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UFR Mathématiques

Thèorie des modeles TD4

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Exercise 0.1 Consider the ordered set $\mathcal{R} = \langle \mathbb{R}, < \rangle$, and the subset $\mathbb{Q} \subseteq \mathbb{R}$. Describe $\mathrm{acl}_{\mathcal{R}}(\mathbb{Q})$.

Solution: Both $\mathbb{R}$ and $\mathbb{Q}$ are dense linear orderings without endpoints, and it is clear that $\mathbb{Q} \subseteq \mathbb{R}$. We will use the fact that DLO eliminates quantifiers to show that $\mathbb{Q} \preceq \mathbb{R}$. Let $\bar{q} \in \mathbb{Q}$ and let $\phi(x, \bar{y})$ be an $\{<\}$-formula. Let $\psi(\bar{y})$ be a quantifier-free formula such that $\mathsf{DLO} \models \forall \bar{y}(\exists x \phi(x, \bar{y}) \leftrightarrow \psi(\bar{y}))$. Then we have that

$$\mathbb{R} \models \exists x \phi(x, \bar{q})$$

$$\mathbb{R} \models \psi(\bar{q})$$

$$\mathbb{Q} \models \psi(\bar{q}) \quad \text{since } \psi \text{ is qf-free}$$

$$\mathbb{Q} \models \exists x \phi(x, \bar{q}) \quad \text{since } \mathbb{Q} \models \mathsf{DLO}$$

So by Tarski-Vaught, $\mathbb{Q} \preceq \mathbb{R}$. Then by a remark made in class, $\mathrm{acl}_{\mathbb{R}}(\mathbb{Q}) = \mathrm{acl}_{\mathbb{Q}}(\mathbb{Q}) = \mathbb{Q}$.

Exercise 0.2 (Tarski's Chain Lemma) Let $(I, <)$ be a directed set. Consider a collection of $\mathcal{L}$-structures $\{\mathcal{M}_i\}_{i \in I}$ such that for all $i < j$, $\mathcal{M}_i \subseteq \mathcal{M}_j$. Let $M = \bigcup_i M_i$.

(1) Turn M into an $\mathcal{L}$-structure $\mathcal{M}$ such that for every $i \in I$, $\mathcal{M}_i \subseteq \mathcal{M}$.

(2) Let T be an $\mathcal{L}$-theory and suppose that for every $i \in I$, $\mathcal{M}_i \models T$. Does $\mathcal{M} \models T$?

(3) Suppose now that for all $i < j$, $\mathcal{M}_i \preceq \mathcal{M}_j$. Show that for all i, $\mathcal{M}_i \preceq \mathcal{M}$.

(4) Suppose that $\{M_i\}_{i \in I}$ is an elementary chain and that $\mathcal{N}$ is an $\mathcal{L}$-structure. If for all i, $\mathcal{M}_i \preceq \mathcal{N}$, then $\mathcal{M} \preceq \mathcal{N}$.

Solution:

(1) Let c be a constant symbol, since its interpretation is the same in every $\mathcal{M}_i$, then take $c^{\mathcal{M}} = c^{\mathcal{M}_i}$. Let f be a function symbol of any arity, and define $f^{\mathcal{M}} = \bigcup_i f^{\mathcal{M}_i}$, this is well defined since if $\bar{a} \in M_i \cup M_j$, take some $k \geq i, j$, and then since $M_i \cup M_j \subseteq M_k$,

$f^{\mathcal{M}_i}(\bar{a}) = f^{\mathcal{M}_k}(\bar{a}) = f^{\mathcal{M}_j}(\bar{a})$. Similarly, for a function symbol R, define $R^{\mathcal{M}} = \bigcup_i R^{\mathcal{M}_i}$. We have that $R^{\mathcal{M}} \cap M_i = \bigcup_j R^{\mathcal{M}_j} \cap M_i = \bigcup_{j \leq i} R^{\mathcal{M}_j} = R^{\mathcal{M}_i}$: this follows from the fact that if $i \leq j, R^{\mathcal{M}_j} \subseteq M_i$ and if $i < j$,$R^{\mathcal{M}_j} \cap M_i = R^{\mathcal{M}_i}$, by hypothesis. The structure given satisfies what is needed by construction.

(2) Not necessarily, consider T the theory of linear orders with both endpoints. The family of models given by $\mathcal{M}_i = \{-i, -i+1, \ldots, 0, \ldots, i-1, i\}$ with the evident ordering, has $\mathbb{Z}$ as its union, which has no endpoints.

(3) Let $\phi(x, \bar{y})$ an $\mathcal{L}$-formula, let $i \in I$, $\bar{a} \in M_i$, and $m \in M$ such that $\mathcal{M} \models \phi(m, \bar{a})$. There is $k \geq i$ such that $m, \bar{a} \in M_k$, so $\mathcal{M}_k \models \phi(m, \bar{a})$ and therefore $\mathcal{M}_i \models \exists x(x, \bar{a})$ by hypothesis. Conversely if $\mathcal{M}_i \models \exists x(x, \bar{a})$ it inmediately follows that $\mathcal{M} \models \exists x(x, \bar{a})$. Hence, $\mathcal{M}_i \preceq \mathcal{M}$.

(4) Let $\bar{a} \in M$ and $\phi(\bar{x})$ any formula, then $\mathcal{N} \models \phi(\bar{a}) \iff \mathcal{M}_i \models \phi(\bar{a}) \iff \mathcal{M} \models \phi(\bar{a})$.

Exercise 1. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ $\mathcal{L}$-structures such that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$. We aim to show $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{A} \preceq \mathcal{C}$ does not imply $\mathcal{B} \preceq \mathcal{C}$.

(1) Start with $\mathcal{A} = \langle \mathbb{Z}, < \rangle$, construct a proper elementary extension $\mathcal{C}$.

(2) Find $\mathcal{B} \subseteq \mathcal{C}$ such that $|C \setminus B| = 1$, together with an isomorphism $\sigma : \mathcal{C} \to \mathcal{B}$ such that $\sigma(a) = a \ \forall a \in A$.

(3) Find an existential formula φ with parameters from B such that $\mathcal{C} \models \phi$ and $\mathcal{B} \models \neg\phi$.

Solution:

(1) Consider the theory

$$T = \mathrm{Diag}_{el}(\mathcal{A}) \cup \{c > a\}_{a \in \mathbb{Z}}$$

where c is a new constant symbol. Any finite part of T has the form

$$T_0 \subseteq \mathrm{Diag}_{el}(\mathcal{A}) \cup \{c > a\}_{a < m}$$

for some $m \in \mathbb{Z}$. So by interpreting c as $m+1$ (the succesor an predecessor of an element can be defined in our language), we have that $\mathcal{A} \models T_0$. Therefore, any model $\mathcal{C} \models T$ is a proper elementary extension of $\mathcal{A}$ (as $\{<\}$-structures).

(2) Consider $\mathcal{B} = \mathcal{C} \setminus \{c\}$, and we interpret $<^{\mathcal{B}} = <^{\mathcal{C}} \cap B^2$. Let $\sigma : \mathcal{C} \to \mathcal{B}$ defined as

$$\sigma(x) = \begin{cases} x & \text{if } x < c \\ x+1 & \text{if not} \end{cases}$$

this is clearly an order-embedding that fixes A.

(3) Consider ϕ as $\exists x\ c-1<x<c+1$.

Exercise 2: Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. Define $\mathrm{dcl}_{\mathcal{M}}(A) = \{b \in M, \{b\}$ is A-definable$\}$.

(1) Show that $\mathrm{dcl}_{\mathcal{M}}$ is a closure operator on $\mathcal{P}(M)$, which has finite character.

(2) Show that every PEM $f : \mathcal{M} \supseteq A \hookrightarrow \mathcal{N}$ has a unique extension to a PEM $\hat{f} : \mathcal{M} \supseteq \mathrm{dcl}_{\mathcal{M}}(A) \hookrightarrow \mathcal{N}$ and that $\mathrm{Im}(\hat{f}) = \mathrm{dcl}_{\mathcal{N}}(\mathrm{Im}(f))$.

(3) Let $b \in \mathrm{dcl}_{\mathcal{M}}(A)$ and $\sigma \in \mathrm{Aut}_A(\mathcal{M}) = \{\sigma \in \mathrm{Aut}(\mathcal{M}) : \sigma(a) = a\ \forall a \in A\}$. What can we say about the orbit of b under the action of σ?

(4) Let $b, c \in M$ and $A \subseteq M$. Show that $c \in \mathrm{dcl}_{\mathcal{M}}(A \cup \{b\})$ if and only if there if $f : M \to N$ A-definable such that $f(b) = c$.

(5) Let T be a theory with built-in Skolem functions and let $\mathcal{M} \models T$. Show that for every $A \subseteq M$, $\mathrm{dcl}_{\mathcal{M}}(A) = \langle A \rangle_{\mathcal{M}}$.

(6) Let T be a theory with definable Skolem functions and let $\mathcal{M} \models T$. Show that $\mathrm{dcl}_{\mathcal{M}}(A) \preceq \mathcal{M}$.

(7) Let $\mathcal{M}$ be an expansion of a total order. Show that $\mathrm{acl}_{\mathcal{M}} = \mathrm{dcl}_{\mathcal{M}}$.

(8) Let $\mathcal{M} \equiv \langle \mathbb{N}, 0, 1, +, \cdot, < \rangle$ and let $\varnothing \neq A \subseteq M$.Show that $\mathrm{dcl}_{\mathcal{M}}(A) \preceq \mathcal{M}$.

Solution:

(1) It is reflexive since for any $a \in A$, we consider the formula $x = a$ which defines $\{a\}$. It is monotonic because if $\{a\}$ is A-definable, and $A \subseteq B$, then automatically $\{a\}$ is B-definable. These two properties imply that $\mathrm{dcl}_{\mathcal{M}}(A) \subseteq \mathrm{dcl}_{\mathcal{M}}(\mathrm{dcl}_{\mathcal{M}}(A))$. To check the other inclusion, let $b \in \mathrm{dcl}_{\mathcal{M}}(\mathrm{dcl}_{\mathcal{M}}(A))$, and let $\varphi(x, \bar{c})$ an $\mathcal{L}$-formula with $\bar{c} \in \mathrm{dcl}_{\mathcal{M}}(A)$ such that $\varphi(\mathcal{M}, \bar{c}) = \{b\}$. For each c_i, let $\phi_i(x, \bar{a}_i)$ be a formula with $\bar{a}_i \in A$ such that $\phi_i(\mathcal{M}, \bar{a}_i) = \{c_i\}$. Then, consider the $\mathcal{L}_A$ formula

$$\psi(x, \bar{y}) = \exists! z \varphi(z, \bar{y}) \wedge \varphi(x, \bar{y}) \wedge \bigwedge_i \phi_i(y_i, \bar{a}_i).$$

Since there is only one tuple $\bar{c}$ such that $\bigwedge_i \phi_i(y_i, \bar{a}_i)$, and only one b such that $\varphi(b, \bar{c})$, we conclude that this formula defines a single tuple $(b, \bar{c})$. Hence, its projection is definable and $\{b\} = \{x, \exists \bar{y} \psi(x, \bar{y})\}$.

(2) Let Ω be the set of PEM functions with domain $A \subseteq A' \subseteq \mathrm{dcl}_{\mathcal{M}}(A)$ and image $B \subseteq B' \subseteq \mathrm{dcl}_{\mathcal{N}}(B)$ ordered by function extension. It is a direct verification that Ω is closed under

taking chains, so by Zorn's Lemma we can get a maximal $g \in \omega$, with domain A_0 and image B_0. We claim that $A_0 = \mathrm{dcl}_{\mathcal{M}}(A)$ and $B_0 = \mathrm{dcl}_{\mathcal{N}}(B)$. Suppose by contradiction that there is $c \in \mathrm{dcl}_{\mathcal{M}}(A) \setminus A_0$, since $A \subseteq A_0$, $c \in \mathrm{dcl}_{\mathcal{M}}(A_0)$. Choose $\varphi(x, \bar{a})$, with $\bar{a} \in A_0$ such that $\varphi(\mathcal{M}, \bar{a}) = \{c\}$. In other words, $\mathcal{M} \models \exists! x\varphi(x, \bar{a})$, and since g_0 is a PEM, $\mathcal{N} \models \exists! x\varphi(x, f(\bar{a}))$. Since $c \notin A_0$, we get that $\varphi(\mathcal{M}, \bar{a}) \cap A_0 = \varnothing$ and hence $\varphi(\mathcal{N}, g_0(\bar{a})) \cap B_0 = \varnothing$. Let d the only element in $\varphi(\mathcal{N}, g_0(\bar{a})) \setminus B_0$ (in particular $d \in \mathrm{dcl}_{\mathcal{N}}(B_0)$). Define $g_1 : A_0 \cup \{c\} \to B_0 \cup \{d\}$ extending g_0 and sending c to d. If we prove g_1 is a PEM, we contradict maximality of g_0. Let $\theta(c, \bar{a}')$ an $\mathcal{L}_{A_0 \cup \{c\}}$-sentence satisfied by $\mathcal{M}$, then $\mathcal{M} \models \theta(c, \bar{a}') \wedge \varphi(c, \bar{a})$, and since $|\theta(\mathcal{M}, \bar{a}') \cap \varphi(\mathcal{M}, \bar{a})| = 1$, we have

$$\mathcal{M} \models \forall x(\varphi(x, \bar{a}) \to \theta(x, \bar{a}')$$

$$\mathcal{N} \models \forall x(\varphi(x, g_0(a)) \to \theta(x, g_0(\bar{a}'))$$

But since $\mathcal{N} \models \varphi(d, g_0(\bar{a}))$, then $\mathcal{N} \models \theta(d, g_0(\bar{a}'))$, and therefore $\mathcal{N} \models \theta(g_1(c), g_1(\bar{a}'))$. Repeating this argument with $\neg\theta$ gives us the other direction to conclude $\mathcal{M}_{A_0 \cup \{c\}} \equiv \mathcal{N}_{g_1(A_0 \cup \{c\})}$. To prove $B_0 = \mathrm{dcl}_{\mathcal{N}}(B)$ we use the same argument but for the PEM g_0^{-1}, if we extend this map, the inverse of this extension will extend g_0 again contradicting maximality. Finally, to check uniqueness, let $c \in \mathrm{dcl}_{\mathcal{M}}(A)$, then there is a $\mathcal{L}_A$-formula $\varphi(x, \bar{a})$ such that $\mathcal{M} \models \exists! x\varphi(x, \bar{a})$, then $\mathcal{N} \models \exists! x\varphi(x, f(\bar{a}))$, so that any two extensions of f into $\mathrm{dcl}_{\mathcal{M}}(A)$ must agree everywhere.

(3) We have that $\{\sigma^m(b), m \in \mathbb{N}\} = \{b\}$: if $\varphi(x, \bar{a})$ is a formula defining $\{b\}$ with $\bar{a} \in A$, then

$$\begin{aligned} & \mathcal{M} \models \varphi(b, \bar{a}) \\ \iff & \mathcal{M} \models \varphi(\sigma(b), \sigma(\bar{a})) \\ \iff & \mathcal{M} \models \varphi(\sigma(b), \bar{a}) \\ \iff & \quad \sigma(b) \in \{b\} \end{aligned}$$

(4) Suppose $c \in \mathrm{dcl}_{\mathcal{M}}(A)$, then there is some formula $\varphi(x, \bar{a}, b)$ such that $\varphi(\mathcal{M}, \bar{a}, b) = \{c\}$. The set $D = \{x, \exists! y\ \varphi(y, \bar{a}, x)\}$ is A-definable. Fix $a \in A$, and define

$$f(m) = \begin{cases} n & \text{such that } \varphi(n, \bar{a}, m) \text{ , if } m \in D \\ a & \text{if not} \end{cases}$$

by definition, $f(b) = c$. Conversely, suppose there is an A-definable function $f : M \to N$ that sends b to c. Let $\theta(x, y, \bar{a})$ be a formula defining the graph of f. Then $\theta(b, \mathcal{M}, \bar{a}) = \{c\}$.

(5) "$\subseteq$": Let $b \in \mathrm{dcl}_{\mathcal{M}}(A)$, and choose $\varphi(x, \bar{a})$ such that $\mathcal{M} \models \exists! y \varphi(y, \bar{a}) \wedge \varphi(b, \bar{a})$, then by hypothesis there is $f \in \mathcal{L}$ a function such that $\mathcal{M} \models \varphi(f(\bar{a}), \bar{a}) \wedge \varphi(b, \bar{a})$, hence $b = f(\bar{a})$ which implies $b \in \langle A \rangle_{\mathcal{M}}$.

"$\supseteq$": Let now $b \in \langle A \rangle_{\mathcal{M}}$, so $b = t(\bar{a})$ for some term t, we show that $b \in \mathrm{dcl}_{\mathcal{M}}(A)$ by induction on terms: the case where t is a variable or a constant is inmediate, so assume $b = f(t_1(\bar{a}), \ldots, t_m(\bar{a}))$ with $t_i(\bar{a}) \in \mathrm{dcl}_{\mathcal{M}}(A)$ and f and $f \in \mathcal{L}$. Consider the formula $\theta(\bar{x}, y)$ that defines the graph of f, so we have

$$\mathcal{M} \models \exists! y\ \theta(t_1(\bar{a}), \ldots, t_m(\bar{a}), y) \wedge \theta(t_1(\bar{a}), \ldots, t_m(\bar{a}), b)$$

so that $b \in \mathrm{dcl}_{\mathcal{M}}(\mathrm{dcl}_{\mathcal{M}}(A)) = \mathrm{dcl}_{\mathcal{M}}(A)$.

(6) Let $\bar{a} \in \mathrm{dcl}_{\mathcal{M}}(A)$ and any formula φ such that $\mathcal{M} \models \exists x\ \varphi(x, \bar{a})$. Then by hypothesis $\mathcal{M} \models \exists z\ \varphi(z, \bar{a}) \wedge \theta_\varphi(z, \bar{a})$, where θ_φ defines the graph of the Skolem function for φ. In particular $|\theta_\varphi(\mathcal{M}, \bar{a})| = 1$, so if $b \in \mathcal{M}$ is such that $\mathcal{M} \models \varphi(\bar{a}, b)$, then there is $b' \in \mathcal{M}$ such that $\mathcal{M} \models \varphi(b', \bar{a}) \wedge \theta_\varphi(b', \bar{a})$, in particular $\mathcal{M} \models \theta_\varphi(b', \bar{a})$, so $b' \in \mathrm{dcl}_{\mathcal{M}}(\mathrm{dcl}_{\mathcal{M}}(A)) = \mathrm{dcl}_{\mathcal{M}}(A)$.

(7) Clearly $\mathrm{dcl}_{\mathcal{M}}(A) \subseteq \mathrm{acl}_{\mathcal{M}}(A)$ since an A-definable set of size 1 has finite size. Let $b \in \mathrm{acl}_{\mathcal{M}}(A)$, then there is φ such that $|\varphi(\mathcal{M}, b)| = n$ and $\mathcal{M} \models \varphi(b, \bar{a})$. Suppose that $\varphi(\mathcal{M}, b) = \{b_1, \ldots, b_n\}$ and without loss of generality $b_1 < \cdots < b_n$. Then for some k , $b = b_k$ and we define $\{b_k\}$ with the formula

$$\varphi(x, \bar{a}) \wedge \exists^k y\ (\varphi(y, \bar{a}) \wedge y < x) \wedge \exists^{n-k} y\ (\varphi(y, \bar{a}) \wedge y > x)$$

(8) It is enough to show that $T = \mathrm{Th}(\mathcal{M})$ has definable Skolem functions. Let $\varphi(\bar{x}, y)$ be a formula such that for $\bar{a} \in \mathcal{M}$, $T \models \exists y \varphi(\bar{a}, y)$, so that $D = \{\bar{b}, \mathcal{M} \models \exists y\ \varphi(\bar{b}, y)\}$ is a

definable, non-empty set. Consider the function

$$f_\varphi(\bar{b}) = \begin{cases} \min\{c, \mathcal{M} \models \varphi(\bar{b}, c)\} & \text{if } b \in D \\ 0 & \text{if not} \end{cases}.$$

The graph of f_φ is defined by the formula $\theta(\bar{x}, y)$ given by

$$(\bar{x} \in D \wedge \varphi(\bar{x}, y) \wedge \forall z \ (\varphi(\bar{x}, z) \to z \geq y)) \vee (\bar{x} \notin D \wedge y = 0).$$

So we can conclude that $\mathcal{M}$ has definable Skolem functions, and by (6), we have the result.

Exercise 3: Let T be an $\mathcal{L}$-theory. The following are equivalent:

(1) For every $\mathcal{M} \models T$ and for every $\mathcal{A}, \mathcal{B} \preceq \mathcal{M}$ we have $\mathcal{A} \cap \mathcal{B} \preceq \mathcal{M}$.

(2) For every $\mathcal{M} \models T$ and for every $C \subseteq M$, we have $\mathrm{acl}_{\mathcal{M}}(C) \preceq \mathcal{M}$.

Solution: To prove (2) implies (1), let $\mathcal{A}, \mathcal{B} \preceq \mathcal{M}$, so in particular $\mathcal{A} \equiv \mathcal{B} \equiv \mathcal{M}$, then we can apply the joint embedding property (twice), to find $\mathcal{S}$ such that $\mathcal{A}, \mathcal{B}, \mathcal{M} \preceq \mathcal{S}$. We can also ask that $\mathrm{acl}_{\mathcal{S}}(\varnothing) = A \cap B$. Since $\varnothing \subseteq \mathcal{M} \preceq \mathcal{S}$, $\mathrm{acl}_{\mathcal{S}}(\varnothing) = \mathrm{acl}_{\mathcal{M}}(\varnothing)$ and hence by hypothesis $\mathcal{A} \cap \mathcal{B} = \mathrm{acl}_{\mathcal{M}}(\varnothing) \preceq \mathcal{M}$.