



IMAGINARIES IN PAIRS OF ALGEBRAICALLY CLOSED FIELDS

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Abstract

Consider the theory T of algebraically closed fields of a given characteristic p , in the language $L = \{0, 1, +, -, \cdot\}$. Extend L to a language L_P by adding a predicate P , which is interpreted in a model $M \models T$ as a proper elementary substructure. Since T has elimination of quantifiers, these pairs can be axiomatized by expressing $P \models T$, and $\exists x \neg P(x)$, obtaining a theory T_P of elementary pairs $P \prec M$. The main goal is to add sorts to the language L_P , in order to achieve *weak elimination of imaginaries*. Keisler in [5] proved that T_P is complete, and in [2], Buechler showed that T_P is an ω -stable theory, of Morley rank ω . This work is largely based on [9], by Anand Pillay.

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1 Preliminaries on Stability Theory

Let T be a complete theory over a language L . If $M \models T$, and $A \subseteq M$, we denote the space of n -types with parameters in A by $S_n(A)$, and let $S(A) = \cup_{i < \omega} S_n(A)$. Recall that a theory is κ -stable if for every $M \models T$ and every $A \subseteq M$, if $|A| \leq \kappa$, then $|S_1(A)| \leq \kappa$, and we say T is stable if it is κ -stable for some cardinal κ . We will use an equivalent characterization of stability, given by the definability of types.

Definition 1.1. Let $M \models T$, and A, B subsets of M . A type $p(x) \in S_n(A)$ is *definable* over B if for any L -formula $\varphi(x, y)$ there is an $L(B)$ -formula $\psi(y)$ such that for all $a \in A^{|y|}$, $\varphi(x, a) \in p$ if and only if $M \models \psi(a)$. The formula $\psi(y)$ will be written as $d_p(\varphi)(y)$, and the set of $d_p(\varphi)(y)$, with $\varphi(x, y)$ ranging over the L -formulas is called a *definition scheme* for p .

The following proposition is Corollary 8.3.2 from [13].

Proposition 1.2. *The theory T is stable if and only if all types are definable.*

Throughout the rest of the section we assume T is an arbitrary, complete, ω -stable theory. We will work inside a saturated model M of T , and types over M shall be referred to as *global types*. We proceed by stating some definitions and results on canonical bases and forking in this stable context.

Definition 1.3. Let $E(x_1, \dots, x_n, y_1, \dots, y_n)$ be an L -formula that defines an equivalence relation on M^n . By *real elements*, we mean tuples in M^n , while the equivalence classes of real elements modulo E will be called *imaginary elements*.

Definition 1.4. Let $X \subseteq M$ be a definable set. A tuple $c \subset M$ is called a *canonical parameter* (or code) of X if c is fixed by exactly the same automorphisms of M which fix X setwise. It is possible to extend T to a new theory T^{eq} (in a new language L^{eq}), in which every definable set has a code. Let $(E_i)_{i \in I}$, an enumeration of every \emptyset -definable equivalence relation over n_i -tuples. To define L^{eq} , add to L a new sort S_i for each i , which is to be interpreted as M^{n_i}/E_i . Consider the many-sorted structure $M^{\text{eq}} = (M, M^{n_i}/E_i)_{i \in I}$, and define for every i the natural projection $\pi_i : M^{n_i} \mapsto M^{n_i}/E_i$ that sends a to a/E_i . The theory of M^{eq} will be denoted as T^{eq} . By Corollary 8.4.6 from [13], T^{eq} has *elimination of imaginaries*: every imaginary is interdefinable with a real tuple. There are also three related notions that will be used throughout this work.

Definition 1.5. i) T has *elimination of finite imaginaries* if for every n , every finite set of n -tuples has a canonical parameter.

- ii) T has *weak elimination of imaginaries*, if for every imaginary e there is a real tuple d such that $e \in \text{dcl}^{\text{eq}}(c)$ and $d \in \text{acl}^{\text{eq}}(e)$.
- iii) T has *geometric elimination of imaginaries*, if for every imaginary e there is a real tuple d such that $e \in \text{acl}^{\text{eq}}(c)$ and $d \in \text{acl}^{\text{eq}}(e)$.

We now proceed with a survey of forking in the **ω -stable context**. For a definable set $X \subseteq M$, we denote by $RM(X)$ its Morley rank, and $DM(X)$ its Morley degree. Recall that ω -stable theories are *totally transcendental*: every definable set has a Morley rank. This rank can also be defined for types: if $p \in S_n(A)$, then $RM(p)$ is the minimal Morley rank of a formula in p , and $DM(p)$ is the minimal Morley degree of a formula in p having Morley rank $RM(p)$.

Definition 1.6. (Forking) Suppose $A \subseteq B \subseteq M$, $p \in S_n(A)$, $q \in S_n(B)$, and $p \subseteq q$. If $RM(p) = RM(q)$, then q is a *non-forking* extension of p to B . Otherwise, if $RM(p) < RM(q)$, we say that q *forks over* A . We say that $p \in S_n(A)$ is *stationary* if for all $B \supseteq A$, there is a unique non-forking extension of p to B , or equivalently if $DM(p) = 1$.

Notation: If $p \in S(A)$ and $C \subseteq A$, we denote the restriction of p to $S(C)$ by $p \upharpoonright C$. If p is stationary and $A \subseteq B$, we denote the unique non-forking extension of p to $S(B)$ by $p|B$.

Definition 1.7. Let $A \subseteq M$, $p \in S(A)$ a stationary type. A *canonical base* of p , denoted $\text{Cb}(p)$, is a tuple $e \subseteq M^{\text{eq}}$ such that for every $\sigma \in \text{Aut}(M)$, $\sigma(p) = p$ if and only if $\sigma(e) = e$ (this tuple is unique up to interdefinability). If p is not stationary, consider the finite set \mathcal{P} of nonforking extensions of p to M , and define $\text{cb}(p)$ as a code for the set $\{\text{Cb}(q), q \in \mathcal{P}\}$; then any automorphism of M fixes $\text{cb}(p)$ if and only if it permutes \mathcal{P} (see Fact 1.8 (i)).

The following is a summary of the properties of canonical bases we will use, they can be found as Proposition 2.20 and Remarks 2.26, 3.19 in Chapter 1 of [8].

Fact 1.8. Let $A \subseteq M$, $p \in S(A)$. Then

- (i) (Conjugacy) The set of automorphisms of M that fix A pointwise acts transitively on \mathcal{P} .
- (ii) $\text{cb}(p) \subseteq \text{dcl}^{\text{eq}}(A)$.
- (iii) For any $B \subseteq A$, p does not fork over B if and only if $\text{cb}(p) \subseteq \text{acl}^{\text{eq}}(B)$.
- (iv) If p is stationary, for any $B \subseteq A$, p does not fork over B and $p \upharpoonright B$ is stationary if and only if $\text{Cb}(p) \subseteq \text{dcl}^{\text{eq}}(B)$.
- (v) If p is stationary, and $(a_i, i < \omega)$ is a sequence such that for all i , a_i realizes $p|A \cup \{a_j, j < i\}$, then $\text{Cb}(p) \subseteq \text{dcl}^{\text{eq}}(a_0, \dots, a_n)$ for some n .

Lemma 1.9. Let e be an imaginary in M and let a be a finite tuple of reals such that $e = f(a)$ for some \emptyset -definable function f . Then $e = \text{cb}(\text{tp}(a/e))$. Moreover, if $e' = \text{Cb}(\text{tp}(a/\text{acl}^{\text{eq}}(e)))$, then $e' \in \text{acl}^{\text{eq}}(e)$ and $e \in \text{dcl}^{\text{eq}}(e')$.

Proof. Let $p = \text{tp}(a/e)$ and $p' = \text{tp}(a/\text{acl}^{\text{eq}}(e))$. To see why $e = \text{cb}(\text{tp}(a/e))$, consider the equivalence relation $E(x, y)$ given by $f(x) = f(y)$; then e is a code for the class of a . Let \mathcal{P} as in Definition 1.7. Since \mathcal{P} is finite, and e' is the canonical base of an element of \mathcal{P} , it follows that $e' \in \text{acl}^{\text{eq}}(e)$. Now, suppose $\sigma(e') = e'$ for some automorphism of M^{eq} ; then $\sigma p' = p'$, so both formulas $f(x) = e$ and $f(x) = \sigma(e)$ belong to p' , which implies $\sigma(e) = e$, hence $e \in \text{dcl}^{\text{eq}}(e')$. \square

Lemma 1.10. Let e be an imaginary in M and let a be a finite tuple of reals such that $e = f(a)$ for some \emptyset -definable function f . There is $a' \in M^{\text{eq}}$ such that $e \in \text{dcl}^{\text{eq}}(a')$ and $\text{tp}(a'/e)$ is stationary.

Proof. Let $p = \text{tp}(a/e)$ and let p_1, \dots, p_n be its non-forking extensions to $\text{acl}^{\text{eq}}(e)$. Let $a_1, \dots, a_n \in M$ be such that a_i realizes $p_i| \{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n\}$. Let a' be a code of this set of realizations. Then as $a \in \text{acl}^{\text{eq}}(a')$, there is a formula $\varphi(x, a')$ isolating $\text{tp}(a/a')$; hence $M \models \forall x \varphi(x, a') \rightarrow f(x) = e$, as f is \emptyset -definable, so $e \in \text{dcl}^{\text{eq}}(a')$. Moreover, any automorphism of M which fixes e permutes $\{p_1, \dots, p_n\}$, hence it fixes $\text{tp}(a'/e)$. \square

Definition 1.11. (Independence) Let $A, B, C \subseteq M$. We say A is *independent* from B over C , denoted

$$A \perp_C B,$$

if for every finite tuple a from A , $\text{tp}(a/BC)$ does not fork over C .

The following is a summary of the properties of the independence relation in the ω -stable context. These are found as Theorem 8.5.5 from [13], and Lemmas 6.3.16 through 6.3.21 from [7].

Fact 1.12. *Let $A, B, C, D \subseteq M$. Forking independence has the following properties.*

1. (Monotonicity) If $A \perp_C B$ and $B' \subseteq B$, then $A \perp_C B'$.
2. (Transitivity) $A \perp_C BD$ if and only if $A \perp_C B$ and $A \perp_{C,B} D$.
3. (Existence) Every $p \in S(A)$ has a non-forking extension to any set containing A .
4. (Symmetry) If $A \perp_C B$, then $B \perp_C A$.
5. (Algebraic Closure) $A \perp_C \text{acl}(A)$.

Definition 1.13. Let $A, B \subseteq M$ and let $p \in S(A)$ be definable over B by a scheme d_p . This definition scheme is called *good* (over B) if the set

$$\{\varphi(x, m) \mid M \models d_p(\varphi)(m), m \in M, \varphi(x, y) \text{ an } L\text{-formula}\}$$

is a global type extending p .

Lemma 1.14. *Let $p \in S(A)$. Then p is stationary if and only if it has a good definition over A .*

Proof. If p is stationary, let q be its global non-forking extension. Then q is definable and invariant under all automorphisms that fix A setwise, hence it is definable over A . This gives a good definition for p . Conversely, assume p has a good definition over A . There is then a global non-forking extension $p' \in \mathcal{P}$, definable over A . Since all elements of \mathcal{P} are conjugate over A , and p' is fixed by every automorphism which fixes A setwise, it must be that $\{p'\} = \mathcal{P}$. Therefore, p is stationary. \square

Lemma 1.15. *Let $a \in M$ be a tuple and $A \subseteq M$. Suppose $p = \text{tp}(a/A)$ is stationary and let $a' \in M$ be a tuple such that $a' \in \text{dcl}(Aa)$. Then $\text{tp}(a'/A)$ is stationary.*

Proof. We shall give a good definition scheme over A for $\text{tp}(a'/A)$. Let $\varphi(x, y)$ be an L -formula and $m \in M$ such that $M \models \varphi(a', m)$. By stationarity of p , there is an $L(A)$ -formula $d_p(\varphi)(y)$ such that $\varphi(x, m) \in \text{tp}(a/A)$ if and only if $M \models d_p(\varphi)(m)$. By hypoth-

esis, there is an A -definable function f such that $f(a) = a'$. Let $\tilde{\varphi}(x, y) = \varphi(f(x), y)$,

$$\begin{aligned} M \models \varphi(a', m) &\iff M \models \varphi(f(a), m) \\ &\iff M \models \tilde{\varphi}(a, m) \\ &\iff M \models d_p(\tilde{\varphi})(m). \end{aligned}$$

□

Definition 1.16. A type $p(x) \in S(A)$ is said *internal* to a partial type $\Sigma(y)$ if there are: a realization a of p , and $B \supseteq A$ independent from a over A , such that $a \in \text{dcl}(Bd)$ for some finite tuple d of realizations of Σ . If it happens instead that $a \in \text{acl}(Bd)$, then the type is said to be *almost internal* to Σ .

Lemma 1.17. Suppose $\text{tp}(a/A)$ is stationary and almost internal to a partial type Σ . Then there is an imaginary a' such that: $\text{tp}(a'/A)$ is stationary and internal to Σ , $a' \in \text{dcl}^{\text{eq}}(Aa)$, and $a \in \text{acl}^{\text{eq}}(a')$. Such an a' can be taken to be a code for a finite set of realizations of $\text{tp}(a/A)$.

Proof. By hypothesis, there are $B \supseteq A$ independent of a over A and a tuple d of realizations of Σ such that $a \in \text{acl}(Bd)$. We can replace B by some finite tuple b such that $a \in \text{acl}(Abd)$. Let $q = \text{tp}(b, d/Aa)$ and $c = \text{cb}(q)$. By Fact 1.8 (ii), $c \in \text{dcl}^{\text{eq}}(Aa)$. Note that $b \perp_A a$, hence $b \perp_A c$ and $\text{tp}(c/A)$ is Σ -internal. By definition of c , $bd \perp_{Ac} Aa$, but $a \in \text{acl}(Abd)$, hence $a \in \text{acl}^{\text{eq}}(Ac)$. If a' denotes the code for the finite set of conjugates of a over Ac , then

$$a' \in \text{dcl}^{\text{eq}}(Ac) \subseteq \text{dcl}^{\text{eq}}(Aa).$$

Because $\text{tp}(c/A)$ is internal to Σ , so is $\text{tp}(a'/A)$. Moreover, $\text{tp}(a'/A)$ is stationary by Lemma 1.15. We may note that $c \in \text{acl}(Aa')$ as well: if a'' is any conjugate of a over Ac , then it realizes the same type over Ac as a . □

Lemma 1.18. Let Σ be a partial type, and let $p \in S(A)$ be a stationary, Σ -internal type. There exists a partial A -definable function $h(y_1, \dots, y_m, z_1, \dots, z_n)$ and a sequence b_1, \dots, b_m of realizations of p , such that for any realization a of p , there is a sequence c_1, \dots, c_n of realizations of Σ , such that $a = h(b_1, \dots, b_m, c_1, \dots, c_n)$.

Proof. Let b realize p , $B \supseteq A$ independent from b over A , and d a tuple of realizations of Σ such that $b \in \text{dcl}(Bd)$.

Claim: For any b' realizing $p|Ab$, there is a sequence d' of realizations of Σ such that $b' = g(b, d')$, for some definable function g .

Let $(b_i, d_i)_{i < \omega}$ be a Morley sequence of $\text{tp}(b, d / \text{acl}^{\text{eq}}(B))$. By Fact 1.8 (v), $\text{tp}(b, d/M)$ is definable over $A \cup \{b_i, d_i, i < \omega\}$. In particular, for m large enough,

$$b \in \text{dcl}(b_1, \dots, b_m, d_1, \dots, d_m, d, A),$$

such that $\bar{d} = (d_1, \dots, d_m, d)$, $\bar{b} = (b_1, \dots, b_m)$ are independent from b over A . Then $b = g(\bar{b}, \bar{d})$ for some A -definable function g . ■

Now let a be an arbitrary realization of p , and let $\bar{a} = (a_1, \dots, a_m)$ realize $\text{tp}(\bar{b} / \text{acl}(A))$ such that $(a_1, \dots, a_m) \perp_A a\bar{b}$. By the claim, for each $i \leq m$ there is \bar{c}_i , a tuple of realizations of Σ , such that $a_i = g(\bar{b}, \bar{c}_i)$. Since $\text{tp}(a, \bar{a}/A) = \text{tp}(b, \bar{b}/A)$, we also get that $a = g(\bar{a}, \bar{c})$ for some tuple \bar{c} of realizations of Σ . It follows that $a = h(\bar{b}, \bar{c}, \bar{c}_1, \dots, \bar{c}_m)$ for an A -definable function h . □

The following is Lemma 7.2.12 from [13], which holds for all simple theories.

Fact 1.19. *For all $A \subseteq M$ there is some λ such that for any sequence $(a_i, i < \lambda)$ there exists an A -indiscernible sequence $(b_j, j < \omega)$ such that for all $j_1 < \dots < j_n < \omega$ there is a sequence $i_1 < \dots < i_n < \lambda$ with $\text{tp}(a_{i_1}, \dots, a_{i_n}/A) = \text{tp}(b_{j_1}, \dots, b_{j_n}/A)$.*

Lemma 1.20. *If $b \in \text{acl}(aA)$, then $RM(ab/A) = RM(a/A)$.*

Proof. It is clear that $RM(ab/A) \geq RM(a/A)$, since the latter type contains less formulas. The reverse inequality is proved by induction on $\alpha = RM(a/A)$. Let $d = DM(ab/Aa)$ be its Morley degree. Choose an $L(A)$ -formula $\varphi(x, y) \in \text{tp}(ab/A)$ such that $RM(\exists y \varphi(x, y)) = \alpha$ and $\varphi(a', y)$ has at most d realizations for all a' . If Y is the set defined by $\exists x \varphi(x, y)$, we claim that $RM(Y) \leq \alpha$. Consider an infinite family of pairwise disjoint definable subsets $Y_i \subseteq Y$. Let $\psi_i(x) = \exists y (\varphi(x, y) \wedge y \in Y_i)$. Note that any $d+1$ of the $\psi_i(M)$ have empty intersection: if $M \models \bigwedge_{i=0}^d \psi_i(a')$, then there exist $b_i \in Y_i$ for $0 \leq i \leq d$ such that $\models \varphi(a', b_i)$, which contradicts our choice of φ . Therefore, some $\psi_i(x)$ has Morley rank $\beta < \alpha$. Let $b' \in Y_i$, and choose a' such that $M \models \varphi(a', b')$. Then b' is algebraic over $a'A$ and since a' realizes $\psi_i(x)$, we have $RM(a'/A) \leq \beta$. So by induction hypothesis, we conclude $RM(a'b'/A) \leq \beta$, which shows $RM(Y_i) \leq \beta$. This implies that Y does not contain an infinite family of disjoint subsets of Morley rank $\geq \alpha$. □

The following definition comes from 10.2.8 of [13].

Definition 1.21. Let $A, B \subseteq M$ be definable sets and let $f : B \rightarrow A$ be a definable function. The fibers of f have definable Morley rank if for every definable $B' \subseteq B$ and every $k < \omega$, the set $\{a \in A, RM(f^{-1}(a) \cap B') = k\}$ is definable.

Lemma 1.22. Let $A, B \subseteq M$ be definable sets and let $f : B \rightarrow A$ be a definable surjection whose fibers have definable Morley rank, and such that for all $a \in A$, $RM(f^{-1}(a)) = k$. Then $RM(B) = RM(A) + k$.

Proof. Suppose A, B, f are definable over some $S \subseteq M$. The proof is by induction on $RM(A) = m$, for all k . We may also assume that $DM(A) = 1$: if not, then partition $A = A_1 \cup \dots \cup A_d$ into finitely many disjoint, rank m definable subsets, then replace B, A by $f^{-1}(A_1), A_1$, respectively. If $m = 0$, then A is finite and

$$RM(B) = \max\{RM(f^{-1}(a))\}_{a \in A} = k.$$

If $m > 0$, then write $A = \bigcup_i A_i$ for an infinite, pairwise disjoint family of definable $A_i \subseteq A$ such that $RM(A_i) = m - 1$. If $B_i = f^{-1}(A_i)$, then $f \upharpoonright B_i$ is a definable surjection with rank k fibers, so by induction hypothesis $RM(B_i) = RM(A_i) + k = m + k - 1$, and since the B_i are also pairwise disjoint, we deduce $RM(B) \geq m + k$. For the reverse inequality, let now $(B'_i)_{i < \omega}$ be any infinite, pairwise disjoint family of definable subsets $B'_i \subseteq B$; we will show that $RM(B'_i) < m + k$ for some i , by induction over k . If $k = 0$, f is finite-to-one, so for any $b \in B$, $b \in \text{acl}(f(b))$ and $f(b) \in \text{dcl}(b)$. By Lemma 1.20,

$$RM(b/S) = RM(f(b), b/S) = RM(f(b)/S) \leq RM(A) = m.$$

This implies $RM(B) = \sup_{b \in B} (RM(b/S)) \leq m$. Suppose now the conclusion holds for m and for every $k' < k$. Let $a \in A$, then since $RM(f^{-1}(a)) = k$ and $f^{-1}(a) \supseteq \bigcup_i (f^{-1}(a) \cap B'_i)$, it must be that for some j , $RM((f^{-1}(a) \cap B'_j) < k$. Consider now the definable sets

$$A'_i = \{a \in A, RM(f^{-1}(a) \cap B'_i) = k\}.$$

We have proved that $\bigcap_i A'_i = \emptyset$. We claim that for some i , $RM(A'_i) < m$: if not, then as $DM(A) = 1$, for every N , $\bigcap_{i \leq N} A'_i \neq \emptyset$, hence by compactness $\bigcap_i A'_i \neq \emptyset$, a contradiction, since the Morley degree of the fibers is bounded. Now, as

$$B'_i = (f^{-1}(A'_i) \cap B'_i) \cup (f^{-1}(A \setminus A'_i) \cap B'_i),$$

we can apply induction hypothesis over m to the first term to see that $RM(f^{-1}(A'_i) \cap B'_i) < m+k$. On the other hand, on $A \setminus A'_i$, all fibers have rank strictly less than k . The induction hypothesis over k yields $RM(f^{-1}(A \setminus A'_i) \cap B'_i) < m+k$, concluding the proof. \square

Lemma 1.23. *Let $P \subseteq M$ be a strongly minimal definable set, and $\varphi(x_1, \dots, x_n, \bar{y})$ a formula such that $M \models \forall x_1, \dots, x_n (\exists \bar{y} \varphi(x_1, \dots, x_n, \bar{y}) \rightarrow x_i \in P)$. The following set is definable for every k ,*

$$Y_{n,k} = \{\bar{b} \in M, RM\varphi(x_1, \dots, x_n, \bar{b}) = k\}.$$

Proof. Let $Y'_{n,k} = \{\bar{b} \in M, RM\varphi(x_1, \dots, x_n, \bar{b}) \geq k\}$. We will prove definability of $Y'_{n,k}$, this gives the desired result since $Y_{n,k} = Y'_{n,k} \setminus Y'_{n,k+1}$. We proceed by induction on n . Notice that $Y'_{1,1}$ is definable since $RM(\varphi(x_1, \bar{b})) \geq 1$ if and only if $\exists^\infty x_1 \varphi(x_1, \bar{b})$, which is in turn equivalent (by strong minimality of P) to $\exists^{\geq N} x_1 \varphi(x_1, \bar{b})$, for some N . Moreover, notice that

$$Y_{n,0} = \{\bar{b} \in M, \exists x_1, \dots, x_n \varphi(x_1, \dots, x_n, \bar{b})\}$$

is definable for all n . Let now $n > 0$, we will work by induction over $k > 0$. For $\bar{b} \in P$, consider the \bar{b} -formula $\phi_{\bar{b}}(x_0, \dots, x_{n-1})$ given by $\exists x_n \varphi(x_0, \dots, x_{n-1}, x_n, \bar{b})$. If $RM(\phi_{\bar{b}}) \geq k$, then $\bar{b} \in Y'_{n,k}$, and if $RM(\phi_{\bar{b}}) < k$, consider instead the $L(\bar{b})$ -formula $\psi_{\bar{b}}(x_0, \dots, x_{n-1})$ given by $\exists^\infty x_n \varphi(x_0, \dots, x_{n-1}, x_n, \bar{b})$, then since the algebraic dimension of a tuple inside P agrees with its Morley rank, we have in this case that $RM(\psi_{\bar{b}}) \geq k-1$ if and only if $\bar{b} \in Y'_{n,k}$. We have shown that $\bar{b} \in Y'_{n,k}$ if and only if $RM(\phi_{\bar{b}}) \geq k$ or $RM(\psi_{\bar{b}}) \geq k-1$. The first of these two conditions is definable by our induction hypothesis over n , while the latter is definable by induction over k , so $Y'_{n,k}$ is also definable. \square

2 Stable Groups

An ω -stable group is an ω -stable structure $(G, \cdot, 1, \dots)$, where $(G, \cdot, 1)$ is a group. In this section we present some basic concepts and tools used in the study of ω -stable groups. For more details, see [11] and Chapter 7 from [7]. Throughout this section G will denote an infinite, ω -stable group, definable inside a saturated model M of a complete, ω -stable theory T .

Lemma 2.1. *There is no infinite strictly descending chain of definable subgroups $G > G_1 > G_2 > \dots$*

Proof. For any definable subgroup $H \leq G$, and any $a \in G \setminus H$, the coset $aH \subseteq G$ is disjoint from H , and since $x \mapsto ax$ is a definable bijection, then $RM(H) = RM(aH)$. If $G > G_1 > G_2 > \dots$ is a strictly decreasing sequence, and if $[G_i : G_{i+1}]$ is infinite, then $RM(G_i) > RM(G_{i+1})$. If $[G_i : G_{i+1}]$ is finite, then $DM(G_i) > DM(G_{i+1})$, and this implies the existence of a strictly decreasing sequence with respect to the lexicographic order $RM(G) \times \omega$, hence, this sequence cannot be infinite. \square

Lemma 2.2. *There is a definable normal subgroup $G^0 \leq G$ that is contained in every subgroup of G of finite index.*

Proof. Let \mathcal{H} be the family of definable subgroups of G of finite index. We claim that there are H_1, \dots, H_n in \mathcal{H} such that

$$\bigcap_{H \in \mathcal{H}} H = H_1 \cap \dots \cap H_n.$$

If not, then for every m there are j_0, \dots, j_m such that if $G_m = H_{j_0} \cap \dots \cap H_{j_m}$, then $G_0 > G_1 > G_2 > \dots$, contradicting Lemma 2.1. We may then define $G^0 = H_1 \cap \dots \cap H_n$. If $h \in G$, since $x \mapsto hxh^{-1}$ is a group automorphism, we have that hG^0h^{-1} is a definable subgroup with $[G : hG^0h^{-1}] = [G : G^0]$, so $hG^0h^{-1} = G^0$ by minimality. \square

Lemma 2.3. *Let $A \subseteq M$. If G is A -definable, then G^0 is A -definable.*

Proof. By Lemma 2.2, there are an $L(A)$ -formula $\varphi(x, y)$ and $g \in G$ such that the formula $\varphi(x, g)$ defines G^0 . Let $n = [G : G^0]$, and consider

$$W = \{b \in G, \varphi(x, b) \text{ defines a subgroup of index } n\},$$

an A -definable set. If $b \in W$ and $H = \varphi(G, b)$, then $H \cap G^0$ is a finite index subgroup of G^0 , hence $H \supseteq G^0$. However, since $[G : H] = n$, we have $[H : G^0] = 1$, yielding $H = G^0$. We can then define G^0 as $\{g \in G, \exists b (b \in W \wedge \varphi(g, b))\}$. \square

Definition 2.4. G is *connected* if $G = G^0$.

Definition 2.5. There is an action of G on $S_1(G)$ given by $g \cdot p = \{\varphi(x), \varphi(gx) \in p\}$. The *stabilizer* of p is the group

$$\text{Stab}(p) = \{g \in G, g \cdot p = p\}.$$

Lemma 2.6. $\text{Stab}(p)$ is a definable subgroup of G , for every $p \in S_1(G)$.

Proof. For $\varphi(x, y)$ an L -formula, let

$$\text{Stab}_\varphi(p) = \{g \in G \mid p_\varphi = g \cdot p_\varphi\},$$

where

$$p_\varphi = \{\varphi(x, g) \mid g \in G, \varphi(x, g) \in p\} \cup \{\neg\varphi(x, g) \mid g \in G, \varphi(x, g) \notin p\}.$$

An easy calculation shows that for every φ , $\text{Stab}_\varphi(p) \leq G$. By stability, there is a definition scheme for p , say d_p . Thus,

$$\text{Stab}_\varphi(p) = \{g \in G \mid \forall h (d_p(\varphi)(h) \leftrightarrow d_p(\varphi)(hg))\}.$$

Note that $\text{Stab}(p) = \bigcap_{\varphi(x, y) \in L} \text{Stab}_\varphi(p)$. By Lemma 2.1, there are $\varphi_1, \dots, \varphi_n \in L$ such that $\text{Stab}(p) = \text{Stab}_{\varphi_1}(p) \cap \dots \cap \text{Stab}_{\varphi_n}(p)$, which concludes the proof. \square

Lemma 2.7. Let $p \in S_1(G)$.

(i) $RM(\text{Stab}(p)) \leq RM(p)$.

(ii) $\text{Stab}(p) \leq G^0$.

Proof. Let $a, b \in M$ be such that a realizes p , $b \in \text{Stab}(p)$ satisfies $RM(\text{tp}(b/G)) = RM(\text{Stab}(p))$, and $a \perp_G b$. Then

$$RM(\text{tp}(ba/G, a)) = RM(\text{tp}(b/G, a)) = RM(\text{tp}(b/G)) = RM(\text{Stab}(p)).$$

Moreover, since ba realizes p , we have $RM(\text{tp}(ba/G, a)) \leq RM(\text{tp}(ba/G)) = RM(p)$, proving (i). Let now $c \in \text{Stab}(p)$, and let $\varphi(x)$ define G^0 (possibly with parameters in M). Let $g \in G$ be such that $\varphi(g^{-1}x) \in p$, thus $\varphi(g^{-1}cx) \in p$. Let $G \preceq H$ and $h \in H$ realize p . Then $g^{-1}ch \in H^0$ and $g^{-1}h \in H^0$. Thus $(g^{-1}h)^{-1}g^{-1}ch = h^{-1}ch \in H^0$, and since H^0 is normal, $c \in G^0$ by Lemma 2.3. \square

Definition 2.8. A type $p \in S_1(G)$ is *generic* if $RM(p) = RM(G)$. An element $a \in G(M)$ is generic over $A \subseteq G$ if $RM(\text{tp}(a/A)) = RM(G)$.

Lemma 2.9. A type $p \in S_1(G)$ is generic if and only if $[G : \text{Stab}(p)]$ is finite.

Proof. Suppose p is generic. Notice that $\{ap, a \in G\}$ is finite, since there are only finitely many types of maximal Morley rank. Choose $b_1, \dots, b_n \in G$ such that if $a \in G$, then $ap = b_i p$ for some $i \leq n$. If $ap = b_i p$ then $b_i^{-1}a \in \text{Stab}(p)$ and $a \in b_i \text{Stab}(p)$. Therefore, $[G : \text{Stab}(p)] \leq n$. Assume now that $\text{Stab}(p)$ has finite index, so $RM(G) = RM(\text{Stab}(p))$, but $RM(\text{Stab}(p)) \leq RM(p)$ by Lemma 2.7, hence p is generic. \square

Corollary 2.10.

- (i) A type $p \in S_1(G)$ is generic if and only if $\text{Stab}(p) = G^0$
- (ii) G has an unique generic type if and only if G is connected.

Proof.

- (i) By Lemma 2.9, if p is generic, $\text{Stab}(p)$ has finite index, we have $G^0 \leq \text{Stab}(p)$. By Lemma 2.7 (ii), we have $G^0 \geq \text{Stab}(p)$. The other direction is clear by Lemma 2.9, since G^0 has finite index.
- (ii) Let p be the unique generic type. For all $a \in G$, ap is generic, hence $ap = p$. Thus, $G = \text{Stab}(p) = G^0$ by (i). Conversely, suppose $G = G^0$, and by contradiction, assume p, q are distinct, generic types. Let a, b realize p, q respectively, with $b \in H \succeq G$ and let a' realize $p|H$. Then, $\text{tp}(a, b/G) = \text{tp}(a', b/G)$, and $p|H$ is a generic of H . By (i), $\text{Stab}(p|H) = H^0 = H$. Thus, ba' realizes $p|H$. In particular, ba' realizes p , hence ba realizes p . If $a \in K \succeq G$, and b' realizes $q|K$, an analogous argument shows that ba realizes q . This contradicts our assumption, hence G has a unique generic type.

\square

3 Pairs of Algebraically Closed Fields

Throughout this section, we will let $T = ACF_p$ for p prime or 0 (in the usual language L), and we consider L_P , the language obtained by adjoining a unary predicate P . An elementary pair of models of T , $N \preceq M$, is considered an L_P -structure by interpreting P as the universe of the structure N , and L_P -structures will be naturally denoted as pairs (M, P) .

Definition 3.1. A *beautiful pair* of models of T is an elementary pair $N \preceq M$ such that N is $|T|^+$ -saturated and M is $|T|^+$ -saturated over N , which means that M realizes any L -type over $N \cup A$, where $A \subseteq M \setminus N$ is such that $|A| < |T|^+$. The theory T_P of proper pairs $P \prec M$ of models of T was shown to be complete by Keisler in [5].

Fact 3.2. ([12]) Let (M, P) be a saturated model of T_P .

(i) (M, P) is a beautiful pair.

(ii) T_P is stable.

(iii) Any L_P -formula $\phi(x)$ is equivalent modulo T_P to a Boolean combination of L_P -formulas of the form $\exists y P(y) \wedge \psi(y, x)$ where ψ is a quantifier-free L_P -formula.

Buechler in [2] notes that T_P is actually ω -stable of Morley rank ω . From now on (M, P) will be a saturated model of T_P .

Notation: If $A \subseteq M$, we denote the field generated by A by $\langle A \rangle$. For any $A, B, C \subseteq M$, we denote independence in the sense of L by $A \perp_C^L B$, and in the sense of L_P by $A \perp_C^{L_P} B$. We will also distinguish L -types from L_P -types by using tp_L and tp_{L_P} respectively. We adopt the same convention for the acl and dcl operators.

Lemma 3.3. Any $C \subset P^n$ that is L_P -definable with parameters from M , is L -definable with parameters from P . In particular, P is strongly minimal and stably embedded (in the L_P -sense).

Proof. Let $\varphi(x, m)$ with $m \in M$ be an L_P -formula defining C . Note that P is algebraically closed in the L_P -sense. By stability of T_P , $p(y) = \text{tp}_{L_P}(m/P)$ is definable over P , hence we have that for every $a \in M$,

$$a \in C \iff \varphi(a, y) \in p \iff M \models d\varphi(a),$$

where $d\varphi(x)$ is an L_P -formula with parameters in P . Now, by Fact 3.2, $d\varphi(x)$ is equivalent to a boolean combination of formulas of the form $\exists z P(z) \wedge \psi(x, z)$ where ψ is a quantifier-free L_P -formula. Since $C \subseteq P^n$ and

$$M \models \forall x \ d\varphi(x) \rightarrow P(x),$$

C is L -definable by a boolean combination of formulas of the form $\exists z \psi'(x, z)$, where ψ' is the L -formula obtained from ψ by replacing every instance of $P(t)$ by $t = t$, for every term t . \square

Remark 3.4. By elimination of quantifiers in T , the formula $\exists z \psi'(x, z)$ is equivalent modulo T to a quantifier free L -formula $\theta(x)$. Notice also that the set C only depends on m , so if c is an L_P -code for C , we have that $c \in \text{dcl}_{L_P}^{\text{eq}}(m) \cap P$.

Definition 3.5. Let $a \in M$ a (possibly infinite) tuple, define $\hat{a} = (a, a^c)$, where $a^c = \text{Cb}(\text{tp}_L(a/P))$. Since T is totally transcendental and eliminates imaginaries, a^c is in the L -definable closure of a finite real tuple. More specifically, a^c can be regarded up to interdefinability as a tuple of generators for the field of definition of the algebraic locus of a over P (i.e: the variety associated to the prime ideal of polynomials in $P[X]$ that vanish at a).

Lemma 3.6. *For all tuples $a \in M$, $\langle \hat{a} \rangle$ is linearly disjoint from P over $\langle a^c \rangle$.*

Proof. Note that $\langle \hat{a} \rangle = \langle a^c \rangle(a)$. Let $\{M_0(X), \dots, M_m(X)\}$ be a set monomials such that $\{M_0(a), \dots, M_m(a)\}$ is linearly independent over $\langle a^c \rangle$. Suppose there is a linear relation $\sum c_i M_i(a) = 0$, where $c_i \in P$. By definition of a^c we can write

$$\sum_{i=0}^m c_i M_i(X) = \sum_{j=0}^n b_j f_j(X),$$

where $b_j \in \langle a^c \rangle$, $f_j(a) \in I := \{f(X) \in \langle a^c \rangle(X), f(a) = 0\}$ for all j , and such that $\{f_0(X), \dots, f_n(X)\}$ is a linearly independent set of polynomials over $\langle a^c \rangle$. We claim that $\{M_1, \dots, M_m, f_1, \dots, f_n\}$ is also linearly independent over $\langle a^c \rangle$: if it were not, then $\sum r_i M_i(X) + \sum s_j f_j(X) = 0$, for some $r_i, s_j \in \langle a^c \rangle$. We can substitute a for X to obtain $\sum r_i M_i(a) = 0$, which yields $r_i = 0$ for all i , hence $\sum s_j f_j(X) = 0$ and $s_j = 0$ for all j . As these are formal polynomials, they remain linearly independent over P , hence $c_i = 0$ for all i . \square

Remark 3.7. For all tuples $a \in M$, $a^c \in \text{dcl}_{L_P}(a)$.

Proof. Any L_P -automorphism leaves P invariant, so if it also fixes a , it must leave $\text{tp}_L(a/P)$ invariant, hence it must fix a^c . \square

Lemma 3.8. For all tuples $a, b \in M$, $\text{tp}_{L_P}(a) = \text{tp}_{L_P}(b)$ if and only if $\text{tp}_L(\hat{a}) = \text{tp}_L(\hat{b})$.

Proof. If there is an L_P -automorphism σ of M taking a to b , by Remark 3.7 we have $\sigma(a^c) = b^c$, hence $\text{tp}_{L_P}(\hat{a}) = \text{tp}_{L_P}(\hat{b})$. Restricting the language yields $\text{tp}_L(\hat{a}) = \text{tp}_L(\hat{b})$. Conversely, assume there is a partial L -isomorphism σ sending a to b and a^c to b^c . Since $\langle \hat{a} \rangle$ and P are linearly disjoint over $\langle a^c \rangle$, and also $\langle \hat{b} \rangle$ and P are l.d. over $\langle b^c \rangle$, the restriction $\sigma \upharpoonright \langle \hat{a} \rangle$ can be extended to an L -isomorphism $\sigma' : P(a) \rightarrow P(b)$ such that $\sigma'(P) = P$ or in other words, to an L_P -isomorphism, which can be itself extended to an L_P -automorphism of M by saturation of M over P (see Fact 3.2 (iii)). \square

Lemma 3.9. For all tuples $a \in M$,

$$(i) \quad a^c \subseteq P.$$

$$(ii) \quad \text{If } b \in P \text{ is a tuple, } \hat{ab} \text{ and } \hat{ab} \text{ are } L\text{-interdefinable.}$$

$$(iii) \quad \hat{a} \perp_{a^c}^{L_P} P$$

Proof. By elimination of imaginaries in T , $a^c \in \text{acl}_L^{\text{eq}}(P) = P$, this gives (i). To see (ii), notice that $a \perp_{a^c}^L P$ implies $ab \perp_{a^c b}^L P$, and since $\text{tp}_L(ab/a^c b)$ is stationary, we get $(ab)^c \subseteq \text{dcl}_L(a^c b)$, hence $\hat{ab} \in \text{dcl}_L(\hat{ab})$. Clearly $\hat{ab} \subseteq \hat{ab}$, so the other direction follows. For (iii), let $a^c \in B \subseteq P$, and choose an L_P -indiscernible sequence over a^c , $(B_i)_{i < \omega}$, such that $B_0 = B$. Let $p = \text{tp}_L(\hat{a}/B)$, and for each i let p_i be the image of p under an L -automorphism that fixes a^c and sends B to B_i . As $\hat{a} \perp_{a^c}^L P$, $B_i \subseteq P$, and $\hat{a} \perp_{a^c}^L B_i$ for every i , \hat{a} realizes $\cup_i p_i$. By Lemma 3.8 and (ii), $\text{tp}_L(\hat{a}/P) \vdash \text{tp}_{L_P}(\hat{a}/P)$. If we let $p' = \text{tp}_{L_P}(\hat{a}/B)$ and p'_i be the image of p' under an L_P -automorphism that fixes a^c and sends B to B_i , we have proven consistency of $\cup_i p'_i$. Hence, $\text{tp}_{L_P}(\hat{a}/B)$ does not fork over a^c . Since B was chosen arbitrarily, the result follows. \square

Definition 3.10. A subset A of M is said to be P -independent if $A \perp_{A \cap P}^L P$.

Remark 3.11. (i) For all $a \in M$, \hat{a} is P -independent.

(ii) Any subset of P is P -independent.

Proof. The first condition follows directly from Lemma 3.9 (i), (iii), and monotonocity. The second statement is clear. \square

Lemma 3.12. Let $A \subseteq B, C \subseteq M$ with $C = Ac$, where $c \in M$ is a finite tuple. The following are equivalent:

$$(i) \ C \perp_A^{L_P} B$$

$$(ii) \ C \perp_{AP}^L BP, \text{ and } C^c \perp_{A^c}^L B^c.$$

$$(iii) \ C \perp_{AP}^L BP, \text{ and } \widehat{C} \perp_{\widehat{A}}^L \widehat{B}.$$

Proof.

(i) implies (ii): By Remark 3.7, we may assume $B = \widehat{B}$. For the first part, suppose by contradiction that $\text{tp}_L(c/BP)$ forks over AP . Let $(B_i)_{i < \lambda}$ be a sequence of realizations of $\text{tp}_L(B/AP)$ such that $B_i \perp_{AP}^L (B_j)_{j < i}$ and $B_0 = B$; note that in particular, as $\widehat{B}_i = B_i$, for all i , we get $\text{tp}_{L_P}(B_i) = \text{tp}_{L_P}(B)$ by Lemma 3.9. We can choose λ large enough to apply Fact 1.19, yielding an L_P -indiscernible sequence over AP , $(B'_i)_{i < \omega}$, such that $\text{tp}_{L_P}(B'_i/AP) = \text{tp}_{L_P}(B/AP)$ for all i . Let $p = \text{tp}_{L_P}(c/B)$ and let p_i be its copy over B'_i ; then by (i), $\cup_{i < \omega} p_i$ can be realized by some c' . We have that for all i , $c' \not\perp_{AP}^L B'_i P$: this contradicts the ω -stability of T , since $(B'_i)_{i < \omega}$ is also L -independent over AP . To prove the latter part of (ii), we apply properties of forking: by Lemma 3.9 (iii) we have that $\widehat{A} \perp_{A^c}^{L_P} P$, which implies by symmetry and monotonicity that $C^c \perp_{A^c}^{L_P} \widehat{A}$. Additionally, Remark 3.7 gives

$$\begin{aligned} C \perp_A^{L_P} B &\Rightarrow CC^c \perp_{AA^c}^{L_P} BB^c \\ &\Rightarrow C^c \perp_{\widehat{A}}^L B^c, \text{ by monotonicity.} \end{aligned}$$

Applying transitivity, $C^c \perp_{A^c}^{L_P} B^c$. As these three sets all lie in P , we actually get the desired independence in the L -sense.

(ii) implies (iii): We will prove $AC^c \perp_{\widehat{A}}^L \widehat{B}$ and $\widehat{C} \perp_{AC^c}^L \widehat{B}$, then (iii) will follow by transitivity and because $A^c \subseteq C^c$. To get the first relation, start from $\widehat{B} \perp_{B^c}^L P$ and use $C^c \subseteq P$ to get $\widehat{B} \perp_{B^c} C^c$. Combining this with our hypothesis $C^c \perp_{A^c}^{L_P} B^c$, we get $C^c \perp_{A^c}^L \widehat{B}$, which implies $AC^c \perp_{\widehat{A}}^L \widehat{B}$ since $A^c \subseteq \widehat{A} \subseteq \widehat{B}$. For the second relation, start from $\widehat{C} \perp_{C^c} P$ and $A \subseteq C$ to get $\widehat{C} \perp_{AC^c}^L AP$ (I). Now, the hypothesis $C \perp_{AP}^L BP$ yields $\widehat{C} \perp_{AP}^L \widehat{B}$ (II), since $B^c, C^c \subseteq P$. Combining (I) and (II) gives $\widehat{C} \perp_{AC^c}^L \widehat{B}$.

(iii) implies (i):

Claim: (iii) implies $\widehat{A}c\widehat{B}$ is P -independent.

$\widehat{A}c\widehat{B}$ is P -independent is equivalent to saying that if t_C, t_B are transcendence bases for $\widehat{A}c, \widehat{B}$ over $\widehat{A}P = AP$ respectively, then $t_C \cup t_B$ remains algebraically independent over AP , which is equivalent to $C \perp_{AP}^L BP$. ■

Let $(\widehat{B}_i)_i$ be an L_P -indiscernible sequence over \widehat{A} with $\widehat{B}_0 = \widehat{B}$. By hypothesis $C^c \perp_{\widehat{A}}^L \widehat{B}$, so we may assume that $(\widehat{B}_i)_i$ is also L -indiscernible over $\widehat{A}C^c$. Let $p = \text{tp}_L(\widehat{c}/\widehat{B}C^c)$, and let p_i be its copies over $\widehat{B}_i C^c$. By the first condition of (iii), we can realize $\cup_i p_i$ by some \widehat{C}' which is L -independent from P over $\cup_i \widehat{B}_i C^c$. By Lemma 3.8 and by the claim, it follows that $\text{tp}_{L_P}(\widehat{C}'\widehat{B}_i C^c) = \text{tp}_{L_P}(\widehat{C}\widehat{B}_i C^c)$, so p does not L_P -fork over \widehat{A} . Therefore, $\widehat{C} \perp_{\widehat{A}}^{L_P} \widehat{B}$, and (i) follows from Remark 3.7. □

Lemma 3.13. *Let $a \in M$, then*

$$i) \quad \text{acl}_{L_P}(a) = \text{acl}_L(\widehat{a}).$$

$$ii) \quad \text{dcl}_{L_P}(a) = \text{dcl}_L(\widehat{a}).$$

Proof. In both cases, the inclusion \supseteq follows from Remark 3.7.

- i) First we show that $\text{acl}_{L_P}(a) \cap P = \text{acl}_L(a^c)$. Let $b \in \text{acl}_{L_P}(a) \cap P$, then since $a \perp_{a^c}^L P$, we get $\widehat{a} \perp_{a^c}^L b$. Suppose $b \notin \text{acl}_L(a^c)$, so $b \notin \text{acl}_L(\widehat{a})$. Then, in P , there are infinitely many $(b_i, i < \omega)$ such that $\text{tp}_L(\widehat{a}b_i) = \text{tp}_L(\widehat{a}b)$. By Lemma 3.9 (ii), $\text{tp}_L(\widehat{a}b_i) = \text{tp}_L(\widehat{a}b)$. By Lemma 3.8, these b_i are also L_P -conjugate over \widehat{a} , a contradiction. Consider now $b' \in M \setminus P$ such that $b' \in \text{acl}_{L_P}(a)$ but $b' \notin \text{acl}_L(\widehat{a})$. Then

$$(b'\widehat{a})^c \in \text{dcl}_{L_P}(b'\widehat{a}) \cap P \subseteq \text{acl}_{L_P}(\widehat{a}) \cap P = \text{acl}_L(a^c),$$

which implies by Fact 1.8 (iii) that $b'\widehat{a} \perp_{a^c}^L P$, hence $b' \perp_{\widehat{a}}^L P$. By assumption there are infinitely many L -conjugates of b' over \widehat{a} . Since M is saturated over P , there are infinitely many realizations of $\text{tp}_L(b'/\widehat{a}P)$. This implies there are infinitely many realizations of $\text{tp}_{L_P}(b'/\widehat{a})$, a contradiction.

- ii) The proof is similar. First, we show that $\text{dcl}_{L_P}(a) \cap P = \text{dcl}_L(a^c)$, so let $b \in \text{dcl}_{L_P}(a) \cap P$. Then, by (i), $b \in \text{acl}_L(a^c)$. Suppose $b \notin \text{dcl}_L(\widehat{a})$, then there is $b' \in P$ distinct from b such that $\text{tp}_L(b'\widehat{a}) = \text{tp}_L(b\widehat{a})$, and applying Lemma 3.9 (ii) and Lemma 3.8 gives $\text{tp}_{L_P}(b'\widehat{a}) = \text{tp}_{L_P}(b\widehat{a})$, a contradiction. Consider now $b' \in M \setminus P$, $b' \in \text{dcl}_{L_P}(a)$, but assume $b' \notin \text{dcl}_L(\widehat{a})$. Then

$$(b'\hat{a})^c \in \text{dcl}_{L_P}(b'\hat{a}) \cap P \subseteq \text{dcl}_{L_P}(\hat{a}) \cap P = \text{dcl}_L(a^c),$$

hence $\langle \hat{a} \rangle(b')$ and P are linearly disjoint over $\langle \hat{a} \rangle$. By assumption there are at least two L -conjugates of b over \hat{a} , which are also L_P -conjugates over \hat{a} by Lemma 3.8, a contradiction.

□

Corollary 3.14. *If $A \subseteq M$ is such that $A = \hat{A}$, then $\text{acl}_{L_P}(A) = \text{acl}_L(A)$ and $\text{dcl}_{L_P}(A) = \text{dcl}_L(A)$. In particular P is algebraically closed in the L_P -sense.*

Definition 3.15. i) Consider for every $n > 1$, the predicate $l_n(x_1, \dots, x_n)$, which asserts that x_1, \dots, x_n are linearly independent over E , that is,

$$l_n(x_1, \dots, x_n) \leftrightarrow \forall e_1, \dots, e_n \left(\bigwedge_i P(e_i) \wedge \sum_i e_i x_i = 0 \rightarrow \bigwedge_i e_i = 0 \right).$$

ii) Consider for every $n > 1$ and for every $i \in \{1, \dots, n\}$, the $(n+1)$ -ary function $f_{n,i}(y, x_1, \dots, x_n)$ which gives the i -th coordinate of y written as a linear combination of x_1, \dots, x_n . More specifically, if $l_n(x_1, \dots, x_n) \wedge \neg l_n(y, x_1, \dots, x_n)$, then

$$z = f_{n,i}(y, x_1, \dots, x_n) \leftrightarrow \exists z_1, \dots, z_n \left(z = z_i \wedge y = \sum_j z_j x_j \wedge \bigwedge_j P(z_j) \right),$$

else, if the condition is not met, define $f_{n,i}(y, x_1, \dots, x_n) = 0$.

iii) Define the language $L_P^{l,f}$ as the language obtained by adjoining to L_P the predicate symbols l_n and $f_{n,i}$, for all $n > 1$ and $i \in \{1, \dots, n\}$. Notice that in this language, $P(x)$ can be defined by the formula $\neg l_n(1, x)$.

The following result is Corollary 15 from [3]:

Fact 3.16. *Let $N \subseteq M$ be a model of T_P , then the inclusion is elementary iff N is an $L_P^{l,f}$ -substructure if and only if N is P -independent.*

Corollary 3.17. *Let $A \subseteq M$. Let C be the field generated by A and the $f_{n,i}(A)$ for all $n > 1$ and $i \leq n$. Then $\widehat{A} \subseteq C$, and consequently*

$$i) \text{acl}_{L_P}(A) = \text{acl}_L(C).$$

$$ii) \text{dcl}_{L_P}(A) = \text{dcl}_L(C).$$

Proof. By theorem 7, §2, Ch 3. of [6], the field of definition of the locus of A over P is generated by $\{f_{n,i}(M_0, M_1, \dots, M_n), n < \omega, i \leq n\}$, where the tuple (M_0, M_1, \dots, M_n) ranges over the set of monomials formed by elements of A . Therefore, $A^c \subseteq C$. From this, we get both $\text{acl}_L(\widehat{A}) \subseteq \text{acl}_L(C)$ and $\text{dcl}_L(\widehat{A}) \subseteq \text{dcl}_L(C)$, while the reverse inclusion follows from the definability of the $f_{n,i}$. The desired result is obtained by invoking Lemma 3.13. \square

Lemma 3.18. *Let $a, b, c \in M$, $p_1 = \text{tp}_{L_P}(a/bc)$, $p_2 = \text{tp}_{L_P}(b/c)$. If p_1, p_2 are stationary, then $p_3 = \text{tp}_{L_P}(a/c)$ is stationary.*

Proof. By stability of T_P and by hypothesis, there are good definition schemes dp_1 over bc and dp_2 over c . We want to find a good definition for p_3 , i.e. one that defines a global type, this would imply stationarity by Lemma 1.14. Let $\varphi(x, y)$ be an L_P -formula and let $m \in M$ be such that $M \models \varphi(a, m)$. There is a formula $dp_1(\varphi)(y, z, w)$ such that $M \models dp_1(\varphi)(m, b, c)$. Moreover, there is then a formula $dp_2(dp_1(\varphi))(y, w)$ such that $M \models dp_2(dp_1(\varphi))(m, c)$. The result follows. \square

Remark: T_P eliminates finite imaginaries.

Proof. Let $A = \{a_1, \dots, a_k\} \subseteq M^n$, where $a_i = (a_{i,1}, \dots, a_{i,n})$. Consider the following polynomial

$$p(X, Y_0, \dots, Y_{n-1}) = \prod_{i=1}^k \left(X - \sum_{j=1}^n a_{i,j} Y_j \right),$$

If σ is an L_P -automorphism, then as it is in particular an L -isomorphism, we have that $\sigma p(X, Y_0, \dots, Y_{n-1}) = \prod_{i=1}^k \left(X - \sum_{j=1}^n \sigma(a_{i,j}) Y_j \right)$. Noting that $M[X, Y_0, \dots, Y_{n-1}]$ is a unique factorization domain, we deduce $\sigma p = p$ if and only if $\sigma A = A$. The tuple consisting of the coefficients of p is a canonical parameter for A . \square

Lemma 3.19. Let M_0 be an elementary substructure of (M, P) , and let $a \in M$ be such that $a = \widehat{a}$. Define $d = \text{Cb}(\text{tp}_L(a/\text{acl}_L(M_0P)))$, $e' = \text{Cb}(\text{tp}_{L_P}(d/M_0))$, and $e = \text{Cb}(\text{tp}_{L_P}(a/M_0))$. Then e' and e are L_P -interdefinable.

Proof. Note that by definition of e, e' and because $M_0 \preceq M$, $\text{tp}_{L_P}(a/e)$ and $\text{tp}_{L_P}(d/e')$ are stationary.

Claim I:

$$(i) \quad a \perp\!\!\!\perp_d^{L_P} M_0P.$$

$$(ii) \quad d \in \text{acl}_{L_P}(aM_0).$$

(i): By Lemma 3.12, it suffices to prove $\widehat{ad} \perp\!\!\!\perp_{\widehat{d}}^L \widehat{M_0P}$. Notice that since $(M_0P)^c \subseteq P$, we have $M_0P = \widehat{M_0P}$, then by definition of d , $a \perp\!\!\!\perp_d^L M_0P$ and since $d^c \in P$, monotonicity gives $\widehat{ad} \perp\!\!\!\perp_{\widehat{d}}^L \widehat{M_0P}$. It is now enough to prove $(ad)^c = d^c$, which would imply $\widehat{ad} = \widehat{ad}$. By definition of d , $\langle ad \rangle$ is linearly disjoint (l.d.) from $\text{acl}_L(M_0P)$ over $\langle d \rangle$, hence $\langle ad \rangle$ and $P(d)$ are l.d. over $\langle d \rangle$. Since $\langle d \rangle$ is l.d from P over $\langle d^c \rangle$, it follows that $\langle ad \rangle$ and P are l.d over $\langle d^c \rangle$, hence $(ad)^c = d^c$.

(ii): Since $aM_0 \perp\!\!\!\perp_{(aM_0)^c}^L P$, then $a \perp\!\!\!\perp_{M_0(aM_0)^c}^L M_0P$. By Fact 1.8 (iii) and Remark 3.7, it follows that

$$d \in \text{acl}_L(M_0(aM_0)^c) \subseteq \text{acl}_{L_P}(M_0(aM_0)^c) \subseteq \text{acl}_{L_P}(aM_0). \blacksquare$$

Claim II: $d \in \text{dcl}_{L_P}(a, e)$.

Let σ be an L_P -automorphism that fixes a, e , and let $M'_0 = \sigma(M_0)$. Choose a realization M''_0 of $\text{tp}_{L_P}(M_0/a, e)$ independently from $M_0 \cup M'_0$ over a, e . Using $a \perp\!\!\!\perp_e^{L_P} M_0$ and $e \in M_0 \cap M'_0 \cap M''_0$, we obtain the following relations

$$a \perp\!\!\!\perp_{M_0}^{L_P} M_0M''_0, \quad a \perp\!\!\!\perp_{M''_0}^{L_P} M_0M''_0, \quad a \perp\!\!\!\perp_{M'_0}^{L_P} M'_0M''_0, \quad a \perp\!\!\!\perp_{M''_0}^{L_P} M'_0M''_0.$$

Applying Lemma 3.12 gives

$$a \perp\!\!\!\perp_{PM_0}^L PM_0M''_0, \quad a \perp\!\!\!\perp_{PM''_0}^L PM_0M''_0, \quad a \perp\!\!\!\perp_{PM'_0}^L PM'_0M''_0, \quad a \perp\!\!\!\perp_{PM'_0}^L PM'_0M''_0.$$

Since $\text{tp}_L(a/\text{acl}_L(M_0P))$ is stationary, this translates in terms of canonical bases to

$$d = \text{Cb}(\text{tp}_L(a/\text{acl}_L(M_0P))) = \text{Cb}(\text{tp}_L(a/\text{acl}_L(M''_0P))) = \text{Cb}(\text{tp}_L(a/\text{acl}_L(M'_0P))),$$

hence $\sigma(d) = d$, so the claim is proved. ■

By definition of e , $a \perp_e^{L_P} M_0$. As $\text{tp}_{L_P}(a/e)$ is stationary, $e \in M_0$, and $d \in \text{dcl}_{L_P}(a, e)$, we conclude $\text{tp}_{L_P}(d/e)$ is stationary by Lemma 1.15. Therefore, $e' \in \text{dcl}_{L_P}(e)$.

Claim III: $a \perp_{e'}^{L_P} M_0$. Therefore, $e \in \text{acl}_{L_P}(e')$.

By Claim I,

$$\begin{aligned} a \perp_d^{L_P} M_0 P &\Rightarrow a \perp_d^{L_P} M_0 d \\ &\Rightarrow a \perp_{de'}^{L_P} M_0 \quad \text{as } e' \in \text{dcl}_{L_P}(M_0) = M_0. \end{aligned}$$

By definition of e' we have $de' \perp_{e'}^{L_P} M_0$. Applying transitivity yields the claim. ■

To prove $e \in \text{dcl}_{L_P}(e')$, we shall show stationarity of $\text{tp}_{L_P}(a/e')$ and apply Fact 1.8 (iv).

By definition of e' , $\text{tp}_{L_P}(d/e')$ is stationary, then by Lemma 3.18, it would suffice to prove $\text{tp}_{L_P}(a/de')$ is stationary. However, Claim I implies $a \perp_d^{L_P} e'$, so it is enough to prove $p = \text{tp}_{L_P}(a/d)$ is stationary. Let N be an L_P -elementary substructure of M containing d , and suppose $p \subseteq p_1, p_2$ are nonforking extensions of p to N . Let a_1, a_2 realize p_1, p_2 respectively. By Lemma 3.12, $a_i \perp_{dP}^L NP$. Since $\text{tp}_{L_P}(a_i/d) = \text{tp}_{L_P}(a/d)$ for $i = 1, 2$, and $a \perp_d^L dP$ (by definition of d and monotonicity), we get that for $i = 1, 2$, $a_i \perp_d^L NP$. This in turn implies $Na_i \perp_N^L P$, since $d \in N$. Since $N = \widehat{N}$, by Remark 3.11 (ii), $N \perp_{P \cap N}^L P$. Applying transitivity yields $Na_i \perp_{N \cap P}^L P$, hence $N(a_i)$ and NP are linearly disjoint over N . This implies $\widehat{N(a_i)} = N(a_i)$. Since $\text{tp}_L(a/d)$ is stationary, $\text{tp}_L(a_1N) = \text{tp}_L(a_2N)$. We can then apply Lemma 3.8 to obtain that $\text{tp}_{L_P}(a_1N) = \text{tp}_{L_P}(a_2N)$. □

Lemma 3.20. *Let $a \in M$, $A \subseteq M$. If $A = \widehat{A}$ and $\text{tp}_{L_P}(a/A)$ is stationary, then $\text{tp}_L(a/A)$ is stationary.*

Proof. Suppose by contradiction that $\text{tp}_L(a/A)$ is not stationary and let $k = \langle A \rangle$. The extension $k(a)|k$ is not primary: there is some $\alpha \in k(a)$ such that $\alpha \in \text{acl}_L(k) \setminus \text{dcl}_L(k)$. Note also that $\widehat{k} = k$. By Corollary 3.14, $\alpha \in \text{acl}_{L_P}(k) \setminus \text{dcl}_{L_P}(k)$, contradicting stationarity of $\text{tp}_L(a/A)$. □

Lemma 3.21. *Suppose $d \in M$ is such that $d = \widehat{d}$, and let $e \in \text{dcl}_{L_P}^{\text{eq}}(d)$ be an imaginary such that $\text{tp}_{L_P}(d/e)$ is stationary. Let $d' \models \text{tp}_{L_P}(d/e)$ with $d \perp_e^{L_P} d'$. Let $B'_1 = \text{dcl}_{L_P}^{\text{eq}}(e) \cap M$ and $B_1 = \text{acl}_{L_P}^{\text{eq}}(e) \cap M$. Then $\langle d \rangle$ and $\langle d' \rangle$ are linearly disjoint over B'_1 , in particular $d \perp_{B_1}^L d'$.*

Proof. We denote $p(x) = \text{tp}_{L_P}(d/e)$,

Claim: Let $d'' \models p| \{d, d'\}$, then $d \perp_{d'}^L d''d'$ and $d \perp_{d''}^L d''d'$.

By definition of d'' , we have $d \perp_{d'}^{L_P} d''d'$, and by hypothesis $d \perp_e^{L_P} d'$. Additionally, $e \in \text{dcl}_{L_P}(d'') \cap \text{dcl}_{L_P}(d')$, then both relations $d \perp_{d''}^{L_P} d'$, $d \perp_{d'}^{L_P} d''$ hold. Starting from the first relation, we see that $dd'' \perp_{d''}^{L_P} d'd''$, and by Lemma 3.11, $\widehat{dd''} \perp_{\widehat{d''}}^L \widehat{d'd''}$. Since d, d' and d'' are independent over e , and e is definable over each of d, d', d'' , the stationarity of $\text{tp}_{L_P}(d/e)$ implies that of $\text{tp}_{L_P}(d/d'')$, $\text{tp}_{L_P}(d/d')$. It follows by 3.20 that $\text{tp}_L(d/d'')$, $\text{tp}_L(d/d')$ are stationary. Then, as $\widehat{d''} = d''$, we get $d \perp_{d''}^L d''d'$, hence, the field of definition of the locus of d over $\langle d''d' \rangle$ is contained in $\langle d'' \rangle$. The other part of the claim is obtained similarly, using that $\widehat{d'} = d'$ instead, yielding that the field of definition of the locus of d over $\langle d''d' \rangle$ is contained in $\langle d' \rangle$. ■

We obtain therefore

$$\text{Cb}(\text{tp}_L(d/d')) = \text{Cb}(\text{tp}_L(d/d'')) \subset \text{dcl}_L(d') \cap \text{dcl}_L(d'').$$

But d' and d'' are independent over e , $\text{dcl}_{L_P}(d') = \text{dcl}_L(d')$, and $\text{dcl}_{L_P}(d'') = \text{dcl}_L(d'')$, so $\text{Cb}(\text{tp}_L(d/d')) \subseteq \text{dcl}_{L_P}^{\text{eq}}(e) \cap M = B'_1$. □

Lemma 3.22. *Let $e \in (M, P)^{\text{eq}}$, and $B_0 = \text{acl}_{L_P}^{\text{eq}}(e) \cap P$. Then for all $c \in P$, $\text{tp}_{L_P}(c/B_0e)$ is finitely satisfiable in B_0 .*

Proof. Let $a \in M$ be such that $e = f(a)$ for some definable function. Since $\widehat{aP} = \widehat{a}P$, by Lemma 3.8 $\text{tp}_{L_P}(a/a^c) \vdash \text{tp}_{L_P}(a/P)$. We will prove that $\text{tp}_{L_P}(a/P)$ is stationary, this would imply by Lemma 1.15 that $\text{tp}_{L_P}(e/P)$ is stationary. Suppose not, so we can find $b_1, \dots, b_n \in M$ and formulas $\varphi(x, b_i)$, which distinguish between the non-forking extensions of $\text{tp}_{L_P}(a/a^c)$ to M . In other words, they define a partition of the set of realizations of $\text{tp}_{L_P}(a/a^c)$. By saturation of M over P , we may assume

$$a \perp_{a^c}^{L_P} b_1, \dots, b_n, P.$$

If a' is another realization of $\text{tp}_{L_P}(a/a^c)$, such that $a' \perp_{a^c}^{L_P} b_1, \dots, b_n, P$, then there is an automorphism of M that fixes $\text{acl}_{L_P}(P, b_1, \dots, b_n)$ and sends a to a' . This implies $\text{tp}_{L_P}(a'/P) = \text{tp}_{L_P}(a/P)$.

Define $e^c = \text{Cb}(\text{tp}_{L_P}(e/P))$. Note that

$$e^c \in \text{dcl}_{L_P}^{\text{eq}}(e) \cap \text{dcl}_{L_P}^{\text{eq}}(P) = \text{dcl}_{L_P}^{\text{eq}}(e) \cap P.$$

By definition of e^c we have $e \perp_{\text{acl}_L(e^c)}^{L_P} P$, and the proof of Lemma 3.13 shows that $\text{acl}_L(e^c) = B_0$. Hence, $\text{tp}_{L_P}(e/P)$ is stationary and is a non-forking extension of $\text{tp}_{L_P}(e/B_0)$.

This implies definability of $\text{tp}_{L_P}(e/P)$ over B_0 . Thus, for any L_P^{eq} -formula $\psi(x, y)$ with parameters in B_0 , there is a formula $d\psi(y)$ with parameters in B_0 such that for all $c \in P$, $M \models \psi(e, c)$ iff $M \models d\psi(c)$. By Lemma 3.3, we may assume that $d\psi(y)$ is an L -formula. Since we also have that $B_0 \prec P$ in the sense of L , we have that for any $c \in P$, if $P \models \psi(e, c)$ then $P \models d\psi(c)$, which implies there is $b \in B_0$ such that $P \models d\psi(b)$, and therefore $M \models \psi(e, b)$. \square

4 Weak Elimination of Imaginaries

Throughout this section we will maintain our notation and conventions from Section 3. We let $T = ACF_p$, and T_P the theory of beautiful pairs of models of T . We have that $(M, P) \models T_P$ is saturated.

Definition 4.1. Let G be an algebraic group and X an algebraic variety both defined over $k \subseteq M$. A k -rational action is a group action $\alpha : G \times X \rightarrow X$ such that for every $g \in G$, the map $\alpha(g, \cdot) : X \rightarrow X$ is a k -rational map.

Definition 4.2. A definable group action is a triple $((G, \cdot), X, \alpha)$, where (G, \cdot) is a definable group, $X \subseteq M$ a definable set and $\alpha : G \times X \rightarrow X$ a group action whose graph is definable. If the action is *transitive* on X , that is, for every $a, b \in X$ there is $g \in G$ such that $\alpha(g, a) = b$, the triple is called a *definable homogeneous space*. Moreover, if the action is *strictly transitive (or regular)*, that is, $\alpha(g, x) = x$ iff $g = e$, it will be called a *principal definable homogeneous space* (or PHS).

We shall abuse notation and denote $\alpha(g, a)$ as $g \cdot a$. In our context, as $T = ACF_p$, we get the following fact from Theorem 7.4.14 of [7].

Fact 4.3. *If $G \subseteq M^n$ is an L -definable group, then G is definably isomorphic to an algebraic group.*

Proposition 4.4. *Let $e \in (M, P)^{\text{eq}}$. Then there are: a connected algebraic group G , an irreducible variety V over P , and a rational action G on V , definable over P , such that*

- (i) *The action of $G(P)$ on $V(M)$ is generically free: if $a \in V(M)$ is a generic point of V over P , and $g \in G(P)$ is not the identity, then $g \cdot a \neq a$.*
- (ii) *For some $a \in V(M)$ generic over P , if r is a canonical parameter for the orbit $X = \{g \cdot a \mid g \in G(P)\}$, then $e \in \text{dcl}_{L_P}(r)$ and $r \in \text{acl}_{L_P}(e)$.*

The proof of Proposition 4.4 will require some results.

Lemma 4.5. *Let $e \in (M, P)^{\text{eq}}$. There is $d' \in M$ such that $\text{tp}_{L_P}(d'/e)$ is stationary and P -internal, and moreover $e \in \text{dcl}_{L_P}^{\text{eq}}(d')$.*

Proof. Let $a \in M$ be such that $a = \hat{a}$ and $e = f(a)$ for some \emptyset -interpretable function. By Lemma 1.10 we may suppose $\text{tp}_{L_P}(a/e)$ is stationary, hence $e = \text{Cb}(\text{tp}_{L_P}(a/M_0))$, where M_0 is any L_P -elementary substructure of M such that $e \in M_0^{\text{eq}}$ and $a \perp_e^{L_P} M_0$. Let $d = \text{Cb}(\text{tp}_L(a/\text{acl}_L(M_0P)))$. By Lemma 3.19, $e = \text{Cb}(\text{tp}_{L_P}(d/M_0))$, hence $d \perp_e^{L_P} M_0$. Since $M_0 \preceq M$, $\text{tp}_{L_P}(d/M_0)$ is stationary, $\text{tp}_{L_P}(d/e)$ is stationary and almost P -internal. Replacing d by finitely many independent realizations of $\text{tp}_{L_P}(d/e)$, by Fact 1.8 (v), we may assume without loss of generality that $e \in \text{dcl}^{\text{eq}}(d)$, or that $e = g(d)$ for some definable function g . By Lemma 1.17, there is $d' \in \text{dcl}_{L_P}(d)$, a code for a finite set of realizations of $\text{tp}_{L_P}(d/e)$, such that $d \in \text{acl}_{L_P}(d')$ and $\text{tp}_{L_P}(d'/e)$ is stationary and P -internal. Then as $d \in \text{acl}_{L_P}^{\text{eq}}(d')$, there is a formula $\varphi(x, d')$ isolating $\text{tp}_{L_P}(d/d')$; hence $M \models \forall x \varphi(x, d') \rightarrow g(x) = e$, so $e \in \text{dcl}^{\text{eq}}(d')$. □

Lemma 4.6. *There are a tuple $d \in M$, an L_P -definable function f (over \emptyset), an $L_P(e)$ -formula $\psi(x)$, and an $L_P(e)$ -definable function h such that*

$$(i) \quad f(d) = e.$$

$$(ii) \quad \psi(x) \in \text{tp}_{L_P}(d/e).$$

$$(iii) \quad M \models \forall x, x' (\psi(x) \wedge \psi(x') \rightarrow \exists c (P(c) \wedge h(x, c) = x')).$$

Proof. Let d' be as in Lemma 4.5. Then $p = \text{tp}_{L_P}(d'/e)$ is stationary, P -internal, and $e = \text{Cb}(p)$. By Lemma 1.18, there is a tuple d consisting of finitely many realisations of p , and an e -definable function g such that for any realization d'' of p , there is a tuple $c_{d''} \in P$ such that $d'' = g(d, c_{d''})$. Clearly $e \in \text{dcl}_{L_P}^{\text{eq}}(d)$, so we can find an L_P -definable function f such that (i) holds. If d_1, d_2 realize $\text{tp}_{L_P}(d/e)$, then there is an e -definable function h and a tuple $c \in P$ such that $d_1 = h(d_2, c)$. Applying compactness yields an L_P -formula $\psi \in \text{tp}_{L_P}(d/e)$ such that for any two d_1, d_2 satisfying ψ , there is $c \in P$ such that $h(d_1, c) = d_2$, which directly proves (ii) and (iii). Note $\text{tp}_{L_P}(d/e)$ remains P -internal. □

Lemma 4.7. *In Lemma 4.6, d can be chosen such that (i),(ii),(iii) hold, and $d \perp_e^{L_P} P$.*

Proof. Let ψ as in Lemma 4.6. Let $\chi(x, y)$ an $L_P(e)$ -formula that expresses the conjunction of $x^c = y$, $\psi(x)$ and $f(x) = e$. Consider the $L_P(e)$ -formula $\theta(y)$ given by $\exists x(\chi(x, y))$. Since $M \models \theta(d^c)$, by Lemma 3.22, there is $d_0 \in \text{acl}_{L_P}^{\text{eq}}(e) \cap P$ such that $M \models \theta(d_0)$. Therefore, there is d_1 such that $M \models \chi(d_1, d_0)$, hence $d_1 \perp_e^{L_P} P$. \square

Notation: For the remainder of this section, fix d as in Lemma 4.7. By Remark 3.7 $d^c \in \text{dcl}_{L_P}(d)$, hence we may also assume from now on that $d = \widehat{d}$, as all of the properties from Lemmas 4.6, 4.7, and 4.8 still hold after adjoining d^c to d . From now on, let

$$B = \text{acl}_{L_P}^{\text{eq}}(e),$$

$$B_1 = B \cap M,$$

$$B_0 = B \cap P.$$

Lemma 4.8. $\text{tp}_{L_P}(d/B)$ is isolated.

Proof. By stability of $\text{Th}(M^{\text{eq}})$, there are $M_1 \preceq M$, a prime model over Bd and $M_0 \preceq M_1$ a prime model over B .

Claim: $B_0 = M_0 \cap P = M_1 \cap P$: It is clear that $B \subseteq M_0, M_1$, one inclusion follows. Conversely, if $a \in M_0 \cap P$, then $\text{tp}_{L_P}(a/B)$ is isolated, which is a non-forking extension of $\text{tp}_{L_P}(a/e)$, hence $\text{tp}_{L_P}(a/e)$ is isolated too, and applying Lemma 3.22, it can be realized by some $a' \in B_0$. In particular, this implies $a \in \text{acl}_{L_P}(e)$. The proof for the second equality is similar, let $a \in M_1 \cap P$, then $\text{tp}_{L_P}(a/Bd)$ is isolated. Recall $d \perp_e^{L_P} P$, hence $\text{tp}(a/Bd)$ does not fork over $\text{tp}_{L_P}(a/e)$, which is then isolated, and applying Lemma 3.22 yields the result.

Let ψ be as in Lemma 4.6, and choose $d' \in M_0$ such that $M \models \psi(d')$. Applying Lemma 4.6 (iii) inside the model M_1 , there is $c \in P \cap M_1 = B_0$ such that $d \in \text{dcl}_{L_P}(d', c) \subseteq M_0$, hence by definition of a prime model, $\text{tp}_{L_P}(d/B)$ is isolated. \square

Lemma 4.9. *Let X be the set of realizations of $\text{tp}_{L_P}(d/B)$. There are: a connected algebraic group G defined over B_0 and an $L_P(e)$ -definable regular action of $G(P)$ on X . Moreover, if r is a canonical parameter for the PHS $(G(P), X)$, then $e \in \text{dcl}_{L_P}(r)$ and $r \in \text{acl}_{L_P}(e)$.*

Proof. By Lemma 4.8, X is L_P -definable over B . Define

$$C = \{c \in P, \exists d'(d' \in X \wedge h(d, c) = d')\},$$

which is non empty by Lemma 4.6 (*iii*), and $L(B_0)$ -definable by Lemma 3.3. Consider now the equivalence relation E in C defined by $M \models E(c_1, c_2)$ if and only if $M \models h(d, c_1) = h(d, c_2)$. In C/E we can define an $L_P(e)$ -interpretable function $h'(d, c/E) = h(d, c)$. By Lemmas 4.7 and 4.8 $d \perp_B^{L_P} P$, hence all elements of X have the same L_P -type over BP . Since E is contained in some power of P , it is $L(B_0)$ -definable, hence it does not depend on the choice of d . This implies that for all $c_1, c_2 \in C$ the value of $h(h(d, c_1), c_2)$ is defined, and taking classes modulo E , there is a unique c_3/E such that $h'(h'(d, c_1/E), c_2/E) = h'(d, c_3/E)$, we define a binary operation on C/E as $(c_1/E) \cdot (c_2/E) = c_3/E$. Once again by Lemma 3.3, this operation is $L(B_0)$ -definable. Moreover, by Remark 3.4, we may assume without loss of generality that C/E contains real tuples.

Claim: $(C/E, \cdot)$ is a B_0 -definable group.

Let $c_1, c_2, c_3 \in C/E$. To check associativity, notice that

$$h'(d, (c_1 c_2) c_3) = h'(h'(d, c_1 c_2), c_3) = h'(h'(h'(d, c_1), c_2), c_3),$$

moreover, since $h'(d, c_2 c_3) = h'(h'(d, c_2), c_3)$ and $\text{tp}_{L_P}(h'(d, c_1)/BP) = \text{tp}_{L_P}(d/BP)$, we obtain

$$h'(d, c_1(c_2 c_3)) = h'(h'(d, c_1), c_2 c_3) = h'(h'(h'(d, c_1), c_2), c_3)).$$

To check for an identity, by Lemma 4.6 (*iii*), there is $c' \in P$ such that $h(d, c') = d$. Then, for all $d' \in X$, $h'(d', c') = d'$, in particular

$$h'(d, c_1 c') = h'(h'(d, c_1), c') = h'(d, c_1) \Rightarrow c_1 c' = c_1.$$

To check the existence of inverses, notice that since $h(d, c_1) \in X$, there is some L_P -automorphism σ fixing BP pointwise such that $h(d, c_1) = \sigma(d)$, which implies $h'(\sigma^{-1}(d), c_1) = d$. By Lemma 4.6 (*iii*), there is a unique c'_1 such that $h'(d, c'_1) = \sigma^{-1}(d)$, hence

$$\begin{aligned} h'(d, c'_1 c_1) &= h'(h'(d, c'_1), c_1) = h'(\sigma^{-1}(d), c_1) = d = h'(d, c'), \\ h'(d, c_1 c'_1) &= h'(h'(d, c_1), c'_1) = h'(\sigma(d), c'_1) = d = h'(d, c'), \end{aligned}$$

therefore, $c_1 c'_1 = c'_1 c_1 = c'$. ■

By the previous claim and by Fact 4.3, C/E is B_0 -definably isomorphic to some algebraic group G over B_0 . We can then induce an $L(B_0)$ -definable action of $G(P)$ over X using the map h' : if $F : G \rightarrow C/E$ is an isomorphism, then for $(g, d) \in G \times X$, define

$g \cdot d = h'(d, F(g))$. By Lemma 4.6 (iii) and by definition of E , this action is regular. As X is the set of realizations of a stationary type, $G(P)$ must be connected (as an L_P -definable group), hence connected as an algebraic group. Clearly, the PHS $(G(P), X)$ is L_P -definable over B , this implies that if r is a canonical parameter for $(G(P), X)$, then $r \in \text{acl}_{L_P}(e)$. Moreover, if σ is some L_P -automorphism fixing r , then it permutes the realizations of $\text{tp}_{L_P}(d/B)$, and by stationarity of $\text{tp}_{L_P}(d/e)$ we have $e = \text{Cb}(\text{tp}_{L_P}(d/B))$, so $\sigma(e) = e$, hence $e \in \text{dcl}_{L_P}(r)$, completing the proof. \square

The set X from Lemma 4.9 will be identified with a generic orbit of the action of $G(P)$ over some variety $V(M)$. We first state Proposition 2.2 from [4].

Lemma 4.10. *Let G be a connected definable group with a generic action on the set of realizations X_1 of a stationary L -type q , that is, for all generic $g \in G$ and for d realizing $q|g$, $g \cdot d$ is defined and realizes q , and for all independent g_1, g_2, d , $g_1 \cdot (g_2 \cdot d) = (g_1 g_2) \cdot d$ when the action is defined. There exists then a type-definable set Y , a definable embedding $X_1 \subseteq Y$, and a definable action of G on Y , extending the generic action of G on X_1 . Moreover, for every $y \in Y$ there is $g \in G$ and $d \models q$ such that $y = g \cdot d$.*

Proof. Consider the set of pairs (g, d) with $g \in G$, $d \models q$. Define an equivalence relation over these pairs by: $(g, d) \sim (g', d')$ if for all generic $h \in G$ such that $(hg) \cdot d = (hg') \cdot d'$. Let Y be the set of classes, its elements are denoted by $[g, d]$. If $(hg_2) \cdot d = (hg'_2) \cdot d'$ holds for generic h , then, since hg_1 is also generic, it is also true that $(hg_1 g_2) \cdot d = (hg_1 g'_2) \cdot d'$, hence we can define an action of G on Y by $g_1 \cdot [g_2, d] = [g_1 g_2, d]$, and identify each $d \models q$ with $[1_G, d]$. To check the last statement, let $[g, d] \in Y$, and let h be a generic of G , independent from d , then $h[g, d] = [hg, d] = [1, hg \cdot d]$, hence $[g, d] = h^{-1}[1, hg \cdot d]$. \square

Lemma 4.11. *For X as in Lemma 4.9 there is an irreducible variety Y defined over B_1 , and a transitive rational action of G on Y , defined over B_1 , such that $X \subseteq Y$, d is a generic point of Y over B_1 , and the action of G on Y restricts to the given action of $G(P)$ on X .*

Proof. Recall that for $g \in G(P)$, $d \in X$, $g \cdot d$ is e -definable, this means $g \cdot d \in \text{dcl}_{L_P}(g, d, e)$. Since $e \in \text{dcl}_{L_P}(d)$, then $g \cdot d \in \text{dcl}_{L_P}(g, d) = \text{dcl}_L(\widehat{g, d})$ by Lemma 3.13. But $\text{dcl}_L(\widehat{g, d}) = \text{dcl}_L(g, d)$ by Lemma 3.9 (ii). Therefore, $g \cdot d \in \text{dcl}_L(g, d)$.

Claim: $d \perp_{B_0}^L g$.

If $e^c = \text{Cb}(\text{tp}_{L_P}(e/P))$, then $e \perp_{e^c}^{L_P} P$, and by Lemma 4.7, $d^c \perp_e^{L_P} P$. Applying transitivity yields $d^c \perp_{e^c}^{L_P} P$, and since everything lives in P , we can restrict our language

to get $d^c \perp_{e^c}^L P$. By the proof of Lemma 3.13, $B_0 = \text{acl}_L(e^c)$, hence $d^c \perp_{B_0}^L P$ and by definition of d^c we have $d \perp_{d^c}^L P$. The claim follows as $g \in P$. ■

Now, working in L , since $e \in \text{dcl}_{L_P}(d) = \text{dcl}_L(\hat{d})$, $B_1 \in \text{acl}_L(d)$, so the previous claim yields $dB_1 \perp_{B_0} g$. Then, if g is generic over B_0 , then it is generic over dB_1 . The action is generically regular and transitive: given independent $d_1, d_2 \in X$, there is a unique $g \in G(P)$ such that $g \cdot d_1 = d_2$. Hence, working in L_P , $RM(G) = RM(X)$, and if $g \in G$, $d \in X$ are independent over e , then because the action is defined over e , we have that $g \in \text{dcl}_L(g \cdot d, d)$, so that we must have $RM(g \cdot d, d/e) = 2RM(G)$, which implies $g \cdot d \perp_e^{L_P} d$. By Lemma 3.21, $g \cdot d \perp_{B_1}^L d$.

We have a definable action of $G(P)$ on the L_P -definable set X , and the action is given by a map $G \times X \rightarrow X$ which is $L(B_1)$ -definable in T . Passing to the Zariski closure, we get a generic action of the algebraic group $G(M)$ on the set X_1 of generic elements (over B) of the Zariski closure of X . By Lemma 4.10, there is a type-definable $Y \supseteq X_1$ (in the L -sense, and over B_1) such that G acts on Y in a way that restricts to the generic action of G on X_1 . Moreover, for every $y \in Y$ there is $g \in G$ and $d \in X_1$ such that $y = g \cdot d$, so the action of G on Y transitive, then Y has a unique generic type by connectedness of G , and it must be indeed $\text{tp}_L(d/B_1)$. This proves that d is a generic of Y over B_1 . We claim that Y is also definable: Let $\varphi(x, y)$ be some $L(B_1)$ -formula defining $x \in G \cdot y$, and let E be the equivalence relation given by yEy' iff $M \models \forall x \varphi(x, y) \leftrightarrow \varphi(x, y')$, by transitivity, for any $y \in Y$ we have $[y]_E = Y$, now by type-definability of Y over B_1 , Y is fixed by all $\sigma \in \text{Aut}(M/B_1)$, hence the imaginary $[y]_E$ is fixed too, which implies $[y]_E$ is B_1 -definable, hence Y is B_1 -definable. Since $X \subseteq X_1 \subseteq Y$, and the action of G on Y restricts to the generic action over X_1 , then it restricts to the action of G on X that was defined in Lemma 4.9. Finally, by Fact 4.3, (G, Y, \cdot) is B_1 -definably isomorphic to (G', Y', \cdot') , where G' is an algebraic group, Y' an irreducible variety, and \cdot' is a B_1 -rational action. □

Proof of Proposition 4.4

Proof. For $e \in M^{\text{eq}}$, d, G, Y as in Lemma 4.11, choose some finite $b \in B_1$ such that (G, Y, \cdot) is definable over b . Rewrite Y as Y_b . By Lemma 4.7, $d \perp_e^{L_P} P$, together with $e \perp_{B_0}^L P$ implies that $bd \perp_{B_0}^L P$ (recall $b \in \text{acl}_{L_P}(e)$). Since $e \perp_{e^c}^L P$, and $(bd)^c \perp_e^L P$, applying transitivity yields $(bd)^c \perp_{e^c}^L P$, and since everything lives in P , we can restrict our language to get $(bd)^c \perp_{e^c}^L P$. By the proof of Lemma 3.13, $B_0 = \text{acl}_L(e^c)$, hence $(bd)^c \perp_{B_0}^L P$ and by definition of $(bd)^c$ we have $bd \perp_{(bd)^c}^L P$, applying transitivity once more yields $bd \perp_{B_0}^L P$. Let V, Z be the loci of bd and b over B_0 , respectively, and consider

the projection $f : V \rightarrow Z$ sending bd to b , then note that $f^{-1}(b) = Y_b$. Then by compactness, there is a Zariski open subset U in Z , also defined over B_0 , such that G acts rationally in $f^{-1}(U)$ and this action restricted to Y_b coincides with the one defined in Lemma 4.9. This proves (i), as any generic $a \in V$ has the same L -type over B_1 as bd , and the action in Lemma 4.9 is regular by construction. Since $f^{-1}(U)$ is still a variety, by shrinking V we may without loss of generality let $V = f^{-1}(U)$, and by $bd \perp_{B_0}^L P$, we conclude bd is a generic point of V over P , therefore (ii) follows by applying Lemma 4.9. \square

We state our main result, which will follow from Proposition 4.4.

Corollary 4.12. *There is a set of sorts $\mathcal{S} \subseteq L^{\text{eq}}$, such that T_P has weak elimination of imaginaries in the language obtained by adjoining \mathcal{S} to L .*

Proof. Let G, V be as in Proposition 4.4, and let $c \in P$ generate a field over which (G, V, \cdot) are defined. There is a variety Z defined over the prime field such that there exist varieties \mathcal{G}, \mathcal{V} , along with surjective regular maps to Z , and for each $b \in Z$, the fiber \mathcal{G}_b is an algebraic group that acts on \mathcal{V}_b , and moreover $\mathcal{G}_c = G$ and $\mathcal{V}_c = V$. For each $e \in M^{\text{eq}}$, we define a sort $S_{(\mathcal{G}, \mathcal{V}, Z, e)}$ in the following manner: let $W_e = \cup\{\mathcal{V}_b, b \in Z(P)\}$, and define an equivalence relation on W as $w_1 \sim w_2$ iff for some $b \in Z(P)$, $w_1, w_2 \in \mathcal{V}_b$ and there exists $g \in \mathcal{G}_b(P)$ such that $w_1 = g \cdot w_2$. We interpret the elements of $S_{(\mathcal{G}, \mathcal{V}, Z, e)}$ as the classes of W modulo \sim , which are in turn representatives of each orbit of the fiberwise action of \mathcal{G} on \mathcal{V} . By Proposition 4.4, for every $e \in M^{\text{eq}}$, there is $r \in S_{(\mathcal{G}, \mathcal{V}, Z, e)}$, such that $e \in \text{dcl}_{L_P}(r)$ and $r \in \text{acl}_{L_P}(e)$. \square

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