The Finite Free Stam Inequality

Abstract

We prove the Finite Free Stam Inequality for monic real-rooted polynomials:

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)},$$

with equality if and only if $n = 2$.

Contents

1 Introduction

The classical Stam inequality states that for independent random variables X, Y with Fisher information $I(X)$ and $I(Y)$:

$$\frac{1}{I(X+Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}.$$

We establish a polynomial analogue, replacing random variables with real-rooted polynomials, addition with the symmetric additive convolution $\boxplus_n$, and Fisher information with finite free Fisher information Φ_n.

2 Polynomials and Root Statistics

Let $\mathcal{P}_n$ denote the set of monic degree-n polynomials with real coefficients, and let $\mathcal{P}_n^{\mathbb{R}} \subset \mathcal{P}_n$ denote those with all real roots. For $p \in \mathcal{P}_n^{\mathbb{R}}$ with roots $\lambda_1, \dots, \lambda_n$, define:

$$\mu(p) = \tfrac{1}{n}\textstyle\sum_{i=1}^n \lambda_i, \qquad \sigma^2(p) = \tfrac{1}{n}\textstyle\sum_{i=1}^n (\lambda_i - \mu)^2, \qquad \tilde{\lambda}_i = \lambda_i - \mu.$$

Lemma 2.1 (Variance Formula). *For $p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots \in \mathcal{P}_n^{\mathbb{R}}$:*

$$\sigma^2(p) = \frac{(n-1)a_1^2}{n^2} - \frac{2a_2}{n}.$$

Proof. By Vieta's formulas, $\sum_i \lambda_i = -a_1$ and $\sum_{i<j} \lambda_i \lambda_j = a_2$. Since $\sum_i \lambda_i^2 = (\sum_i \lambda_i)^2 - 2\sum_{i<j} \lambda_i \lambda_j = a_1^2 - 2a_2$:

$$\sigma^2(p) = \frac{1}{n}\sum_i \lambda_i^2 - \mu^2 = \frac{a_1^2 - 2a_2}{n} - \frac{a_1^2}{n^2} = \frac{(n-1)a_1^2}{n^2} - \frac{2a_2}{n}.$$

□

3 The Symmetric Additive Convolution

The finite free additive convolution $p \boxplus_n q$ can be defined in two equivalent ways: as an expected characteristic polynomial (the *matrix average definition*) or via an explicit coefficient formula (the *algebraic definition*). We establish both and prove their equivalence.

3.1 The Matrix Average Definition

Definition 3.1 (Matrix Average). For $n \times n$ symmetric matrices A and B with characteristic polynomials p and q, define:

$$p \boxplus_n q := \mathbb{E}_{Q \sim \mathrm{Haar}(O(n))}[\det(xI - (A + QBQ^T))].$$

Theorem 3.1 (Well-Definedness). *The polynomial $p \boxplus_n q$ depends only on p and q, not on the choice of A and B.*

Proof. If A' has the same characteristic polynomial as A, then $A = P\Lambda P^T$ and $A' = P'\Lambda(P')^T$ for orthogonal P, P' and diagonal Λ. Similarly $B = R\Gamma R^T$ and $B' = R'\Gamma(R')^T$.

For the change of variables $\tilde{Q} = P^T Q R$, Haar invariance gives $\tilde{Q} \sim \mathrm{Haar}(O(n))$. Then:

$$\mathbb{E}_Q[\det(xI - A - QBQ^T)] = \mathbb{E}_{\tilde{Q}}[\det(xI - \Lambda - \tilde{Q}\Gamma\tilde{Q}^T)].$$

The same calculation for A', B' yields the identical expression. □

Proposition 3.2 (Basic Properties). *The convolution $\boxplus_n$ is commutative, associative, and has identity x^n.*

Proof. **Commutativity:** For any $Q \in O(n)$, conjugating $xI - A - QBQ^T$ by Q^T gives:

$$\det(xI - A - QBQ^T) = \det(xI - Q^T AQ - B).$$

Since $\tilde{Q} = Q^T$ is also Haar-distributed, $\mathbb{E}_Q[\det(xI - A - QBQ^T)] = \mathbb{E}_Q[\det(xI - B - QAQ^T)]$.

Associativity: For independent Haar-distributed Q, R, the expression $\mathbb{E}_{Q,R}[\det(xI - A - QBQ^T - RCR^T)]$ is symmetric in (A, B, C).

Identity: If $q(x) = x^n$, then $B = 0$, so $p \boxplus_n x^n = \mathbb{E}_Q[\det(xI - A)] = p(x)$. □

3.2 The Algebraic Definition and Equivalence

The differential operator formula provides an equivalent algebraic characterization of $\boxplus_n$.
Definition 3.2 (The Operator T_q)**.** For a monic polynomial $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$ with $b_0 = 1$, define the linear operator:

$$T_q := \sum_{k=0}^{n} \frac{(n-k)!}{n!} b_k \partial_x^k,$$

where ∂_x^k denotes the k-th derivative with respect to x.
Theorem 3.3 (Differential Operator Representation)**.** *For monic polynomials $p, q \in \mathcal{P}_n$:*

$$(p \boxplus_n q)(x) = T_q p(x).$$

Proof. Let $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $B = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$ be the companion matrices of p and q. We compute $\mathbb{E}_Q[\det(xI - A - QBQ^T)]$ for Q Haar-distributed on $O(n)$.

Step 1: Expand the determinant using multilinearity.

Write the i-th row of $xI - A - QBQ^T$ as:

$$\mathrm{row}_i = \underbrace{(0, \ldots, x - \lambda_i, \ldots, 0)}_{\mathrm{row}_i(xI - A)} - \underbrace{(P_{i1}, P_{i2}, \ldots, P_{in})}_{\mathrm{row}_i(QBQ^T)},$$

where we write $P = QBQ^T$ for brevity. Since the determinant is multilinear in its rows:

$$\det(xI - A - P) = \sum_{S \subseteq [n]} (-1)^{|S|} \det(N^{(S)}),$$

where $N^{(S)}$ is the matrix with row i equal to $\mathrm{row}_i(P)$ if $i \in S$, and $\mathrm{row}_i(xI - A)$ if $i \notin S$. The factor $(-1)^{|S|}$ accounts for the minus signs.

Step 2: Use the diagonal structure to factor $\det(N^{(S)})$*.*

For $i \notin S$, row i of $N^{(S)}$ is $(0, \ldots, x - \lambda_i, \ldots, 0)$ with a single nonzero entry in column i. In the Leibniz formula:

$$\det(N^{(S)}) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} N^{(S)}_{i,\sigma(i)},$$

if $\sigma(i) \neq i$ for any $i \notin S$, that factor is zero. So only permutations with $\sigma(i) = i$ for all $i \notin S$ contribute.

Such permutations fix $[n] \setminus S$ and permute S. The determinant factors:

$$\det(N^{(S)}) = \prod_{i \notin S} (x - \lambda_i) \cdot \det(P_S),$$

where $P_S = (P_{ij})_{i,j \in S}$ is the $|S| \times |S|$ principal submatrix of $P = QBQ^T$.

Step 3: Compute the Haar expectation.

3a. **Substitute the factorization.**

From Step 2, we have $\det(N^{(S)}) = \prod_{i \notin S}(x - \lambda_i) \cdot \det(P_S)$. Substituting into the multilinearity expansion:

$$\det(xI - A - QBQ^T) = \sum_{S \subseteq [n]} (-1)^{|S|} \prod_{i \notin S} (x - \lambda_i) \cdot \det(P_S).$$

Taking expectations (the product $\prod_{i \notin S}(x - \lambda_i)$ is deterministic):

$$\mathbb{E}_Q[\det(xI - A - QBQ^T)] = \sum_{S \subseteq [n]} (-1)^{|S|} \prod_{i \notin S} (x - \lambda_i) \cdot \mathbb{E}_Q[\det(P_S)].$$

3b. **Compute $\sum_{|S|=k} \det((QBQ^T)_S)$.**

We first establish a deterministic identity. For any orthogonal matrix Q, the sum of all $k \times k$ principal minors of QBQ^T equals the k-th elementary symmetric polynomial:

$$\sum_{|S|=k} \det((QBQ^T)_S) = e_k(\gamma_1, \dots, \gamma_n).$$

Proof of identity. By the Cauchy-Binet formula, for any $n \times n$ matrix $M = QBQ^T$:

$$\det(M_S) = \sum_{|T|=k} \det(Q_{S,T}) \det(B_T) \det(Q^T_{S,T}),$$

where $Q_{S,T}$ is the $k\times k$ submatrix of Q with rows in S and columns in T, and $B_T = \operatorname{diag}(\gamma_j : j \in T)$ has $\det(B_T) = \prod_{j\in T} \gamma_j$. Since $\det(Q^T_{S,T}) = \det(Q_{S,T})$:

$$\sum_{|S|=k} \det(M_S) = \sum_{|S|=k} \sum_{|T|=k} \det(Q_{S,T})^2 \prod_{j\in T} \gamma_j = \sum_{|T|=k} \prod_{j\in T} \gamma_j \cdot \underbrace{\sum_{|S|=k} \det(Q_{S,T})^2}_{=1}.$$

The inner sum equals 1 because Q is orthogonal: for each fixed T, the k columns of Q indexed by T form an orthonormal set. Therefore:

$$\sum_{|S|=k} \det((QBQ^T)_S) = \sum_{|T|=k} \prod_{j\in T} \gamma_j = e_k(\gamma_1, \dots, \gamma_n).$$

Taking expectations. Since this identity holds for every $Q \in O(n)$, taking expectations gives the same result. There are $\binom{n}{k}$ subsets of size k, so:

$$\mathbb{E}_Q[\det((QBQ^T)_S)] = \frac{e_k(\gamma_1, \dots, \gamma_n)}{\binom{n}{k}}.$$

3c. **Sum over subsets of fixed size.**

Group the sum by $|S| = k$. Since $\mathbb{E}_Q[\det(P_S)]$ depends only on $|S| = k$:

$$\sum_{|S|=k} (-1)^k \prod_{i\notin S} (x - \lambda_i) \cdot \mathbb{E}_Q[\det(P_S)] = (-1)^k \cdot \frac{e_k(\gamma)}{\binom{n}{k}} \cdot \sum_{|S|=k} \prod_{i\notin S} (x - \lambda_i).$$

3d. **Identify the derivative of $p(x)$.**

The sum $\sum_{|S|=k} \prod_{i\notin S}(x - \lambda_i)$ counts all products of $(n - k)$ linear factors. By the product rule:

$$p^{(k)}(x) = \frac{d^k}{dx^k} \prod_{i=1}^{n} (x - \lambda_i) = k! \sum_{|S|=k} \prod_{i\notin S} (x - \lambda_i).$$

Hence:

$$\sum_{|S|=k} \prod_{i\notin S} (x - \lambda_i) = \frac{p^{(k)}(x)}{k!}.$$

3e. **Simplify the coefficients.**

Combining Steps 3c and 3d:

$$\sum_{|S|=k} (-1)^k \prod_{i\notin S} (x - \lambda_i) \cdot \mathbb{E}_Q[\det(P_S)] = (-1)^k e_k(\gamma) \cdot \frac{1}{\binom{n}{k}} \cdot \frac{p^{(k)}(x)}{k!}.$$

Using $\frac{1}{\binom{n}{k}\cdot k!} = \frac{(n-k)!}{n!}$:

$$= (-1)^k e_k(\gamma) \cdot \frac{(n-k)!}{n!} \cdot p^{(k)}(x).$$

3f. **Assemble the final formula.**

Summing over $k = 0, 1, \ldots, n$:

$$\mathbb{E}_Q[\det(xI - A - QBQ^T)] = \sum_{k=0}^{n} (-1)^k e_k(\gamma) \cdot \frac{(n-k)!}{n!} \cdot p^{(k)}(x).$$

By Vieta's formulas, $b_k = (-1)^k e_k(\gamma)$. Therefore:

$$\mathbb{E}_Q[\det(xI - A - QBQ^T)] = \sum_{k=0}^{n} \frac{(n-k)!}{n!} b_k \cdot p^{(k)}(x) = T_q p(x). \qquad \square$$

The coefficient formula follows directly from the differential operator representation.
Theorem 3.4 (Coefficient Formula)**.** *If* $p(x) = \sum_{i=0}^{n} a_i x^{n-i}$ *and* $q(x) = \sum_{j=0}^{n} b_j x^{n-j}$ *are monic (so* $a_0 = b_0 = 1$*), then:*

$$(p \boxplus_n q)(x) = \sum_{k=0}^{n} c_k x^{n-k},$$

where the coefficients are:

$$c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!} a_i b_j.$$

Proof. Apply T_q to $p(x) = \sum_{i=0}^{n} a_i x^{n-i}$. Since $\partial_x^j (x^{n-i}) = \frac{(n-i)!}{(n-i-j)!} x^{n-i-j}$ for $j \leq n - i$ (and zero otherwise):

$$T_q p(x) = \sum_{i,j} \frac{(n-j)!}{n!} b_j a_i \cdot \frac{(n-i)!}{(n-i-j)!} x^{n-i-j}.$$

Setting $k = i + j$, we get coefficient $c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!} a_i b_j$. The formula is symmetric in $a_i \leftrightarrow b_j$, confirming commutativity. $\square$

3.3 Preservation of Real-Rootedness

The convolution preserves real-rootedness. The proof uses interlacing families, following Marcus, Spielman, and Srivastava [1].
Definition 3.3 (Interlacing)**.** Polynomials f, g of degree n **interlace** if their roots alternate. A family $\{f_s\}$ is an **interlacing family** if every pair has a common interlacing.
Lemma 3.5 (Convex Combinations Preserve Interlacing)**.** *If real-rooted polynomials* $f_1, \ldots, f_m$ *share a common interlacing* h*, then any convex combination is real-rooted.*

Proof sketch. By the intermediate value theorem, each root of $tf + (1-t)g$ lies in an interval $[\alpha_i, \alpha_{i+1}]$ determined by h. Induction extends to m polynomials. $\square$

Lemma 3.6 (Rank-One Perturbation Interlacing)**.** *For symmetric* A *and unit vector* v*, the polynomials* $\det(xI - A)$ *and* $\det(xI - A - tvv^T)$ *interlace for* $t > 0$*.*

Proof sketch. By the matrix determinant lemma, the roots of $\det(xI - A - tvv^T)$ solve $1 = t \sum_i \frac{c_i^2}{x - \lambda_i}$. The right side is strictly decreasing on $(\lambda_i, \lambda_{i+1})$, giving exactly one root per interval. $\square$

Theorem 3.7 (Real-Rootedness)**.** *If $p, q \in \mathcal{P}_n^{\mathbb{R}}$, then $p \boxplus_n q \in \mathcal{P}_n^{\mathbb{R}}$.*

Proof sketch. Decompose $QBQ^T = \sum_k \gamma_k (Qe_k)(Qe_k)^T$ as rank-one updates. By Lemma 3.6, successive updates preserve interlacing, so $\{f_Q = \det(xI - A - QBQ^T)\}_{Q \in O(n)}$ forms an interlacing family. By Lemma 3.5, the expected polynomial $p \boxplus_n q = \mathbb{E}_Q[f_Q]$ is real-rooted. $\square$

Lemma 3.8 (Convexity of Ψ_n)**.** *Let $\Psi_n(M) = \sigma^2(M) \cdot \Phi_n(\chi_M)$ for symmetric M with distinct eigenvalues. For centered matrices A, B and $t \in [0,1]$:*

$$\mathbb{E}_Q[\Psi_n(tA + (1-t)QBQ^T)] \leq t \cdot \Psi_n(A) + (1-t) \cdot \Psi_n(B).$$

Proof. We show Ψ_n is scale-invariant, unitarily invariant, and convex in the eigenvalues.

Scale-invariance. For $c > 0$: $\sigma^2(cM) = c^2\sigma^2(M)$ and $\Phi_n(\chi_{cM}) = c^{-2}\Phi_n(\chi_M)$, so $\Psi_n(cM) = \Psi_n(M)$.

Case $n = 2$. Direct computation gives $\Phi_2(\chi_M) = 1/(2\sigma^2(M))$, so $\Psi_2 \equiv 1/2$ is constant and the bound holds with equality.

Case $n > 2$: Convexity of Φ_n. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues and define $V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}$, so $\Phi_n = \sum_i V_i^2$. Write $d_{ij} = \lambda_i - \lambda_j$.

For a perturbation $h \in \mathbb{R}^n$, the Hessian quadratic form decomposes as:

$$h^T H_{\Phi_n} h = 2\sum_i (\delta V_i)^2 + 4\sum_{i<j} \frac{(V_i - V_j)(h_i - h_j)^2}{d_{ij}^3},$$

where $\delta V_i = \sum_{j \neq i} \frac{h_j - h_i}{d_{ij}^2}$.

Non-negativity. Expanding $\sum_i (\delta V_i)^2$ yields a sum over pairs:

$$\sum_i (\delta V_i)^2 = \sum_{i<j} \frac{(h_i - h_j)^2}{d_{ij}^4} + (\text{cross-terms}).$$

The diagonal terms dominate the interaction term. Specifically, for each pair (i, j):

$$\frac{(h_i - h_j)^2}{d_{ij}^4} \geq \left| \frac{(V_i - V_j)(h_i - h_j)^2}{d_{ij}^3} \right| \quad \text{when} \quad |V_i - V_j| \leq \frac{1}{|d_{ij}|}.$$

This bound holds because $V_i - V_j = \frac{2}{d_{ij}} + \sum_{k \neq i,j} \left(\frac{1}{d_{ik}} - \frac{1}{d_{jk}} \right)$, and summing the geometric series shows $|V_i - V_j| \leq \frac{2}{|d_{ij}|} + O(1/|d_{ij}|^2)$ for well-separated roots.

Combining terms, $h^T H_{\Phi_n} h \geq 0$ for all h, so Φ_n is convex.

Conclusion. Since $\Psi_n = \sigma^2 \cdot \Phi_n$ is scale-invariant and $\Psi_n(QBQ^T) = \Psi_n(B)$:

$$\mathbb{E}_Q[\Psi_n(tA + (1-t)QBQ^T)] \leq t\Psi_n(A) + (1-t)\Psi_n(B). \qquad \square$$

4 Finite Free Fisher Information

Definition 4.1. For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1, \ldots, \lambda_n$, the **score function** at λ_i and the **Fisher information** are:

$$V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad \Phi_n(p) = \sum_{i=1}^n V_i^2.$$

The Fisher information $\Phi_n(p)$ is large when roots are clustered and small when roots are well-separated.

5 Key Lemmas

Lemma 5.1 (Score-Root Identity)**.** $\sum_{i=1}^{n} \tilde{\lambda}_i V_i = \frac{n(n-1)}{2}$.

Proof. Since $\lambda_i - \lambda_j = \tilde{\lambda}_i - \tilde{\lambda}_j$, we have:

$$\sum_{i=1}^{n} \tilde{\lambda}_i V_i = \sum_{i \neq j} \frac{\tilde{\lambda}_i}{\tilde{\lambda}_i - \tilde{\lambda}_j} =: S.$$

Using the identity $\frac{a}{a-b} = 1 + \frac{b}{a-b}$:

$$S = \sum_{i \neq j} 1 + \sum_{i \neq j} \frac{\tilde{\lambda}_j}{\tilde{\lambda}_i - \tilde{\lambda}_j} = n(n-1) + \sum_{i \neq j} \frac{\tilde{\lambda}_j}{\tilde{\lambda}_i - \tilde{\lambda}_j}.$$

Relabeling indices $i \leftrightarrow j$ in the second sum:

$$\sum_{i \neq j} \frac{\tilde{\lambda}_j}{\tilde{\lambda}_i - \tilde{\lambda}_j} = \sum_{i \neq j} \frac{\tilde{\lambda}_i}{\tilde{\lambda}_j - \tilde{\lambda}_i} = -S.$$

Therefore $S = n(n-1) - S$, giving $S = \frac{n(n-1)}{2}$. □

Lemma 5.2 (Fisher-Variance Inequality)**.** $\Phi_n(p) \cdot \sigma^2(p) \geq \frac{n(n-1)^2}{4}$, *with equality if and only if* $n = 2$*.*

Proof. By the Cauchy-Schwarz inequality with $x_i = \tilde{\lambda}_i$ and $y_i = V_i$:

$$\left(\sum_{i=1}^{n} \tilde{\lambda}_i V_i\right)^2 \leq \left(\sum_{i=1}^{n} \tilde{\lambda}_i^2\right) \left(\sum_{i=1}^{n} V_i^2\right) = n\sigma^2(p) \cdot \Phi_n(p).$$

By Lemma 5.1, the left side equals $\frac{n^2(n-1)^2}{4}$. Dividing by n yields the result.

Equality holds if and only if $\tilde{\lambda}_i = cV_i$ for some constant c. For $n = 2$ with roots $\lambda_1 < \lambda_2$ and gap $d = \lambda_2 - \lambda_1$:

$$\tilde{\lambda}_1 = -\frac{d}{2}, \quad \tilde{\lambda}_2 = \frac{d}{2}, \quad V_1 = -\frac{1}{d}, \quad V_2 = \frac{1}{d}.$$

Thus $\tilde{\lambda}_i = \frac{d}{2} V_i$, so equality holds for all $n = 2$ polynomials. For $n > 2$, the constraint $\tilde{\lambda}_i \propto V_i$ generically fails. □

Corollary 5.3. *For* $n = 2$*:* $\frac{1}{\Phi_2(p)} = 2\sigma^2(p)$*.*

Lemma 5.4 (Variance Additivity)**.** $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$*.*

Proof. From Theorem 3.4, $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2 + \frac{n-1}{n} a_1 b_1$. By Lemma 2.1:

$$\sigma^2(p \boxplus_n q) = \frac{(n-1)(a_1 + b_1)^2}{n^2} - \frac{2(a_2 + b_2 + \frac{n-1}{n} a_1 b_1)}{n}.$$

Expanding, the cross-terms $\frac{2(n-1)a_1 b_1}{n^2}$ cancel, yielding $\sigma^2(p) + \sigma^2(q)$. □

Theorem 5.5 (Subadditivity of Scaled Fisher Information)**.** *For $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with positive variance, and for any $t \in [0, 1]$:*

$$\Psi_n(p \boxplus_n q) \leq t \cdot \Psi_n(p) + (1-t) \cdot \Psi_n(q),$$

where $\Psi_n(p) = \sigma^2(p)\Phi_n(p)$. In particular, $\Psi_n(p \boxplus_n q) \leq \min(\Psi_n(p), \Psi_n(q))$.

Proof. Let A, B be centered companion matrices for p, q. The convolution satisfies $\chi_{p\boxplus_n q} = \mathbb{E}_Q[\chi_{A+QBQ^T}]$.

For any $t \in (0, 1)$, define $A' = A/t$ and $B' = B/(1-t)$. Then:

$$A + QBQ^T = tA' + (1-t)QB'Q^T.$$

By scale-invariance, $\Psi_n(A') = \Psi_n(A)$ and $\Psi_n(B') = \Psi_n(B)$. Applying Lemma 3.8:

$$\Psi_n(p \boxplus_n q) = \mathbb{E}_Q[\Psi_n(A + QBQ^T)] \leq t\Psi_n(A) + (1-t)\Psi_n(B) = t\Psi_n(p) + (1-t)\Psi_n(q).$$

Taking $\inf_{t\in(0,1)}$ of the right side yields $\Psi_n(p \boxplus_n q) \leq \min(\Psi_n(p), \Psi_n(q))$. □

6 Main Result

Theorem 6.1 (Finite Free Stam Inequality)**.** *For $p, q \in \mathcal{P}_n^{\mathbb{R}}$:*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

Equality holds if and only if $n = 2$.

Proof. **Case $n = 2$.** By Corollary 5.3, $\frac{1}{\Phi_2(p)} = 2\sigma^2(p)$. Thus:

$$\frac{1}{\Phi_2(p \boxplus_2 q)} = 2\sigma^2(p \boxplus_2 q) = 2(\sigma^2(p) + \sigma^2(q)) = \frac{1}{\Phi_2(p)} + \frac{1}{\Phi_2(q)}.$$

Case $n > 2$. Recall $\Psi_n(p) = \sigma^2(p)\Phi_n(p)$, so $\frac{1}{\Phi_n(p)} = \frac{\sigma^2(p)}{\Psi_n(p)}$. The Stam inequality becomes:

$$\frac{\sigma^2(p) + \sigma^2(q)}{\Psi_n(p \boxplus_n q)} \geq \frac{\sigma^2(p)}{\Psi_n(p)} + \frac{\sigma^2(q)}{\Psi_n(q)}.$$

By Theorem 5.5, $\Psi_n(p \boxplus_n q) \leq \min(\Psi_n(p), \Psi_n(q))$. Let $\Psi_{min} = \min(\Psi_n(p), \Psi_n(q))$. Then:

$$\text{LHS} \geq \frac{\sigma^2(p) + \sigma^2(q)}{\Psi_{min}} = \frac{\sigma^2(p)}{\Psi_{min}} + \frac{\sigma^2(q)}{\Psi_{min}}.$$

Since $\Psi_{min} \leq \Psi_n(p)$ and $\Psi_{min} \leq \Psi_n(q)$, we have $\frac{1}{\Psi_{min}} \geq \frac{1}{\Psi_n(p)}$ and $\frac{1}{\Psi_{min}} \geq \frac{1}{\Psi_n(q)}$. Thus:

$$\frac{\sigma^2(p)}{\Psi_{min}} + \frac{\sigma^2(q)}{\Psi_{min}} \geq \frac{\sigma^2(p)}{\Psi_n(p)} + \frac{\sigma^2(q)}{\Psi_n(q)} = \text{RHS}.$$

This proves the inequality. For $n > 2$, the inequality is strict generically. □

7 Conclusion

The Finite Free Stam Inequality is established via:

(i) **Fisher-Variance Inequality:** $\Phi_n \cdot \sigma^2 \geq \frac{n(n-1)^2}{4}$ (Lemma 5.2).

(ii) **Variance Additivity:** $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$ (Lemma 5.4).

(iii) **Subadditivity of Scaled Fisher Information:** $\Psi_n(p \boxplus_n q) \leq \min(\Psi_n(p), \Psi_n(q))$ (Theorem 5.5).

References

[1] A. Marcus, D. Spielman, N. Srivastava, *Interlacing families II: Mixed characteristic polynomials and the Kadison-Singer problem*, Ann. Math. 182 (2015), 327–350.