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UFR Mathématiques

Thèorie des ensembles, Devoir Maison

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Exercise 1. Prove that ω_1 with the order topology is not metrizable.

Solution.

Suppose it is metrizable, with a metric function $d : \omega_1 \times \omega_1 \to \mathbb{R}$. We denote the *sphere of center α and radius r* by

$$B_r(\alpha) = \{\beta \in \omega_1, d(\alpha, \beta) < r\}.$$

Recall that if X is a topological space and $A \subseteq X$, we say A is *dense* if its topological closure $\bar{A} = X$ and we say that X is *separable* if it contains some countable dense subset.We first see that ω_1 is not separable, let $C \subseteq \omega_1$ countable, and let $\alpha = \sup C < \omega_1$. Then we have $C \subseteq [0, \alpha] \subsetneq \omega_1$ so that C is contained in a proper closed set, therefore it cannot be dense.

Lemma: There is $\varepsilon > 0$ and an uncountable set $C \subseteq \omega_1$ such that for every $x \neq y \in C$, $d(x, y) > \varepsilon$.

Proof: We will construct C and ϵ. First, we define a sequence $\{\alpha_\beta, \beta < \omega_1\}$ and a function $k : \omega_1 \to \mathbb{N}$ inductively by

- $\alpha_0 = 0$, $k_0 = 0$.
- Suppose α_γ has been defined for all $\gamma < \beta$. Then $X = \{\alpha_\gamma, \gamma < \beta\}$ is countable, so it cannot be dense. Pick $\alpha_\beta \in \omega_1 \setminus \bar{X}$, and pick k_β such that $B_{1/k_\beta}(\alpha_\beta) \cap X = \varnothing$. This implies that for all $\gamma \leq \beta$, $d(\alpha_\gamma, \alpha_\beta) > 1/k_\beta$.

Since k can be regarded as a map from ω_1 to ω, then k is regressive, so there is $K \in \mathbb{N}$ such that $C = \{\alpha_\gamma \in \omega_1, k_\gamma = K\}$ is uncountable. Given any $\alpha_{\gamma_1}, \alpha_{\gamma_2} \in C$ with $\gamma_1 > \gamma_2$, by construction, $d(\alpha_{\gamma_1}, \alpha_{\gamma_2}) > 1/k_{\gamma_2} = 1/K$. We can then take $\varepsilon = 1/K$.

To show the result, consider any sequence $\{\alpha_n, n < \omega\}$ in C, since it is countable, it converges in order topology (more precisely $\alpha = \sup \alpha_n < \omega_1$), but since we are assuming that the metric topology coincides with the order, we must have that

$$\lim_{n\to\infty} d(\alpha_n, \alpha) = 0.$$

Indeed for all $\epsilon > 0$ as the topologies coincide, $B_\epsilon(\alpha)$ contains a tail of α, so in particular contains some α_n. By the triangle inequality, $d(\alpha_m, \alpha_n) \leq d(\alpha_n, \alpha) + d(\alpha_m, \alpha)$, so we get

$$\lim_{n,m\to\infty} d(\alpha_n, \alpha_m) = 0$$

this contradicts that any two members of C are at least ε apart. Our initial assumption was therefore incorrect, ω_1 cannot be metrizable.

Exercise 2. Let α be a countable ordinal.

(1) Prove that the order topology in α coincides with the subspace topology $\alpha \subseteq \omega_1$.
(2) Prove that the order topology in α is metrizable.
(3) Conclude that if $X \subseteq \omega_1$, X is metrizable with the induced order topology.

Solution.

(1) Let τ_o, τ_s be the order and subspace topology, respectively. It is clear that $\tau_o \subseteq \tau_s$ since every open interval in α is also open in ω_1. To see the other direction, consider the τ_s-open set $U = \alpha \cap (\beta, \gamma)$ for some open interval (β, γ) in ω_1. If $\beta \geq \alpha$, then $U \cap \alpha = \varnothing$ so it is open, otherwise $U \cap \alpha = (\beta, \min(\alpha, \gamma))$, which is τ_o-open.
(2) Let $\phi : \alpha \to \mathbb{Q}$ be an order-embedding. We will prove that α is homeomorphic to its image under ϕ. Since ϕ is injective, we don't have to prove bijectivity. Notice that for any $\gamma_1 < \gamma_2 < \alpha$, we have $\phi[(\gamma_1, \gamma_2)] = (\phi(\gamma_1), \phi(\gamma_2))$ (because ϕ preserves order), and for all $p < q \in \mathbb{Q}$, that $\phi^{-1}((p, q) \cap \mathrm{Im}(\phi)) = (\gamma_p, \delta_p)$ where $\gamma_p = \min\{\beta < \alpha, \phi(\beta) > p\}$ and $\delta_p = \sup\{\beta < \alpha, \phi(\beta) < q\}$. So, we have that ϕ and ϕ^{-1} both preserve open sets under preimage. Then, we can metrize α as $d(\beta, \gamma) = |\phi(\beta) - \phi(\gamma)|$ (we copy the metric of the homeomorphic image of α in $\mathbb{Q}$).
(3) Lastly, if $X \subseteq \omega_1$ is bounded, it is countable (otherwise $\sup X$ would be some uncountable ordinal under ω_1), and therefore metrizable by the preceding arguments.

Exercise 3. Let $X \subseteq \omega_1$. Show that if X is a club, then $X \approx \omega_1$ (homeomorphic spaces). Conclude that clubs are not metrizable.

Solution. Consider the function $f : \omega_1 \to X$ defined inductively by

- $f(0) = \min X$.
- $f(\alpha + 1) = \min\{x \in X, x > f(\alpha)\}$.
- If α is a limit and $f(\beta)$ is defined for all $\beta < \alpha$, set $f(\alpha) = \sup_{\beta<\alpha} f(\beta)$. This is well-defined since X is a club, so we can take sup and stay inside X.

Notice that, by construction f respects suprema, and therefore f is continuous (this was proved in TD1). Clearly f is injective, let's prove that f is surjective by contradiction: let $x = \min(X \setminus f[\omega_1])$. Let $A = \{\beta < \omega_1, f(\beta) < x\}$, notice that A is bounded by x since for every β , $f(\beta) \geq \beta$. Let $\alpha = \sup A$, so then we have that $f(\alpha) < x$ by monotonicity, and also $f(\alpha + 1) > x$, because otherwise $\alpha + 1$ would be in A (the inequalities are both strict since x is not in the image of f). We then have $f(\alpha) < x < f(\alpha + 1)$ so that x is less than the minimum element of X bigger than $f(\alpha)$ (definition of $f(\alpha + 1)$), so we have reached a contradiction, and f is therefore surjective. Finally, since f is also an order-embedding, for all $\gamma < \beta < \omega_1$, $f[(\gamma, \beta)] = (f(\gamma), f(\beta)) \cap X$ which means f is an open map. All of these show that f is a homeomorphism. To conclude, if X were metrizable, then ω_1 would be too, a contradiction to ex1, since homeomorphisms preserve metrizability (as they essentially copy the topology).

Exercise 4. If $S \subseteq \omega_1$ is stationary, then it is not paracompact.

Solution. Let us show first that if C is a club, then $S \cap C$ is stationary: indeed if C' is any club, since $C \cap C'$ is also a club, then $(S \cap C) \cap C' = S \cap (C \cap C')$ is non-empty, so $S \cap C$ is a club. This in particular allows us to assume S only contains limit points, since we can restrict ourselves to the intersection of S and the club of limit ordinals. Consider the covering of S given by the family $U_\alpha = [0, \alpha + 1)$ for $\alpha \in S$. Assume by contradiction that there is a cover $\mathcal{V}$ that is a locally finite refinement of $\mathcal{U}$. Then by definition, for every $\alpha \in S$ there is a neighborhood of α intersecting only finitely many elements of $\mathcal{V}$, more specifically, since open neighborhoods contain tails of limit elements, there is $\beta(\alpha) < \alpha$ such that $[\beta(\alpha), \alpha + 1)$ intersects finitely many elements of $\mathcal{V}$. The function $\alpha \mapsto \beta(\alpha)$ is therefore regressive, and by Fodor's Pressing Down Lemma, there is a fixed $\beta < \omega_1$ and some stationary $T \subseteq \omega_1$ such that for all $\alpha \in T$, $[\beta, \alpha+1)$ intersects finitely many elements of $\mathcal{V}$.

We construct a sequence $\{\alpha_n, n < \omega\} \subseteq T$ by taking $\alpha_0 \in T$ any element bigger than β. If α_n is defined, suppose $[\beta, \alpha_n + 1)$ intersects $m_n < \omega$ many elements of $\mathcal{V}$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, there is γ_n such that the union these m_n sets in $\mathcal{V}$ is contained in $[0, \gamma_n)$. Choose $\alpha_{n+1} > \gamma_n$ in T. Then the interval $[0, \alpha_{n+1} + 1]$ intersects at least $m_n + 1$ elements in $\mathcal{V}$, because α_{n+1} has to be in some $V \in \mathcal{V}$ not included among the other m_n ones (thanks to $\mathcal{V}$ also being a covering). Letting $\alpha = \sup \alpha_n$ we observe that $[\beta, \alpha + 1)$ intersects infinitely many sets in $\mathcal{V}$, but $\alpha \in T$, this is a contradiction.

Exercise 5. If X is a metrizable topological space, X is paracompact. Conclude that stationary sets in ω_1 are not metrizable.

Solution. Let $\{U_\alpha, \alpha < \kappa\}$ be an open cover of X and suppose d is a metric function. We define for $n > 1$ (by induction) the sets $U_{\alpha,n}$ to be the union of all spheres of the form $B_{2^{-n}}(x)$ where x satisfies the following:

(1) α is the least ordinal such that $x \in U_\alpha$.
(2) For all $\beta \in \kappa$, $x \notin U_{\beta,i}$ if $i < n$.
(3) $B_{3 \cdot 2^{-n}}(x) \subseteq U_\alpha$.

We will show that this family is a finitely local refinement of U that is also a cover. Notice also that $U_{\alpha,n}$ are all open sets, being unions of spheres.

First, it is clear that it is a refinement since each of the spheres that compose $U_{\alpha,n}$ are contained in U_α by (3). Second, to check that it is a covering, let $x \in X$, and let α be minimal such that $x \in U_\alpha$. Since U_α is open, we can choose n big enough so that $B_{3 \cdot 2^{-n}}(x) \subseteq U_\alpha$. So we have (1) and (3), if x satisfies (2) then automatically $x \in U_{\alpha,n}$, and otherwise $x \in U_{\beta,i}$ for some β and some $i < n$. Either way this shows that this family covers X.

To prove that it is locally finite, let $x \in X$ be contained in some $U_{\alpha,n}$, and pick k large enough so that $B_{2^{-k}}(x) \subseteq U_{\alpha,n}$. We claim that $B_{2^{-k-n}}(x)$ intersects finitely many $U_{\beta,i}$.

(Case 1: $i < n + k$) We will show that for this case, $B_{2^{-k-n}}(x)$ can intersect at most one of the $U_{\beta,i}$. We will show this by proving that any element of $U_{\beta,i}$ is at least 2^{-i} distance away from any element in $U_{\gamma,i}$, for any $\beta < \gamma$: indeed let x_1, x_2 satisfying (1),(2),(3) such that if $a \in B_{2^{-i}}(x_1) \subseteq U_{\beta,i}$ and $b \in B_{2^{-i}}(x_2) \subseteq U_{\gamma,i}$. Then, by (3) $B_{3 \cdot 2^{-i}}(x_1) \subseteq U_{\beta,i}$ and by (1), $x_2 \notin U_\beta$ (by minimality of γ).

Therefore $d(x_1, x_2) \geq 3 \cdot 2^{-i}$ and consequently $d(a, b) \geq 2^{-i}$: otherwise if $d(a, b) < 2^{-i}$, we would have by triangle inequality that $d(x_1, x_2) \leq d(a, x_1) + d(a, b) + d(b, x_2) < 3 \cdot 2^{-i}$ (this is best seen with a drawing). But $i \leq n + k - 1$, hence $d(a, b) \geq 2^{-n-k+1}$ so our sphere $B_{2^{-k-n}}(x)$ cannot possibly intersect both $U_{\beta,i}$ and $U_{\gamma,i}$.
(Case 2: $i \geq n + k$) For this case, we will show that $B_{2^{-k-n}}(x)$ does not intersect any other $U_{\beta,i}$. Let $U_{\beta,i}$ be the union of spheres of the form $B_{2^{-i}}(y)$ for y satisfying (1),(2),(3). By (2), and because $i \geq n$, $y \notin U_{\alpha,n}$. Now, since $B_{2^{-k}}(x) \subseteq U_{\alpha,n}$, we have that $d(x, y) \geq 2^{-k}$, this implies that

$$B_{2^{-k-n}}(x) \cap B_{2^{-i}}(y) = \varnothing$$

because the radius of each sphere is less than half the distance between their centers. Taking union over all of the y satisfying (1),(2),(3), we have shown that $B_{2^{-k-n}}(x) \cap U_{\beta,i} = \varnothing$.
Finally, to conclude: if some stationary $X \subseteq \omega_1$ is metrizable, then it is paracompact by these arguments, but this contradicts exercise 4.

Exercise 6. Show that if $S \subseteq \omega_1$ is non-stationary, it has a σ-locally finite base. Conclude that $S \subseteq \omega_1$ is metrizable if and only if S is nonstationary.

Solution. We start with a lemma from topology.
Lemma : If X is a T_3 topological space, then every $Y \subseteq X$ is also T_3 with the subpsace topology.
Proof: Let $y \in Y$ and $C \cap Y$ with $C \subseteq X$ closed such that $y \notin C$. Since X is T_3, pick U_1, U_2 open disjoint sets containing y and C respectively. Then $U_1 \cap Y$ and $U_2 \cap Y$ are Y-open and separate y and $C \cap Y$.
First, let's show that ω_1 is T_3 (every ordinal is T_3). Every point is closed since for all $\alpha \in \omega_1$, $\omega_1 \setminus \{\alpha\} = [0, \alpha) \cup (\alpha, \omega_1)$. Let $\alpha \in \omega_1$ and $C \subseteq \omega_1$ closed such that $\alpha \notin C$. We know that succesor ordinals are isolated points in the order topology (this is clear from $\{\gamma + 1\} = (\gamma, \gamma + 2)$), so we can assume α is a limit and write $C = C' \cup C''$ where C' contains the succesor elements in C and C'' the limit elements, also C' is open since it contains only isolated points. For all $\beta \in C''$ we can find γ_β such that $\alpha \notin (\gamma_\beta, \beta + 1)$ (since $\{\alpha\}$ is closed) and by the same reason we can find γ_α such that $(\gamma_\alpha, \alpha + 1) \cap C = \varnothing$. Then take as separating open sets

$$U_1 = \bigcup_{\beta \in C''} (\gamma_\beta, \beta + 1) \cup C' \ , \ U_2 = (\gamma_\alpha, \alpha + 1).$$

We now show that if $S \subseteq \omega_1$ is non-stationary, then it has a σ-locally finite basis. Since $\omega_1 \setminus S$ contains some club C, then S is contained in $\omega_1 \setminus C$, an open set. We can then carry on the following construction in $\omega_1 \setminus C$, or just assume S is open and do it for S. Let's assume S is open, then by definition of our topology, $S = \cup_{\alpha \in \kappa} S_\alpha$ for some family of open intervals S_α. Since the union of open intervals with non-empty intersection is itself an open interval, we may assume (up to merging some S_α's) that S is the union of disjoint open intervals. Also, if any of the S_α is unbounded, we would get that for some $\alpha < \omega_1$, $[\alpha, \omega_1) \subseteq S$, which is impossible since $\omega_1 \setminus S$ is unbounded, therefore every S_α must be bounded and therefore countable.

By ex3, every S_α is metrizable (and countable by boundedness), so consider the countable basis $S_{\alpha,n}$ consisting of all $B_{1/n}(x) \cap S_\alpha$ for $x \in S_\alpha$ and $n \in \mathbb{N}$. This family is indeed a basis: it clearly covers S_α, and also if $z \in B_{r_1}(x) \cap B_{r_2}(y) \cap S_\alpha$, by choosing n such that $1/n < \min\{r_1 - d(z,x), r_2 - d(z,y)\}$, we have that $B_{1/n}(z) \cap S_\alpha \subseteq B_{r_1}(x) \cap B_{r_2}(y) \cap S_\alpha$. Enumerate $S_{\alpha,n} = \{S^1_{\alpha,n}, S^2_{\alpha,n}, \dots\}$. Now, define

$$\mathcal{B} = \bigcup_{k<\omega} \{S^k_{\alpha,n}, \alpha \in \kappa\}.$$

Each of the $\{S^k_{\alpha,n}, \alpha \in \kappa\}$ is locally finite, since if $x \in S^k_{\alpha,n}$, we can always find an open neighborhood of x contained in $S^k_{\alpha,n}$ that only intersects one set of index k ($S^k_{\alpha,n}$ itself), because the S_α are taken open and disjoint. Since $\mathcal{B}$ contains bases for every S_α, this means that $\mathcal{B}$ is a σ-locally finite basis for S. To conclude, if S is nonstationary, then it has a σ-locally finite basis and since we proved S is also T_3 , by Nagata-Smirnov theorem, S is metrizable. Conversely if S is metrizable but also stationary, then by ex3, $\omega_1 \approx S$ which implies ω_1 is metrizable, contradicting ex1.