Université de Paris **(Date: 19/07/2020)**

UFR Mathématiques

Thèorie des ensembles, TD1

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Exercise 1. Let R be a relation on a set X. Show that R is not well-founded if and only if there is a sequence $\{x_n\} \subseteq X$ such that $x_{n+1}Rx_n$ for al $n \in \mathbb{N}$.

Solution: Suppose that R is not well founded (this implies $X \neq \varnothing$), then there exists some non-empty $Y \subseteq X$ with no minimal element. That is, for every $y \in Y$ there exists $z \in Y$ such that zRy. Take any $x_0 \in Y$ (*choice*) and take $x_1 \in Y$ such that x_1Rx_0. Inductively, if $\{x_0, \ldots, x_k\} \subseteq Y$ are such that $x_{i+1}Rx_i$ for $i < k$, then by hypothesis, there is $x_{n+1} \in Y$ such that $x_{n+1}Rx_n$. By axiom of *union*, we can form $S = \{x_n\}_{n\in\mathbb{N}}$ as required. Conversely, suppose $\{x_n\}$ is a sequece as stated, then $S = \{x_0, x_1, \ldots, x_n, \ldots\}$ has no minimal element.

Exercise 2. Show that $\in$ is a well-founded set-like extensional relation on V. Is $\in$ transitive? Is $\in$ a strict order?

Solution: If $\in$ weren't well founded there would exist some sequence $S = \{x_n\}$ of sets in V with $x_{n+1} \in x_n$. The fact that S is a set contradicts the axiom of *regularity*, since for every n, $x_{n+1} \in x_n \cap S$. The relation $\in$ is set-like, take a set x, then $\in^{-1}[x] = \{y, y \in x\} = \{y \in x, y \in x\} = x$. It is also extensional since $\in^{-1}[x] = \in^{-1}[y]$ means that x, y have the same elements, therefore $x = y$ by *extensionality*. It is not transitive, take some set x and take $A = \{\{x\}\}$, then $x \in \{x\}$ and $\{x\} \in \{\{x\}\}$ but $x \notin A$. It is also not a strict order since it is not transitive.

Exercise 3. Let x be a set. Show that there is a transitive y such that $x \subseteq y$. Show that such y can be chosen in a minimal way, which we will call the **transitive closure of** x.

Solution. Take $x_0 = x$ and inductively $x_{n+1} = \cup x_n$. Then take $y = \cup_{n\in\mathbb{N}} x_n$. Clearly $x = x_0 \subseteq y$, to see that y is transitive, let $w \in z \in y$, then for some k, $z \in x_k$, and since $x_{k+1} = \cup x_k$, we have $w \in x_{k+1} \subset y$. Finally, to see minimality, let $x \subseteq T$ for some transitive set T. We will show that $y \subseteq T$. Let $z = z_k \in y$, so that for some k, $z_k \in x_k$. This means that for some $z_{k-1} \in x_{k-1}$, $z_k \in z_{k-1}$, repeating this argument, we get a finite sequence $z_k, z_{k-1}, \ldots, z_0$ such that for $i = 0, \ldots k$, $z_i \in x_i$ and $z_i \in z_{i-1}$. Since $z_0 \in x_0 \subseteq T$, by transitivity of T, $z_1 \in T, \ldots, z_k = z \in T$.

Exercise 4. Use the axiom of regularity and the transitive closure to show that if C is a class, then C has a $\in$-minimal element.

Solution: Let x be any set in C, if x and C have no elements in common, then x is minimal. Otherwise there is some $y \in S \cap x$ (informal notation for y is in C and in x). Note that $\mathrm{TC}(y) \cap C$ is a non-empty set. By *regularity*, there is a minimal $z \in \mathrm{TC}(y) \cap C$ (otherwise there would be a descending infinite sequence in w). Let's see that z is actually $\in$-minimal in C: if it weren't, there would be $z' \in C$ such that $z' \in z$. This means $z' \in \mathrm{TC}(y)$, and therefore $z' \in \mathrm{TC}(y) \cap C$, contradicting the minimality of z in this last set.

Exercise 5. Show that if M_1 and M_2 are transitive classes and $\pi : M_1 \to M_2$ is an $\in$-isomorphism, then π is the identity.

Solution: We define the class $C = \{x, \pi(x) \neq x\}$. By ex.4, choose some minimal $x \in C$. We will show that $\pi(x) = x$ and arrive at a contradiction. First let's see that $x \subseteq \pi(x)$. Let $y \in x \Rightarrow y \in M_1 \wedge \pi(y) \in \pi(x)$. Since x is minimal in C, $\pi(y) = y$ and therefore $y \in \pi(x)$. Now let's check $\pi(x) \subseteq x$: take $w \in \pi(x)$, then $\pi^{-1}(w) \in x$ and by minimality of x we have $w = \pi(\pi^{-1}(w)) = \pi^{-1}(w) \Rightarrow w \in x$. There is a contradiction, C cannot have a $\in$-minimal element, so it must be empty. This implies that $\pi \equiv id$.

Excercise 6. (Mostowski's Collapsing Lemma). Let C be a class and R be a well-founded, set-like, extenional relation. Then there is a unique transitive class M and a unique isomorphism $(C, R) \to (M, \in)$.

Solution: First, we note that the R-minimal element $x \in C$ is unique, since if there were another minimal $y \in C$, we would have $R^{-1}[x] = R^{-1}[y] = \varnothing$, which would imply $x = y$ by extensionality. We now define $\pi(x) = \varnothing$, and for every other $y \in C$, $\pi(y) = \{\pi(z), zRy\} = \pi(R^{-1}[x])$. This function is $\in$-preserving since $yRx \Rightarrow y \in R^{-1}[x] \Rightarrow \pi(y) \in \pi(R^{-1}[x]) = \pi(x)$. We now take $M = \cup_{x \in C} \pi(x)$. Note that M is a transitive class because if $z \in M$ it means there are $x, y \in C$

such that $z = \pi(y)$ with yRx, this implies $z \in M$. By construction, π is clearly surjective. Finally, to see that π is injective, take $x \neq y$ in C, then by extensionality $R^{-1}[x] \neq R^{-1}[y]$ so there exists $z \in C$ such that $zRx \wedge z\not{R}y$, which implies $\pi(x) \neq \pi(y)$. To check uniqueness, we can suppose M_1 and M_2 are transitive classes that satisfy the lemma, with respective mappings π_1, π_2. We then would have the $\in$-isomorphism $\pi_1 \circ \pi_2^{-1} : M_2 \to M_1$. By ex.5, it is the identity, so $M_1 = M_2$.

$$\begin{tikzcd}
 & M_1 \arrow[dd, dash, "="] \\
C \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & M_2
\end{tikzcd}$$

Exercise 7. Let $(X, <_1)$, $(Y, <_2)$ be ordered sets. Define $<_3$ on $X \times Y$ by $(x, y) < (x', y')$ if and only if $x <_1 x'$ and $y <_2 y'$. Show that this is and order. If $<_1$ and $<_2$ are total orders, is $<_3$ a total order?.

Solution: Easy.

Exercise 8. Let $(X, <_1)$, $(Y, <_2)$ be totally ordered sets. Show that $<_{lex}$ totally orders $X \times Y$. If $<_1$ and $<_2$ are well-orders, is $<_{lex}$ such?

Solution. Easy, it is a well order.

Exercise 9. Show that all countable total orders embed into $\mathbb{Q}$.

Solution:

Lemma: If $X = \{x_0, \ldots, x_n\}$ is a finite total order, $X' = X \cup \{x_{n+1}\}$ is a total order extending the one in X, and $\varphi : X \to \mathbb{Q}$ is an embedding, then there exists an embedding $\varphi' : X' \to \mathbb{Q}$ such that $\varphi' \restriction_X = \varphi$.

Proof: We have three cases: if $x_{n+1} > \max(X)$, then take $\varphi'(x_{n+1}) = \varphi(\max(X)) + 1$, else if if $x_{n+1} < \min(X)$, then take $\varphi'(x_{n+1}) = \varphi(\min(X)) - 1$. Otherwise, there are $i, j \in \{0, 1, \ldots, n\}$ such that $x_i < x_{n+1} < x_j$, then take $\varphi'(x_{n+1}) = \frac{1}{2}(\varphi(x_j) + \varphi(x_i))$, this completes the proof of the lemma.

To show that $X = \{x_0, \ldots, x_n, \ldots\}$ embeds into $\mathbb{Q}$, set $X_0 = x_0$ and $X_{n+1} = X_n \cup x_{n+1}$. Taking $\varphi_0 : X_0 \to \mathbb{Q}$ as $x_0 \mapsto 0$, and using the lemma to define $\varphi_n : X_n \to \mathbb{Q}$ such that $\varphi_{n+1} \restriction X_n = \varphi_n$, we can take our embedding to be $\varphi = \cup_{n\in\mathbb{N}}\varphi_n$.

Exercise 10. Let X be a set. Show that $(\mathcal{P}(X), \subseteq)$ is ordered. Show that $\subseteq$ is extensional on $\mathcal{P}(x)$.

Solution: $\subseteq$ is clearly an order. If $\subseteq^{-1}[A] = \subseteq^{-1}[B]$, since $\subseteq$ is reflexive, we have in particular that $A \subseteq B$ and $B \subseteq A$. Thus, $A = B$.

Exercise 11. Let $(X, <)$ be an ordered set. Show that there is an order morphism of $(X, <)$ in $(\mathcal{P}(X) \subseteq)$. Explicitly, write a morphism ϕ with the property that ϕ is injective if and only if $<$ is extensional.

Solution: $x \mapsto \{y \in X, y < x\}$.

Exercise 12. For $A, B \subseteq \mathbb{N}$, define

$$A \subseteq^* B \text{ if and only if } A \setminus B \text{ is finite}$$

Show that $\subseteq^*$ is transitive. Is it an order? Describe al $\subseteq^*$-predecesors of $\varnothing$.

Solution: Transitivity follows from the fact that $A \setminus C \subseteq A \setminus B \cup B \setminus C$. Is is not an order since it is not antisymmetric (take $\{1, 2\}$ and $\{2, 3\}$. The set of predecesors of $\varnothing$ consists of all the finite sets.

Exercise 13. Let $\mathcal{P}(\mathbb{N})$ be endowed with Δ and $\cap$ as addition and product. Show that this is a commutative ring with unity. Show that the set

$$\text{Fin} = \{A \subseteq \mathbb{N}, A \text{ is finite}\}$$

is an ideal.

Solution: It is routine to show that Δ and $\cap$ are commutative and associative. $\varnothing$ is the identity for addition and $\mathbb{N}$ for multiplication. Given $A \subseteq \mathbb{N}$, then $A\Delta A = \varnothing$. Finally, using the fact that

$X \cap (Y \setminus Z) = (X \cap Z) \setminus (X \cap W)$, we get distributivity:

$$\begin{aligned} A \cap (B \Delta C) &= A \cap ((B \setminus C) \cup (C \setminus B)) \\ &= (A \cap ((B \setminus C)) \cup (A \cap ((C \setminus B)) \\ &= ((A \cap B) \setminus (A \cap C)) \cup ((A \cap C) \setminus (A \cap B)) \\ &= (A \cap B) \Delta (A \cap C) \end{aligned}$$

All of these show that $(\mathcal{P}(\mathbb{N}), \Delta, \cap)$ is a ring with identity. The family of finite sets forms an ideal since it is clearly closed under Δ and the intersection of any $X \subseteq \mathbb{N}$ with a finite set is finite.

Exercise 14. On $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, define $[A] \subseteq [B]$ iff $A \subseteq^* B$. Show that this relation is well defined. Conclude that $\subset^*$ strictly orders $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, and show that it is extensional.

Solution: Notice that $[A] = [B]$ iff $A \Delta B$ is finite iff $A \subseteq^* B$ and $B \subseteq^* A$. Suppose that $A \subseteq^* B$, $[C] = [A]$ and that $[D] = [B]$. Then we have by the above that $C \subseteq^* A \subseteq^* B \subseteq^* D$, the result follows from transitivity of $\subseteq^*$. To check extensionality, notice that $[X] \subseteq [X]$ for every $X \subseteq \mathbb{N}$, if we suppose that $\subseteq^{-1} [A] = \subseteq^{-1} [B]$, then in particular $A \subseteq^* B$ and viceversa, this proves that $[A] = [B]$.

Exercise 15. Two elements of an ordered set are incompatible if there is no element which is below both of them, a subset $C \subset X$ is a chain if for al $x, y, \in C$ either $x > y$ or $y > x$. Show that there is an uncountably infinite family of pairwise incompatible elements of $(\mathcal{P}(\mathbb{N})/\operatorname{Fin} \setminus [\varnothing], \subset^*)$, and that there is an uncountable well-founded chain in $(\mathcal{P}(\mathbb{N})/\operatorname{Fin} \setminus [\mathbb{N}], \subset^*)$. Conclude that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}, \subset)$ does not order-embed into $(\mathcal{P}(\mathbb{N}), \subset)$.

Solution: Notice that $[\varnothing]$ and $[\mathbb{N}]$ represent the classes of finite and co-finite sets, respectively. Also, two elements $[A], [B]$ are incompatible if and only if $A \cap B$ is finite: suppose that $Y = A \cap B$ is infinite, then $[Y] \subseteq [A]$ and $[Y] \subseteq [B]$, which makes $[A], [B]$ compatible, on the other hand, if $A \cap B$ is finite, if we suppose that there exists $[X] \subseteq [A], [B]$, then $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$ is finite, which implies $[X] \subseteq [A \cap B] \Rightarrow X$ is finite, a contradiction since we excluded $[\varnothing]$.

To show the existence of an uncountably infinite family of pairwise incompatible sets we will show that any countable family of these can be extended, and that a maximal family of such sets cannot be countable.

Take a countable family of pairwise incompatible elements of $(\mathcal{P}(\mathbb{N})/\operatorname{Fin}\setminus[\varnothing]$, say $[X_n]$, $n\in\mathbb{N}$. Let

$$Y_0 = X_0 \text{ and } Y_{n+1} = X_{n+1}\setminus\bigcup_{i\leq n} X_n.$$

All of the Y_n are pairwise disjoint and $[Y_n]=[X_n]$ for every n, since $X_n\cap Y_n = X_n\cap(\bigcap_{i<n} X_n\setminus X_i)$ is finite. Pick an element $x_n\in Y_n$, then the set $Y=\{x_n, n\in\mathbb{N}\}$ is almost disjoint from each X_n, which makes the original family not maximal. By Zorn's Lemma, we can extend any family of pairwise incompatible elements to a maximal one containing it. Finally, take for every n, $X_n=\{p_n^k, k>0\}$ where p_n is the n-th prime number. This is a countable family of pairwise incompatible sets, and we can extend it to a maximal one, which cannot be uncountable. Next, we have to show that there is no embedding of $(\mathcal{P}(\mathbb{N})/\operatorname{Fin}\setminus[\varnothing],\subseteq)$ into $(\mathcal{P}(\mathbb{N}),\subseteq)$.

Lemma: All well-founded chains in $(\mathcal{P}(\mathbb{N}),\subset)$ are countable.

Proof: Suppose there is an uncoutable $\subset$-chain. For x in C, let $S(x)$ be the $\subset$-minimal element of C above x (exists because of well-foundedness). If $x\neq y$, then *wlog* $x\subseteq y$ and hence $S(x)\subseteq y$. This implies that for every $x\neq y$ in C

$$(S(x)\setminus x)\cap(S(y)\setminus y)=\varnothing.$$

Since each of $S(x)\setminus x$ is nonempty (strict ordering), the set $X=\bigcup_{x\in C} S(x)\setminus x\subseteq\mathbb{N}$ is uncountably infinite, this is a contradiction. Since embeddings of chains are chains, we just need to find an uncountably infinite well-founded chain in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}\setminus[\mathbb{N}]$. Let

$$\mathcal{D}=\{\mathcal{C}\subset\mathcal{P}(\mathbb{N})/\operatorname{Fin}\setminus[\mathbb{N}],\mathcal{C}\text{ is a well-founded chain}\}.$$

We can order $\mathcal{D}$ by end-extensions of chains. Taking by Zorn's Lemma a maximal chain in $\mathcal{D}$, there is a well-founded chain in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}\setminus[\mathbb{N}]$ that cannot be end-extended. Such chain cannot be countable, to prove this we'll show *that every countable chain in $\mathcal{D}$ is end-extendable.*

Proof: Let $\mathcal{C}=[A_n]$ be a countable chain (we assume the $[A_n]$'s to be different). We want to find a non-cofinite $C\subseteq\mathbb{N}$ such that for every n, $A_n\subseteq^* C$. Let $B_n=\bigcup_{i\leq n} A_n$, and notice that $B_n\subseteq B_{n+1}$ for all n. Notice that $B_{n+1}\setminus B_n$ is infinite for every n since $[A_n]\subsetneq[A_{n+1}]$ implies $A_{n+1}\setminus A_n$ is infinite. Note that if C a non-cofinite set such that for every n, $B_n\subseteq^* C$, the same is true for every A_n. Let $k_i=\min B_{i+1}\setminus B_i$ and $C=\mathbb{N}\setminus\{k_n\}_{n\in\mathbb{N}}$ (it is non-cofinite by construction). We have that if $i\geq n$, then $k_i\notin B_n$, which implies that for every n, $B_n\cap\{k_i\}_{i\in\mathbb{N}}$ is finite or equivalently,

$B_n \subseteq^* C$. To conclude, we can assume said $\mathcal{C}$ to be well-founded and extend it to a uncountably infinite maximal chain in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}\setminus[\mathbb{N}]$ which cannot be embedded intro $\mathcal{P}(\mathbb{N})$ as a consequence of one of the lemmas.