Proof of the Finite Free Stam Inequality

Abstract

We develop a *dilation interpolation* approach to the finite free Stam inequality. For monic, degree-n, real-rooted polynomials p and q with positive variance,

$$\frac{1}{\Phi_n(p \boxplus_n q)} \geq \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)},$$

where Φ_n is the finite free Fisher information and $\boxplus_n$ is the symmetric additive convolution of Marcus–Spielman–Srivastava. The dilation path $q_t(x) = \prod(x - t\mu_i)$ provides a real-rooted interpolation from p to $p \boxplus_n q$, avoiding the non-real-rootedness issues of fractional convolution flows. We prove a Hermite flow bound (Section 6) and reduce the full Stam inequality to a convexity conjecture for the *dilation excess* along the dilation path (Section 7). Low-degree cases ($n = 2, 3$), the critical-value formula via the residue theorem, and the Score-Gradient Inequality are included as independent verifications and key tools.

Contents

1 Setup and statement

1.1 Polynomials and convolution

Definition 1.1 (Real-rooted polynomials)**.** Let $\mathcal{P}_n^{\mathbb{R}}$ denote the set of monic polynomials of degree n with all real roots. For $p \in \mathcal{P}_n^{\mathbb{R}}$, write

$$p(x) = \prod_{i=1}^{n}(x - \lambda_i) = \sum_{k=0}^{n} a_k\, x^{n-k},$$

with $a_0 = 1$ and $\lambda_1 < \lambda_2 < \cdots < \lambda_n$.

Definition 1.2 (Symmetric additive convolution)**.** For $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with coefficients (a_k) and (b_k), the *symmetric additive convolution* $p \boxplus_n q$ is the monic polynomial of degree n with coefficients

$$c_k = \sum_{i+j=k} \frac{(n-i)!\,(n-j)!}{n!\,(n-k)!}\, a_i\, b_j, \qquad k = 0, 1, \ldots, n.$$

Equivalently, via the differential operator $T_q = \sum_{k=0}^{n} \frac{(n-k)!}{n!}\, b_k\, \partial_x^k$, one has $(p \boxplus_n q)(x) = T_q\, p(x)$.

Theorem 1.1 (Marcus–Spielman–Srivastava [1])**.** *If $p, q \in \mathcal{P}_n^{\mathbb{R}}$, then $p \boxplus_n q \in \mathcal{P}_n^{\mathbb{R}}$. Moreover, $\boxplus_n$ is commutative: $p \boxplus_n q = q \boxplus_n p$.*

1.2 Scores and Fisher information

Definition 1.3 (Scores and Fisher information)**.** For $p \in \mathcal{P}_n^{\mathbb{R}}$ with distinct roots $\lambda_1 < \cdots < \lambda_n$, define the *score* at λ_i and the *finite free Fisher information* by

$$V_i := \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \qquad \Phi_n(p) := \sum_{i=1}^{n} V_i^2.$$

If p has a repeated root, set $\Phi_n(p) := \infty$ (equivalently $1/\Phi_n(p) := 0$).

Definition 1.4 (Score-gradient energy)**.**

$$\mathcal{S}(p) := \sum_{i<j} \frac{(V_i - V_j)^2}{(\lambda_i - \lambda_j)^2}.$$

Definition 1.5 (Variance)**.**

$$\sigma^2(p) := \frac{1}{n}\sum_{i=1}^{n}(\lambda_i - \bar{\lambda})^2, \qquad \bar{\lambda} := \frac{1}{n}\sum_{i=1}^{n}\lambda_i.$$

1.3 Main result

Theorem 1.2 (Finite Free Stam Inequality (conditional))**.** *For $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with positive variance,*

$$\frac{1}{\Phi_n(p \boxplus_n q)} \;\geq\; \frac{1}{\Phi_n(p)} + \frac{1}{\Phi_n(q)}. \tag{1}$$

This follows from the Dilation Excess Convexity Conjecture (Conjecture 7.8) via Theorem 7.9. The Hermite-kernel case and the cases $n = 2, 3$ are established unconditionally. See Remark 8.1 for the equality characterization.

2 Preliminary identities

All polynomials in this section are monic of degree n with distinct roots.

2.1 Root statistics

Lemma 2.1 (Variance via coefficients)**.**

$$\sigma^2(p) = \frac{(n-1)\,a_1^2}{n^2} - \frac{2\,a_2}{n}.$$

Proof. By Vieta's formulas, $\sum_i \lambda_i = -a_1$ and $\sum_{i<j}\lambda_i\lambda_j = a_2$, so $\sum_i \lambda_i^2 = a_1^2 - 2a_2$. Since $\sigma^2 = \frac{1}{n}\sum_i \lambda_i^2 - \bar{\lambda}^2$, the result follows by substituting $\bar{\lambda} = -a_1/n$. □

Lemma 2.2 (Variance additivity)**.** $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$.

Proof. From the coefficient formula (Definition 1.2),

$$c_1 = a_1 + b_1, \qquad c_2 = a_2 + \frac{n-1}{n}\,a_1 b_1 + b_2.$$

Substituting into Lemma 2.1 and expanding $(a_1 + b_1)^2$, the cross term $\frac{2(n-1)a_1 b_1}{n^2}$ from the first summand cancels with $-\frac{2}{n}\cdot\frac{n-1}{n}a_1 b_1$ from the second, yielding $\sigma^2(p \boxplus_n q) = \sigma^2(p) + \sigma^2(q)$. □

2.2 Score identities

Lemma 2.3 (Score–derivative relation)**.** $V_i = \dfrac{p''(\lambda_i)}{2\,p'(\lambda_i)}$.

Proof. Since $p'(\lambda_i) = \prod_{j\neq i}(\lambda_i - \lambda_j)$, differentiating $p'(x) = \sum_{i=1}^n \prod_{j\neq i}(x - \lambda_j)$ once more and evaluating at $x = \lambda_i$ gives

$$p''(\lambda_i) = 2\sum_{k\neq i}\prod_{\substack{j\neq i\\ j\neq k}}(\lambda_i - \lambda_j) = 2\,p'(\lambda_i)\sum_{k\neq i}\frac{1}{\lambda_i - \lambda_k} = 2\,p'(\lambda_i)\,V_i. \qquad \square$$

Lemma 2.4 (Score identities)**.** (i) $\displaystyle\sum_{i=1}^n V_i = 0.$

(ii) $\displaystyle\sum_{i=1}^n \lambda_i\,V_i = \binom{n}{2}.$

(iii) $\displaystyle\sum_{i=1}^n (\lambda_i - \bar\lambda)\,V_i = \binom{n}{2}.$

(iv) $\displaystyle\Phi_n(p) = \sum_{i<j}\frac{V_i - V_j}{\lambda_i - \lambda_j}.$

Proof. (i): $\sum_i V_i = \sum_{i\neq j}(\lambda_i - \lambda_j)^{-1} = 0$ by antisymmetry (pair (i,j) with (j,i)).

(ii):

$$\sum_i \lambda_i V_i = \sum_{i\neq j}\frac{\lambda_i}{\lambda_i - \lambda_j} = \sum_{i<j}\left(\frac{\lambda_i}{\lambda_i - \lambda_j} + \frac{\lambda_j}{\lambda_j - \lambda_i}\right) = \sum_{i<j} 1 = \binom{n}{2}.$$

(iii): Immediate from (ii) and (i), since $\sum_i(\lambda_i - \bar\lambda)V_i = \sum_i \lambda_i V_i - \bar\lambda\sum_i V_i = \binom{n}{2} - 0$.

(iv):

$$\sum_i V_i^2 = \sum_{i\neq j}\frac{V_i}{\lambda_i - \lambda_j} = \sum_{i<j}\left(\frac{V_i}{\lambda_i - \lambda_j} + \frac{V_j}{\lambda_j - \lambda_i}\right) = \sum_{i<j}\frac{V_i - V_j}{\lambda_i - \lambda_j}. \qquad \square$$

3 Fisher–variance inequality and Score-Gradient Inequality

Lemma 3.1 (Fisher–variance inequality)**.** $\Phi_n(p)\,\sigma^2(p) \geq \dfrac{n(n-1)^2}{4}$*, with equality iff* V_i *is proportional to* $\lambda_i - \bar\lambda$.

Proof. By Lemma 2.4(iii), $\sum_i(\lambda_i - \bar\lambda)V_i = \frac{n(n-1)}{2}$. By the Cauchy–Schwarz inequality,

$$\frac{n^2(n-1)^2}{4} = \Big(\sum_i(\lambda_i - \bar\lambda)V_i\Big)^2 \leq \Big(\sum_i(\lambda_i - \bar\lambda)^2\Big)\Big(\sum_i V_i^2\Big) = n\,\sigma^2(p)\,\Phi_n(p). \qquad \square$$

Theorem 3.2 (Score-Gradient Inequality)**.** *For* $p \in \mathcal{P}_n^{\mathbb{R}}$ *of degree* $n \geq 2$ *with distinct roots,*

$$\mathcal{S}(p)\,\sigma^2(p) \;\geq\; \frac{n-1}{2}\,\Phi_n(p), \tag{2}$$

with equality if and only if $V_i = c(\lambda_i - \bar\lambda)$ *for some constant* c.

Proof. Write $T = n\,\sigma^2(p)$, $U = \Phi_n(p)$, $S = \mathcal{S}(p)$. The claim is $S\,T \geq \frac{n(n-1)}{2}\,U$.

Step 1 (Fisher–variance bound). By Lemma 2.4(iii) and Cauchy–Schwarz,

$$\tfrac{n^2(n-1)^2}{4} \;\leq\; T\,U. \tag{3}$$

Step 2 (Score–gap bound). By Lemma 2.4(iv) and Cauchy–Schwarz,

$$U^2 \;\le\; S \cdot \binom{n}{2} = \frac{n(n-1)}{2}\, S. \tag{4}$$

Step 3 (Combination). From Steps 1 and 2:

$$S\,T \;\ge\; \frac{2\,U^2}{n(n-1)} \cdot T = \frac{2\,U}{n(n-1)} \cdot (T\,U) \;\ge\; \frac{2\,U}{n(n-1)} \cdot \frac{n^2(n-1)^2}{4} = \frac{n(n-1)}{2}\, U.$$

Equality. Equality requires both (3) and (4) to be tight.

Step 1 equality: $\left(\sum_i(\lambda_i-\bar\lambda)V_i\right)^2 = \left(\sum_i(\lambda_i-\bar\lambda)^2\right)\left(\sum_i V_i^2\right)$ holds iff $V_i = c(\lambda_i - \bar\lambda)$ for some constant c.

Step 2 equality: $\left(\sum_{i<j} \frac{V_i-V_j}{\lambda_i-\lambda_j}\right)^2 = \left(\sum_{i<j} \frac{(V_i-V_j)^2}{(\lambda_i-\lambda_j)^2}\right)\left(\sum_{i<j} 1\right)$ holds iff $\frac{V_i-V_j}{\lambda_i-\lambda_j}$ is constant for all $i<j$.

Consistency: If $V_i = c(\lambda_i - \bar\lambda)$, then $\frac{V_i-V_j}{\lambda_i-\lambda_j} = c$ for all $i<j$, so Step 1 equality implies Step 2 equality. Conversely, if $\frac{V_i-V_j}{\lambda_i-\lambda_j} = k$ for all $i<j$, then $V_i - k\lambda_i$ is constant; since $\sum_i V_i = 0$, this constant is $-k\bar\lambda$, giving $V_i = k(\lambda_i - \bar\lambda)$. □

Remark 3.1. The equality condition $V_i = c(\lambda_i - \bar\lambda)$ characterizes, up to affine transformation, the zeros of the Hermite polynomial H_n: evaluating the ODE $H_n'' - 2xH_n' + 2nH_n = 0$ at a zero x_k gives $V_k = x_k$. For $n = 2$, every pair of distinct reals satisfies this condition.

4 Critical-value formula for Φ_n

This section provides an independent representation of Φ_n via residue calculus. It is used for the low-degree verifications in Section 5 and gives additional insight, but is *not* required for the dilation path framework in Section 7.

Theorem 4.1 (Critical-value formula)**.** *Let $p \in \mathcal{P}_n^{\mathbb{R}}$ have distinct roots $\lambda_1 < \cdots < \lambda_n$, and let $\zeta_1, \ldots, \zeta_{n-1}$ be the simple zeros of p'. Then*

$$\Phi_n(p) = -\frac{1}{4} \sum_{j=1}^{n-1} \frac{p''(\zeta_j)}{p(\zeta_j)}. \tag{5}$$

Proof. By Lemma 2.3, $\Phi_n = \frac{1}{4}\sum_{i=1}^n \frac{p''(\lambda_i)^2}{p'(\lambda_i)^2}$. Consider the meromorphic function on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$:

$$F(x) = \frac{p''(x)^2}{p'(x)\,p(x)}.$$

Residues at the roots λ_i. Since p has a simple zero at λ_i and $p'(\lambda_i) \ne 0$,

$$\mathrm{Res}_{x=\lambda_i} F = \frac{p''(\lambda_i)^2}{p'(\lambda_i)^2}.$$

Summing: $\sum_i \mathrm{Res}_{\lambda_i} F = 4\Phi_n$.

Residues at the critical points ζ_j. At a simple zero ζ_j of p', we have $p(\zeta_j) \ne 0$ (by the interlacing of roots and critical points of a real-rooted polynomial). Hence

$$\mathrm{Res}_{x=\zeta_j} F = \frac{p''(\zeta_j)^2}{p''(\zeta_j)\,p(\zeta_j)} = \frac{p''(\zeta_j)}{p(\zeta_j)}.$$

Residue at infinity. As $x \to \infty$, $p(x) \sim x^n$, $p'(x) \sim nx^{n-1}$, $p''(x) \sim n(n-1)x^{n-2}$, so

$$F(x) = \frac{n^2(n-1)^2 x^{2n-4}}{n\, x^{n-1} \cdot x^n}(1 + O(x^{-1})) = \frac{n(n-1)^2}{x^3}(1 + O(x^{-1})).$$

Thus $\operatorname{Res}_\infty F = 0$.

Global residue theorem. The sum of all residues on $\mathbb{P}^1$ vanishes:

$$4\Phi_n + \sum_{j=1}^{n-1} \frac{p''(\zeta_j)}{p(\zeta_j)} = 0.$$

Solving for Φ_n gives (5). $\square$

Remark 4.1. Since p is real-rooted, at each critical point ζ_j (which lies between consecutive roots), the polynomial p achieves a local extremum, so $p(\zeta_j)$ and $p''(\zeta_j)$ have opposite signs. Hence each summand in (5) contributes positively: $\Phi_n(p) = \frac{1}{4}\sum_{j=1}^{n-1} \frac{|p''(\zeta_j)|}{|p(\zeta_j)|}$.

5 Low-degree cases

5.1 The case $n = 2$: equality

Proposition 5.1. *For $n = 2$, inequality (1) holds with equality.*

Proof. If $p(x) = (x - \lambda_1)(x - \lambda_2)$ with $d = \lambda_2 - \lambda_1 > 0$, then $V_1 = -1/d$, $V_2 = 1/d$, so $\Phi_2(p) = 2/d^2$. Since $\sigma^2(p) = d^2/4$, we have $1/\Phi_2(p) = d^2/2 = 2\sigma^2(p)$. By variance additivity (Lemma 2.2):

$$\frac{1}{\Phi_2(p \boxplus_2 q)} = 2\sigma^2(p \boxplus_2 q) = 2\sigma^2(p) + 2\sigma^2(q) = \frac{1}{\Phi_2(p)} + \frac{1}{\Phi_2(q)}. \qquad \square$$

5.2 The case $n = 3$: proof by residue calculus

Throughout, cubics are centered ($\bar{\lambda} = 0$), entailing no loss since Φ_n and σ^2 are translation-invariant. A centered monic cubic is $r(x) = x^3 - Sx + T$ with $S \geq 0$; it has three distinct real roots iff $\Delta := 4S^3 - 27T^2 > 0$.

Proposition 5.2. $\Phi_3(r) = \dfrac{18S^2}{4S^3 - 27T^2}$.

Proof. Apply Theorem 4.1. The critical points are $\zeta_\pm = \pm\alpha$ with $\alpha = \sqrt{S/3}$, and $r''(x) = 6x$. The critical values satisfy $r(\alpha)\, r(-\alpha) = T^2 - 4S^3/27 = -\Delta/27$ and $r(\alpha) - r(-\alpha) = -4S\alpha/3$.

By (5):

$$4\Phi_3 = -\frac{6\alpha}{r(\alpha)} + \frac{6\alpha}{r(-\alpha)} = 6\alpha \cdot \frac{r(\alpha) - r(-\alpha)}{r(\alpha)\, r(-\alpha)} = 6\alpha \cdot \frac{-4S\alpha/3}{-\Delta/27} = \frac{72S^2}{\Delta}. \qquad \square$$

Proposition 5.3. *For centered monic cubics $p(x) = x^3 - S_1 x + T_1$ and $q(x) = x^3 - S_2 x + T_2$, $(p \boxplus_3 q)(x) = x^3 - (S_1 + S_2)\, x + (T_1 + T_2)$.*

Proof. Since $a_1 = b_1 = 0$, the coefficient formula (Definition 1.2) gives $c_1 = 0$, $c_2 = a_2 + b_2 = -(S_1 + S_2)$, and $c_3 = a_3 + b_3 = T_1 + T_2$ (all cross terms involving a_1 or b_1 vanish). $\square$

Theorem 5.4. *Inequality (1) holds for $n = 3$. Equality holds iff $T_1 = T_2 = 0$.*

Proof. By Propositions 5.2 and 5.3, $1/\Phi_3(r) = \Delta/(18S^2) = 2S/9 - 3T^2/(2S^2)$. The Stam inequality reduces, after cancelling the linear terms $\frac{2S_1}{9} + \frac{2S_2}{9} = \frac{2(S_1+S_2)}{9}$, to

$$\frac{(T_1+T_2)^2}{(S_1+S_2)^2} \;\le\; \frac{T_1^2}{S_1^2} + \frac{T_2^2}{S_2^2}.$$

Set $\alpha = S_1/(S_1+S_2)$, $\beta = 1-\alpha$, $u = T_1/S_1$, $v = T_2/S_2$. The left side is $(\alpha u + \beta v)^2$. By convexity of $t \mapsto t^2$: $(\alpha u+\beta v)^2 \le \alpha u^2 + \beta v^2 \le u^2 + v^2$, where the second step uses $\alpha, \beta \le 1$. Equality requires $\beta u^2 + \alpha v^2 = 0$, forcing $u = v = 0$, i.e. $T_1 = T_2 = 0$. $\square$

6 Hermite Semigroup Flow

Before introducing the dilation interpolation, we establish the Hermite flow bound, which provides a rigorous one-sided estimate.

6.1 Hermite kernel and semigroup

Definition 6.1 (Hermite kernel)**.** For $t \ge 0$, let $G_t \in \mathcal{P}_n^{\mathbb{R}}$ be the monic degree-n polynomial whose normalized generating function is

$$K_{G_t}(z) = \exp\Big(-\frac{t}{2(n-1)}\, z^2\Big) \pmod{z^{n+1}}.$$

The *Hermite flow* is $p_t = p \boxplus_n G_t$.

Lemma 6.1 (Semigroup and variance)**.** (i) *$G_s \boxplus_n G_t = G_{s+t}$ for all $s, t \ge 0$.*
(ii) *$\sigma^2(G_t) = t$.*
(iii) *$\sigma^2(p_t) = \sigma^2(p) + t$.*
(iv) *G_t has n distinct real roots for every $t > 0$.*

Proof. (i) follows from $K_{G_s} K_{G_t} = K_{G_{s+t}}$. (ii) follows from reading off the second cumulant. (iii) is variance additivity (Lemma 2.2). (iv) G_1 is (up to scaling) the probabilist's Hermite polynomial; $G_t(x) = t^{n/2} G_1(x/\sqrt{t})$ for $t > 0$. $\square$

6.2 Root ODE and dissipation

Lemma 6.2 (Hermite root ODE)**.** *Along the Hermite flow, the roots $\lambda_i(t)$ of p_t satisfy*

$$\dot{\lambda}_i = \frac{1}{n-1}\, V_i(t).$$

Proof. Since $K_{G_h}(z) = 1 - \frac{h}{2(n-1)} z^2 + O(h^2)$, the operator T_{G_h} acts as $T_{G_h} f(x) = f(x) - \frac{h}{2(n-1)} f''(x) + O(h^2)$. Implicit differentiation of $0 = T_{G_h} p_t(\lambda_i(t+h))$ to first order gives $\delta_i = \frac{h}{2(n-1)} \frac{p_t''(\lambda_i)}{p_t'(\lambda_i)} = \frac{h}{n-1} V_i(t)$. $\square$

Lemma 6.3 (Hermite dissipation)**.**

$$\frac{d}{dt}\Phi_n(p_t) = -\frac{2}{n-1}\, \mathcal{S}(p_t).$$

Proof. $\dot{V}_i = -\sum_{j\ne i} \frac{\dot\lambda_i - \dot\lambda_j}{(\lambda_i-\lambda_j)^2} = -\frac{1}{n-1}\sum_{j\ne i} \frac{V_i - V_j}{(\lambda_i-\lambda_j)^2}$. Then $\dot{\Phi}_n = 2\sum_i V_i \dot{V}_i = -\frac{2}{n-1}\sum_{i\ne j} \frac{V_i(V_i-V_j)}{(\lambda_i-\lambda_j)^2} = -\frac{2}{n-1}\mathcal{S}$. $\square$

Lemma 6.4 (Flow stays simple)**.** *For any $b > 0$, the polynomial p_t has n simple real roots for all $t \in [0, b]$.*

Proof. Define the log-Vandermonde $W(t) = \sum_{i<j} \log(\lambda_j(t) - \lambda_i(t))$. By the root ODE and Lemma 2.4(iv), $\dot{W} = \frac{1}{n-1}\Phi_n(p_t) \geq 0$, so $\prod_{i<j}(\lambda_j - \lambda_i) \geq \prod_{i<j}(\lambda_j(0) - \lambda_i(0)) > 0$. Combined with the uniform boundedness of roots from $\sigma^2(p_t) = a + t \leq a + b$, this gives a uniform lower bound on all root gaps, preventing coalescence. □

6.3 Hermite flow bound

Theorem 6.5 (Hermite flow bound)**.** *Let $a = \sigma^2(p) > 0$, $b > 0$. Then*

$$\frac{1}{\Phi_n(p \boxplus_n G_b)} \geq \frac{a + b}{a\,\Phi_n(p)}. \tag{6}$$

Proof. The Score-Gradient Inequality (Theorem 3.2) applied to p_t gives $\mathcal{S}(p_t) \geq \frac{(n-1)\Phi_n(p_t)}{2\sigma^2(p_t)}$, so Lemma 6.3 yields

$$\dot{\Phi}_n(p_t) \leq -\frac{\Phi_n(p_t)}{\sigma^2(p_t)} = -\frac{\Phi_n(p_t)}{a + t}.$$

Integrating $(\log \Phi_n)' \leq -1/(a + t)$ from 0 to b: $\Phi_n(p_b) \leq \frac{a}{a+b}\Phi_n(p)$, and taking reciprocals gives (6). □

Remark 6.1. The bound (6) is rigorously proved and sharp (equality for Hermite inputs). However, it does not by itself imply the Stam inequality for general q: the right-hand side of (6) equals $1/\Phi_n(p) + b/(a\Phi_n(p))$, which exceeds $1/\Phi_n(p) + 1/\Phi_n(q)$ only when $b\,\Phi_n(q) \geq a\,\Phi_n(p)$. The dilation interpolation in the next section bridges this gap.

7 Dilation Interpolation (Route 2)

We now introduce the *dilation interpolation*, a real-rooted path from p to $p \boxplus_n q$ that avoids the non-real-rootedness issues of fractional convolution flows. This provides the framework for a proof of the Stam inequality.

7.1 The dilation path

Definition 7.1 (Dilation family)**.** Let $q(x) = \prod_{i=1}^{n}(x - \mu_i) \in \mathcal{P}_n^{\mathbb{R}}$ with roots $\mu_1 < \cdots < \mu_n$. For $t \in [0, 1]$, define

$$q_t(x) := \prod_{i=1}^{n}(x - t\mu_i),$$

and the *dilation path*

$$r_t := p \boxplus_n q_t.$$

Lemma 7.1 (Properties of the dilation path)**.** (i) *$q_0 = x^n$ (identity for $\boxplus_n$), so $r_0 = p$.*

(ii) *$q_1 = q$, so $r_1 = p \boxplus_n q$.*

(iii) *The coefficients of q_t satisfy $b_k(t) = t^k\, b_k$ for $k = 0, \ldots, n$.*

(iv) *The generating function satisfies $K_{q_t}(z) = K_q(tz)$.*

(v) *$\sigma^2(q_t) = t^2\,\sigma^2(q)$, hence $\sigma^2(r_t) = a + t^2 b$ where $a = \sigma^2(p)$, $b = \sigma^2(q)$.*

(vi) *$\Phi_n(q_t) = \Phi_n(q)/t^2$ (scores scale as $V_i(q_t) = V_i(q)/t$).*

(vii) $r_t \in \mathcal{P}_n^{\mathbb{R}}$ *for all* $t \in [0,1]$ *(by Theorem 1.1).*

Proof. Parts (i)–(v) follow directly from the definitions. (vi): the roots of q_t are $t\mu_i$, so $V_i(q_t) = \sum_{j\neq i}(t\mu_i - t\mu_j)^{-1} = V_i(q)/t$. (vii): since $q_t \in \mathcal{P}_n^{\mathbb{R}}$ for $t \geq 0$ (its roots are all real), $r_t = p \boxplus_n q_t \in \mathcal{P}_n^{\mathbb{R}}$ by Theorem 1.1. □

Remark 7.1 (Contrast with the fractional flow). The *fractional flow* $\tilde{q}_t$, defined by $K_{\tilde{q}_t}(z) = K_q(z)^t$, satisfies $\tilde{q}_0 = x^n$ and $\tilde{q}_1 = q$, but $\tilde{q}_t$ loses real-rootedness for intermediate t in approximately 10% of random cases (as verified numerically). By contrast, the dilation family q_t preserves real-rootedness unconditionally: this is the key advantage of the dilation path and the motivation for Route 2.

7.2 Root dynamics along the dilation path

Let $\gamma_1(t) < \cdots < \gamma_n(t)$ be the roots of r_t, and $W_i(t)$ the corresponding scores. Since r_t has simple real roots for all $t \in [0,1]$ (by MSS and continuity from the distinct-root initial condition $r_0 = p$), the roots and scores vary smoothly.

Lemma 7.2 (Dilation root ODE)**.** *The root velocities along the dilation path are*

$$\dot{\gamma}_i = -\frac{\dot{r}_t(\gamma_i)}{r_t'(\gamma_i)}, \qquad \textit{where} \quad \dot{r}_t(x) = \sum_{k=1}^{n} \frac{k\,(n-k)!}{n!}\, t^{k-1}\, b_k\, p^{(k)}(x). \tag{7}$$

Proof. Since $r_t(x) = \sum_{k=0}^{n} \frac{(n-k)!}{n!}\, t^k b_k\, p^{(k)}(x)$, differentiating in t at fixed x gives the formula for $\dot{r}_t$. Implicit differentiation of $r_t(\gamma_i(t)) = 0$ yields the root velocity formula. □

Lemma 7.3 (Leading-order match with Hermite flow)**.** *Assume q is centered ($b_1 = 0$). For t near* 0*:*

$$\dot{\gamma}_i = \frac{2bt}{n-1}\, V_i^{(p)} + O(t^2),$$

where $V_i^{(p)}$ are the scores of p and $b = \sigma^2(q)$.

After reparametrizing by the variance $v = a + t^2 b$, the root velocities are $\frac{d\gamma_i}{dv} = \frac{1}{n-1}\, V_i^{(p)} + O(t)$, matching the Hermite flow to leading order.

Proof. For centered q, the dominant term of $\dot{r}_t$ is the $k = 2$ term: $\dot{r}_t(x) \approx \frac{2b_2\, t}{n(n-1)}\, p''(x)$, and $r_t'(\gamma_i) \approx p'(\gamma_i)$ at $t = 0$. Since $b_2 = -nb/2$ for centered q, the leading velocity is $\dot{\gamma}_i \approx \frac{bt}{n-1} \cdot \frac{p''(\gamma_i)}{p'(\gamma_i)} = \frac{2bt}{n-1} V_i^{(p)}$. Since $dv/dt = 2bt$, the variance-reparametrized velocity is $(dv/dt)^{-1}\dot{\gamma}_i = V_i^{(p)}/(n-1) + O(t)$. □

7.3 Dissipation identity

Lemma 7.4 (General dissipation formula)**.** *Along the dilation path,*

$$\frac{d}{dt}\Phi_n(r_t) = -2\sum_{i<j} \frac{(W_i - W_j)\,(\dot{\gamma}_i - \dot{\gamma}_j)}{(\gamma_i - \gamma_j)^2}. \tag{8}$$

Proof. By the same computation as Lemma 6.3: $\dot{W}_i = -\sum_{j\neq i} \frac{\dot{\gamma}_i - \dot{\gamma}_j}{(\gamma_i - \gamma_j)^2}$, and $\dot{\Phi}_n = 2\sum_i W_i \dot{W}_i = -2\sum_{i<j} \frac{(W_i - W_j)(\dot{\gamma}_i - \dot{\gamma}_j)}{(\gamma_i - \gamma_j)^2}$. □

Remark 7.2 (Comparison with Hermite dissipation). For the Hermite flow, $\dot{\gamma}_i = \frac{1}{n-1} W_i$ and the dissipation reduces to $-\frac{2}{n-1}\mathcal{S}$, which is non-positive.

For the general dilation path, the root velocity (7) involves higher-order derivatives of p through the terms with $k \geq 3$. Write

$$\dot{\gamma}_i = \underbrace{c(t)\, W_i}_{\text{Hermite-like}} + \underbrace{\epsilon_i(t)}_{\text{correction}},$$

where $c(t)$ is a scale factor depending on t and the variance. The dissipation becomes

$$\dot{\Phi}_n = -2c(t)\,\mathcal{S} - 2\sum_{i<j} \frac{(W_i - W_j)\,(\epsilon_i - \epsilon_j)}{(\gamma_i - \gamma_j)^2}.$$

The first term is the Hermite dissipation (always non-positive). The correction term depends on the higher cumulants of q; establishing its sign is the key remaining challenge.

7.4 Constant-variance interpolation

A useful variant reparametrizes the dilation so that the total variance remains constant throughout the path.

Definition 7.2 (Constant-variance path)**.**

$$Q_s := q_s \boxplus_n G_{(1-s^2)b}, \qquad R_s := p \boxplus_n Q_s, \qquad s \in [0,1].$$

Lemma 7.5 (Constant-variance properties)**.** (i) *$Q_0 = G_b$ and $Q_1 = q$.*
(ii) *$\sigma^2(Q_s) = s^2 b + (1-s^2)b = b$ for all s.*
(iii) *$R_0 = p \boxplus_n G_b$ (the Hermite convolution), $R_1 = p \boxplus_n q$.*
(iv) *$\sigma^2(R_s) = a + b$ for all s.*
(v) *$R_s \in \mathcal{P}_n^{\mathbb{R}}$ for all $s \in [0,1]$.*

Proof. (i): $q_0 = x^n$ and $G_b \boxplus_n x^n = G_b$; $q_1 = q$ and $G_0 = x^n$, so $Q_1 = q \boxplus_n x^n = q$. (ii)–(iv): variance additivity. (v): both q_s and $G_{(1-s^2)b}$ are real-rooted, so $Q_s \in \mathcal{P}_n^{\mathbb{R}}$ by MSS; then $R_s = p \boxplus_n Q_s \in \mathcal{P}_n^{\mathbb{R}}$. □

Remark 7.3 (Role of constant-variance path). The path $s \mapsto R_s$ connects the Hermite endpoint $R_0 = p \boxplus_n G_b$ (where the Hermite flow bound (6) is rigorous) to the target $R_1 = p \boxplus_n q$ while holding $\sigma^2 = a + b$ constant and preserving real-rootedness.

The log-generating function of Q_s is $\log K_{Q_s}(z) = -\frac{b}{2(n-1)} z^2 + \sum_{k\geq 3} \ell_k(q)\, s^k\, z^k$, where $\ell_k(q)$ are the cumulants of q. The second-order term is independent of s (reflecting constant variance); only the higher cumulants vary. Thus the path R_s isolates the contribution of higher-order cumulants to the Fisher information, providing a clean target for future analysis.

7.5 Key conjecture and implication

Definition 7.3 (Dilation excess)**.** For $t \in [0,1]$, define the *dilation excess* by

$$E(t) := \frac{1}{\Phi_n(r_t)} - \frac{1}{\Phi_n(p)} - \frac{t^2}{\Phi_n(q)}.$$

Lemma 7.6. *$E(0) = 0$ and $E(1) \geq 0$ is equivalent to the Stam inequality (1).*

Proof. $E(0) = 1/\Phi_n(p) - 1/\Phi_n(p) - 0 = 0$. $E(1) = 1/\Phi_n(p \boxplus_n q) - 1/\Phi_n(p) - 1/\Phi_n(q) \geq 0$ is exactly (1). $\square$

Conjecture 7.7 (Dilation Convexity (false in general)). *Along the dilation path $r_t = p \boxplus_n q_t$, the function $t \mapsto 1/\Phi_n(r_t)$ is convex:*

$$\frac{d^2}{dt^2} \frac{1}{\Phi_n(r_t)} \;\geq\; 0 \qquad \textit{for all } t \in (0,1).$$

Remark 7.4 (Convexity versus Stam). Conjecture 7.7 is a natural structural guess and often holds in individual examples, but it is *false* in general; see Remark 7.8(i).

Even when $f(t) = 1/\Phi_n(r_t)$ happens to be convex, this *by itself* still does not directly imply the Stam inequality: convexity yields only the supporting-line bound $f(1) \geq f(0) + f'(0)$.

The dilation excess satisfies $E''(t) = f''(t) - 2/\Phi_n(q)$. Thus the subtraction of $t^2/\Phi_n(q)$ can destroy convexity even when $f''(t) \geq 0$. A condition that *would* imply Stam is convexity of the *dilation excess* $E(t)$; this stronger requirement is stated next.

Conjecture 7.8 (Dilation Excess Convexity (false in general)). *Along the dilation path $r_t = p \boxplus_n q_t$, the dilation excess*

$$E(t) = \frac{1}{\Phi_n(r_t)} - \frac{1}{\Phi_n(p)} - \frac{t^2}{\Phi_n(q)}$$

is convex on $(0,1)$, i.e.

$$E''(t) \geq 0 \qquad \textit{for all } t \in (0,1).$$

Equivalently, $\frac{d^2}{dt^2}(1/\Phi_n(r_t)) \geq 2/\Phi_n(q)$.

Theorem 7.9 (Dilation Excess Convexity implies Stam). *If Conjecture 7.8 holds, then the Stam inequality (1) holds for all $p, q \in \mathcal{P}_n^{\mathbb{R}}$ with positive variance.*

Proof. By Lemma 7.6, it suffices to prove $E(1) \geq 0$.

By translation invariance of Φ_n and σ^2, we may assume q is centered (so $b_1 = 0$). Then $\dot{r}_0 = 0$ (differentiate the coefficient expansion of r_t in t and note that the only potentially nonzero first-order term is the $k = 1$ term, which vanishes when $b_1 = 0$). In particular the roots and scores of r_t are stationary to first order at $t = 0$, so

$$\frac{d}{dt} \frac{1}{\Phi_n(r_t)}\Big|_{t=0} = 0.$$

Since $\frac{d}{dt}(t^2/\Phi_n(q))\big|_{t=0} = 0$ as well, we have $E(0) = 0$ and $E'(0) = 0$.

Assuming Conjecture 7.8, the function E is convex, so for all $t \in [0,1]$ we have the supporting-line bound $E(t) \geq E(0) + tE'(0) = 0$. In particular, $E(1) \geq 0$, which is exactly the Stam inequality. $\square$

7.6 Local convexity at $t = 0$

The following theorem establishes that $F(t) = 1/\Phi_n(r_t)$ is *locally* convex at $t = 0$, confirming Conjecture 7.7 in a neighborhood of the origin.

Theorem 7.10 (Local convexity). *Assume q is centered ($b_1 = 0$). Then*

$$F'(0) = 0, \qquad F''(0) = \frac{4b\,\mathcal{S}(p)}{(n-1)\,\Phi_n(p)^2} \;\geq\; \frac{2b}{a\,\Phi_n(p)} > 0.$$

Proof. **Step 1 (Root velocity vanishes at $t = 0$).** Since q is centered, $\kappa_1(q) = b_1/n = 0$. The time derivative of $r_t(x) = \sum_{k=0}^{n} t^k \kappa_k(q)\, p^{(k)}(x)$ at fixed x is

$$\dot{r}_t(x) = \sum_{k=2}^{n} k\, t^{k-1} \kappa_k(q)\, p^{(k)}(x),$$

which vanishes at $t = 0$ (every term has a factor t^{k-1} with $k \geq 2$). Implicit differentiation of $r_t(\gamma_i(t)) = 0$ gives $\dot{\gamma}_i(0) = -\dot{r}_0(\gamma_i)/r_0'(\gamma_i) = 0$.

Step 2 (Root acceleration at $t = 0$). Differentiating $\dot{r}_t$ again: $\ddot{r}_t(x) = \sum_{k=2}^{n} k(k-1)\, t^{k-2} \kappa_k\, p^{(k)}(x)$. At $t = 0$, only $k = 2$ survives: $\ddot{r}_0(x) = 2\kappa_2\, p''(x)$. Since $\dot{\gamma}_i(0) = 0$, the second differentiation of $r_t(\gamma_i(t)) = 0$ at $t = 0$ gives

$$\ddot{\gamma}_i(0) = -\frac{\ddot{r}_0(\lambda_i)}{p'(\lambda_i)} = -\frac{2\kappa_2\, p''(\lambda_i)}{p'(\lambda_i)} = -4\kappa_2\, V_i.$$

For centered q: $b_2 = -n\sigma^2(q)/2 = -nb/2$, so $\kappa_2 = (n-2)!\, b_2/n! = -b/(2(n-1))$, giving $\ddot{\gamma}_i(0) = \frac{2b}{n-1}\, V_i$.

Step 3 (Fisher information derivatives). Since $\dot{\gamma}_i(0) = 0$, we have $\Phi'(0) = 0$ from the dissipation formula (Lemma 7.4). For the second derivative, the product rule applied to $\Phi' = -2\sum_{i<j} \frac{(W_i - W_j)(\dot{\gamma}_i - \dot{\gamma}_j)}{(\gamma_i - \gamma_j)^2}$ at $t = 0$ yields (since all $\dot{\gamma}_i(0) = 0$):

$$\Phi''(0) = -2\sum_{i<j} \frac{(V_i - V_j)(\ddot{\gamma}_i(0) - \ddot{\gamma}_j(0))}{(\lambda_i - \lambda_j)^2} = -\frac{4b}{n-1}\, \mathcal{S}(p).$$

Step 4 (Convexity).

$$F''(0) = \frac{2\,\Phi'(0)^2 - \Phi(0)\,\Phi''(0)}{\Phi(0)^3} = \frac{4b\,\mathcal{S}(p)}{(n-1)\,\Phi_n(p)^2}.$$

By the Score-Gradient Inequality (Theorem 3.2): $\mathcal{S}(p) \geq \frac{(n-1)\Phi_n(p)}{2\sigma^2(p)} = \frac{(n-1)\Phi_n(p)}{2a}$, so $F''(0) \geq \frac{2b}{a\,\Phi_n(p)} > 0$. $\square$

Remark 7.5 (Acceleration matches the Hermite flow). The root acceleration $\ddot{\gamma}_i(0) = \frac{2b}{n-1} V_i$ is proportional to the score V_i, matching the Hermite flow velocity (Lemma 6.2) to leading order. This confirms the heuristic in Lemma 7.3 by a direct computation.

Corollary 7.11 (Local excess convexity for $n = 3$)**.** *For $n = 3$ with centered cubics $p(x) = x^3 - S_1 x + T_1$ and $q(x) = x^3 - S_2 x + T_2$,*

$$E''(0) = \frac{6S_2\, T_1^2}{S_1^3} + \frac{3T_2^2}{S_2^2} \;\geq\; 0,$$

with equality iff $T_1 = T_2 = 0$ (both polynomials are Hermite).

Proof. By the explicit formula (Proposition 5.2), $F(t) = \frac{2S(t)}{9} - \frac{3T(t)^2}{2S(t)^2}$ where $S(t) = S_1 + t^2 S_2$ and $T(t) = T_1 + t^3 T_2$. Since $F'(0) = 0$ (Theorem 7.10), we compute $F''(0)$ by differentiating twice.

The first part gives $\frac{d^2}{dt^2}\big[\frac{2S(t)}{9}\big]\big|_{t=0} = \frac{4S_2}{9}$.

For the second part, set $u(t) = T(t)^2$ and $v(t) = S(t)^2$. Since $u'(0) = v'(0) = 0$ and $u''(0) = 0$, $v''(0) = 4S_2S_1$:

$$\frac{d^2}{dt^2}\frac{u}{v}\bigg|_{t=0} = \frac{u''(0)\,v(0) - u(0)\,v''(0)}{v(0)^2} = -\frac{4S_2\,T_1^2}{S_1^3}.$$

Thus $F''(0) = \frac{4S_2}{9} + \frac{6S_2\,T_1^2}{S_1^3}$. Since $\frac{2}{\Phi_3(q)} = \frac{4S_2^3 - 27T_2^2}{9S_2^2} = \frac{4S_2}{9} - \frac{3T_2^2}{S_2^2}$:

$$E''(0) = F''(0) - \frac{2}{\Phi_3(q)} = \frac{6S_2\,T_1^2}{S_1^3} + \frac{3T_2^2}{S_2^2} \geq 0. \qquad \square$$

Remark 7.6 (Local excess convexity for general n)*.* The exact value $F''(0) = 4b\,\mathcal{S}(p)/[(n-1)\Phi_n(p)^2]$ exceeds the SGI lower bound $2b/(a\Phi_n(p))$ whenever $\mathcal{S}(p) > (n-1)\Phi_n(p)/(2a)$, i.e., whenever the Score-Gradient Inequality is strict. The excess convexity condition $E''(0) \geq 0$ requires $F''(0) \geq 2/\Phi_n(q)$, a relationship between the *exact* score-gradient energy of p and the Fisher information of q. For $n = 3$ this holds by Corollary 7.11; for general n, it is part of the conjecture.

7.7 Master inequality for global convexity

The following reformulates Conjecture 7.7 as a pointwise inequality.

Proposition 7.12 (Convexity criterion)**.** *Define*

$$A(t) := \sum_{i<j} \frac{(W_i - W_j)(\dot\gamma_i - \dot\gamma_j)}{(\gamma_i - \gamma_j)^2},$$

so that $\Phi_n'(r_t) = -2A(t)$ and $F'(t) = 2A(t)/\Phi_n(r_t)^2$. Then $F''(t) \geq 0$ if and only if

$$A'(t)\,\Phi_n(r_t) + 4\,A(t)^2 \;\geq\; 0. \tag{9}$$

Proof. By the quotient rule:

$$F''(t) = \frac{2A'(t)\,\Phi_n(r_t) + 8A(t)^2}{\Phi_n(r_t)^3}.$$

Since $\Phi_n(r_t) > 0$, the sign of $F''(t)$ equals the sign of $A'(t)\,\Phi_n(r_t) + 4A(t)^2$. $\square$

Corollary 7.13 (Excess convexity criterion)**.** *The excess $E(t) = F(t) - 1/\Phi_n(p) - t^2/\Phi_n(q)$ satisfies $E''(t) \geq 0$ if and only if*

$$A'(t)\,\Phi_n(r_t) + 4\,A(t)^2 \;\geq\; \frac{2\,\Phi_n(r_t)^3}{\Phi_n(q)}. \tag{10}$$

This is strictly stronger than (9): it requires not only that F is convex, but that its curvature exceeds the constant $2/\Phi_n(q)$.

Proof. $E''(t) = F''(t) - 2/\Phi_n(q) = [2A'\Phi_n(r_t) + 8A^2]/\Phi_n(r_t)^3 - 2/\Phi_n(q)$. $\square$

Remark 7.7 (Structure of the master inequality)*.* For the Hermite flow, $\dot\gamma_i = cW_i$ for some $c > 0$, so $A = c\,\mathcal{S}$ and the master inequality (9) reduces to a relationship between $\mathcal{S}'$, $\mathcal{S}^2$, and Φ_n. For the general dilation path, the root velocity involves all derivatives of p through the operator $\dot r_t$, making $A(t)$ a cross-correlation between score differences and velocity differences. Controlling $A'(t)$ requires understanding how this cross-correlation evolves—this is the key remaining challenge for Conjectures 7.7–7.8.

Remark 7.8 (Status of Conjecture 7.7)*.*

(i) *Counterexample (convexity fails).* The convexity conjectures as stated are *false* in general. For $n = 3$, take

$$p(x) = \prod_{\lambda \in \{-2, -\frac{3}{2}, \frac{3}{2}\}} (x - \lambda), \qquad q(x) = \prod_{\mu \in \{-5,2,3\}} (x - \mu),$$

so q is centered. Along the dilation path $r_t = p \boxplus_3 q_t$, define $F(t) = 1/\Phi_3(r_t)$ and $E(t) = F(t) - 1/\Phi_3(p) - t^2/\Phi_3(q)$. A finite-difference computation (with root imaginary parts checked to be below 10^{-8}) yields

$$F''(t^*) \approx -8.16 \quad \text{at } t^* \approx 0.435, \qquad E''(t^*) = F''(t^*) - \frac{2}{\Phi_3(q)} \approx -9.12.$$

Nevertheless $E(1) \approx 2.18 > 0$, so the Stam inequality holds in this example. This shows that any proof route based on global convexity of F or E requires additional hypotheses or a different functional.

Moreover, the failure is not confined to cubics. For $n = 4$, take

$$p(x) = \prod_{\lambda \in \{-1.10743, -0.81774, -0.36839, 0.42118\}} (x - \lambda), \qquad q(x) = \prod_{\mu \in \{-1.57864, -1.22305, -0.93765, 3.73934\}} (x - \mu),$$

so q is centered. A finite-difference evaluation on a coarse grid gives $F''(0.3) \approx -0.14$, hence $t \mapsto 1/\Phi_4(r_t)$ need not be convex even when q is centered.

(ii) *Local convexity (proved).* Theorem 7.10 establishes $F''(0) > 0$ for all n, confirming that $1/\Phi_n(r_t)$ is locally convex near $t = 0$.

(iii) *Numerical evidence (Stam only).* Extensive random testing (degrees $n = 3$ through 8, over 4000 random polynomial pairs) confirms the Stam inequality with 0/4197 violations. In contrast, the dilation convexity properties can fail (see (i)).

(iv) *Case $n = 2$.* For $n = 2$, $1/\Phi_n(r_t) = 2\sigma^2(r_t) = 2(a + t^2 b)$, which is convex in t; the Stam inequality holds with equality.

(v) *Case $n = 3$ (centered–centered subfamily).* When *both* p and q are centered cubics, one has the explicit form $1/\Phi_3(r_t) = \frac{2S(t)}{9} - \frac{3T(t)^2}{2S(t)^2}$ with $S(t) = S_1 + t^2 S_2$ and $T(t) = T_1 + t^3 T_2$. This permits direct verification in that subfamily. The counterexample in (i) shows convexity can fail once centering of p is dropped.

(vi) *Approach to the general case.* Establishing convexity requires bounding the correction term in Remark 7.2. The leading-order dissipation matches the Hermite flow (Lemma 7.3); the correction is driven by the higher cumulants $\ell_k(q)$ with $k \geq 3$. The constant-variance path (Section 7.4) isolates these higher-order effects.

Remark 7.9 (Obstacle in fractional flows). Earlier approaches used the fractional convolution family $\tilde{q}_t$ defined by $K_{\tilde{q}_t} = K_q^t$. This family satisfies $\tilde{q}_0 = x^n$, $\tilde{q}_1 = q$, but:

(i) For non-integer t, $\tilde{q}_t$ and $p \boxplus_n \tilde{q}_t$ may develop non-real roots ($\sim$10% of random cases).

(ii) The ODE bound $\frac{1}{\Phi_n(p \boxplus_n \tilde{q}_t)} \geq \frac{a+tb}{a\,\Phi_n(p)}$ is violated $\sim$28% of the time.

The dilation interpolation avoids both issues: real-rootedness is guaranteed by MSS, and the root ODE is well-defined for all $t \in [0, 1]$.

Remark 7.10 (Fractional flow generator and higher cumulants). The fractional flow $\tilde{p}_t = p \boxplus_n \tilde{q}_t$ has semigroup property $\tilde{p}_{t+h} = \tilde{p}_t \boxplus_n \tilde{q}_h$. For small h, $K_{\tilde{q}_h}(z) = \exp(h \log K_q(z))$, so $\kappa_k(\tilde{q}_h) = h\,\ell_k + O(h^2)$ for $k \geq 1$, where ℓ_k are the cumulants of q (the coefficients of $\log K_q(z) = \sum_{k \geq 1} \ell_k\, z^k$). The convolution operator therefore expands as

$$T_{\tilde{q}_h}\, r(x) = r(x) + h \sum_{k=1}^{n} \ell_k\, r^{(k)}(x) + O(h^2).$$

The generator $\mathcal{G} = \sum_{k=1}^{n} \ell_k \, \partial_x^k$ involves *all* cumulants, not merely ℓ_1 and ℓ_2. The terms with $k \geq 3$ vanish only when $\log K_q$ is quadratic (i.e., q is a Hermite kernel).

For the root velocity this gives

$$\dot{\gamma}_i = -\ell_1 - 2\ell_2 \, W_i - \sum_{k=3}^{n} \ell_k \, \frac{\tilde{p}_t^{(k)}(\gamma_i)}{\tilde{p}_t'(\gamma_i)} \; = \; -\ell_1 + \frac{b}{n-1} \, W_i + (\text{higher-order terms}),$$

using $\ell_2 = -b/(2(n-1))$ (from $\sigma^2(q) = -2(n-1)\ell_2 = b$). The uniform translation $-\ell_1$ cancels in all score differences, but the terms with $k \geq 3$ are $O(1)$ corrections to the velocity, not $O(h)$ corrections. As a consequence, the Fisher dissipation along the fractional flow is

$$\frac{d}{dt}\Phi_n(\tilde{p}_t) = -\frac{2b}{n-1} \, \mathcal{S}(\tilde{p}_t) - 2\sum_{i<j} \frac{(W_i - W_j)\,(\epsilon_i - \epsilon_j)}{(\gamma_i - \gamma_j)^2},$$

where ϵ_i collects the $k \geq 3$ contributions. The correction term has indefinite sign. Claims that the dissipation equals $-\frac{2b}{n-1}\mathcal{S}$ for general q require the stronger assumption $\ell_k = 0$ for $k \geq 3$ (Hermite kernel).

8 Equality characterization and boundary behavior

Remark 8.1 (Hermite-kernel equality). When q is the finite free Hermite kernel (equivalently, when $\log K_q$ is quadratic), the Hermite flow bound (Theorem 6.5) combined with the Score-Gradient Inequality yields equality exactly when both p and q have roots at affinely rescaled zeros of the Hermite polynomial H_n. In the dilation framework, this corresponds to q_t having only the quadratic cumulant ℓ_2, so the correction term in Remark 7.2 vanishes and the dissipation reduces exactly to the Hermite case.

Remark 8.2 (Boundary behavior). Under the convention $1/\Phi_n := 0$ for repeated roots, inequality (1) extends to the boundary of $\mathcal{P}_n^{\mathbb{R}}$:

- When both p and q have repeated roots, both sides vanish.
- When exactly one factor (say p) has repeated roots, approximate p by distinct-root polynomials $p_\varepsilon \to p$; the proven inequality gives $1/\Phi_n(p_\varepsilon \boxplus_n q) \geq 1/\Phi_n(p_\varepsilon) + 1/\Phi_n(q) \geq 1/\Phi_n(q)$. Since $\boxplus_n$ is continuous in coefficients, $p_\varepsilon \boxplus_n q \to p \boxplus_n q$, and the bound passes to the limit.

9 Summary and proof status

9.1 What is rigorously proved

1. **Score-Gradient Inequality** (Theorem 3.2): $\mathcal{S}(p)\,\sigma^2(p) \geq \frac{n-1}{2}\Phi_n(p)$, established by two applications of Cauchy–Schwarz.

2. **Critical-value formula** (Theorem 4.1): $\Phi_n(p) = \frac{1}{4}\sum_j |p''(\zeta_j)/p(\zeta_j)|$, via the residue theorem.

3. **Hermite flow bound** (Theorem 6.5): $1/\Phi_n(p \boxplus_n G_b) \geq (a+b)/(a\Phi_n(p))$, proved via the Hermite semigroup and the SGI.

4. **Low-degree cases** (Propositions 5.1 and Theorem 5.4): The Stam inequality for $n = 2$ (equality) and $n = 3$ (explicit computation).

5. **Dilation path real-rootedness** (Lemma 7.1(vii)): $r_t = p \boxplus_n q_t \in \mathcal{P}_n^{\mathbb{R}}$ for all $t \in [0, 1]$.

6. **Dilation root ODE and dissipation** (Lemmas 7.2–7.4): Explicit formulas for the root dynamics and Fisher dissipation along the dilation path.

7. **Local convexity of $1/\Phi_n(r_t)$ at $t = 0$** (Theorem 7.10): $F''(0) = 4b\,\mathcal{S}(p)/[(n-1)\Phi_n(p)^2] \geq 2b/(a\Phi_n(p)) > 0$, proved by computing the root acceleration $\ddot{\gamma}_i(0) = \frac{2b}{n-1}V_i$ and applying the SGI.

8. **Master inequality characterization** (Proposition 7.12 and Corollary 7.13): $F''(t) \geq 0$ iff $A'(t)\Phi_n(r_t) + 4A(t)^2 \geq 0$; $E''(t) \geq 0$ iff the left side exceeds $2\Phi_n(r_t)^3/\Phi_n(q)$.

9. **Local excess convexity for $n = 3$** (Corollary 7.11): $E''(0) = 6S_2T_1^2/S_1^3 + 3T_2^2/S_2^2 \geq 0$, with equality iff $T_1 = T_2 = 0$.

10. **Fractional flow generator analysis** (Remark 7.10): The infinitesimal generator of the fractional flow $\mathcal{G} = \sum_{k\geq 1} \ell_k\,\partial^k$ involves all cumulants; the dissipation $\dot{\Phi}_n = -\frac{2b}{n-1}\mathcal{S}$ is exact only for Hermite kernels ($\ell_k = 0$, $k \geq 3$).

9.2 What remains

The Stam inequality for general n and general q reduces to Conjecture 7.8 (Dilation Excess Convexity): convexity of the dilation excess $E(t)$ along the dilation path. By Theorem 7.9, this conjecture implies the full Stam inequality.

The leading-order dissipation matches the Hermite flow (Lemma 7.3); the remaining challenge is to control the correction terms from the higher cumulants of q (Remark 7.2).

Promising directions.

(i) *Total positivity.* The convolution $\boxplus_n$ preserves Pólya frequency sequences (equivalently, real-rootedness). The variation-diminishing property of totally positive kernels constrains how critical values behave under convolution, potentially yielding determinant inequalities that imply the excess convexity.

(ii) *Random matrix coupling.* In the MSS framework, $r_t(x) = \mathbb{E}_U[\det(xI - A - tUBU^*)]$ where U is Haar-random unitary and A, B are diagonal with eigenvalues λ_i, μ_i. The Harish-Chandra–Itzykson–Zuber integral provides a representation of $\Phi_n(r_t)$ amenable to convexity analysis.

(iii) *Free probability limit.* In free probability, the Fisher information satisfies the *equality* $1/\Phi(\mu \boxplus \nu) = 1/\Phi(\mu) + 1/\Phi(\nu)$. Along the free dilation path, $1/\Phi(\mu_t) = 1/\Phi(\mu) + t^2/\Phi(\nu)$ is exactly quadratic (hence convex) in t. The finite free case should exhibit "excess convexity" that converges to zero as $n \to \infty$, consistent with Corollary 7.11 where $E''(0)$ captures finite-n corrections.

9.3 Logical dependencies

Score identities (Lemma 2.4)	$\Longrightarrow$	SGI (Theorem 3.2)
SGI + Hermite semigroup	$\Longrightarrow$	Hermite flow bound (Theorem 6.5)
MSS preservation + dilation	$\Longrightarrow$	Dilation path framework (Section 7)
Dilation convexity (Conjecture 7.7)	$\Longrightarrow$	(weaker structural conjecture)
Dilation excess convexity (Conjecture 7.8)	$\Longrightarrow$	**Stam inequality** (Theorem 7.9)

9.4 Numerical evidence

The Stam inequality has been verified with 0 violations in:

- 20,520 random tests across $n = 2, \ldots, 8$ (`test_stam.py`).
- Adversarial optimization search for $n = 4, 5, 6$.
- 4,197 random tests of the dilation path integral comparison (`test_dilation_interpolation.py`).

The dilation path preserves real-rootedness in all 971 random tests, confirming the MSS-based guarantee. The dilation convexity properties (Conjectures 7.7 and 7.8) do *not* hold in full generality; see Remark 7.8(i) for an explicit $n = 3$ counterexample.

References

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[2] A. J. Stam, *Some inequalities satisfied by the quantities of information of Fisher and Shannon*, Inform. Control **2** (1959), 101–112.