Université de Paris **(Date: 21/09/2020)**

UFR Mathématiques

Théorie des modèles TD1

Professor: T. Servi — Juan Ignacio Padilla, M2 LMFI

Exercise 0.1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $m, n \in \mathbb{N}$ and $A \subseteq M^{n+m}$ be definable in $\mathcal{M}$. For $\bar{b} \in M^m$, let $A_{\bar{b}} = \{\bar{a} \in M^n, (\bar{a}, \bar{b}) \in A\}$ the fiber of A over b. Let $k \in \mathbb{N}$. Show that the set $\{\bar{b} \in M^n, |A_{\bar{b}}| < k\}$ is definable. (*) Is the set $\{\bar{b} \in M^n, |A_{\bar{b}}| < \infty\}$ definable?

Solution 0.1. If $A \subseteq M^{n+m}$ is definable, then there is some $\bar{s} \in M$, and some formula $\phi(\bar{x}, \bar{y}, \bar{z})$ such that $A = \{(\bar{x}, \bar{y}) \in M^{n+m}, \mathcal{M} \models \phi(\bar{x}, \bar{y}, \bar{s})\}$. The following formula states that the $A_{\bar{b}}-$fiber has less than k elements.

$$\phi_k(y_1, \ldots, y_m) = \forall \bar{x_1} \ldots \bar{x_k} \left(\bigwedge_{i=1}^{k} \phi(\bar{x_i}, \bar{y}) \Rightarrow \bigvee_{1 \le i \ne j \le k} \bar{x_i} = \bar{x_j} \right).$$

We see that $\mathcal{M} \models \phi_k(\bar{b})$ if and only if $A_{\bar{b}}$ has less than k elements.
(*) Consider $\mathcal{N} = (\mathbb{N}, <)$, and let $\mathcal{U}$ a non-principal ultrafilter over $\mathbb{N}$. Let $\mathcal{M} = \mathcal{N}^{\mathcal{U}}$. We can identify each $n \in \mathbb{N}$ with $[n, n, \ldots,]_{\mathcal{U}} \in \mathcal{M}$. Notice that for every $n \in \mathbb{N}$,

$$\omega = [0, 1, 2, \ldots, n, n+1, \ldots]_{\mathcal{U}} > n$$

We call elements bigger than every n, *infinite.* Suppose now the set $\{\bar{b} \in M^n, |A_{\bar{b}}| < \infty\}$ is definable by some formula $\phi(x, \bar{m})$ with parameters $\bar{m} \in \mathcal{M}$. In other words, $\mathcal{M} \models \phi(x, \bar{m})$ if and only if there is a finite number of elements below x (in the usual finite sense), we show this implies that x is necessarily finite: suppose not, then for every $n, \mathcal{U}$-almost everywhere, $x_i \neq n$. We want to prove that actually $x_i \geq n$: if it were not the case, then again, $\mathcal{U}$-almost everywhere $x_i < n \Rightarrow x_i \in \{0, 1, \ldots, n-1\} = \bigcup_{j=0}^{n-1}\{j\}$. In other words, this means that

$$\bigcup_{j=0}^{n-1} \{i, x_i = j\} \in \mathcal{U}.$$

By ultrafilter properties, (if $A \cup B \in \mathcal{U}$ then either $A \in \mathcal{U}$ of $B \in \mathcal{U}$) we conclude that for some k, $[x] = [k]$, which is a contradiction since x is infinite. We consider now $\Sigma(x, \bar{m}) = \{\neg\phi_k(x, \bar{m})\}_{k \in \mathbb{N}} \cup \{\phi(x, \bar{m})\}$. It is finitely consistent, since if $\Sigma_N(x, \bar{m}) = \{\neg\phi_k(x, \bar{m})\}_{k<N} \cup \{\phi(x, \bar{m})\}$ is a finite part of Σ, then

$\mathcal{M} \models \Sigma_N(N, \bar{m})$ (N has at least k elements below it for every $k > N$, and also has a finite number of elements below it since it is finite). By compactness, there is $N' \in \mathcal{M}$ such that $\mathcal{M} \models \Sigma_N(N', \bar{m})$. We conclude that N' has at least k elements below it for every k, and that N is finite by the above. This is a contradiction, so $\{\bar{b} \in M^n, |A_{\bar{b}}| < \infty\}$ is not definable.

Exercise 2. Let M be a set and $\mathcal{D} = \bigcup_n D_n$ be a collection of subsets of $\bigcup_n M^n$ containing $\varnothing$, M^n for every n, the diagonals, and closed under permutation of the coordinates, cartesian products, the boolean set operations and linear projections. Show that $\mathcal{D} = \mathrm{Def}(\mathcal{M}, \varnothing)$, for some language $\mathcal{L}$ and some $\mathcal{L}$-structure $\mathcal{M}$.

Solution 0.2. Take $\mathcal{C} = \varnothing$, and $\mathcal{R} = \bigcup_{n\in\mathbb{N}}\{(x_1, \dots, x_n), (x_1, \dots, x_n) \in D_n\}$. In other words, take no constants and set each of the D_n's to be a predicate. For each n, if for every $(x_1, \dots, x_n) \in M^n$, there exists $y \in M$ such that $(x_1, \dots, x_n, y) \in D_{n+1}$, we set $f : M^n \to M$ which sends $(x_1, \dots, x_n)$ to y. We may have to do this (possibly infinitely) many times since such y may not be unique.

Exercise 0.3. Let $\mathcal{M}$ be an expansion of a total order equipped with the *order topology*. Let $A \subseteq M^n$ and $f : A \to M$ both definable.

(1) Show that $A^\circ, \overline{A}$ and $\mathrm{bd}(A)$ are all definable.
(2) Show that the set of discontinuity points of f is definable.
(3) Show that the following properties are definable: A is discrete, A is bounded.
(4) What about A is compact and connected?

Solution 0.3. We use the following abbreviations: $\bar{x} < \bar{y}$ for $x_i < y_i$ for each i and if ϕ is a formula then $Qx \in A$ (ϕ) (where Q is a quantifier) for $Qx(x \in A \Rightarrow \phi)$.

(1) $\bar{x} \in A^\circ$ if and only if $\exists \bar{y}, \bar{z} \in A$ $(\bar{z} < \bar{x} < \bar{y})$.
    $\bar{x} \in \bar{A}$ if and only if $\forall \bar{y}, \bar{z}((\bar{z} < \bar{x} < \bar{y}) \Rightarrow \exists \bar{w} \in A(\bar{z} < \bar{w} < \bar{y}))$
    $\bar{x} \in \mathrm{bd}(A)$ if and only if $\bar{x} \notin A^\circ \wedge \bar{x} \in \bar{A}$.
(2) $\bar{x}$ is a discontinuity point of f if and only if

$$\exists r \exists s((r < f(\bar{x}) < s) \wedge \forall \bar{y}, \bar{z} \in A((\bar{z} < \bar{x} < \bar{y}) \Rightarrow \exists \bar{w}((\bar{z} < \bar{w} < \bar{y}) \wedge (f(\bar{w}) < r \vee f(\bar{w}) > s))$$

(3) We say A is discrete if $\forall \bar{x} \in A$ $\exists \bar{y} \in A(\bar{x} < \bar{y} \Rightarrow \nexists \bar{z} \in A(\bar{x} < \bar{z} < \bar{y}))$. We say A is bounded if $\forall \bar{x} \in A$ $\exists \bar{y}, \bar{z}(\bar{x} < \bar{y} \wedge \bar{x} > \bar{z})$.

(4) Consider $\mathcal{U}$ a non-principal ultrafilter on $\mathcal{N}$ and let $\mathbb{R}^* = \mathbb{R}^{\mathcal{U}}$. Consider

$$\varepsilon = [1, 1/2, 1/3, \dots, 1/n, \dots]_{\mathcal{U}}.$$

Notice that for $i \geq n$, $\varepsilon_i < 1/n$, and since $\{n, n+1, \dots\} \in \mathcal{U}$ (it is cofinite), we conclude that for every n, $\varepsilon < 1/n$. This proves that $\mathbb{R}^*$ is not archimedean, and in particular this also proves that archimedianity for a field is not axiomatizable, since if it were by, say, some theory T, we would have $\mathbb{R} \models T$ and $\mathbb{R}^* \not\models T$, contradicting Łos' theorem. We will show that connectedness and compactness are not 1st order expressible: consider E the set of *infinitesimal* elements in $\mathbb{R}^*$, i.e the set of elements smaller than every $1/n$.

E is closed: Let $\varepsilon \in \bar{E}$, then for every $x, y \in \mathbb{R}^*$ such that $x < \varepsilon < y$ there is $\epsilon \in E$ such that $x < \epsilon < y$. If $\varepsilon \geq 1/n$ for some n, then we can find some infinitesimal $1/n < \epsilon < 1$, a contradiction.

E is open: For any $\varepsilon \in E$ we have $\varepsilon/2 < \varepsilon < 2\varepsilon$. We have to show these are infinitesimal: for $\varepsilon/2$ is is trivial since it is below an infinitesimal. Now if $2\varepsilon \notin E$, we can find n such that $1/n < 2\varepsilon \Rightarrow 1/2n < \varepsilon$, which is impossible.

E has no supremum: Let $r = \sup E$. We have that, $r \notin E$ (otherwise $r < 2r \in E$), now let $\epsilon \in E$, and we claim $r - \epsilon$ bounds E: suppose not, so there is $\varepsilon \in E$ such that

$$r - \epsilon < \varepsilon < r \Rightarrow r < \varepsilon + \epsilon < r + \epsilon.$$

Notice also that $\epsilon + \varepsilon \in E$ because, for any $n \in \mathbb{N}$ since $\epsilon, \varepsilon < 1/2n$, then $\epsilon + \varepsilon < 1/n$. We have that $r \leq$some infinitesimal, a contradiction. E cannot have a supremum.

E is not compact: the sequence

$$\varepsilon < 2\varepsilon < \dots < n\varepsilon < \dots$$

is contained in E (by the above argument), it is strictly increasing and is bounded above by 1. The above argument can be used to show it has no limit point. This proves E is not compact. If compactness of some set A was given by some sentence ϕ_A, then we would have $\mathbb{R} \models \phi_{[0,1]}$ but $\mathbb{R}^* \not\models \phi_{[0,1]}$, contradicting Łos' theorem.

Finaly, since E is clopen and it is neither $\varnothing$ nor $[0, 1]$, we conclude that $[0, 1]$ is not connected in $\mathbb{R}^*$, and we can infer that connectedness is also not 1st order expressible.

Exercise 1. Let $\mathcal{L} = \varnothing$ and $\mathcal{M}$ be an $\mathcal{L}$-structure. Show that $A \subseteq M$ is definable in $\mathcal{M}$ if and only if A is either finite or cofinite.

Solution 1.

Lemma 0.1. *If A is S-definable, then every automorphism σ that fixes S pointwise fixes A pointiwse.*

Proof. Let $\psi(x, \bar{s})$ be a formula defining A, since automorphisms preserve formula, then we have

$$\mathcal{M} \models \psi(x, \bar{s}) \iff \mathcal{M} \models \psi(\sigma(x), \sigma(\bar{s})) \iff \mathcal{M} \models \psi(\sigma(x), s)$$

so that $\sigma(X) = X$. $\square$

If there was some definable A which is neither finite nor cofinite, then we can choose infinite sets

$$\{a_0, a_1 \ldots, a_n, \ldots\} \in A \setminus S$$

$$\{b_0, b_1 \ldots, b_n, \ldots\} \in (M \setminus A) \setminus S$$

Then the bijection which sends a_i to b_i and fixes everything else (in particular S) is an automorphism that doesn't fix A, a contradiction. Conversely, if $A = \{a_0 \ldots, a_n\}$ is finite, the formula

$$\phi(x, \bar{a}) = \bigvee_{i=0}^{n} x = a_i$$

defines A. If A is cofinite repeat this argument for $M \setminus A$.

Exercise 2. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, let $m, n \in \mathbb{N}$. A collection $\mathcal{A} = \{A_{\bar{b}}\}_{\bar{b} \in M^m}$ of subsets of M^n is a *definable family* if there exists $S \subseteq M$ and a formula $\phi \in \mathcal{F}_{n+m}(\mathcal{L}_S)$ such that $A_{\bar{b}} = \{\bar{a} \in M^n, \mathcal{M}_S \models \phi(\bar{a}, \bar{b})\}$. Let $D \subseteq M$ be a finite set. Given a D-definable family $\mathcal{A} = \{A_{\bar{b}}\}_{\bar{b} \in M^m}$, let $A = \cup \mathcal{A}$ and let $f : A \to M$ a function.

(1) Show that f is D-definable if and only if all restrictions $f \restriction A_{\bar{b}}$ are D-definable.

(2) What can we say if D is infinite?

Solution 2. Suppose there is $\phi \in \mathcal{F}_{n+1}(\mathcal{L}_D)$ such that $f(a_1, \ldots, a_n) = y$ if and only if $\mathcal{M}_D \models \phi(\bar{a}, y)$. Then $f \restriction_{A_{\bar{b}}} (\bar{a}) = y$ if and only if $\mathcal{M}_D \models \varphi(\bar{a}, \bar{b}) \wedge \phi(\bar{a}, y)$, where φ is the formula which defines the family

$\{A_{\bar{b}}\}_{\bar{b}\in D^m}$. Conversely if there is $\phi_{\bar{b}}$ which defines each $f\restriction_{A_{\bar{b}}}$, we have that $f(\bar{a}) = y$ if and only if

$$\mathcal{M}_D \models \bigvee_{\bar{b}\in D^m} \varphi(\bar{a},\bar{b}) \wedge \phi_{\bar{b}}(\bar{a},y).$$

In case D is infinite, the first direction holds, but the converse may not, for instance take $\mathcal{M} = \langle \mathbb{C}, +, -, \times, 0, 1\rangle$ and $D = \mathbb{R}$. Take $A_b = \{a \in \mathbb{R}, a = b\} = \{b\}$, and $f : A_b \to \mathbb{C}$ as the identity. Since $A = \mathbb{R}$, and the inclusion $\mathbb{R} \subseteq \mathbb{C}$ is not definable, even though its restrictions are.

Exercise 3. Let $\bar{\mathbb{R}} = \langle \mathbb{R}, 0, 1, -, +, \cdot, <\rangle$ be the real ordered field. Let f be a unary symbol and $\mathcal{L} = \mathcal{L}_{OR} \cup \{f\}$.

(1) Show that the $\mathcal{L}$-structures $\langle \bar{\mathbb{R}}, \sin\left(\frac{1}{1+x^2}\right)\rangle$ and $\langle \bar{\mathbb{R}}, \arctan x\rangle$
(2) Show that $\bar{\mathbb{R}}$ is definable int he structure $\langle \mathbb{R}, +, \exp(x)\rangle$.
(3) Let $\mathbb{R}_{\exp} = \langle \bar{\mathbb{R}}, \exp\rangle$ be the real ordered exponential field. An *exponential polynomial* is a function $F : \mathbb{R}^n \to \mathbb{R}$ such that there exists a polynomial $P \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ such that $F(x_1, \ldots, x_n) = P(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$. Show that every set $A \in \mathbb{R}^m$ existentially definable in $\mathbb{R}_{\exp}$ is a linear projection of the zero-set of some exponential polynomial F_A.

Solution 3.

(1) We first define $\sin(x)\restriction_{[0,1]}$ from $\sin\left(\frac{1}{1+x^2}\right)$:

$$\operatorname{graph}\sin(x)\restriction_{[0,1]} = \left\{(x,y), (x = 0 \wedge y = 0) \vee \exists z (x(1+x^2) = 1 \wedge y = \sin\left(\frac{1}{1+z^2}\right)\right\}.$$

We can now define

$$x = \frac{\pi}{2} \iff 2\sin^2\restriction_{[0,1]}(x/2) = 2$$

And define for $0 < x < \pi/2$,

$$\tan(x) = \frac{2\sin\restriction_{[0,1]}(x/2)\sqrt{1-\sin^2\restriction_{[0,1]}(x/2)}}{1-2\sin^2\restriction_{[0,1]}(x/2)}.$$

And for $-\pi/2 < x < 0$

$$\tan(x) = -\tan(-x).$$

Then finally set

$$y = \arctan(x) \iff \tan(x) = y.$$

For the other direction we can define $\tan x$ from $\arctan x$ and set

$$\sin\restriction_{[0,1]} x = \frac{\tan x}{\sqrt{1+\tan^2 x}}$$

define $\sin(1/(1+x^2))$ as above.

(2) (a) $x = 0 \iff x + x = x$

(b) $1 = e^0$

(c) $y = -x \iff x + y = 0$

(d) $x > 0 \iff \exists y e^y = x$

(e) $xy = \exp(\log x + \log y)$

(3) Let $\varphi(\bar{x}, \bar{c})$ and existential formula in $\mathcal{L}_{\exp}$ with parameters $\bar{c} \in \mathbb{R}$. We can assume φ has the form

$$\varphi(\bar{x}, \bar{c}) = \exists z_1, \ldots, \exists z_n \bigvee_{i=1}^{l} \bigwedge_{j=1}^{s} \theta_{ij}(\bar{x}, \bar{z}, \bar{c})$$

where θ_{ij} is atomic or $\neg$-atomic. We know that atomic formulas have the form $t_1 = t_2, t_1 < t_2, t_1 = 0$ or $t_1 < 0$ for t_1, t_2 terms with parameters $\bar{c}$. We can replace in θ_{ij}, $t \neq 0$ for $t < 0 \wedge -t < 0$ and $t < 0$ for $\exists y t y^2 + 1 = 0$ $t \neq 0$, and $t_1 = t_2$ for $t_1 - t_2 = 0$. In other words, we can assumme θ_{ij} to be of the form $t = 0$. We know show by induction on $t(\bar{x}, \bar{c})$ that any term can be replaced by a conjunction of existential formulas containing only terms of the form $F(\bar{x}, \bar{y}, \bar{c})$, where F is an exponential polynomial and $\bar{y}$ are new variables. In other words, $t(\bar{x}, \bar{c}) = 0$ becomes a system of exponential polynomial equations on variables $\bar{y}$.

If $t = c$ then $c = 0$ is already of the form we want.

If $t = t_1 + t_2$ then, since the sum of exponential polynomials is also an exponential polynomial, we can just add each of the rows of each system of equations to get one for $t = 0$.

The case $t_1 t_2$ is similar.

If $t(\bar{x}, \bar{c}) = e^{t_1(\bar{x}, \bar{c})}$, then we can replace

$$t(\bar{x}, \bar{c}) = 0 \iff \exists w e^w = 0 \wedge w = t_1(\bar{x}, \bar{c})$$

and then we can apply induction on t_1, adding the variable w to our exponential polynomial.

We can then suppose (renaming variables and reindexing) that θ_{ij} has the form $F_{ij}(\bar{x}, \bar{z}, \bar{c}) = 0$ for some exponential polynomial. So that

$$\varphi(\bar{x}, \bar{c}) = \exists z_1, \ldots, \exists z_n \bigvee_{i=1}^{l} \bigwedge_{j=1}^{s} F_{ij}(\bar{x}, \bar{z}, \bar{c}) = 0.$$

It is clear that the set of zeros of

$$F = \sum_{i=1}^{l} \left(\prod_{j=1}^{s} F_{ij}(\bar{x}, \bar{c}) \right)^2$$

defines the same set as $\varphi(\bar{x}, \bar{c})$.