Differential Equations
Course Notes

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Contents

1

Foundations

1.1 Introduction

Welcome to the differential equations course! As an introduction, I will share some general details and observations about the course. All of the following information will be detailed in the student letter. Our main objective in this course will be to develop the mathematical skills necessary to solve differential equations.

The formal prerequisites for this course are MA-1002 Calculus II and MA-1004 Linear Algebra.

1.2 A Brief Review

1.2.1 Derivatives

We will work primarily with differentiable functions $f : \mathbb{R} \to \mathbb{R}$, that is, real-valued functions of a single variable whose derivatives (first, second, third, ...) exist. The first fundamental concept we must remember is:

What is a derivative?

Throughout mathematics courses, we have learned several ways to understand the derivative of a function. More specifically, if $f(x)$ is a function, we can give at least 3 interpretations of what $f'(x)$ means, which we also denote $\frac{df}{dx}$:

1. $f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$. Our original definition of a derivative is in the form of a limit. However, this definition does not give us much geometric or physical meaning of the function f.

2. $f'(x)$ represents the **slope** of the tangent line to the graph of f at the point x. For example, we know that the tangent line to the parabola $f(x) = 1 - x^2$ at the point $x = 0$ is given by the line $y = 1$, which has slope 0. Indeed, $f'(x) = -2x$, so $f'(0) = 0$.

3. $f'(x)$ also tells us in some sense **how fast** $f(x)$ **grows** as we move from left to right along the x-axis. For example: we know intuitively that the function $f(x) = e^x$ grows faster than the function $g(x) = x + 2$ when $x > 0$. This is evident by computing $f'(x) = e^x$ and $g'(x) = 1$. Since $f'(x)$ is greater than 1 (whenever $x > 0$), we can conclude that $f(x)$ grows faster than $g(x)$. Note that f grows faster than g, but this does not imply that $f(x) > g(x)$ for all x, since for example $f(1) = e \approx 2.7$, while $g(1) = 3$.

We must always keep these 3 concepts in mind, especially when solving application problems.

In simple words, the derivative of a function $f(x)$ tells us **how** $f(x)$ **changes** when x changes.

The main motivation for studying differential equations is that even if we do not know the value of some function $f(x)$, knowing how it changes (i.e., knowing its derivative) is enough to deduce quite a bit of information. Let us see this with an example:

> **Example 1.1**
>
> **There is 1 bacterium in a jar. We know that every minute, the bacteria in the jar double. So, how many bacteria will there be after m minutes?**
> In this example, we are asked to find the function $f(m)$ that takes as input the minutes that have passed and returns how many bacteria are in the jar. We only have as information the fact that each minute, whatever the quantity $f(m)$, the next minute there will be double, that is:
> $$f(m+1) = 2f(m)$$
> Knowing that at minute 0 there is 1 bacterium, it is not difficult to deduce that the function that describes this situation is precisely
> $$f(m) = 2^m.$$

We have been able to deduce the exact value of f just by knowing how it changes (although we have not explicitly used its derivative). However, we have used an additional piece of information, namely, the **initial quantity** of bacteria. Let us think about what would happen if at minute 0, instead of 1 bacterium, there were 2 bacteria. Then the solution to the problem would be
$$f(m) = 2^{m+1}.$$

Exercises

1. How many bacteria are there at minute m if we start with some number n of bacteria?

The idea we should keep in mind is that, in general, the only ingredients we need to solve this type of problem (which are essentially differential equations) are:

- How our function f changes.
- Where our function f starts.

We can even think of it this way: If we know that a train leaves at a certain time, and we know its speed, we can easily calculate its position at any moment.

Throughout the course, knowledge of how to calculate derivatives and their properties will be absolutely necessary. As a summary, I will include some of the most common derivatives and some important properties. If you as a student feel that you do not master (or do not remember) some of these topics, I recommend reviewing them, as they will be used every day.

$f(x)$	$f'(x)$
C	0
x	1
$x^n \quad (n \neq 0)$	nx^{n-1}
e^x	e^x
$a^x \quad (a \in \mathbb{R})$	$a^x \ln(a)$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$	$\frac{1}{x\ln(a)}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sqrt[n]{x}$	$\frac{1}{n\sqrt[n]{x^{n-1}}}$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc^2(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$

Let f, g be differentiable functions and $C \in \mathbb{R}$. Then:

- **Linearity:** $(Cf(x) + g(x))' = Cf'(x) + g'(x)$.
- **Product rule:** $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.
- **Chain rule:** $(f(g(x)))' = f'(g(x))g'(x)$.

1.2.2 Integrals

Just like the concept of differentiation, integration will be of great use throughout the course. Similarly, if we have a function $f(x)$, we can summarize the concept of the integral (or antiderivative) of $f(x)$ in 2 ways:

- $\int f(x)dx$ is a **function** whose derivative is exactly $f(x)$. This concept is known as the indefinite integral.
- $\int_a^b f(x)dx$ is a **number**, which corresponds to the area under the graph of f, starting to measure from $x = a$ and ending at $x = b$. We call this concept the definite integral.

Integration will be tool #1 in solving differential equations. Therefore, it is essential that we master all the integration methods we have learned since the first calculus course.

Exercises

1. Review all integration methods.

As with derivatives, I will give a very brief summary of some indefinite integrals, along with basic integration techniques.

$f(x)$	$\int f(x)dx$
1	$x + C$
A	$Ax + C$
$x^n \quad n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$\ln(x) + C$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\tan(x)$	$-\ln(\cos(x)) + C$
$\cot(x)$	$\ln(\sin(x)) + C$
$\sec(x)$	$\ln(\sec(x) + \tan(x)) + C$
$\csc(x)$	$-\ln(\csc(x) + \cot(x)) + C$
e^x	$e^x + C$
a^x	$\frac{a^x}{\ln(a)} + C$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin(\frac{x}{a}) + C$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan(\frac{x}{a}) + C$

Linearity: If f, g are integrable functions, and $C \in \mathbb{R}$ then

$$\int Cf(x) + g(x)dx = C\int f(x)dx + \int g(x)dx.$$

1.2.3 Integration Methods

Below is a summary of the various integration methods. Once again, for understanding the course, it is essential that the student **master all the methods well**. I recommend solving each of the integrals used as examples.

Substitution

This works when we need to solve an integral of the type

$$\int f(u(t))u'(t)dt.$$

That is, when within the integral, we find an expression whose derivative is also within the integral (in this case u). For example, to solve the integral

$$\int \frac{\ln(t)}{t}dt$$

the substitution $u = \ln(t)$ is very helpful.

Integration by Parts

This helps when we need to integrate a product of functions of the form $u(x)v'(x)$, using the identity:

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx + C$$

Or written more simply:

$$\int udv = uv - \int vdu$$

When facing an integration by parts, the correct choice of u and v is fundamental. A good rule for choosing them is that u should be easy to differentiate and v should be easy to integrate. For example, to evaluate the integral

$$\int xe^{2x}dx$$

by parts, one can take $u = x$ and $dv = e^{2x}$.

Partial Fractions

This method allows us to evaluate integrals of the type

$$\int \frac{P(x)}{Q(x)}dx$$

where $P(x)$ and $Q(x)$ are polynomials. This method has many variants, but the general steps are the same:

- **Step 0:** If the degree of P is greater than the degree of Q, then it is necessary to perform polynomial division. Otherwise, proceed to step 1.
- **Step 1:** Factor $Q(x)$, either using inspection, completing the square, or synthetic division.
- **Step 2:** Use the factorization obtained in step 1 to decompose the fraction to be integrated into a sum of simpler fractions to integrate, usually fractions whose denominator is a polynomial of degree 1 or 2. Remember that it is necessary to solve for the numerators of said fractions, as they appear as unknowns when performing the decomposition.
- **Step 3:** Integrate each fraction separately, using the other known techniques.

For example, to solve

$$\int \frac{1}{x^2-16}$$

we can apply the decomposition

$$\frac{1}{x^2-16} = \frac{1}{(x+4)(x-4)} = \frac{A}{x+4} + \frac{B}{x-4}.$$

Or to solve

$$\int \frac{1}{x(x^2+2x+5)}$$

we can use a variant:

$$\frac{1}{x(x^2+2x+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+5}$$

since the discriminant of the quadratic factor in the denominator is negative.

Throughout the course we will encounter many integrals of this type, so it will be helpful to review all the variants of this method.

Trigonometric Substitution

This type of substitution will be useful mainly in 3 cases:

- For integrals of the form $\int \sqrt{b^2-x^2}$, use the substitution $x = b\sin(\theta)$.
- For integrals of the form $\int \sqrt{b^2+x^2}$, use the substitution $x = b\tan(\theta)$.
- For integrals of the form $\int \sqrt{x^2-b^2}$, use the substitution $x = b\sec(\theta)$.

We perform this review prior to starting the course, because solving differential equations extensively includes solving integrals, so it is very important that we have no problem when integrating functions, as this is not the objective of the course. Later we will revisit more integration methods, but I consider that those presented in this lesson are a good foundation to start the course.

1.3 Basic Concepts of Differential Equations

In this section we will define the concept of a differential equation, and we will also give some definitions that will help us identify the different types of differential equations. The ability to correctly identify the type of equation we are facing is the first step in solving it. First we must answer the question:

What is a differential equation?

Just like the classical equations we studied in school, in differential equations we will be trying to isolate or find the value of an unknown (or several). The main difference is that in classical equations, the value to be found, or **solution**, is a number, while in differential equations, it is a function. An example:

- **Classical equation:** $x^2 + 2x + 1 = 0$ has as solution $x = -1$, a **real number**.
- **Differential equation:** We are asked to find a function $y(x)$ that satisfies the equation $y' = y$. One solution is $y(x) = e^x$, a **function**.

More specifically,

Definition 1.1 Differential Equation.

A **differential equation** is an equation where the following may appear:

- Independent variables, (x,t, ...)
- Dependent variables, which will be the unknowns to be solved. They are usually denoted by y, but we must always remember that it depends on x, so it is actually $y(x)$.
- Derivative (or derivatives) of the dependent variable: y', y'', y''', etc.

Example 1.2

- $y' = e^x$
- $y' + y'' = \cos(x)$
- $x^2y'' + xy + 1 = 0$
- $(y')^2 + \dfrac{1}{2\sin(x)} = \sqrt{xy}$
- $\dfrac{dy}{dx} + \dfrac{d^2y}{dx^2} = y\cos(x)$.

As mentioned earlier, it is convention that y is a function of x, although we could have equations where the independent variable is t, and the dependent one is x, or other cases, for example

- $x' = t$
- $\dfrac{dx}{dt} + \dfrac{d^2x}{dt^2} = x\cos(t)$.
- $z'(v) = z(v) + v^2$

are all differential equations. Little by little we will study methods to solve them.

1.3.1 Classification

The study of differential equations is very broad, and there are many ways to classify and study them. We begin with the first definition:

Definition 1.2 Ordinary Differential Equation (ODE).

An **ordinary** differential equation (ODE) is one where the solution is a function of **one variable.** For example:

$$y'x = 1$$

Is an ODE with solution $y(x) = \ln(x)$.

Definition 1.3 Partial Differential Equation (PDE).

A **partial** differential equation (PDE) is one where the solution is a function of **several variables.** In this case, partial derivatives of the function to be solved also appear. For example:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0$$

has as solution the function of two variables $f(x, y) = x - y$.

During this course, we will focus mainly on the study of ODEs. In the last two weeks we will give some methods to solve the most basic PDEs. Therefore, we can momentarily forget about partial differential equations and work only with ODEs.

A more formal definition of an ordinary differential equation is the following.

Definition 1.4 ODE (Formal).

An ODE is an equation of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

Where $F : \mathbb{R}^{n+1} \to \mathbb{R}$ is a function.

The previous definition simply summarizes more concisely the concept of a differential equation: an expression where we have: variables (x), functions of said variables (y), and their respective derivatives ($y', y'', \dots$).

Next, we will give two definitions that will help us identify the different types of ODEs. In a way, they will serve to classify equations by their difficulty.

Definition 1.5 Order.

The **order** of an ODE is the order of the highest derivative that appears in said equation.

Definition 1.6 Degree.

The **degree** of an ODE is the exponent to which the highest order derivative is raised.

We must not confuse these definitions! The order is about how many times we have differentiated the unknown function, while the degree is simply the exponent to which the highest derivative is raised.

Example 1.3

- $xy' = yx^2$ has order 1 and degree 1.
- $(1 + y')^3 = x$ has order 1 and degree 3.
- $(y'')^3 + (y')^7 = 1 + \ln(x)$ has order 2 and degree 3.
- $y^{(9)} + y^{(8)} + \cdots + y' + y = 0$ has order 9 and degree 1

NOTE: When our equation has radicals, it is necessary to eliminate them to know its degree, for example

$$\sqrt{\frac{dy}{dx}} = y + x$$

Should be rewritten as

$$\frac{dy}{dx} = (y + x)^2.$$

From which we deduce that it is an ODE of order 1 and degree 1.

Some ODEs are presented in **differential form**:

$$M(x, y)dx + N(x, y)dy = 0$$

which may seem unfamiliar. This is simply a rewriting of an ordinary ODE. Let us see an example, the equation

$$(y - x)dx + 4xdy = 0$$

can be transformed into

$$\frac{dy}{dx} = \frac{x-y}{4x}$$

by a simple rearrangement. Note that to perform this rearrangement, it was necessary to move the quantity dx to divide. The formal justification of this fact will not concern us in this course.

Next we will define the concept of a linear equation, which will be extensively studied, as they are among the simplest to solve.

Definition 1.7 Linear Differential Equation.

A **linear** ordinary differential equation is an ODE of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

where $g(x), a_1(x), a_2(x), \ldots, a_n(x)$ are functions that **only depend on** x (they can even be constant functions).

Another way to identify if a differential equation is linear is the following:

1. The unknown y and all its derivatives appear with exponent 1. That is, all linear ODEs have degree 1.
2. The unknown y and all its derivatives appear multiplied **only** by functions of the variable x (or constants).

In a linear ODE, the functions $a_i(x)$ are called **coefficients**.

Example 1.4

- $\cos(x)y^{(3)} + e^x y'' - \frac{y'}{x} + y = \tan(x)$ is a linear ODE of order 3
- $y^{(5)} = y$ is a linear ODE of order 5. Note that all coefficients are constants, many of which are 0.

Note that linear ODEs have a slight resemblance to polynomials, since in general, a polynomial has the form

$$p(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

Only in this case, the coefficients are real numbers, not functions, and the variable t represents a number, not a function as in the case of ODEs.

Types of linear equations: There are two types of linear ODEs, which are very easy to identify, but the solution methods for each are different. Let

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

be a linear ODE. If $g(x) = 0$, we say it is a **homogeneous** equation, otherwise (if $g(x) \neq 0$), it would be **non-homogeneous**.

Example 1.5

- The equation $x^4 y^{(4)} + y'' + xy = 0$ is linear, homogeneous, of order 4.
- The equation $y'' + \sin(x) y = x$ is linear non-homogeneous, of order 2.

Exercises

1. For each of the following ODEs, determine the degree, the order, and if it is linear, specify whether it is homogeneous or not.
   - $v'(t) + \dfrac{v(t)}{5} = \dfrac{t}{5}$
   - $\dfrac{dT}{dt} = 9(200 - T)$
   - $\dfrac{dy}{dx} = \dfrac{-x \pm \sqrt{x^2 + y^2}}{y}$
   - $y'' - (1 - y^2)y' + y = 0$
   - $yy'y''y''' = x$
   - $y^{(50)} = 1$

1.4 Solutions of an ODE

Definition 1.8 Solution.

Let

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0 \tag{1.1}$$

be an ordinary differential equation. A **solution** of (1.1) is a function $f(x)$ that satisfies

$$F\left(x, f(x), f'(x), \ldots, f^{(n)}(x)\right) = 0.$$

This definition may seem a bit redundant, but let us look at it more closely. Let us return to the example of classical equations. If we have, for example, to solve the equation

$$t^4 = 81$$

We are looking for a real number, which when substituted in place of t, makes the equality true. It is easy to see that 3 is a solution, since

$$3^4 = 81.$$

In the case of ODEs the situation is similar, let us see an example. We have the equation

$$y' = x\sqrt{y} \tag{1.2}$$

The proposed solution is $y = \dfrac{x^4}{16}$. To verify that our candidate works, we must substitute it into the equation. Note that unlike the previous example, we need to differentiate y to be able to substitute, otherwise we will have an error.

Note that

$$y' = \frac{x^3}{4}$$

So when substituting into (1.2) it should hold that

$$\frac{x^3}{4} \stackrel{?}{=} x\sqrt{\frac{x^4}{16}}$$

which becomes evident after removing the root from the right side. It is clear then that the function $y(x) = \frac{x^4}{16}$ is a solution of equation (1.2).

Let us see another example:

$$y'' - 2y' + y = 0. \tag{1.3}$$

We propose the solution $y = xe^x$. To verify our solution we need to differentiate 2 times. Let us see that

$$\begin{aligned} y' &= e^x(1+x) \\ y'' &= e^x(2+x) \end{aligned}$$

When substituting into (1.3), we must verify if

$$\begin{aligned} & (2+x)e^x - 2(1+x)e^x + xe^x \stackrel{?}{=} 0 \\ \iff & 2e^x + xe^x - 2e^x - 2xe^x + xe^x \stackrel{?}{=} 0 \\ \iff & 0 \stackrel{?}{=} 0 \end{aligned}$$

Which holds. So the function $y(x) = xe^x$ is a solution of (1.3). Note also that it is very easy to verify that the function $y(x) = 0$ is also a solution.

In mathematics it is not only important to find solutions, it is common to ask ourselves: Are there more solutions? How can we find all the solutions to my problem? The study of differential equations is no exception: a differential equation can have:

- A unique solution, for example the equation $(y')^2 + y^2 = 0$ has as its unique solution $y = 0$.
- Infinitely many solutions, for example the equation $y' = y$ has as solutions $y = e^x, 2e^x, 3e^x, -e^x$ and in general Ce^x where C is any real number.
- Zero solutions, for example, the equation $(y')^2 = -x^2 - 1$ has no solutions that are real functions.

Exercises

1. For each of the following equations, verify whether the proposed function is a solution or not.

   (a) $y(x) = x^2 + C$, for the equation $y' = x$

   (b) $y(x) = x^2 + Cx$, for the equation
   $$x\left(\frac{dy}{dx}\right) = x^2 + y.$$

   (c) $y = A\sin(5x) + B\cos(5x)$, for the equation $y'' + 25y = 0$

   (d) $y(t) = 8t^5 + 3t^2 + 5$, for the equation $\dfrac{d^2y}{dt^2} - 6 = 160t^3$.

We can also ask the inverse question, that is, given a function $y(x)$, is it possible to find some ODE for which $y(x)$ is a solution?

Example 1.6

Find a differential equation whose solution is $y = \sin(x)$.
Solution: Since we must find a differential equation, the idea is to calculate the derivatives of our function, and look for some relationship between them and the function. Observe that $y' = \cos(x)$ and $y''(x) = -\sin(x)$. From this information we see that the second derivative of y is precisely $-y$. That is, y is a solution to the equation

$$y'' = -y.$$

Example 1.7

Find a differential equation whose solution is $y = e^{2x}$.
Solution: Observe that $y' = 2e^{2x}$. We see immediately that a possible equation would be

$$y' = 2y.$$

NOTE: this process can be done in many different ways, and there may be many correct answers, for example, the equation $y''' - 8y = 0$ also has as solution $y = e^{2x}$.

Exercises

1. For each of the following functions, find a differential equation for which they are a solution.

   (a) $y = C_1e^x + C_2e^{-x}$

   (b) $y = \tan(4x + c)$

   (c) $y = (x - C_1)^2 + y^2 + C_2^2$

1.4.1 Types of Solutions

As we mentioned before, an ODE can have zero, one, or infinitely many solutions. We will classify them as follows:

- **General solution:** Describes simultaneously a family of solutions that only differ from each other by a parameter. In other words, the general solution gives us the form of each possible solution. For example
  - The simple equation $y' = 1$ has as general solution $y = x + C$, where $C \in \mathbb{R}$. That is, any value that C takes generates a different solution.
  - The general solution of the equation $x''(t) + 16x(t) = 0$ is the function $x(t) = A\cos(4t) + B\sin(4t)$. In this case we have 2 parameters, A and B.

  In general, the number of parameters that appear in the general solution corresponds to the order of the differential equation.

- **Particular solution:** It is simply a particular case of the general solution, where we give a concrete value to the parameters of the solution (we can even give them the value 0). For example
  - A particular solution of $y' = 1$ is $y = x + 20$.
  - The functions $\cos(4t)$, $5\sin(4t)$ and $2\cos(4t) + 9\sin(4t)$ are all particular solutions of the equation $x''(t) + 16x(t) = 0$.

- **Singular solution:** These are solutions that cannot be obtained from the general solution. That is, no matter what value we give to the parameters, we cannot produce said solution. For example: the ODE $(y')^2 = 4y$ has as general solution $y = (x + C)^2$. All solutions we will obtain by choosing values of C will be parabolas. However, the student can verify that the function $y = 0$ is also a solution of the equation, which does not obey the form dictated by the general solution. It is a singular solution. *Not all differential equations have singular solutions.*

When a solution (whether general, particular, or singular) can be expressed **only in terms of the dependent variable**, we say it is an **explicit** solution. Otherwise, when we cannot isolate the criterion of our solution, we say it is an **implicit solution.**

Example 1.8

- A particular explicit solution of the equation $xy' + y = 0$ would be $y(x) = 1/x$. Note that we can express the criterion of y explicitly as a function of x.
- Consider the equation $\dfrac{dy}{dx} = -\dfrac{x}{y}$. Let us see that an implicit solution is given by $x^2 + y^2 = 25$ (an expression in which it is not possible to isolate y, since we would have two possible isolations $y = \pm\sqrt{25 - x^2}$). To verify that our curve is

indeed a solution, we must resort to **implicit differentiation**. Observe that

$$\begin{aligned} x^2 + y^2 &= 25 \\ \Rightarrow 2x + 2yy' &= 0 \\ \Rightarrow y' &= -\frac{2x}{2y} = -\frac{x}{y} \end{aligned}$$

Another way to define a solution to an ODE is piecewise. For example, consider the equation

$$xy' - 4y = 0$$

whose general solution is $y = cx^4$. We can then define the solution

$$y(x) = \begin{cases} x^4 & \text{if } x > 0 \\ -x^4 & \text{if } x \leq 0 \end{cases}$$

by choosing $c = 1$ on the positive axis and $c = -1$ on the negative axis.

1.5 Initial Value Problems

Suppose we want to solve the equation

$$y' = y.$$

The general solution of this equation is $y(x) = Ce^x$. Here we are actually talking about infinitely many solutions, one for each value of C. Now, we can ask ourselves the following question: Which of all those solutions satisfies $y(0) = 5$? If we know that the solution must have the form Ce^x, we only need to verify if $Ce^0 = 5$. Therefore, we obtain that when $C = 5$, the particular solution $y = 5e^x$ solves our problem.

This type of problem is known as **initial value problems** (or Cauchy problems). Initial value problems are frequently applied in modeling real-life phenomena, as they have a specific solution, not a family of infinitely many solutions. Here is where we return to what was mentioned in section 2: a differential equation can have many solutions, but once we specify its initial value, we obtain a concrete solution, instead of an infinite family. The formal definition of this situation is presented.

Definition 1.9 Initial Value Problem.

An initial value problem is a system of the type

$$\begin{cases} F(x,y,y',\dots,y^{(n)}) & = 0 \\ y(x_0) & = y_0 \\ y'(x_1) & = y_1 \\ y''(x_2) & = y_2 \\ & \vdots \\ y^{(n-1)}(x_{n-1}) & = y_{n-1} \end{cases}$$

where $x_0, x_1, \dots, x_{n-1}, y_0, y_1, \dots, y_{n-1}$ are real numbers. When all the x_i's are equal, we call it a **Cauchy problem**.

In its general form, an initial value problem, in addition to including a differential equation, includes information about all the derivatives of the function. This information will help us give concrete values to each parameter in the general solution. Let us explain this with several examples:

Example 1.9

Consider the problem

$$\begin{cases} y' + y & = x \\ y(0) & = 1 \end{cases}$$

The general solution of the equation is $y(x) = Ce^{-x} + x - 1$. Now, to ensure that $y(0) = 1$ we must choose an appropriate value for the parameter C. We need

$$\begin{aligned} Ce^0 + 0 - 1 &= 1 \\ \Rightarrow \qquad C - 1 &= 1 \\ \Rightarrow \qquad C &= 2 \end{aligned}$$

Therefore, the solution of our initial value problem is $y(x) = 2e^{-x} + x - 1$.

Example 1.10

Consider the problem

$$\begin{cases} y'' + 2y' + y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0 \end{cases}$$

The general solution of the equation is $y(x) = C_1 e^{-x} + C_2 x e^{-x}$, it has 2 parameters since it is a second order equation. Now, to ensure that $y(0) = 1$ and $y'(0) = 0$ we must solve for both parameters. First, since $y(0) = 1$, that implies

$$\begin{aligned} C_1 e^0 + C_2 0 e^0 &= 1 \\ \Rightarrow \qquad\qquad C_1 &= 1 \end{aligned}$$

Now, we calculate $y'(x) = -e^{-x} + C_2 e^{-x}(1-x)$ (where we already used that $C_1 = 1$). Now, since $y'(0) = 0$, that implies

$$\begin{aligned} -e^0 + C_2 e^0 (1-0) &= 0 \\ \Rightarrow \qquad\qquad -1 + C_2 &= 0 \\ \Rightarrow \qquad\qquad C_2 &= 1 \end{aligned}$$

So finally, the solution to our problem is $y(x) = e^{-x} + xe^{-x}$. Note that this last problem is a Cauchy problem, since both y and y' appear evaluated at $x = 0$ in the initial conditions.

Example 1.11

Consider the problem

$$\begin{cases} y'' + 9y &= 0 \\ y\left(\frac{\pi}{12}\right) &= 0 \\ y'\left(\frac{\pi}{9}\right) &= 1 \end{cases}$$

The general solution is $y = A\sin(3x) + B\cos(3x)$, once again we have 2 parameters. Let us calculate at once $y'(x) = 3A\cos(3x) - 3B\sin(3x)$. Note that this problem is an initial value problem, but not a Cauchy problem, since the initial conditions are evaluated at different points ($\pi/12$ and $\pi/9$). First, since $y\left(\frac{\pi}{12}\right) = 0$, we obtain that

$$A\sin\left(\frac{3\pi}{12}\right) + B\cos\left(\frac{3\pi}{12}\right) = A\frac{\sqrt{2}}{2} + B\frac{\sqrt{2}}{2} = 0$$

while with the other condition $y'\left(\frac{\pi}{9}\right) = 1$ we obtain that

$$3A\cos\left(\frac{3\pi}{9}\right) - 3B\sin\left(\frac{3\pi}{9}\right) = A\frac{3}{2} - B\frac{3\sqrt{3}}{2} = 1$$

To solve for A and B, we must then solve a system of equations:

$$\begin{cases} A\frac{\sqrt{2}}{2} + B\frac{\sqrt{2}}{2} = 0 \\ A\frac{3}{2} - B\frac{3\sqrt{3}}{2} = 1 \end{cases}$$

which should not represent any difficulty for the student. The solution is

$$A = \frac{2}{3+3\sqrt{3}} \quad ; \quad B = -\frac{2}{3+3\sqrt{3}}$$

so the solution to the problem is

$$y(x) = \frac{2}{3+3\sqrt{3}}(\sin(3x) - \cos(3x)).$$

Exercises

1. Several initial value problems are presented, with their respective general solutions. Find the value of the parameters in each case, to solve the problem.

   (a) $\begin{cases} y' - 2y = 3x \\ y(0) = 1 \end{cases}$, with general solution $y = Ce^{2x} - \dfrac{3x}{2} + \dfrac{3}{4}$.

   (b) $\begin{cases} y'' - 6y' + 5y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$, with general solution $y = C_1e^x + C_2e^{5x}$.

   (c) $\begin{cases} y'' + 25y = 0 \\ y\left(\frac{\pi}{15}\right) = 1 \\ y'\left(\frac{\pi}{20}\right) = 0 \end{cases}$, with general solution $y = A\sin(5x) + B\cos(5x)$.

   (d) $\begin{cases} y''' = 8 \\ y(0) = 1 \\ y'(1) = 0 \\ y''(0) = 0 \end{cases}$, with general solution $y = \dfrac{4x^3}{3} + Ax^2 + Bx + C$.

We conclude this section with an important theorem, which tells us when an initial value problem has a unique solution.

Theorem 1.1 Existence and Uniqueness.

Let

$$\begin{cases} y' = F(x, y) \\ y(x_0) = y_0 \end{cases}$$

be a Cauchy problem. If the function F and all its partial derivatives are continuous, then there exists a unique solution for said problem.

This theorem is very easy to apply:

Example 1.12

For the problem

$$\begin{cases} y' = y + x\sin(y) \\ y(x_0) = y_0 \end{cases}$$

We have $F(x, y) = y + x\sin(y)$, which is clearly a continuous function. Let us also see that

$$\frac{\partial F}{\partial x} = \sin(y)$$

and also

$$\frac{\partial F}{\partial y} = 1 + x\cos(y).$$

Both partial derivatives of F are continuous, so, although we do not know how to solve the system, we can assure thanks to the theorem that there is a unique solution.

1.6 Direction Fields and Isoclines

Our last section before starting in earnest with solution methods corresponds to a more geometric interpretation of differential equations. Consider for example the differential equation

$$f'(x) = f(x)$$

Geometrically, the previous equation tells us that for every point x, the slope of the tangent line to the curve of $f(x)$ is precisely $f(x)$. That is, it is not necessary to solve the equation to have an idea of what its graph looks like. Unfortunately, since we do not have initial conditions, we will not obtain a single curve, but many.

Visually, these curves coincide with the general solution of the equation: $f(x) = Ce^x$, a family of exponential functions. Although at the beginning of the problem we did not have the solution, by drawing many tangent segments, we can get an idea of what the solution curve looks like, this is the main idea behind the concept of direction fields.

Definition 1.10 Direction Field.

Let

$$y' = F(x, y)$$

be a differential equation. A **direction field** (or slope field) is a family of tangent lines, each passing through each point (a, b) of the domain of $F(x, y)$, that is, they are **all possible tangent lines of all particular solutions**. An **isocline** is a curve of the form $F(x, y) = C$, where C is a constant.

Geometrically, to obtain an isocline of an equation, we first take any constant C, and in the direction field, we look for all the points where the slope is precisely C. The collection of these points is an isocline of the equation. Obviously, there are infinitely many isoclines (one for each C).

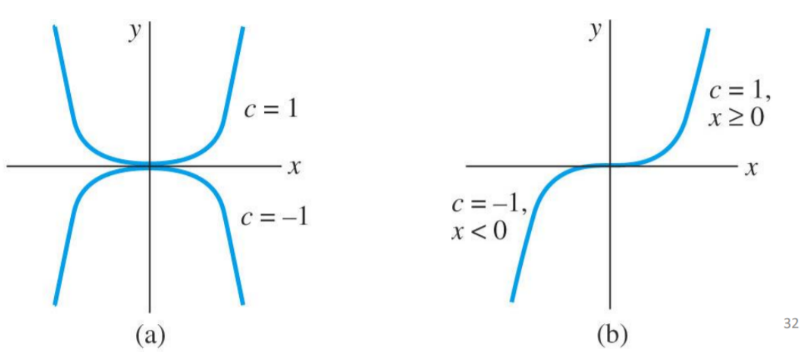

Figure 1.1: Solution curves of a differential equation. Solutions for $c = 1$ and $c = -1$ are shown.

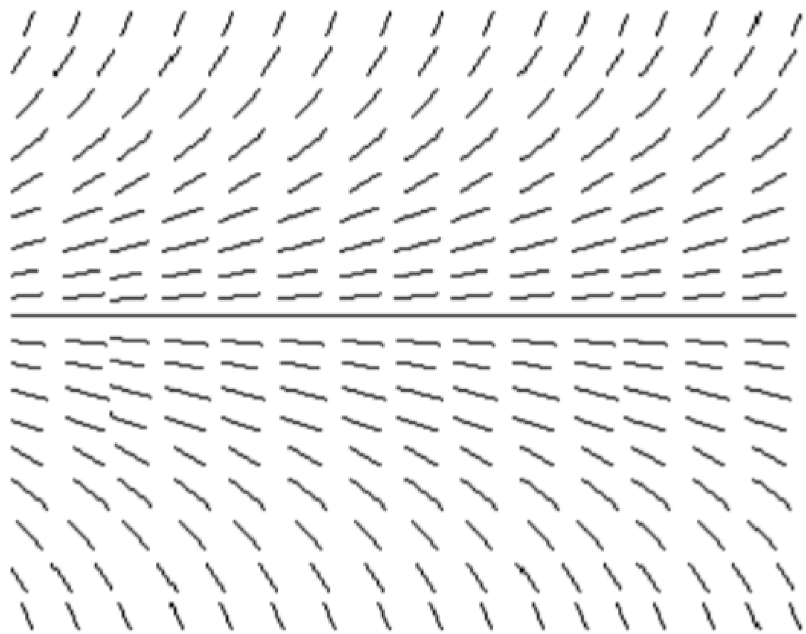

Figure 1.2: Direction field of a differential equation. Each segment shows the slope of the solution at that point.

Example 1.13

Consider the differential equation

$$y' = -\frac{x}{y}$$

To calculate its direction field, we simply take many points in the plane (x_0, y_0) and calculate the slope that the tangent curve to the solution passing through said point (x_0, y_0) would have using the formula $-x_0/y_0$. The more points we calculate, the better resolution we will have of the field. A table with some values is attached:

x_0	y_0	**slope** $= -\frac{x_0}{y_0}$
1	1	-1
-1	1	1
1	-1	1
-1	-1	-1
0	1	0 (horizontal)
1	0	∞ (vertical)

We also see that, without having solved the equation, the graphs of the solutions appear to be circles! Indeed, as seen in a previous example, the general solution of this equation is given implicitly by the equation

$$x^2 + y^2 = K.$$

By changing the value of K, different solutions (different circles) are obtained.

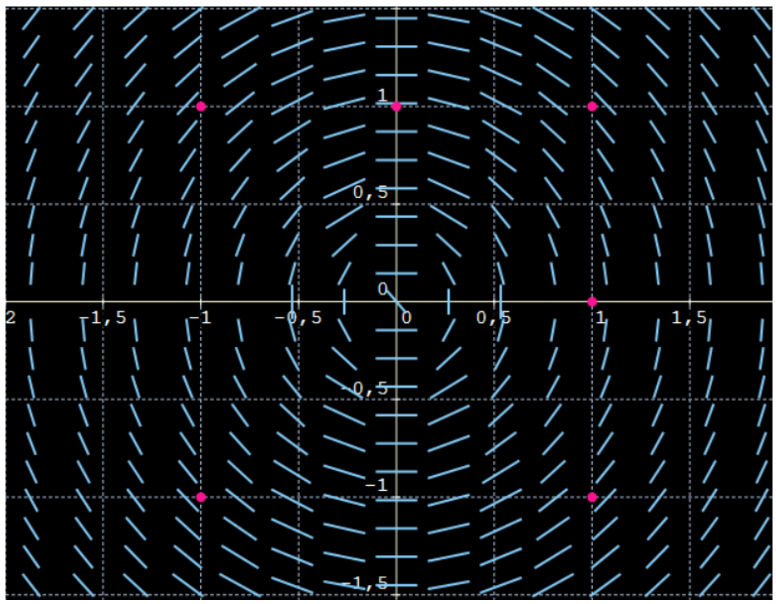

Figure 1.3: Direction field of the equation $y' = -\frac{x}{y}$. The solution curves are circles centered at the origin.

Now, remember that the isoclines of an equation are given by the family $F(x, y) = C$. For this specific case, the family of isoclines is

$$-\frac{x}{y} = C.$$

That is, the family of isoclines are all the lines that pass through the origin.
Geometrically, returning to the image, we could ask ourselves: In the direction field, which points have **horizontal** slope (i.e., slope $C = 0$)? Thanks to the image, it is easy to observe that only those points that are on the y axis satisfy this. Equivalently, given our family of isoclines, we look for those points whose slope is $C = 0$, that is, that satisfy the equation

$$-\frac{x}{y} = 0.$$

So $x = 0$ (the y axis) are all the points we are looking for. Similarly, if now $C = 1$, the isocline of points with slope 1 is precisely the line $y = -x$. **Note:** Isoclines are NOT solutions of the differential equation.

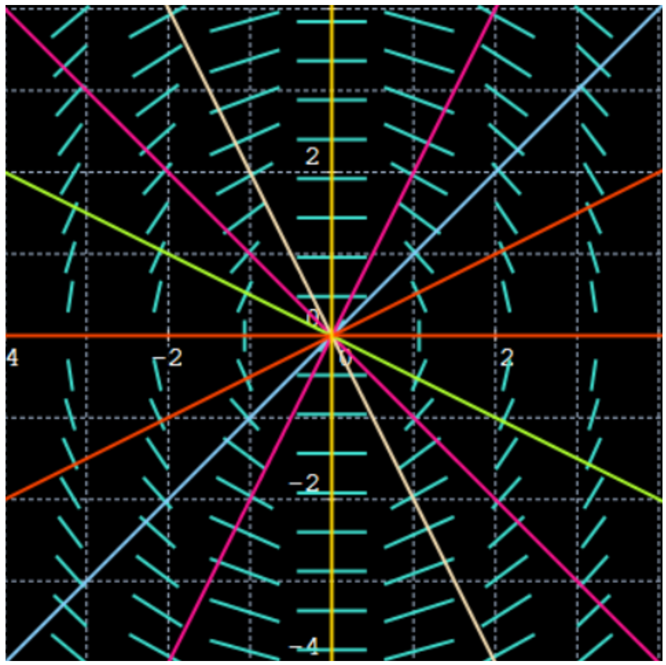

Figure 1.4: Direction field of $y' = -\frac{x}{y}$ with isoclines (lines through the origin) superimposed.

Example 1.14

The equation

$$y' = e^{-x^2}$$

cannot be solved in terms of elementary functions (although it does have a solution, namely $y(x) = \int_0^x e^{-t^2} dt$). However, using the direction field, we can get a visual idea of what its general solution looks like.

x_0	y_0	**slope** $= e^{-x_0^2}$
-1	0	$e^{-1} \approx 0.368$
0	0	1
1	0	$e^{-1} \approx 0.368$
0	1	1
0	-1	1

Furthermore, the family of isoclines is given by $e^{-x^2} = C$, from which we obtain

$$x = \sqrt{-\ln C}.$$

That is, all isoclines are vertical lines. In this image it is very easy to see that on the isoclines, all the small line segments that pass through have the **same inclination**. Hence the name.

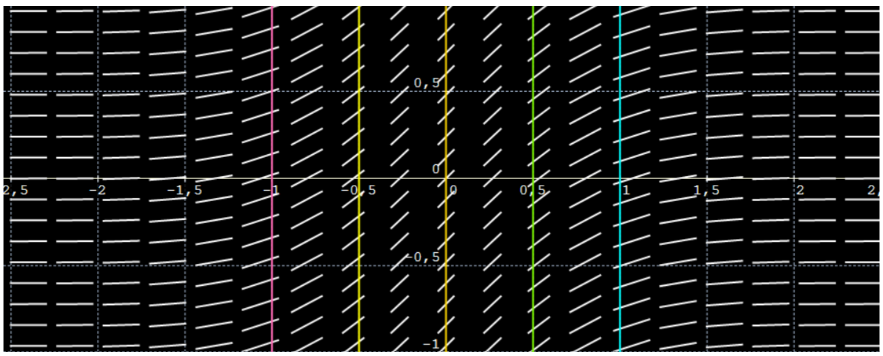

Figure 1.5: Direction field of $y' = e^{-x^2}$ with vertical isoclines.

Definition 1.11 Integral Curve.

An **integral curve** (or solution curve) is simply the graph of a particular solution of a differential equation.

Example 1.15

Consider the differential equation

$$yy' = \cos(x)$$

whose solution is given implicitly by $y^2 = 2(C + \sin(x))$.

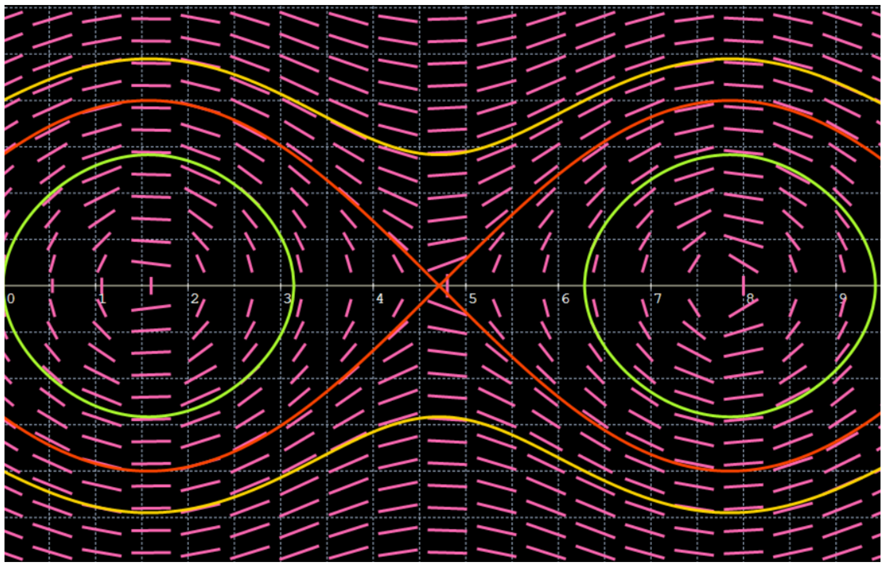

Figure 1.6: Direction field and integral curves of the equation $yy' = \cos(x)$.

In summary, we can say that the **direction field** is an idea of what all the solutions of a given differential equation look like (it is visualized as a set of line segments, which give an idea of each of the solutions). **Isoclines** are curves whose points have the same slope assigned. Finally, **integral curves** are the graphs of the particular solutions to the equation.

Now that we have all the basic concepts of ODEs, we can present the different methods for solving differential equations. That is, we are going to answer the question:

Given a differential equation, how can we find its general solution?

2

First Order Equations

2.1 Separation of Variables Method

The first method is known as separation of variables, and it will allow us to solve any equation of the form

$$f(x)dx = g(y)dy.$$

That is, if by means of algebraic manipulations (treating the differentials dx and dy as variables), we manage to have on one side of the equation only x and on the other side only y, this method will work. Once this form is achieved, we simply integrate on both sides:

$$\int f(x)dx = \int g(y)dy.$$

Example 2.1

To solve the equation

$$\frac{dy}{dx} = \frac{x^2}{y(1+x^3)}$$

We can easily rearrange to obtain

$$ydy = \frac{x^2}{1+x^3}dx.$$

So we only have to integrate both sides to obtain

$$\int y dy = \int \frac{x^2}{1+x^3} dx$$

This integral can be solved using the substitution $u = (1 + x^3)$. Finally, the general solution is obtained implicitly:

$$\frac{y^2}{2} = \frac{\ln(1+x^3)}{3} + C$$

Where $C \in \mathbb{R}$ is a parameter.

NOTE: Strictly speaking, we should obtain 2 constants of integration, one on each side of the equality, obtaining the general solution

$$\frac{y^2}{2} = \frac{\ln(1+x^3)}{3} + C_1 - C_2$$

However, since both C_1 and C_2 are any real numbers, the quantity $C_1 - C_2$ will also be any real number, so we summarize it simply as a new parameter C. We will abuse this fact a lot throughout the resolution of ODEs and initial value problems.

Example 2.2

We can also solve initial value problems:

$$x dx + y e^{-x} dy = 0 \quad , \quad y(0) = 1.$$

First we must find the general solution, to be able to solve for the parameter. Separating the variables we obtain

$$y dy = -x e^{x} dx$$

and integrating on both sides we get the (implicit) solution

$$\frac{y^2}{2} = e^x(1-x) + C$$

or equivalently

$$y^2 = 2e^x(1-x) + C$$

where we again abuse "$2C = C$". Now we just need to solve using the initial condition. Since $y(0) = 1$, we need

$$1 = 2e^0(1-0) + C$$

from which we deduce that $C = -1$. The solution to the problem is then

$$y^2 = 2e^x(1-x) - 1$$

Exercises

Find the general solution of the following ODEs

1. $e^x y' = 2x$
2. $dy + 2xydx = 0$
3. $\dfrac{dQ}{dt} = 300(Q - 70)$
4. $\dfrac{1+x^2}{\sqrt{1-y^2}}dy = \dfrac{dx}{\arcsin(y)}$
5. $y' + y\tan(x) = 0$
6. $\sec^2 x dy + \csc y dx = 0$

Summary: This method simply reduces to "separating" the variables on each side of the equation, with their respective dx and dy, and then integrating both sides. The greatest difficulty that can arise is a complicated integral at the end, which is why we must master the integration methods.

2.2 Variable Substitutions

Just as when we studied integration methods, in solving differential equations, a good substitution can turn a problem that seemed impossible into a simpler one. Substitutions must always be made respecting the chain rule. Let us see an example.

Example 2.3

The equation

$$\frac{dy}{dx} = \frac{1}{x+y}$$

is not separable. Consider now the substitution $z = x + y$. Differentiating (just as would be done in the substitution of an integral)

$$dz = dx + dy \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$$

which means, that in this particular case, we can rewrite the equation only in terms of x and $z(x)$:

$$\frac{dz}{dx} - 1 = \frac{1}{z}$$

which is separable. Upon separating its variables we obtain

$$\frac{z}{1+z}dz = dx.$$

Integrating on both sides we obtain the implicit solution but in terms of $z(x)$.

$$z - \ln(1+z) = x + C$$

so we just need to undo the substitution

$$x + y - \ln(1 + x + y) = x + C.$$

The general solution is given implicitly:

$$y - \ln(1 + x + y) = C.$$

Example 2.4

Sometimes it is necessary to perform more than one substitution. Consider the equation

$$xy' = y\cos(xy)$$

which we rewrite as

$$xdy = y\cos(xy)dx.$$

Let us begin by first taking $x = e^t$ (this will change the independent variable, now it will be t). We then have $dx = e^t dt$ so when substituting into the equation we obtain

$$e^t dy = y\cos(ye^t)e^t dt$$

which simplifies to

$$dy = y\cos(ye^t)dt.$$

This equation is still not separable, so now we consider the substitution $u = ye^t$. Differentiating (using the product rule) we have

$$\begin{aligned} du &= e^t dy + ye^t dt \\ &= e^t dy + udt \\ \Rightarrow dy &= (du - udt)e^{-t}. \end{aligned}$$

So our equation can be expressed in terms of $u(t)$ and t

$$(du - udt)e^{-t} = ue^{-t}\cos(u)dt$$

which is separable. We are not interested for now in the rest of the solution, since the integral that results at the end cannot be expressed elementarily.

When making substitutions we must always be attentive to whether our variables are dependent or independent. Remember that since we work with **ordinary** equations, we can only have at all times **one** independent variable, and **one** dependent (and its derivatives).

NOTE: The substitution from the previous example actually works for any equation of the form

$$xy' = yf(xy)$$

2.2.1 Linear Substitution

This substitution is used to solve equations of the form

$$y' = f(ax + by + c)$$

The substitution to use is $z = ax + by + c$, from which we deduce that $dz = adx + bdy$.

Example 2.5

$$y' = \tan(x + y + 3).$$

Let $z = x + y + 3$, which implies that $dz = dx + dy$. Then, remembering that our equation can be seen as $dy = \tan(x + y + 3)dx$, upon substituting we obtain

$$\begin{aligned} (dz - dx) &= \tan(z)dx \\ \Rightarrow \quad dz &= (1 + \tan(z))dx \end{aligned}$$

Therefore, our last obstacle to solving this equation is solving the integral

$$\int \frac{dz}{1 + \tan(z)}.$$

I recommend the substitution $u = \tan(z)$, followed by a partial fraction decomposition. The general solution of the equation is

$$\ln(\tan(x + y + 3) + 1) + y + \ln(\cos(x + y + 3)) = x + C$$

2.3 Equations Containing Homogeneous Functions

Before proceeding with this method, we are going to define the concept of a **homogeneous function**. This should NOT be confused with the concept of a homogeneous linear ODE, as they are completely different.

Definition 2.1 Homogeneous Function.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of 2 variables. We say that $f(x, y)$ is **homogeneous of degree** k if for all $t \in \mathbb{R}$ it holds that

$$f(tx, ty) = t^k f(x, y).$$

Example 2.6

- The function $f(x,y) = x^2 + y^2$ is homogeneous of degree 2, since if $t \in \mathbb{R}$, we have
$$f(tx,ty) = (tx)^2 + (ty)^2 = t^2x^2 + t^2y^2 = t^2(x^2+y^2) = t^2 f(x,y)$$
- The function $f(x,y) = \dfrac{x^3+y^3}{x^3-y^3}$ is homogeneous of degree 0, since if $t \in \mathbb{R} \setminus \{0\}$, we have
$$f(tx,ty) = \frac{(tx)^3+(ty)^3}{(tx)^3-(ty)^3} = \frac{t^3(x^3+y^3)}{t^3(x^3-y^3)} = \frac{x^3+y^3}{x^3-y^3} = t^0 f(x,y)$$
- The function $f(x,y) = x^4y^4$ is homogeneous of degree 8, since if $t \in \mathbb{R}$, we have
$$f(tx,ty) = (tx)^4(ty)^4 = t^8(xy)^4 = t^8 f(x,y)$$
- The function $f(x,y) = \cos(x+y)$ is not homogeneous, since there is no number k such that
$$\cos(t(x+y)) \neq t^k \cos(x+y)$$

Exercises

1. Verify if the function $g(x,y) = \dfrac{1}{\sqrt{x+y}}$ is homogeneous. Find its degree.

In this subsection we will see the method of solution for equations that have the form
$$y' = f(x,y) \tag{2.1}$$
where f is a homogeneous function **of degree 0**. Or equivalently, equations of the form
$$M(x,y)dx + N(x,y)dy = 0$$
where $M(x,y)$ and $N(x,y)$ are homogeneous functions **of the same degree**.

The substitution we will use to solve (2.1), when $f(x,y)$ is homogeneous of degree 0 will be
$$u = \frac{y}{x}$$
or equivalently $y = ux$, so
$$dy = udx + xdu.$$

This converts our equation into

$$udx + xdu = f(x, ux)dx$$

which by homogeneity becomes

$$udx + xdu = f(1, u)dx$$

a separable equation.

Example 2.7

Solve the equation

$$y' = \frac{x^2 + 3y^2}{2xy}.$$

Observe that the right side of the equation is a homogeneous function of degree 0. Therefore taking $y = ux$ and $dy = udx + xdu$ we obtain

$$\begin{aligned} xdu + udx &= \frac{x^2 + 3(ux)^2}{2x(ux)}dx \\ &= \frac{1 + 3u^2}{2u}dx \end{aligned}$$

which, after separating variables, becomes

$$\begin{aligned} xdu &= \left(\frac{1 + 3u^2}{2u} - u\right) dx \\ \Rightarrow \quad xdu &= \left(\frac{1 + u^2}{2u}\right) dx \\ \Rightarrow \frac{2u}{1 + u^2} &= \frac{dx}{x} \end{aligned}$$

Integrating on both sides, we obtain

$$\ln(1 + u^2) = \ln(x) + C.$$

Undoing the substitution, we obtain the general solution

$$\ln\left(1 + \frac{y^2}{x^2}\right) = \ln(x) + C$$

which can be simplified by applying the exponential function to both sides (and using the abuse "$e^C = C$")

$$1 + \frac{y^2}{x^2} = Cx \quad , C > 0$$

Example 2.8

Consider the differential equation

$$xdy - (\sqrt{y^2 - x^2} + y)dx = 0.$$

We are facing an equation of the form $M(x,y)dx + N(x,y)dy = 0$. It is easy to verify that both $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree (1). Applying the substitution $xu = y$, we obtain

$$x(udx + xdu) - (\sqrt{x^2u^2 - x^2} + ux)dx = 0$$

which, upon dividing everything by x becomes

$$udx + xdu - (u + \sqrt{u^2 - 1})dx = 0$$

which is a separable equation. After combining like terms and separating the variables, we obtain

$$\frac{du}{\sqrt{u^2 - 1}} = \frac{dx}{x}.$$

Integrating on both sides, we obtain the solution

$$\ln(\sqrt{u^2 - 1} + u) = \ln(x) + C.$$

Finally, undoing the substitution and exponentiating both sides, we obtain the general solution.

$$\sqrt{\frac{y^2}{x^2} - 1} + \frac{y}{x} = Cx \quad , C > 0.$$

As an exercise, it is recommended to complete the details in this example, including the integral.

Exercises

Solve the following initial value problems.

1. $y' = \dfrac{y(2xy+1)}{x(xy-1)}$, with the condition $y(1) = 1$. Use the substitution $z = xy$.
2. $y' = \sin(-x + y - 2\pi)$, with the condition $y(0) = 0$
3. $(x + y)dx + xdy = 0$, with the condition $y(1) = 0$.
4. $y' = \dfrac{y - x}{y + x}$, with the condition $y(1) = 1$.
5. $\left(y + x\cot\dfrac{y}{x}\right) - xdy = 0$, with the condition $y(1) = -\pi$.
6. $\dfrac{x+y+1}{x-y-1}$, with the condition $y(1) = 1$.

2.4 Exact Differential Equations

A first order ODE in the form

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be **exact** if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Example 2.9

The equation

$$(2xy - 9x^2)dx + (2y + x^2 + 1)dy = 0$$

is exact. For taking $M(x,y) = 2xy - 9x^2$ and $N(x,y) = 2y + x^2 + 1$, we can note that

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Note: Remember that to calculate the partial derivative with respect to a variable, we treat the other variables as constants.

Example 2.10

The equation

$$\cos(x+y)dx + ydy = 0$$

is not exact, since if $M(x,y) = \cos(x+y)$ and $N(x,y) = y$, we see that

$$\frac{\partial M}{\partial y} = -\sin(x+y) \neq 1 = \frac{\partial N}{\partial x}.$$

To solve an exact differential equation, we have to find a **potential function**, that is, a function $F(x,y)$ that satisfies

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad , \quad \frac{\partial F(x,y)}{\partial y} = N(x,y).$$

Once such a function is found, the general solution is given simply by

$$F(x,y) = C$$

where C is the parameter.

Example 2.11

Let us solve step by step the equation

$$(2xy^2+4)dx - 2(3-x^2y)dy = 0.$$

If we wish to use this method, we must first verify that it is an exact equation. Let us therefore take $M(x,y) = (2xy^2+4)$ and $N(x,y) = -2(3-x^2y)$. We then have

$$\frac{\partial M}{\partial y} = 4xy$$

while

$$\frac{\partial N}{\partial x} = -2(-2xy) = 4xy.$$

So our derivatives match. What remains is to find our potential function. We are looking for some function $F(x,y)$ that satisfies

$$\begin{cases} \frac{\partial F(x,y)}{\partial x} = M(x,y) = 2xy^2+4 \\ \frac{\partial F(x,y)}{\partial y} = N(x,y) = -2(3-x^2y) \end{cases}$$

To find it, we must integrate twice, once with respect to x and another with respect to y.

$$\begin{aligned} \frac{\partial F(x,y)}{\partial x} &= 2xy^2+4 \\ \Rightarrow \int \frac{\partial F(x,y)}{\partial x} dx &= \int (2xy^2+4)dx \\ \Rightarrow \qquad F(x,y) &= x^2y^2+4x+C(y) \end{aligned}$$

In the integral on the right, since we work with respect to x, we treat y as a number, so it comes out of the integral. Also, note that instead of a constant of integration, we add a function that only depends on y. This is because, when differentiating the entire expression again with respect to x, we have $\frac{\partial C(y)}{\partial x} = 0$, so $C(y)$ actually behaves analogously to the constant of integration. Similarly, integrating the second equality,

$$\begin{aligned} \frac{\partial F(x,y)}{\partial y} &= -2(3-x^2y) \\ \Rightarrow \int \frac{\partial F(x,y)}{\partial y} dy &= \int -2(3-x^2y)dy \\ \Rightarrow \qquad F(x,y) &= -6y+x^2y^2+K(x) \end{aligned}$$

where once again, $K(x)$ is the "constant of integration" which actually depends on x (since this time we integrated with respect to y). We therefore have enough information about $F(x,y)$:

$$F(x,y) = x^2y^2 + 4x + C(y)$$
$$F(x,y) = x^2y^2 + K(x) - 6y$$

From here, we can see by inspection that $K(x) = 4x$ and that $C(y) = -6y$. That is, while each of the integrals does not show us who F is, by combining the information, we arrive at the function, so $F(x,y) = x^2y^2 + 4x - 6y$. Finally, the general solution of the equation is $F(x,y) = C$, that is

$$x^2y^2 + 4x - 6y = C$$

Example 2.12

Solve the following initial value problem

$$(3y^3e^{3xy} - 1)dx + (2ye^{3xy} + 3xy^2e^{3xy})dy = 0\ ;\ y(0) = 1$$

We then have $M(x,y) = 3y^3e^{3xy} - 1$ and $N(x,y) = 2ye^{3xy} + 3xy^2e^{3xy}$. Differentiating, we obtain

$$\frac{\partial M}{\partial y} = 9y^2e^{3xy} + 9xy^3e^{3xy} = \frac{\partial N}{\partial x}$$

so our equation is exact. We now have to find $F(x,y)$ such that

$$\begin{cases} \dfrac{\partial F(x,y)}{\partial x} = 3y^3e^{3xy} - 1 \\[2ex] \dfrac{\partial F(x,y)}{\partial y} = 2ye^{3xy} + 3xy^2e^{3xy} \end{cases}$$

for which we integrate with respect to x:

$$\begin{aligned} F(x,y) &= \int (3y^3e^{3xy} - 1)dx \\ \Rightarrow \qquad &= y^2e^{3xy} - x + C(y). \end{aligned}$$

Now we can integrate with respect to y, but instead, we will use a shortcut. We know [a] that $F_y = N(x,y)$. That is, if we differentiate the expression we have for F with respect to y, we should obtain $N(x,y)$, in other words

$$\begin{aligned} \frac{\partial}{\partial y}\left(y^2e^{3xy} - x + C(y)\right) &= 2ye^{3xy} + 3xy^2e^{3xy} \\ \Rightarrow 2ye^{3xy} + 3xy^2e^{3xy} + C'(y) &= 2ye^{3xy} + 3xy^2e^{3xy} \\ \Rightarrow \qquad C'(y) &= 0 \end{aligned}$$

This tells us that $C(y)$ is actually a constant, say $C(y) = k$. We then have our potential function, without having done the second integral with respect to y.

$$F(x,y) = y^2 e^{3xy} - x + k.$$

The general solution to the problem is

$$y^2 e^{3xy} - x = C.$$

where we once again use the abuse "$C - k = C$". We just need to solve the initial condition: since $y(0) = 1$, we have

$$1e^0 - 0 = C \Rightarrow C = 1$$

so the solution to the problem is finally

$$y^2 e^{3xy} - x = 1.$$

[a]Recall the abbreviation $F_z = \dfrac{\partial F}{\partial z}$

2.5 Solution by Integrating Factors

Not all equations of the form

$$M(x,y)dx + N(x,y)dy = 0$$

are exact. However, sometimes it is possible to multiply the entire equation by some function $\mu(x,y) \neq 0$ so that the new equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact. When this is possible, we say that the function $\mu(x,y)$ is an **integrating factor** of the differential equation.

Example 2.13

The equation

$$\left(1 + \frac{y^2}{x}\right) - 2ydy = 0$$

is not exact, since if $M(x,y) = \left(1 + \frac{y^2}{x}\right)$ and $N(x,y) = -2y$, we easily see that $M_y = \frac{2y}{x} \neq 0 = N_x$. However, if we multiply by $\mu(x) = \frac{1}{x}$ (the integrating factor can even be a function of one variable), we obtain the equation

$$\left(\frac{1}{x} + \frac{y^2}{x^2}\right) - \frac{2y}{x}dy = 0$$

in which, if we now take $P = \frac{1}{x} + \frac{y^2}{x^2}$ and $Q = -\frac{2y}{x}$, we see that

$$P_y = \frac{2y}{x^2} = Q_x,$$

which implies that our new equation is exact. We then have that the potential function is

$$F(x,y) = \int P dx = \ln(x) - \frac{y^2}{x} + C(y)$$

and since $F_y = Q$, differentiating this last expression with respect to y, we obtain

$$-\frac{2y}{x} + C'(y) = -\frac{2y}{x} \Rightarrow C'(y) = 0$$

meaning that $C(y) = C$ (is constant). After simplifying the additional constants, the general solution of the equation is

$$\ln(x) - \frac{y^2}{x} = C.$$

In general, finding an integrating factor is difficult, as there are not many formulas that help us find them. However, we have a resource that works for certain types of differential equations. Suppose we wish to solve

$$M(x,y)dx + N(x,y)dy = 0$$

using some integrating factor $\mu(x,y)$. Then, it holds that

- If the function $\varphi = \dfrac{M_y - N_x}{N}$ is only a function of x, then an integrating factor for the differential equation is $\mu(x) = e^{\int \varphi(x)}$.
- If the function $\psi = \dfrac{N_x - M_y}{M}$ is only a function of y, then an integrating factor for the differential equation is $\mu(y) = e^{\int \psi(y)}$.

Example 2.14

Find the general solution of the equation

$$(y\ln(y) + ye^x)dx + (x + y\cos y)dy = 0.$$

This equation is not exact. Let us try to calculate an integrating factor taking $M = (y\ln(y) + ye^x)$ and $N = (x + y\cos y)$. We then have

$$M_y = \ln(y) + e^x + 1, \quad N_x = 1.$$

We can then observe that the function

$$\psi = \frac{N_x - M_y}{M} = \frac{\ln(y) - e^x}{y(\ln(y) + e^x)} = \frac{-1}{y}$$

only depends on y. Therefore, our integrating factor is

$$\mu(y) = e^{-\int \frac{1}{y}dy} = e^{-\ln y} = \frac{1}{y}.$$

Multiplying the original equation by μ, we obtain the exact equation

$$(\ln(y) + e^x)dx + \left(\frac{x}{y} + \cos(y)\right)dy = 0$$

which we can now solve with our tools. The general solution of the equation is

$$e^x + x\ln(y) + \sin(x) = C.$$

2.6 First Order Linear Equations

Recall that a first order linear ODE has the form

$$a(x)y' + b(x)y = c(x)$$

where $a(x) \neq 0$. We can even divide the entire equation by $a(x)$ to arrive at one of the form

$$y' + p(x)y = q(x). \tag{2.2}$$

In this section we will give a **general formula** to solve any equation of this type. First note that we can rewrite equation (2.2) as

$$(p(x)y - q(x))dx + dy = 0.$$

Now taking $M = p(x)y - q(x)$ and $N = 1$, we can see that

$$\frac{M_y - N_x}{N} = \frac{p(x)}{1} = p(x)$$

only depends on x. We can take the integrating factor $\mu(x) = e^{\int p(x)dx}$. Thus from (2.2) we obtain the equation

$$y'e^{\int p(x)dx} + p(x)ye^{\int p(x)dx} = q(x)e^{\int p(x)dx}$$

whose left side is simply the derivative of μy:

$$(ye^{\int p(x)dx})' = q(x)e^{\int p(x)dx}.$$

This equation is solved simply by integrating both sides with respect to x (it can also be done by separating variables), to obtain

$$\begin{aligned} \int (\mu(x)y(x))'dx &= \int q(x)\mu(x)dx \\ \Rightarrow \qquad \mu(x)y &= \int q(x)\mu(x)dx + C \end{aligned}$$

Therefore, we have shown that the general solution of equation (2.2) is

$$y = \frac{1}{\mu(x)}\left(\int q(x)\mu(x)dx + C\right)$$

where C is a parameter and $\mu(x) = e^{\int p(x)dx}$.

Example 2.15

To solve the equation $y' + y\tan(x) = x^2\cos(x)$, we simply take $p(x) = \tan(x)$, $q(x) = x^2\cos(x)$. We calculate

$$\mu(x) = e^{\int p(x)dx} = e^{-\ln(\cos(x))} = \sec(x).$$

Then our general solution is given by

$$y = \frac{1}{\sec(x)}\left(\int x^2\cos(x)\sec(x)dx + C\right) = \cos(x)\left(\frac{x^3}{3} + C\right).$$

The general formula for linear equations is a very useful tool, as it saves us all the work of finding integrating factors. To finish the section, we will see the method of solution for two more types of equations, in which a substitution leads us to a linear equation.

2.6.1 Bernoulli Equation

These are ODEs of the form

$$y' + p(x)y = q(x)y^n, \quad \text{with } n \in \mathbb{R}$$

which can be rewritten as

$$y'y^{-n} + p(x)y^{1-n} = q(x).$$

To solve these equations, we use the substitution $u = y^{1-n}$, so $du = (1-n)y^{-n}dy$. After substituting into the rewritten equation, we obtain

$$\frac{u'}{1-n} + up(x) = q(x)$$

which is a linear differential equation, we just need to apply the general formula.

> **Example 2.16**
>
> Solve the following initial value problem
>
> $$y' - 5y = e^{-2x}y^{-2} \,;\, y(0) = 1$$
>
> We are in the presence of a Bernoulli equation. Taking then $u = y^3$, we see that $du = 3y^2 dy$, so, substituting we obtain
>
> $$\frac{u'}{3} - 5u = e^{-2x} \Rightarrow u' - 15u = 3e^{-2x}.$$
>
> This equation has as integrating factor $\mu(x) = e^{-15x}$, so thanks to the general formula, we have
>
> $$u = e^{15x}\left(\int 3e^{-15x}e^{-2x}dx + C\right) = 3e^{15x}\left(\int e^{-17x}dx + C\right) = 3e^{15x}\left(-\frac{e^{-17x}}{17} + C\right).$$
>
> Undoing our substitution, we have that the general solution is
>
> $$y^3 = -\frac{3}{17}\left(e^{-2x} + Ce^{15x}\right).$$
>
> Finally, since $y(0) = 1$, we have
>
> $$1 = -\frac{3}{17}(1 + C).$$
>
> So $C = -\frac{21}{3}$.

2.6.2 Riccati Equation

These are equations that have the form

$$y' + a(x)y + b(x)y^2 = c(x).$$

Although we will not see the general solution method, we will see a way to deduce a solution from another. If a solution y_1 is known, then we can apply the substitution $z = y - y_1$, where $z' = y' - y_1'$ to obtain

$$\begin{aligned}
& y_1' + z' + a(x)(y_1 + z) + b(x)(y_1 + z)^2 = c(x) \\
\Rightarrow\ & y_1' + z' + a(x)y_1 + a(x)z + b(x)y_1^2 + 2b(x)y_1z + b(x)z^2 = c(x) \\
\Rightarrow & (y_1' + a(x)y_1 + b(x)y_1^2) + z' + a(x)z + 2b(x)y_1z + b(x)z^2 = c(x)
\end{aligned}$$

Now, since we know that y_1 is a solution, this means that

$$y_1' + a(x)y_1 + b(x)y_1^2 = c(x)$$

so substituting this into the previous equation we obtain

$$\begin{aligned} c(x) + z' + a(x)z + 2b(x)y_1 z + b(x)z^2 &= c(x) \\ \Rightarrow \qquad z' + a(x)z + 2b(x)y_1 z + b(x)z^2 &= 0 \\ \Rightarrow \qquad z' + (a(x) + 2b(x)y_1)z + b(x)z^2 &= 0 \end{aligned}$$

This last equation is a Bernoulli equation, and can be solved with the method from the previous subsection.

Example 2.17

Solve the equation

$$y' + y^2 \sin(x) = 2\tan(x)\sec(x)$$

if it is known that $y_1 = \sec x$ is a solution of the equation. First note that in this equation, $a(x) = 0$, $b(x) = \sin(x)$, and $c(x) = 2\tan(x)\sec(x)$. Taking the substitution $z = y - \sec(x)$, we can save ourselves all the process done above to arrive at our equation becoming

$$z' + 2\sin(x)\sec(x)z + \sin(x)z^2 = 0$$

which is a Bernoulli equation, which can be rewritten as

$$z'z^{-2} + 2\sin(x)\sec(x)z^{-1} = -\sin(x).$$

The substitution $u = z^{-1}$ converts this equation into a linear one:

$$-u' + 2\sin(x)\sec(x)u = -\sin(x)$$

which is solved thanks to the integrating factor

$$\mu(x) = e^{-2\int \sin(x)\sec(x)dx} = e^{-2\int \tan(x)dx} = e^{2\ln(\cos(x))} = \cos^2(x).$$

The solution is given thanks to the general formula

$$u = -\sec^2(x)\left(\int \sin(x)\cos^2(x)dx + C\right) = \frac{-\cos(x)}{3} - C\sec^2(x).$$

Finally, undoing the substitutions $u = z^{-1}$, and $z = y - \sec(x)$, we arrive at the general solution

$$\frac{1}{\sec(x) - y} = \frac{\cos(x)}{3} + C\sec^2(x).$$

Exercises

Solve the following exact ODEs. If they are not exact, use an integrating factor.

1. $2(y-1)e^x dx + 2(e^x - 2y)dy = 0$.
2. $x \arctan(y)dx + \dfrac{x^2}{2(1+y^2)}dy = 0$.
3. $(3x^2 + y\cos(x))dx + (\sin(x) - 4y^3)dy = 0$.
4. $\dfrac{2y}{x}y' = \dfrac{x+y^2}{x^2}$
5. $(1+y^2)dx + xydy = 0$.
6. $\left(e^x - \dfrac{y^2}{2}\right)dx + (e^y - xy)dy = 0$.
7. $(t^2-1)x'(t) = -t - 2x(t)$.

Exercises

Solve the following linear, Bernoulli and Riccati ODEs.

1. $y' - 2xy = e^{x^2}$.
2. $2r'(\theta) + r\sec(\theta) = \cos(\theta)$.
3. $xy' + 2y + x^5y^3e^x = 0$.
4. $y' = y\cot(x) + y^3\csc(x)$.
5. $\dfrac{dy}{dx} = -2 - y + y^2$, if it is known that a solution is $y_1 = 2$.
6. $\dfrac{dy}{dx} = \dfrac{2\cos^2(x) - \sin^2(x) + y^2}{2\cos(x)}$, if it is known that a solution is $y_1 = \sin(x)$.

3

Higher Order Equations

We have already studied several solution methods for first order differential equations (those where the highest order derivative is the first). In this section we will mainly study the solution of **linear** equations of order greater than or equal to 2. Recall that such ODEs have the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = f(x) \tag{3.1}$$

where $a_n, a_{n+1}, \ldots, a_1, a_0, f$ are functions of a variable x, and also $a_n(x) \neq 0$, with $n \geq 2$. Recall also that if $f(x) = 0$, we call equation (3.1) **homogeneous**. Finally, when $a_n, a_{n-1}, \ldots, a_1, a_0$ are constant functions, that is, numbers, we say that equation (3.1) has **constant coefficients.** First we present a theorem that generalizes the last one from section 5.

Theorem 3.1 Existence and Uniqueness.

Consider the Cauchy problem

$$\begin{cases} a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = f(x) \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \\ y''(x_0) = y_2 \\ \vdots \\ y^{(n-1)}(x_0) = y_{n-1} \end{cases}$$

When all the functions $a_n, a_{n+1}, \dots, a_1, a_0, f$ are continuous on some $A \subseteq \mathbb{R}$, then the Cauchy problem has a unique solution.

Example 3.1

The problem

$$\frac{y''}{x^2} - \sqrt{x}y = 0 \ , \ y(2) = 1 \ , \ y'(3) = 0$$

has a unique solution, as long as $x > 0$, thanks to the fact that the coefficients $\frac{1}{x^2}$, and $-\sqrt{x}$ are continuous in this region. The application of the theorem is direct in this case.

Example 3.2

Every Cauchy problem whose differential equation is linear with constant coefficients and homogeneous, that is of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \quad , \quad a_i \in \mathbb{R} \text{ for } i = 0, 1, \dots, n$$

has a unique solution, since obviously all constant functions are continuous.

The importance of existence and uniqueness theorems for differential equations is more theoretical than practical, since they are not really ways to calculate the solution of a given problem, but only tools to prove that a solution exists (which makes it worthwhile to try to find it in the first place).

3.1 Vector Spaces

We have seen that when we solve a first order differential equation, for example $y' = y$, its general solution is given by a *family* of functions, in this case $y' = Ce^x$, where C is any real number. For higher order equations, we will have a similar situation (the higher the order, the more parameters we need). To make this intuition mathematically precise, we need to return to the notion of a **vector space**. From now on, we will allow ourselves to use complex numbers ($\mathbb{C}$) when convenient. The use of complex numbers will allow us to solve even more differential equations.

Definition 3.1 Linear Combination.

If $f_1(x), \dots, f_n(x) : \mathbb{R} \to \mathbb{R}$ are functions, then those functions of the form

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)$$

for $c_i \in \mathbb{C}$, are called **linear combinations** of $f_1, \ldots, f_n$. Note that linear combinations of a single function $g(x)$, are all of the form $\lambda g(x)$, with $\lambda \in \mathbb{C}$, that is, they are all **multiples** of $g(x)$.

Example 3.3

- The function $f(x) = x^2 + 3x + 1$ is a linear combination of the functions $x^2, x, 1$.
- The function $e^{iz} = \cos(z) + i\sin(z)$ is a linear combination of $\sin(z)$ and $\cos(z)$.
- The function $\cos(x)$ cannot be written as a multiple of $\sin(x)$.

The following theorem is known as the *superposition principle* for homogeneous equations, and tells us that the solution set of a homogeneous linear ODE is indeed a vector space.

Theorem 3.2 Superposition Principle.

If $y_1, y_2, \ldots, y_k$ are solutions of a homogeneous linear ODE, then any linear combination of them is also a solution.

This basically tells us that adding solutions of a homogeneous linear ODE produces new solutions of the same equation.

Example 3.4

The functions $y_1 = x^2$ and $y_2 = x^2 \ln x$ are solutions of the homogeneous linear equation $x^3 y''' - 2xy' + 4y = 0$. By the superposition principle, we have that the functions $y_3 = -x^2 + 5x^2 \ln(x)$, $y_4 = \frac{2}{7}x^2 - \sqrt{3}x^2 \ln(x)$ are also solutions to the same equation. More generally, for any $C_1, C_2 \in \mathbb{C}$, the function

$$y = C_1 x^2 + C_2 x^2 \ln(x)$$

is a solution to the differential equation.

We see then that when dealing with a homogeneous linear ODE, regardless of the degree, its *family* of solutions is actually a vector space. Recall that to describe a vector space, it suffices to describe a basis of it. We will then look for a basis of the solution space.

Definition 3.2 Linear Independence.

A set of functions $\{f_1(x), \ldots, f_n(x)\}$ is said to be **linearly independent** if there are no constants $c_1, c_2, \ldots, c_n \in \mathbb{C}$ **not all zero**, that make

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all $x \in \mathbb{R}$.

Verifying whether a set of functions is l.i. can be complicated, especially if we have many functions. For this, we borrow a resource from linear algebra, the determinant.

Definition 3.3 Wronskian.

The **Wronskian** of the functions $f_1(x), \ldots, f_n(x)$ is defined by the formula

$$W(f_1(x), \ldots, f_n(x))(x) = \det \begin{pmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{pmatrix}$$

Theorem 3.3.

If $f_1, \ldots, f_n$ are functions that are **l.d.**, then, for all $x \in \mathbb{R}$,

$$W(f_1(x), f_2(x), \ldots, f_n(x))(x) = 0.$$

That is, the Wronskian of the functions is the constant 0.

We can also see the same theorem written in another form:

Theorem 3.4.

If $f_1, \ldots, f_n$ are functions, and there exists some x such that

$$W(f_1(x), \ldots, f_n(x))(x) \neq 0,$$

then the functions are **l.i**. That is, when the Wronskian is not the constant 0, the functions will be l.i.

Example 3.5

Let us prove again that the functions x, x^2 are l.i. Note that

$$W(x, x^2)(x) = \det \begin{pmatrix} x & x^2 \\ 1 & 2x \end{pmatrix} = (x)(2x) - (1)(x^2) = x^2.$$

Since the Wronskian is not the constant 0, we conclude thanks to the theorem that x

and x^2 are l.i. The proof is easier than before.

Theorem 3.5.

The family of solutions of a homogeneous linear ODE of order n, that is, of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0 \tag{3.2}$$

is a vector space of dimension n. In other words, there exist n linearly independent functions $y_1, \ldots, y_n$ that satisfy that, for any other solution y, there exist (unique) scalars $c_1, c_2, \ldots, c_n \in \mathbb{C}$ such that for all x,

$$y(x) = c_1y_1(x) + \cdots + c_ny_n(x).$$

Said in a third way, the solution set of a homogeneous linear ODE of order n, has a *basis* with n elements.

The previous theorem simply tells us that if we are trying to solve a homogeneous linear ODE of order n, and we already have n solutions that are l.i., then we already have the general solution.

3.2 Homogeneous Differential Equations with Constant Coefficients

In this subsection we are going to solve equations of the form

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0 \tag{3.3}$$

where the a_i are all real numbers. To solve this equation, we are going to construct a new equation, called the **characteristic equation**, which is not an ODE, but a classical polynomial equation. This is obtained by changing in (3.3) the $y^{(k)}$ by m^k for all $k \in \{0, 1, \ldots, n\}$. That is, the characteristic equation is

$$a_nm^n + a_{n-1}m^{n-1} + \cdots + a_1m + a_0 = 0. \tag{3.4}$$

The roots of this equation are n complex numbers (since every polynomial of degree n has exactly n roots in $\mathbb{C}$). Suppose that $\alpha_1, \ldots, \alpha_n$ are the roots of (3.4). Then it is possible to show that the functions $y_i(x) = e^{\alpha_i x}$ are solutions of equation (3.3). From them we can then directly construct the general solution of equation (3.3).

Case 1: If all $\alpha_1, \ldots, \alpha_n$ are real and distinct, then the general solution is

$$y(x) = C_1e^{\alpha_1 x} + C_2e^{\alpha_2 x} + \cdots + C_{n-1}e^{\alpha_{n-1} x} + C_ne^{\alpha_n x}.$$

This is because if all the roots are distinct, all the $y_i = e^{\alpha_i x}$ are l.i, and since there are exactly n, they are a basis of the solution space.

Case 2: If among the $\alpha_1, \ldots, \alpha_n$ there are some repeated (whether real or complex), then we must count their **multiplicities** (that is, how many times each one appears repeated). Then for example, if the root α_1 has a multiplicity of k, this root is associated with the solution

$$y_{\alpha_1}(x) = C_0 e^{\alpha_1 x} + C_1 x e^{\alpha_1 x} + C_2 x^2 e^{\alpha_1 x} + \ldots C_{k_1} x^{k-1} e^{\alpha_1 x}.$$

That is, to obtain the solution associated with a root repeated k times, we sum starting from $e^{\alpha_1 x}$, adding $x e^{\alpha_1 x}$, and so on until reaching the power $k-1$ in x. Repeating this for each **distinct** solution of the characteristic equation, we have that the general solution is

$$y(x) = y_{\alpha_1}(x) + \cdots + y_{\alpha_n}(x)$$

Case 3: When any of the roots is a complex number (not real), of the form $\alpha + \beta i$, we know that $\alpha - \beta i$ (its complex conjugate) is also a root of (3.4). This tells us that the functions $e^{\alpha + \beta i}$ and $e^{\alpha - \beta i}$ are both solutions of (3.3). Applying the superposition principle and some complex number identities, we can summarize the solution associated with both roots $\alpha + \beta i$ and $\alpha - \beta i$, as a real-argument function, given by

$$y_{\alpha+\beta i} = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)) \quad \text{with } c_1, c_2 \in \mathbb{C}.$$

Example 3.6

The equation

$$y'' - 3y' - 10y = 0$$

is a homogeneous linear ODE with constant coefficients. So we can apply this method. Its characteristic equation is

$$m^2 - 3m - 10 = 0.$$

By inspection or the general formula, we can see that the roots of this equation are $m_1 = -2$ and $m_2 = 5$. Since they are distinct real solutions, we can apply case 1 of the method and simply say that the general solution is

$$y(x) = C_1 e^{-2x} + C_2 e^{5x}.$$

Example 3.7

The equation

$$y'' + 10y' + 25y = 0$$

has characteristic equation

$$m^2 + 10m + 25 = 0.$$

which factors as

$$(m+5)^2 = 0.$$

> This tells us that the root $m_1 = -5$ has **multiplicity 2**, so we must apply case 2 of the method. The solution associated with $m_1 = -5$ is $y_{m_1} = C_1 e^{-5x} + C_2 x e^{-5x}$, and since there are no more roots, the general solution is given by
>
> $$y(x) = y_{m_1} = C_1 e^{-5x} + C_2 x e^{-5x}.$$

Exercises

Find the general solution of each of the following ODEs.

1. $y'' + 4y' - y = 0.$
2. $y''' - 3y'' + 3y' - 3y = 0$
3. $y'' + 5y = 4y'.$
4. $16\dfrac{d^4y}{dx^4} + 24\dfrac{d^2y}{dx^2} + 9y = 0.$
5. $u^{(5)} + 5u^{(4)} - 2u^{(3)} - 10u'' + u' + 5u = 0.$

Exercises

Solve the following initial value problems.

1. $8x''(t) - 2x'(t) - 15x(t) = 0$ with the conditions $y(0) = y'(0) = 1$.
2. $y^{(6)} - 2y^{(5)} - 2y^{(4)} + 2y^{(3)} + y'' + 4y' + 4y = 0$, with the condition $y(0) = y'(0) = \cdots = y^{(5)}(0) = 0.$

3.3 Differential Operators

We are going to introduce a new way to denote the derivative of a function. We define the *differential operator* D as the function

$$D(f(x)) = f'(x).$$

Thus, we have $D(x^3) = 3x^2$, and $D(\sin(x)) = \cos(x)$ for example. We denote Df instead of $D(f(x))$ for convenience. Note that higher order derivatives can also be expressed with this operator

$$D^2y = D(D(y)) = D(y') = y''\text{ , and in general }\; D^n y = y^{(n)}.$$

Furthermore, we can form polynomial expressions using this operator. Any linear ODE can be rewritten using the differential operator D. For example, the equation

$$y'' + 5y' + 6y = 5x$$

becomes

$$(D^2 + 5D + 6)y = 5x.$$

To solve non-homogeneous equations, we are going to define differential annihilators.

Definition 3.4 Differential Annihilator.

A *differential annihilator* of a function f is a polynomial in D, called $H(D)$, that satisfies

$$H(D)f = 0.$$

For example, $D-1$ is an annihilator of e^x, D^2+1 is an annihilator of $\sin(x)$. Below is a list of the annihilators we will need most often.

Function f	**Differential Annihilator**
$a_n x^n + \cdots + a_1 x + a_0$	D^{n+1}
$Ae^{\alpha x}$	$D-\alpha$
$Ax^n e^{\alpha x}$	$(D-\alpha)^{n+1}$
$A\cos(\omega x) + B\sin(\omega x)$	$D^2+\omega^2$
$A\cosh(\omega x) + B\sinh(\omega x)$	$D^2-\omega^2$
$P_n(x)\cos(\omega x) + Q_m(x)\sin(\omega x)$	$(D^2+\omega^2)^{N+1}\ \ N = \max n, m$
$e^{\alpha x}(P_n(x)\cos(\omega x) + Q_m(x)\sin(\omega x))$	$((D-\alpha)^2 + w^2)^{N+1}\ ,\ N = \max n, m$

3.4 Non-homogeneous Linear ODEs

We are going to consider equations of the form

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 = f(x) \tag{3.5}$$

Where $a_i \in \mathbb{R}$ and $f(x) \neq 0$. This equation can be seen as

$$(a_n D^n + \cdots + a_1 D + a_0)y = f(x)$$

To solve this equation we must find a differential annihilator H of $f(x)$. Once found, we multiply the entire equation by H, to obtain

$$H(a_n D^n + \cdots + a_1 D + a_0)y = Hf(x) = 0.$$

which is a homogeneous linear equation. Once this is done, we must compare the general solution y_h of this homogeneous equation with the **complementary solution** y_c of

$$(a_n D^n + \cdots + a_1 D + a_0)y = 0$$

that is, of the original equation without the term f. This comparison will allow us to find a particular solution y_p of equation (3.5). The general solution is finally given by

$$y = y_c + y_p.$$

Example 3.8

Find the general solution of the equation

$$y'' - 2y + y = x.$$

The first thing we must do is solve the equation as if it were homogeneous, to find y_c. We already know how to solve this equation, so

$$y_c = C_1 e^x + C_2 x e^x.$$

Now, we rewrite the equation in terms of the operator D, to obtain

$$(D^2 - 2D + 1)y = (D - 1)^2 y = x,$$

since the annihilator of x is D^2, we apply it to the equation on both sides.

$$D^2(D - 1)^2 y = 0.$$

We also know how to solve this equation (we simply solve the characteristic equation $m^2(m-1)^2 = 0$. This tells us that

$$y_h = C_3 + C_4 x + C_5 e^x + C_6 x e^x.$$

To obtain y_p we must eliminate the terms *similar* to those of y_c, in the solution y_h, that is, the terms that are equal except for parameters. In this case, the similar terms are e^x and xe^x, so, eliminating them, we look for a particular solution of the form

$$y_p = C_3 + C_4 x.$$

Since we are looking for a *particular* solution, we must find the values of C_3 and C_4, for this we simply differentiate our candidate, and substitute it into the original equation.

$$\begin{aligned} (C_3 + C_4 x)'' - 2(C_3 + C_4 x)' + (C_3 + C_4 x) &= x \\ \Rightarrow \qquad -2C_4 + C_3 + C_4 x &= x \end{aligned}$$

Comparing coefficients on each side, we deduce that $C_4 = 1$, and therefore $C_3 = 2$. This implies that $y_p = 2 + x$. Finally, the general solution of the equation is

$$y = y_c + y_p = C_1 e^x + C_2 x e^x + 2 + x.$$

Exercises

Find the general solution of the following differential equations.

1. $y'' + 3y' + 2y = 4x^2$.
2. $y'' - 5y' + 6y = e^{-3x} + \sin(2x)$.
3. $y''' + y'' = e^x \sin(x)$

3.4.1 Method of Undetermined Coefficients

Once again, we are interested in solving equations of the form (3.5). We will use the same technique of finding the complementary solution y_c, and from it a particular solution y_p, from which it follows that the general solution is $y = y_c + y_p$. However, to find y_p, we will use a new method, which consists of a kind of *guess* we make based on the form that $f(x)$ has.

Note: This method only works when the term $f(x)$ of equation (3.5) is a polynomial, a sine or cosine, an exponential, or any sum and/or product of these.

Example 3.9

Find the general solution of

$$y'' + 4y - 2y = 2x^2 - 3x + 6$$

using undetermined coefficients. We start just as with the annihilator method, solving the equation without the term $f(x)$,

$$y'' + 4y' - 2y = 0,$$

whose solution is

$$y_c = C_1 e^{-(2+\sqrt{6})x} + C_2 e^{(-2+\sqrt{6})x}.$$

Now, since the function $f(x)$ is a polynomial of degree 2, the method consists of **assuming** that the particular solution must also be a polynomial of degree 2:

$$y_p = Ax^2 + Bx + C,$$

from which we only need to solve for A, B, C. Differentiating our candidate, we obtain

$$\begin{aligned} & y_p'' + 4y_p' - 2y_p = 2x^2 - 3x + 6 \\ \Rightarrow & (Ax^2 + Bx + C)'' + 4(Ax^2 + Bx + C)' - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6 \\ \Rightarrow & 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6 \end{aligned}$$

Comparing the coefficients of each polynomial, we must solve the system

$$\begin{cases} -2A = 2 \\ 8A - 2B = -3 \\ 2A + 4B - 2C = 6 \end{cases}.$$

The solution is $A = -1$, $B = -\frac{5}{2}$, $C = -9$. This tells us that

$$y_p = -x^2 - \frac{5}{2}x - 9$$

so finally the general solution is

$$y = y_c + y_p = C_1 e^{-(2+\sqrt{6})x} + C_2 e^{(-2+\sqrt{6})x} + -x^2 - \frac{5}{2}x - 9.$$

Exercise 3.1

Complete the following table.

$f(x)$	**Form of y_p**
1 (or any constant)	A
$5x + 5$	Ax+B
$3x^2 - 2$	
$x^4 - 2x$	
$\sin(4x)$	
$\cos(x) + 1$	
e^{5x}	
$9xe^{5x}$	
$x^2 \sin(3x)$	
$e^{3x} \sin(4x)$	
$5x^2 + x \sin(4x)$	
$xe^x \sin(x)$	

3.4.2 Method of Variation of Parameters

In this subsection we are interested in solving second order equations, of the form

$$y'' + p(x)y' + q(x)y = f(x) \tag{3.6}$$

Where $f(x) \neq 0$ and is also not in the appropriate form to apply annihilators or undetermined coefficients. The inspiration for this method comes from the solution of the homogeneous. Suppose for a moment that the equation is homogeneous, that is, $f(x) = 0$. Then there exist two functions, $y_1(x)$ and $y_2(x)$, that are l.i. and that make the homogeneous solution

$$y_c = C_1 y_1(x) + C_2 y_2(x)$$

where C_1, C_2 are scalars. The idea of this method is not to consider C_1, C_2 as numbers but as **functions** of x. This will help us find the particular solution which will have the form

$$y_p = C_1(x)y_1(x) + C_2(x)y_2(x).$$

Theorem 3.6 Variation of Parameters.

Consider the differential equation (3.6), with fundamental system of solutions $\{y_1, y_2\}$. The general solution of the equation is given by

$$y(x) = Ay_1(x) + By_2(x) + C_1(x)y_1(x) + C_2(x)y_2(x)$$

where $A, B \in \mathbb{C}$, and

$$C_1(x) = \int \frac{-y_2(x)f(x)}{W(y_1,y_2)(x)}dx \quad , \quad C_2(x) = \int \frac{y_1(x)f(x)}{W(y_1,y_2)(x)}dx.$$

Example 3.10

Consider the equation

$$y'' - 2y' + y = e^x \ln(x).$$

We begin as always, since it has constant coefficients, it is easy to calculate the complementary solution

$$y_c = Ae^x + Bxe^x$$

from which we are only interested in taking the fundamental system:

$$y_1 = e^x \; ; \; y_2 = xe^x.$$

We can calculate the Wronskian at once

$$W(e^x, xe^x)(x) = \det \begin{pmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{pmatrix} = e^{2x}.$$

Now, we can apply variation of parameters, using the formulas for C_1 and C_2.

$$C_1 = \int \frac{-y_2(x)f(x)}{W(y_1,y_2)(x)}dx = \int \frac{-xe^xe^x\ln(x)}{e^{2x}}dx = \int -x\ln(x)dx = \frac{x^2(1-2\ln(x))}{4}$$
$$C_2 = \int \frac{y_1(x)f(x)}{W(y_1,y_2)(x)}dx = \int \frac{e^xe^x\ln(x)}{e^{2x}} = \int \ln(x)dx) = x\ln(x) - x$$

Both integrals are calculated using integration by parts. Also, we omit the constants of integration since in case of adding them, they would be redundant with the final parameters A and B. Thanks to the theorem, we quickly find the general solution to the equation

$$y = y_c + y_p = Ae^x + Bxe^x + \left(\frac{x^2(1-2\ln(x))}{4}\right)e^x + (x\ln(x) - x)xe^x$$

which, after grouping a bit, becomes

$$y = Ae^x + Bxe^x - \frac{3}{4}x^2e^x + \frac{1}{2}x^2\ln(x)e^x.$$

Exercises

Solve the following differential equations and initial value problems, using the method of variation of parameters.

1. $y'' - 2y' + y = \dfrac{e^x}{1+x^2}$.
2. $y'' - y = \frac{1}{x}$. *Note:* if non-elementary integrals appear, you can leave them in integral form.
3. $x''(t) + 3x'(t) + 2x(t) = \sin(e^{-t})$.
4. $y'' + y = \cot(x)$, under the condition $y(\pi/4) = y'(\pi/4) = 0$.
5. $x^2y'' - xy' + y = \dfrac{x}{1+\ln^2(x)}$ with the conditions $y(1) = y(e) = 0$, knowing that a particular solution is $y = x$. *Hint:* This equation does not have constant coefficients, to find the other l.i. solution you can use Abel's formula.

3.4.3 Cauchy-Euler Equation

To conclude the topic of higher order ODEs, we will study the solution method for *Cauchy-Euler* equations, which have the form

$$a_n x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1 xy' + a_0 y = 0. \tag{3.7}$$

Where $a_0, a_1, \dots, a_n$ are constants. That is, these equations have variable coefficients, but these coefficients have a pattern, since the coefficient accompanying the k-th derivative of y is simply x^k. To solve this type of equation, we must apply the substitution $x = e^u$, where $\frac{dx}{du} = x$, which converts the equation into one with constant coefficients, which must be solved by one of the other methods we have studied.

Example 3.11

Consider the second order equation

$$x^2y'' + bxy' + cy = 0.$$

Taking the substitution $x = e^u$ (or equivalently $u = \ln(x)$), we must now calculate the first two derivatives of y with respect to u. We must use the chain rule,

$$\frac{dy}{du} = \frac{dy}{dx}\frac{dx}{du} = x\frac{dy}{dx} = xy',$$

and differentiating again with respect to u (we must apply the product rule, and the chain rule again),

$$\frac{d^2y}{du^2} = \frac{dx}{du}\frac{dy}{dx} + x\frac{d^2y}{dx^2}\frac{dx}{du} = xy' + x^2y''.$$

Using these two equalities, we can solve for y'' and y', to obtain

$$y' = \frac{1}{x}\frac{dy}{du}$$
$$y'' = \frac{1}{x^2}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right)$$

After substituting and simplifying, this converts our equation into

$$\frac{d^2y}{du^2} + (b-1)\frac{dy}{du} + cy = 0$$

which is a linear ODE with constant coefficients. Note that this substitution and the manipulations to the differentials will always be the same, regardless of which equation we have (as long as it is second order).

Exercise 3.2

- $x^2y'' - 3xy' + 3y = 2x^4e^x$.
- $3x^2y'' - xy' + 2y = 0$.
- $x^3y'' - 4x^2y' + 6xy = \ln(x^{x^4})$.
- $x^3y''' + 5x^2y'' + 7xy' + 8y = 0$.

4

Power Series Solutions

In this section we are going to introduce a new method for solving differential equations. To apply this method, we must revisit the concept of a **power series**, which will allow us to find solutions *analytically.* Recall that a power series centered at a number a is an infinite sum of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots \tag{4.1}$$

where $a, c_n \in \mathbb{R}$ (or even $\mathbb{C}$).

Power series help us approximate functions, in fact, we say that a power series **converges to a function** $f(x)$ if on some interval $I \subseteq \mathbb{R}$ it holds that

$$f(x) = \lim_{K\to\infty} \sum_{n=0}^{K} c_n(x-a)^n, \quad \text{for all } x \in I$$

or in other words, if the partial sums of the sum are getting closer and closer to a value $f(x)$. We give here a quick review of the key concepts about convergence of power series. Let us assume that our series is defined only on the real numbers.

Definition 4.1 Convergence.

Consider the power series (4.1).

- **Interval of convergence:** It is the interval of real numbers x where the series converges.
- **Radius of convergence:** The interval of convergence I can be expressed as $I =$

$[a - R, a + R]$, where $R \geq 0$. We call this number R the radius of convergence (if $R = \infty$, we say that $I = \mathbb{R}$).

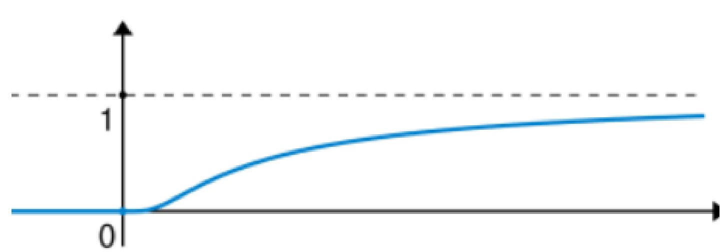

Figure 4.1: Asymptotic behavior of a convergent power series.

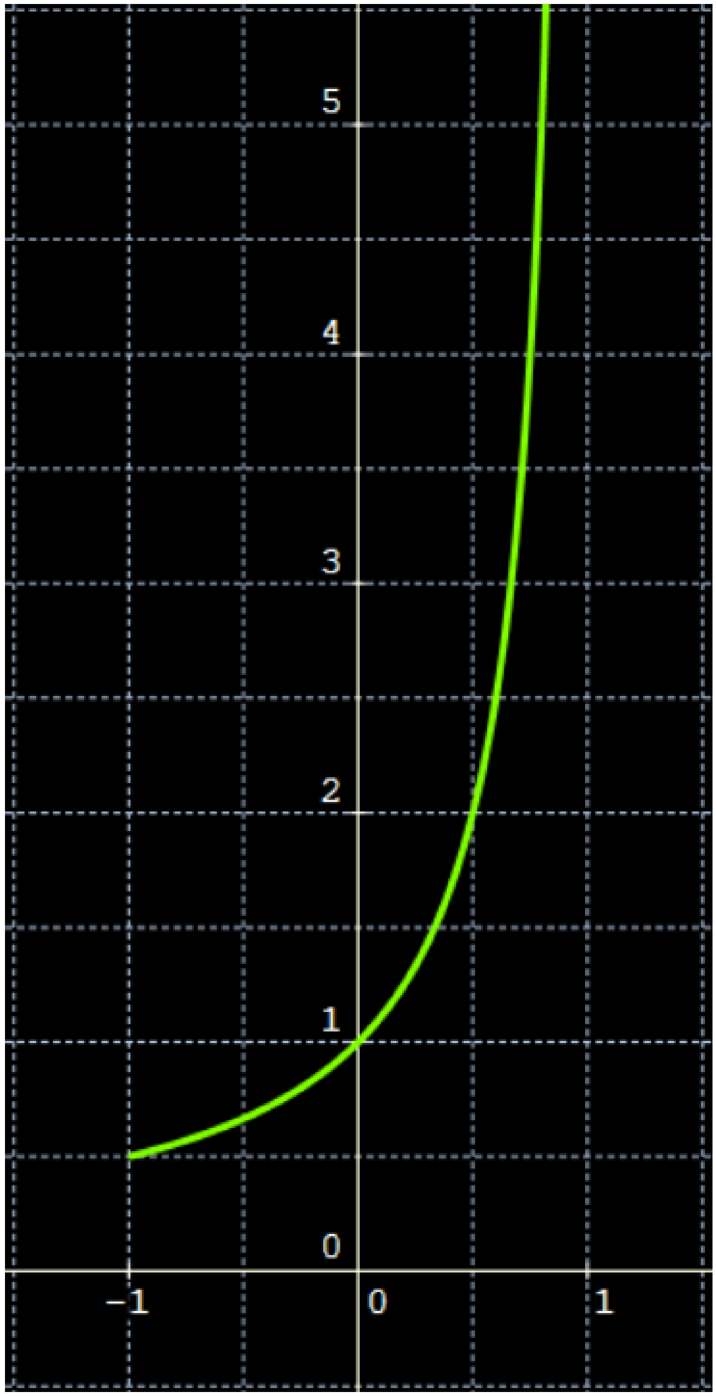

Figure 4.2: Exponential function and its power series approximation.

A special case of power series are Taylor (and Maclaurin) series.

Definition 4.2 Taylor Series.

Given an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$, we define its Taylor series centered at a as the sum

$$S(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

When $a = 0$, we call it a Maclaurin series.

We are going to connect these concepts from differential calculus with our interest in differential equations with the following definition.

Definition 4.3 Ordinary Point.

Consider the second order linear differential equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{4.2}$$

We say that a point x_0 is an **ordinary point** of equation (4.2) if both functions P and Q are analytic at x_0. Those points that are not ordinary are called **singular points**.

Example 4.1

Consider the equation

$$x^2y'' - 2y' + xy = 0$$

(note that none of the methods we have can solve this equation). The point $x = 1$ is an ordinary point of the equation, since the functions

$$P(x) = -\frac{2}{x^2}\,,\ Q(x) = \frac{1}{x}$$

are both analytic at 1. However, the point $x = 0$ is singular, since $P'(0)$ and $Q'(0)$ cannot even be defined (for a function to be analytic, it must be at least infinitely differentiable).

4.1 Solution in a Neighborhood of an Ordinary Point

We then have the first theorem that helps us solve equations using series.

Theorem 4.1.

If x_0 is an ordinary point of an equation in the form (4.2), we can find two linearly independent solutions y_1, y_2 in the form of power series centered at x_0. These series converge for all x that is between x_0 and the nearest singular point.

This theorem tells us that, if a is an ordinary point, we can declare a candidate solution of the form

$$y(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

And therefore, if we manage to solve for the coefficients c_n in some way, we will have our solution in series form.

Example 4.2

Solve the equation

$$y'' + xy = 0.$$

Note that none of the methods we have studied so far works to solve this equation. We will proceed by power series: since there are no singular points, we can choose any point (we will choose 0 for convenience) as the center of our power series, and this same series will converge on all of $\mathbb{R}$ by the previous theorem. We then assume that our solution has the form of a series centered at 0

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots.$$

The idea of this method is the following: since we are assuming that y is a solution, by differentiating it and substituting it into the differential equation, we will have enough information to solve for the coefficients c_n.

$$2c_2 + \sum_{n=1}^{\infty} [c_{n+2}(n+2)(n+1) + c_{n-1}]x^n = 0.$$

From here, we must compare coefficients on each side of the equality, to find the values of c_n.

$$\begin{cases} c_2 = 0 \\ c_{n+2}(n+2)(n+1) + c_{n-1} = 0, \text{ for all } n \geq 1. \end{cases}$$

From here, to solve for the other c_n's, we must use a **recurrence relation**: Note that c_{n+2} can be found from c_{n-1}, in the form

$$c_{n+2} = \frac{-c_{n-1}}{(n+2)(n+1)}$$

The general solution to the equation $y'' + xy = 0$ is then

$$y = c_0 y_0(x) + c_1 y_1(x)$$

where

$$y_0(x) = \sum_{n=0}^{\infty} \frac{4 \cdot 7 \cdot 10 \cdots (3n-2)}{(3n)!} (-1)^n x^{3n}$$
$$y_1(x) = \sum_{n=0}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdots (3n-1)}{(3n+1)!} (-1)^n x^{3n+1}$$

In general it is difficult to go from the series expression to an elementary expression (trigonometric, exponential, logarithm, etc). In fact, most analytic solutions do not have an elementary expression.

Example 4.3

Solve the equation

$$y'' + (\cos x)y = 0.$$

Once again, we take $y = \sum_{n=0}^{\infty} c_n x^n$. We will center around 0 for convenience, since there are no singular points. We then have to use the Taylor series for $\cos(x)$, which is analytic.

$$\begin{aligned} y'' + (\cos x)y &= \sum_{n=2} n(n-1)c_n x^{n-2} + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) \sum_{n=0}^{\infty} c_n x^n = 0 \\ \Rightarrow \qquad &= 2c_2 + 6c_3 x + 12c_4 x^2 + \dots + \left(1 - \frac{x^2}{2!} + \dots\right)(c_0 + c_1 x + \dots) = 0 \end{aligned}$$

Then, comparing coefficients, we obtain the system of equations

$$\begin{cases} 2c_2 + c_0 = 0 \\ 6c_3 + c_1 = 0 \\ 12c_4 + c_2 - \frac{1}{2}c_0 = 0 \\ 20c_5 + c_3 - \frac{1}{2}c_1 = 0 \end{cases}$$

whose solution is $c_2 = -c_0/2$, $c_3 = -c_1/6$, $c_4 = c_0/12$, and $c_5 = c_1/30$. Grouping the terms of the series, we finally obtain that the general solution (as far as we could calculate it), is

$$y = c_0 y_0(x) + c_1 y_1(x)$$

where

$$y_0(x) = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 + \dots \text{ and } y_2(x) = x - \frac{1}{6}x^3 + \frac{1}{30}x^5 + \dots$$

4.2 Solution in a Neighborhood of a Regular Singular Point

We can also solve differential equations of the form (4.2) centering our series around a singular point, but we must take certain precautions.

Definition 4.4 Regular Singular Point.

Let x_0 be a singular point of the equation $y'' + P(x)y' + Q(x)y = 0$. We say that x_0 is a **regular singular point** if the functions

$$p(x) = (x - x_0)P(x)\ ,\ q(x) = (x - x_0)^2 Q(x)$$

are both analytic at x_0. Singular points that are not regular are called **irregular** points.

Theorem 4.2 Frobenius Method.

If $x = x_0$ is a regular singular point of the differential equation

$$y'' + P(x)y' + Q(x)y = 0,$$

then there exists at least one solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}.$$

The series will converge in some interval $(x_0 - R, x_0 + R)$, with $R > 0$. To find r, we must solve the **indicial equation**, given by

$$r(r-1) + p_0 r + q_0 = 0$$

where

$$p_0 = \lim_{x \to x_0} (x - x_0)p(x) \quad \text{and} \quad q_0 = \lim_{x \to x_0} (x - x_0)^2 q(x).$$

Depending on the two solutions of the indicial equation, the solutions are constructed in different ways:

- **Case 1:** When $r_1 - r_2 \notin \mathbb{Z}$, we have two l.i. solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

- **Case 2:** If $r_1 - r_2 \in \mathbb{Z}$, the solutions are

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \quad \text{and} \quad y_2(x) = A y_1(x) \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

  where A is a constant to be determined (which could even be 0). In the special case when $r_1 = r_2$, we have $A = 1$.

Example 4.4

For the equation

$$(x^2-4)^2y''+3(x-2)y'+5y=0$$

it is evident that $x_0=2$ and $x_1=-2$ are both singular points. However, note that, studying first $x_0=2$,

$$p(x)=(x-2)P(x)=\frac{3(x-2)^2}{(x^2-4)^2}=\frac{3}{(x+2)^2}$$
$$q(x)=(x-2)^2Q(x)=\frac{5(x-2)^2}{(x^2-4)^2}=\frac{5}{(x+2)^2}.$$

Since both functions $p(x), q(x)$ are analytic at $x=2$, we conclude that 2 is a regular singular point. Let's see what happens with -2,

$$p(x)=(x+2)P(x)=\frac{3(x+2)(x-2)}{(x^2-4)^2}=\frac{3}{(x+2)(x-2)}.$$

This function is not analytic at $x=-2$, so the point -2 is an irregular singular point (we don't need to check $q(x)$).

Example 4.5

Solve the following ODE

$$3xy''+y'-y=0.$$

First, we rewrite:

$$y''+\frac{y'}{3x}-\frac{y}{3x}=0,$$

from which we can immediately observe that $x_0=0$ is a regular singular point, indeed:

$$p(x)=\frac{x}{3x}=\frac{1}{3}$$
$$q(x)=-\frac{x^2}{3x}=-\frac{x}{3}$$

both functions analytic at 0 (and on all of $\mathbb{R}$). Then we proceed to solve the indicial equation. Since

$$p_0=\lim_{x\to 0}p(x)=\frac{1}{3}\quad\text{and}\quad q_0=\lim_{x\to 0}q(x)=0,$$

we must solve the equation

$$r(r-1)+\frac{r}{3}=0.$$

The roots of this equation are $r_1 = 0$ and $r_2 = 2/3$, so we must proceed as in case 1. We propose two solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+2/3}.$$

Differentiating y_1 and substituting into the equation, we obtain the recurrence

$$c_0 = c_1 \text{ and } c_{n+1} = \frac{c_n}{(n+1)(3n+1)}$$

so that, in general, for all $n \geq 1$,

$$c_n = \frac{c_0}{(n!) \cdot 1 \cdot 4 \cdot 7 \cdot (3n-2)}$$

and therefore

$$y_1(x) = \left(1 + \sum_{n=1}^{\infty} \frac{x^n}{(n!) \cdot 1 \cdot 4 \cdot 7 \cdots (3n-2)}\right).$$

For y_2, we obtain the recurrence

$$b_n = \frac{b_{n-1}}{n(3n+2)} \text{ for } n \geq 1.$$

which gives

$$b_n = \frac{b_0}{(n!) \cdot 5 \cdot 8 \cdot 11 \cdots (3n+2)} \text{ for } n \geq 1$$

from which we obtain

$$y_2(x) = x^{2/3}\left(1 + \sum_{n=1}^{\infty} \frac{x^n}{(n!) \cdot 5 \cdot 8 \cdot 11 \cdots (3n+2)}\right).$$

The general solution is $y(x) = Ay_1(x) + By_2(x)$.

Example 4.6

Consider the equation

$$x(x-1)y'' + 3xy' + y = 0$$

The point $x_0 = 0$ is a regular singular point (exercise: verify). The indicial equation is $r(r-1) = 0$, with roots $r_1 = 0$, $r_2 = 1$. Since we are in case 2, we take $y_1(x) = \sum_{n=0}^{\infty} c_n x^n$. Substituting we obtain the recurrence $c_0 = 0$ and $c_{n+1} = c_n \frac{n+1}{n}$, from which $c_n = nc_1$, which gives us

$$y_1(x) = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

For y_2, we can use Abel's formula:

$$y_2(x) = y_1(x) \int \frac{\exp(-\int \frac{3dx}{x-1})}{y_1^2(x)} dx = -\frac{1+x\ln x}{(1-x)^2}.$$

The general solution is

$$y = C_1 \frac{x}{(1-x)^2} + C_2 \frac{1+x\ln(x)}{(1-x)^2}.$$

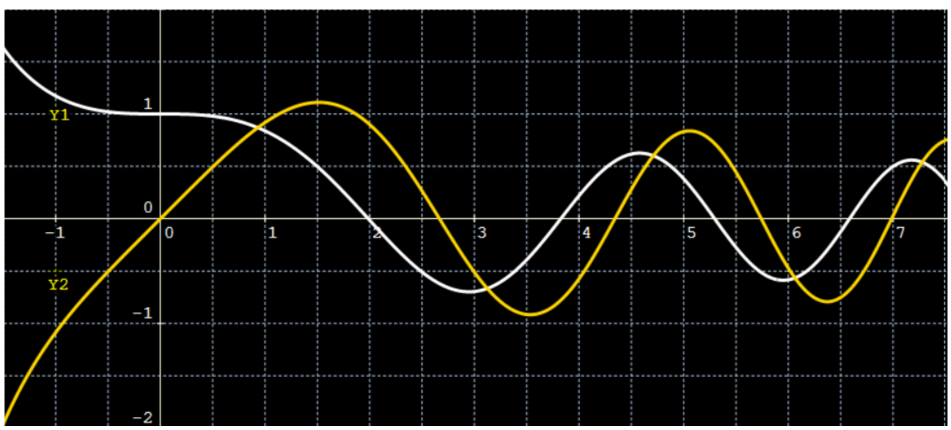

Figure 4.3: Bessel functions $J_0(t)$ and $J_1(t)$.

Exercises

1. Find the general solution of $y' = 3y$ using a power series expansion centered at $x = 0$.
2. Find the two fundamental solutions of the *Airy* equation $y'' = xy$.
3. Find the general solution of $(x^2+1)y'' + xy' - y = 0$.
4. Prove that the function

$$J_0(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{t}{2}\right)^m$$

is a solution of the equation $t^2x''(t) + tx'(t) + t^2x(t) = 0$. This function is known as the *Bessel function of order* 0. (See Figure 4.3)

5. Prove using power series, that the general solution of $y'' + y = 0$ is given by $y = A\cos(x) + B\sin(x)$.
6. Find the first 5 terms of the power series expansion of each fundamental solution of the equation $y'' + \sin(x) + e^x y = 0$.

7. Find a fundamental solution of the equation $x^2y'' - xy' + (1-x)y = 0$. Express it as a power series around the point $x = 0$.

5

Systems of Differential Equations

So far, we have been studying differential equations that involve a single unknown function $y(x)$. However, in practice (in physics, biology, chemistry, etc.), it is common to encounter problems where several quantities depend on each other, and their rates of change are interrelated. A system of first-order linear differential equations has the form

$$\begin{aligned} x_1'(t) &= a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t) \\ x_2'(t) &= a_{21}(t)x_1(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t) \\ &\vdots \\ x_n'(t) &= a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t) \end{aligned}$$

Where $x_1, \ldots, x_n$ are the unknown functions, and $a_{ij}(t)$ and $f_i(t)$ are given functions. This system can be written much more conveniently using matrix notation. If we define

$$X(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \; A(t) = \begin{pmatrix} a_{11}(t) & \ldots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \ldots & a_{nn}(t) \end{pmatrix}, \; F(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

then the system can be viewed as

$$X'(t) = A(t)X(t) + F(t). \tag{5.1}$$

When $F(t) = \mathbf{0}$, we say the system is **homogeneous**.

Theorem 5.1 Existence and Uniqueness.

If the entries of the matrix $A(t)$ and the vector $F(t)$ are continuous functions on an interval I, then for any $t_0 \in I$ and any initial vector $X_0 \in \mathbb{R}^n$, there exists a **unique** solution to the initial value problem

$$\begin{cases} X'(t) = A(t)X(t) + F(t) \\ X(t_0) = X_0 \end{cases}$$

defined on the entire interval I.

This theorem is analogous to the one we had for first and higher order equations.

Theorem 5.2.

The solution set of a homogeneous linear system ($F = \mathbf{0}$) is a vector space of dimension n. This means that there exist n linearly independent (vector) solutions, $X_1, \ldots, X_n$, such that any other solution can be written as

$$X(t) = c_1 X_1(t) + \cdots + c_n X_n(t).$$

That is, to solve a homogeneous system, it suffices to find n linearly independent solutions.

Definition 5.1 Wronskian.

Given n solutions $X_1, \ldots, X_n$, we define their Wronskian as the determinant of the matrix formed by placing the column vectors side by side.

$$W(X_1, \ldots, X_n)(t) = \det(X_1 \ldots X_n).$$

As in previous sections, if $X_1, \ldots, X_n$ are solutions of a homogeneous system, then their Wronskian is either always zero (if they are l.d.) or never zero (if they are l.i.).

Example 5.1

Consider the homogeneous linear system

$$X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X.$$

It can be verified that the vectors

$$X_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} \quad \text{and} \quad X_2(t) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$$

are both solutions of the system. Furthermore, their Wronskian is

$$W(X_1, X_2)(t) = \det \begin{pmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{pmatrix} = 5e^{4t} - (-3e^{4t}) = 8e^{4t} \neq 0$$

so they are l.i. This means that $\{X_1, X_2\}$ is a **fundamental set of solutions**, and the general solution is

$$X(t) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}.$$

Exercises

Verify in each case that the given functions are solutions of the system, calculate their Wronskian and write the general solution.

1. $X' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} X$, $X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$, $X_2 = \begin{pmatrix} 1+t \\ -t \end{pmatrix} e^{2t}$

2. $X' = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 0 & 6 \\ -4 & -2 & -3 \end{pmatrix} X$, $X_1 = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} e^{-t}$, $X_2 = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} e^{-t}$, $X_3 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} e^{2t}$

Exercises

Write the following systems in matrix form.

1. $\begin{cases} x'(t) = 4x - 5y \\ y'(t) = -4x + 9y \end{cases}$

2. $\begin{cases} x' = 1 + x + y + z \\ y' = 3y + 5z - 2y + t^2 \\ z' = z + x + e^{\sin t} \end{cases}$

3. $\begin{cases} x' = \ln tx - 2\cos(t)y + 1/t \\ y' = tx + ty + t^2 \end{cases}$

5.1 Substitution Method

Some of the simplest systems can be solved simply by integrating and substituting the result of one equation into another.

Example 5.2

The system

$$\begin{cases} x'(t) = 1 + y^2(t) \\ y'(t) = \sec^2(t) \end{cases}$$

is not linear, so our theorems do not apply. However, we can integrate the second equation with respect to t (since it does not depend on $x(t)$) to obtain

$$y(t) = \tan(t),$$

and substituting this into the first equation, we see that

$$x'(t) = 1 + \tan^2(t) = \sec^2(t)$$

Which implies that $x(t) = \int \sec^2(t)dt = \tan(t)$, so a solution of the system is given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \tan(t) \\ \tan(t) \end{pmatrix}.$$

Example 5.3

Consider now the system

$$\begin{cases} x' = y \\ y' = x \end{cases}$$

which is linear and is in normal form. Differentiating the second equation, we can see that

$$y'' = x' = y,$$

but we already know that the general solution of the equation $y'' = y$ is

$$y = C_1 e^t + C_2 e^{-t}.$$

Finally, since $x = y'$, we have the general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} e^t \\ e^t \end{pmatrix} + C_2 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}.$$

These first examples are simple, we will not often encounter systems that can be solved in this way. However, they serve to illustrate that the same techniques we have developed for ordinary equations can work to solve systems of equations, when used with a certain degree of ingenuity.

5.2 Gaussian Elimination

The Gaussian elimination method that we learned in linear algebra also works for linear systems. We can see an example for a 2×2 system.

Example 5.4

The system with independent variable t:

$$\begin{cases} x' = -9y \\ y' = -4x \end{cases}$$

We can use differential operator notation to convert this system into

$$\begin{cases} Dx + 9y = 0 \\ 4x + Dy = 0 \end{cases}$$

The elimination method is analogous to the one studied in linear algebra. First, we multiply the first equation by D and the second by -9 to obtain

$$\begin{cases} D^2x + 9Dy = 0 \\ -36x + -9Dy = 0 \end{cases}$$

From here, we can add both equations, eliminating the variable y, and leaving a second order equation with variable $x(t)$,

$$D^2x - 36x = 0.$$

We already know how to calculate the solution of this equation,

$$x(t) = C_1e^{6t} + C_2e^{-6t}.$$

Next we have two options: first, substitute this function into the second equation and integrate (we must be careful since new integration constants will appear), the second option is to repeat the elimination process, this time for x. We will prefer the second option: multiplying the first equation now by 4 and the second by $-D$, we arrive at

$$\begin{cases} 4Dx + 36y = 0 \\ -4Dx + -D^2y = 0 \end{cases}$$

Then, adding both equations we arrive at

$$(36 - D^2)y = 0,$$

whose solution is

$$y(t) = C_3 e^{6t} + C_4 e^{-6t}.$$

Note that the solution is almost identical to the equation for $x(t)$, however the parameters must be new since it is nevertheless another equation. We see then that we have found $x(t)$ and $y(t)$, however, we have not arrived at the general solution, since we have 4 parameters, and the theorems we studied in the previous section tell us that we can only have 2. This means we must solve for C_3 and C_4 in terms of the other two (actually we can solve for any two of the C_i's in terms of the other two). We can do this in several ways. For example, we can differentiate x to obtain

$$x'(t) = 6C_1 e^{6t} - 6C_2 e^{-6t},$$

and since the first equation tells us that x' must equal $-9y$, we simply compare the resulting expressions

$$6C_1 e^{6t} - 6C_2 e^{-6t} = -9C_3 e^{6t} - 9C_4 e^{-6t}$$

From here we can deduce that, comparing coefficients on each side, that $C_3 = -2/3C_1$ and that $C_4 = 2/3C_2$. So we can see that

$$y(t) = -\frac{2}{3} C_1 e^{6t} + \frac{2}{3} C_2 e^{-6t}.$$

The other way to do this solving is similar, but using the second equation of the system, both approaches will yield the same result. Finally, we express the general solution in vector form,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} e^{6t} \\ -\frac{2}{3} e^{6t} \end{pmatrix} + C_2 \begin{pmatrix} e^{-6t} \\ \frac{2}{3} e^{-6t} \end{pmatrix}.$$

We can also apply this method to solve non-homogeneous systems.

Example 5.5

Consider the initial value problem

$$\begin{cases} x' = x + y + e^t \\ y' - x = 2y \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

Which can be written using differential operator notation as

$$\begin{cases} (D-1)x - y = e^t \\ -x + (D-2)y = 0 \end{cases}$$

First, we apply elimination to eliminate y, this is equivalent to multiplying the first equation by $(D-2)$ and the second by 1. After adding, we obtain

$$(D-1)(D-2)x - x = (D-2)e^t$$

which simplifies to

$$(D^2 - 3D + 1)x = -e^t.$$

This equation can be solved using annihilators or undetermined coefficients, its solution is

$$x(t) = C_1 e^{(3-\sqrt{5})\frac{t}{2}} + C_2 e^{(3+\sqrt{5})\frac{t}{2}} - e^t.$$

Next, to eliminate y, we multiply the first equation by 1 and the second by $(D-1)$ to obtain, after adding, that

$$(D-1)(D-2)y - y = e^t \Rightarrow (D^2 - 3D + 1)y = e^t,$$

so, again

$$y(t) = C_3 e^{(3-\sqrt{5})\frac{t}{2}} + C_4 e^{(3+\sqrt{5})\frac{t}{2}} - e^t.$$

We must then solve for C_3 and C_4 in terms of C_1 and C_2. Let us now denote $\alpha = \frac{3-\sqrt{5}}{2}$ and $\beta = \frac{3+\sqrt{5}}{2}$. Differentiating, we see that

$$x'(t) = \alpha C_1 e^{\alpha t} + \beta C_2 e^{\beta t} - e^t$$

and since it must hold that $x' = x + y + e^t$, it must then hold that

$$\alpha C_1 e^{\alpha t} + \beta C_2 e^{\beta t} - e^t = e^{\alpha t}(C_1 + C_3) + e^{\beta t}(C_2 + C_4) - e^t$$

so, comparing terms and solving the resulting system, we obtain that

$$C_3 = C_1(\alpha - 1) = \frac{1-\sqrt{5}}{2} C_1$$
$$C_4 = C_2(\beta - 1) = \frac{1+\sqrt{5}}{2} C_2$$

So the general solution of the system is

$$x(t) = C_1 e^{(3-\sqrt{5})\frac{t}{2}} + C_2 e^{(3+\sqrt{5})\frac{t}{2}} - e^t$$
$$y(t) = \frac{1-\sqrt{5}}{2} C_1 e^{(3-\sqrt{5})\frac{t}{2}} + \frac{1+\sqrt{5}}{2} C_2 e^{(3+\sqrt{5})\frac{t}{2}} - e^t$$

Finally, since $x(0) = 1$ and $y(0) = 0$, we have that

$$\begin{cases} C_1 + C_2 - 1 = 1 \\ \frac{1-\sqrt{5}}{2}C_1 + \frac{1+\sqrt{5}}{2}C_2 - 1 = 0 \end{cases}$$

from which we obtain that $C_1 = -\frac{1}{\sqrt{5}}$, and that $C_2 = \frac{1}{\sqrt{5}}$. Finally, we have the solution of the initial value problem:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{5}}e^{(3-\sqrt{5})\frac{t}{2}} \\ \frac{\sqrt{5}-1}{10}e^{(3-\sqrt{5})\frac{t}{2}} \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{5}}e^{(3+\sqrt{5})\frac{t}{2}} \\ \frac{\sqrt{5}+1}{10}e^{(3+\sqrt{5})\frac{t}{2}} \end{pmatrix} - \begin{pmatrix} e^t \\ e^t \end{pmatrix}$$

Exercises

Find the general solution of the following systems:

1. $\begin{cases} x' = 2y \\ y' = x - y \end{cases}$
2. $\begin{cases} 3x' + 2y' = x - y \\ x' - y' = x + 2y \end{cases}$
3. $\begin{cases} x' = x + 2y + e^{-t} \\ y' = 3y \end{cases}$
4. $\begin{cases} x'' + x' + y = e^t \\ x' + y' = 1 \end{cases}$

Exercises

Solve the following initial value problems

1. $X' = \begin{pmatrix} 8 & -1 \\ 4 & 12 \end{pmatrix} X$, with condition $X\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
2. $X' = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} X$, with condition $X\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

5.3 Solution by Eigenvalues

Recall that if A is an $n \times n$ matrix, we say that a number $\lambda \in \mathbb{C}$ is an **eigenvalue** (or characteristic value) of A, if there exists some vector $v \neq 0$ such that $Av = \lambda v$ (v is called the **eigenvector** associated with λ).
Consider now the system of equations

$$X'(t) = AX(t), \tag{5.2}$$

where A is a matrix with constant entries (A does not have functions as entries). Let λ be an eigenvalue of A, with $\mathbf{v}$ its associated eigenvector. Then it holds that the function

$$X(t) = \begin{pmatrix} e^{\lambda t} v_1 \\ e^{\lambda t} v_2 \\ \vdots \\ e^{\lambda t} v_n \end{pmatrix} = e^{\lambda t}\mathbf{v}$$

is a solution of system (5.2), since

$$X'(t) = \begin{pmatrix} \lambda e^{\lambda t} v_1 \\ \lambda e^{\lambda t} v_2 \\ \vdots \\ \lambda e^{\lambda t} v_n \end{pmatrix} = \lambda e^{\lambda t}\mathbf{v},$$

while

$$AX(t) = Ae^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v} = e^{\lambda t}\lambda\mathbf{v} = \lambda e^{\lambda t}\mathbf{v}.$$

Note that $Ae^{\lambda t}\mathbf{v} = e^{\lambda t}A\mathbf{v}$, since $e^{\lambda t}$ is a scalar. In summary, to solve system (5.2), we only need to find the eigenvalues and eigenvectors of the matrix A. In general, four things can happen, which we exemplify below.

Example 5.6

Case 1: Distinct real eigenvalues. Consider the system of equations

$$\begin{cases} x' = 3x - 2y \\ y' = -3x + 2y \end{cases}$$

We could apply elimination to solve it, but we are going to calculate the eigenvalues of the system matrix

$$A = \begin{pmatrix} 3 & -2 \\ -3 & 2 \end{pmatrix}.$$

Using the *characteristic polynomial* of matrix A, we see that

$$det(A - xI) = \det\begin{pmatrix} 3-x & -2 \\ -3 & 2-x \end{pmatrix} = (3-x)(2-x) - 6 = x^2 - 5x.$$

This tells us that the eigenvalues are the roots of this polynomial, that is, 0 and 5. We must then calculate an eigenvector associated with each one, first we solve the system $(A - 5I)v = 0$,

$$\begin{pmatrix} -2 & -2 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This system has infinitely many solutions, but we only need one, and any vector where $a = -b$ will suffice, for example $v_1 = (1, -1)^t$. We now solve the system $(A - 0I)v = 0$,

$$\begin{pmatrix} 3 & -2 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For this case we need some vector where $b = 3a/2$, so we can take $v_2 = (1, \frac{3}{2})^t$. We then have two solutions to the differential equation:

$$X_1(t) = e^{5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{5t} \\ -e^{5t} \end{pmatrix}$$
$$X_2(t) = e^{0t} \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}$$

It is easy to verify, using the Wronskian criterion, that these solutions are linearly independent. Therefore, we have 2 l.i. solutions of a 2×2 system. By the general solution theorem, we conclude that the general solution of this system is

$$X(t) = C_1 X_1 + C_2 X_2 = C_1 \begin{pmatrix} e^{5t} \\ -e^{5t} \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}.$$

Example 5.7

Case 2: Repeated real roots, with correct multiplicity. Consider the system

$$X' = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix} X$$

We calculate the zeros of the characteristic polynomial of A,

$$\det(A - xI) = \det \begin{pmatrix} 5-x & 4 & 2 \\ 4 & 5-x & 2 \\ 2 & 2 & 2-x \end{pmatrix} = -x^3 + 12x^2 - 21x + 10 = (10 - x)(x - 1)^2,$$

(we factor using synthetic division). The eigenvalues are then 10 and 1, but note that 1 appears with multiplicity 2, so it must be treated differently. Let us begin by solving the system $(A - 10I)v = 0$

$$\begin{pmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

we can use any method (elimination, substitution) to see that a possible solution is $v_1 = (2,2,1)^t$ (Important: in general these systems cannot be solved by calculator, since these matrices have determinant 0). This produces the solution of the system

$$X_1(t) = \begin{pmatrix} 2e^{10t} \\ 2e^{10t} \\ e^{10t} \end{pmatrix}.$$

To find the other solutions, we must now solve the system $(A - I)v = 0$. However, since 1 is an eigenvalue of multiplicity 2, we must find two linearly independent solutions v_2, v_3 instead of one. This is only possible if the dimension of the solution space is 2 (which is indeed the case), for when this is not fulfilled, we have case 3. Once again, it should be noted that the choice of v_2, v_3 can be made in many ways, since the system $(A - I)v = 0$ has infinitely many solutions. We have then

$$\begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In this system, we see that in fact all rows are multiples of the third, that is, it gives us all the information we need. More specifically, we only need two vectors of the form $v = (a, b, c)$ that satisfy $2a + 2b + c = 0$ and are l.i. Two options are $v_2 = (-1, 1, 0)^t$ and $v_3 = (-1, 0, 2)^t$ (it is easy to see they are l.i. since neither is a multiple of the other). This produces then the other two fundamental solutions of the system

$$X_2 = \begin{pmatrix} -e^t \\ e^t \\ 0 \end{pmatrix}, \; X_3 = \begin{pmatrix} -e^t \\ 0 \\ 2e^t \end{pmatrix}.$$

The reader can verify that X_1, X_2, X_3 are linearly independent, so the general solution is given by

$$X(t) = C_1 \begin{pmatrix} 2e^{10t} \\ 2e^{10t} \\ e^{10t} \end{pmatrix} + C_2 \begin{pmatrix} -e^t \\ e^t \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} -e^t \\ 0 \\ 2e^t \end{pmatrix}.$$

We have in summary a method to solve systems of type (5.2), when the eigenvalues are real numbers: we simply calculate each eigenvalue λ, and if its multiplicity is k, we must find k l.i. vectors that satisfy the equation $(A - \lambda I)v = 0$. Once said vectors v are found, the solution $e^{\lambda t}v$ is associated to each one. This procedure generates then a total of n l.i. solutions of the system, which leads us immediately to the general solution. In case an eigenvalue λ has multiplicity k, but the solution space of $(A - \lambda I)v = 0$ has dimension less than k (this means we cannot find k l.i. eigenvectors associated to λ), we must vary the solution slightly, finding generalized eigenvectors.

Example 5.8

Case 3: Repeated real roots, with incorrect multiplicity. Solve the system

$$X' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} X.$$

The characteristic polynomial is $(x-2)^2$, so the only eigenvalue is 2, with multiplicity 2. We would like to now find 2 l.i. solutions for the system

$$(A-2I)v = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0.$$

Unfortunately, the rank of this matrix is 1, which makes it impossible to find 2 l.i. solutions. (since if the rank is 1, the dimension of the solution space of the system is 1, meaning we have at most one l.i. solution). The l.i. solution we can find is $v_0 = (-1,1)^t$. To finish solving the system, we proceed then to find **generalized eigenvectors**, that is, instead of solving the system $(A-2I)v = 0$, we solve $(A-2I)v = v_0$,

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

which admits the solution $v_1 = (1,0)^t$. Then we can associate the solutions to each vector

$$X_1(t) = e^{2t} v_0 = \begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix}$$

To find $X_2(t)$ however, we must use the formula

$$X_2(t) = e^{2t}(tv_0 + v_1) = e^{2t}\left(\begin{pmatrix} -t \\ t \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} (1-t)e^{2t} \\ te^{2t} \end{pmatrix}$$

The general solution is then expressed in the same form.

$$X(t) = C_1 X_1 + C_2 X_2.$$

We will now explain in general how to proceed in this method. Suppose we have an eigenvalue λ with multiplicity k, but the solution space of the system $(A-\lambda I)v$ does not provide us with enough l.i. solutions. (it must always provide at least one). Then we take a solution v_0 of said system, and construct generalized eigenvectors, solving the systems

$$\begin{aligned}(A-\lambda I)v &= v_0, \text{ whose solution we call } v_1\\ (A-\lambda I)v &= v_1, \text{ whose solution we call } v_2\\ &\vdots\end{aligned}$$

until we have the k vectors we need. Once these vectors are constructed, the solution of the system associated to each one is constructed, in the form

$$\begin{aligned}X_0(t) &= e^{\lambda t}v_0\\ X_1(t) &= e^{\lambda t}(v_1 + tv_0)\\ X_2(t) &= e^{\lambda t}\left(v_2 + tv_1 + \frac{t^2}{2}v_0\right)\\ X_3(t) &= e^{\lambda t}\left(v_3 + tv_2 + \frac{t^2}{2!}v_1 + \frac{t^3}{3!}v_0\right)\\ &\vdots\end{aligned}$$

The pattern to follow is the same one given by the Taylor series of e^x (this is no coincidence, we will see in the next section what happened here). Finally, the general solution is expressed as always:

$$X(t) = C_0X_0 + \cdots + C_{k-1}X_{k-1}.$$

Next we will see how to proceed in case some eigenvalue is not a real number. Let us remember that although the eigenvalues are complex, we are working with matrices A that only have real numbers as entries. We can use the following theorem.

Theorem 5.3.

Let A be an $n \times n$ matrix whose entries are real numbers. If $\lambda_1 = a + bi$ is an eigenvalue with associated eigenvector v_1, then the number $\lambda_2 = a - bi$ (the conjugate of λ_1) is also an eigenvalue of A, and its associated eigenvector is $\bar{v}$ (the vector obtained by conjugating each entry of v).

Example 5.9

Consider the system

$$X'(t) = \begin{pmatrix} 3 & 9\\ -4 & -3\end{pmatrix}X.$$

The characteristic polynomial of this matrix is $x^2 + 27$, so the eigenvalues are $\pm 3\sqrt{3}i$. Thanks to the previous theorem, we only have to find one eigenvector. Let us solve the system $(A - 3\sqrt{3}iI)v = 0$.

$$\begin{pmatrix} 3-3\sqrt{3}i & 9 \\ -4 & -3-3\sqrt{3}i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

We can apply the substitution $b = -\frac{1}{3}(1-\sqrt{3}i)a$ to the first equation, so a possible choice would be taking $a = 3$, to obtain $v_1 = (3, -1+\sqrt{3}i)^t$. This choice associates the first solution of the system

$$X(t) = e^{3\sqrt{3}it} \begin{pmatrix} 3 \\ -1+\sqrt{3}i \end{pmatrix}.$$

We can then apply Euler's formula to obtain

$$\begin{aligned} X(t) &= (\cos(3\sqrt{3}t) + i\sin(3\sqrt{3}t)) \begin{pmatrix} 3 \\ -1+\sqrt{3}i \end{pmatrix} \\ &= \begin{pmatrix} 3\cos(3\sqrt{3}t) + 3i\sin(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - i\sin(3\sqrt{3}t) + \sqrt{3}i\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} \end{aligned}$$

From here, we can separate the real part from the imaginary part, to be able to write in the form $X_1(t) + iX_2(t)$.

$$X(t) = \underbrace{\begin{pmatrix} 3\cos(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix}}_{X_1} + i \underbrace{\begin{pmatrix} 3\sin(3\sqrt{3}t) \\ -\sin(3\sqrt{3}t) - \sqrt{3}\cos(3\sqrt{3}t) \end{pmatrix}}_{X_2}.$$

However, thanks to the previous theorem (and a series of algebraic manipulations), we can extract the general solution at once, without having to go through the other eigenvalue. In fact, the general solution is given directly by

$$\begin{aligned} X(t) &= C_1X_1(t) + C_2X_2(t) \\ &= C_1 \begin{pmatrix} 3\cos(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{pmatrix} + C_2 \begin{pmatrix} 3\sin(3\sqrt{3}t) \\ -\sin(3\sqrt{3}t) - \sqrt{3}\cos(3\sqrt{3}t) \end{pmatrix}. \end{aligned}$$

5.4 The Matrix Exponential Function

Let us return to chapter 1 for a moment. When solving the equation

$$y' = ay,$$

with $a \in \mathbb{R}$, we can arrive almost immediately at the solution $y = Ce^{ax}$. Let now A be an $n \times n$ matrix of constant coefficients, and consider the system

$$X' = AX.$$

In a way, we are going to extend the exponential function e^x to be able to calculate e^A (the result of this operation will be a matrix of the same size as A). When we manage to do this, we will be able, analogously to first order equations, to affirm that the solution to the system is precisely

$$X = e^{At}C,$$

where $C = (C_1, \ldots, C_n)^t$ is our column vector of parameters (we must multiply on the left for size reasons). To be able to define the operation e^A, we borrow the Taylor series.

Definition 5.2 Matrix Exponential.

For a matrix A $n \times n$ with constant entries, we define e^A as

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k.$$

The matrix exponential is easy to calculate when some power of our matrix is the null matrix.

Example 5.10

Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that

$$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so

$$e^A = I + A + 0 + 0 + \cdots = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Note: The exponential of a matrix A is **not** obtained by exponentiating each entry of A.

We can also define the matrix exponential when we multiply all entries of A by the variable t, repeating the definition.

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k,$$

which produces a matrix function. The derivative of this function is calculated the same way as with real variable functions,

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

It is thanks to this property that we see that indeed, the general solution of the system $X' = AX$ is precisely the matrix $e^{At}C$, where C is a parameter vector. In other words, the fundamental matrix of the system $X' = AX$ is $\Phi(t) = e^{At}$.

Theorem 5.4 Properties of the Matrix Exponential.

The matrix exponential function satisfies the following properties:

- $e^O = I$, where O denotes the null matrix.
- $e^{At}e^{As} = e^{A(t+s)}$, where t and s are scalars.
- When $AB = BA$, then $e^A e^B = e^{A+B}$. This can be false if A and B do not commute.
- $(e^{At})^{-1} = e^{-tA}$, where t is a scalar.
- When A is a diagonal matrix, we can calculate e^A simply by exponentiating each entry. That is,

$$\exp\begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} e^{a_{11}} & 0 & \dots & 0 \\ 0 & e^{a_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{a_{nn}} \end{pmatrix}.$$

These properties help us calculate some exponentials more easily.

5.5 Variation of Parameters

So far, we have not solved any non-homogeneous system, that is, of the form

$$X'(t) = A(t)X(t) + F(t) \tag{5.3}$$

where $F(t)$ is not null. Recall that the general procedure for solving non-homogeneous systems is to find the complementary solution of the homogeneous X_c and then a particular solution X_p. The general solution would be given by $X_c + X_p$, as we already know. To find X_p we will use variation of parameters.

Theorem 5.5 Variation of Parameters for Systems.

To solve system (6.1), we take the fundamental matrix $\Phi(t)$, and calculate the particular solution X_p using the formula

$$X_p(t) = \Phi(t) \int \Phi^{-1}(t)F(t)dt,$$

where $\Phi^{-1}(t)$ denotes the inverse of the fundamental matrix, and the integral is calculated entry by entry.

Example 5.11

Solve the system

$$X' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix} X + \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}.$$

First we have to solve

$$X' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix} X,$$

which we do by means of eigenvalues. The eigenvalues of this matrix are -2 and -5, and their associated eigenvectors are $v_1 = (1,1)^t$ and $v_2 = (1,-2)$ (exercise). Then we have that

$$X_c = C_1 \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{-5t} \\ -2e^{-5t} \end{pmatrix}.$$

The fundamental matrix of this system is therefore

$$\Phi(t) = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}.$$

Recall that the inverse of a 2×2 matrix is very easy to calculate, by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

This helps us calculate

$$\Phi^{-1}(t) = \begin{pmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{pmatrix}.$$

Then, we only need to apply the variation of parameters formula.

$$\begin{aligned}
X_p &= \Phi(t) \int \Phi^{-1}(t) F(t) dt \\
&= \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix} \int \begin{pmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{pmatrix} \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix} dt \\
&= \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix} \int \begin{pmatrix} 2te^{2t} + \frac{1}{3}e^{t} \\ te^{5t} - \frac{1}{3}e^{4t} \end{pmatrix} dt \\
&= \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix} \begin{pmatrix} te^{2t} - \frac{1}{2}e^{2t} + \frac{1}{3}e^{t} \\ \frac{1}{5}te^{5} - \frac{1}{25}e^{5t} - \frac{1}{12}e^{4t} \end{pmatrix} \\
&= \begin{pmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{1}{4}e^{-t} \\ \frac{3}{5}t - \frac{21}{50} - \frac{1}{2}e^{-t} \end{pmatrix}.
\end{aligned}$$

Remember that to evaluate the integral of a vector, we simply integrate each of its entries separately. So we have that the general solution of the equation is

$$X = X_c + X_p = C_1 \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{-5t} \\ -2e^{-5t} \end{pmatrix} + \begin{pmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{1}{4}e^{-t} \\ \frac{3}{5}t - \frac{21}{50} - \frac{1}{2}e^{-t} \end{pmatrix}.$$

5.6 Higher Order Equations and Systems

Throughout this section, we have noted many similarities with the section on higher order equations: solution space, Wronskian criterion, complementary solution, particular solution. Practically all concepts have their counterpart in the topic of higher order ODEs. The reason for this is very simple: **Every higher order linear differential equation can be converted into a system of first order linear equations.** The way to do this is very simple, we simply introduce additional variables, one for each derivative.

Example 5.12

Consider the differential equation

$$x(t)''' + 3x(t)'' + 3x(t)' + x(t) = 0.$$

We know, by the theory of linear ODEs with constant coefficients, that the general solution is

$$x = C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t}.$$

We are going to convert it into a first order system. Define the variables $x_1 = x$, $x_2 = x'$, $x_3 = x''$. Then the original equation becomes

$$x_3' = -x_1 - 3x_2 - 3x_3.$$

Now, differentiating the definitions of the new variables, we obtain enough equations to convert the higher order equation into the system of equations

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -x_1 - 3x_2 - 3x_3 \end{cases}.$$

Taking $X = (x_1, x_2, x_3)^t$, we must solve the system

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{pmatrix} X.$$

The only eigenvalue of this matrix is -1. We solve the system

$$(A+I)v = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -3 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

We find that the only l.i. eigenvector of this matrix is $v_0 = (1, -1, 1)^t$. Then we solve the system $(A+I)v = v_0$, whose solution is v_1, and then the system $(A+I)v = v_1$ whose solution is v_2. In summary, the generalized eigenvectors are $v_1 = (1, 0, -1)^t$ and $v_2 = (1, 0, 0)^t$. We can follow the formula we studied for this case, to find the solutions

$$X_1 = e^{-t} v_0 = \begin{pmatrix} e^{-t} \\ -e^{-t} \\ e^{-t} \end{pmatrix}$$

$$X_2 = e^{-t}(v_1 + t v_0) = e^{-t} \begin{pmatrix} 1+t \\ -t \\ -1+t \end{pmatrix}$$

$$X_3 = e^{-t}(v_2 + t v_1 + \frac{t^2}{2} v_0) = e^{-t} \begin{pmatrix} 1+t+\frac{t^2}{2} \\ -\frac{t^2}{2} \\ -t+\frac{t^2}{2} \end{pmatrix}$$

So the general solution of the system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = K_1 \begin{pmatrix} e^{-t} \\ -e^{-t} \\ e^{-t} \end{pmatrix} + K_2 e^{-t} \begin{pmatrix} 1+t \\ -t \\ -1+t \end{pmatrix} + K_3 e^{-t} \begin{pmatrix} 1+t+\frac{t^2}{2} \\ -\frac{t^2}{2} \\ -t+\frac{t^2}{2} \end{pmatrix}.$$

Since $x_1 = x$, the solution of the original equation is given by the first entry of this vector,

$$x = K_1 e^{-t} + K_2 e^{-t}(1+t) + K_3 e^{-t}(1+t+\frac{t^2}{2}).$$

To arrive at the same solution, we must make a small renaming of constants. Take $C_1 = K_1 + K_2 + K_3$, $C_2 = K_2 + K_3$, and $C_3 = \frac{K_3}{2}$, to arrive at the solution we obtained at the beginning.

$$x = C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t}.$$

It is clear however, that in most cases this method is not efficient to apply. It is much simpler to just find the zeros of the characteristic equation.

6

The Laplace Transform

In this section we will study the Laplace Transform (LT), a very powerful tool that will allow us to solve differential equations much more easily.

Definition 6.1 Laplace Transform.

Let $f : [0, \infty) \to \mathbb{R}$ be a piecewise continuous function. We define its **Laplace transform** as the improper integral

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt \tag{6.1}$$

for all s where the integral converges.

Let us emphasize the nature of the Laplace transform: it receives a function $f(t)$ and returns another function of a different (and new) variable s. Normally, for notation purposes, we denote the transform $\mathcal{L}\{f\}(s)$ simply as $F(s)$.

Example 6.1

The transform of $f(t) = t$. Let us calculate $F(s)$:

$$F(s) = \int_0^\infty te^{-st}dt = \left[-\frac{1}{s}te^{-st}\right]_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}dt = \frac{1}{s^2}$$

by integration by parts, useful whenever $s > 0$. We then have that $F(s) = s^{-2}$, that is, $\mathcal{L}\{t\}(s) = \frac{1}{s^2}$.

Example 6.2

The transform of $f(t) = e^{at}$.

$$\mathcal{L}\{e^{at}\}(s) = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{(a-s)t}dt = \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^\infty$$

In this case, when evaluating at ∞, the limit is the function $e^{(a-s)t}$, which only vanishes when $a - s < 0$. Therefore, when $s > a$, we have

$$\frac{1}{a-s}(0-1) = \frac{1}{s-a}.$$

We then have that $\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}$.

A table with many of the Laplace transforms we need is given below:

$f(t)$	$\mathcal{L}\{f\}(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$
$t\sin(kt)$	$\frac{2ks}{(s^2+k^2)^2}$
$t\cos(kt)$	$\frac{s^2-k^2}{(s^2+k^2)^2}$

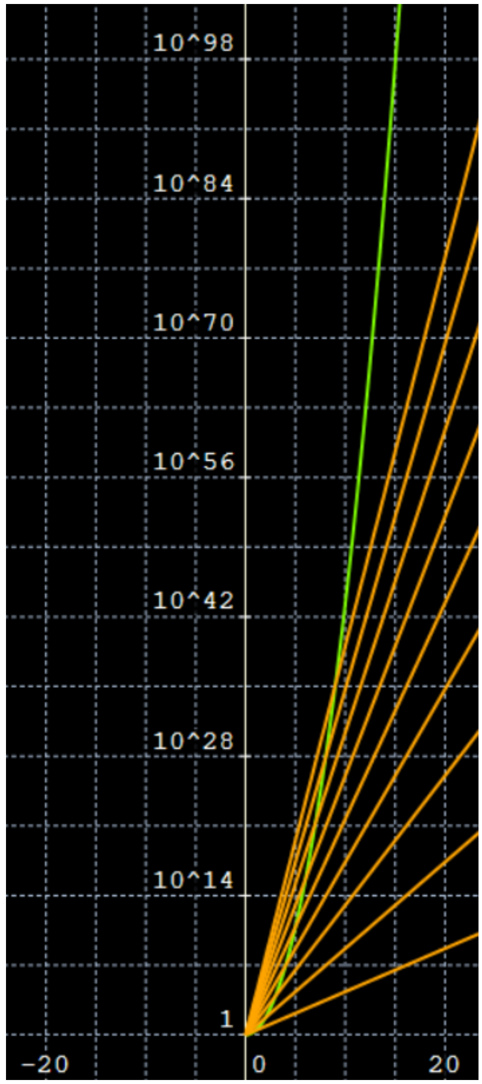

Figure 6.1: Graphs of exponential functions showing typical Laplace transform behavior.

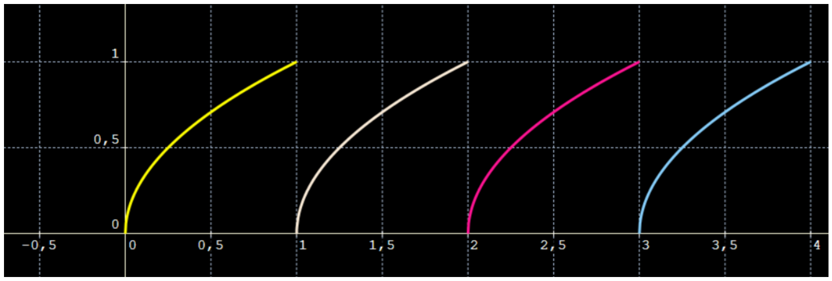

Figure 6.2: Shifted Heaviside-type functions for different values of a.

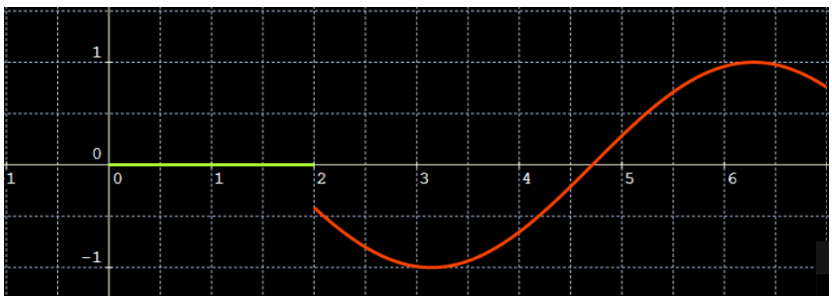

Figure 6.3: Combination of step function (Heaviside) with sinusoidal function.

6.1 Properties

The Laplace transform is linear:

$$\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}.$$

With this property alone, we can already calculate several more transforms:

Example 6.3

The transform $\mathcal{L}\{5e^{3t}\}$ is, by linearity,

$$\mathcal{L}\{5e^{3t}\} = 5\mathcal{L}\{e^{3t}\} = \frac{5}{s-3}.$$

The transform $\mathcal{L}\{4t^3 + 3\sin(2t)\}$ is, by linearity,

$$\mathcal{L}\{4t^3 + 3\sin(2t)\} = 4\frac{3!}{s^4} + 3\frac{2}{s^2+4} = \frac{24}{s^4} + \frac{6}{s^2+4}.$$

The following theorem summarizes most of the important properties of the Laplace Transform. We will assume that $F(s) = \mathcal{L}\{f(t)\}(s)$.

Identity	Result
Translation in s	$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
Derivative of F	$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$
Transform of the derivative	$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
Transform of the n-th derivative	$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
Transform of the integral	$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{F(s)}{s}$

The properties related to the transform of derivatives are the most important. They will be the key to converting differential equations into algebraic equations, which we can solve.

6.2 Special Functions

In this section we are going to study the properties of some special functions that we have not yet used in the course.

6.2.1 The Heaviside Function

The **unit step function**, or Heaviside step function, is defined as

$$\mathcal{U}(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

This function will help us define other functions by cases.

A translated version of this step function is $\mathcal{U}(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$

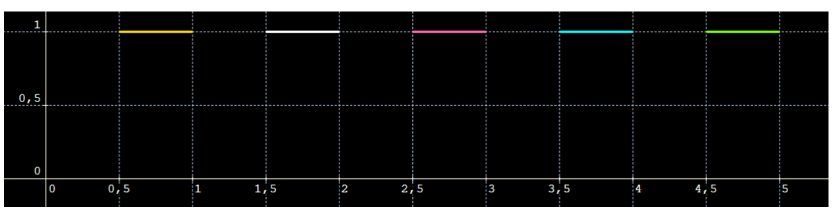

Figure 6.4: Heaviside functions $\mathcal{U}(t-a)$ for different values of a.

Example 6.4

The function $f(t) = \begin{cases} g(t) & t < a \\ h(t) & t \geq a \end{cases}$ can be rewritten as $f(t) = g(t) + (h(t) - g(t))\mathcal{U}(t - a)$.

In fact, a function of three or more cases can also be written using the function $\mathcal{U}$, simply by adding more copies of $\mathcal{U}$.

Example 6.5

The function $f(t) = \begin{cases} 2 & 0 \leq t < 4 \\ -1 & 4 \leq t < 6 \\ 3 & t \geq 6 \end{cases}$ can be rewritten as $f(t) = 2 - 3\mathcal{U}(t-4) + 4\mathcal{U}(t-6)$.

The Laplace transform of $\mathcal{U}(t-a)$ is $\boxed{\mathcal{L}\{\mathcal{U}(t-a)\}(s) = \frac{e^{-as}}{s}}$. Furthermore, just as e^{at} is a factor that allows us to transform our function with respect to s, the factor $\mathcal{U}$ allows us to transform the argument of the function. That is to say,

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\}(s) = e^{-as}F(s).$$

This property helps us calculate the transform of piecewise functions.

Example 6.6

We can calculate the Laplace transform of $f(t) = \begin{cases} 0 & t < 2\pi \\ \cos t & t \geq 2\pi \end{cases}$ as follows:

$$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{\cos(t-2\pi)\mathcal{U}(t-2\pi)\}(s) = e^{-2\pi s}\mathcal{L}\{\cos(t)\}(s) = \frac{se^{-2\pi s}}{s^2+1}.$$

6.2.2 The Dirac Delta

The next function we will study will help us model instantaneous phenomena, for example, an explosion, or an electric shock.

Definition 6.2 Dirac Delta δ.

The **Dirac delta** $\delta(t)$ is a function whose value is

$$\delta(t) = \begin{cases} \infty & : t = 0 \\ 0 & : t \neq 0 \end{cases}$$

and furthermore (magically?) $\int_{-\infty}^{\infty} \delta(t)dt = 1$. Actually, the function is an idealization of a *distribution*, which would explain this last property. However rigorously, the Dirac delta is a generalized function. For a more formal definition, we would need to study measure theory.

This function also has a translation $\delta(t-t_0)$. Let us now calculate its Laplace transform.

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^{\infty} \delta(t-t_0)e^{-st}dt.$$

Using the property that the only non-zero value of $\delta(t - t_0)$ occurs at $t = t_0$, we can see that this integral reduces to e^{-st_0}, that is

$$\boxed{\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}.}$$

Particularly, $\mathcal{L}\{\delta(t)\} = 1$.

6.2.3 The Gamma Function

We also introduce the Gamma function, since it helps us with the Laplace transform.

Definition 6.3 Gamma Function Γ.

The **Gamma function** is defined as the integral

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt \quad , \quad n > 0.$$

Functions defined as integrals are sometimes called *special functions*. Particularly, the Gamma function allows us to generalize the factorial $n!$ to any real number n: $\Gamma(n+1) = n!$. It also follows that $\Gamma(n+1) = n \cdot \Gamma(n)$ for all $n > 0$.

With the Gamma function, we can generalize the identity $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ for any $n > 0$ (not necessarily an integer):

$$\mathcal{L}\{t^n\}(s) = \frac{\Gamma(n+1)}{s^{n+1}} \text{ for all } n > 0.$$

6.3 Convolution

The convolution is a binary operation on functions, similar in some sense to a kind of product. However, we will be interested in how this operation interacts with the Laplace transform.

Definition 6.4 Convolution.

Let f, g be piecewise continuous functions in $[0, \infty)$. The **convolution** of f and g is defined as

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau) d\tau.$$

Theorem 6.1 Convolution Theorem.

If $\mathcal{L}\{f\} = F$ and $\mathcal{L}\{g\} = G$, then

$$\mathcal{L}\{f * g\}(s) = F(s)G(s).$$

Example 6.7

Since the transforms of t and $\sin(t)$ are $\frac{1}{s^2}$ and $\frac{1}{s^2+1}$ respectively, it must hold that $\mathcal{L}\{t * \sin(t)\}(s) = \frac{1}{s^2(s^2+1)}$.

6.4 Inverse Transform

To be able to use the Laplace transform, we also need its inverse, that is, to calculate from a function $F(s)$ a function $f(t)$.

Definition 6.5 Inverse Laplace Transform.

Let $F(s)$ be the Laplace transform of some function $f(t)$. We denote the inverse transform as $\mathcal{L}^{-1}\{F\}(t) = f(t)$.

In principle, to calculate the inverse Laplace, we need to be able to resolve a function $F(s)$ as an expression of known transforms. For this we make extensive use of the technique of **partial fractions,** which we already know from the integration lesson.

Example 6.8

Calculate $\mathcal{L}^{-1}\left\{\frac{1}{s^2-9}\right\}$..
First, let us decompose into partial fractions. Note that $(s-3)(s+3) = s^2-9$, so we have

$$\frac{1}{s^2-9} = \frac{A}{s+3} + \frac{B}{s-3}$$

for some constants A, B. From here we deduce that $A = -1/6$ and $B = 1/6$, so

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-9}\right\} = \mathcal{L}^{-1}\left\{\frac{-1/6}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{1/6}{s-3}\right\} = \frac{1}{6}e^{3t} - \frac{1}{6}e^{-3t}.$$

Example 6.9

Calculate $\mathcal{L}^{-1}\left\{\frac{s}{(s-4)(s^2+4s+5)}\right\}$.
This is a case with a non-factorable quadratic denominator. In this case, the partial fraction decomposition has the form

$$\frac{s}{(s-4)(s^2+4s+5)} = \frac{A}{s-4} + \frac{Bs+C}{s^2+4s+5}$$

from where the constants can be deduced as $A = 4/37$, $B = -4/37$ and $C = -6/37$. The inverse Laplace transform in this case is a bit more complicated, since we must use the technique of completing the square to evaluate the inverse of $\frac{Bs+C}{s^2+4s+5}$, since

$$\frac{Bs+C}{s^2+4s+5} = \frac{Bs+C}{(s+2)^2+1}.$$

We leave the details to the reader, the answer is

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-4)(s^2+4s+5)}\right\} = \frac{4}{37}e^{4t} + \frac{1}{37}e^{-2t}(2\sin(t) - 4\cos(t)).$$

6.5 Solving Differential Equations

We are now in a position to give the method for solving initial value problems using the Laplace Transform.

1. Apply the LT to both sides of the equation, using the identity
$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$
2. Substitute the initial values and isolate the transform $\mathcal{L}\{f\}$.
3. Use the inverse LT to find the solution f.

Example 6.10

Solve the problem $\begin{cases} y'' - 4y' + 4y = t^3e^{2t} \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

Applying the LT to both sides:

$$\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\{t^3e^{2t}\}$$
$$\implies (s^2Y - sy(0) - y'(0)) - 4(sY - y(0)) + 4Y = \frac{6}{(s-2)^4}$$

where $Y = \mathcal{L}\{y\}$. Substituting the initial conditions (which are 0) and isolating Y:

$$s^2Y - 4sY + 4Y = \frac{6}{(s-2)^4}$$
$$\implies Y = \frac{6}{(s-2)^4(s^2-4s+4)} = \frac{6}{(s-2)^6}$$

So we must invert the transform, which thanks to the table gives us

$$\mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\} = \frac{1}{20}t^5e^{2t}.$$

The solution to the problem is $y(t) = \frac{t^5}{20}e^{2t}$.

Exercises

Solve the following differential equations using Laplace transforms.

1. $\begin{cases} y'' - 6y = 0 \\ y(0) = 1 \\ y'(0) = -1 \end{cases}$

2. $\begin{cases} y - y'' = 1 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

3. $\begin{cases} y'' + 4y' + 3y = 1 - \mathcal{U}(t-2) - \mathcal{U}(t-4) + \mathcal{U}(t-6) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

4. The ODE $y'' + y = \delta(t - 2\pi)$ with $y(0) = 1$ and $y'(0) = 0$.

6.6 Integral and Integro-differential Equations

A somewhat more exotic application of the Laplace transform is to solve differential equations where both derivatives and integrals of the unknown function appear.

Definition 6.6 Integro-differential Equation.

An **integro-differential** equation for a function y is an equation where both derivatives and integrals of the function appear, for example

$$y' + 6y + 9\int_0^t y dt = 1$$

is an integro-differential equation. An **integral equation** is an equation of this type, where only integrals of the function appear. For example

$$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}d\tau.$$

is an integral equation.

Example 6.11

Solve the problem

$$\begin{cases} y' + 6y + 9\int_0^t y dt = 1 \\ y(0) = 0 \end{cases}$$

Recall that the LT of the integral is $\mathcal{L}\{\int_0^t y dt\} = \frac{Y}{s}$.
When applying the LT, we have

$$sY - y(0) + 6Y + \frac{9Y}{s} = \frac{1}{s}$$

$$\implies sY + 6Y + \frac{9Y}{s} = \frac{1}{s}$$

Multiplying by s and isolating Y:

$$s^2Y + 6sY + 9Y = 1 \implies Y(s^2 + 6s + 9) = 1 \implies Y = \frac{1}{(s+3)^2}$$

Inverting, we have $y(t) = te^{-3t}$ which is the solution to the problem.

Example 6.12

Solve the integral equation

$$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}d\tau.$$

First observe that under the integral we have a convolution: $\int_0^t f(\tau)e^{t-\tau}d\tau = (f * e^{-t})(t)$. So we can apply the LT directly

$$F = \frac{6}{s^3} - \frac{1}{s+1} - \frac{F}{s+1}$$

Multiplying by $(s+1)$:

$$F(s+1) = \frac{6(s+1)}{s^3} - 1 - F$$

$$\implies F(s+2) = \frac{6s+6}{s^3} - 1 = \frac{6s+6-s^3}{s^3}$$

$$\implies F = \frac{6s+6-s^3}{s^3(s+2)}$$

The partial fraction decomposition is $F = -\frac{7}{4(s+2)} + \frac{1}{4s} + \frac{3}{s^2} - \frac{3}{s^3}$ so the solution to the integral equation is $f(t) = -\frac{7}{4}e^{-2t} + \frac{1}{4} + 3t - \frac{3}{2}t^2$.

6.7 Systems of ODEs

To conclude the course, we are going to study an alternative method to solve systems of differential equations, which uses the Laplace transform.

Example 6.13

Solve the system

$$\begin{cases} x' = -x + y \\ y' = 2x \\ x(0) = 0 \\ y(0) = 1 \end{cases}$$

Applying the LT to each equation, and substituting the initial values, we have (denoting $X = \mathcal{L}\{x\}$, $Y = \mathcal{L}\{y\}$):

$$\begin{cases} sX = -X + Y \\ sY - 1 = 2X \end{cases}$$

This is a **linear** system of equations in X and Y. We must solve it to find the values of the transforms.
From the first equation: $(s+1)X = Y$. Substituting into the second: $s(s+1)X - 1 = 2X \implies X(s^2 + s - 2) = 1$. From where $X = \frac{1}{(s+2)(s-1)}$. We can then also calculate $Y = (s+1)X = \frac{s+1}{(s+2)(s-1)}$.
Using partial fractions, we can now invert the transforms.

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1/3}{s+2} + \frac{1/3}{s-1}\right\} = \frac{1}{3}(-e^{-2t} + e^t)$$
$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+2)(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1/3}{s+2} + \frac{2/3}{s-1}\right\} = \frac{1}{3}(-e^{-2t} + 2e^t)$$

So the solution to the system is

$$\mathbf{X}(t) = \frac{1}{3}\begin{pmatrix} -e^{-2t} + e^t \\ -e^{-2t} + 2e^t \end{pmatrix}.$$

Exercises

Solve the following systems using the Laplace transform.

1. $\begin{cases} \dot{x} = 2y + e^t \\ \dot{y} = 8x - t \\ x(0) = y(0) = 0 \end{cases}$.

2. $\begin{cases} \dot{x} = 4x - 2y \\ \dot{y} = 3x - y \\ x(0) = 1,\ y(0) = -1 \end{cases}$.