The Denavit-Hartenberg Model

We take the coordinates with respect to frame i as $\begin{pmatrix} x_i \\ y_i \\ z_i \\ 1 \end{pmatrix}$

- **We can reduce the required number of transformation parameters from six to four.** These four parameters have been given names as delineated below:
  - **Joint Angle,** θ_i: The angle by which we need to rotate about the z_{i-1} axis to align the x_{i-1} axis with the x_i axis.
  - **Link Offset,** d_i: The displacement required along the z_{i-1} axis to cause the already aligned x_{i-1} and x_i axes to come onto the same line/intersect.
  - **Link Length,** a_i: The displacement along the x_i axis so that the two origins O_{i-1} and O_i coincide.
  - **Link Twist,** α_i: The rotation angle about the x_i axis to align the z_{i-1} and z_i axes. There we go! We have completed the transition from the *frame i-1* to *frame i*. The four parameters θ_i, i, i, and α_i are associated with the transformation from the *frame i* to *frame i-1*.

We define $A_i^{i-1}(\theta_i, d_i, a_i, \alpha_i)$ to be the matrix which transforms the coordinate system from frame i to frame $i-1$.

$$\begin{pmatrix} x_{i-1} \\ y_{i-1} \\ z_{i-1} \\ 1 \end{pmatrix} = R_z(\theta_i) \cdot O_z(d_i) \cdot O_x(d_i) \cdot R_x(\alpha_i) \cdot \begin{pmatrix} x_i \\ y_i \\ z_i \\ 1 \end{pmatrix}$$

$$A_i^{i-1}(\theta_i, d_i, a_i, \alpha_i) = R_z(\theta_i) \cdot O_z(d_i) \cdot O_x(d_i) \cdot R_x(\alpha_i)$$

where

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ R_z(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$O_x(d) = \begin{pmatrix} 1 & 0 & 0 & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; O_z(d) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This gives

$$A_i^{i-1}(\theta_i, d_i, a_i, \alpha_i) = \begin{pmatrix} \cos(\theta_i) & -\cos(\alpha_i)\sin(\theta_i) & \sin(\alpha_i)\sin(\theta_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\alpha_i)\cos(\theta_i) & -\sin(\alpha_i)\cos(\theta_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 1

In 2D, there can only be one rotation matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, for the transformation matrix which converts the coordinate system from link 0 to link 1 is:

$$
\begin{aligned}
\begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix} &= R(\theta_1) \cdot O_x(a_1) \cdot \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} \\
&= O_x^{-1}(a_1) \cdot R^{-1}(\theta_1) \cdot \begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix} \\
&= O_x(-a_1) \cdot R(-\theta_1) \cdot \begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -a_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) & -a_1 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix}
\end{aligned}
$$

Similarly, for converting the coordinate system from link 1 to link 2 is:

$$
\begin{aligned}
\begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} &= R(\theta_2) \cdot O_x(a_2) \cdot \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} \\
&= O_x^{-1}(a_2) \cdot R^{-1}(\theta_2) \cdot \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} \\
&= O_x(-a_2) \cdot R(-\theta_2) \cdot \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -a_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & \sin(\theta_2) & 0 \\ -\sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta_2) & \sin(\theta_2) & -a_2 \\ -\sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix}
\end{aligned}
$$

Hence, the transformation matrix for transforming link 0 coordinate system to link

2 coordinate system:

$$A_0^2(\theta_1, \theta_2, a_1, a_2) = \begin{pmatrix} \cos(\theta_2) & \sin(\theta_2) & -a_2 \\ -\sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) & -a_1 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_0^2(\theta_1, \theta_2, a_1, a_2) = \begin{pmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & -a_1\cos(\theta_2) - a_2 \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & a_1\sin(\theta_2) \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & -a_1\cos(\theta_2) - a_2 \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & a_1\sin(\theta_2) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix}$$

Note that

$$\begin{aligned} A_2^0(\theta_1, \theta_2, a_1, a_2) &= (A_0^2(\theta_1, \theta_2, a_1, a_2))^{-1} \\ &= \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & a_1\cos(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) & a_1\sin(\theta_1) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) & a_2\cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & a_2\sin(\theta_2) \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & a_2\cos(\theta_1+\theta_2) + a_1\cos(\theta_1) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & a_2\sin(\theta_1+\theta_2) + a_1\sin(\theta_1) \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & a_2\cos(\theta_1+\theta_2) + a_1\cos(\theta_1) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & a_2\sin(\theta_1+\theta_2) + a_1\sin(\theta_1) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix}$$

Hence, the coordinates of origin of link 2 frame in base frame is:

$$\begin{aligned} \begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix} &= \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & a_2\cos(\theta_1+\theta_2) + a_1\cos(\theta_1) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & a_2\sin(\theta_1+\theta_2) + a_1\sin(\theta_1) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} a_2\cos(\theta_1+\theta_2) + a_1\cos(\theta_1) \\ a_2\sin(\theta_1+\theta_2) + a_1\sin(\theta_1) \\ 1 \end{pmatrix} \end{aligned}$$

Hence,

$$\boxed{\begin{aligned} x_0 &= a_1\cos(\theta_1) + a_2\cos(\theta_1+\theta_2) \\ y_0 &= a_1\sin(\theta_1) + a_2\sin(\theta_1+\theta_2) \end{aligned}}$$

Problem 2

Joint 1 is a revolute joint.
Joint 2 is a prismatic joint.
Joint 3 is a prismatic joint.

$$A_1^0(\theta_1, 0, 0, 0) = \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_2^1\left(0, d_2, 0, -\frac{\pi}{2}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_3^2(0, d_3, 0, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence, the equation for transforming the end-effector coordinate system to base frame coordinate system is:

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} = A_1^0 \cdot A_2^1 \cdot A_3^2 \cdot \begin{pmatrix} x_3 \\ y_3 \\ z_3 \\ 1 \end{pmatrix}$$

This gives

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1) & 0 & -\sin(\theta_1) & -d_3\sin(\theta_1) \\ \sin(\theta_1) & 0 & \cos(\theta_1) & d_3\cos(\theta_1) \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \\ 1 \end{pmatrix}$$

$$\boxed{A_3^0 = \begin{pmatrix} \cos(\theta_1) & 0 & -\sin(\theta_1) & -d_3\sin(\theta_1) \\ \sin(\theta_1) & 0 & \cos(\theta_1) & d_3\cos(\theta_1) \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

Problem 3

$$A_4^3\left(\frac{\pi}{2}+\theta_4, 0, 0, -\frac{\pi}{2}\right) = \begin{pmatrix} -\sin(\theta_4) & 0 & -\cos(\theta_4) & 0 \\ \cos(\theta_4) & 0 & -\sin(\theta_4) & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_5^4\left(\theta_5-\frac{\pi}{2}, 0, 0, \frac{\pi}{2}\right) = \begin{pmatrix} \sin(\theta_5) & 0 & -\cos(\theta_5) & 0 \\ -\cos(\theta_5) & 0 & -\sin(\theta_5) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_6^5(\theta_6, d, 0, 0) = \begin{pmatrix} \cos(\theta_6) & -\sin(\theta_6) & 0 & 0 \\ \sin(\theta_6) & \cos(\theta_6) & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformation matrix A_6^3 is given by:

$$\begin{pmatrix} x_3 \\ y_3 \\ z_3 \\ 1 \end{pmatrix} = A_6^3 \begin{pmatrix} x_6 \\ y_6 \\ z_6 \\ 1 \end{pmatrix} = A_4^3 \cdot A_5^4 \cdot A_6^5 \begin{pmatrix} x_6 \\ y_6 \\ z_6 \\ 1 \end{pmatrix}$$

$$\boxed{A_6^3 = \begin{pmatrix} -\mathrm{s4s5c6} - \mathrm{c4s6} & \mathrm{s4s5s6} - \mathrm{c4c6} & \mathrm{s4c5} & d \cdot \mathrm{s4c5} \\ \mathrm{c4s5c6} - \mathrm{s4s6} & -\mathrm{c4s5s6} - \mathrm{s4c6} & -\mathrm{c4c5} & -d \cdot \mathrm{c4c5} \\ \mathrm{c5c6} & -\mathrm{c5s6} & \mathrm{s5} & d \cdot \mathrm{s5} \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

Problem 4

The final transformation matrix is just the product of the answers from Problem 2 and Problem 3:

$$A_6^0 = A_3^0 \cdot A_6^3$$