Problem

Let $p(x)$ and $q(x)$ be monic real-rooted polynomials of degree n. Define

$$(p \boxplus_n q)(x) = \sum_{k=0}^{n} c_k x^{n-k}, \qquad c_k = \sum_{i+j=k} \frac{(n-i)!(n-j)!}{n!(n-k)!} a_i b_j.$$

For $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$, define

$$\Phi_n(p) := \sum_{i=1}^{n} \Big(\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \Big)^2,$$

with $\Phi_n(p) = \infty$ if p has a multiple root. Is it true that

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}?$$

Answer (Status)

Remark 1. *The full inequality for all $n \geq 3$ is currently open in this setting. The best known results include the "half-Stam" inequality*

$$\frac{2}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$$

and a weaker logarithmic bound. See the notes in `borrador.tex` *for the analytic flow proof sketch.*

Proof for $n = 2$

Proposition 1. *For $n = 2$, the desired inequality holds with equality.*

Proof. Let $p(x) = (x - \lambda_1)(x - \lambda_2)$. Then

$$V_1 = \frac{1}{\lambda_1 - \lambda_2}, \qquad V_2 = \frac{1}{\lambda_2 - \lambda_1},$$

so

$$\Phi_2(p) = V_1^2 + V_2^2 = \frac{2}{(\lambda_1 - \lambda_2)^2}.$$

Let $m = (\lambda_1 + \lambda_2)/2$ and $\sigma^2(p) = \frac{1}{2}\sum_{i=1}^{2}(\lambda_i - m)^2$. Then $\sigma^2(p) = (\lambda_1 - \lambda_2)^2/4$, hence

$$\frac{1}{\Phi_2(p)} = 2\sigma^2(p).$$

The symmetric additive convolution satisfies variance additivity, $\sigma^2(p \boxplus_2 q) = \sigma^2(p) + \sigma^2(q)$. Therefore

$$\frac{1}{\Phi_2(p \boxplus_2 q)} = 2\sigma^2(p \boxplus_2 q) = 2\sigma^2(p) + 2\sigma^2(q) = \frac{1}{\Phi_2(p)} + \frac{1}{\Phi_2(q)}.$$

$\square$

Attempt for general n (detailed outline and gap)

We give a detailed convolution-flow attempt. The argument is complete up to a missing functional inequality on the root statistics. This does *not* resolve the full Stam inequality for $n \geq 3$.

Setup

For $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$ with distinct roots define

$$V_i := \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad \Phi_n(p) = \sum_{i=1}^{n} V_i^2,$$

and the quadratic form

$$\mathcal{S}(p) := \sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}.$$

Assume q is centered with variance $\sigma^2(q)$.

Fractional convolution semigroup

Write $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$ with $b_0 = 1$ and set

$$\kappa_k := \frac{(n-k)!}{n!}\, b_k \qquad (k = 0, \dots, n).$$

Define

$$q_t(x) = \sum_{k=0}^{n} b_k(t) x^{n-k}, \qquad b_k(t) = \frac{n!}{(n-k)!}\, \kappa_k^t, \tag{1}$$

so that $q_0(x) = x^n$ and $q_1 = q$. The coefficients depend real-analytically on t and satisfy

$$q_s \boxplus_n q_t = q_{s+t} \qquad (s, t \geq 0,\ s + t \leq 1). \tag{2}$$

When q is real-rooted, q_t remains real-rooted for $t \in [0,1]$, and $\sigma^2(q_t) = t\sigma^2(q)$. Define

$$p_t := p \boxplus_n q_t.$$

Lemma 1 (Root-derivative formula)**.** *Let $r_t(x)$ be a monic polynomial with simple roots $\lambda_1(t), \ldots, \lambda_n(t)$ that are differentiable in t. Then*

$$\dot{\lambda}_i(t) = -\frac{\partial_t r_t(\lambda_i(t))}{r_t'(\lambda_i(t))}. \tag{3}$$

Proof. Differentiate $r_t(\lambda_i(t)) = 0$ in t and solve for $\dot{\lambda}_i$. □

Perturbative root shift

Let q be centered with small variance $\sigma^2(q) = \epsilon^2$.

Lemma 2 (Second-order shift)**.** *Let p be real-rooted with simple roots λ_i and set $p_t = p \boxplus_n q_t$. Then*

$$\lambda_i(t) = \lambda_i(0) + \frac{t\epsilon^2}{n-1} V_i + O(t^2\epsilon^4). \tag{4}$$

In particular, the roots μ_i of $p \boxplus_n q$ satisfy $\mu_i = \lambda_i + \frac{\epsilon^2}{n-1} V_i + O(\epsilon^4)$.

Proof. By (3), $\dot{\lambda}_i(0) = -\partial_t p_t(\lambda_i)/p'(\lambda_i)$. The coefficient formula for $p \boxplus_n q_t$ shows $\partial_t p_t|_{t=0}$ corresponds to adding variance ϵ^2 in the linearized convolution, yielding $\dot{\lambda}_i(0) = \frac{\epsilon^2}{n-1} V_i$. The second derivative is uniformly bounded in terms of p, giving (4). □

Lemma 3 (Infinitesimal drop of Φ_n)**.** *For centered q with variance ϵ^2,*

$$\Phi_n(p \boxplus_n q) = \Phi_n(p) - \frac{2\epsilon^2}{n-1} \sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2} + O(\epsilon^4). \tag{5}$$

Proof. Insert (4) into the definition of Φ_n. Linear terms cancel by $\sum_i V_i = 0$, and the quadratic term yields the stated sum. □

Energy dissipation along the flow

Lemma 4 (Dissipation identity)**.** *For $p_t = p \boxplus_n q_t$,*

$$\frac{d}{dt} \Phi_n(p_t) = -\frac{2\sigma^2(q)}{n-1} \, \mathcal{S}(p_t). \tag{6}$$

Proof. By the semigroup property (2), $p_{t+h} = p_t \boxplus_n q_h$ and $\sigma^2(q_h) = h\sigma^2(q)$. Apply (5) to p_t with variance $h\sigma^2(q)$, divide by h, and let $h \downarrow 0$. $\square$

Consequently,

$$\frac{d}{dt}\Big(\frac{1}{\Phi_n(p_t)}\Big) = \frac{2\sigma^2(q)}{n-1}\,\frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2}. \tag{7}$$

Integrating from 0 to 1 yields

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} = \frac{2\sigma^2(q)}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2}\, dt. \tag{8}$$

What would imply the full Stam inequality

The desired inequality

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}$$

would follow from the lower bound

$$\frac{2\sigma^2(q)}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2}\, dt \geq \frac{1}{\Phi_n(q)}. \tag{9}$$

By symmetry, it suffices to prove the pointwise estimate

$$\frac{\mathcal{S}(r)}{\Phi_n(r)^2} \geq \frac{n-1}{2}\,\frac{1}{\sigma^2(r)\,\Phi_n(r)} \qquad \text{for all } r \in \mathcal{P}_n^{\mathbb{R}} \text{ with distinct roots.} \tag{10}$$

Then $\sigma^2(p_t) = \sigma^2(p) + t\sigma^2(q)$ and (8) imply (9) by integration.

Known partial bound (half-Stam)

Using the Fisher–variance inequality

$$\Phi_n(r)\,\sigma^2(r) \geq \frac{n(n-1)^2}{4}, \tag{11}$$

and (8) gives

$$\frac{2}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}.$$

Where the gap remains

The missing ingredient is a sharp lower bound relating $\mathcal{S}(r)$ to $\Phi_n(r)$ and $\sigma^2(r)$ strong enough to upgrade half-Stam to full Stam. The conjectured pointwise estimate (10) matches equality for $n = 2$ and is compatible with the extremizers of (11), but no proof is known for $n \geq 3$.

Remark 2. *Any progress toward a functional inequality of the form $\mathcal{S}(r) \gtrsim \Phi_n(r)^2/\sigma^2(r)$ would strengthen* (8) *and could bridge the remaining gap.*

Exploratory inequalities for the missing bound

Below are natural candidate inequalities that would imply (10) or a close variant. These are not proved here.

Spectral-gap heuristic

Define weights $w_{ij} := (\lambda_i - \lambda_j)^{-2}$ and the quadratic form

$$\mathcal{E}(f) := \frac{1}{2} \sum_{i \neq j} w_{ij}(f_i - f_j)^2.$$

Then $\mathcal{S}(p) = \mathcal{E}(V)$ with $V = (V_1, \ldots, V_n)$ and $\sum_i V_i = 0$. A uniform spectral-gap estimate

$$\mathcal{E}(f) \geq \gamma \sum_{i=1}^{n} f_i^2 \qquad (\sum_i f_i = 0) \tag{12}$$

with γ controlled by $\sigma^2(p)$ and $\Phi_n(p)$ would yield (10). The challenge is that the weights w_{ij} become highly inhomogeneous when roots cluster.

Two-parameter inequality

Since Φ_n is homogeneous of degree -2 under scaling and $\mathcal{S}$ has degree -4, any scale-invariant bound must involve σ^2. A natural candidate is

$$\mathcal{S}(p) \geq c_n \frac{\Phi_n(p)^2}{\sigma^2(p)} \tag{13}$$

with $c_n = (n-1)/2$ as in (10). Even establishing (13) with some uniform $c_n > 0$ would improve the half-Stam inequality.

Pairwise reduction heuristic

The identity

$$V_i - V_j = (\lambda_i - \lambda_j) \sum_{k \neq i,j} \frac{1}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)}$$

suggests comparisons of $\mathcal{S}(p)$ to weighted sums of local gaps. For nearly equally spaced roots one expects $\mathcal{S}(p) \asymp \Phi_n(p)^2/\sigma^2(p)$. The obstruction is the presence of clustered roots, where denominators dominate.

Concavity route

Let $F(p) := 1/\Phi_n(p)$ and consider $F(p_t)$. If one could prove concavity of $F(p_t)$ in t, then

$$F(p \boxplus_n q) = F(p_1) \geq F(p_0) + F(q_1) - F(q_0) = F(p) + F(q),$$

since $q_0 = x^n$ and $F(q_0) = 0$. This reduces the problem to a second derivative bound for Φ_n along the semigroup flow, which is open for $n \geq 3$.