# On a Finite Free Stam Inequality: Progress Report

## Abstract

We investigate the analog of Stam's inequality from information theory in the setting of finite free probability. Given degree-$n$ real-rooted monic polynomials $p$ and $q$ with positive variance, denote by $\Phi_n$ the finite free Fisher information and by $\boxplus_n$ the symmetric additive convolution of Marcus, Spielman, and Srivastava. We analyze the following inequality:

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

Using a Score–Gradient Inequality and a genuine second-order generator, we prove a Hermite semigroup flow bound. We also point out the gap in the proof via fractional flows for general $q$, and outline an alternative route based on a critical point identity.

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# 1  Introduction

## 1.1  Background and Motivation

Stam's inequality [2] in information theory asserts: if $X$ and $Y$ are independent random variables with finite Fisher information $I(X)$ and $I(Y)$, then

$$\frac{1}{I(X+Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}.$$

This fundamental inequality—equivalent to the Shannon–Stam entropy power inequality—captures the essential nature that disorder strictly increases after convolution of independent sources.

Finite free probability, introduced by Marcus, Spielman, and Srivastava [1], is a polynomial analog of free probability: random variables are replaced by real-rooted polynomials, and addition is replaced by a deterministic convolution operation $\boxplus_n$. Within this framework, a natural question arises:

> *Does the finite free additive convolution also satisfy a Stam inequality?*

This article aims to develop partial results and outline several feasible routes for a proof in the general case.

## 1.2  Statement of the Main Result

Let $\mathcal{P}_n$ denote the space of degree-$n$ monic polynomials with real coefficients, and $\mathcal{P}_n^{\mathbb{R}} \subset \mathcal{P}_n$ the subset of all real-rooted polynomials. For $p \in \mathcal{P}_n^{\mathbb{R}}$, if its roots are distinct $\lambda_1 < \cdots < \lambda_n$, define the *scores*

$$V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}$$

and the *finite free Fisher information*

$$\Phi_n(p) = \sum_{i=1}^{n} V_i^2.$$

The *symmetric additive convolution* $p \boxplus_n q$ will be reviewed in Section 2.

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*Conjecture* 1 (Finite Free Stam Inequality). For $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with positive variance,

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}. \tag{1}$$

Using a Score–Gradient Inequality (Theorem 9) and a dissipative identity for the Hermite semigroup (Lemma 19), we prove an exact Hermite flow bound. The general case is reduced to an explicit comparison problem outlined in Section 6.

**Convention.** When $p$ has repeated roots, we set $\Phi_n(p) := \infty$ (equivalently $1/\Phi_n(p) := 0$). The results below establish the Hermite flow bound under the assumption that $p$ has distinct roots; Remark 28 discusses how a proof of Conjecture 1 for distinct roots can be extended to the boundary of $\mathcal{P}_n^{\mathbb{R}}$.

## 2 Preliminaries

### 2.1 Root Statistics

Let

$$p(x) = \prod_{i=1}^{n} (x - \lambda_i) = \sum_{k=0}^{n} a_k \, x^{n-k}$$

with $a_0 = 1$. Then the mean and variance of the root distribution are

$$\bar{\lambda} = \frac{1}{n} \sum_{i=1}^{n} \lambda_i, \qquad \sigma^2(p) = \frac{1}{n} \sum_{i=1}^{n} (\lambda_i - \bar{\lambda})^2.$$

**Lemma 2.**

$$\sigma^2(p) = \frac{(n-1) \, a_1^2}{n^2} - \frac{2 \, a_2}{n}.$$

*Proof.* By Vieta's formulas, $\sum_i \lambda_i = -a_1$, $\sum_{i<j} \lambda_i \lambda_j = a_2$, hence

$$\sum_{i=1}^{n} \lambda_i^2 = a_1^2 - 2a_2.$$

Combining with $\sigma^2 = \frac{1}{n} \sum_i \lambda_i^2 - \bar{\lambda}^2$ yields the result. $\qquad \square$

### 2.2 Symmetric Additive Convolution

Let $A$ and $B$ be real symmetric matrices with characteristic polynomials $p$ and $q$ respectively. Finite free additive convolution is defined via orthogonal group averaging:

$$(p \boxplus_n q)(x) = \int_{O(n)} \det\bigl(xI - (A + QBQ^T)\bigr) \, d\mu_{\text{Haar}}(Q).$$

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According to the MSS theorem [1], this operation can be expressed in differential operator form: if $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$, then

$$(p \boxplus_n q)(x) = T_q p(x), \qquad T_q = \sum_{k=0}^{n} \frac{(n-k)!}{n!} b_k \partial_x^k. \tag{2}$$

Let $r = p \boxplus_n q$, $r(x) = \sum_k c_k x^{n-k}$; its coefficients satisfy

$$c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!} a_i b_j. \tag{3}$$

We will repeatedly use the following two basic properties:

**Theorem 3** ([1]). *If* $p, q \in \mathcal{P}_n^{\mathbb{R}}$, *then* $p \boxplus_n q \in \mathcal{P}_n^{\mathbb{R}}$.

**Lemma 4** (Variance Additivity). $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

*Proof.* From (3) we obtain

$$c_1 = a_1 + b_1 \quad \text{and} \quad c_2 = a_2 + \frac{n-1}{n} a_1 b_1 + b_2.$$

Substituting into Lemma 2 and expanding $(a_1 + b_1)^2$, the cross term $\frac{2(n-1)a_1 b_1}{n^2}$ cancels with $-\frac{2(n-1)a_1 b_1}{n^2}$, yielding $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$. $\qquad\square$

### 2.3 Scores and Fisher Information

**Definition 5.** For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1 < \cdots < \lambda_n$, define the *score* at root $\lambda_i$ and the *finite free Fisher information* as

$$V_i = \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad \Phi_n(p) = \sum_{i=1}^{n} V_i^2.$$

The *score–gradient energy* is defined by

$$\mathcal{S}(p) = \sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}.$$

**Lemma 6.** $V_i = \dfrac{p''(\lambda_i)}{2\,p'(\lambda_i)}$.

*Proof.* Since $p'(\lambda_i) = \prod_{j \neq i}(\lambda_i - \lambda_j)$, differentiating once more gives

$$p''(\lambda_i) = 2 \sum_{k \neq i} \prod_{\substack{j \neq i \\ j \neq k}} (\lambda_i - \lambda_j) = 2\,p'(\lambda_i)\,V_i.$$

$\qquad\square$

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**Lemma 7** (Score Identities). (i) $\displaystyle\sum_{i=1}^{n} V_i = 0.$

(ii) $\displaystyle\sum_{i=1}^{n} \lambda_i V_i = \binom{n}{2}.$

(iii) $\displaystyle\sum_{i=1}^{n} (\lambda_i - \bar{\lambda}) V_i = \binom{n}{2}.$

(iv) $\displaystyle\Phi_n(p) = \sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j}.$

*Proof.* (i): $\sum_i V_i = \sum_{i \neq j} (\lambda_i - \lambda_j)^{-1} = 0$ (by antisymmetry).

(ii): $\sum_i \lambda_i V_i = \sum_{i \neq j} \frac{\lambda_i}{\lambda_i - \lambda_j} = \sum_{i<j} \left( \frac{\lambda_i}{\lambda_i - \lambda_j} + \frac{\lambda_j}{\lambda_j - \lambda_i} \right) = \sum_{i<j} 1 = \binom{n}{2}$.

(iii): Follows immediately from (ii) and (i).

(iv): $\sum_i V_i^2 = \sum_{i \neq j} \frac{V_i}{\lambda_i - \lambda_j} = \sum_{i<j} \frac{V_i - V_j}{\lambda_i - \lambda_j}$. $\qquad\square$

**Lemma 8** (Fisher–Variance Inequality). $\Phi_n(p)\, \sigma^2(p) \geq \dfrac{n(n-1)^2}{4}.$

*Proof.* Apply the Cauchy–Schwarz inequality to Lemma 7(iii):

$$\frac{n^2(n-1)^2}{4} \;\leq\; \left( \sum_i (\lambda_i - \bar{\lambda})^2 \right) \left( \sum_i V_i^2 \right) = n\, \sigma^2(p)\, \Phi_n(p). \qquad\square$$

# 3 Score–Gradient Inequality

The following algebraic inequality is a key input for the general proof.

**Theorem 9** (Score–Gradient Inequality). *For $n \geq 2$ and $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots,*

$$\mathcal{S}(p)\, \sigma^2(p) \;\geq\; \frac{n-1}{2}\, \Phi_n(p), \tag{4}$$

*with equality iff there exists a constant $c$ such that $V_i = c(\lambda_i - \bar{\lambda})$.*

*Proof.* Let $T = n\, \sigma^2(p)$, $U = \Phi_n(p)$, $S = \mathcal{S}(p)$. We need to prove $ST \geq \frac{n(n-1)}{2} U$.

**Step 1.** By Lemma 7(iii) and the Cauchy–Schwarz inequality,

$$\frac{n^2(n-1)^2}{4} \;\leq\; T U. \tag{5}$$

**Step 2.** By Lemma 7(iv) and the Cauchy–Schwarz inequality,

$$U^2 \;\leq\; S \cdot \binom{n}{2}. \tag{6}$$

**Step 3.** Combine:

$$S\,T \geq \frac{2U^2}{n(n-1)} \cdot T = \frac{2U}{n(n-1)} \cdot T\,U \geq \frac{2U}{n(n-1)} \cdot \frac{n^2(n-1)^2}{4} = \frac{n(n-1)}{2}\,U.$$

**Equality condition.** Equality $S\,T = \frac{n(n-1)}{2}\,U$ requires equality in both Step 2 and Step 1.

*Equality in Step 1:* Cauchy–Schwarz in (5) is $\left(\sum_i(\lambda_i-\bar{\lambda})V_i\right)^2 \leq \left(\sum_i(\lambda_i-\bar{\lambda})^2\right)\left(\sum_i V_i^2\right)$; equality holds iff $V_i = c(\lambda_i - \bar{\lambda})$ for some constant $c$.

*Equality in Step 2:* Cauchy–Schwarz in (6) is $\left(\sum_{i<j}\frac{V_i-V_j}{\lambda_i-\lambda_j}\cdot 1\right)^2 \leq \left(\sum_{i<j}\frac{(V_i-V_j)^2}{(\lambda_i-\lambda_j)^2}\right)\left(\sum_{i<j}1\right)$; equality holds iff $\frac{V_i-V_j}{\lambda_i-\lambda_j}$ is constant $k$ for all $i < j$.

*Consistency:* If $V_i = c(\lambda_i - \bar{\lambda})$, then for all $i < j$, $\frac{V_i-V_j}{\lambda_i-\lambda_j} = c$, so equality in Step 1 automatically implies equality in Step 2. Conversely, if $\frac{V_i-V_j}{\lambda_i-\lambda_j} = k$ for all $i < j$, then $V_i - k\lambda_i$ is independent of $i$; using $\sum_i V_i = 0$ (Lemma 7(i)), this constant equals $-k\bar{\lambda}$, hence $V_i = k(\lambda_i - \bar{\lambda})$. Thus the two equality conditions are equivalent. $\square$

*Remark* 10. The equality condition $V_i = c(\lambda_i - \bar{\lambda})$ (up to affine transformation) characterizes the zeros of the Hermite polynomial $H_n$: evaluating the ODE $H_n'' - 2xH_n' + 2nH_n = 0$ at its zeros $x_k$ gives $V_k = x_k$. For $n = 2$, this condition holds for any distinct root configuration.

# 4 Hermite Semigroup Flow

## 4.1 Hermite Kernel and Flow

Fix $p \in \mathcal{P}_n^{\mathbb{R}}$, let $a = \sigma^2(p) > 0$. We define a semigroup $(G_t)_{t\geq 0}$ via its generating polynomial.

**Definition 11** (Hermite Kernel)**.** Let

$$K_{G_t}(z) = \exp\left(-\frac{t}{2(n-1)}z^2\right) \quad (\mathrm{mod}\ z^{n+1}). \tag{7}$$

Define $G_t \in \mathcal{P}_n$ as the degree-$n$ polynomial whose normalized coefficients equal those of $K_{G_t}(z)$. The *Hermite flow* is

$$p_t = p \boxplus_n G_t.$$

**Lemma 12** (Semigroup Property)**.** *For any $s, t \geq 0$,*

$$G_s \boxplus_n G_t = G_{s+t}.$$

*Proof.* By multiplicativity of $K$ under $\boxplus_n$,

$$K_{G_s \boxplus_n G_t}(z) = K_{G_s}(z)\,K_{G_t}(z) = \exp\left(-\frac{s+t}{2(n-1)}z^2\right) \quad (\mathrm{mod}\ z^{n+1}),$$

which by definition is $K_{G_{s+t}}(z)$. $\square$

**Lemma 13** (Real-rootedness and Variance). *For each $t \geq 0$, the polynomial $G_t$ has $n$ distinct real roots, and*

$$\sigma^2(G_t) = t.$$

*Proof.* $G_1$ is, up to affine scaling, the Hermite polynomial $H_n$ in probability theory; its roots are all simple and real. Via scaling, for $t > 0$, $G_t(x) = t^{n/2} G_1(x/\sqrt{t})$ still has simple real roots, and $G_0 = x^n$. Since $K_{G_t}(z) = \exp(-\frac{t}{2(n-1)} z^2)$, we have $\kappa_1(G_t) = 0$, $\kappa_2(G_t) = -\frac{t}{2(n-1)}$; substituting into Lemma 2 yields $\sigma^2(G_t) = t$. $\square$

**Lemma 14** (Variance of the Flow). *For any $t \geq 0$, $\sigma^2(p_t) = a + t$.*

*Proof.* By variance additivity (Lemma 4) and Lemma 13, $\sigma^2(p_t) = \sigma^2(p) + \sigma^2(G_t) = a + t$. $\square$

## 4.2 Perturbation Analysis

**Lemma 15.** *Let $\lambda_i(t)$ be the roots of $p_t$. Then*

$$\lambda_i(t + h) = \lambda_i(t) + \frac{h}{n-1} V_i(t) + O(h^2).$$

*Proof.* By the semigroup property, $p_{t+h} = p_t \boxplus_n G_h$ and $\sigma^2(G_h) = h$. Since $K_{G_h}(z) = \exp(-\frac{h}{2(n-1)} z^2)$, the operator $T_{G_h}$ contains no first-order term and only the second-order term at order $h$, thus

$$T_{G_h} r(x) = r(x) - \frac{h}{2(n-1)} r''(x) + O(h^2).$$

Set $\lambda_i(t + h) = \lambda_i(t) + \delta_i$ and substitute into $T_{G_h} p_t(\lambda_i(t + h)) = 0$; solving to first order gives

$$\delta_i = \frac{h}{2(n-1)} \cdot \frac{p_t''(\lambda_i)}{p_t'(\lambda_i)} + O(h^2)$$

$$= \frac{h}{n-1} V_i(t) + O(h^2)$$

(using Lemma 6 in the last step). $\square$

**Lemma 16.**
$$\Phi_n(p_{t+h}) = \Phi_n(p_t) - \frac{2h}{n-1} \mathcal{S}(p_t) + O(h^2).$$

*Proof.* Let $\epsilon = h/(n-1)$, temporarily suppress the $t$ dependence. By Lemma 15, the perturbed score is

$$V_i(h) = \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j) + \epsilon(V_i - V_j) + O(h^2)} = V_i - \epsilon \sum_{j \neq i} \frac{V_i - V_j}{(\lambda_i - \lambda_j)^2} + O(h^2).$$

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Squaring and summing:

$$\Phi_n(p_{t+h}) = \sum_i V_i^2 - 2\epsilon \sum_{i \neq j} \frac{V_i(V_i - V_j)}{(\lambda_i - \lambda_j)^2} + O(h^2).$$

Pairing $(i,j)$ and $(j,i)$:

$$\sum_{i \neq j} \frac{V_i(V_i - V_j)}{(\lambda_i - \lambda_j)^2} = \sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2} = \mathcal{S}(p_t).$$

$\square$

*Remark* 17 (Consistency of Error Terms). The $O(h^2)$ remainders in Lemma 15 and Lemma 16 depend on the minimal root spacing $\delta_{\min}(t) = \min_i(\lambda_{i+1}(t) - \lambda_i(t))$: the implicit constant grows like $\delta_{\min}(t)^{-m}$ for some $m$ depending on $n$. Lemma 18 below shows that for a fixed $b > 0$, there exists a uniform lower bound $\delta_{\min}(t) \geq \delta_* > 0$ on $t \in [0, b]$, so the error terms are uniformly bounded on $[0, b]$, allowing passage to the derivatives $\dot{\lambda}_i = \frac{1}{n-1} V_i$, $\dot{\Phi}_n = -\frac{2}{n-1} \mathcal{S}$.

**Lemma 18.** *For any $b > 0$, the polynomials $p_t$ have $n$ simple real roots for $t \in [0, b]$.*

*Proof.* The coefficients of $p_t$ are smooth in $t$ (Definition 11), so the roots $\lambda_i(t)$ vary continuously. Since $p_0 = p$ has simple real roots, there exists a maximal interval $[0, T)$ where $p_t$ remains simple real-rooted; continuity ensures $T > 0$.

**Lyapunov function.** On $[0, T)$, define the logarithmic Vandermonde function

$$W(t) = \sum_{i<j} \log\big(\lambda_j(t) - \lambda_i(t)\big).$$

By Lemma 15, on $[0, T)$ the roots satisfy $\dot{\lambda}_i = \frac{1}{n-1} V_i$, hence

$$\dot{W}(t) = \sum_{i<j} \frac{\dot{\lambda}_j - \dot{\lambda}_i}{\lambda_j - \lambda_i} = \frac{1}{n-1} \sum_{i<j} \frac{V_j - V_i}{\lambda_j - \lambda_i}$$

$$= \frac{1}{n-1} \Phi_n(p_t) \geq 0,$$

where the penultimate equality follows from Lemma 7(iv). Therefore for any $t \in [0, T)$, $W(t) \geq W(0)$, i.e.,

$$\prod_{i<j}\big(\lambda_j(t) - \lambda_i(t)\big) \geq \prod_{i<j}\big(\lambda_j(0) - \lambda_i(0)\big) =: D_0 > 0. \tag{8}$$

**Uniform spacing lower bound.** Since $\sigma^2(p_t) = a + t \leq a + b$ for $t \in [0, b]$, the roots satisfy $\sum_i(\lambda_i - \bar{\lambda})^2 = n(a + t) \leq n(a + b)$, hence all roots lie within a distance $R = \sqrt{n(a+b)}$ from $\bar{\lambda}(t)$. Any two roots differ by at most $2R$. For a

given adjacent spacing $\delta_k(t) = \lambda_{k+1}(t) - \lambda_k(t)$, we can bound all other factors in (8) above by $2R$:

$$D_0 \leq \prod_{i<j} \big(\lambda_j(t) - \lambda_i(t)\big) \leq (2R)^{\binom{n}{2}-1} \delta_k(t),$$

yielding the uniform lower bound

$$\delta_k(t) \geq \delta_* := D_0 \, (2R)^{1-\binom{n}{2}} > 0$$

for each $k$ and each $t \in [0, T)$.

**Extendability.** Since all spacings are uniformly greater than $\delta_* > 0$ on $[0, T)$, continuity implies that the roots of $p_T$ remain distinct. This contradicts the maximality of $T$, unless $T \geq b$. $\qquad\square$

## 4.3  Dissipation and Integral Identity

**Lemma 19** (Dissipation).

$$\frac{d}{dt} \Phi_n(p_t) = -\frac{2}{n-1} \, \mathcal{S}(p_t).$$

*Proof.* From Lemma 16, $\frac{\Phi_n(p_{t+h}) - \Phi_n(p_t)}{h} = -\frac{2}{n-1} \mathcal{S}(p_t) + O(h)$. By Lemma 18, $p_t$ has distinct roots for $t \in [0, b]$, hence the scores $V_i(t)$ and $\mathcal{S}(p_t)$ are continuous in $t$. Letting $h \to 0$ gives the result. $\qquad\square$

**Corollary 20** (Integral Identity).

$$\frac{1}{\Phi_n(p \boxplus_n G_b)} - \frac{1}{\Phi_n(p)} = \frac{2}{n-1} \int_0^b \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} \, dt. \tag{9}$$

*Proof.* Let $f(t) = 1/\Phi_n(p_t)$. By the chain rule and Lemma 19,

$$f'(t) = \frac{2}{n-1} \cdot \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2} \geq 0.$$

By the fundamental theorem of calculus, $f(b) - f(0) = \int_0^b f'(t) \, dt$. Substituting $f(0) = 1/\Phi_n(p_0) = 1/\Phi_n(p)$, $f(b) = 1/\Phi_n(p_b) = 1/\Phi_n(p \boxplus_n G_b)$ yields (9). $\qquad\square$

## 5  Obstacles in General Fractional Flows

The fractional family based on $K_q(z)^t$ fails to generate an effective second-order generator for general $q$. When $\log K_q$ contains non-zero coefficients of degree $\geq 3$, the first-order expansion of $T_{q_h}$ involves higher-order derivatives, and the root ODE and dissipative identity used in the Hermite flow proof cease to hold. Moreover, for non-integer $t$, $q_t$ and $p \boxplus_n q_t$ may lose real-rootedness, making the Lyapunov argument concerning root simplicity inapplicable.

*Remark* 21 (Identified Obstacles). The following points summarize the main issues encountered in existing flow-based proofs and natural comparison attempts:

(i) The generator of a fractional flow is second-order iff $\log K_q$ is purely quadratic; this uniquely singles out the Hermite kernel.

(ii) For non-integer $t$, fractional families do not preserve real-rootedness; both $q_t$ and $p \boxplus_n q_t$ may develop non-real roots, rendering the Lyapunov argument invalid.

(iii) Asymmetric forward/reverse bounds derived from fractional flows do not constitute a universal inequality—even under the anticipated case distinction. This conclusion is drawn from experimental observation and has not been used in any formal argument.

(iv) Naive monotonicity comparison $\Phi_n(p \boxplus_n q) \leq \Phi_n(p \boxplus_n G_b)$ fails numerically; therefore the Hermite bound cannot be directly extended via substitution.

# 6 Another Route: Critical Point Identities

We outline an alternative approach that does not rely on flows. The critical point identity

$$\Phi_n(p) = -\frac{1}{4} \sum_{p'(\zeta)=0} \frac{p''(\zeta)}{p(\zeta)}$$

suggests that controlling $\Phi_n(T_q p)$ can be achieved by comparing $(T_q p)''/(T_q p)$ with $p''/p$ at the critical points of $T_q p$.

## 6.1 Critical Point Identity

**Lemma 22.** *Let $p \in \mathcal{P}_n^{\mathbb{R}}$ have distinct roots, and let $\zeta_1, \ldots, \zeta_{n-1}$ be the simple zeros of $p'$. Then*

$$\Phi_n(p) = -\frac{1}{4} \sum_{j=1}^{n-1} \frac{p''(\zeta_j)}{p(\zeta_j)}. \tag{10}$$

*Proof.* By Lemma 6, $\Phi_n = \frac{1}{4} \sum_{i=1}^{n} \frac{p''(\lambda_i)^2}{p'(\lambda_i)^2}$. Consider the meromorphic function

$$F(x) = \frac{p''(x)^2}{p'(x)\, p(x)}.$$

*Poles at roots.* Since $p$ has a simple zero at $\lambda_i$ and $p'(\lambda_i) \neq 0$, $\operatorname{Res} x = \lambda_i F = p''(\lambda_i)^2/p'(\lambda_i)^2$. Summing gives $\sum_i \operatorname{Res} \lambda_i F = 4\Phi_n$.

*Poles at critical points.* At a simple zero $\zeta_j$ of $p'$, by the interlacing property $p(\zeta_j) \neq 0$, hence $\operatorname{Res}_{x=\zeta_j} F = p''(\zeta_j)/p(\zeta_j)$.

*Pole at infinity.* As $x \to \infty$, $F(x) = n(n-1)^2/x^3 + O(x^{-4})$, therefore $\operatorname{Res}_\infty F = 0$.

The sum of all residues on $\mathbb{P}^1$ is zero, i.e., $4\Phi_n + \sum_j p''(\zeta_j)/p(\zeta_j) = 0$. $\square$

## 6.2 Comparison Scheme

Let $r = T_q p$. By Lemma 22,

$$\Phi_n(r) = -\frac{1}{4} \sum_{r'(\xi)=0} \frac{r''(\xi)}{r(\xi)}.$$

Thus, if we can compare $r''/r$ pointwise with $p''/p$ at the critical points of $r$, this translates into a global bound on $\Phi_n(r)$.

**Lemma 23** (Comparison Target). *There exist constants $\alpha(q)$ and $\beta(q)$, depending only on the variance and low-order normalized coefficients of $q$, such that for any real-rooted polynomial $p$ with distinct roots and any critical point $\xi$ of $T_q p$,*

$$\frac{(T_q p)''(\xi)}{(T_q p)(\xi)} \leq \alpha(q) + \beta(q) \frac{p''(\xi)}{p(\xi)}.$$

If Lemma 23 can be established with $\alpha(q) = 0$, $\beta(q) \leq 1/\sigma^2(q)$, then summing over critical points yields a bound on $\Phi_n(p \boxplus_n q)$ that implies Conjecture 1. Proving this comparison is the central unsolved problem of this route.

*Remark* 24 (Simplification Steps). The comparison lemma can be approached progressively through the following steps:

(i) Express $r''/r$ as a rational function of the derivatives of $p$:

$$\frac{r''}{r} = \frac{\sum_k c_k \, p^{(k+2)}}{\sum_k c_k \, p^{(k)}}$$

where the coefficients $c_k$ are determined by $q$.

(ii) Using interlacing and sign regularity of $p^{(k)}$ evaluated between roots, obtain inequalities for the ratios $p^{(k+2)}/p^{(k)}$ at the critical points of $r$.

(iii) Show that the right-hand side can be controlled by an affine function of $p''/p$ depending only on low-order normalized coefficients of $q$.

This route would not rely on any flow and could directly prove Conjecture 1.

# 7 Hermite Flow Bound

**Theorem 25** (Hermite Flow Bound). *Let $p \in \mathcal{P}_n^{\mathbb{R}}$, $a = \sigma^2(p) > 0$, $b > 0$. Then*

$$\frac{1}{\Phi_n(p \boxplus_n G_b)} \geq \frac{a+b}{a \, \Phi_n(p)}. \tag{11}$$

*Proof.* Let $a = \sigma^2(p)$.

**Step 1 (Differential inequality).** Applying the Score–Gradient Inequality (Theorem 9) to $p_t$ yields

$$\mathcal{S}(p_t) \geq \frac{(n-1)\, \Phi_n(p_t)}{2\, \sigma^2(p_t)}.$$

Substituting into Lemma 19:

$$\frac{d}{dt}\Phi_n(p_t) \;\leq\; -\frac{1}{a+t}\,\Phi_n(p_t).$$

Integrating $(\log \Phi_n(p_t))' \leq -1/(a+t)$ from 0 to $t$ gives

$$\frac{1}{\Phi_n(p_t)} \;\geq\; \frac{a+t}{a\,\Phi_n(p)}. \tag{12}$$

**Step 2 (Forward bound).** By Corollary 20 and the Score–Gradient Inequality:

$$\frac{1}{\Phi_n(p \boxplus_n G_b)} - \frac{1}{\Phi_n(p)} \;\geq\; \int_0^b \frac{dt}{(a+t)\,\Phi_n(p_t)}.$$

Substituting (12), the factors $(a+t)$ cancel:

$$\frac{1}{\Phi_n(p \boxplus_n G_b)} \;\geq\; \frac{a+b}{a\,\Phi_n(p)}. \tag{13}$$

$\square$

*Remark* 26. The Hermite bound (11) is sharp for Hermite inputs, but does not itself imply Conjecture 1. The asymmetric reverse bound used in fractional flow proofs has not been successfully justified for general inputs.

*Remark* 27. Strict inequality typically holds. Equality in the Hermite bound (11) occurs iff the roots of $p$ are an affine scaling of the zeros of the Hermite polynomial $H_n$. For $n = 2$, this condition is satisfied for any distinct root configuration.

*Remark* 28 (Boundary Behavior). Under the convention $1/\Phi_n := 0$ (for repeated roots), if Conjecture 1 is proved for the distinct-root case, it can be extended to the boundary of $\mathcal{P}_n^{\mathbb{R}}$ as follows. When both $p$ and $q$ have repeated roots, both sides of the inequality are zero and it holds trivially. When exactly one factor (say $p$) has repeated roots, the inequality reduces to $\Phi_n(q) \geq \Phi_n(p \boxplus_n q)$, a monotonicity statement. To verify this, approximate $p$ by polynomials $p_\varepsilon \to p$ with distinct roots; the proven inequality gives $1/\Phi_n(p_\varepsilon \boxplus_n q) \geq 1/\Phi_n(p_\varepsilon) + 1/\Phi_n(q) \geq 1/\Phi_n(q)$. Since convolution is continuous in coefficients, $p_\varepsilon \boxplus_n q \to p \boxplus_n q$. If $p \boxplus_n q$ has distinct roots, then $\Phi_n(p_\varepsilon \boxplus_n q) \to \Phi_n(p \boxplus_n q)$ and the bound carries over to the limit. The remaining case—$p \boxplus_n q$ itself has repeated roots while $q$ has distinct roots—cannot occur: in the matrix model $(p \boxplus_n q)(x) = \int_{O(n)} \det(xI - (A + QBQ^T))\,d\mu(Q)$, if $B$ has at least two distinct eigenvalues, then for $\mu$-almost every $Q$, the polynomial in the integrand has $n$ distinct real roots; the averaged polynomial (which is real-rooted by [1]) can acquire new repeated roots only at the boundaries of its interlacing families, and assuming $\sigma^2(q) > 0$ (positive variance) excludes such boundary cases.

# References

[1] A. Marcus, D. A. Spielman, and N. Srivastava, *Interlacing families II: Mixed characteristic polynomials and the Kadison–Singer problem*, Ann. of Math. **182** (2015), 327–350.

[2] A. J. Stam, *Some inequalities satisfied by the quantities of information of Fisher and Shannon*, Inform. Control **2** (1959), 101–112.