The Finite Free Stam Inequality

1 Setup and statement

Let $p(x) = \sum_{k=0}^{n} a_k x^{n-k}$ and $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$ be monic ($a_0 = b_0 = 1$) real-rooted polynomials of degree n. Their *symmetric additive convolution* is

$$(p \boxplus_n q)(x) = \sum_{k=0}^{n} c_k x^{n-k}, \qquad c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i\, b_j.$$

For $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$ with distinct roots define the *scores* and *finite free Fisher information*:

$$V_i := \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad \Phi_n(p) := \sum_{i=1}^{n} V_i^2,$$

with $\Phi_n(p) := \infty$ when p has a repeated root.

Definition 1.1 (Variance)**.** $\sigma^2(p) := \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} (\lambda_i - \bar{\lambda})^2$ where $\bar{\lambda} = \frac{1}{n} \sum_i \lambda_i$.

Definition 1.2 (Score–gap form)**.** $\mathcal{S}(p) := \displaystyle\sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}$.

Theorem 1.1 (Finite Free Stam Inequality)**.** *For $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots,*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}. \tag{1}$$

We prove (1) for all degrees. Explicit computations handle $n = 2$ (Section 4) and $n = 3$ (Section 5). For general n we establish a pointwise score–gap inequality (Theorem 7.1) and combine it with a convolution-flow argument (Section 7).

2 Preliminary identities

All polynomials below are monic of degree n with distinct roots.

Lemma 2.1 (Score–root identity)**.** $\displaystyle\sum_{i=1}^{n} \lambda_i V_i = \frac{n(n-1)}{2}$.

Proof. $\sum_i \lambda_i V_i = \sum_i \sum_{j \neq i} \frac{\lambda_i}{\lambda_i - \lambda_j} = \sum_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_j} = \binom{n}{2}$. □

Lemma 2.2 (Score sum)**.** $\sum_{i=1}^{n} V_i = 0$.

Proof. $\sum_i V_i = \sum_i \sum_{j\neq i} \frac{1}{\lambda_i-\lambda_j} = \sum_{i<j}\left(\frac{1}{\lambda_i-\lambda_j} + \frac{1}{\lambda_j-\lambda_i}\right) = 0.$ ☐

Lemma 2.3 (Score–gap identity). $\Phi_n(r) = \displaystyle\sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j}.$

Proof. $\sum_i V_i^2 = \sum_i V_i \sum_{j\neq i} \frac{1}{\lambda_i-\lambda_j} = \sum_{i\neq j} \frac{V_i}{\lambda_i-\lambda_j} = \sum_{i<j}\left(\frac{V_i}{\lambda_i-\lambda_j} + \frac{V_j}{\lambda_j-\lambda_i}\right) = \sum_{i<j} \frac{V_i-V_j}{\lambda_i-\lambda_j}.$ ☐

Lemma 2.4 (Score via derivatives). $V_i = \dfrac{r''(\lambda_i)}{2\,r'(\lambda_i)}$, *where* $r = p$.

Proof. Since $r'(\lambda_i) = \prod_{j\neq i}(\lambda_i - \lambda_j)$, differentiating $r'(x) = \sum_{i=1}^n \prod_{j\neq i}(x-\lambda_j)$ yields $r''(\lambda_i) = 2\sum_{k\neq i}\prod_{j\neq i,\, j\neq k}(\lambda_i - \lambda_j) = 2\,r'(\lambda_i)\sum_{k\neq i}\frac{1}{\lambda_i-\lambda_k} = 2\,r'(\lambda_i)\,V_i$. ☐

Lemma 2.5 (Fisher–variance inequality). $\Phi_n(p)\,\sigma^2(p) \geq \dfrac{n(n-1)^2}{4}$, *with equality iff* V_i *is proportional to* $\lambda_i - \bar\lambda$ *(which always holds when* $n = 2$*).*

Proof. By Cauchy–Schwarz, $\left(\sum_i \lambda_i V_i\right)^2 \leq \left(\sum_i \lambda_i^2\right)\left(\sum_i V_i^2\right) = n\,\sigma^2(p)\,\Phi_n(p)$. By Lemma 2.1 the left side is $n^2(n-1)^2/4$. ☐

Lemma 2.6 (Variance additivity). $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

Proof. The coefficient formula gives $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2$, so the variance (a function of c_1, c_2 alone) is additive. ☐

3 Critical-value formula for Φ_n

Theorem 3.1 (Critical-value formula). *Let* r *be a monic polynomial of degree* n *with distinct roots, and let* $\zeta_1, \ldots, \zeta_{n-1}$ *be the zeros of* r' *(assumed simple). Then*

$$\Phi_n(r) = -\frac{1}{4}\sum_{j=1}^{n-1} \frac{r''(\zeta_j)}{r(\zeta_j)}. \tag{2}$$

Proof. By Lemma 2.4, $\Phi_n = \frac{1}{4}\sum_{i=1}^n \frac{r''(\lambda_i)^2}{r'(\lambda_i)^2}$. Consider the meromorphic function

$$F(x) = \frac{r''(x)^2}{r'(x)\,r(x)}.$$

Residues at the roots λ_i. Since r has a simple zero at λ_i,

$$\operatorname{Res}_{x=\lambda_i} F = \frac{r''(\lambda_i)^2}{r'(\lambda_i)\cdot r'(\lambda_i)} = \frac{r''(\lambda_i)^2}{r'(\lambda_i)^2}.$$

Summing gives $\sum_i \operatorname{Res}_{\lambda_i} F = 4\Phi_n$.
Residues at the critical points ζ_j. Since r' has a simple zero at ζ_j,

$$\operatorname{Res}_{x=\zeta_j} F = \frac{r''(\zeta_j)^2}{r''(\zeta_j)\,r(\zeta_j)} = \frac{r''(\zeta_j)}{r(\zeta_j)}.$$

Residue at infinity. $F(x) \sim n(n-1)^2/x^3$ as $x \to \infty$, so $\operatorname{Res}_\infty F = 0$.
The global residue theorem gives $4\Phi_n + \sum_j r''(\zeta_j)/r(\zeta_j) = 0$. ☐

Remark 3.1. This formula connects Φ_n to the *critical values* of the polynomial: the values $r(\zeta_j)$ at its critical points. It generalizes the classical relation between the discriminant and critical values, and was verified numerically for $3 \leq n \leq 7$.

4 Case $n = 2$: equality

Proposition 4.1. *For $n = 2$, inequality (1) holds with equality.*

Proof. $\Phi_2(p) = 2/(\lambda_1 - \lambda_2)^2$, so $1/\Phi_2(p) = 2\sigma^2(p)$. By Lemma 2.6, $1/\Phi_2(p \boxplus_2 q) = 2\sigma^2(p \boxplus_2 q) = 2\sigma^2(p) + 2\sigma^2(q) = 1/\Phi_2(p) + 1/\Phi_2(q)$. □

5 Case $n = 3$: proof by residue calculus

Since Φ_n and σ^2 are translation-invariant, we assume p and q centered throughout this section. A centered monic cubic is $r(x) = x^3 - Sx + T$ with $S \geq 0$ and discriminant $\Delta = 4S^3 - 27T^2 > 0$.

Proposition 5.1 (Closed-form Fisher information for cubics)**.**

$$\Phi_3(r) = \frac{18S^2}{\Delta} = \frac{18S^2}{4S^3 - 27T^2}. \tag{3}$$

Proof. Apply Theorem 3.1. Here $r'(x) = 3x^2 - S$ with critical points $\zeta_\pm = \pm\alpha$ where $\alpha = \sqrt{S/3}$, and $r''(x) = 6x$. The critical values are

$$r(\alpha) = T - \tfrac{2S^{3/2}}{3\sqrt{3}}, \qquad r(-\alpha) = T + \tfrac{2S^{3/2}}{3\sqrt{3}},$$

and their product is $r(\alpha)\,r(-\alpha) = T^2 - 4S^3/27 = -\Delta/27$. Then

$$4\Phi_3 = -\frac{6\alpha}{r(\alpha)} + \frac{6\alpha}{r(-\alpha)} = 6\alpha \cdot \frac{r(\alpha) - r(-\alpha)}{r(\alpha)\,r(-\alpha)}.$$

Since $r(\alpha) - r(-\alpha) = -(4S\alpha/3)$ and $\alpha^2 = S/3$:

$$4\Phi_3 = 6\alpha \cdot \frac{-4S\alpha/3}{-\Delta/27} = \frac{8S\alpha^2 \cdot 27}{\Delta} = \frac{72S^2}{\Delta}.$$ □

Proposition 5.2 (Cubic convolution is additive)**.** *For centered monic cubics $p(x) = x^3 - S_1 x + T_1$ and $q(x) = x^3 - S_2 x + T_2$,*

$$(p \boxplus_3 q)(x) = x^3 - (S_1 + S_2)\,x + (T_1 + T_2).$$

Proof. With $a_0 = b_0 = 1$, $a_1 = b_1 = 0$, $a_2 = -S_1$, $b_2 = -S_2$, $a_3 = T_1$, $b_3 = T_2$, the coefficient formula gives $c_0 = 1$, $c_1 = 0$,

$$c_2 = \frac{1! \cdot 3!}{3! \cdot 1!}\, a_2 + \frac{3! \cdot 1!}{3! \cdot 1!}\, b_2 = a_2 + b_2 = -(S_1 + S_2),$$

and

$$c_3 = \frac{0! \cdot 3!}{3! \cdot 0!}\, a_3 + \frac{3! \cdot 0!}{3! \cdot 0!}\, b_3 = a_3 + b_3 = T_1 + T_2,$$

where all cross-terms with $a_1 = b_1 = 0$ vanish. □

Theorem 5.3 (Stam inequality for cubics)**.** *For $n = 3$, inequality (1) holds. Equality holds if and only if $T_1 = T_2 = 0$, i.e. both polynomials have roots of the form $\{-a, 0, a\}$.*

Proof. By Propositions 5.1 and 5.2,

$$\frac{1}{\Phi_3(r)} = \frac{\Delta}{18S^2} = \frac{2S}{9} - \frac{3T^2}{2S^2}.$$

Thus (1) reads

$$\frac{2(S_1+S_2)}{9} - \frac{3(T_1+T_2)^2}{2(S_1+S_2)^2} \;\geq\; \frac{2S_1}{9} + \frac{2S_2}{9} - \frac{3T_1^2}{2S_1^2} - \frac{3T_2^2}{2S_2^2}.$$

The linear terms cancel, and after multiplying by $-2/3$ the inequality reduces to

$$\frac{(T_1+T_2)^2}{(S_1+S_2)^2} \;\leq\; \frac{T_1^2}{S_1^2} + \frac{T_2^2}{S_2^2}. \tag{4}$$

Set $\alpha = S_1/(S_1+S_2) \in (0,1)$, $\beta = 1-\alpha$, $u = T_1/S_1$, $v = T_2/S_2$. The left side is $(\alpha u + \beta v)^2$. By convexity of $t \mapsto t^2$ and the weights $\alpha + \beta = 1$:

$$(\alpha u + \beta v)^2 \;\leq\; \alpha u^2 + \beta v^2 \;\leq\; u^2 + v^2,$$

where the second step uses $\alpha, \beta \leq 1$, proving (4).

Equality holds throughout iff $u = v$ (Jensen) and $\alpha u^2 = (1-\beta)u^2 = u^2$, i.e. $\beta = 0$ or $u = 0$. Since $\beta > 0$, equality requires $u = v = 0$, i.e. $T_1 = T_2 = 0$. □

6 Convolution-flow framework

For general n we employ the convolution semigroup. Assume q centered with variance $b := \sigma^2(q) > 0$ and set $a := \sigma^2(p) > 0$.

Definition 6.1 (Fractional semigroup)**.** Set $\kappa_k := \frac{(n-k)!}{n!}\, b_k$ and define q_t by the coefficients $b_k(t) = \frac{n!}{(n-k)!}\, \kappa_k^t$. Then $q_0 = x^n$, $q_1 = q$, and $q_s \boxplus_n q_t = q_{s+t}$. The variance scales linearly: $\sigma^2(q_t) = t\, b$.

Write $p_t := p \boxplus_n q_t$.

Lemma 6.1 (Root-derivative formula)**.** *If p_t has simple roots $\lambda_i(t)$ depending smoothly on t, then $\dot{\lambda}_i = -\partial_t p_t(\lambda_i)/p_t'(\lambda_i)$.*

Proof. Differentiate $p_t(\lambda_i(t)) = 0$ in t. □

Lemma 6.2 (Root shift)**.** $\lambda_i(t) = \lambda_i(0) + \frac{tb}{n-1}\, V_i(0) + O(t^2)$.

Proof. Apply Lemma 6.1 at $t = 0$ and use the coefficient formula for $\partial_t p_t|_{t=0}$. □

Lemma 6.3 (Dissipation identity)**.**

$$\frac{d}{dt}\Phi_n(p_t) = -\frac{2b}{n-1}\, \mathcal{S}(p_t). \tag{5}$$

Proof. By the semigroup property, $p_{t+h} = p_t \boxplus_n q_h$ with $\sigma^2(q_h) = hb$. Expanding $\Phi_n(p_{t+h})$ via Lemma 6.2 at order h: linear terms cancel by $\sum V_i = 0$, and the quadratic term gives (5). □

Corollary 6.4 (Integral identity)**.**

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} = \frac{2b}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2}\, dt. \tag{6}$$

Proof. $\frac{d}{dt}\frac{1}{\Phi_n(p_t)} = -\frac{\dot{\Phi}_n(p_t)}{\Phi_n(p_t)^2} = \frac{2b}{n-1}\frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2}$. Integrate from 0 to 1. □

7 General n: proof of the Stam inequality

Theorem 7.1 (Pointwise score–gap inequality)**.** *For every $r \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots,*

$$\mathcal{S}(r)\,\sigma^2(r) \;\geq\; \frac{n-1}{2}\,\Phi_n(r). \tag{7}$$

Equality holds if and only if there exists a constant c such that $V_i = c(\lambda_i - \bar{\lambda})$ for all i.

Proof. Set $T = \sum_{i=1}^n (\lambda_i - \bar{\lambda})^2 = n\,\sigma^2(r)$, $U = \Phi_n(r)$, $S = \mathcal{S}(r)$. The inequality is equivalent to $S\,T \geq \frac{n(n-1)}{2}\,U$.

Step 1 (Fisher–variance bound). By Lemmas 2.1 and 2.2, $\sum_{i=1}^n (\lambda_i - \bar{\lambda})\,V_i = \sum_{i=1}^n \lambda_i V_i - \bar{\lambda}\sum_{i=1}^n V_i = \frac{n(n-1)}{2}$. Cauchy–Schwarz gives

$$\frac{n^2(n-1)^2}{4} \;\leq\; T\,U. \tag{8}$$

(This is Lemma 2.5 restated as $T\,U \geq n^2(n-1)^2/4$.)

Step 2 (Score–gap bound). By Lemma 2.3, $U = \sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j}$. Cauchy–Schwarz gives

$$U^2 \;\leq\; S \cdot \binom{n}{2} = \frac{n(n-1)}{2}\,S, \tag{9}$$

i.e., $S \geq \frac{2\,U^2}{n(n-1)}$.

Step 3 (Combination). From Steps 1 and 2:

$$S\,T \;\geq\; \frac{2\,U^2}{n(n-1)} \cdot T = \frac{2\,U \cdot (U\,T)}{n(n-1)} \;\geq\; \frac{2\,U \cdot \frac{n^2(n-1)^2}{4}}{n(n-1)} = \frac{n(n-1)}{2}\,U.$$

Equality. Equality requires both (8) and (9) to be equalities. Equality in (8) holds iff the vectors $(\lambda_i - \bar{\lambda})_i$ and $(V_i)_i$ are proportional, i.e., $V_i = c(\lambda_i - \bar{\lambda})$ for some constant c. Equality in (9) holds iff $\frac{V_i - V_j}{\lambda_i - \lambda_j}$ is constant for all $i < j$. If the first condition holds, then $\frac{V_i - V_j}{\lambda_i - \lambda_j} = c$, so the second is automatic. Conversely, if the second holds with constant k, then $V_i - k\lambda_i$ is the same for all i; since $\sum_i V_i = 0$, this yields $V_i = k(\lambda_i - \bar{\lambda})$. □

Remark 7.1. The equality condition $V_i = c(\lambda_i - \bar{\lambda})$ characterizes affine images of the roots of the Hermite polynomial $H_n(x)$. Indeed, at a root x_i of H_n the differential equation $H_n'' - 2xH_n' + 2nH_n = 0$ gives $V_i = H_n''(x_i)/(2H_n'(x_i)) = x_i$, so the scores are proportional to the (centered) roots. For $n = 2$ every pair of distinct reals is an affine image of the roots of H_2, so equality always holds. For $n = 3$ the equality case is $\{-a, 0, a\}$, consistent with Theorem 5.3.

Theorem 7.2 (Stam inequality — general case)**.** *The Stam inequality* (1) *holds for every degree $n \geq 2$.*

Proof. Write $a = \sigma^2(p)$ and $b = \sigma^2(q)$.

Step 1 (ODE bound). Applying (7) to p_t, $\mathcal{S}(p_t) \geq \frac{n-1}{2}\frac{\Phi_n(p_t)}{\sigma^2(p_t)}$. The dissipation identity (5) then gives

$$\frac{d}{dt}\Phi_n(p_t) \leq -\frac{b}{a+tb}\,\Phi_n(p_t).$$

Integrating $(\log \Phi_n(p_t))' \le -b/(a+tb)$ from 0 to t:

$$\frac{1}{\Phi_n(p_t)} \ge \frac{a+tb}{a\,\Phi_n(p)}. \tag{10}$$

Step 2 (Integral bound from the p-flow). From (6), using (7) and $\sigma^2(p_t) = a + tb$:

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} \ge b \int_0^1 \frac{dt}{(a+tb)\,\Phi_n(p_t)}.$$

Substituting (10): the factor $(a+tb)$ cancels and

$$\frac{1}{\Phi_n(p \boxplus_n q)} \ge \frac{1}{\Phi_n(p)} + \frac{b}{a\,\Phi_n(p)} = \frac{a+b}{a\,\Phi_n(p)}. \tag{11}$$

Step 3 (Symmetric bound from the q-flow). Repeating Steps 1–2 with the roles of p and q exchanged (flowing $\hat{q}_s := q \boxplus_n p_s$ from $s=0$ to $s=1$):

$$\frac{1}{\Phi_n(p \boxplus_n q)} \ge \frac{a+b}{b\,\Phi_n(q)}. \tag{12}$$

Step 4 (Case split). Exactly one of the following holds:

(a) $b\,\Phi_n(q) \ge a\,\Phi_n(p)$. Then $\frac{b}{a\,\Phi_n(p)} \ge \frac{1}{\Phi_n(q)}$, so (11) gives (1).

(b) $a\,\Phi_n(p) \ge b\,\Phi_n(q)$. Then $\frac{a}{b\,\Phi_n(q)} \ge \frac{1}{\Phi_n(p)}$, so (12) gives (1).

□

Remark 7.2. The case-split exploits both the p-flow and the q-flow. It is crucial that $\boxplus_n$ is commutative: $p \boxplus_n q = q \boxplus_n p$.

8 Summary of results

The Stam inequality (Theorem 1.1) is now proved in full generality. The argument combines three ingredients:

1. **Pointwise score–gap inequality** (Theorem 7.1): $\mathcal{S}(r)\,\sigma^2(r) \ge \frac{n-1}{2}\,\Phi_n(r)$, established by two applications of Cauchy–Schwarz via the score–root identity ($\sum \lambda_i V_i = \binom{n}{2}$) and the score–gap identity ($\Phi_n = \sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j}$).

2. **Convolution-flow dissipation** (Corollary 6.4): the integral identity expressing $1/\Phi_n(p \boxplus_n q) - 1/\Phi_n(p)$ in terms of $\mathcal{S}(p_t)/\Phi_n(p_t)^2$ along the flow.

3. **Case-split argument** (Theorem 7.2): applying the ODE bound from both the p-flow and q-flow directions and exploiting commutativity of $\boxplus_n$.

Equality in the pointwise inequality holds if and only if the scores are proportional to the centered roots, characterizing affine images of Hermite polynomial roots.