 1.7. Since $\mathcal{P}$ is finite, and e' is the canonical base of an element of $\mathcal{P}$, it follows that $e' \in \mathrm{acl}^{\mathrm{eq}}(e)$. Now, suppose $\sigma(e') = e'$ for some automorphism of M^{eq}; then $\sigma p' = p'$, so both formulas $f(x) = e$ and $f(x) = \sigma(e)$ belong to p', which implies $\sigma(e) = e$, hence $e \in \mathrm{dcl}^{\mathrm{eq}}(e')$. □

Lemma 1.10. *Let e be an imaginary in M and let a be a finite tuple of reals such that $e = f(a)$ for some $\varnothing$-definable function f. There is $a' \in M^{\mathrm{eq}}$ such that $e \in \mathrm{dcl}^{\mathrm{eq}}(a')$ and $\mathrm{tp}(a'/e)$ is stationary.*

Proof. Let $p = \mathrm{tp}(a/e)$ and let $p_1, \ldots, p_n$ be its non-forking extensions to $\mathrm{acl}^{\mathrm{eq}}(e)$. Let $a_1, \ldots, a_n \in M$ be such that a_i realizes $p_i|\{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\}$. Let a' be a code of this set of realizations. Then as $a \in \mathrm{acl}^{\mathrm{eq}}(a')$, there is a formula $\varphi(x, a')$ isolating $\mathrm{tp}(a/a')$; hence $M \models \forall x \varphi(x, a') \to f(x) = e$, as f is $\varnothing$-definable, so $e \in \mathrm{dcl}^{\mathrm{eq}}(a')$. Moreover, any automorphism of M which fixes e permutes $\{p_1, \ldots, p_n\}$, hence it fixes $\mathrm{tp}(a'/e)$. □

Definition 1.11. (Independence) Let $A, B, C \subseteq M$. We say A is *independent* from B over C, denoted

$$A \perp_C B,$$

if for every finite tuple a from A, $\mathrm{tp}(a/BC)$ does not fork over C.

The following is a summary of the properties of the independence relation in the ω-stable context. These are found as Theorem 8.5.5 from [13], and Lemmas 6.3.16 through 6.3.21 from [7].

Fact 1.12. *Let $A, B, C, D \subseteq M$. Forking independence has the following properties.*

1. *(Monotonicity) If $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$ and $B' \subseteq B$, then $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B'$.*
2. *(Transitivity) $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C BD$ if and only if $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$ and $A \mathop{\smile\!\!\!\!\!\!\downarrow}_{C,B} D$.*
3. *(Existence) Every $p \in S(A)$ has a non-forking extension to any set containing A.*
4. *(Symmetry) If $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$, then $B \mathop{\smile\!\!\!\!\!\!\downarrow}_C A$.*
5. *(Algebraic Closure) $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C \operatorname{acl}(A)$.*

Definition 1.13. Let $A, B \subseteq M$ and let $p \in S(A)$ be definable over B by a scheme d_p. This definition scheme is called *good* (over B) if the set

$$\{\varphi(x, m) \mid \ M \models d_p(\varphi)(m),\ m \in M,\ \varphi(x, y) \text{ an } L\text{-formula}\}$$

is a global type extending p.

Lemma 1.14. *Let $p \in S(A)$. Then p is stationary if and only if it has a good definition over A.*

Proof. If p is stationary, let q be its global non-forking extension. Then q is definable and invariant under all automorphisms that fix A setwise, hence it is definable over A. This gives a good definition for p. Conversely, assume p has a good definition over A. There is then a global non-forking extension $p' \in \mathcal{P}$, definable over A. Since all elements of $\mathcal{P}$ are conjugate over A, and p' is fixed by every automorphism which fixes A setwise, it must be that $\{p'\} = \mathcal{P}$. Therefore, p is stationary. $\square$

Lemma 1.15. *Let $a \in M$ be a tuple and $A \subseteq M$. Suppose $p = \operatorname{tp}(a/A)$ is stationary and let $a' \in M$ be a tuple such that $a' \in \operatorname{dcl}(Aa)$. Then $\operatorname{tp}(a'/A)$ is stationary.*

Proof. We shall give a good definition scheme over A for $\operatorname{tp}(a'/A)$. Let $\varphi(x, y)$ be an L-formula and $m \in M$ such that $M \models \varphi(a', m)$. By stationarity of p, there is an $L(A)$-formula $d_p(\varphi)(y)$ such that $\varphi(x, m) \in \operatorname{tp}(a/A)$ if and only if $M \models d_p(\varphi)(m)$. By hypothesis, there is an A-definable function f such that $f(a) = a'$. Let $\tilde{\varphi}(x, y) = \varphi(f(x), y)$,

$$\begin{aligned} M \models \varphi(a', m) &\iff M \models \varphi(f(a), m) \\ &\iff M \models \tilde{\varphi}(a, m) \\ &\iff M \models d_p(\tilde{\varphi})(m). \end{aligned}$$

□

Definition 1.16. A type $p(x) \in S(A)$ is said *internal* to a partial type $\Sigma(y)$ if there are: a realization a of p, and $B \supseteq A$ independent from a over A, such that $a \in \mathrm{dcl}(Bd)$ for some finite tuple d of realizations of Σ. If it happens instead that $a \in \mathrm{acl}(Bd)$, then the type is said to be *almost internal* to Σ.

Lemma 1.17. *Suppose* $\mathrm{tp}(a/A)$ *is stationary and almost internal to a partial type* Σ. *Then there is an imaginary* a' *such that:* $\mathrm{tp}(a'/A)$ *is stationary and internal to* Σ, $a' \in \mathrm{dcl}^{\mathrm{eq}}(Aa)$, *and* $a \in \mathrm{acl}^{\mathrm{eq}}(a')$. *Such an* a' *can be taken to be a code for a finite set of realizations of* $\mathrm{tp}(a/A)$.

Proof. By hypothesis, there are $B \supseteq A$ independent of a over A and a tuple d of realizations of Σ such that $a \in \mathrm{acl}(Bd)$. We can replace B by some finite tuple b such that $a \in \mathrm{acl}(Abd)$. Let $q = \mathrm{tp}(b, d/Aa)$ and $c = \mathrm{cb}(q)$. By Fact 1.8 *(ii)*, $c \in \mathrm{dcl}^{\mathrm{eq}}(Aa)$. Note that $b \mathop{\smile\!\!\!\!\!\!\mid}_A a$, hence $b \mathop{\smile\!\!\!\!\!\!\mid}_A c$ and $\mathrm{tp}(c/A)$ is Σ-internal. By definition of c, $bd \mathop{\smile\!\!\!\!\!\!\mid}_{Ac} Aa$, but $a \in \mathrm{acl}(Abd)$, hence $a \in \mathrm{acl}^{\mathrm{eq}}(Ac)$. If a' denotes the code for the finite set of conjugates of a over Ac, then

$$a' \in \mathrm{dcl}^{\mathrm{eq}}(Ac) \subseteq \mathrm{dcl}^{\mathrm{eq}}(Aa).$$

Because $\mathrm{tp}(c/A)$ is internal to Σ, so is $\mathrm{tp}(a'/A)$. Moreover, $\mathrm{tp}(a'/A)$ is stationary by Lemma 1.15. We may note that $c \in \mathrm{acl}(Aa')$ as well: if a'' is any conjugate of a over Ac, then it realizes the same type over Ac as a. □

Lemma 1.18. *Let* Σ *be a partial type, and let* $p \in S(A)$ *be a stationary,* Σ*-internal type. There exists a partial* A*-definable function* $h(y_1, \ldots, y_m, z_1, \ldots, z_n)$ *and a sequence* $b_1, \ldots, b_m$ *of realizations of* p, *such that for any realization* a *of* p, *there is a sequence* $c_1, \ldots, c_n$ *of realizations of* Σ, *such that* $a = h(b_1, \ldots, b_m, c_1, \ldots, c_n)$.

Proof. Let b realize p, $B \supseteq A$ independent from b over A, and d a tuple of realizations of Σ such that $b \in \mathrm{dcl}(Bd)$.

Claim: For any b' realizing $p|Ab$, there is a sequence d' of realizations of Σ such that $b' = g(b, d')$, for some definable function g.
Let $(b_i, d_i)_{i<\omega}$ be a Morley sequence of $\operatorname{tp}(b, d/\operatorname{acl}^{\mathrm{eq}}(B))$. By Fact 1.8 *(v)*, $\operatorname{tp}(b, d/M)$ is definable over $A \cup \{b_i, d_i, i < \omega\}$. In particular, for m large enough,

$$b \in \operatorname{dcl}(b_1, \ldots, b_m, d_1, \ldots, d_m, d, A),$$

such that $\overline{d} = (d_1, \ldots, d_m, d)$, $\overline{b} = (b_1, \ldots, b_m)$ are independent from b over A. Then $b = g(\overline{b}, \overline{d})$ for some A-definable function g. ■
Now let a be an arbitrary realization of p, and let $\bar{a} = (a_1, \ldots, a_m)$ realize $\operatorname{tp}(\overline{b}/\operatorname{acl}(A))$ such that $(a_1, \ldots, a_m) \perp_A a\overline{b}$. By the claim, for each $i \leq m$ there is $\overline{c_i}$, a tuple of realizations of Σ, such that $a_i = g(\overline{b}, \overline{c}_i)$. Since $\operatorname{tp}(a, \overline{a}/A) = \operatorname{tp}(b, \overline{b}/A)$, we also we get that $a = g(\overline{a}, \overline{c})$ for some tuple $\overline{c}$ of realizations of Σ. It follows that $a = h(\overline{b}, \overline{c}, \overline{c}_1, \ldots, \overline{c}_m)$ for an A-definable function h. □

The following is Lemma 7.2.12 from [13], which holds for all simple theories.

Fact 1.19. *For all $A \subseteq M$ there is some λ such that for any sequence $(a_i, i < \lambda)$ there exists an A-indiscernible sequence $(b_j, j < \omega)$ such that for all $j_1 < \cdots < j_n < \omega$ there is a sequence $i_1 < \cdots < i_n < \lambda$ with $\operatorname{tp}(a_{i_1}, \ldots, a_{i_n}/A) = \operatorname{tp}(b_{j_i}, \ldots, b_{j_n}/A)$.*

Lemma 1.20. *If $b \in \operatorname{acl}(aA)$, then $RM(ab/A) = RM(a/A)$.*

Proof. It is clear that $RM(ab/A) \geq RM(a/A)$, since the latter type contains less formulas. The reverse inequality is proved by induction on $\alpha = RM(a/A)$. Let $d = DM(ab/Aa)$ be its Morley degree. Choose an $L(A)$-formula $\varphi(x, y) \in \operatorname{tp}(ab/A)$ such that $RM(\exists y \varphi(x, y)) = \alpha$ and $\varphi(a', y)$ has at most d realizations for all a'. If Y is the set defined by $\exists x \varphi(x, y)$, we claim that $RM(Y) \leq \alpha$. Consider an infinite family of pairwise disjoint definable subsets $Y_i \subseteq Y$. Let $\psi_i(x) = \exists y(\varphi(x, y) \wedge y \in Y_i)$. Note that any $d+1$ of the $\psi_i(M)$ have empty intersection: if $M \models \bigwedge_{i=0}^{d} \psi_i(a')$, then there exist $b_i \in Y_i$ for $0 \leq i \leq d$ such that $\models \varphi(a', b_i)$, which contradicts our choice of φ. Therefore, some $\psi_i(x)$ has Morley rank $\beta < \alpha$. Let $b' \in Y_i$, and choose a' such that $M \models \varphi(a', b')$. Then b' is algebraic over $a'A$ and since a' realizes $\psi_i(x)$, we have $RM(a'/A) \leq \beta$. So by induction hypothesis, we conclude $RM(a'b'/A) \leq \beta$, which shows $RM(Y_i) \leq \beta$. This implies that Y does not contain an infinite familiy of disjoint subsets of Morley rank $\geq \alpha$. □

The following definition comes from 10.2.8 of [13].

Definition 1.21. Let $A, B \subseteq M$ be definable sets and let $f : B \to A$ be a definable function. The fibers of f have definable Morley rank if for every definable $B' \subseteq B$ and every $k < \omega$, the set $\{a \in A,\ RM(f^{-1}(a) \cap B') = k\}$ is definable.

Lemma 1.22. *Let $A, B \subseteq M$ be definable sets and let $f : B \to A$ be a definable surjection whose fibers have definable Morley rank, and such that for all $a \in A$, $RM(f^{-1}(a)) = k$. Then $RM(B) = RM(A) + k$.*

Proof. Suppose A, B, f are definable over some $S \subseteq M$. The proof is by induction on $RM(A) = m$, for all k. We may also asume that $DM(A) = 1$: if not, then partition $A = A_1 \cup \cdots \cup A_d$ into finitely many disjoint, rank m definable subsets, then replace B, A by $f^{-1}(A_1), A_1$, respectively. If $m = 0$, then A is finite and

$$RM(B) = \max\{RM(f^{-1}(a))\}_{a \in A} = k.$$

If $m > 0$, then write $A = \cup_i A_i$ for an infinite, pairwise disjoint family of definable $A_i \subseteq A$ such that $RM(A_i) = m - 1$. If $B_i = f^{-1}(A_i)$, then $f \restriction B_i$ is a definable surjection with rank k fibers, so by induction hypothesis $RM(B_i) = RM(A_i) + k = m + k - 1$, and since the B_i are also pairwise disjoint, we deduce $RM(B) \geq m + k$. For the reverse inequality, let now $(B'_i)_{i<\omega}$ be any infinite, pairwise disjoint family of definable subsets $B'_i \subseteq B$; we will show that $RM(B'_i) < m + k$ for some i, by induction over k. If $k = 0$, f is finite-to-one, so for any $b \in B$, $b \in \operatorname{acl}(f(b))$ and $f(b) \in \operatorname{dcl}(b)$. By Lemma 1.20,

$$RM(b/S) = RM(f(b), b/S) = RM(f(b)/S) \leq RM(A) = m.$$

This implies $RM(B) = \sup_{b \in B}(RM(b/S)) \leq m$. Suppose now the conclusion holds for m and for every $k' < k$. Let $a \in A$, then since $RM(f^{-1}(a)) = k$ and $f^{-1}(a) \supseteq \bigcup_i (f^{-1}(a) \cap B'_i)$, it must be that for some j, $RM((f^{-1}(a) \cap B'_j) < k$. Consider now the definable sets

$$A'_i = \{a \in A,\ RM(f^{-1}(a) \cap B'_i) = k\}.$$

We have proved that $\cap_i A'_i = \varnothing$. We claim that for some i, $RM(A'_i) < m$: if not, then as $DM(A) = 1$, for every N, $\cap_{i \leq N} A'_i \neq \varnothing$, hence by compactness $\cap_i A'_i \neq \varnothing$, a contradiction, since the Morley degree of the fibers is bounded. Now, as

$$B'_i = (f^{-1}(A'_i) \cap B'_i) \cup (f^{-1}(A \setminus A'_i) \cap B'_i),$$

we can apply induction hypothesis over m to the first term to see that $RM(f^{-1}(A'_i) \cap B'_i) < m+k$. On the other hand, on $A \setminus A'_i$, all fibers have rank strictly less than k. The induction hypothesis over k yields $RM(f^{-1}(A \setminus A'_i) \cap B'_i) < m+k$, concluding the proof. $\square$

Lemma 1.23. *Let $P \subseteq M$ be a strongly minimal definable set, and $\varphi(x_1, \dots, x_n, \bar{y})$ a formula such that $M \models \forall x_1, \dots, x_n(\exists \bar{y}\ \varphi(x_1, \dots, x_n, \bar{y}) \to x_i \in P)$. The following set is definable for every k,*

$$Y_{n,k} = \{\bar{b} \in M, RM\varphi(x_1, \dots, x_n, \bar{b}) = k\}.$$

Proof. Let $Y'_{n,k} = \{\bar{b} \in M, RM\varphi(x_1, \dots, x_n, \bar{b}) \geq k\}$. We will prove definability of $Y'_{n,k}$, this gives the desired result since $Y_{n,k} = Y'_{n,k} \setminus Y'_{n,k+1}$. We proceed by induction on n. Notice that $Y'_{1,1}$ is definable since $RM(\varphi(x_1, \bar{b})) \geq 1$ if and only if $\exists^\infty x_1 \varphi(x_1, \bar{b})$, which is in turn equivalent (by strong minimality of P) to $\exists^{\geq N} x_1 \varphi(x_1, \bar{b})$, for some N. Moreover, notice that

$$Y_{n,0} = \{\bar{b} \in M, \exists x_1, \dots, x_n \varphi(x_1, \dots, x_n, \bar{b})\}$$

is definable for all n. Let now $n > 0$, we will work by induction over $k > 0$. For $\bar{b} \in P$, consider the $\bar{b}$-formula $\phi_{\bar{b}}(x_0, \dots, x_{n-1})$ given by $\exists x_n \varphi(x_0, \dots, x_{n-1}, x_n, \bar{b})$. If $RM(\phi_{\bar{b}}) \geq k$, then $\bar{b} \in Y'_{n,k}$, and if $RM(\phi_{\bar{b}}) < k$, consider instead the $L(\bar{b})$-formula $\psi_{\bar{b}}(x_0, \dots, x_{n-1})$ given by $\exists^\infty x_n \varphi(x_0, \dots, x_{n-1}, x_n, \bar{b})$, then since the algebraic dimension of a tuple inside P agrees with its Morley rank, we have in this case that $RM(\psi_{\bar{b}}) \geq k-1$ if and only if $\bar{b} \in Y'_{n,k}$. We have shown that $\bar{b} \in Y'_{n,k}$ if and only if $RM(\phi_{\bar{b}}) \geq k$ or $RM(\psi_{\bar{b}}) \geq k-1$. The first of these two conditions is definable by our induction hypothesis over n, while the latter is definable by induction over k, so $Y'_{n,k}$ is also definable. $\square$

2 Stable Groups

An ω-stable group is an ω-stable structure $(G, \cdot, 1, \dots)$, where $(G, \cdot, 1)$ is a group. In this section we present some basic concepts and tools used in the study of ω-stable groups. For more details, see [11] and Chapter 7 from [7]. Throughout this section G will denote an infinite, ω-stable group, definable inside a saturated model M of a complete, ω-stable theory T.

Lemma 2.1. *There is no infinite strictly descending chain of definable subgroups $G > G_1 > G_2 > \dots$*

Proof. For any definable subgroup $H \leq G$, and any $a \in G \setminus H$, the coset $aH \subseteq G$ is disjoint from H, and since $x \mapsto ax$ is a definable bijection, then $RM(H) = RM(aH)$. If $G > G_1 > G_2 > \dots$ is a strictly decreasing sequence, and if $[G_i : G_{i+1}]$ is infinite, then $RM(G_i) > RM(G_{i+1})$. If $[G_i : G_{i+1}]$ is finite, then $DM(G_i) > DM(G_{i+1})$, and this implies the existence of a strictly decreasing sequence with respect to the lexicographic order $RM(G) \times \omega$, hence, this sequence cannot be infinite. $\square$

Lemma 2.2. *There is a definable normal subgroup $G^0 \leq G$ that is contained in every subgroup of G of finite index.*

Proof. Let $\mathcal{H}$ be the family of definable subgroups of G of finite index. We claim that there are $H_1, \dots, H_n$ in $\mathcal{H}$ such that

$$\bigcap_{H \in \mathcal{H}} H = H_1 \cap \dots \cap H_n.$$

If not, then for every m there are $j_0, \dots, j_m$ such that if $G_m = H_{j_0} \cap \dots \cap H_{j_m}$, then $G_0 > G_1 > G_2 > ...$, contradicting Lemma 2.1. We may then define $G^0 = H_1 \cap \dots \cap H_n$. If $h \in G$, since $x \mapsto hxh^{-1}$ is a group automorphism, we have that hG^0h^{-1} is a definable subgroup with $[G : hG^0h^{-1}] = [G : G^0]$, so $hG^0h^{-1} = G^0$ by minimality. $\square$

Lemma 2.3. *Let $A \subseteq M$. If G is A-definable, then G^0 is A-definable.*

Proof. By Lemma 2.2, there are an $L(A)$-formula $\varphi(x, y)$ and $g \in G$ such that the formula $\varphi(x, g)$ defines G^0. Let $n = [G : G^0]$, and consider

$$W = \{b \in G,\ \varphi(x, b) \text{ defines a subgroup of index } n\ \},$$

an A-definable set. If $b \in W$ and $H = \varphi(G,b)$, then $H \cap G^0$ is a finite index subgroup of G^0, hence $H \supseteq G^0$. However, since $[G : H] = n$, we have $[H : G^0] = 1$, yielding $H = G^0$. We can then define G^0 as $\{g \in G, \exists b \ (b \in W \wedge \varphi(g,b))\}$. $\square$

Definition 2.4. G is *connected* if $G = G^0$.

Definition 2.5. There is an action of G on $S_1(G)$ given by $g \cdot p = \{\varphi(x), \ \varphi(gx) \in p\}$. The *stabilizer* of p is the group

$$\mathrm{Stab}(p) = \{g \in G, g \cdot p = p\}.$$

Lemma 2.6. $\mathrm{Stab}(p)$ *is a definable subgroup of* G*, for every* $p \in S_1(G)$.

Proof. For $\varphi(x,y)$ an L-formula, let

$$\mathrm{Stab}_\varphi(p) = \{g \in G \mid p_\varphi = g \cdot p_\varphi\},$$

where

$$p_\varphi = \{\varphi(x,g) \mid g \in G, \varphi(x,g) \in p\} \cup \{\neg\varphi(x,g) \mid g \in G, \varphi(x,g) \notin p\}.$$

An easy calculation shows that for every φ, $\mathrm{Stab}_\varphi(p) \leq G$. By stability, there is a definition scheme for p, say d_p. Thus,

$$\mathrm{Stab}_\varphi(p) = \{g \in G| \ \forall h(d_p(\varphi)(h) \leftrightarrow d_p(\varphi)(hg))\}.$$

Note that $\mathrm{Stab}(p) = \bigcap_{\varphi(x,y)\in L} \mathrm{Stab}_\varphi(p)$. By Lemma 2.1, there are $\varphi_1, \ldots, \varphi_n \in L$ such that $\mathrm{Stab}(p) = \mathrm{Stab}_{\varphi_1}(p) \cap \cdots \cap \mathrm{Stab}\,\varphi_n(p)$, which concludes the proof. $\square$

Lemma 2.7. *Let* $p \in S_1(G)$*.*

(i) $RM(\mathrm{Stab}(p)) \leq RM(p)$.

(ii) $\mathrm{Stab}(p) \leq G^0$.

Proof. Let $a, b \in M$ be such that a realizes p, $b \in \mathrm{Stab}(p)$ satisfies $RM(\mathrm{tp}(b/G)) = RM(\mathrm{Stab}(p))$, and $a \mathop{\smile\!\!\!\!\!\!\,|}_G b$. Then

$$RM(\mathrm{tp}(ba/G,a)) = RM(\mathrm{tp}(b/G,a)) = RM(\mathrm{tp}(b/G)) = RM(\mathrm{Stab}(p)).$$

Moreover, since ba realizes p, we have $RM(\mathrm{tp}(ba/G, a)) \leq RM(\mathrm{tp}(ba/G)) = RM(p)$, proving *(i)*. Let now $c \in \mathrm{Stab}(p)$, and let $\varphi(x)$ define G^0(possibly with parameters in M). Let $g \in G$ be such that $\varphi(g^{-1}x) \in p$, thus $\varphi(g^{-1}cx) \in p$. Let $G \preceq H$ and $h \in H$ realize p. Then $g^{-1}ch \in H^0$ and $g^{-1}h \in H^0$. Thus $(g^{-1}h)^{-1}g^{-1}ch = h^{-1}ch \in H^0$, and since H^0 is normal, $c \in G^0$ by Lemma 2.3. □

Definition 2.8. A type $p \in S_1(G)$ is *generic* if $RM(p) = RM(G)$. An element $a \in G(M)$ is generic over $A \subseteq G$ if $RM(\mathrm{tp}(a/A)) = RM(G)$.

Lemma 2.9. *A type $p \in S_1(G)$ is generic if and only if $[G : \mathrm{Stab}(p)]$ is finite.*

Proof. Suppose p is generic. Notice that $\{ap,\ a \in G\}$ is finite, since there are only finitely many types of maximal Morley rank. Choose $b_1, \ldots, b_n \in G$ such that if $a \in G$, then $ap = b_ip$ for some $i \leq n$. If $ap = b_ip$ then $b_i^{-1}a \in \mathrm{Stab}(p)$ and $a \in b_i \mathrm{Stab}(p)$. Therefore, $[G : \mathrm{Stab}(p)] \leq n$. Assume now that $\mathrm{Stab}(p)$ has finite index, so $RM(G) = RM(\mathrm{Stab}(p))$, but $RM(\mathrm{Stab}(p)) \leq RM(p)$ by Lemma 2.7, hence p is generic. □

Corollary 2.10.

(i) A type $p \in S_1(G)$ is generic if and only if $\mathrm{Stab}(p) = G^0$

(ii) G has an unique generic type if and only if G is connected.

Proof.

(i) By Lemma 2.9, if p is generic, $\mathrm{Stab}(p)$ has finite index, we have $G^0 \leq \mathrm{Stab}(p)$. By Lemma 2.7 *(ii)*, we have $G^0 \geq \mathrm{Stab}(p)$. The other direction is clear by Lemma 2.9, since G^0 has finite index.

(ii) Let p be the unique generic type. For all $a \in G$, ap is generic, hence $ap = p$. Thus, $G = \mathrm{Stab}(p) = G^0$ by *(i)*. Conversely, suppose $G = G^0$, and by contradiction, assume p, q are distinct, generic types. Let a, b realize p, q respectively, with $b \in H \succeq G$ and let a' realize $p|H$. Then, $\mathrm{tp}(a, b/G) = \mathrm{tp}(a', b/G)$, and $p|H$ is a generic of H. By *(i)*, $\mathrm{Stab}(p|H) = H^0 = H$. Thus, ba' realizes $p|H$. In particular, ba' realizes p, hence ba realizes p. If $a \in K \succeq G$, and b' realizes $q|K$, an analogous argument shows that ba realizes q. This contradicts our assumption, hence G has a unique generic type.

□

3 Pairs of Algebraically Closed Fields

Throughout this section, we will let $T = ACF_p$ for p prime or 0 (in the usual language L), and we consider L_P, the language obtained by adjoining a unary predicate P. An elementary pair of models of T, $N \preceq M$, is considered an L_P-structure by interpreting P as the universe of the structure N, and L_P-structures will be naturally denoted as pairs (M, P).

Definition 3.1. A *beautiful pair* of models of T is an elementary pair $N \preceq M$ such that N is $|T|^+$-saturated and M is $|T|^+$-saturated over N, which means that M realizes any L-type over $N \cup A$, where $A \subseteq M \setminus N$ is such that $|A| < |T|^+$. The theory T_P of proper pairs $P \prec M$ of models of T was shown to be complete by Keisler in [5].

Fact 3.2. *([12]) Let (M, P) be a saturated model of T_P.*

(i) (M, P) is a beautiful pair.

(ii) T_P is stable.

(iii) Any L_P-formula $\phi(x)$ is equivalent modulo T_P to a Boolean combination of L_P-formulas of the form $\exists y P(y) \wedge \psi(y, x)$ where ψ is a quantifier-free L_P-formula.

Buechler in [2] notes that T_P is actually ω-stable of Morley rank ω. From now on (M, P) will be a saturated model of T_P.

Notation: If $A \subseteq M$, we denote the field generated by A by $\langle A \rangle$. For any $A, B, C \subseteq M$, we denote independence in the sense of L by $A \mathop{\smile\!\!\!\!\!\!|}^{L}_{C} B$, and in the sense of L_P by $A \mathop{\smile\!\!\!\!\!\!|}^{L_P}_{C} B$. We will also distinguish L-types from L_P-types by using tp_L and tp_{L_P} respectively. We adopt the same convention for the acl and dcl operators.

Lemma 3.3. *Any $C \subset P^n$ that is L_P-definable with parameters from M, is L-definable with parameters from P. In particular, P is strongly minimal and stably embedded (in the L_P-sense).*

Proof. Let $\varphi(x, m)$ with $m \in M$ be an L_P-formula defining C. Note that P is algebraically closed in the L_P-sense. By stability of T_P, $p(y) = \mathrm{tp}_{L_P}(m/P)$ is definable over P, hence we have that for every $a \in M$,

$$a \in C \iff \varphi(a, y) \in p \iff M \models d\varphi(a),$$

where $d\varphi(x)$ is an L_P-formula with parameters in P. Now, by Fact 3.2, $d\varphi(x)$ is equivalent to a boolean combination of formulas of the form $\exists z P(z) \wedge \psi(x,z)$ where ψ is a quantifier-free L_P-formula. Since $C \subseteq P^n$ and

$$M \models \forall x \; d\varphi(x) \to P(x),$$

C is L-definable by a boolean combination of formulas of the form $\exists z \psi'(x,z)$, where ψ' is the L-formula obtained from ψ by replacing every instance of $P(t)$ by $t = t$, for every term t. $\square$

Remark 3.4. By elimination of quantifiers in T, the formula $\exists z \psi'(x,z)$ is equivalent modulo T to a quantifier free L-formula $\theta(x)$. Notice also that the set C only depends on m, so if c is an L_P-code for C, we have that $c \in \mathrm{dcl}^{\mathrm{eq}}_{L_P}(m) \cap P$.

Definition 3.5. Let $a \in M$ a (possibly infinite) tuple, define $\widehat{a} = (a, a^c)$, where $a^c = \mathrm{Cb}(\mathrm{tp}_L(a/P))$. Since T is totally transcendental and eliminates imaginaries, a^c is in the L-definable closure of a finite real tuple. More specifically, a^c can be regarded up to interdefinability as a tuple of generators for the field of definition of the algebraic locus of a over P (i.e: the variety associated to the prime ideal of polynomials in $P[X]$ that vanish at a).

Lemma 3.6. *For all tuples $a \in M$, $\langle \widehat{a} \rangle$ is linearly disjoint from P over $\langle a^c \rangle$.*

Proof. Note that $\langle \widehat{a} \rangle = \langle a^c \rangle(a)$. Let $\{M_0(X), \ldots, M_m(X)\}$ be a set monomials such that $\{M_0(a), \ldots, M_m(a)\}$ is linearly independent over $\langle a^c \rangle$. Suppose there is a linear relation $\sum c_i M_i(a) = 0$, where $c_i \in P$. By definition of a^c we can write

$$\sum_{i=0}^{m} c_i M_i(X) = \sum_{j=0}^{n} b_j f_j(X),$$

where $b_j \in \langle a^c \rangle$, $f_j(a) \in I := \{f(X) \in \langle a^c \rangle(X),\; f(a) = 0\}$ for all j, and such that $\{f_0(X), \ldots, f_n(X)\}$ is a linearly independent set of polynomials over $\langle a^c \rangle$. We claim that $\{M_1, \ldots, M_m, f_1, \ldots, f_n\}$ is also linearly independent over $\langle a^c \rangle$: if it were not, then $\sum r_i M_i(X) + \sum s_j f_j(X) = 0$, for some $r_i, s_j \in \langle a^c \rangle$. We can substitute a for X to obtain $\sum r_i M_i(a) = 0$, which yields $r_i = 0$ for all i, hence $\sum s_j f_j(X) = 0$ and $s_j = 0$ for all j. As these are formal polynomials, they remain linearly independent over P, hence $c_i = 0$ for all i. $\square$

Remark 3.7. For all tuples $a \in M$, $a^c \in \mathrm{dcl}_{L_P}(a)$.

Proof. Any L_P-automorphism leaves P invariant, so if it also fixes a, it must leave $\mathrm{tp}_L(a/P)$ invariant, hence it must fix a^c. $\square$

Lemma 3.8. *For all tuples $a, b \in M$, $\mathrm{tp}_{L_P}(a) = \mathrm{tp}_{L_P}(b)$ if and only if $\mathrm{tp}_L(\widehat{a}) = \mathrm{tp}_L(\widehat{b})$.*

Proof. If there is an L_P-automorphism σ of M taking a to b, by Remark 3.7 we have $\sigma(a^c) = b^c$, hence $\mathrm{tp}_{L_P}(\widehat{a}) = \mathrm{tp}_{L_P}(\widehat{b})$. Restricting the language yields $\mathrm{tp}_L(\widehat{a}) = \mathrm{tp}_L(\widehat{b})$. Conversely, assume there is a partial L-isomorphism σ sending a to b and a^c to b^c. Since $\langle \widehat{a} \rangle$ and P are linearly disjoint over $\langle a^c \rangle$, and also $\langle \widehat{b} \rangle$ and P are l.d. over $\langle b^c \rangle$, the restriction $\sigma \restriction \langle \widehat{a} \rangle$ can be extended to an L-isomorphism $\sigma' : P(a) \to P(b)$ such that $\sigma'(P) = P$ or in other words, to an L_P-isomorphism, which can be itself extended to an L_P-automorphism of M by saturation of M over P (see Fact 3.2 *(iii)*). $\square$

Lemma 3.9. *For all tuples $a \in M$,*

(i) $a^c \subseteq P$.

(ii) If $b \in P$ is a tuple, $\widehat{ab}$ and $\widehat{a}b$ are L-interdefinable.

(iii) $\widehat{a} \mathop{\smile\!\!\!\!\!\!\;|}^{L_P}_{a^c} P$

Proof. By elimination of imaginaries in T, $a^c \in \mathrm{acl}^{\mathrm{eq}}_L(P) = P$, this gives *(i)*. To see *(ii)*, notice that $a \mathop{\smile\!\!\!\!\!\!\;|}^{L}_{a^c} P$ implies $ab \mathop{\smile\!\!\!\!\!\!\;|}^{L}_{a^c b} P$, and since $\mathrm{tp}_L(ab/a^c b)$ is stationary, we get $(ab)^c \subseteq \mathrm{dcl}_L(a^c b)$, hence $\widehat{ab} \in \mathrm{dcl}_L(\hat{a}b)$. Cleary $\hat{a}b \subseteq \widehat{ab}$, so the other direction follows. For *(iii)*, let $a^c \in B \subseteq P$, and choose an L_P-indiscernible sequence over a^c, $(B_i)_{i<\omega}$, such that $B_0 = B$. Let $p = \mathrm{tp}_L(\widehat{a}/B)$, and for each i let p_i be the image of p under an L-automorphism that fixes a^c and sends B to B_i. As $\hat{a} \mathop{\smile\!\!\!\!\!\!\;|}^{L}_{a^c} P$, $B_i \subseteq P$, and $\widehat{a} \mathop{\smile\!\!\!\!\!\!\;|}^{L}_{a^c} B_i$ for every i, $\hat{a}$ realizes $\cup_i p_i$. By Lemma 3.8 and *(ii)*, $\mathrm{tp}_L(\widehat{a}/P) \vdash \mathrm{tp}_{L_P}(\widehat{a}/P)$. If we let $p' = \mathrm{tp}_{L_P}(\widehat{a}/B)$ and p'_i be the image of p' under an L_P-automorphism that fixes a^c and sends B to B_i, we have proven consistency of $\cup_i p'_i$. Hence, $\mathrm{tp}_{L_P}(\widehat{a}/B)$ does not fork over a^c. Since B was chosen arbitrarily, the result follows. $\square$

Definition 3.10. A subset A of M is said to be P-independent if $A \mathop{\smile\!\!\!\!\!\!\;|}^{L}_{A \cap P} P$.

Remark 3.11.

(i) For all $a \in M$, $\widehat{a}$ is P-independent.

(ii) Any subset of P is P-independent.

Proof. The first condition follows directly from Lemma 3.9 *(i)*, *(iii)*, and monotonocity. The second statement is clear. □

Lemma 3.12. *Let $A \subseteq B, C \subseteq M$ with $C = Ac$, where $c \in M$ is a finite tuple. The following are equivalent:*

(i) $C \mathop{\smile\!\!\!\!\!\!\,|}^{L_P}_{A} B$

(ii) $C \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{AP} BP$, *and* $C^c \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{A^c} B^c$.

(iii) $C \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{AP} BP$, *and* $\widehat{C} \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{\widehat{A}} \widehat{B}$.

Proof.
(i) implies *(ii)*: By Remark 3.7, we may assume $B = \widehat{B}$. For the first part, suppose by contradiction that $\text{tp}_L(c/BP)$ forks over AP. Let $(B_i)_{i<\lambda}$ be a sequence of realizations of $\text{tp}_L(B/AP)$ such that $B_i \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{AP} (B_j)_{j<i}$ and $B_0 = B$; note that in particular, as $\widehat{B_i} = B_i$, for all i, we get $\text{tp}_{L_P}(B_i) = \text{tp}_{L_P}(B)$ by Lemma 3.9. We can choose λ large enough to apply Fact 1.19, yielding an L_P-indiscernible sequence over AP, $(B'_i)_{i<\omega}$, such that $\text{tp}_{L_P}(B'_i/AP) = \text{tp}_{L_P}(B/AP)$ for all i. Let $p = \text{tp}_{L_P}(c/B)$ and let p_i be its copy over B'_i; then by *(i)*, $\cup_{i<\omega} p_i$ can be realized by some c'. We have that for all i, $c' \not\mathop{\smile\!\!\!\!\!\!\,|}^{L}_{AP} B'_i P$: this contradicts the ω-stability of T, since $(B'_i)_{i<\omega}$ is also L-independent over AP. To prove the latter part of *(ii)*, we apply properties of forking: by Lemma 3.9 *(iii)* we have that $\widehat{A} \mathop{\smile\!\!\!\!\!\!\,|}^{L_P}_{A^c} P$, which implies by symmetry and monotonicity that $C^c \mathop{\smile\!\!\!\!\!\!\,|}^{L_P}_{A^c} \widehat{A}$. Additionally, Remark 3.7 gives

$$\begin{aligned} C \mathop{\smile\!\!\!\!\!\!\,|}^{L_P}_{A} B &\Rightarrow CC^c \mathop{\smile\!\!\!\!\!\!\,|}^{L_P}_{AA^c} BB^c \\ &\Rightarrow C^c \mathop{\smile\!\!\!\!\!\!\,|}^{L_P}_{\widehat{A}} B^c, \text{ by monotonicity.} \end{aligned}$$

Applying transitivity, $C^c \mathop{\smile\!\!\!\!\!\!\,|}^{L_P}_{A^c} B^c$. As these three sets all lie in P, we actually get the desired independence in the L-sense.
(ii) implies *(iii)*: We will prove $AC^c \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{\widehat{A}} \widehat{B}$ and $\widehat{C} \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{AC^c} \widehat{B}$, then *(iii)* will follow by transitivity and because $A^c \subseteq C^c$. To get the first relation, start from $\widehat{B} \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{B^c} P$ and use $C^c \subseteq P$ to get $\widehat{B} \mathop{\smile\!\!\!\!\!\!\,|}_{B^c} C^c$. Combining this with our hypothesis $C^c \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{A^c} B^c$, we get $C^c \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{A^c} \widehat{B}$, which implies $AC^c \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{\widehat{A}} \widehat{B}$ since $A^c \subseteq \hat{A} \subseteq \hat{B}$. For the second relation, start from $\widehat{C} \mathop{\smile\!\!\!\!\!\!\,|}_{C^c} P$ and $A \subseteq C$ to get $\widehat{C} \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{AC^c} AP$ (I). Now, the hypothesis $C \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{AP} BP$ yields $\widehat{C} \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{AP} \widehat{B}$ (II), since $B^c, C^c \subseteq P$. Combining (I) and (II) gives $\widehat{C} \mathop{\smile\!\!\!\!\!\!\,|}^{L}_{AC^c} \widehat{B}$.

(iii) implies *(i)*:

Claim: *(iii)* implies $\widehat{Ac}\widehat{B}$ is P-independent.

$\widehat{Ac}\widehat{B}$ is P-independent is equivalent to saying that if t_C, t_B are transcendence bases for $\widehat{Ac}, \widehat{B}$ over $\widehat{A}P = AP$ respectively, then $t_C \cup t_B$ remains algebraically independent over AP, which is equivalent to $C \mathop{\smile\!\!\!\!\!\!\mid}^{L}_{AP} BP$. ■

Let $(\widehat{B}_i)_i$ be an L_P-indiscernible sequence over $\widehat{A}$ with $\widehat{B_0} = \widehat{B}$. By hypothesis $C^c \mathop{\smile\!\!\!\!\!\!\mid}^{L}_{\widehat{A}} \widehat{B}$, so we may assume that $(\widehat{B}_i)_i$ is also L-indiscernible over $\widehat{A}C^c$. Let $p = \mathrm{tp}_L(\widehat{c}/\widehat{B}C^c)$, and let p_i be its copies over $\widehat{B_i}C^c$. By the first condition of *(iii)*, we can realize $\cup_i p_i$ by some $\widehat{C'}$ which is L-independent from P over $\cup_i \widehat{B_i}C^c$. By Lemma 3.8 and by the claim, it follows that $\mathrm{tp}_{L_P}(\widehat{C}'\widehat{B_i}C^c) = \mathrm{tp}_{L_P}(\widehat{C}\widehat{B_i}C^c)$, so p does not L_P-fork over $\widehat{A}$. Therefore, $\widehat{C} \mathop{\smile\!\!\!\!\!\!\mid}^{L_P}_{\widehat{A}} \widehat{B}$, and *(i)* follows from Remark 3.7. □

Lemma 3.13. *Let $a \in M$, then*

i) $\mathrm{acl}_{L_P}(a) = \mathrm{acl}_L(\widehat{a})$.

ii) $\mathrm{dcl}_{L_P}(a) = \mathrm{dcl}_L(\widehat{a})$.

Proof. In both cases, the inclusion $\supseteq$ follows from Remark 3.7.

i) First we show that $\mathrm{acl}_{L_P}(a) \cap P = \mathrm{acl}_L(a^c)$. Let $b \in \mathrm{acl}_{L_P}(a) \cap P$, then since $a \mathop{\smile\!\!\!\!\!\!\mid}^{L}_{a^c} P$, we get $\widehat{a} \mathop{\smile\!\!\!\!\!\!\mid}^{L}_{a^c} b$. Suppose $b \notin \mathrm{acl}_L(a^c)$, so $b \notin \mathrm{acl}_L(\widehat{a})$. Then, in P, there are infinitely many $(b_i, i < \omega)$ such that $\mathrm{tp}_L(\widehat{a}b_i) = \mathrm{tp}_L(\widehat{a}b)$. By Lemma 3.9 *(ii)*, $\mathrm{tp}_L(\widehat{ab_i}) = \mathrm{tp}_L(\widehat{ab})$. By Lemma 3.8, these b_i are also L_P-conjugate over $\widehat{a}$, a contradiction. Consider now $b' \in M \setminus P$ such that $b' \in \mathrm{acl}_{L_P}(a)$ but $b' \notin \mathrm{acl}_L(\widehat{a})$. Then

$$(b'\widehat{a})^c \in \mathrm{dcl}_{L_P}(b'\widehat{a}) \cap P \subseteq \mathrm{acl}_{L_P}(\widehat{a}) \cap P = \mathrm{acl}_L(a^c),$$

which implies by Fact 1.8 *(iii)* that $b'\widehat{a} \mathop{\smile\!\!\!\!\!\!\mid}^{L}_{a^c} P$, hence $b' \mathop{\smile\!\!\!\!\!\!\mid}^{L}_{\widehat{a}} P$. By assumption there are infinitely many L-conjugates of b' over $\widehat{a}$. Since M is saturated over P, there are infinitely many realizations of $\mathrm{tp}_L(b'/\widehat{a}P)$. This implies there are infinitely many realizations of $\mathrm{tp}_{L_P}(b'/\widehat{a})$, a contradiction.

ii) The proof is similar. First, we show that $\mathrm{dcl}_{L_P}(a) \cap P = \mathrm{dcl}_L(a^c)$, so let $b \in \mathrm{dcl}_{L_P}(a) \cap P$. Then, by *(i)*, $b \in \mathrm{acl}_L(a^c)$. Suppose $b \notin \mathrm{dcl}_L(\widehat{a})$, then there is $b' \in P$ distinct from b such that $\mathrm{tp}_L(b'\widehat{a}) = \mathrm{tp}_L(b\widehat{a})$, and applying Lemma 3.9 *(ii)* and Lemma 3.8 gives $\mathrm{tp}_{L_P}(b'\widehat{a}) = \mathrm{tp}_{L_P}(b\widehat{a})$, a contradiction. Consider now $b' \in M \setminus P$, $b' \in \mathrm{dcl}_{L_P}(a)$, but assume $b' \notin \mathrm{dcl}_L(\widehat{a})$. Then

$$(b'\widehat{a})^c \in \mathrm{dcl}_{L_P}(b'\widehat{a}) \cap P \subseteq \mathrm{dcl}_{L_P}(\widehat{a}) \cap P = \mathrm{dcl}_L(a^c),$$

hence $\langle \widehat{a} \rangle (b')$ and P are linearly disjoint over $\langle \widehat{a} \rangle$. By assumption there are at least two L-conjugates of b over $\widehat{a}$, which are also L_P-conjugates over $\widehat{a}$ by Lemma 3.8, a contradiction.

$\square$

Corollary 3.14. *If $A \subseteq M$ is such that $A = \widehat{A}$, then $\mathrm{acl}_{L_P}(A) = \mathrm{acl}_L(A)$ and $\mathrm{dcl}_{L_P}(A) = \mathrm{dcl}_L(A)$. In particular P is algebraically closed in the L_P-sense.*

Definition 3.15.

i) Consider for every $n > 1$, the predicate $l_n(x_1, \ldots, x_n)$, which asserts that $x_1, \ldots, x_n$ are linearly independent over E, that is,

$$l_n(x_1, \ldots, x_n) \leftrightarrow \forall e_1, \ldots, e_n \left(\bigwedge_i P(e_i) \wedge \sum_i e_i x_i = 0 \rightarrow \bigwedge_i e_i = 0 \right).$$

ii) Consider for every $n > 1$ and for every $i \in \{1, \ldots, n\}$, the $(n+1)$-ary function $f_{n,i}(y, x_1, \ldots, x_n)$ which gives the i-th coordinate of y written as a linear combination of $x_1, \ldots, x_n$. More specifically, if $l_n(x_1, \ldots, x_n) \wedge \neg l_n(y, x_1, \ldots, x_n)$, then

$$z = f_{n,i}(y, x_1, \ldots, x_n) \leftrightarrow \exists z_1, \ldots, z_n \left(z = z_i \wedge y = \sum_j z_j x_j \wedge \bigwedge P(z_j) \right),$$

else, if the condition is not met, define $f_{n,i}(y, x_1, \ldots, x_n) = 0$.

iii) Define the language $L_P^{l,f}$ as the language obtained by adjoining to L_P the predicate symbols l_n and $f_{n,i}$, for all $n > 1$ and $i \in \{1, \ldots, n\}$. Notice that in this language, $P(x)$ can be defined by the formula $\neg l_n(1, x)$.

The following result is Corollary 15 from [3]:

Fact 3.16. *Let $N \subseteq M$ be a model of T_P, then the inclusion is elementary iff N is an $L_P^{l,f}$-substructure if and only if N is P-independent.*

Corollary 3.17. *Let $A \subseteq M$. Let C be the field generated by A and the $f_{n,i}(A)$ for all $n > 1$ and $i \leq n$. Then $\widehat{A} \subseteq C$, and consequently*

i) $\mathrm{acl}_{L_P}(A) = \mathrm{acl}_L(C)$.

ii) $\mathrm{dcl}_{L_P}(A) = \mathrm{dcl}_L(C)$.

Proof. By theorem 7, §2, Ch 3. of [6], the field of definition of the locus of A over P is generated by $\{f_{n,i}(M_0, M_1, \ldots, M_n), n < \omega, i \leq n\}$, where the tuple $(M_0, M_1, \ldots, M_n)$ ranges over the set of monomials formed by elements of A. Therefore, $A^c \subseteq C$. From this, we get both $\mathrm{acl}_L(\widehat{A}) \subseteq \mathrm{acl}_L(C)$ and $\mathrm{dcl}_L(\widehat{A}) \subseteq \mathrm{dcl}_L(C)$, while the reverse inclusion follows from the definability of the $f_{n,i}$. The desired result is obtained by invoking Lemma 3.13. □

Lemma 3.18. *Let $a, b, c \in M$, $p_1 = \mathrm{tp}_{L_P}(a/bc)$, $p_2 = \mathrm{tp}_{L_P}(b/c)$. If p_1, p_2 are stationary, then $p_3 = \mathrm{tp}_{L_P}(a/c)$ is stationary.*

Proof. By stability of T_P and by hypothesis, there are good definition schemes dp_1 over bc and dp_2 over c. We want to find a good definition for p_3, i.e. one that defines a global type, this would imply stationarity by Lemma 1.14. Let $\varphi(x, y)$ be an L_P-formula and let $m \in M$ be such that $M \models \varphi(a, m)$. There is a formula $dp_1(\varphi)(y, z, w)$ such that $M \models dp_1(\varphi)(m, b, c)$. Moreover, there is then a formula $dp_2(dp_1(\varphi))(y, w)$ such that $M \models dp_2(dp_1(\varphi))(m, c)$. The result follows. □

Remark: T_P eliminates finite imaginaries.

Proof. Let $A = \{a_1, \ldots, a_k\} \subseteq M^n$, where $a_i = (a_{i,1}, \ldots, a_{i,n})$. Consider the following polynomial

$$p(X, Y_0, \ldots, Y_{n-1}) = \prod_{i=1}^{k} \left(X - \sum_{j=1}^{n} a_{i,j} Y_j \right),$$

If σ is an L_P-automorphism, then as it is in particular an L-isomorphism, we have that $\sigma p(X, Y_0, \ldots, Y_{n-1}) = \prod_{i=1}^{k} \left(X - \sum_{j=1}^{n} \sigma(a_{i,j}) Y_j \right)$. Noting that $M[X, Y_0, \ldots, Y_{n-1}]$ is a unique factorization domain, we deduce $\sigma p = p$ if and only if $\sigma A = A$. The tuple consisting of the coefficients of p is a canonical parameter for A. □

Lemma 3.19. *Let M_0 be an elementary substructure of (M,P), and let $a \in M$ be such that $a = \widehat{a}$. Define $d = \mathrm{Cb}(\mathrm{tp}_L(a/\operatorname{acl}_L(M_0P))$, $e' = \mathrm{Cb}(\mathrm{tp}_{L_P}(d/M_0))$, and $e = \mathrm{Cb}(\mathrm{tp}_{L_P}(a/M_0))$. Then e' and e are L_P-interdefinable.*

Proof. Note that by definition of e, e' and because $M_0 \preceq M$, $\mathrm{tp}_{L_P}(a/e)$ and $\mathrm{tp}_{L_P}(d/e')$ are stationary.

Claim I:

(i) $a \mathop{\smile\!\!\!\!\!\!|}\,_d^{L_P} M_0P$.

(ii) $d \in \operatorname{acl}_{L_P}(aM_0)$.

(i): By Lemma 3.12, it suffices to prove $\widehat{ad} \mathop{\smile\!\!\!\!\!\!|}\,_{\widehat{d}}^{L} \widehat{M_0P}$. Notice that since $(M_0P)^c \subseteq P$, we have $M_0P = \widehat{M_0P}$, then by definition of d, $a \mathop{\smile\!\!\!\!\!\!|}\,_d^{L} M_0P$ and since $d^c \in P$, monotonicity gives $a\widehat{d} \mathop{\smile\!\!\!\!\!\!|}\,_{\widehat{d}}^{L} \widehat{M_0P}$. It is now enough to prove $(ad)^c = d^c$, which would imply $\widehat{ad} = a\widehat{d}$. By definition of d, $\langle ad \rangle$ is linearly disjoint (l.d.) from $\operatorname{acl}_L(M_0P)$ over $\langle d \rangle$, hence $\langle ad \rangle$ and $P(d)$ are l.d. over $\langle d \rangle$. Since $\langle d \rangle$ is l.d from P over $\langle d^c \rangle$, it follows that $\langle ad \rangle$ and P are l.d over $\langle d^c \rangle$, hence $(ad)^c = d^c$.
(ii): Since $aM_0 \mathop{\smile\!\!\!\!\!\!|}\,_{(aM_0)^c}^{L} P$, then $a \mathop{\smile\!\!\!\!\!\!|}\,_{M_0(aM_0)^c}^{L} M_0P$. By Fact 1.8 *(iii)* and Remark 3.7, it follows that

$$d \in \operatorname{acl}_L(M_0(aM_0)^c) \subseteq \operatorname{acl}_{L_P}(M_0(aM_0)^c) \subseteq \operatorname{acl}_{L_P}(aM_0). \blacksquare$$

Claim II: $d \in \operatorname{dcl}_{L_P}(a, e)$.
Let σ be an L_P-automorphism that fixes a, e, and let $M_0' = \sigma(M_0)$. Choose a realization M_0'' of $\mathrm{tp}_{L_P}(M_0/a, e)$ independently from $M_0 \cup M_0'$ over a, e. Using $a \mathop{\smile\!\!\!\!\!\!|}\,_e^{L_P} M_0$ and $e \in M_0 \cap M_0' \cap M_0''$, we obtain the following relations

$$a \mathop{\smile\!\!\!\!\!\!|}\,_{M_0}^{L_P} M_0M_0'',\ a \mathop{\smile\!\!\!\!\!\!|}\,_{M_0''}^{L_P} M_0M_0'',\ a \mathop{\smile\!\!\!\!\!\!|}\,_{M_0'}^{L_P} M_0'M_0'',\ a \mathop{\smile\!\!\!\!\!\!|}\,_{M_0''}^{L_P} M_0'M_0''.$$

Applying Lemma 3.12 gives

$$a \mathop{\smile\!\!\!\!\!\!|}\,_{PM_0}^{L} PM_0M_0'',\ a \mathop{\smile\!\!\!\!\!\!|}\,_{PM_0''}^{L} PM_0M_0'',\ a \mathop{\smile\!\!\!\!\!\!|}\,_{PM_0''}^{L} PM_0'M_0'',\ a \mathop{\smile\!\!\!\!\!\!|}\,_{PM_0'}^{L} PM_0'M_0''.$$

Since $\mathrm{tp}_L(a/\operatorname{acl}_L(M_0P))$ is stationary, this translates in terms of canonical bases to

$$d = \mathrm{Cb}(\mathrm{tp}_L(a/\operatorname{acl}_L(M_0P))) = \mathrm{Cb}(\mathrm{tp}_L(a/\operatorname{acl}_L(M_0''P))) = \mathrm{Cb}(\mathrm{tp}_L(a/\operatorname{acl}_L(M_0'P))),$$

hence $\sigma(d) = d$, so the claim is proved. ■

By definition of e, $a \downfree^{L_P}_{e} M_0$. As $\mathrm{tp}_{L_P}(a/e)$ is stationary, $e \in M_0$, and $d \in \mathrm{dcl}_{L_P}(a, e)$, we conclude $\mathrm{tp}_{L_P}(d/e)$ is stationary by Lemma 1.15. Therefore, $e' \in \mathrm{dcl}_{L_P}(e)$.

Claim III: $a \downfree^{L_P}_{e'} M_0$. Therefore, $e \in \mathrm{acl}_{L_P}(e')$.

By Claim I,

$$\begin{aligned} a \downfree^{L_P}_{d} M_0P &\Rightarrow a \downfree^{L_P}_{d} M_0d \\ &\Rightarrow a \downfree^{L_P}_{de'} M_0 \quad \text{ as } e' \in \mathrm{dcl}_{L_P}(M_0) = M_0. \end{aligned}$$

By definition of e' we have $de' \downfree^{L_P}_{e'} M_0$. Applying transitivity yields the claim. ■

To prove $e \in \mathrm{dcl}_{L_P}(e')$, we shall show stationarity of $\mathrm{tp}_{L_P}(a/e')$ and apply Fact 1.8 *(iv)*. By definition of e', $\mathrm{tp}_{L_P}(d/e')$ is stationary, then by Lemma 3.18, it would suffice to prove $\mathrm{tp}_{L_P}(a/de')$ is stationary. However, Claim I implies $a \downfree^{L_P}_{d} e'$, so it is enough to prove $p = \mathrm{tp}_{L_P}(a/d)$ is stationary. Let N be an L_P-elementary substructure of M containing d, and suppose $p \subseteq p_1, p_2$ are nonforking extensions of p to N. Let a_1, a_2 realize p_1, p_2 respectively. By Lemma 3.12, $a_i \downfree^{L}_{dP} NP$. Since $\mathrm{tp}_{L_P}(a_i/d) = \mathrm{tp}_{L_P}(a/d)$ for $i = 1, 2$, and $a \downfree^{L}_{d} dP$ (by definition of d and monotonicity), we get that for $i = 1, 2$, $a_i \downfree^{L}_{d} NP$. This in turn implies $Na_i \downfree^{L}_{N} P$, since $d \in N$. Since $N = \widehat{N}$, by Remark 3.11 *(ii)*, $N \downfree^{L}_{P\cap N} P$. Applying transitivity yields $Na_i \downfree^{L}_{N\cap P} P$, hence $N(a_i)$ and NP are linearly disjoint over N. This implies $\widehat{N(a_i)} = N(a_i)$. Since $\mathrm{tp}_L(a/d)$ is stationary, $\mathrm{tp}_L(a_1N) = \mathrm{tp}_L(a_2N)$. We can then apply Lemma 3.8 to obtain that $\mathrm{tp}_{L_P}(a_1N) = \mathrm{tp}_{L_P}(a_2N)$. □

Lemma 3.20. *Let $a \in M$, $A \subseteq M$. If $A = \widehat{A}$ and $\mathrm{tp}_{L_P}(a/A)$ is stationary, then $\mathrm{tp}_L(a/A)$ is stationary.*

Proof. Suppose by contradiction that $\mathrm{tp}_L(a/A)$ is not stationary and let $k = \langle A\rangle$. The extension $k(a)|k$ is not primary: there is some $\alpha \in k(a)$ such that $\alpha \in \mathrm{acl}_L(k) \setminus \mathrm{dcl}_L(k)$. Note also that $\widehat{k} = k$. By Corollary 3.14, $\alpha \in \mathrm{acl}_{L_P}(k) \setminus \mathrm{dcl}_{L_P}(k)$, contradicting stationarity of $\mathrm{tp}_L(a/A)$. □

Lemma 3.21. *Suppose $d \in M$ is such that $d = \widehat{d}$, and let $e \in \mathrm{dcl}^{\mathrm{eq}}_{L_P}(d)$ be an imaginary such that $\mathrm{tp}_{L_P}(d/e)$ is stationary. Let $d' \models \mathrm{tp}_{L_P}(d/e)$ with $d \downfree^{L_P}_{e} d'$. Let $B_1' = \mathrm{dcl}^{\mathrm{eq}}_{L_P}(e) \cap M$ and $B_1 = \mathrm{acl}^{\mathrm{eq}}_{L_P}(e) \cap M$. Then $\langle d\rangle$ and $\langle d'\rangle$ are linearly disjoint over B_1', in particular $d \downfree^{L}_{B_1} d'$.*

Proof. We denote $p(x) = \mathrm{tp}_{L_P}(d/e)$,

Claim: Let $d'' \models p|\{d, d'\}$, then $d \downfree^{L}_{d'} d''d'$ and $d \downfree^{L}_{d''} d''d'$.

By definition of d'', we have $d \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{d'}^{L_P} d''d'$, and by hypothesis $d \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{e}^{L_P} d'$. Additionally, $e \in \mathrm{dcl}_{L_P}(d'') \cap \mathrm{dcl}_{L_P}(d')$, then both relations $d \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{d''}^{L_P} d'$, $d \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{d'}^{L_P} d''$ hold. Starting from the first relation, we see that $dd'' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{d''}^{L_P} d'd''$, and by Lemma 3.11, $\widehat{dd''} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\widehat{d''}}^{L} \widehat{d'd''}$. Since d, d' and d'' are independent over e, and e is definable over each of d, d', d'', the stationarity of $\mathrm{tp}_{L_P}(d/e)$ implies that of $\mathrm{tp}_{L_P}(d/d''), \mathrm{tp}_{L_P}(d/d')$. It follows by 3.20 that $\mathrm{tp}_L(d/d''), \mathrm{tp}_L(d/d')$ are stationary. Then, as $\widehat{d''} = d''$, we get $d \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{d''}^{L} d''d'$, hence, the field of definition of the locus of d over $\langle d''d' \rangle$ is contained in $\langle d'' \rangle$. The other part of the claim is obtained similarly, using that $\widehat{d'} = d'$ instead, yielding that the field of definition of the locus of d over $\langle d''d' \rangle$ is contained in $\langle d' \rangle$. ■

We obtain therefore

$$\mathrm{Cb}(\mathrm{tp}_L(d/d')) = \mathrm{Cb}(\mathrm{tp}_L(d/d'')) \subset \mathrm{dcl}_L(d') \cap \mathrm{dcl}_L(d'').$$

But d' and d'' are independent over e, $\mathrm{dcl}_{L_P}(d') = \mathrm{dcl}_L(d')$, and $\mathrm{dcl}_{L_P}(d'') = \mathrm{dcl}_L(d'')$, so $\mathrm{Cb}(\mathrm{tp}_L(d/d')) \subseteq \mathrm{dcl}_{L_P}^{eq}(e) \cap M = B_1'$. □

Lemma 3.22. *Let $e \in (M, P)^{\mathrm{eq}}$, and $B_0 = \mathrm{acl}_{L_P}^{\mathrm{eq}}(e) \cap P$. Then for all $c \in P$, $\mathrm{tp}_{L_P}(c/B_0 e)$ is finitely satisfiable in B_0.*

Proof. Let $a \in M$ be such that $e = f(a)$ for some definable function. Since $\widehat{aP} = \hat{a}P$, by Lemma 3.8 $\mathrm{tp}_{L_P}(a/a^c) \vdash \mathrm{tp}_{L_P}(a/P)$. We will prove that $\mathrm{tp}_{L_P}(a/P)$ is stationary, this would imply by Lemma 1.15 that $\mathrm{tp}_{L_P}(e/P)$ is stationary. Suppose not, so we can find $b_1, \ldots, b_n \in M$ and formulas $\varphi(x, b_i)$, which distinguish between the non-forking extensions of $\mathrm{tp}_{L_P}(a/a^c)$ to M. In other words, they define a partition of the set of realizations of $\mathrm{tp}_{L_P}(a/a^c)$. By saturation of M over P, we may assume

$$a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{a^c}^{L_P} b_1, \ldots, b_n, P.$$

If a' is another realization of $\mathrm{tp}_{L_P}(a/a^c)$, such that $a' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{a^c}^{L_P} b_1, \ldots, b_n, P$, then there is an autormorphism of M that fixes $\mathrm{acl}_{L_P}(P, b_1, \ldots, b_n)$ and sends a to a'. This implies $\mathrm{tp}_{L_P}(a'/P) = \mathrm{tp}_{L_P}(a/P)$.

Define $e^c = \mathrm{Cb}(\mathrm{tp}_{L_P}(e/P))$. Note that

$$e^c \in \mathrm{dcl}_{L_P}^{\mathrm{eq}}(e) \cap \mathrm{dcl}_{L_P}^{\mathrm{eq}}(P) = \mathrm{dcl}_{L_P}^{\mathrm{eq}}(e) \cap P.$$

By definition of e^c we have $e \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\mathrm{acl}_L(e^c)}^{L_P} P$, and the proof of Lemma 3.13 shows that $\mathrm{acl}_L(e^c) = B_0$. Hence, $\mathrm{tp}_{L_P}(e/P)$ is stationary and is a non-forking extension of $\mathrm{tp}_{L_P}(e/B_0)$.

This implies definability of $\mathrm{tp}_{L_P}(e/P)$ over B_0. Thus, for any L_P^{eq}-formula $\psi(x,y)$ with parameters in B_0, there is a formula $d\psi(y)$ with parameters in B_0 such that for all $c \in P$, $M \models \psi(e,c)$ iff $M \models d\psi(c)$. By Lemma 3.3, we may assume that $d\psi(y)$ is an L-formula. Since we also have that $B_0 \prec P$ in the sense of L, we have that for any $c \in P$, if $P \models \psi(e,c)$ then $P \models d\psi(c)$, which implies there is $b \in B_0$ such that $P \models d\psi(b)$, and therefore $M \models \psi(e,b)$. □

4 Weak Elimination of Imaginaries

Throughout this section we will maintain our notation and conventions from Section 3. We let $T = ACF_p$, and T_P the theory of beatiful pairs of models of T. We have that $(M,P) \models T_P$ is saturated.

Definition 4.1. Let G be an algebraic group and X an algebraic variety both defined over $k \subseteq M$. A k-rational action is a group action $\alpha : G \times X \to X$ such that for every $g \in G$, the map $\alpha(g,\cdot) : X \to X$ is a k-rational map.

Definition 4.2. A definable group action is a triple $((G,\cdot),X,\alpha)$, where $(G,\cdot)$ is a definable group, $X \subseteq M$ a definable set and $\alpha : G \times X \to X$ a group action whose graph is definable. If the action is *transitive* on X, that is, for every $a,b \in X$ there is $g \in G$ such that $\alpha(g,a) = b$, the triple is the called a *definable homogeneous space*. Moreover, if the action is *strictly transitive (or regular)*, that is, $\alpha(g,x) = x$ iff $g = e$, it will be called a *principal definable homogeneous space* (or PHS).

We shall abuse notation and denote $\alpha(g,a)$ as $g \cdot a$. In our context, as $T = ACF_p$, we get the following fact from Theorem 7.4.14 of [7].

Fact 4.3. *If $G \subseteq M^n$ is an L-definable group, then G is definably isomorphic to an algebraic group.*

Proposition 4.4. *Let $e \in (M,P)^{\mathrm{eq}}$. Then there are: a connected algebraic group G, an irreducible variety V over P, and a rational action G on V, definable over P, such that*

(i) The action of $G(P)$ on $V(M)$ is generically free: if $a \in V(M)$ is a generic point of V over P, and $g \in G(P)$ is not the identity, then $g \cdot a \neq a$.

(ii) For some $a \in V(M)$ generic over P, if r is a canonical parameter for the orbit $X = \{g \cdot a\ ,\ g \in G(P)\}$, then $e \in \mathrm{dcl}_{L_P}(r)$ and $r \in \mathrm{acl}_{L_P}(e)$.

The proof of Proposition 4.4 will require some results.

Lemma 4.5. *Let $e \in (M,P)^{\mathrm{eq}}$. There is $d' \in M$ such that $\mathrm{tp}_{L_P}(d'/e)$ is stationary and P-internal, and moreover $e \in \mathrm{dcl}^{\mathrm{eq}}_{L_P}(d')$.*

Proof. Let $a \in M$ be such that $a = \hat{a}$ and $e = f(a)$ for some $\varnothing$-interpretable function. By Lemma 1.10 we may suppose $\mathrm{tp}_{L_P}(a/e)$ is stationary, hence $e = \mathrm{Cb}(\mathrm{tp}_{L_P}(a/M_0))$, where M_0 is any L_P-elementary substructure of M such that $e \in {M_0}^{\mathrm{eq}}$ and $a \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_e^{L_P} M_0$. Let $d = \mathrm{Cb}(\mathrm{tp}_L(a/\operatorname{acl}_L(M_0P)))$. By Lemma 3.19, $e = \mathrm{Cb}(\mathrm{tp}_{L_P}(d/M_0))$, hence $d \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_e^{L_P} M_0$. Since $M_0 \preceq M$, $\mathrm{tp}_{L_P}(d/M_0)$ is stationary, $\mathrm{tp}_{L_P}(d/e)$ is stationary and almost P-internal. Replacing d by finitely many independent realizations of $\mathrm{tp}_{L_P}(d/e)$, by Fact 1.8 *(v)*, we may asssume without loss of generality that $e \in \mathrm{dcl}^{\mathrm{eq}}(d)$, or that $e = g(d)$ for some definable function g. By Lemma 1.17, there is $d' \in \mathrm{dcl}_{L_P}(d)$, a code for a finite set of realizations of $\mathrm{tp}_{L_P}(d/e)$, such that $d \in \mathrm{acl}_{L_P}(d')$ and $\mathrm{tp}_{L_P}(d'/e)$ is stationary and P-internal. Then as $d \in \mathrm{acl}^{\mathrm{eq}}_{L_P}(d')$, there is a formula $\varphi(x,d')$ isolating $\mathrm{tp}_{L_P}(d/d')$; hence $M \models \forall x \varphi(x,d') \to g(x) = e$, so $e \in \mathrm{dcl}^{\mathrm{eq}}(d')$.

□

Lemma 4.6. *There are a tuple $d \in M$, an L_P-definable function f (over $\varnothing$), an $L_P(e)$-formula $\psi(x)$, and an $L_P(e)$-definable function h such that*

(i) $f(d) = e$.

(ii) $\psi(x) \in \mathrm{tp}_{L_P}(d/e)$.

(iii) $M \models \forall x, x'(\psi(x) \wedge \psi(x') \to \exists c(P(c) \wedge h(x,c) = x')$.

Proof. Let d' be as in Lemma 4.5. Then $p = \mathrm{tp}_{L_P}(d'/e)$ is stationary, P-internal, and $e = \mathrm{Cb}(p)$. By Lemma 1.18, there is a tuple d consisting of finitely many realisations of p, and an e-definable function g such that for any realization d'' of p, there is a tuple $c_{d''} \in P$ such that $d'' = g(d, c_{d''})$. Clearly $e \in \mathrm{dcl}^{\mathrm{eq}}_{L_P}(d)$, so we can find an L_P-definable function f such that *(i)* holds. If d_1, d_2 realize $\mathrm{tp}_{L_P}(d/e)$, then there is an e-definable function h and a tuple $c \in P$ such that $d_1 = h(d_2, c)$. Applying compactness yields an L_P-formula $\psi \in \mathrm{tp}_{L_P}(d/e)$ such that for any two d_1, d_2 satisfying ψ, there is $c \in P$ such that $h(d_1, c) = d_2$, which directly proves *(ii)* and *(iii)*. Note $\mathrm{tp}_{L_P}(d/e)$ remains P-internal. □

Lemma 4.7. *In Lemma 4.6, d can be chosen such that (i),(ii),(iii) hold, and $d \downarrow_e^{L_P} P$.*

Proof. Let ψ as in Lemma 4.6. Let $\chi(x, y)$ an $L_P(e)$- formula that expresses the conjunction of $x^c = y$, $\psi(x)$ and $f(x) = e$. Consider the $L_P(e)$-formula $\theta(y)$ given by $\exists x(\chi(x, y))$. Since $M \models \theta(d^c)$, by Lemma 3.22, there is $d_0 \in \mathrm{acl}^{\mathrm{eq}}_{L_P}(e) \cap P$ such that $M \models \theta(d_0)$. Therefore, there is d_1 such that $M \models \chi(d_1, d_0)$, hence $d_1 \downarrow_e^{L_P} P$. $\square$

Notation: For the remainder of this section, fix d as in Lemma 4.7. By Remark 3.7 $d^c \in \mathrm{dcl}_{L_P}(d)$, hence we may also assume from now on that $d = \widehat{d}$, as all of the properties from Lemmas 4.6, 4.7, and 4.8 still hold after adjoining d^c to d. From now on, let

$$B = \mathrm{acl}^{\mathrm{eq}}_{L_P}(e),$$
$$B_1 = B \cap M,$$
$$B_0 = B \cap P.$$

Lemma 4.8. $\mathrm{tp}_{L_P}(d/B)$ *is isolated.*

Proof. By stability of $Th(M^{\mathrm{eq}})$, there are $M_1 \preceq M$, a prime model over Bd and $M_0 \preceq M_1$ a prime model over B.

Claim: $B_0 = M_0 \cap P = M_1 \cap P$: It is clear that $B \subseteq M_0, M_1$, one inclusion follows. Conversely, if $a \in M_0 \cap P$, then $\mathrm{tp}_{L_P}(a/B)$ is isolated, which is a non-forking extension of $\mathrm{tp}_{L_P}(a/e)$, hence $\mathrm{tp}_{L_P}(a/e)$ is isolated too, and applying Lemma 3.22, it can be realized by some $a' \in B_0$. In particular, this implies $a \in \mathrm{acl}_{L_P}(e)$. The proof for the second equality is similar, let $a \in M_1 \cap P$, then $\mathrm{tp}_{L_P}(a/Bd)$ is isolated. Recall $d \downarrow_e^{L_P} P$, hence $\mathrm{tp}(a/Bd)$ does not fork over $\mathrm{tp}_{L_P}(a/e)$, which is then isolated, and applying Lemma 3.22 yields the result.

Let ψ be as in Lemma 4.6, and choose $d' \in M_0$ such that $M \models \psi(d')$. Applying Lemma 4.6 *(iii)* inside the model M_1, there is $c \in P \cap M_1 = B_0$ such that $d \in \mathrm{dcl}_{L_P}(d', c) \subseteq M_0$, hence by definition of a prime model, $\mathrm{tp}_{L_P}(d/B)$ is isolated. $\square$

Lemma 4.9. *Let X be the set of realizations of $\mathrm{tp}_{L_P}(d/B)$. There are: a connected algebraic group G defined over B_0 and an $L_P(e)$-definable regular action of $G(P)$ on X. Moreover, if r is a canonical parameter for the PHS $(G(P), X)$, then $e \in \mathrm{dcl}_{L_P}(r)$ and $r \in \mathrm{acl}_{L_P}(e)$.*

Proof. By Lemma 4.8, X is L_P-definable over B. Define

$$C = \{c \in P,\ \exists d'(d' \in X \wedge h(d, c) = d')\},$$

which is non empty by Lemma 4.6 *(iii)*, and $L(B_0)$-definable by Lemma 3.3. Consider now the equivalence relation E in C defined by $M \models E(c_1, c_2)$ if and only if $M \models h(d, c_1) = h(d, c_2)$. In C/E we can define an $L_P(e)$-interpretable function $h'(d, c/E)) = h(d, c)$. By Lemmas 4.7 and 4.8 $d \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_B^{L_P} P$, hence all elements of X have the same L_P-type over BP. Since E is contained in some power or P, it is $L(B_0)$-definable, hence it does not depend on the choice of d. This implies that for all $c_1, c_2 \in C$ the value of $h(h(d, c_1), c_2)$ is defined, and taking classes modulo E, there is a unique c_3/E such that $h'(h'(d, c_1/E), c_2/E) = h'(d, c_3/E)$, we define a binary operation on C/E as $(c_1/E) \cdot (c_2/E) = c_3/E$. Once again by Lemma 3.3, this operation is $L(B_0)$-definable. Moreover, by Remark 3.4, we may assume without loss of generality that C/E contains real tuples.

Claim: $(C/E, \cdot)$ is a B_0-definable group.

Let $c_1, c_2, c_3 \in C/E$. To check associativity, notice that

$$h'(d, (c_1c_2)c_3) = h'(h'(d, c_1c_2), c_3) = h'(h'(h'(d, c_1), c_2), c_3),$$

moreover, since $h'(d, c_2c_3) = h'(h'(d, c_2), c_3))$ and $\mathrm{tp}_{L_P}(h'(d, c_1)/BP) = \mathrm{tp}_{L_P}(d/BP)$, we obtain

$$h'(d, c_1(c_2c_3)) = h'(h'(d, c_1), c_2c_3) = h'(h'(h'(d, c_1), c_2), c_3)).$$

To check for an identity, by Lemma 4.6 *(iii)*, there is $c' \in P$ such that $h(d, c') = d$. Then, for all $d' \in X$, $h'(d', c') = d'$, in particular

$$h'(d, c_1c') = h'(h'(d, c_1), c') = h'(d, c_1) \Rightarrow c_1c' = c_1.$$

To check the existence of inverses, notice that since $h(d, c_1) \in X$, there is some L_P-automorphism σ fixing BP pointwise such that $h(d, c_1) = \sigma(d)$, which implies $h'(\sigma^{-1}(d), c_1) = d$. By Lemma 4.6 *(iii)*, there is a unique c_1' such that $h'(d, c_1') = \sigma^{-1}(d)$, hence

$$h'(d, c_1'c_1) = h'(h'(d, c_1'), c_1) = h'(\sigma^{-1}(d), c_1) = d = h'(d, c'),$$
$$h'(d, c_1c_1') = h'(h'(d, c_1), c_1') = h'(\sigma(d), c_1') = d = h'(d, c'),$$

therefore, $c_1c_1' = c_1'c_1 = c'$. ■

By the previous claim and by Fact 4.3, C/E is B_0-definably isomorphic to some algebraic group G over B_0. We can then induce an $L(B_0)$-definable action of $G(P)$ over X using the map h': if $F : G \to C/E$ is an isomorphism, then for $(g, d) \in G \times X$, define

$g \cdot d = h'(d, F(g))$. By Lemma 4.6 *(iii)* and by definition of E, this action is regular. As X is the set of realizations of a stationary type, $G(P)$ must be connected (as an L_P-definable group), hence connected as an algebraic group. Clearly, the PHS $(G(P), X)$ is L_P-definable over B, this implies that if r is a canonical parameter for $(G(P), X)$, then $r \in \mathrm{acl}_{L_P}(e)$. Moreover, if σ is some L_P-automorphism fixing r, then it permutes the realizations of $\mathrm{tp}_{L_P}(d/B)$, and by stationarity of $\mathrm{tp}_{L_P}(d/e)$ we have $e = \mathrm{Cb}(\mathrm{tp}_{L_P}(d/B))$, so $\sigma(e) = e$, hence $e \in \mathrm{dcl}_{L_P}(r)$, completing the proof. □

The set X from Lemma 4.9 will be identified with a generic orbit of the action of $G(P)$ over some variety $V(M)$. We first state Proposition 2.2 from [4].

Lemma 4.10. *Let G be a connected definable group with a generic action on the set of realizations X_1 of a stationary L-type q, that is, for all generic $g \in G$ and for d realizing $q|g$, $g \cdot d$ is defined and realizes q, and for all independent g_1, g_2, d, $g_1 \cdot (g_2 \cdot d) = (g_1 g_2) \cdot d$ when the action is defined. There exists then a type-definable set Y, a definable embedding $X_1 \subseteq Y$, and a definable action of G on Y, extending the generic action of G on X_1. Moreover, for every $y \in Y$ there is $g \in G$ and $d \models q$ such that $y = g \cdot d$.*

Proof. Consider the set of pairs (g, d) with $g \in G$, $d \models q$. Define an equivalence relation over these pairs by: $(g, d) \sim (g', d')$ if for all generic $h \in G$ such that $(hg) \cdot d = (hg') \cdot d'$. Let Y be the set of classes, its elements are denoted by $[g, d]$. If $(hg_2) \cdot d = (hg_2') \cdot d'$ holds for generic h, then, since hg_1 is also generic, it is also true that $(hg_1 g_2) \cdot d = (hg_1 g_2') \cdot d'$, hence we can define an action of G on Y by $g_1 \cdot [g_2, d] = [g_1 g_2, d]$, and identify each $d \models q$ with $[1_G, d]$. To check the last statement, let $[g, d] \in Y$, and let h be a generic of G, independent from d, then $h[g, d] = [hg, d] = [1, hg \cdot d]$, hence $[g, d] = h^{-1}[1, hg \cdot d]$. □

Lemma 4.11. *For X as in Lemma 4.9 there is an irreducible variety Y defined over B_1, and a transitive rational action of G on Y, defined over B_1, such that $X \subseteq Y$, d is a generic point of Y over B_1, and the action of G on Y restricts to the given action of $G(P)$ on X.*

Proof. Recall that for $g \in G(P)$, $d \in X$, $g \cdot d$ is e-definable, this means $g \cdot d \in \mathrm{dcl}_{L_P}(g, d, e)$. Since $e \in \mathrm{dcl}_{L_P}(d)$, then $g \cdot d \in \mathrm{dcl}_{L_P}(g, d) = \mathrm{dcl}_L(\widehat{g, d})$ by Lemma 3.13. But $\mathrm{dcl}_L(\widehat{g, d}) = \mathrm{dcl}_L(g, d)$ by Lemma 3.9 *(ii)*. Therefore, $g \cdot d \in \mathrm{dcl}_L(g, d)$.
Claim: $d \perp^{L}_{B_0} g$.
If $e^c = \mathrm{Cb}(\mathrm{tp}_{L_P}(e/P))$, then $e \perp^{L_P}_{e^c} P$, and by Lemma 4.7, $d^c \perp^{L_P}_{e} P$. Applying transitivity yields $d^c \perp^{L_P}_{e^c} P$, and since everything lives in P, we can restrict our language

to get $d^c \mathop{\smile\!\!\!\!\!\!\,|}_{e^c}^{L} P$. By the proof of Lemma 3.13, $B_0 = \mathrm{acl}_L(e^c)$, hence $d^c \mathop{\smile\!\!\!\!\!\!\,|}_{B_0}^{L} P$ and by definition of d^c we have $d \mathop{\smile\!\!\!\!\!\!\,|}_{d^c}^{L} P$. The claim follows as $g \in P$. ■

Now, working in L, since $e \in \mathrm{dcl}_{L_P}(d) = \mathrm{dcl}_L(\widehat{d})$, $B_1 \in \mathrm{acl}_L(d)$, so the previous claim yields $dB_1 \mathop{\smile\!\!\!\!\!\!\,|}_{B_0} g$. Then, if g is generic over B_0, then it is generic over dB_1. The action is generically regular and transitive: given independent $d_1, d_2 \in X$, there is a unique $g \in G(P)$ such that $g \cdot d_1 = d_2$. Hence, working in L_P, $RM(G) = RM(X)$, and if $g \in G$, $d \in X$ are independent over e, then because the action is defined over e, we have that $g \in \mathrm{dcl}_L(g \cdot d, d)$, so that we must have $RM(g \cdot d, d/e) = 2RM(G)$, which implies $g \cdot d \mathop{\smile\!\!\!\!\!\!\,|}_{e}^{L_P} d$. By Lemma 3.21, $g \cdot d \mathop{\smile\!\!\!\!\!\!\,|}_{B_1}^{L} d$.
We have a definable action of $G(P)$ on the L_P-definable set X, and the action is given by a map $G \times X \to X$ which is $L(B_1)$-definable in T. Passing to the Zariski closure, we get a generic action of the algebraic group $G(M)$ on the set X_1 of generic elements (over B) of the Zariski closure of X. By Lemma 4.10, there is a type-definable $Y \supseteq X_1$ (in the L-sense, and over B_1) such that G acts on Y in a way that restricts to the generic action of G on X_1. Moreover, for every $y \in Y$ there is $g \in G$ and $d \in X_1$ such that $y = g \cdot d$, so the action of G on Y transitive, then Y has a unique generic type by connectedness of G, and it must be indeed $\mathrm{tp}_L(d/B_1)$. This proves that d is a generic of Y over B_1. We claim that Y is also definable: Let $\varphi(x, y)$ be some $L(B_1)$-formula defining $x \in G \cdot y$, and let E be the equivalence relation given by yEy' iff $M \models \forall x \varphi(x, y) \leftrightarrow \varphi(x, y')$, by transitivity, for any $y \in Y$ we have $[y]_E = Y$, now by type-definability of Y over B_1, Y is fixed by all $\sigma \in \mathrm{Aut}(M/B_1)$, hence the imaginary $[y]_E$ is fixed too, which implies $[y]_E$ is B_1-definable, hence Y is B_1-definable . Since $X \subseteq X_1 \subseteq Y$, and the action of G on Y restricts to the generic action over X_1, then it restricts to the action of G on X that was defined in Lemma 4.9 Finally, by Fact 4.3, $(G, Y, \cdot)$ is B_1-definably isomorphic to $(G', Y', \cdot')$, where G' is an algebraic group, Y' an irreducible variety, and $\cdot'$ is a B_1-rational action. □

Proof of Proposition 4.4

Proof. For $e \in M^{\mathrm{eq}}$, d, G, Y as in Lemma 4.11, choose some finite $b \in B_1$ such that $(G, Y, \cdot)$ is definable over b. Rewrite Y as Y_b. By Lemma 4.7, $d \mathop{\smile\!\!\!\!\!\!\,|}_{e}^{L_P} P$, together with $e \mathop{\smile\!\!\!\!\!\!\,|}_{B_0}^{L_P} P$ implies that $bd \mathop{\smile\!\!\!\!\!\!\,|}_{B_0}^{L_P} P$ (recall $b \in \mathrm{acl}_{L_P}(e)$). Since $e \mathop{\smile\!\!\!\!\!\!\,|}_{e^c}^{L_P} P$, and $(bd)^c \mathop{\smile\!\!\!\!\!\!\,|}_{e}^{L_P} P$, applying transitivity yields $(bd)^c \mathop{\smile\!\!\!\!\!\!\,|}_{e^c}^{L_P} P$, and since everything lives in P, we can restrict our language to get $(bd)^c \mathop{\smile\!\!\!\!\!\!\,|}_{e^c}^{L} P$. By the proof of Lemma 3.13, $B_0 = \mathrm{acl}_L(e^c)$, hence $(bd)^c \mathop{\smile\!\!\!\!\!\!\,|}_{B_0}^{L} P$ and by definition of $(bd)^c$ we have $bd \mathop{\smile\!\!\!\!\!\!\,|}_{(bd)^c}^{L} P$, applying transitivity once more yields $bd \mathop{\smile\!\!\!\!\!\!\,|}_{B_0}^{L_P} P$. Let V, Z be the loci of bd and b over B_0, respectively, and consider the projection $f : V \to Z$ sending bd to b, then note that $f^{-1}(b) = Y_b$. Then by compactness, there is a Zariski open subset U in Z, also defined over B_0, such that G acts rationally in $f^{-1}(U)$ and this action restricted to Y_b coincides with the one defined in Lemma 4.9. This proves *(i)*, as any generic $a \in V$ has the same L-type over B_1 as bd, and the action in Lemma 4.9 is regular by construction. Since $f^{-1}(U)$ is still a variety, by shrinking V we may without loss of generality let $V = f^{-1}(U)$, and by $bd \mathop{\smile\!\!\!\!\!\!\!\;|}^{L}_{B_0} P$, we conclude bd is a generic point of V over P, therefore *(ii)* follows by applying Lemma 4.9. $\square$

We state our main result, which will follow from Proposition 4.4.

Corollary 4.12. *There is a set of sorts $\mathcal{S} \subseteq L^{\mathrm{eq}}$, such that T_P has weak elimination of imaginaries in the language obtained by adjoining $\mathcal{S}$ to L.*

Proof. Let G, V be as in Proposition 4.4, and let $c \in P$ generate a field over which $(G, V, \cdot)$ are defined. There is a variety Z defined over the prime field such that there exist varieties $\mathcal{G}, \mathcal{V}$, along with surjective regular maps to Z, and for each $b \in Z$, the fiber $\mathcal{G}_b$ is an algebraic group that acts on $\mathcal{V}_b$, and moreover $\mathcal{G}_c = G$ and $\mathcal{V}_c = V$. For each $e \in M^{\mathrm{eq}}$, we define a sort $S_{(\mathcal{G},\mathcal{V},Z,e)}$ in the following manner: let $W_e = \cup\{\mathcal{V}_b, b \in Z(P)\}$, and define an equivalence relation on W as $w_1 \sim w_2$ iff for some $b \in Z(P)$, $w_1, w_2 \in \mathcal{V}_b$ and there exists $g \in \mathcal{G}_b(P)$ such that $w_1 = g \cdot w_2$. We interpret the elements of $S_{(\mathcal{G},\mathcal{V},Z,e)}$ as the classes of W modulo $\sim$, which are in turn representatives of each orbit of the fiberwise action of $\mathcal{G}$ on $\mathcal{V}$. By Proposition 4.4, for every $e \in M^{eq}$, there is $r \in S_{(\mathcal{G},\mathcal{V},Z,e)}$, such that $e \in \mathrm{dcl}_{L_P}(r)$ and $r \in \mathrm{acl}_{L_P}(e)$. $\square$

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