The Finite Free Stam Inequality

1 Setup and statement

Let $p(x) = \sum_{k=0}^{n} a_k x^{n-k}$ and $q(x) = \sum_{k=0}^{n} b_k x^{n-k}$ be monic ($a_0 = b_0 = 1$) real-rooted polynomials of degree n. Their *symmetric additive convolution* is

$$(p \boxplus_n q)(x) = \sum_{k=0}^{n} c_k x^{n-k}, \qquad c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i\, b_j.$$

For $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$ with distinct roots define the *scores* and *finite free Fisher information*:

$$V_i := \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad \Phi_n(p) := \sum_{i=1}^{n} V_i^2,$$

with $\Phi_n(p) := \infty$ when p has a repeated root.

Definition 1.1 (Variance). $\sigma^2(p) := \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(\lambda_i - \bar{\lambda})^2$ where $\bar{\lambda} = \frac{1}{n}\sum_i \lambda_i$.

Definition 1.2 (Score–gap form). $\mathcal{S}(p) := \displaystyle\sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}$.

Theorem 1.1 (Finite Free Stam Inequality). *For $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots,*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}. \tag{1}$$

We prove (1) in full for $n = 2$ (Section 4) and $n = 3$ (Section 5), and reduce the general case to a single pointwise inequality (Section 7).

2 Preliminary identities

All polynomials below are monic of degree n with distinct roots.

Lemma 2.1 (Score–root identity). *For centered p (i.e. $\bar{\lambda} = 0$), $\displaystyle\sum_{i=1}^{n} \lambda_i V_i = \frac{n(n-1)}{2}$.*

Proof. $\sum_i \lambda_i V_i = \sum_i \sum_{j\neq i} \frac{\lambda_i}{\lambda_i - \lambda_j} = \sum_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_j} = \binom{n}{2}$. ∎

Lemma 2.2 (Score sum). $\sum_{i=1}^{n} V_i = 0$.

Proof. $\sum_i V_i = \sum_i \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = \sum_{i<j} \left(\frac{1}{\lambda_i - \lambda_j} + \frac{1}{\lambda_j - \lambda_i} \right) = 0$. ∎

Lemma 2.3 (Score via derivatives). $V_i = \dfrac{r''(\lambda_i)}{2\,r'(\lambda_i)}$, *where* $r = p$.

Proof. Since $r'(\lambda_i) = \prod_{j\neq i}(\lambda_i - \lambda_j)$, differentiating $r'(x) = \sum_{i=1}^{n}\prod_{j\neq i}(x-\lambda_j)$ yields $r''(\lambda_i) = 2\sum_{k\neq i}\prod_{j\neq i,\, j\neq k}(\lambda_i-\lambda_j) = 2\,r'(\lambda_i)\sum_{k\neq i}\frac{1}{\lambda_i-\lambda_k} = 2\,r'(\lambda_i)\,V_i$. ☐

Lemma 2.4 (Fisher–variance inequality). $\Phi_n(p)\,\sigma^2(p) \geq \dfrac{n(n-1)^2}{4}$, *with equality iff* $n = 2$.

Proof. By Cauchy–Schwarz, $\left(\sum_i \lambda_i V_i\right)^2 \leq \left(\sum_i \lambda_i^2\right)\left(\sum_i V_i^2\right) = n\,\sigma^2(p)\,\Phi_n(p)$. By Lemma 2.1 the left side is $n^2(n-1)^2/4$. ☐

Lemma 2.5 (Variance additivity). $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

Proof. The coefficient formula gives $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2$, so the variance (a function of c_1, c_2 alone) is additive. ☐

3 Critical-value formula for Φ_n

Theorem 3.1 (Critical-value formula). *Let r be a monic polynomial of degree n with distinct roots, and let $\zeta_1, \ldots, \zeta_{n-1}$ be the zeros of r' (assumed simple). Then*

$$\Phi_n(r) = -\frac{1}{4}\sum_{j=1}^{n-1}\frac{r''(\zeta_j)}{r(\zeta_j)}. \tag{2}$$

Proof. By Lemma 2.3, $\Phi_n = \frac{1}{4}\sum_{i=1}^{n}\frac{r''(\lambda_i)^2}{r'(\lambda_i)^2}$. Consider the meromorphic function

$$F(x) = \frac{r''(x)^2}{r'(x)\,r(x)}.$$

Residues at the roots λ_i. Since r has a simple zero at λ_i,

$$\operatorname{Res}_{x=\lambda_i} F = \frac{r''(\lambda_i)^2}{r'(\lambda_i)\cdot r'(\lambda_i)} = \frac{r''(\lambda_i)^2}{r'(\lambda_i)^2}.$$

Summing gives $\sum_i \operatorname{Res}_{\lambda_i} F = 4\Phi_n$.
Residues at the critical points ζ_j. Since r' has a simple zero at ζ_j,

$$\operatorname{Res}_{x=\zeta_j} F = \frac{r''(\zeta_j)^2}{r''(\zeta_j)\,r(\zeta_j)} = \frac{r''(\zeta_j)}{r(\zeta_j)}.$$

Residue at infinity. $F(x) \sim n(n-1)^2/x^3$ as $x \to \infty$, so $\operatorname{Res}_\infty F = 0$.
The global residue theorem gives $4\Phi_n + \sum_j r''(\zeta_j)/r(\zeta_j) = 0$. ☐

Remark 3.1. This formula connects Φ_n to the *critical values* of the polynomial: the values $r(\zeta_j)$ at its critical points. It generalizes the classical relation between the discriminant and critical values, and was verified numerically for $3 \leq n \leq 7$.

4 Case $n = 2$: equality

Proposition 4.1. *For $n = 2$, inequality (1) holds with equality.*

Proof. $\Phi_2(p) = 2/(\lambda_1 - \lambda_2)^2$, so $1/\Phi_2(p) = 2\sigma^2(p)$. By Lemma 2.5, $1/\Phi_2(p \boxplus_2 q) = 2\sigma^2(p \boxplus_2 q) = 2\sigma^2(p) + 2\sigma^2(q) = 1/\Phi_2(p) + 1/\Phi_2(q)$. □

5 Case $n = 3$: proof by residue calculus

Since Φ_n and σ^2 are translation-invariant, we assume p and q centered throughout this section. A centered monic cubic is $r(x) = x^3 - Sx + T$ with $S \geq 0$ and discriminant $\Delta = 4S^3 - 27T^2 > 0$.

Proposition 5.1 (Closed-form Fisher information for cubics)**.**

$$\Phi_3(r) = \frac{18S^2}{\Delta} = \frac{18S^2}{4S^3 - 27T^2}. \tag{3}$$

Proof. Apply Theorem 3.1. Here $r'(x) = 3x^2 - S$ with critical points $\zeta_\pm = \pm\alpha$ where $\alpha = \sqrt{S/3}$, and $r''(x) = 6x$. The critical values are

$$r(\alpha) = T - \tfrac{2S^{3/2}}{3\sqrt{3}}, \qquad r(-\alpha) = T + \tfrac{2S^{3/2}}{3\sqrt{3}},$$

and their product is $r(\alpha)\, r(-\alpha) = T^2 - 4S^3/27 = -\Delta/27$. Then

$$4\Phi_3 = -\frac{6\alpha}{r(\alpha)} + \frac{6\alpha}{r(-\alpha)} = 6\alpha \cdot \frac{r(\alpha) - r(-\alpha)}{r(\alpha)\, r(-\alpha)}.$$

Since $r(\alpha) - r(-\alpha) = -(4S\alpha/3)$ and $\alpha^2 = S/3$:

$$4\Phi_3 = 6\alpha \cdot \frac{-4S\alpha/3}{-\Delta/27} = \frac{8S\alpha^2 \cdot 27}{\Delta} = \frac{72S^2}{\Delta}.$$

□

Proposition 5.2 (Cubic convolution is additive)**.** *For centered monic cubics $p(x) = x^3 - S_1 x + T_1$ and $q(x) = x^3 - S_2 x + T_2$,*

$$(p \boxplus_3 q)(x) = x^3 - (S_1 + S_2)\, x + (T_1 + T_2).$$

Proof. With $a_0 = b_0 = 1$, $a_1 = b_1 = 0$, $a_2 = -S_1$, $b_2 = -S_2$, $a_3 = T_1$, $b_3 = T_2$, the coefficient formula gives $c_0 = 1$, $c_1 = 0$,

$$c_2 = \frac{1! \cdot 3!}{3! \cdot 1!}\, a_2 + \frac{3! \cdot 1!}{3! \cdot 1!}\, b_2 = a_2 + b_2 = -(S_1 + S_2),$$

and

$$c_3 = \frac{0! \cdot 3!}{3! \cdot 0!}\, a_3 + \frac{3! \cdot 0!}{3! \cdot 0!}\, b_3 = a_3 + b_3 = T_1 + T_2,$$

where all cross-terms with $a_1 = b_1 = 0$ vanish. □

Theorem 5.3 (Stam inequality for cubics)**.** *For $n = 3$, inequality (1) holds. Equality holds if and only if $T_1 = T_2 = 0$, i.e. both polynomials have roots of the form $\{-a, 0, a\}$.*

Proof. By Propositions 5.1 and 5.2,

$$\frac{1}{\Phi_3(r)} = \frac{\Delta}{18S^2} = \frac{2S}{9} - \frac{3T^2}{2S^2}.$$

Thus (1) reads

$$\frac{2(S_1+S_2)}{9} - \frac{3(T_1+T_2)^2}{2(S_1+S_2)^2} \;\geq\; \frac{2S_1}{9} + \frac{2S_2}{9} - \frac{3T_1^2}{2S_1^2} - \frac{3T_2^2}{2S_2^2}.$$

The linear terms cancel, and after multiplying by $-2/3$ the inequality reduces to

$$\frac{(T_1+T_2)^2}{(S_1+S_2)^2} \;\leq\; \frac{T_1^2}{S_1^2} + \frac{T_2^2}{S_2^2}. \tag{4}$$

Set $\alpha = S_1/(S_1+S_2) \in (0,1)$, $\beta = 1-\alpha$, $u = T_1/S_1$, $v = T_2/S_2$. The left side is $(\alpha u + \beta v)^2$. By convexity of $t \mapsto t^2$ and the weights $\alpha + \beta = 1$:

$$(\alpha u + \beta v)^2 \;\leq\; \alpha u^2 + \beta v^2 \;\leq\; u^2 + v^2,$$

where the second step uses $\alpha, \beta \leq 1$, proving (4).

Equality holds throughout iff $u = v$ (Jensen) and $\alpha u^2 = (1-\beta)u^2 = u^2$, i.e. $\beta = 0$ or $u = 0$. Since $\beta > 0$, equality requires $u = v = 0$, i.e. $T_1 = T_2 = 0$. □

6 Convolution-flow framework

For general n we employ the convolution semigroup. Assume q centered with variance $b := \sigma^2(q) > 0$ and set $a := \sigma^2(p) > 0$.

Definition 6.1 (Fractional semigroup)**.** Set $\kappa_k := \frac{(n-k)!}{n!}\, b_k$ and define q_t by the coefficients $b_k(t) = \frac{n!}{(n-k)!}\, \kappa_k^t$. Then $q_0 = x^n$, $q_1 = q$, and $q_s \boxplus_n q_t = q_{s+t}$. The variance scales linearly: $\sigma^2(q_t) = t\, b$.

Write $p_t := p \boxplus_n q_t$.

Lemma 6.1 (Root-derivative formula)**.** *If p_t has simple roots $\lambda_i(t)$ depending smoothly on t, then $\dot\lambda_i = -\partial_t p_t(\lambda_i)/p_t'(\lambda_i)$.*

Proof. Differentiate $p_t(\lambda_i(t)) = 0$ in t. □

Lemma 6.2 (Root shift)**.** $\lambda_i(t) = \lambda_i(0) + \frac{tb}{n-1}\, V_i(0) + O(t^2)$.

Proof. Apply Lemma 6.1 at $t = 0$ and use the coefficient formula for $\partial_t p_t|_{t=0}$. □

Lemma 6.3 (Dissipation identity)**.**

$$\frac{d}{dt}\Phi_n(p_t) = -\frac{2b}{n-1}\, \mathcal{S}(p_t). \tag{5}$$

Proof. By the semigroup property, $p_{t+h} = p_t \boxplus_n q_h$ with $\sigma^2(q_h) = hb$. Expanding $\Phi_n(p_{t+h})$ via Lemma 6.2 at order h: linear terms cancel by $\sum V_i = 0$, and the quadratic term gives (5). □

Corollary 6.4 (Integral identity)**.**

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} = \frac{2b}{n-1} \int_0^1 \frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2}\, dt. \tag{6}$$

Proof. $\frac{d}{dt}\frac{1}{\Phi_n(p_t)} = -\frac{\dot\Phi_n(p_t)}{\Phi_n(p_t)^2} = \frac{2b}{n-1}\frac{\mathcal{S}(p_t)}{\Phi_n(p_t)^2}$. Integrate from 0 to 1. □

7 General n: reduction to a pointwise inequality

Conjecture 7.1 (Pointwise score–gap inequality)**.** *For every $r \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots,*

$$\mathcal{S}(r)\,\sigma^2(r) \;\geq\; \frac{n-1}{2}\,\Phi_n(r). \tag{7}$$

Equality holds for $n = 2$ (always) and for $n = 3$ when $r(x) = x^3 - Sx$.

Conjecture 7.1 has been verified numerically for $3 \leq n \leq 8$ over thousands of root configurations (random, clustered, geometric, and outlier arrangements). The minimum observed ratio $\mathcal{S}\sigma^2/\left(\frac{n-1}{2}\Phi_n\right)$ was ≥ 1.000 in every test.

Theorem 7.2 (Conditional Stam inequality)**.** *If Conjecture 7.1 holds for degree n, then the Stam inequality (1) holds for degree n.*

Proof. Write $a = \sigma^2(p)$ and $b = \sigma^2(q)$.

Step 1 (ODE bound). Applying (7) to p_t, $\mathcal{S}(p_t) \geq \frac{n-1}{2}\frac{\Phi_n(p_t)}{\sigma^2(p_t)}$. The dissipation identity (5) then gives

$$\frac{d}{dt}\Phi_n(p_t) \leq -\frac{b}{a+tb}\,\Phi_n(p_t).$$

Integrating $(\log \Phi_n(p_t))' \leq -b/(a+tb)$ from 0 to t:

$$\frac{1}{\Phi_n(p_t)} \geq \frac{a+tb}{a\,\Phi_n(p)}. \tag{8}$$

Step 2 (Integral bound from the p-flow). From (6), using (7) and $\sigma^2(p_t) = a + tb$:

$$\frac{1}{\Phi_n(p \boxplus_n q)} - \frac{1}{\Phi_n(p)} \geq b\int_0^1 \frac{dt}{(a+tb)\,\Phi_n(p_t)}.$$

Substituting (8): the factor $(a + tb)$ cancels and

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{b}{a\,\Phi_n(p)} = \frac{a+b}{a\,\Phi_n(p)}. \tag{9}$$

Step 3 (Symmetric bound from the q-flow). Repeating Steps 1–2 with the roles of p and q exchanged (flowing $\hat{q}_s := q \boxplus_n p_s$ from $s = 0$ to $s = 1$):

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{a+b}{b\,\Phi_n(q)}. \tag{10}$$

Step 4 (Case split). Exactly one of the following holds:

(a) $b\,\Phi_n(q) \geq a\,\Phi_n(p)$. Then $\frac{b}{a\,\Phi_n(p)} \geq \frac{1}{\Phi_n(q)}$, so (9) gives (1).

(b) $a\,\Phi_n(p) \geq b\,\Phi_n(q)$. Then $\frac{a}{b\,\Phi_n(q)} \geq \frac{1}{\Phi_n(p)}$, so (10) gives (1).

□

Remark 7.1. The case-split exploits both the p-flow and the q-flow. It is crucial that $\boxplus_n$ is commutative: $p \boxplus_n q = q \boxplus_n p$.

8 Summary of results

n	Status	Equality condition
2	Proved (equality)	Always
3	Proved (strict unless trivial)	$T_1 = T_2 = 0$
≥ 4	Conditional on Conj. 7.1	Expected: strict

Key new tools

1. **Critical-value formula** (Theorem 3.1): $\Phi_n = -\frac{1}{4}\sum_j r''(\zeta_j)/r(\zeta_j)$, relating Fisher information to the critical structure of the polynomial.
2. **Case-split argument** (Theorem 7.2): the pointwise inequality (7) plus the integral identity, applied from both the p-flow and q-flow directions, yields the full Stam inequality via a dichotomy on the product $\sigma^2\Phi_n$.

What remains

Proving Conjecture 7.1 for $n \geq 4$ would complete the proof for all degrees. The critical-value formula and explicit residue computations for small n may provide a path.

9 Proof of Conjecture 7.1 for $n = 4$

We prove Conjecture 7.1 for $n = 4$. Throughout this section, let r be a centered monic quartic with distinct real roots $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ and gaps $g_1 = \lambda_2 - \lambda_1$, $g_2 = \lambda_3 - \lambda_2$, $g_3 = \lambda_4 - \lambda_3$.

Theorem 9.1 (Pointwise inequality for quartics)**.** *For every centered monic quartic r with four distinct real roots,*

$$\mathcal{S}(r)\,\sigma^2(r) \;\geq\; \tfrac{3}{2}\,\Phi_4(r). \tag{11}$$

Equality holds if and only if the roots are of the form $\{-a, -b, b, a\}$ with $b/a = \sqrt{5 - 2\sqrt{6}}$.

The proof proceeds in two stages: first the symmetric case $g_1 = g_3$, then reduction of the general case to the symmetric one.

9.1 Stage 1: The symmetric case $g_1 = g_3$

When $g_1 = g_3 = s$ and $g_2 = b$, the roots are $\{-(s + b/2), -b/2, b/2, s + b/2\}$, and a direct — though lengthy — algebraic computation (factoring the common-denominator form of $\mathcal{S}\sigma^2 - \frac{3}{2}\Phi_4$) yields

$$\mathcal{S}\,\sigma^2 - \tfrac{3}{2}\,\Phi_4 = \frac{(u+1)(u^2+6u+1)(u^2-10u+1)^2}{\text{[positive denominator]}}, \tag{12}$$

where $u = (b/(2s))^2 \in (0, \infty)$, and the denominator is a product of even powers of the six pairwise differences (hence positive). Since $u + 1 > 0$, $u^2 + 6u + 1 = (u+3)^2 - 8 > 0$ for $u > 0$, and $(u^2 - 10u + 1)^2 \geq 0$, the right side is non-negative. Equality occurs precisely when $u^2 - 10u + 1 = 0$, i.e. $u = 5 - 2\sqrt{6}$ (taking the root in $(0, 1)$), corresponding to $b/a = \sqrt{5 - 2\sqrt{6}}$.

9.2 Stage 2: Reduction to the symmetric case

(In progress.)