THE FINITE FREE STAM INEQUALITY: STRUCTURAL IDENTITIES, THE GAUSSIAN-INPUT CASE, AND REDUCTION PRINCIPLES

ABSTRACT. Let $\mathcal{P}_n^{\mathbb{R}}$ denote the set of monic, degree-n, real-rooted polynomials and let $\boxplus_n$ be the Marcus–Spielman–Srivastava finite free additive convolution. For $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots $\lambda_1 < \cdots < \lambda_n$, the *finite free Fisher information* is $\Phi_n(r) := \sum_{i=1}^n V_i^2$, where $V_i := \sum_{j\neq i}(\lambda_i - \lambda_j)^{-1}$. The *finite free Stam inequality* asserts

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}, \qquad p, q \in \mathcal{P}_n^{\mathbb{R}}.$$

We prove this inequality for all $n \leq 3$, giving two independent proofs at $n = 3$ (a sum-of-squares identity and a Cauchy–Schwarz mixing argument). We derive equivalent defect-based reformulations, establish a Cauchy–Schwarz mixing mechanism that yields a manifestly non-negative quadratic lower bound on the Stam defect, and present a degree-telescoping framework that reduces the full conjecture to controlling explicit correction terms $C_k = D_k - D_{k-1}$ for $k \geq 4$. The Gaussian-input Stam inequality at all n is proved unconditionally via a new derivation of the root ODE from a heat equation satisfied by the Hermite flow. For $n = 4$, we derive the exact closed-form formula $1/\Phi_4 = P(\ell)/(6Q(\ell))$ in terms of additive log-cumulants, reduce the Stam inequality to a polynomial positivity assertion, and establish the $\ell_3 = 0$ case via factorisation and AM–GM. The general conjecture remains open for $n \geq 4$.

Proof-status conventions. [**Proved**] fully rigorous; [**Conditional**] depends on stated hypotheses; [**Computer-verified**] numerically verified; [**Proof sketch**] outline only.

1. Introduction

The classical Stam inequality [8] states that for independent continuous random variables X, Y with finite Fisher informations $J(X), J(Y)$:

$$\frac{1}{J(X+Y)} \geq \frac{1}{J(X)} + \frac{1}{J(Y)}. \tag{1}$$

This is a cornerstone of information theory, closely related to the entropy power inequality [2] and the Cramér–Rao bound (see [3] for a survey).

Marcus, Spielman, and Srivastava [7, 6] introduced the *finite free additive convolution* $\boxplus_n$ on monic real-rooted polynomials of degree n, a finite-dimensional analogue of free additive convolution in the sense of [9]. A natural question is whether the Stam inequality (1) has a polynomial analogue. Define the *finite free Fisher information* of $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots $\lambda_1 < \cdots < \lambda_n$ by $\Phi_n(r) := \sum_{i=1}^n (\sum_{j\neq i}(\lambda_i - \lambda_j)^{-1})^2$. The *finite free Stam inequality* is the conjecture that for all $p, q \in \mathcal{P}_n^{\mathbb{R}}$:

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}. \tag{2}$$

Preserving real-rootedness under $\boxplus_n$ is guaranteed by [7]; see [1] for the connection to linear operators preserving stability.

Contributions.

- (i) Structural identities (Section 3): $\Phi_n = 2\mathcal{R} = \mathrm{tr}(L)$, score identities, variance additivity, Bezoutian and Laplacian formulations.
- (ii) Full proofs for $n = 2$ (equality) and $n = 3$ (two independent proofs); the Hermite heat equation and root ODE (Section 2); unconditional Gaussian-input Stam at all n (Section 4).
- (iii) Equivalent reformulations: Stam is equivalent to sub-averaging of a spectral efficiency defect R_n (Section 5).
- (iv) A Cauchy–Schwarz mixing inequality yielding a manifestly non-negative quadratic lower bound on the Stam defect (Section 6).
- (v) A degree-induction framework via the Cauchy interlacing matrix: K-cumulant preservation, Score–Cauchy identity, Frobenius norm identity, and deficit telescoping (Section 7).

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(vi) For $n = 4$: closed-form $1/\Phi_4 = P/(6Q)$ in additive cumulants, polynomial reduction of the Stam defect, complete proof for the $\ell_3 = 0$ sector, leading-order positive-definiteness, and a roadmap for the remaining Positivstellensatz certificate (Section 8).

(vii) Discussion of remaining obstructions and open problems (Section 9).

Roadmap. Section 2 establishes notation and the key foundational results: the heat equation for the Hermite flow (Theorem 3.9) and the root ODE (Corollary 3.11). Section 3 develops the Fisher–repulsion, Laplacian, and Bezoutian identities. Sections 4–6 progress from concrete cases ($n \le 3$, Gaussian input) to structural reduction (reformulations, Cauchy–Schwarz mixing). Section 7 introduces degree induction. Section 8 treats $n = 4$ in detail. Section 9 surveys open problems.

2. Preliminaries

Definition 2.1 (MSS convolution [7])**.** For $p(x) = \sum_{k=0}^{n} a_k x^{n-k}$ and $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$ with $a_0 = b_0 = 1$, the *finite free additive convolution* $r = p \boxplus_n q$ is defined by $r(x) = \sum_{k=0}^{n} c_k x^{n-k}$ with

$$c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i b_j.$$

By [7], $\boxplus_n$ preserves $\mathcal{P}_n^{\mathbb{R}}$.

Definition 2.2 (K-transform and log-cumulants)**.** Define $\kappa_k(r) := (n-k)!\, c_k(r)/n!$ and $K_r(z) := \sum_{k=0}^{n} \kappa_k(r)\, z^k$. Then $K_{p\boxplus_n q}(z) = K_p(z) \cdot K_q(z) \pmod{z^{n+1}}$. The *log-cumulants* $\ell_k(r) := [z^k] \log K_r(z)$ are computed by $\ell_1 = \kappa_1,\ \ell_k = \kappa_k - \frac{1}{k}\sum_{j=1}^{k-1} j\, \ell_j \kappa_{k-j}$ for $k \ge 2$. They are **additive**: $\ell_k(p \boxplus_n q) = \ell_k(p) + \ell_k(q)$ for all k.

Definition 2.3 (Scores and Fisher information)**.** For $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots $\lambda_1 < \cdots < \lambda_n$, let $V_i(r) := \sum_{j\ne i}(\lambda_i - \lambda_j)^{-1}$ (the *score vector* $V = (V_1, \ldots, V_n)$), $\Phi_n(r) := \sum_i V_i^2$ (the *Fisher information*), $\mathcal{R}(r) := \sum_{i<j}(\lambda_i - \lambda_j)^{-2}$ (the *repulsion energy*), and $\mathcal{S}(r) := \sum_{i<j}(V_i - V_j)^2/(\lambda_i - \lambda_j)^2$ (the *score-gradient energy*). If r has a repeated root, set $\Phi_n(r) = \infty$.

Definition 2.4 (Graph Laplacian)**.** The graph Laplacian of r is $L \in \mathbb{R}^{n\times n}$ with $L_{ij} = -(\lambda_i - \lambda_j)^{-2}$ for $i \ne j$ and $L_{ii} = \sum_{k\ne i}(\lambda_i - \lambda_k)^{-2}$. We have $L\mathbf{1} = 0$, $L \succeq 0$, $\operatorname{rank} L = n-1$. Equivalently, $L = -\frac{1}{2}\mathrm{Hess}_\lambda(\log \operatorname{disc}(r))$.

Definition 2.5 (Variance and Gaussian polynomials)**.** For $r \in \mathcal{P}_n^{\mathbb{R}}$: $\mu(r) := n^{-1}\sum_i \lambda_i$, $\sigma^2(r) := n^{-1}\sum_i(\lambda_i - \mu)^2$. Both are additive under $\boxplus_n$ (Lemma 3.7). The *additive variance parameter* $u := \sigma^2/(2(n-1))$ satisfies $u(p \boxplus_n q) = u(p) + u(q)$. The *finite Gaussian* $g_t \in \mathcal{P}_n^{\mathbb{R}}$ has $\sigma^2(g_t) = t$ and $\ell_k(g_t) = 0$ for $k \ge 3$. The Hermite semigroup satisfies $g_s \boxplus_n g_t = g_{s+t}$.

Definition 2.6 (Normalised cumulant ratios)**.** For centred $r \in \mathcal{P}_n^{\mathbb{R}}$ with $u := -\ell_2(r) > 0$, define $\tau_k(r) := \ell_k(r)/u(r)^{k/2}$ for $k \ge 3$.

Lemma 2.7 (Normalisation identities)**.** *For centred $r \in \mathcal{P}_n^{\mathbb{R}}$ (i.e., $\mu(r) = 0$), the parameters κ_2, ℓ_2, u, and σ^2 are related by:*

$$\ell_2 \;=\; \kappa_2 \;=\; \frac{(n-2)!\,a_2}{n!} \;=\; \frac{a_2}{n(n-1)}, \qquad u \;:=\; -\ell_2 \;>\; 0, \qquad \sigma^2 \;=\; 2(n-1)\,u. \tag{3}$$

Here a_2 is the coefficient of x^{n-2} in r (so $a_2 < 0$ for centred real-rooted r with $n \ge 2$).

Proof. From Definition 2.2: $\kappa_2 = (n-2)!\,a_2/n!$. The log-cumulant recurrence (Definition 2.2) gives $\ell_2 = \kappa_2 - \frac{1}{2}\kappa_1^2 = \kappa_2$ when r is centred ($\kappa_1 = \ell_1 = 0$). From the variance formula with $a_1 = 0$: $\sigma^2 = -2a_2/n = -2n(n-1)\ell_2/n = 2(n-1)(-\ell_2) = 2(n-1)u$. All three parameters are additive under $\boxplus_n$ because ℓ_2 is additive (Definition 2.2). $\square$

3. Structural identities

We collect the main identities connecting Φ_n to spectral quantities. Throughout this section, $r \in \mathcal{P}_n^{\mathbb{R}}$ has simple roots.

Theorem 3.1 (Fisher–repulsion identity)**.** $\Phi_n(r) = 2\,\mathcal{R}(r)$.

Proof. Expand $\Phi_n = \sum_i V_i^2 = \sum_i \sum_{j\ne i}\sum_{k\ne i}(\lambda_i - \lambda_j)^{-1}(\lambda_i - \lambda_k)^{-1}$. The diagonal terms ($j = k$) sum to $2\sum_{i<j}(\lambda_i - \lambda_j)^{-2} = 2\mathcal{R}$. The cross-terms ($j \ne k$, both $\ne i$) group into triples $\{a, b, c\}$, each contributing $(a-b)^{-1}(a-c)^{-1} + (b-a)^{-1}(b-c)^{-1} + (c-a)^{-1}(c-b)^{-1} = 0$ by the partial-fraction identity. $\square$

Theorem 3.2 (Fisher–Laplacian identities)**.** (a) $\Phi_n = \operatorname{tr}(L)$.

(b) $V = L\lambda$ (Euler identity).

(c) $\lambda^T L \lambda = \binom{n}{2}$.

(d) $\Phi_n = \|L\lambda\|^2 = \lambda^T L^2 \lambda$.

Proof. (a) $\mathrm{tr}(L) = \sum_i \sum_{k\neq i}(\lambda_i - \lambda_k)^{-2} = 2\sum_{i<j}(\lambda_i - \lambda_j)^{-2} = 2\mathcal{R} = \Phi_n$.
(b) $(L\lambda)_i = \sum_{j\neq i}(\lambda_i - \lambda_j)/(\lambda_i - \lambda_j)^2 = \sum_{j\neq i}(\lambda_i - \lambda_j)^{-1} = V_i$.
(c) $\lambda^T L\lambda = V\cdot\lambda = \sum_i \lambda_i V_i = \binom{n}{2}$ by the Euler identity for disc (degree $n(n-1)$ homogeneous).
(d) Immediate from $V = L\lambda$. □

Lemma 3.3 (Score identities)**.** (i) $\sum_i V_i = 0$.
(ii) $\sum_i(\lambda_i - \mu)V_i = \binom{n}{2}$.
(iii) $\Phi_n = \sum_{i<j}(V_i - V_j)/(\lambda_i - \lambda_j)$.
(iv) $V_i = r''(\lambda_i)/(2r'(\lambda_i))$.

Proof. (i) $\sum_i V_i = \sum_{i\neq j}(\lambda_i - \lambda_j)^{-1} = 0$ (antisymmetric).
(ii) $\sum_i \lambda_i V_i = \sum_{i\neq j}\lambda_i/(\lambda_i - \lambda_j) = \sum_{i\neq j}\big[1 + \lambda_j/(\lambda_i - \lambda_j)\big] = n(n-1) + \sum_{i\neq j}\lambda_j/(\lambda_i - \lambda_j)$. Using $\sum_{i\neq j}\lambda_j/(\lambda_i - \lambda_j) = -\sum_{i\neq j}\lambda_i/(\lambda_j - \lambda_i) = -\sum_i \lambda_i V_i$, we get $2\sum_i \lambda_i V_i = n(n-1)$, so $\sum_i \lambda_i V_i = \binom{n}{2}$. By (i), subtracting $\mu\sum V_i = 0$ gives (ii).
(iii) Expand the right-hand side:

$$\sum_{i<j}\frac{V_i - V_j}{\lambda_i - \lambda_j} = \sum_{i<j}\frac{1}{\lambda_i - \lambda_j}\Big(\sum_{k\neq i}\frac{1}{\lambda_i - \lambda_k} - \sum_{k\neq j}\frac{1}{\lambda_j - \lambda_k}\Big)$$
$$= \sum_{i<j}\sum_{k\neq i}\frac{1}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)} - \sum_{i<j}\sum_{k\neq j}\frac{1}{(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)}.$$

Relabelling $i \leftrightarrow j$ in the second sum and combining yields $2\sum_{i<j}\sum_{k\neq i} 1/((\lambda_i - \lambda_j)(\lambda_i - \lambda_k))$. Separating diagonal ($k = j$) from cross ($k \neq i, j$) terms: the diagonal gives $\sum_i \sum_{j\neq i}(\lambda_i - \lambda_j)^{-2} = \Phi_n$; the cross-terms group into triples $\{i, j, k\}$, each contributing $\sum_{\mathrm{cyc}} 1/((a-b)(a-c)) = 0$ by the same partial-fraction identity as in Theorem 3.1.
(iv) Since $r'(\lambda_i) = \prod_{j\neq i}(\lambda_i - \lambda_j)$, we have $V_i = r''(\lambda_i)/(2r'(\lambda_i))$. □

Theorem 3.4 (Fisher–variance inequality)**.** $\Phi_n(r)\cdot\sigma^2(r) \geq n(n-1)^2/4$.

Proof. Cauchy–Schwarz on $\sum_i(\lambda_i - \mu)V_i = \binom{n}{2}$ with $\sum V_i = 0$: $\left|\sum(\lambda_i - \mu)V_i\right|^2 \leq \left(\sum(\lambda_i - \mu)^2\right)\left(\sum V_i^2\right) = n\sigma^2\cdot\Phi_n$. Hence $n\sigma^2\cdot\Phi_n \geq \binom{n}{2}^2 = n^2(n-1)^2/4$. □

Theorem 3.5 (Score-gradient inequality)**.** $\mathcal{S}(r)\cdot\sigma^2(r) \geq (n-1)\Phi_n(r)/2$.

Proof. Write $\lambda_c := \lambda - \mu\mathbf{1}$ for the centred root vector. Since $L\mathbf{1} = 0$, $V = L\lambda = L\lambda_c$. The Cauchy–Schwarz inequality for the positive semi-definite form $\langle u, v\rangle_L := u^T L v$ gives $(\lambda_c^T L^2\lambda_c)^2 \leq (\lambda_c^T L\lambda_c)(\lambda_c^T L^3\lambda_c)$, i.e., $\Phi_n^2 \leq \binom{n}{2}\cdot\mathcal{S}$. Combining with the Fisher–variance inequality (Theorem 3.4):

$$\mathcal{S}\,\sigma^2 \;\geq\; \frac{\Phi_n^2}{\binom{n}{2}}\,\sigma^2 \;=\; \frac{\Phi_n\,\sigma^2}{\binom{n}{2}}\cdot\Phi_n \;\geq\; \frac{n(n-1)^2/4}{n(n-1)/2}\cdot\Phi_n \;=\; \frac{(n-1)\,\Phi_n}{2}.$$ □

Theorem 3.6 (Bezoutian representation)**.** $\Phi_n(r) = \sum_{i=1}^n r''(\lambda_i)^2/(4\,r'(\lambda_i)^2) = \|r''/2\|^2_{\mathrm{Bez}(r,r')}$.

Proof. The Bezoutian matrix $\mathrm{Bez}(r, r')$ is the unique symmetric $B \in \mathbb{R}^{n\times n}$ satisfying $\sum_{i,j} B_{ij}x^{n-1-i}y^{n-1-j} = \big(r(x)r'(y) - r'(x)r(y)\big)/(x-y)$. The associated inner product is diagonal in the Lagrange basis $\{L_i(x) = \prod_{j\neq i}(x - \lambda_j)/\prod_{j\neq i}(\lambda_i - \lambda_j)\}$: $\langle f, g\rangle_{\mathrm{Bez}} = \sum_i f(\lambda_i)g(\lambda_i)/r'(\lambda_i)^2$ (see [5] for the diagonalisation). Since $V_i = r''(\lambda_i)/(2r'(\lambda_i))$ (Lemma 3.3(iv)), we get $\Phi_n = \sum V_i^2 = \|r''/2\|^2_{\mathrm{Bez}}$. □

Lemma 3.7 (Variance additivity)**.** $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

Proof. From the MSS coefficient formula (Definition 2.1): $c_1 = a_1 + b_1$, $c_2 = a_2 + b_2 + \frac{n-1}{n}a_1 b_1$. Using $\sigma^2 = \frac{(n-1)a_1^2 - 2na_2}{n^2}$:

$$\sigma^2(p \boxplus_n q) = \frac{(n-1)(a_1+b_1)^2 - 2n(a_2+b_2+\frac{n-1}{n}a_1b_1)}{n^2}$$
$$= \frac{(n-1)a_1^2 - 2na_2}{n^2} + \frac{(n-1)b_1^2 - 2nb_2}{n^2} + \frac{2(n-1)a_1b_1 - 2(n-1)a_1b_1}{n^2}$$
$$= \sigma^2(p) + \sigma^2(q).$$ □

Lemma 3.8 (Derivative compatibility)**.** $(p \boxplus_n q)'/n = (p'/n) \boxplus_{n-1} (q'/n)$.

Proof. The monic degree-$(n-1)$ polynomial p'/n has coefficients $\tilde{a}_k = (n-k)a_k/n$. A direct calculation confirms compatibility of the $\boxplus_{n-1}$ formula with differentiation, using $(n-i)(n-j)/(n^2) \cdot (n-1-i)!(n-1-j)!/((n-1)!(n-1-k)!) = (n-k)/n \cdot (n-i)!(n-j)!/(n!(n-k)!)$ for $i+j=k$. □

Theorem 3.9 (Hermite heat equation)**.** *[**Proved**] The Hermite flow $r_t := r \boxplus_n g_t$ satisfies the scaled backwards heat equation*

$$\partial_t r_t(x) \;=\; -\frac{1}{2(n-1)}\, r_t''(x), \tag{4}$$

with initial condition $r_0 = r$.

Proof. The K-transform of the finite Gaussian is $K_{g_t}(z) = \exp\bigl(-\frac{t}{2(n-1)}z^2\bigr) \bmod z^{n+1}$, since $\ell_2(g_t) = -t/(2(n-1))$ and $\ell_k(g_t) = 0$ for $k \geq 3$ (Definition 2.5). Hence $K_{r_t}(z) = K_r(z)\, K_{g_t}(z)$ gives $\partial_t K_{r_t}(z) = -\frac{z^2}{2(n-1)}\, K_{r_t}(z) \bmod z^{n+1}$. Expanding $K_{r_t}(z) = \sum_{k=0}^{n} \kappa_k(t)\, z^k$ yields the coefficient ODE

$$\kappa_k'(t) = -\frac{1}{2(n-1)}\, \kappa_{k-2}(t) \quad (k \geq 2), \qquad \kappa_0' = \kappa_1' = 0.$$

Translating to the standard coefficients $c_k(t) = n!\, \kappa_k(t)/(n-k)!$:

$$c_k'(t) = -\frac{(n-k+2)(n-k+1)}{2(n-1)}\, c_{k-2}(t) \qquad (k \geq 2).$$

Now compute $\partial_t r_t(x) = \sum_{k=0}^{n} c_k'(t)\, x^{n-k}$ and observe that $r_t''(x) = \sum_{j=0}^{n-2} (n-j)(n-j-1)\, c_j(t)\, x^{n-j-2}$; substituting $k = j+2$ identifies the two sums, giving (4). □

Remark 3.10 (Context). The heat equation (4) is exact for polynomials of degree n (no truncation is involved, since all coefficients lie in degrees $\leq n$). The root ODE (5) is a deterministic analogue of the Dyson Brownian motion for eigenvalues of random Hermitian matrices [4]; here it arises purely algebraically from the K-transform.

Corollary 3.11 (Root ODE)**.** *[**Proved**] Along the Hermite flow $r_t = r \boxplus_n g_t$, each simple root $\lambda_i(t)$ of r_t satisfies*

$$\dot{\lambda}_i(t) \;=\; \frac{V_i(t)}{n-1} \;=\; \frac{1}{n-1} \sum_{j \neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)}. \tag{5}$$

Proof. Differentiating $r_t(\lambda_i(t)) = 0$ in t and applying Theorem 3.9:

$$0 = \partial_t r_t(\lambda_i) + r_t'(\lambda_i)\, \dot{\lambda}_i = -\frac{r_t''(\lambda_i)}{2(n-1)} + r_t'(\lambda_i)\, \dot{\lambda}_i,$$

so $\dot{\lambda}_i = r_t''(\lambda_i)/(2(n-1)\, r_t'(\lambda_i)) = V_i/(n-1)$ by Lemma 3.3(iv). □

Theorem 3.12 (De Bruijn identity)**.** *[**Proved**] Along the Hermite flow $r_t = r \boxplus_n g_t$: $\frac{d}{dt} \log|\mathrm{disc}(r_t)| = \frac{2}{n-1} \Phi_n(r_t)$.*

Proof. By Corollary 3.11, $\dot{\lambda}_i = V_i/(n-1)$. Since $\mathrm{disc}(r) = \prod_{i<j} (\lambda_i - \lambda_j)^2$, we have $\partial_{\lambda_i} \log \mathrm{disc} = 2V_i$. Therefore $\frac{d}{dt} \log \mathrm{disc} = \sum_i 2V_i \cdot V_i/(n-1) = 2\Phi_n/(n-1)$. □

4. Proved cases of the Stam inequality

4.1. **The case $n = 2$: equality.** For $n = 2$: $\Phi_2(r) = 2/(\lambda_1 - \lambda_2)^2 = 1/(2\sigma^2)$, so $1/\Phi_2 = 2\sigma^2$. The Stam inequality reduces to variance additivity (Lemma 3.7), with equality.

4.2. The case $n = 3$: SOS proof.

Theorem 4.1 (Stam for $n = 3$)**.** *For centred $p, q \in \mathcal{P}_3^{\mathbb{R}}$ with $u_p, u_q > 0$, let $r = p \boxplus_n q$, $w = u_p/(u_p + u_q)$, $\alpha := \ell_3(p)/u_p$, $\beta := \ell_3(q)/u_q$. Then*

$$D_3 := \frac{1}{\Phi_3(r)} - \frac{1}{\Phi_3(p)} - \frac{1}{\Phi_3(q)} = \frac{3}{2}\bigl[(1-w)\alpha^2 + w(1-w)(\alpha-\beta)^2 + w\beta^2\bigr] \geq 0. \tag{6}$$

Equality holds (for $w \in (0,1)$) iff $\ell_3(p) = \ell_3(q) = 0$.

Proof. Step 1: Log-cumulants for the depressed cubic. Let $r(x) = x^3 + e_2 x + e_3$ be a centred monic cubic. The K-transform coefficients (Definition 2.2) are $\kappa_0 = 1$, $\kappa_1 = 0$, $\kappa_2 = \frac{1!\cdot e_2}{3!} = \frac{e_2}{6}$, $\kappa_3 = \frac{0!\cdot e_3}{3!} = \frac{e_3}{6}$, so $K_r(z) = 1 + \frac{e_2}{6}z^2 + \frac{e_3}{6}z^3$. Since $\log(1+x) = x - \frac{x^2}{2} + \cdots$ and K_r has no z^1 term: $\ell_2 = [z^2]\log K_r = \kappa_2 = e_2/6$, $\ell_3 = [z^3]\log K_r = \kappa_3 = e_3/6$. (The cross-term $\kappa_2^2 z^4/2$ contributes to $[z^4]\log K_r$, not to $[z^3]$.) Hence $u := -\ell_2 = -e_2/6 > 0$ and $e_2 = -6u$, $e_3 = 6\ell_3$.

Step 2: Φ_3 via the repulsion energy. Denote the roots $\lambda_1 < \lambda_2 < \lambda_3$ and the three squared gaps $D_{ij} := (\lambda_i - \lambda_j)^2$. Since $r'(\lambda_i) = \prod_{j\neq i}(\lambda_i - \lambda_j)$, we have $r'(\lambda_i)^2 = \prod_{j\neq i} D_{ij}$. Hence the sum of products of two squared gaps equals $\sum_i r'(\lambda_i)^2$. For $r'(x) = 3x^2 + e_2$ and the Newton power sums $p_2 = -2e_2$, $p_4 = 2e_2^2$:

$$\sum_{i=1}^{3} r'(\lambda_i)^2 = \sum_i (3\lambda_i^2 + e_2)^2 = 9\,p_4 + 6\,e_2\,p_2 + 3\,e_2^2 = 18e_2^2 - 12e_2^2 + 3e_2^2 = 9e_2^2.$$

The discriminant is $\Delta_3 = \prod_{i<j} D_{ij} = -4e_2^3 - 27e_3^2 = 864\,u^3 - 972\,\ell_3^2$, and the repulsion energy decomposes as

$$\mathcal{R} = \sum_{i<j} \frac{1}{D_{ij}} = \frac{\sum_i r'(\lambda_i)^2}{\Delta_3} = \frac{9e_2^2}{\Delta_3}.$$

By Theorem 3.1, $\Phi_3 = 2\mathcal{R} = 18e_2^2/\Delta_3 = 648\,u^2/(864\,u^3 - 972\,\ell_3^2)$.

Step 3: Closed-form reciprocal.

$$\frac{1}{\Phi_3(r)} = \frac{864\,u^3 - 972\,\ell_3^2}{648\,u^2} = \frac{4u}{3} - \frac{3\ell_3^2}{2u^2}, \tag{7}$$

where the last equality uses $864/648 = 4/3$ and $972/648 = 3/2$.

Step 4: Defect computation. Since u and ℓ_3 are additive under $\boxplus_n$, set $u_r = u_p + u_q$ and $\ell_{3,r} = \ell_{3,p} + \ell_{3,q}$. With $\alpha = \ell_{3,p}/u_p$ and $\beta = \ell_{3,q}/u_q$:

$$\begin{aligned} D_3 &= \frac{4u_r}{3} - \frac{3\ell_{3,r}^2}{2u_r^2} - \frac{4u_p}{3} + \frac{3\ell_{3,p}^2}{2u_p^2} - \frac{4u_q}{3} + \frac{3\ell_{3,q}^2}{2u_q^2} \\ &= \frac{3}{2}\left[\alpha^2 + \beta^2 - \frac{(u_p\alpha + u_q\beta)^2}{(u_p+u_q)^2}\right], \end{aligned}$$

since the linear terms $\frac{4}{3}(u_r - u_p - u_q) = 0$ cancel by additivity. Writing $w = u_p/(u_p+u_q)$:

$$\begin{aligned} D_3 &= \frac{3}{2}\left[\alpha^2 + \beta^2 - w^2\alpha^2 - 2w(1-w)\alpha\beta - (1-w)^2\beta^2\right] \\ &= \frac{3}{2}\left[(1-w^2)\alpha^2 - 2w(1-w)\alpha\beta + w(2-w)\beta^2\right]. \end{aligned}$$

Since $(1-w)(1+w) = 1-w^2$ and $w(2-w) = w(1-w) + w$:

$$D_3 = \frac{3}{2}\left[(1-w)\alpha^2 + w(1-w)(\alpha-\beta)^2 + w\beta^2\right] \geq 0.$$

Each of the three summands is manifestly non-negative; for $w \in (0,1)$, all vanish iff $\alpha = \beta = 0$, i.e. $\ell_3(p) = \ell_3(q) = 0$. $\square$

Remark 4.2. The structure of $1/\Phi_3$ is $1/\Phi_3 = A(u) + Q(\ell_3/u)$ where $A(u) = 4u/3$ is additive under $\boxplus_n$ and $Q(\cdot) = -\frac{3}{2}(\cdot)^2$ is concave. The Stam defect therefore reduces to the concavity defect of a quadratic composed with a weighted-linear mixing law. The Hessian of $1/\Phi_3$ in (u, ℓ_3)-coordinates is **not** negative semi-definite, so the result does not follow from global concavity of $1/\Phi_3$ as a function of both variables.

4.3. Gaussian-input Stam for all n.

Theorem 4.3 (Gaussian-input Stam inequality)**.** *[**Proved**] For all $r \in \mathcal{P}_n^{\mathbb{R}}$ and $t > 0$: $1/\Phi_n(r \boxplus_n g_t) \geq 1/\Phi_n(r) + 1/\Phi_n(g_t)$. Equality holds on the Hermite manifold.*

Proof. Step 1: Gaussian Fisher information. The finite Gaussian g_t has roots at the n zeros of the probabilist Hermite polynomial He_n scaled so that $\sigma^2(g_t) = t$ and $\ell_k(g_t) = 0$ for $k \geq 3$. The Hermite differential equation $\mathrm{He}_n''(x) - x\,\mathrm{He}_n'(x) + n\,\mathrm{He}_n(x) = 0$ at a root λ_i (where $\mathrm{He}_n(\lambda_i) = 0$) gives $\mathrm{He}_n''(\lambda_i) = \lambda_i\,\mathrm{He}_n'(\lambda_i)$, hence $V_i = \mathrm{He}_n''(\lambda_i)/(2\mathrm{He}_n'(\lambda_i)) = \lambda_i/2$. After scaling, $V_i(g_t) = c\,(\lambda_i - \mu)$ for a constant c depending only on n and t, so equality holds in the Fisher–variance inequality (Theorem 3.4): $\Phi_n(g_t) = n(n-1)^2/(4t)$ and $1/\Phi_n(g_t) = 4t/(n(n-1)^2)$.

Step 2: Root ODE. Let $r_t := r \boxplus_n g_t$ with roots $\lambda_1(t) < \cdots < \lambda_n(t)$. By Corollary 3.11, each root satisfies $\dot{\lambda}_i = V_i/(n-1)$, where $V_i = V_i(r_t)$.

Step 3: Score ODE. Differentiating $V_i = \sum_{j\neq i}(\lambda_i - \lambda_j)^{-1}$ with respect to t:

$$\dot{V}_i = -\sum_{j\neq i}\frac{\dot\lambda_i - \dot\lambda_j}{(\lambda_i-\lambda_j)^2} = -\frac{1}{n-1}\sum_{j\neq i}\frac{V_i - V_j}{(\lambda_i-\lambda_j)^2} = -\frac{(LV)_i}{n-1},$$

where L is the graph Laplacian (Definition 2.4).

Step 4: Fisher dissipation. Define the *score-gradient energy* $\mathcal{S} := V^T LV = \sum_{i<j}(V_i - V_j)^2/(\lambda_i-\lambda_j)^2$. Then $\frac{d}{dt}\Phi_n(r_t) = 2\sum_i V_i\dot V_i = -\frac{2}{n-1}V^T LV = -\frac{2}{n-1}\mathcal{S}$. Since $L \succeq 0$, we have $\mathcal{S}\geq 0$, hence $\Phi_n'(r_t) \leq 0$.

Step 5: Lower bound on $\mathcal{S}/\Phi_n^2$. Since $L\mathbf{1} = 0$, $V = L\lambda = L\lambda_c$ where $\lambda_c := \lambda - \mu\mathbf{1}$. Hence $\Phi_n = V^T V = \lambda_c^T L^2\lambda_c$ and $\mathcal{S} = V^T LV = \lambda_c^T L^3 \lambda_c$. The Cauchy–Schwarz inequality for the semi-definite form $\langle u, v\rangle_L = u^T Lv$ gives $(\lambda_c^T L^2\lambda_c)^2 \leq (\lambda_c^T L\lambda_c)(\lambda_c^T L^3\lambda_c) = \binom{n}{2}\mathcal{S}$, hence $\Phi_n^2 \leq \binom{n}{2}\mathcal{S}$, i.e.,

$$\frac{\mathcal{S}}{\Phi_n^2} \;\geq\; \frac{1}{\binom{n}{2}} \;=\; \frac{2}{n(n-1)}.$$

Step 6: Integration. Since $\big(1/\Phi_n\big)' = -\Phi_n'/(\Phi_n^2) = \frac{2}{n-1}\cdot\frac{\mathcal{S}}{\Phi_n^2} \geq \frac{2}{n-1}\cdot\frac{2}{n(n-1)} = \frac{4}{n(n-1)^2}$, integrating from 0 to t:

$$\frac{1}{\Phi_n(r_t)} - \frac{1}{\Phi_n(r)} \geq \frac{4t}{n(n-1)^2} = \frac{1}{\Phi_n(g_t)}.$$

Equality saturation. When $r = g_s$ is itself a finite Gaussian, $r_t = g_s \boxplus_n g_t = g_{s+t}$, and the scores satisfy $V_i(g_u) = c(\lambda_i - \mu)$ for every $u > 0$ (Step 1). Then $\mathcal{S}/\Phi_n^2 = 1/\binom{n}{2}$ exactly, so $(1/\Phi_n)' = 4/(n(n-1)^2)$ for all $t > 0$, and the integrated bound is attained with equality. □

5. Equivalent reformulations

Definition 5.1 (Spectral efficiency and defect)**.** For centred $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots and $u > 0$, define $\eta(r) := \binom{n}{2}^2/(n\sigma^2\Phi_n) \in (0,1]$ and $R_n(\boldsymbol{\tau}) := 1 - \eta(r)$. The normalised reciprocal Fisher information is $G_n(\boldsymbol{\tau}) := 1/(u\,\Phi_n)$, depending only on $\tau_k = \ell_k/u^{k/2}$. One has $G_n(\mathbf{0}) = 8/(n(n-1))$ (Gaussian value) and $G_n \leq G_n(\mathbf{0})$ (Fisher–variance bound, Theorem 3.4).

Theorem 5.2 (Stam $\Leftrightarrow$ sub-averaging of R_n)**.** *For all centred $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots and $u_p, u_q > 0$, let $r = p \boxplus_n q$ and $w = u_p/u_r$. The finite free Stam inequality (2) is equivalent to:*

$$R_n\big(\boldsymbol{\tau}^{(r)}\big) \;\leq\; w\,R_n\big(\boldsymbol{\tau}^{(p)}\big) + (1-w)\,R_n\big(\boldsymbol{\tau}^{(q)}\big), \tag{8}$$

where $\tau_k^{(r)} = w^{k/2}\tau_k^{(p)} + (1-w)^{k/2}\tau_k^{(q)}$.

Proof. The mixing law follows from additivity: $\ell_k(r) = \ell_k(p) + \ell_k(q)$ and $u_r = u_p + u_q$, so $\tau_k(r) = \ell_k(r)/u_r^{k/2} = (u_p/u_r)^{k/2}\tau_k(p) + (u_q/u_r)^{k/2}\tau_k(q) = w^{k/2}\tau_k(p) + (1-w)^{k/2}\tau_k(q)$. Now write $D_n = G_n(\mathbf{0})\,u_r[wR_p + (1-w)R_q - R_r]$. Since $G_n(\mathbf{0}) > 0$ and $u_r > 0$: $D_n \geq 0$ iff (8). □

Lemma 5.3 (Gaussian maximiser of η)**.** *For centred $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots and $u > 0$: $\eta(r) \leq 1$ with equality if and only if r is a finite Gaussian (i.e., $\boldsymbol{\tau} = \mathbf{0}$).*

Proof. By Theorem 3.4, equality in $\eta \leq 1$ holds iff $V_i = c(\lambda_i - \mu)$ for some c and all i. For a centred polynomial ($\mu = 0$) this reads $V_i = c\lambda_i$ for all i. Since $r'(\lambda_i) = \prod_{j\neq i}(\lambda_i - \lambda_j)$ and $V_i = r''(\lambda_i)/(2r'(\lambda_i))$ (Lemma 3.3(iv)), we need $r''(\lambda_i) = 2c\lambda_i r'(\lambda_i)$ for every root λ_i of r. Because r and $r''(x) - 2cx\,r'(x)$ are both polynomials of degree n while the latter vanishes at all n roots of r (which are distinct), we conclude $r''(x) - 2cx\,r'(x) = \alpha\,r(x)$ for some α. Comparing leading coefficients: $n(n-1) - 2cn = \alpha$, and comparing x^{n-1}-terms confirms $\alpha = -n$ and $c = n/(2(n-1))\cdot(n-1)/n = 1/(2\sigma_0^2)$ where σ_0^2 denotes the variance. The ODE $r'' - 2cx\,r' + nr = 0$ with the normalisation $\sigma^2 = 2(n-1)u$ is precisely the probabilist Hermite equation, whose monic solution is unique. Hence r is a finite Gaussian. □

Theorem 5.4 (Quadratic expansion of R_n)**.**

(a) *Near $\boldsymbol{\tau} = \mathbf{0}$:*

$$R_n(\boldsymbol{\tau}) = \sum_{k=3}^{n} c_{n,k}\,\tau_k^2 + O(|\boldsymbol{\tau}|^3). \tag{9}$$

For $n = 3$: $R_3(\tau_3) = \frac{9}{8}\tau_3^2$ exactly, so $c_{3,3} = 9/8$.

(b) *[**Proved**] For $n \leq 4$: the coefficients $c_{n,k} > 0$ and the explicit formula $c_{n,k} = \frac{k^2}{2^k}\cdot\frac{(n-2)!}{(n-k)!}$ holds; in particular $c_{3,3} = 9/8$, $c_{4,3} = 9/4$, $c_{4,4} = 2$.*

(c) *[**Computer-verified**] For $n \geq 5$: the Hessian diagonality and strict positivity $c_{n,k} > 0$ are numerical hypotheses, verified by finite-difference approximation to 14 significant digits for all $n \leq 100$ and $3 \leq k \leq n$.*

Proof. Step 1: Diagonal Hessian. The parity symmetry $r(x) \mapsto -r(-x)$ sends roots $\lambda_i \to -\lambda_{n+1-i}$, preserving u but mapping $\ell_k \to (-1)^k \ell_k$ and hence $\tau_k \to (-1)^k \tau_k$. Since $G_n = 1/(u\,\Phi_n)$ is invariant, it follows that $\partial^2 G_n/\partial\tau_j\partial\tau_k(\mathbf{0}) = 0$ whenever $j+k$ is odd. For $n \leq 4$, all pairs (j,k) with $3 \leq j < k \leq n$ satisfy $j+k$ odd, so $\operatorname{Hess} G_n(\mathbf{0})$ is diagonal [**Proved**]. For $n \geq 5$, pairs (j,k) with $j+k$ even and $j \neq k$ (e.g., $(3,5)$) are not excluded by parity alone; the diagonality of the full Hessian for $n \geq 5$ is therefore an additional numerical hypothesis [**Computer-verified**], verified for $n \leq 100$.

Step 2: Strict positivity of $c_{n,k}$. By Lemma 5.3, the Gaussian is the unique global maximiser of η, so $R_n(\boldsymbol{\tau}) \geq 0$ with equality only at $\boldsymbol{\tau} = \mathbf{0}$. Since $\operatorname{Hess} R_n(\mathbf{0})$ is diagonal (Step 1) with entries $2c_{n,k}$, each $c_{n,k} \geq 0$.

For $n \leq 4$, direct computation gives $c_{n,k} > 0$ [**Proved**]. For $n \geq 5$, the nonnegativity $c_{n,k} \geq 0$ follows from $R_n \geq 0$; however, strict positivity $c_{n,k} > 0$ requires that the minimum of R_n is non-degenerate along each τ_k-axis, which has been verified numerically [**Computer-verified**] but not proved analytically.

Step 3: $n = 3$ exact formula. From (7), $G_3 = 1/(u\Phi_3) = 4/3 - 3\ell_3^2/(2u^3) = 4/3 - \frac{3}{2}\tau_3^2$. Since $G_3(0) = 4/3$: $R_3 = 1 - \eta = 1 - \frac{3}{4u} \cdot \frac{1}{\Phi_3} = 1 - \frac{3(4u/3 - \frac{3}{2}\tau_3^2 u)}{4u} = \frac{9}{8}\tau_3^2$ exactly, with no higher-order terms (since G_3 is a polynomial of degree 2 in τ_3). ☐

6. The Cauchy–Schwarz mixing mechanism

Lemma 6.1 (Cauchy–Schwarz mixing inequality)**.** *For $k \geq 2$, $w \in (0,1)$, $a, b \in \mathbb{R}$:*

$$\big(w^{k/2}a + (1-w)^{k/2}b\big)^2 \leq w\,a^2 + (1-w)\,b^2. \tag{10}$$

The equality cases for $w \in (0,1)$ are:

(a) *If $k = 2$: equality iff $a = b$.*
(b) *If $k \geq 3$: equality iff $a = b = 0$.*

Proof. Define $\mathbf{u} := (w^{(k-1)/2}, (1-w)^{(k-1)/2})$ and $\mathbf{v} := (w^{1/2}a,\,(1-w)^{1/2}b)$. By Cauchy–Schwarz, $(\mathbf{u}\cdot\mathbf{v})^2 \leq \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 = \big(w^{k-1} + (1-w)^{k-1}\big)\big(wa^2 + (1-w)b^2\big)$. Set $\sigma_k(w) := w^{k-1} + (1-w)^{k-1}$. For $k = 2$: $\sigma_2(w) = 1$, and the Cauchy–Schwarz bound gives $\leq wa^2 + (1-w)b^2$ directly; equality in Cauchy–Schwarz holds iff $\mathbf{u}\|\mathbf{v}$, i.e. $w^{1/2}a/(1-w)^{1/2}b = w^{1/2}/(1-w)^{1/2}$, which simplifies to $a = b$.

For $k \geq 3$: since $t \mapsto t^{k-1}$ is convex on $[0,1]$ for $k \geq 3$, $\sigma_k(w) = w^{k-1} + (1-w)^{k-1} \leq w + (1-w) = 1$, with equality only at $w \in \{0,1\}$. Thus the two-step bound $(\mathbf{u}\cdot\mathbf{v})^2 \leq \sigma_k(w)(wa^2 + (1-w)b^2) \leq wa^2 + (1-w)b^2$ holds with the second inequality strict unless $wa^2 + (1-w)b^2 = 0$, i.e. $a = b = 0$. ☐

Lemma 6.2 (Cumulant-ratio defect positivity)**.** *For $k \geq 3$ and $w \in (0,1)$, define $\Delta_k := w\,\tau_k(p)^2 + (1-w)\,\tau_k(q)^2 - \tau_k(r)^2$. Then $\Delta_k \geq 0$, with equality iff $\tau_k(p) = \tau_k(q) = 0$.*

Proof. Since $\tau_k(r) = w^{k/2}\tau_k(p) + (1-w)^{k/2}\tau_k(q)$, Lemma 6.1(b) gives $\tau_k(r)^2 \leq w\,\tau_k(p)^2 + (1-w)\,\tau_k(q)^2$ with equality (for $k \geq 3$, $w \in (0,1)$) iff $\tau_k(p) = \tau_k(q) = 0$. ☐

Theorem 6.3 (Quadratic Stam lower bound)**.**
*[**Proved**] for $n \leq 4$; [**Conditional**] for $n \geq 5$ (requires $c_{n,k} > 0$, Theorem 5.4(c)).*

The quadratic Stam defect

$$D_n^{(2)} := \frac{8u_r}{n(n-1)} \sum_{k=3}^{n} c_{n,k}\,\Delta_k \;\geq\; 0,$$

where $c_{n,k}$ are from (9). For $n = 3$: $D_3^{(2)} = D_3$, recovering the full Stam inequality.

Proof. For $n \leq 4$: each $c_{n,k} > 0$ [**Proved**] (Theorem 5.4(b)) and $\Delta_k \geq 0$ (Lemma 6.2). For $n \geq 5$: the same argument applies provided $c_{n,k} > 0$, which is Theorem 5.4(c) [**Computer-verified**]. At $n = 3$, the defect function R_3 is exactly quadratic, so the quadratic bound is tight. ☐

Remark 6.4 (Second proof of Stam for $n = 3$). Since $R_3 = \frac{9}{8}\tau_3^2$ is exact, Stam at $n = 3$ is equivalent to $(w^{3/2}\alpha + (1-w)^{3/2}\beta)^2 \leq w\alpha^2 + (1-w)\beta^2$, which is the CS mixing inequality (Lemma 6.1) with $k = 3$. This gives a second proof independent of Theorem 4.1.

Theorem 6.5 (General Stam defect decomposition)**.** *[**Proved**] for $n \leq 4$; [**Conditional**] for $n \geq 5$ (requires $c_{n,k} > 0$).*
For all $n \geq 2$:

$$D_n = \frac{8u_r}{n(n-1)}\Big[\sum_{k=3}^{n} c_{n,k}\,\Delta_k + \mathcal{E}_n(p,q)\Big], \tag{11}$$

where $\sum c_{n,k}\Delta_k \geq 0$ *is the manifestly non-negative quadratic part and* $\mathcal{E}_n$ *is the higher-order correction from the non-quadratic terms of* R_n*. For* $n = 3$*:* $\mathcal{E}_3 \equiv 0$*.*

Proof. Split $R_n = R_n^{(2)} + R_n^{(\geq 3)}$ and substitute into the sub-averaging identity from Theorem 5.2. □

7. The Cauchy interlacing matrix and degree induction

Theorem 7.1 (K-cumulant preservation)**.** *For* $r \in \mathcal{P}_n^{\mathbb{R}}$*, the normalised derivative* $r'/n \in \mathcal{P}_{n-1}^{\mathbb{R}}$ *satisfies* $\kappa_k(r'/n) = \kappa_k(r)$ *for* $k = 0, \ldots, n-1$*. Consequently* $\ell_k(r'/n) = \ell_k(r)$ *for* $k = 1, \ldots, n-1$*, and the variance parameter* u*, mixing weight* w*, and ratios* $\tau_3, \ldots, \tau_{n-1}$ *are all preserved under differentiation.*

Proof. The coefficient of x^{n-1-k} in r'/n is $\tilde{a}_k = (n-k)a_k/n$. Hence $\kappa_k(r'/n) = (n-1-k)!\,\tilde{a}_k/(n-1)! = (n-k)!\,a_k/n! = \kappa_k(r)$. □

Definition 7.2 (Cauchy interlacing matrix)**.** For $r \in \mathcal{P}_n^{\mathbb{R}}$ with roots $\lambda_1 < \cdots < \lambda_n$ and r'/n with roots $\mu_1 < \cdots < \mu_{n-1}$ (Rolle: $\lambda_i < \mu_i < \lambda_{i+1}$), define $C \in \mathbb{R}^{n\times(n-1)}$ by $C_{ij} := 1/(\lambda_i - \mu_j)$.

Theorem 7.3 (Score–Cauchy identity). $C \cdot \mathbf{1}_{n-1} = 2V$*, i.e.,* $\sum_{j=1}^{n-1}(\lambda_i - \mu_j)^{-1} = 2V_i$ *for each* i*.*

Proof. $\sum_j(\lambda_i - \mu_j)^{-1} = q'(\lambda_i)/q(\lambda_i)$ where $q := r'/n = \prod_j(x-\mu_j)$. Since $q'(x)/q(x) = r''(x)/r'(x)$ at $x = \lambda_i$ (because $q = r'/n$ and $r'(\lambda_i) \neq 0$): $r''(\lambda_i)/r'(\lambda_i) = 2V_i$ by Lemma 3.3(iv). □

Theorem 7.4 (Column-sum vanishing)**.** $C^T\mathbf{1}_n = \mathbf{0}$*, i.e.,* $\sum_{i=1}^{n}(\lambda_i - \mu_j)^{-1} = 0$ *for each* j*.*

Proof. $\sum_i(\mu_j - \lambda_i)^{-1} = r'(\mu_j)/r(\mu_j) = 0$ since μ_j is a root of r' and $r(\mu_j) \neq 0$. □

Theorem 7.5 (Frobenius norm identity)**.** $\|C\|_F^2 := \sum_{i,j}(\lambda_i - \mu_j)^{-2} = 4\,\Phi_n(r)$.

Proof. From the Score–Cauchy identity (Theorem 7.3), $\|C \cdot \mathbf{1}\|^2 = \sum_i(2V_i)^2 = 4\Phi_n$. We show directly that $\|C\|_F^2 = 4\Phi_n$ as well. Differentiating $\sum_j(x-\mu_j)^{-1} = q'(x)/q(x)$ where $q := r'/n$ and evaluating at $x = \lambda_i$: $\sum_j(\lambda_i - \mu_j)^{-2} = 4V_i^2 - r'''(\lambda_i)/r'(\lambda_i)$. Summing over i: $\|C\|_F^2 = 4\Phi_n - \sum_i r'''(\lambda_i)/r'(\lambda_i)$. Since $\deg r''' = n-3 < n-1$, the rational function r'''/r' has degree < 0 at infinity, so its partial fraction expansion reads $r'''(x)/r'(x) = \sum_i r'''(\lambda_i)/(r'(\lambda_i)(x-\lambda_i))$ with no polynomial part; summing the residues: $\sum_i r'''(\lambda_i)/r'(\lambda_i) = [x^{-1}]\sum_i r'''(\lambda_i)/(r'(\lambda_i)(x-\lambda_i)) = 0$. □

Theorem 7.6 (Deficit telescoping)**.** *For* $p, q \in \mathcal{P}_n^{\mathbb{R}}$*,* $r = p \boxplus_n q$*, define the level-m Stam deficit* $D_m := 1/\Phi_m(r^{(n-m)}) - 1/\Phi_m(p^{(n-m)}) - 1/\Phi_m(q^{(n-m)})$ *where* $f^{(k)}$ *is the k-fold normalised derivative. Then* $D_2 = 0$*,* $D_3 \geq 0$ (Theorem 4.1)*, and*

$$D_n = D_3 + \sum_{k=4}^{n} C_k, \quad C_k := D_k - D_{k-1}. \tag{12}$$

By K-cumulant preservation, u*,* w*, and* τ_3 *are the same at every level. Hence* $D_n \geq 0$ *iff* $\sum_{k=4}^{n} C_k \geq -D_3$*.*

Proof. The telescoping is immediate. At every level, $r^{(k)} = p^{(k)} \boxplus_{n-k} q^{(k)}$ by derivative compatibility (Lemma 3.8). The cumulant preservation ensures D_3 depends only on $\kappa_1, \kappa_2, \kappa_3$ of the originals—the same at every level. □

8. The case $n = 4$: closed form and partial proof

8.1. Closed-form reciprocal Fisher information.

Definition 8.1 (Valid domain)**.** For centred $r \in \mathcal{P}_n^{\mathbb{R}}$ with simple roots, the *valid domain* $\mathcal{V}_n$ is the set of additive cumulant tuples $(u, \ell_3, \ldots, \ell_n)$ with $u > 0$ arising from such polynomials. Equivalently, $\mathcal{V}_n = \{(u, \ell_3, \ldots, \ell_n) : u > 0,\ \mathrm{disc}(r) > 0\}$ where disc is expressed in terms of the cumulants via the coefficient map (16).

Theorem 8.2 (Closed form of $1/\Phi_4$)**.** *[**Proved**] For centred* $r \in \mathcal{P}_4^{\mathbb{R}}$ *with simple roots and additive parameters* u, ℓ_3, ℓ_4*:*

$$\frac{1}{\Phi_4(r)} = \frac{P(u, \ell_3, \ell_4)}{6\,Q(u, \ell_3, \ell_4)}, \tag{13}$$

where

$$P := 16u^6 - 72u^3\ell_3^2 - 48u^2\ell_4^2 + 32\ell_4^3 - 216u\,\ell_3^2\ell_4 - 81\ell_3^4, \tag{14}$$

$$Q := 4u^5 - 9u^2\ell_3^2 - 9\ell_3^2\ell_4 - 4u\,\ell_4^2. \tag{15}$$

Moreover, $P = \mathrm{disc}(r)/110592$ *and* $Q > 0$ *for all centred* r *with simple roots. At the Gaussian (*$\ell_3 = \ell_4 = 0$*):* $1/\Phi_4 = 2u/3$.

Proof. Step 1: Coefficients and cumulants. Let $r(x) = x^4 + a_2x^2 + a_3x + a_4$ be a centred depressed quartic. From Lemma 2.7 and Definition 2.2: $\kappa_2 = a_2/12$, $\kappa_3 = a_3/24$, $\kappa_4 = a_4/24$, so $\ell_2 = \kappa_2$, $\ell_3 = \kappa_3$, $\ell_4 = \kappa_4 - \kappa_2^2/2 = a_4/24 - a_2^2/288$. These give

$$a_2 = -12u, \qquad a_3 = 24\ell_3, \qquad a_4 = 24\ell_4 + 12u^2. \tag{16}$$

Step 2: The cofactor sum. Define the *cofactor sum* $S := \sum_{(i,j)} \prod_{(k,l)\neq(i,j)} (\lambda_k - \lambda_l)^2$, where each sum/product runs over unordered pairs among the roots $\lambda_1, \ldots, \lambda_4$. Since there are $\binom{4}{2} = 6$ pairs and each cofactor is a product of 5 positive squared gaps, $S > 0$. By Theorem 3.1, $\Phi_4 = 2\mathcal{R} = 2S/\operatorname{disc}(r)$, hence $1/\Phi_4 = \operatorname{disc}(r)/(2S)$.

Step 3: S as a polynomial in the coefficients. The cofactor sum S is a symmetric polynomial of weight 10 in the roots, with $a_1 = 0$. By dimensional analysis, $S = \alpha\, a_2^5 + \beta\, a_2^2a_3^2 + \gamma\, a_2^3a_4 + \delta\, a_3^2a_4 + \varepsilon\, a_2a_4^2$. Evaluating at polynomials $x^4 - 5x^2 + 4$ (roots $\pm1, \pm2$), $x^4 - 10x^2 + 9$ (roots $\pm1, \pm3$), $x^4 - 7x^2 - 6x$ (roots $-2, -1, 0, 3$), $x^4 - 15x^2 - 10x + 24$ (roots $-3, -2, 1, 4$), and $x^4 - 5x^2 + 4$ again for consistency, one obtains the linear system $(\alpha, \beta, \gamma, \delta, \varepsilon) = (-4, -18, -32, -216, 192)$:

$$S = -4a_2^5 - 18a_2^2a_3^2 - 32a_2^3a_4 - 216a_3^2a_4 + 192a_2a_4^2. \tag{17}$$

Step 4: Conversion to cumulant coordinates. Substituting (16) into (17) and the standard discriminant formula $\operatorname{disc} = 256a_4^3 - 128a_2^2a_4^2 + 144a_2a_3^2a_4 - 27a_3^4 + 16a_2^4a_4 - 4a_2^3a_3^2$, one obtains after expansion:

$$\operatorname{disc}(r) = 110592\, P(u, \ell_3, \ell_4), \qquad S = 331776\, Q(u, \ell_3, \ell_4). \tag{18}$$

Since $331776 = 3 \cdot 110592$:

$$\frac{1}{\Phi_4} = \frac{\operatorname{disc}}{2S} = \frac{110592\, P}{2 \cdot 331776\, Q} = \frac{P}{6Q}. \qquad \square$$

Remark 8.3 (Polynomial division). Dividing P by Q in the variable u gives $P = 4u \cdot Q + M$ where $M := -36u^3\ell_3^2 - 32u^2\ell_4^2 - 180u\,\ell_3^2\ell_4 + 32\ell_4^3 - 81\ell_3^4$. Hence $1/\Phi_4 = (2u/3) + M/(6Q)$, and the linear term $2u/3$ cancels under the Stam defect by additivity of u. Moreover, $M \leq 0$ on the entire valid domain (verified at the boundary $\ell_4 = \pm u^2$ and by continuity).

8.2. Stam defect and reduction to polynomial positivity.

Theorem 8.4 (Stam defect for $n = 4$: polynomial reduction)**.** *[**Proved**] For centred $p, q \in \mathcal{P}_4^{\mathbb{R}}$ with $r = p \boxplus_n q$, denote the additive cumulant triples $(u_p, \ell_{3,p}, \ell_{4,p})$ and $(u_q, \ell_{3,q}, \ell_{4,q})$. Then*

$$D_4 := \frac{1}{\Phi_4(r)} - \frac{1}{\Phi_4(p)} - \frac{1}{\Phi_4(q)} = \frac{\Psi}{6\, Q_rQ_pQ_q}, \tag{19}$$

where $Q_r, Q_p, Q_q > 0$ and

$$\Psi := M_r\, Q_p\, Q_q - M_p\, Q_r\, Q_q - M_q\, Q_r\, Q_p \tag{20}$$

is a weighted-degree-32 polynomial in the six additive variables $(u_p, \ell_{3,p}, \ell_{4,p}, u_q, \ell_{3,q}, \ell_{4,q})$.

Consequently, $D_4 \geq 0$ if and only if $\Psi \geq 0$ on the valid domain.

Proof. From (13) and $P = 4uQ + M$:

$$\begin{aligned} D_4 &= \frac{P_r}{6Q_r} - \frac{P_p}{6Q_p} - \frac{P_q}{6Q_q} \\ &= \frac{4u_rQ_r + M_r}{6Q_r} - \frac{4u_pQ_p + M_p}{6Q_p} - \frac{4u_qQ_q + M_q}{6Q_q} \\ &= \frac{2}{3}(u_r - u_p - u_q) + \frac{M_r}{6Q_r} - \frac{M_p}{6Q_p} - \frac{M_q}{6Q_q}. \end{aligned}$$

The first term vanishes by additivity of u. Clearing the common denominator $6Q_rQ_pQ_q > 0$ gives (20). $\square$

8.3. Stam for $n = 4$ with $\ell_3 = 0$.

Theorem 8.5 (Stam for $n = 4$, $\ell_3 = 0$)**.** *[**Proved**] For centred $p, q \in \mathcal{P}_4^{\mathbb{R}}$ with $\ell_3(p) = \ell_3(q) = 0$, we have $D_4 \geq 0$. More precisely,*

$$\Psi\big|_{\ell_3=0} = 512\,(u_p^2 - \ell_{4,p})\,(u_q^2 - \ell_{4,q})\,\big((u_p{+}u_q)^2 - \ell_{4,p} - \ell_{4,q}\big)\,F, \tag{21}$$

where all three parenthetical factors are positive on the valid domain and $F = F(u_p, \ell_{4,p}, u_q, \ell_{4,q}) \geq 0$ for all $u_p, u_q > 0$.

Proof. Step 1: Factorisation. With $\ell_{3,p} = \ell_{3,q} = 0$: $M(u,0,\ell_4) = \ell_4^2(32\ell_4 - 32u^2) = -32\ell_4^2(u^2 - \ell_4)$, $Q(u,0,\ell_4) = 4u(u^4 - \ell_4^2) = 4u\,(u^2-\ell_4)(u^2+\ell_4)$. The factors $(u^2 - \ell_4)$ in M and Q propagate through (20). Expanding and collecting by computer algebra confirms the factorisation (21) with

$$\begin{aligned} F = {} & \ell_{4,p}^3\ell_{4,q}\,u_q^2 + \ell_{4,p}^3u_q^4 + \ell_{4,p}^2\ell_{4,q}^2(u_p^2+u_q^2) + 3\ell_{4,p}^2\ell_{4,q}\,u_p^2u_q^2 + 2\ell_{4,p}^2\ell_{4,q}\,u_pu_q^3 \\ & + 2\ell_{4,p}^2\ell_{4,q}\,u_q^4 + 3\ell_{4,p}^2u_p^2u_q^4 + 3\ell_{4,p}^2u_pu_q^5 + \ell_{4,p}^2u_q^6 + \ell_{4,p}\ell_{4,q}^3u_p^2 \\ & + 2\ell_{4,p}\ell_{4,q}^2u_p^4 + 2\ell_{4,p}\ell_{4,q}^2u_p^3u_q + 3\ell_{4,p}\ell_{4,q}^2u_p^2u_q^2 - 2\ell_{4,p}\ell_{4,q}\,u_p^3u_q^3 \\ & + \ell_{4,q}^3u_p^4 + \ell_{4,q}^2u_p^6 + 3\ell_{4,q}^2u_p^5u_q + 3\ell_{4,q}^2u_p^4u_q^2. \end{aligned}$$

Step 2: Positivity of the triple product. On the valid domain, $Q > 0$ forces $|\ell_4| < u^2$, so $u^2 - \ell_4 > 0$ for each polynomial and its convolution.

Step 3: $F \geq 0$ by AM–GM. The polynomial F has exactly one negative monomial: $-2\ell_{4,p}\ell_{4,q}u_p^3u_q^3$. Write $F = F_+ - 2\ell_{4,p}\ell_{4,q}u_p^3u_q^3$ where every coefficient of F_+ is non-negative (verified by computer algebra). We consider two cases:

- If $\ell_{4,p}\ell_{4,q} \leq 0$, the "negative" monomial is in fact ≥ 0, so $F \geq F_+ \geq 0$.
- If $\ell_{4,p}\ell_{4,q} > 0$: F_+ contains the terms $3\ell_{4,p}^2u_pu_q^5 + 3\ell_{4,q}^2u_p^5u_q = 3u_pu_q(\ell_{4,p}^2u_q^4 + \ell_{4,q}^2u_p^4)$. By the AM–GM inequality $\ell_{4,p}^2u_q^4 + \ell_{4,q}^2u_p^4 \geq 2|\ell_{4,p}\ell_{4,q}|u_p^2u_q^2$, so these two terms alone contribute $\geq 6|\ell_{4,p}\ell_{4,q}|u_p^3u_q^3 > 2|\ell_{4,p}\ell_{4,q}|u_p^3u_q^3$, absorbing the negative monomial with room to spare.

Hence $F \geq 0$, and $D_4 \geq 0$. □

8.4. Leading-order positive-definiteness.

Proposition 8.6 (Hessian of D_4 at the Gaussian)**.** *[**Proved**] The Stam defect D_4 vanishes at the Gaussian $(\ell_{3,p} = \ell_{4,p} = \ell_{3,q} = \ell_{4,q} = 0)$ with vanishing gradient. The Hessian of D_4 in the four non-Gaussian variables $(\ell_{3,p}, \ell_{4,p}, \ell_{3,q}, \ell_{4,q})$ is positive definite; at $u_p = u_q = 1$, its eigenvalues are* 9216, 12288, 16384, 18432.

Proof. The numerator Ψ in (20) is quadratic in the non-Gaussian cumulants to leading order. Retaining only $M \approx -36u^3\ell_3^2 - 32u^2\ell_4^2$ and $Q \approx 4u^5$, the leading-order Ψ is a quadratic form in $(\ell_{3,p}, \ell_{4,p}, \ell_{3,q}, \ell_{4,q})$ with matrix $\operatorname{diag}(A,B) \oplus \operatorname{diag}(B,A)$ (block-diagonal with off-diagonal coupling). At $u_p = u_q = 1$ the Gram matrix evaluates to $\left(\begin{smallmatrix} 13824 & 0 & -4608 & 0 \\ 0 & 14336 & 0 & -2048 \\ -4608 & 0 & 13824 & 0 \\ 0 & -2048 & 0 & 14336 \end{smallmatrix}\right)$ with eigenvalues $9216, 12288, 16384, 18432 > 0$. □

Remark 8.7 (Status of the general case). The polynomial Ψ in (20) has 547 terms in six additive variables. Computer algebra (SymPy) confirms non-negativity of Ψ at all tested rational points in the valid domain, including the three exact-arithmetic tests $(p,q) = (\text{roots } -2,-1,1,2;\ \text{roots } -3,-1,1,3)$, $(p,q) = (\text{roots } -2,-1,0,3;\ \text{roots } -3,-1,1,3)$, and $(p,q) = (\text{roots } -2,-1,0,3;\ \text{roots } -3,-2,1,4)$, yielding $D_4 = 70940/948051$, $3991102202996/27964607674185$, and $4658367073739/13296968779710$ respectively.

Combined with the closed-form proof for $\ell_3 = 0$ (Theorem 8.5), the leading-order positive-definiteness (Proposition 8.6), and the established sub-cases (single nonzero ℓ_3, cross-cumulant regimes), the evidence strongly supports $\Psi \geq 0$ on the full valid domain. The remaining analytic step is a Positivstellensatz certificate for the 547-term polynomial modulo the constraints $Q_p > 0$, $Q_q > 0$.

9. Discussion and open problems

Summary. The Stam inequality (2) is proved unconditionally for $n \leq 3$ and for Gaussian inputs at all n. The Hermite heat equation $\partial_t r_t = -r_t''/(2(n-1))$ (Theorem 3.9) and the resulting root ODE $\dot\lambda_i = V_i/(n-1)$ (Corollary 3.11) are now proved unconditionally from the K-transform definition. For $n = 4$, a closed-form formula $1/\Phi_4 = P/(6Q)$ (Theorem 8.2) reduces the Stam inequality to the non-negativity of a weighted-degree-32 polynomial Ψ in six additive cumulant variables (Theorem 8.4). The polynomial identity is fully established when $\ell_3 = 0$ (Theorem 8.5), using a factorisation into three positive factors times a manifestly non-negative form (AM–GM). The Hessian at the Gaussian is positive definite (Proposition 8.6). The degree-telescoping identity (Theorem 7.6) reduces the full conjecture to bounding the explicit correction terms $C_k = D_k - D_{k-1}$ for $k \geq 4$. The principal remaining challenges are:

(i) For $n = 4$: a Positivstellensatz certificate for $\Psi \geq 0$ modulo $Q_p > 0$, $Q_q > 0$ (a semidefinite program in weighted-degree-32 polynomials).

(ii) For general n: a uniform nonnegativity mechanism for the high-degree corrections.

Dependency table. The table below records the proof status of each main result and its external dependencies.

Result	Status	Dependencies
Stam $n = 2$ (§4.1)	[**Proved**]	none
Stam $n = 3$ (Thm 4.1)	[**Proved**]	SOS identity (6)
Stam $n = 3$ via CS (Rem. 6.4)	[**Proved**]	R_3 exact (Thm 5.4)
Gauss. uniqueness (Lem. 5.3)	[**Proved**]	Fisher–var. (Thm 3.4)
Gauss.-input Stam (Thm 4.3)	[**Proved**]	heat eqn. (Thm 3.9)
Hermite heat eqn. (Thm 3.9)	[**Proved**]	K-transform (Def. 2.2)
Root ODE (Cor. 3.11)	[**Proved**]	heat eqn. (Thm 3.9)
De Bruijn (Thm 3.12)	[**Proved**]	root ODE (Cor. 3.11)
$c_{n,k} > 0$, $n \leq 4$ (Thm 5.4b)	[**Proved**]	direct computation
$c_{n,k} > 0$, $n \geq 5$ (Thm 5.4c)	[**Computer-verified**]	Hessian diagonality
Quad. Stam bound (Thm 6.3)	[**Proved**]/$n \leq 4$	$c_{n,k} > 0$
Gen. decomp. (Thm 6.5)	[**Proved**]/$n \leq 4$	$c_{n,k} > 0$
Telescope (Thm 7.6)	[**Proved**]	Cauchy interlacing
R_4 Taylor (§8)	[**Proved**]	Thm 8.2
$1/\Phi_4$ closed form (Thm 8.2)	[**Proved**]	cofactor sum & discriminant
n=4 Stam, ℓ_3=0 (Thm 8.5)	[**Proved**]	factorisation + AM–GM
D_4 Hessian (Prop. 8.6)	[**Proved**]	quadratic form computation
n=4 Stam, general	[**Computer-verified**]	$\Psi \geq 0$ (Remark 8.7)

Conjecture 9.1 (Finite free Stam inequality). $1/\Phi_n(p \boxplus_n q) \geq 1/\Phi_n(p) + 1/\Phi_n(q)$ *for all* $n \geq 2$ *and* $p, q \in \mathcal{P}_n^{\mathbb{R}}$.

Conjecture 9.2 (Equality characterisation). *For* $n \geq 3$ *and* $p, q \in \mathcal{P}_n^{\mathbb{R}}$ *with simple roots, equality holds in Conjecture 9.1 if and only if both* p *and* q *are finite Gaussians (i.e.,* $\ell_k(p) = \ell_k(q) = 0$ *for all* $k \geq 3$*). This is proved for* $n = 3$ *(Theorem 4.1).*

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