Complex Analysis

Chapter I

Preliminaries to Complex Analysis

1. Complex numbers.

$z = x + iy$, $x = \mathrm{Re}(z)$ real part, $y = \mathrm{Im}(z)$ imaginary part

$x = 0$: purely imaginary
$y = 0$: real

1.1. Addition and multiplication.

$$z_1 + z_2 \stackrel{def}{=} (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 \cdot z_2 \stackrel{def}{=} (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) = \langle \bar{z}_1, z_2\rangle + i[\bar{z}_1, z_2],$$

$\langle z_1, z_2\rangle \stackrel{def}{=} x_1x_2 + y_1y_2$
$[z_1, z_2] \stackrel{def}{=} x_1y_2 - x_2y_1$

Properties:

- addition : addition of vectors in $\mathbb{C}$
- multipl. : rotation (proper orth. tr.) + dilation (homothety)
- commutativity, associativity, distributivity
- $\mathbb{C} \simeq \mathbb{R}^2$ as a vector space $(1 \mapsto (1,0), i \mapsto (0,1))$

1.2. Absolute value.

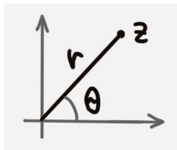
$$|z| = \sqrt{x^2 + y^2}$$

Properties:

- triangle inequality : $|z_1 + z_2| \leq |z_1| + |z_2|$
- $|\mathrm{Re}(z)| \leq |z|$ $\quad|\mathrm{Im}(z)| \leq |z|$
- $||z_1| - |z_2|| \leqslant |z_1 - z_2|$

1.3. Polar form.

$$z = re^{i\theta}, \quad r > 0, \; z \neq 0$$



Properties:

- $r = |z|$ $\quad \theta = \arg z$
- $re^{i\theta} = r(\cos\theta + i\sin\theta)$
- $z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
- $z^n = r^n(\cos n\theta + i \sin n\theta)$
- $\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i \sin\frac{\theta + 2\pi k}{n}\right), k \in \mathbb{Z}$

1.4. Complex conjugate.

$$\bar{z} = x - iy$$

Properties:

- comp. conj. : reflection across the real axis
- $\mathrm{Re}(z) = 0$: $\bar{z} = -z$ $\quad \mathrm{Im}(z) = 0$: $\bar{z} = z$
- $x = \frac{z + \bar{z}}{2}$ $\quad y = \frac{z - \bar{z}}{2i}$
- $|z|^2 = z \cdot \bar{z}$
- $\frac{1}{z} = \frac{1}{|z|^2}\bar{z}$ $(z \neq 0)$

1.5. Sequences.

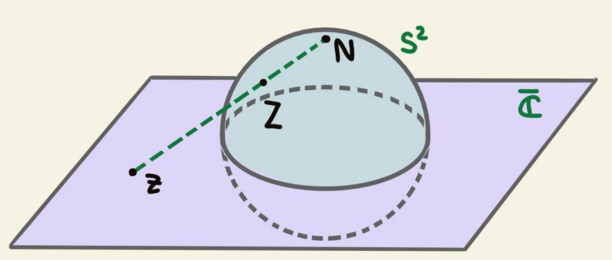
$\{z_n\}$ converges to ω : $\lim_{n\to\infty} |z_n - \omega| = 0$, i.e. $\omega = \lim_{n\to\infty} z_n$

- $z_n \to \omega \iff$ sequence of points in $\mathbb{C}$ conv. to ω
- $z_n \to \omega \iff x_n \to \mathrm{Re}(\omega)$ and $y_n \to \mathrm{Im}(\omega)$ (since $\max\{|x_n - x_\omega|, |y_n - y_\omega|\} \leqslant |z_n - \omega| \leqslant \leqslant |x_n - x_\omega| + |y_n - y_\omega|$)
- $\{z_n\}$ is a Cauchy sequence : $|z_n - z_m| \to 0$ as $n, m \to \infty$

Th.1.1:
$\mathbb{C}$ is complete, i.e. every Cauchy seq. converges.

1.6. Stereographic projection.

One can compactify $\mathbb{C}$ by adding the point $z = \infty$ to it (however, algebraic operations are not defined for it). To visually represent the compactified complex plane $\overline{\mathbb{C}}$ one can use the stereographic projection on the Riemann sphere : $\underset{z}{(x, y)} \mapsto \underset{Z}{(\xi, \eta, \zeta)}$. Then the point of infinity corresponds to the north pole of S^2.



S^2: $\xi^2 + \eta^2 + \zeta^2 = 1$

$\overline{\mathbb{C}} \to S^2$: $\xi = \frac{2x}{1+|z|^2}$ $\eta = \frac{2y}{1+|z|^2}$ $\zeta = \frac{|z|^2 - 1}{1+|z|^2}$

$S^2 \to \mathbb{C}$: $x = \frac{\xi}{1-\zeta}$ $y = \frac{\eta}{1-\zeta}$

(2) Topology of $\mathbb{C}$.

2.1. Paths and curves.

$\gamma: I \to \mathbb{C}$ is a **curve**: $\gamma \in C^0(I)$, where $I \subset \mathbb{R}$ is a non-degenerate interval

$\gamma: [a,b] \to \mathbb{C}$ is a **path**: γ is a curve with $I = [a,b]$, $\gamma(a)$: initial point, $\gamma(b)$: terminal point

$\gamma: [a,b] \to \mathbb{C}$ is a **loop** (closed curve): γ is a path s.t. $\gamma(a) = \gamma(b)$

curve $\gamma: I \to \mathbb{C}$ is **simple**: γ is bijective (i.e. no self-intersections and missing points)

curve $\gamma: I \to \mathbb{C}$ is **Jordan**: γ is simple and closed

curve $\gamma: I \to \mathbb{C}$ is **smooth**: $\gamma \in C^1(I)$

curve $\gamma: I \to \mathbb{C}$ is **regular**: $\gamma \in C^1(I)$ and $\gamma'(t) \neq 0 \ \forall t \in I$

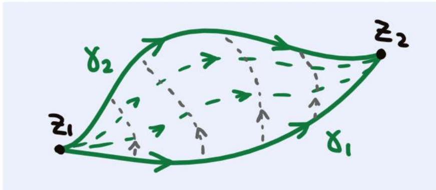
curve $\gamma: I \to \mathbb{C}$ is **piecewise-smooth**: $\gamma \in C^1([a_k, a_{k+1}])$, where $a = a_1 < \dots < a_n = b$

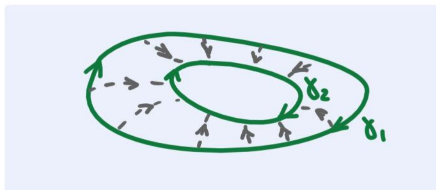
$\gamma_1: I \to \mathbb{C}$, $\gamma_2: J \to \mathbb{C}$ are **equivalent**: $\exists\, p: J \to I$ (reparametrization) s.t.
1. p is a C^1-diffeomorphism
2. $\gamma_2(\tau) = \gamma_1(p(\tau))$
3. $p'(\tau) > 0 \ \forall \tau \in J$ (i.e. orientation is preserved)

$\gamma_1: I \to \mathbb{C}$, $\gamma_2: I \to \mathbb{C}$ (common ends) are **homotopic** $\gamma_1 \sim \gamma_2$:
1. $\gamma_1(a) = \gamma_2(a) = z_1 \quad \gamma_1(b) = \gamma_2(b) = z_2$
2. $\exists$ cont. $H: \underset{t}{I} \times \underset{s}{[0,1]} \to \mathbb{C}$ s.t. 1) $H(t,0) = \gamma_1(t) \quad H(t,1) = \gamma_2(t)$; 2) $H(a,s) = z_1 \quad H(b,s) = z_2$

$\gamma_1: I \to \mathbb{C}$, $\gamma_2: I \to \mathbb{C}$ (closed) are **homotopic** $\gamma_1 \sim \gamma_2$:
1. $\gamma_1(a) = \gamma_1(b) \quad \gamma_2(a) = \gamma_2(b)$
2. $\exists$ cont. $H: \underset{t}{I} \times \underset{s}{[0,1]} \to \mathbb{C}$ s.t. 1) $H(t,0) = \gamma_1(t) \quad H(t,1) = \gamma_2(t)$; 2) $H(a,s) = H(b,s)$

- the relation of reparemetrization defines equivalence classes of curves (arcs).
- the relation of homotopy defines equivalence classes of curves (homotopy types).





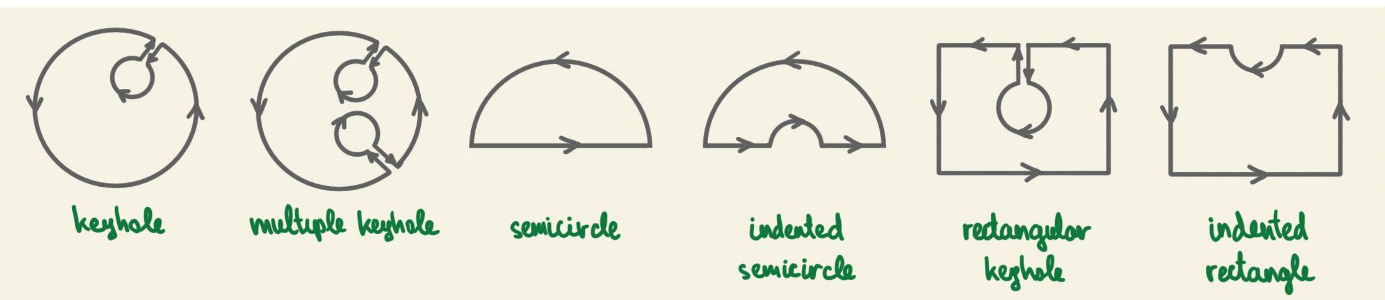
e.g. $H(t,s) = (1-s)\gamma_1(t) + s\,\gamma_2(t)$

Examples:

1) $C_r(z_0)$: $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi)$ pos. orientation

$\gamma(t) = z_0 + re^{-it}$, $t \in [0, 2\pi)$ neg. orientation

2) Toy contours (will be useful later):



2.2. Domains and compact sets.

$D_r(z_0) \overset{def}{=} \{z\in\mathbb{C} : |z-z_0|<r,\ r>0\ z_0\in\mathbb{C}\}$ open disc

$\overline{D}_r(z_0) \overset{def}{=} \{z\in\mathbb{C} : |z-z_0|\le r,\ r>0\ z_0\in\mathbb{C}\}$ closed disc

$C_r(z_0) \overset{def}{=} \{z\in\mathbb{C} : |z-z_0|=r,\ r>0\ z_0\in\mathbb{C}\}$ circle (boundary of a disk)

$\mathbb{D} \overset{def}{=} \{z\in\mathbb{C} : |z|<1\}$ unit disc

$U(z_0)$: neighbourhood of z_0

$\mathring{U}(z_0)$: deleted/punctured neighbourhood of z_0

z_0 is an interior point of $\Omega\subset\mathbb{C}$: $\exists D_r(z_0)\subset\Omega$

z_0 is an exterior point of $\Omega\subset\mathbb{C}$: z_0 is an int. point of $\Omega^c=\mathbb{C}\setminus\Omega$

z_0 is a limit point of $\Omega\subset\mathbb{C}$: $\forall D_r(z_0)$ $\Omega\cap D_r(z_0)$ is an inf. set

(interior (exterior) of $\Omega\subset\mathbb{C}$: all int. (ext.) points of Ω)

Ω is open : $\forall z\in\Omega$ is an int. point

Ω is closed : $\Omega^c=\mathbb{C}\setminus\Omega$ is open

closure of Ω : $\overline{\Omega} \overset{def}{=} \Omega\cup\{\text{all limit points of } \Omega\}$

boundary of Ω : $\partial\Omega \overset{def}{=} \overline{\Omega}\setminus\Omega$

Ω is bounded : $\exists M>0 : \forall z\in\Omega\ |z|<M$ (or $\operatorname{diam}(\Omega)\overset{def}{=}\sup\limits_{z,\omega\in\Omega}|z-\omega|<\infty$)

Ω is compact : Ω is closed and bounded

Ω is connected : $\nexists\ \Omega_1,\Omega_2 : \Omega=\Omega_1\cup\Omega_2$, where Ω_1,Ω_2 are
1. disjoint
2. both open (closed)
3. non-empty

Ω is path-connected : $\forall z_1,z_2\in\Omega\ \exists\gamma:[0,1]\to\Omega : \gamma(0)=z_1\ \gamma(1)=z_2$

Ω is simply-connected : Ω is connected and $\forall$ closed $\gamma\subset\Omega : \gamma\sim 0$

Ω is multiply-connected : Ω is connected but not simply-connected

Ω is a domain (region) : Ω is open and simply-connected

Propositions:

- $\Omega\subset\mathbb{C}$ is compact $\iff$ $\forall\{z_n\}\subset\Omega$ has a subseq. that converges to $z_0\in\Omega$.
- $\Omega\subset\mathbb{C}$ is compact $\iff$ $\forall$ open covering of Ω has a finite subcovering.
  ($\{U_\alpha\}$ is an open covering of Ω : $\Omega\subset\bigcup\limits_\alpha U_\alpha$, where U_α are open)
- If $\{\Omega_n\}$ is a sequence of non-empty compact nested sets s.t. $\operatorname{diam}(\Omega_n)\to 0$, then $\exists!\ z_0\in\mathbb{C} : z_0\in\Omega_n\ \forall n$.
- Ω is path-connected $\Rightarrow$ Ω is connected.
- Ω is simply-connected $\Rightarrow$ $\partial\Omega$ is connected.
- Ω is a domain $\Rightarrow$ $\partial\Omega$ is closed.

3. Functions on $\mathbb{C}$.

3.1. Complex functions.

$$f:\mathbb{C}\to\mathbb{C} \iff \begin{cases} u:\mathbb{C}\to\mathbb{R} \\ v:\mathbb{C}\to\mathbb{R}\end{cases} \overset{f(z)\neq 0}{\underset{f(z)\neq\infty}{\iff}} \begin{cases} \rho:\mathbb{C}\to\mathbb{R}^+ \\ \psi:\mathbb{C}\to[0,2\pi)\end{cases}$$

complex functions can be multi-valued

$$f(z)=u+iv \qquad f(z)=\rho e^{i(\psi+2\pi k)},\ k=0,\pm1,\pm2,\dots$$

3.2. Continuity.

Consider $f:\Omega\to\mathbb{C}$, $\Omega\subset\mathbb{C}$, then using the usual definition of a limit of a function ($\lim\limits_{z\to z_0} f(z)=A: \forall V(A)\ \exists \mathring{U}(z_0): f(\mathring{U}(z_0))\subset V(A)$) we can introduce the notion of continuity:

f is continious at $z_0\in\Omega$: $\lim\limits_{z\to z_0} f(z)=f(z_0)$ (f is cont. on Ω: f is cont. $\forall z\in\Omega$)

f is uniformly continious on Ω: $\forall\varepsilon>0\ \exists\delta>0: \rho(z_1,z_2)<\delta \Rightarrow \rho(f(z_1),f(z_2))<\varepsilon\ \forall z_1,z_2\in\Omega$

f attains max (min) at $z_0\in\Omega$: $|f(z)|\leq|f(z_0)|$ ($|f(z)|\geq|f(z_0)|$) $\forall z\in\Omega$

Properties:

- $f\in C^0(\Omega,\mathbb{C}) \Rightarrow F\in C^0(\Omega,\mathbb{R}^2)$
- $f,g\in C^0(\Omega) \Rightarrow$ 1. $f+g\in C^0(\Omega)$ 2. $f\cdot g\in C^0(\Omega)$
- $f\in C^0(\Omega)\Rightarrow |f|\in C^0(\Omega)$
- $f\in C^0(\Omega)$, where Ω is compact $\Rightarrow$
  1. f is bounded on Ω
  2. f attains max and min on Ω
  3. f is uniformly cont.

3.3. Differentiability.

Consider $f:\Omega\to\mathbb{C}$, where $\Omega\subset\mathbb{C}$ is an open set, then:

f is holomorphic at $z_0\in\Omega$: $\exists!\ \lim\limits_{h\to0}\dfrac{f(z_0+h)-f(z_0)}{h} =: f'(z_0)$, $h\in\mathbb{C}$ $z_0+h\in\Omega$ (i.e. h may approach 0 from any direction)

- f is hol. on Ω: f is hol. $\forall z\in\Omega$
- f is hol. on closed $K\in\mathbb{C}$: $\exists$ open $\Omega\supset K$: f is hol. on Ω
- f is entire: f is hol. on $\mathbb{C}$
- sometimes the terms regular or complex differentiable are used instead

Properties:

- f is hol. at $z_0\in\Omega \iff f(z_0+h)-f(z_0)=f'(z_0)h+o(h)$
- f is hol. $\Rightarrow$ f is cont.
- $f:\Omega\to\mathbb{C}$ $g:\Omega\to\mathbb{C}$ and f,g are hol., then:

  $(f+g)'=f'+g'$ $\quad (f\cdot g)'=f'g+fg'$ $\quad (f/g)'=\dfrac{f'g-fg'}{g^2}$
- $f:\Omega\to U$ $g:U\to\mathbb{C}$ and f,g are hol., then:

  $(g\circ f)'(z)=g'(f(z))f'(z)$
- usual differentiation rules (e.g $(z^n)'=nz^{n-1}$)

Examples:

1) $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ is entire

$p'(z) = a_1 + \ldots + n a_n z^{n-1}$

2) $f(z) = \frac{1}{z}$ is hol. on $\forall$ open Ω s.t. $0 \notin \Omega$

$f'(z) = -\frac{1}{z^2}$

3) $f(z) = \bar{z}$ is not hol. :

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{\bar{h}}{h} \qquad \left.\begin{array}{l} \operatorname{Im}(h) = 0 : \lim\limits_{h \to 0} \frac{\bar{h}}{h} = 1 \\ \operatorname{Re}(h) = 0 : \lim\limits_{h \to 0} \frac{\bar{h}}{h} = -1 \end{array}\right| \Rightarrow \nexists f'(z)$$

4. Holomorphicity.

4.1. Complex and real derivatives.

Let's clarify the difference between the complex and real derivatives:

Consider $z \overset{f}{\mapsto} \bar{z}$ (see ex.3 from §3), it's not holomorphic but it's differentiable as $(x,y) \overset{F}{\mapsto} (x,-y)$, i.e. $F' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = F$ (i.e. indefinitely differentiable). Therefore the existence of real derivative need not guaranty holomorphicity.

Let's associate the mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$, where $F(x,y) = (u(x,y), v(x,y))$, to each $f: \mathbb{C} \to \mathbb{C}$, where $f(z) = u(z) + i v(z)$. Now let's find a connection between the derivatives of F and f:

1. Differetiability of F

$$F(P_0 + H) - F(P_0) = J(P_0) H + \alpha(P_0, H), \text{ where } \alpha(P_0,H) \overset{H \to 0}{=\!=} o(H)$$

$$J(P_0) H = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} du \\ dv \end{pmatrix}$$, i.e. $F'(x,y)$ is a 2x2 real matrix

2. Holomorphicity of f

$f(z) = f(x,y) \quad h = h_1 + i h_2$, where $h_1, h_2 \in \mathbb{R}$

$$h_1 = 0: \quad f'(z_0) = \lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{i h_2} = -i \frac{\partial f}{\partial y}(z_0) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$$h_2 = 0: \quad f'(z_0) = \lim_{h_1 \to 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1} = \frac{\partial f}{\partial x}(z_0) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Th. 4.1: If f is holomorphic at $z = z_0$, then:

1. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ Cauchy-Riemann
2. $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$
3. $f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2 \frac{\partial u}{\partial z}(z_0)$
4. $\det J_F(x_0, y_0) = |f'(z_0)|^2$

$$\frac{\partial}{\partial z} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$$

Wirtinger derivatives

Proof:

1. f is hol. $\Rightarrow \frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y} \Rightarrow$ C-R eqns.

2. $$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \overset{\text{C-R}}{=\!=} 0$$

3. $$f'(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - i \frac{\partial f}{\partial y}(z_0)\right) = \frac{\partial f}{\partial z}(z_0)$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \overset{\text{C-R}}{=\!=} \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = 2 \frac{\partial u}{\partial z}$$

4. $$J_F(x_0, y_0) \overset{\text{C-R}}{=\!=} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} (x_0, y_0)$$, then:

$$\det J_F(x_0, y_0) = \left(\frac{\partial u}{\partial x}(x_0, y_0)\right)^2 + \left(\frac{\partial u}{\partial y}(x_0, y_0)\right)^2 = \left|2 \frac{\partial u}{\partial z}(z_0)\right|^2 = |f'(z_0)|^2$$

Th. 4.2: Consider $f:\Omega\to\mathbb{C}$, where $f=u+iv$. If $u,v\in C^1(\Omega)$ and satisfy C-R eqns. on Ω, then f is holomorphic and $f'(z)=\frac{\partial f}{\partial z}$.

Proof:

$$\left.\begin{aligned} u(x+h_1,y+h_2)-u(x,y) &= \frac{\partial u}{\partial x}h_1+\frac{\partial u}{\partial y}h_2+o(|h|)\\ v(x+h_1,y+h_2)-v(x,y) &= \frac{\partial v}{\partial x}h_1+\frac{\partial v}{\partial y}h_2+o(|h|)\end{aligned}\right\}\text{, where } h=h_1+ih_2$$

$$\begin{aligned} f(z+h)-f(z) &= u(x+h_1,y+h_2)+iv(x+h_1,y+h_2)-u(x,y)-iv(x,y)=\\ &=\frac{\partial u}{\partial x}h_1+\frac{\partial u}{\partial y}h_2+i\frac{\partial v}{\partial x}h_1+i\frac{\partial v}{\partial y}h_2+o(h)=\\ &=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)h_1+\left(\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}\right)ih_2+o(h)=\\ &\overset{C-R}{=}\left(\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\right)h_1+\left(\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\right)ih_2+o(h)=\\ &=\left(\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}\right)(h_1+ih_2)+o(h)=2\frac{\partial u}{\partial z}h+o(h)=\frac{\partial f}{\partial z}h+o(h)\end{aligned}$$

4.2. Chain rule.

Consider $f:U\to V$ and $g:V\to\mathbb{C}$, then let's find the derivative of $h=g\circ f$:

$$h(z,\bar z)=g(f(z,\bar z),\bar f(z,\bar z))$$

$$\frac{\partial h}{\partial z}=\left.\frac{\partial g(\omega,\bar\omega)}{\partial\omega}\right|_{\omega=f(z,\bar z)}\cdot\frac{\partial f(z,\bar z)}{\partial z}+\left.\frac{\partial g(\omega,\bar\omega)}{\partial\bar\omega}\right|_{\bar\omega=\bar f(z,\bar z)}\cdot\frac{\partial\bar f(z,\bar z)}{\partial z}=\left(\frac{\partial g}{\partial z}\circ f\right)\frac{\partial f}{\partial z}+\left(\frac{\partial g}{\partial\bar z}\circ f\right)\frac{\partial\bar f}{\partial z}$$

$$\frac{\partial h}{\partial\bar z}=\left.\frac{\partial g(\omega,\bar\omega)}{\partial\omega}\right|_{\omega=f(z,\bar z)}\cdot\frac{\partial f(z,\bar z)}{\partial\bar z}+\left.\frac{\partial g(\omega,\bar\omega)}{\partial\bar\omega}\right|_{\bar\omega=\bar f(z,\bar z)}\cdot\frac{\partial\bar f(z,\bar z)}{\partial\bar z}=\left(\frac{\partial g}{\partial z}\circ f\right)\frac{\partial f}{\partial\bar z}+\left(\frac{\partial g}{\partial\bar z}\circ f\right)\frac{\partial\bar f}{\partial\bar z}$$

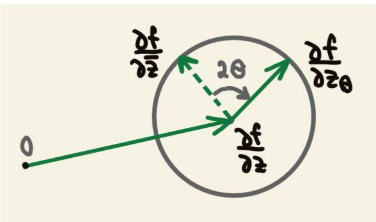
4.3. Harmonicity.

Using the fact that $\Delta=\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}=4\frac{\partial}{\partial z}\frac{\partial}{\partial\bar z}=4\frac{\partial}{\partial\bar z}\frac{\partial}{\partial z}$ one can prove that if $f:\Omega\to\mathbb{C}$ is holomorphic, then $u=\mathrm{Re}(f)$ and $v=\mathrm{Im}(f)$ are harmonic functions, i.e. $\Delta u=0$ and $\Delta v=0$.

4.4. Geometric meaning of complex derivative.

Consider $f:\Omega\to\mathbb{C}$, let's fix a point $z_0\in\Omega$ and investigate the notion of the directional derivative (θ will specify the direction):

$$\frac{\partial f}{\partial z_\theta}=\lim_{\substack{h\to0\\ \arg h=\theta}}\frac{\Delta f}{h}=\lim_{\substack{h\to0\\ \arg h=\theta}}\frac{1}{h}\left(\frac{\partial f}{\partial z}h+\frac{\partial f}{\partial\bar z}\bar h\right)=\lim_{\substack{h\to0\\ \arg h=\theta}}\left(\frac{\partial f}{\partial z}+\frac{\partial f}{\partial\bar z}e^{-2i\theta}\right)=\frac{\partial f}{\partial z}+\frac{\partial f}{\partial\bar z}e^{-2i\theta}$$



When z_0 is fixed and θ changes between 0 and π the point $\frac{\partial f}{\partial z_\theta}$ traverses a circle centered at $\frac{\partial f}{\partial z}$ with radius $\left|\frac{\partial f}{\partial\bar z}\right|$. If f is holomorphic, then $\frac{\partial f}{\partial\bar z}=0$ and all directional derivatives are the same and $\frac{\partial f}{\partial z_\theta}=\frac{\partial f}{\partial z}=f'(z)$.

4.5. Conformity.

Consider $f: \Omega \to \mathbb{C}$: $df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar z} d\bar z$ and $J_f = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar z}\right|^2$ (i.e. f is locally homeomorphic)

If f is hol., then $J_f(z) = \left|\frac{\partial f}{\partial z}(z)\right|^2 > 0$ at any non-critical point.

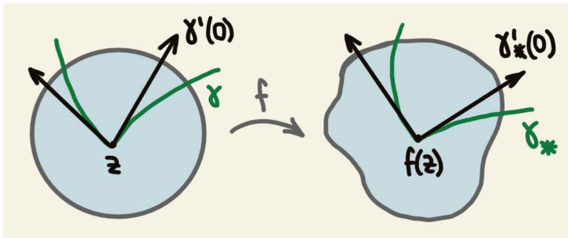
f is conformal at non-crit. $z_0 \in \Omega$: df is a rotation + dilation $\forall z \in U_\varepsilon(z_0)$

Properties:

- f is hol. $\iff$ f is conf.
- Consider a smooth curve $\gamma: [0,1] \to U(z_0)$ s.t. $\gamma(0) = z_0$ and $\gamma'(t) \neq 0 \ \forall t \in [0,1]$. Then its image $\gamma_* = f \circ \gamma$ is also smooth since $\gamma_*'(t) = f'(\gamma(t))\gamma'(t)$.

1. $|f'(z_0)| = \frac{|\gamma_*'(0)|}{|\gamma'(0)|} = \frac{ds_*}{ds}$, i.e the dilation coef. at z_0 under f $\forall \gamma$

2. $\arg f'(z_0) = \arg \gamma_*'(0) - \arg \gamma'(0)$, i.e the rotation angle at z_0 under f $\forall \gamma$



4.6. Hydrodynamic interpretation of holomorphicity.

Consider a steady 2-D flow $v = v_1(x,y) + i v_2(x,y)$ that is irrotational and incompressible:

$$\operatorname{curl} v = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0 \implies \exists \varphi \text{ (potential)}: v_1 = \frac{\partial \varphi}{\partial x} \quad v_2 = \frac{\partial \varphi}{\partial y}$$

$$\operatorname{div} v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \implies \exists \psi \text{ (stream)}: v_2 = -\frac{\partial \psi}{\partial x} \quad v_1 = \frac{\partial \psi}{\partial y}$$

$$\implies \boxed{\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}}$$

C-R eqns. for $f = \varphi + i\psi$

We have $d\psi = -v_2 dx + v_1 dy = 0$ along the level set of ψ and thus $\frac{dy}{dx} = \frac{v_2}{v_1}$.
So the level set of ψ is an integral curve of v.

4.7. Differential forms.

f is holomorphic $\iff$ $\omega = f\,dz$ is closed, i.e. $d\omega = 0$.

5. Infinite series.

5.1. Numerical series

$\sum_{n=1}^{\infty} z_n$: series of complex numbers

$\sum_{n=1}^{\infty} z_n$ conv. absolutely : $\sum_{n=1}^{\infty} |z_n|$ conv.

Th. 5.1 (Cauchy cr.):
$\sum_{n=1}^{\infty} z_n$ conv. $\Leftrightarrow \forall \varepsilon > 0\ \exists N \in \mathbb{N}: \forall k \geq \ell > N \quad |z_\ell + \ldots + z_k| < \varepsilon$

- $\sum_{n=1}^{\infty} z_n$ conv. $\Rightarrow \lim_{n\to\infty} z_n = 0$ (nec. cond. for conv.)
- $\sum_{n=1}^{\infty} z_n$ abs. conv. $\Rightarrow \sum_{n=1}^{\infty} z_n$ conv. (since $|z_\ell + \ldots + z_k| \leq |z_\ell| + \ldots + |z_k| = ||z_\ell| + \ldots + |z_k||$
- $\sum_{n=1}^{\infty} z_n$ abs. conv. $\Rightarrow \forall$ permutation σ $\sum_{n=1}^{\infty} z_n = S$
- $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ abs. conv. $\Rightarrow \sum_{n=1}^{\infty} a_n b_n$ abs. conv. and $S^* = A \cdot B$

5.2. Power series.

$\sum_{n=0}^{\infty} a_n z^n$: power series $(a_n \in \mathbb{C})$

Th. 5.2:
$\sum_{n=1}^{\infty} z_n$ conv. abs. at $z_0 \in \mathbb{C} \Rightarrow \sum_{n=1}^{\infty} z_n$ conv. in the disc $|z| \leq |z_0|$.

Th. 5.3 (Cauchy-Hadamard):
Given $\sum_{n=1}^{\infty} a_n z^n$, $\exists R = \dfrac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}} \geq 0$:
1. $|z| < R$: series conv. abs.
2. $|z| > R$: series diverges
3. $|z| = R$: should be investigated

radius of convergence

Proof: (sketch)
Consider $\sum_{n=0}^{\infty} |a_n z^n|$, then $\alpha = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n||z|^n} = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|} \cdot |z|$. Series converges whenever $\alpha < 1$, or $|z| < 1 / \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}$.

Examples:

1) $\exp(z) \stackrel{def}{=} \sum_{n=0}^{\infty} \frac{z^n}{n!}$ exponential function
   - conv. abs. $\forall z \in \mathbb{C}$ $(R = \infty)$: $\sum_{n=0}^{\infty} \left|\frac{z^n}{n!}\right| = \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = \exp(|z|)$
   - conv. uniformly $\forall D_r(z) \subset \mathbb{C}$
   - entire and $(\exp(z))' = \sum_{n=0}^{\infty} n \frac{z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \exp(z)$

2) $\sum_{n=0}^{\infty} z^n$ geometric series
   - conv. abs. for $|z| < 1$ (i.e. $R = 1$) and $S = \frac{1}{1-z}$.
   - holomorphic on $\mathbb{C} \setminus \{1\}$

3) $\cos z \stackrel{def}{=} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$

   $\sin z \stackrel{def}{=} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

   trigonometric series

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}}$$

Euler's formulas

5.3. Analyticity.

Th. 5.4:
$f(z)=\sum_{n=0}^{\infty} a_n z^n$ is hol. in its disc of convergence and $f'(z)=\sum_{n=0}^{\infty} n a_n z^{n-1}$ is also a power series with the same disc of convergence.

- Therefore, we can conclude that $\sum_{n=0}^{\infty} a_n z^n$ is infinitely complex-differentiable in its disc of convergence, and its higher derivatives are also power series.
- Power series can be centered at any point $z_0 \in \mathbb{C}$, i.e. $f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$. All previously obtained results hold since this case is translated to the old form with an appropriate substitution, i.e. $z-z_0 \mapsto \omega$ $f(z) \mapsto g(\omega)$.

$f: \Omega \to \mathbb{C}$ is analytic at $z_0 \in \Omega$: $\exists \sum_{n=0}^{\infty} a_n (z-z_0)^n = f(z) \quad \forall z \in D_r(z_0)$

- Th. 5.4 implies that every analytic function is holomorphic (converse is also true and will be showed later). So in complex analysis the notions of holomorphicity, analiticity, and conformity are equivalent.
- Convergence of the power series on the boundary of the disc of convergence is not related to it being holomorphic at those points.

  Examples:

  1) $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

     $|z|<1$: disc of conv.

     $|z|=1$: series diverges, but $\frac{1}{1-z}$ is hol. on $|z|=1$ except $z=1$.

  2) $\sum_{n=1}^{\infty} \frac{z^n}{n^2} = f(z)$

     $|z|<1$: disc of conv.

     $|z|=1$: $\frac{z^n}{n^2} < \frac{1}{n^2} \quad \forall |z| \leq 1 \Rightarrow \sum_{n=1}^{\infty} \frac{z^n}{n^2}$ conv. on $|z|=1$

     $f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n}$ is unbounded as $z \to 1 \Rightarrow f(z)$ is not hol. at $z=1$.

⑥ Integration along curves.

Consider a smooth curve γ parametrized by $z:[a,b]\to\mathbb{C}$ and $f\in C^1(\gamma)$, then:

$$\boxed{\int_\gamma f(z)\,dz \overset{\text{def}}{=} \int_a^b f(z(t))\,z'(t)\,dt} = \int_a^b (g_1(t)+ig_2(t))\,dt = \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy$$

- This definition is independent of the parametrization of γ.
- If γ is piecewise-smooth, then $\int_\gamma f(z)\,dz = \sum_{k=0}^{n-1}\int_{a_k}^{a_{k+1}} f(z)\,z'(t)\,dt$.
- Length of a smooth curve: $\ell(\gamma) \overset{\text{def}}{=} \int_a^b |z'(t)|\,dt$.

Properties:

1. $\int_\gamma (\alpha f+\beta g)\,dz = \alpha\int_\gamma f\,dz + \beta\int_\gamma g\,dz$
2. $\int_\gamma f\,dz = -\int_{\gamma^-} f\,dz$, $\gamma^-(t) = \gamma(b+a-t)$ reverse orientation
3. $\int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz = \int_{\gamma_1\cup\gamma_2} f\,dz$
4. $\left|\int_\gamma f(z)\,dz\right| \leq \sup_{z\in\gamma}|f(z)|\cdot\ell(\gamma)$

$F:\Omega\to\mathbb{C}$ is a primitive for $f:\Omega\to\mathbb{C}$: $F'(z)=f(z)\ \forall z\in\Omega$

Th.6.1:
If $\exists F:\Omega\to\mathbb{C}$ for $f\in C^0(\Omega)$, then $\forall\gamma\subset\Omega$ $\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$.

Corollaries:

- If F and G are primitives for f, then $F-G=\text{const}$.
- If $\exists F:\Omega\to\mathbb{C}$ for $f\in C^0(\Omega)$, then $\forall$ closed $\gamma\subset\Omega$ $\oint_\gamma f(z)\,dz = 0$.
- If $f:\Omega\to\mathbb{C}$ is hol. and $f'(z)=0\ \forall z\in\Omega$, then $f=\text{const}$ (since $\int_\gamma f'(z)\,dz = f(z_2)-f(z_1)$).
- If $f:\Omega\to\mathbb{C}$ is hol. and $\mathrm{Re}(f)=\text{const}$ (or $\mathrm{Im}(f)$, or $|f|$), then $f=\text{const}$.
  ($|f|=\text{const}$: $|f|^2 = f\cdot\bar f = \text{const} \Rightarrow f'\bar f = 0 \Rightarrow f'|f|^2=0 \Rightarrow f'=0 \Rightarrow f=\text{const}$)

Examples:

1) $f(z)=(z-z_0)^n$ $\quad\gamma(t) = z_0 + re^{it}$

$$\left.\begin{aligned}\gamma'(t) &= ire^{it}\\ f(\gamma(t)) &= r^n e^{int}\end{aligned}\right| \Rightarrow \oint_\gamma f(z)\,dz = ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt = \begin{cases} n\neq -1: \dfrac{r^{n+1}}{n+1}\left(e^{i(n+1)2\pi}-1\right)=0\\ n=-1:\ i\,(2\pi-0) = 2\pi i\end{cases}$$

$\oint_\gamma \frac{dz}{z} = 2\pi i \neq 0 \Rightarrow f(z)=\frac{1}{z}$ doesn't have a primitive.

2) $f(z)=z^n$, $n\neq -1$ $\quad\gamma(t)$ is arbitrary, but $\gamma(t)\neq 0$ for $n<0$.

$\frac{d}{dt}\gamma^{n+1}(t) = (n+1)\,\gamma^n(t)\,\gamma'(t)$, then:

$$\int_\gamma f(z)\,dz = \int_a^b \gamma^n(t)\gamma'(t)\,dt = \frac{1}{n+1}\left(\gamma^{n+1}(b)-\gamma^{n+1}(a)\right),$$ i.e. is independent of γ

$a=b$: $\oint_\gamma z^n\,dz = 0\ \forall$ closed γ

Chapter II
Cauchy's theorem and its applications

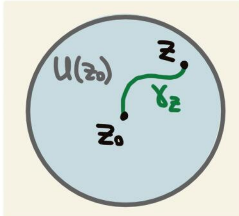
1. Cauchy's theorem.

One can show that locally every holomorphic function has a primitive. The existence of a global primitive, however, is a more comlicated question and depends on topology of Ω. But one can always glue a primitive along a given curve out of local primitives.

Consider $f: \Omega \to \mathbb{C}$, where Ω is a domain, then:

Th. 1.1:
If f is hol., then $\forall z_0 \in \Omega \ \exists F: U(z_0) \to \mathbb{C}$ s.t. $F'(z) = f(z)$, where $F(z) = \int_{\gamma_z} f(\xi)\, d\xi$

Th. 1.2 (Cauchy):
If f is hol., then $\forall$ closed γ $\oint_\gamma f(z)\, dz = 0$.



Proof:
f is hol. $\overset{\text{Th 1.1.}}{\Longrightarrow} \exists F: \Omega \to \mathbb{C} : F'(z) = f(z) \overset{\text{Ch. I §6}}{\Longrightarrow} \oint_\gamma f(z)\, dz = 0 \ \forall$ closed γ

It turns out that the value of an integral along a curve is defined not by the curve itself but by its homotopy class. This result is especially useful for simply-connected sets since on such sets:

- $\forall\ \gamma_1, \gamma_2$ s.t. $\gamma_1(a) = \gamma_2(a)$ $\gamma_1(b) = \gamma_2(b)$ are homotopic
- $\forall\ \gamma_1, \gamma_2$ s.t. $\gamma_1(a) = \gamma_1(b)$ $\gamma_2(a) = \gamma_2(b)$ are homotopic
  (or $\forall$ closed γ is homotopic to a point, i.e. $\gamma \sim 0$)

As a consequence, simple-connectedness of Ω also allows to construct a global primitive for a hol. function f, so this primitive is defined on all of Ω.

Th. 1.3 (general Cauchy th.):
If $f: \Omega \to \mathbb{C}$, where Ω is open, is hol. and $\gamma_1 \sim \gamma_2$, then $\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$.

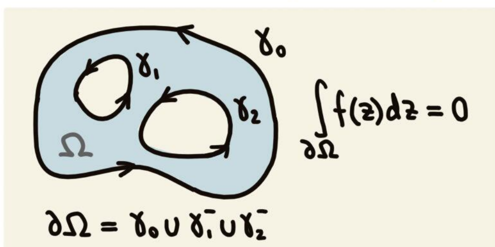
Corollaries:

- If $f: \Omega \to \mathbb{C}$, where Ω is open, is hol. and $\gamma \sim 0$, then $\oint_\gamma f(z) dz = 0$.
- If $f: \Omega \to \mathbb{C}$, where Ω is a domain, is hol., then $\oint_\gamma f(z) dz = 0 \ \ \forall \gamma \sim 0$.
- If $f: \Omega \to \mathbb{C}$, where Ω is a domain, is hol., then $\exists F: \Omega \to \mathbb{C}$ s.t. $F'(z) = f(z) \ \forall z \in \Omega$

Now consider $f: \Omega \to \mathbb{C}$, where Ω is compact and multiply-connected. Let $\partial\Omega$ consist of a finite number of closed curves and let f be hol. on Ω (incl. $\partial\Omega$).

Th. 1.4 (general Cauchy th. for multiply-connected sets):
If $f: \Omega \to \mathbb{C}$ is hol., then $\oint_{\gamma_0} f(z)\, dz = \sum_{i=1}^{n} \oint_{\gamma_i} f(z)\, dz$

γ_0 : external boundary
γ_i : internal boundaries



A converse of Cauchy's theorem is the next result:

Th. 1.3 (Morera):
If $f:\Omega\to\mathbb{C}$, where Ω is open, is cont. and $\oint_\gamma f(z)\,dz = 0$ $\forall$ closed γ, then f is hol.

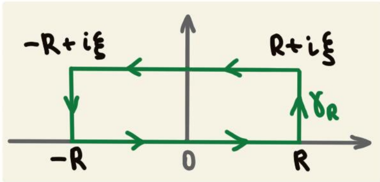
Cor. 1.4:
If $f:\Omega\to\mathbb{C}$, where Ω is open, is cont. and $\exists F:\Omega\to\mathbb{C}$ s.t. $F'(z)=f(z)$, then f is hol.

<u>Example</u>:

Cauchy's th. helps to evaluate some integrals, e.g. $\int_{-\infty}^{\infty}\exp(-\pi x^2)\exp(-2\pi i x\xi)\,dx = \exp(-\pi\xi^2)$

$\xi=0$: $\int_{-\infty}^{\infty}\exp(-\pi x^2)\,dx = 1$ (from real analysis) $\qquad x,\xi\in\mathbb{R}$

$\xi>0$: Consider $f(z)=\exp(-\pi z^2)$, it's entire. Let's choose a rectangular contour γ_R and apply Cauchy's th. to it:



$$\oint_{\gamma_R} f(z)\,dz = 0$$

1) $I_0(R)=\int_{-R}^{R}\exp(-\pi x^2)$ $\qquad \lim_{R\to\infty} I_0(R) = 1$

2) $I_1(R) = \int_0^{\xi} f(R+iy)\,i\,dy = \int_0^{\xi}\exp(-\pi(R^2+2iRy-y^2))\,i\,dy$

$|I_1(R)| \le C\cdot\exp(-\pi R^2) \implies \lim_{R\to\infty} I_1(R) = 0$

$I_2(R) = \int_{\xi}^{0} f(-R+iy)\,i\,dy = \int_{\xi}^{0}\exp(-\pi(R^2-2iRy-y^2))\,i\,dy$

$|I_2(R)| \le C\cdot\exp(-\pi R^2) \implies \lim_{R\to\infty} I_2(R) = 0$

3) $I_3 = \int_{\infty}^{-\infty} f(x+i\xi)\,dx = \int_{\infty}^{-\infty}\exp(-\pi(x^2+2ix\xi-\xi^2))\,dx = -\exp(\pi\xi^2)\int_{-\infty}^{\infty}\exp(-\pi x^2)\exp(-2\pi i x\xi)\,dx$

$I_0+I_1+I_2+I_3=0 \implies \int_{-\infty}^{\infty}\exp(-\pi x^2)\exp(-2\pi i x\xi)\,dx = \exp(-\pi\xi^2)$

$\xi<0$: analogous

2. Miracles of complex analysis.

2.1. Cauchy's integral formulas.

It turns out that the values of a hol. function (and its derivatives) inside a compact domain are completely determined by its values on the boundary of this domain:

Th. 2.1:
If $f: \Omega \to \mathbb{C}$ is hol. and $\bar{D} \subset \Omega$ is compact, then $\forall z \in D$:
1. $f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{\xi - z} d\xi$
2. $f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi$
3. $|f^{(n)}(z)| \leq \frac{n! \, \|f\|_{\partial D}}{R^n}$

Th. 2.2:
If $f: \Omega \to \mathbb{C}$ is hol. and $\bar{D}_R(z_0) \subset \Omega$, then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \;\; \forall z \in D_R(z_0)$, i.e. f is analytic and $a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$.

One useful consequence of Cauchy's integral formulas is the fact that the value of a hol. function in a particular point equals to the average value of this function on any sufficiently small circle centered at this point:

Cor. 2.3:
If $f: \Omega \to \mathbb{C}$ is hol. and $\bar{D} \subset \Omega$ is compact, then $\forall z \in D$ $f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it}) \, dt$.

2.2. Liouville's th. and its applications.

Th. 2.4. (Liouville):
If $f: \mathbb{C} \to \mathbb{C}$ is entire and bounded, then $f = \text{const}$.

So the properties of holomorphicity and boundedness in $\mathbb{C}$ are realized at the same time only in the trivial case ($f = \text{const}$).

Proof:

$|f'(z)| \leq \frac{\|f\|_{\mathbb{C}}}{R} \;\; \forall z \in \mathbb{C} \;\; \forall R > 0 \qquad R \to \infty: \; |f'(z)| = 0 \implies f = \text{const}$

Liouville's theorem provides a simple proof of the fundamental th. of algebra:

Cor. 2.5.:
Every non-constant polynomial $P(z) = a_n z^n + \ldots + a_0$ $(a_1, \ldots, a_n \in \mathbb{C})$ has a root in $\mathbb{C}$.

Proof:

Suppose $P(z)$ has no roots. Consider $\frac{P(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}\right) \;\; \forall z \neq 0$

$|z| \to \infty: \; \left|\frac{P(z)}{z^n}\right| \to |a_n| \implies \exists R > 0: \; |P(z)| \geq \frac{|a_n|}{2} |z|^n \;\; \forall |z| > R$, so:

1) $P(z)$ is bounded for $|z| > R$
2) $P(z)$ is cont. for $|z| \leq R \implies P(z)$ is bounded for $|z| \leq R$

1) + 2) $\overset{\text{Liouville}}{\implies}$ $P = \text{const}$, which contradicts the assumptions of a theorem.

Cor 2.6. (fundamental th. of algebra):

Every polynomial $P(z) = a_n z^n + \ldots + a_0$ $(a_1, \ldots, a_n \in \mathbb{C})$ has precisely n roots in $\mathbb{C}$ and can be factorized as $a_n(z-z_1)\cdot\ldots\cdot(z-z_n)$, where $z_1, \ldots, z_n$ are the roots of $P(z)$.

Proof:

Let z_1 be a root of $P(z)$ (Cor. 2.4.), then let's make a substitution $z = (z-z_1) + z_1$:

1) $P(z) = \ldots = b_n (z-z_1)^n + \ldots + b_1(z-z_1) + b_0$, $b_n = a_n$

$P(z_1) = 0$: $b_0 = 0 \Rightarrow P(z) = (z-z_1)(b_n(z-z_1)^{n-1} + \ldots + b_1) = (z-z_1)Q(z)$

2) repeat the process for $Q(z)$ and so on until $P(z) = a_n(z-z_1)\cdot\ldots\cdot(z-z_n)$

2.3. Analytic continuation.

Th. 2.7:

If $f: \Omega \to \mathbb{C}$ is hol., where Ω is a domain, and $f(z) = 0$ $\forall z \in \{\omega_n : \omega_i \neq \omega_k \, \forall i,k\} \subset \Omega$ s.t. $\lim\limits_{n\to\infty} \omega_n \subset \Omega$, then $f = 0$.

Proof:

Suppose $\lim\limits_{n\to\infty} \omega_n = z_0$ and $f(\omega_n) = 0$ $\forall n$. Let's show that $f(z) = 0$ $\forall z \in D_R(z_0) \subset \Omega$:

$f(z) = \sum\limits_{n=0}^{\infty} a_n (z-z_0)^n$ $\forall z \in D_R(z_0)$

If $f \neq 0$, then $\exists m \in \mathbb{N}$: $a_m \neq 0$, so $f(z) = a_m(z-z_0)^m + o((z-z_0)^m)$, but:

$z = \omega_n$: $f(z) = 0$, i.e. contradiction $\Rightarrow f = 0$

Cor. 2.8.:

If $f: \Omega \to \mathbb{C}$ and $g: \Omega' \to \mathbb{C}$ are hol., where $\Omega \subset \Omega'$ are domains, and $f(z) = g(z)$ $\forall z \in \{\omega_n : \omega_i \neq \omega_k \, \forall i,k\} \subset \Omega$ s.t. $\lim\limits_{n\to\infty} \omega_n \subset \Omega$, then $f(z) = g(z)$ $\forall z \in \Omega$.

So hol. functions that coincide on a converging sequence in the domain where they are hol. have to be equal in the whole domain.

Suppose $f: \Omega \to \mathbb{C}$ and $F: \Omega' \to \mathbb{C}$ are hol., where $\Omega \subset \Omega'$ are domains, then:

F is **analytic continuation** of f into Ω': $F(z) = f(z)$ $\forall z \in \Omega$.

Cor 2.8. guarantees that there can only be one such analytic continuation, i.e. F is uniquely determined by f.

③ Further applications.

3.1. Sequences of hol. functions.

Consider $\{f_n : \Omega \to \mathbb{C} \,;\, n \in \mathbb{N}\}$, where f_n is hol. $\forall n$, and $f : \Omega \to \mathbb{C}$, then:

Th. 3.1 (Weierstrass):
If $f_n \underset{\bar{D}}{\rightrightarrows} f$ (i.e. conv. uniformly) $\forall$ compact $\bar{D} \subset \Omega$, then f is hol.

Th. 3.2:
If $f_n \underset{\bar{D}}{\rightrightarrows} f$ $\forall$ compact $\bar{D} \subset \Omega$, then $f_n' \underset{\bar{D}}{\rightrightarrows} f'$ $\forall$ compact $\bar{D} \subset \Omega$.

- These results don't hold for real analysis, e.g. every cont. function on $[0,1]$ can be approximated uniformly by polynomials, yet not every cont. function is differentiable.
- Th. 3.1 is often used to construct hol. functions with prescribed properties as a series $\sum_{n=1}^{\infty} f_n(z)$ (e.g. the Riemann zeta function).

3.2. Defining hol. functions with integrals.

A number of special functions are defined as:

$f(z) = \int_0^1 F(z,s)\,ds$, where $F : \underset{z}{\Omega} \times \underset{s}{[0,1]} \to \mathbb{C}$ is cont. and Ω is open

Th. 3.3:
If $F : \Omega \times [0,1] \to \mathbb{C}$ is hol $\forall s \in [0,1]$, then $f : \Omega \to \mathbb{C}$ is hol.

3.3. Schwarz reflection principle.

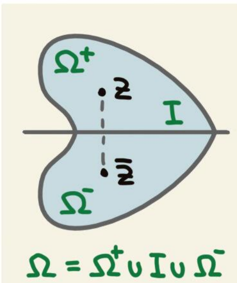
In real analysis there exist several techniques that extend a cont. function from a given set to a larger one. The difficulty of a technique increases with the number of conditions that are imposed on an extension. The situation is very different for holomorphic functions due to the following facts:

1. Hol. functions are "rigid", e.g. $\exists$ hol. functions in a disc D which are cont. on $\bar{D}$, but which can't be analytically continued into any region larger than D.
2. Hol. functions must be 0 if they vanish on small open sets.

These properties provide a simple extension technique that is useful in applications:
Consider open $\Omega \subset \mathbb{C}$ s.t. $z \in \Omega \iff \bar{z} \in \Omega$

Th. 3.4 (symmetry principle):
If $f^+ : \Omega^+ \to \mathbb{C}$ and $f^- : \Omega^- \to \mathbb{C}$ are hol. and cont. extend to I so $f^+(x) = f^-(x)$ $\forall x \in I$, then

$$f = \begin{cases} f^+ & , z \in \Omega^+ \\ f^+ = f^- & , z \in I \\ f^- & , z \in \Omega^- \end{cases}$$ is hol. on Ω.



$\Omega = \Omega^+ \cup I \cup \Omega^-$

Th. 3.5 (Schwarz):
If $f : \Omega^+ \to \mathbb{C}$ is hol. and cont. extends to I s.t. $f : I \to \mathbb{R}$, then $\exists F : \Omega \to \mathbb{C}$ s.t. F is hol. and $F(z) = f(z)$ $\forall z \in \Omega^+$ ($F(z) = \bar{f}(\bar{z})$ for $z \in \Omega^-$).

Chapter III

Singularities, residues and meromorphic functions

1. The Laurent series.

1.1. Functions holomorphic in annuli.

Taylor expansion gives a representation of a function hol. in a disc (neighbourhood of a point), Laurent expansion represents a function hol. in an annulus (or deleted neighbourhood of a point). Such expansion is useful for describing behaviour of a function near singular points (see §2).

Th 1.1 (Laurent):
If $f: V \to \mathbb{C}$, $V = \underbrace{\{r < |z - z_0| < R\}}^{\text{annulus}}$, is hol., then $f(z) = \sum_{-\infty}^{\infty} c_n (z - z_0)^n \ \forall z \in V$, where $c_n = \frac{1}{2\pi i} \oint_{\gamma_s} \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}}$ $(r < s < R)$ and are uniqely determined.

Proof:
Consider $V' = \{r' < |z - z_0| < R'\}$ s.t. $V' \subset V$, and fix an arbitrary $z \in V'$, then:

$$f(z) = \frac{1}{2\pi i} \int_{\partial V'} \frac{f(\xi)\, d\xi}{\xi - z} = \frac{1}{2\pi i} \oint_{\Gamma'} \frac{f(\xi)\, d\xi}{\xi - z} - \frac{1}{2\pi i} \oint_{\gamma'} \frac{f(\xi)\, d\xi}{\xi - z}, \text{ where } \begin{array}{l} \Gamma' = \{|z - z_0| = R'\} \\ \gamma' = \{|z - z_0| = r'\} \end{array}$$

1) Since $\left|\frac{z - z_0}{\xi - z_0}\right| = q < 1 \ \forall \xi \in \Gamma'$ we have:

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0)\left(1 - \frac{z - z_0}{\xi - z_0}\right)} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}},$$ converges uniformly and absolutely $\forall \xi \in \Gamma'$.

$$\frac{1}{2\pi i} \oint_{\Gamma'} \frac{f(\xi)\, d\xi}{\xi - z} = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \oint_{\Gamma'} \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}}\right] (z - z_0)^n = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

2) Since $\left|\frac{\xi - z_0}{z - z_0}\right| = q < 1 \ \forall \xi \in \gamma'$ we have:

$$-\frac{1}{\xi - z} = \frac{1}{(z - z_0)\left(1 - \frac{\xi - z_0}{z - z_0}\right)} = \sum_{n=1}^{\infty} \frac{(\xi - z_0)^{n-1}}{(z - z_0)^n},$$ converges uniformly and absolutely $\forall \xi \in \gamma'$.

$$-\frac{1}{2\pi i} \oint_{\gamma'} \frac{f(\xi)\, d\xi}{\xi - z} = \sum_{n=1}^{\infty} \left[\frac{1}{2\pi i} \oint_{\gamma'} f(\xi)(\xi - z_0)^{n-1}\, d\xi\right] \frac{1}{(z - z_0)^n} =$$

$$= \sum_{n=-1}^{-\infty} \left[\frac{1}{2\pi i} \oint_{\gamma'} f(\xi)(\xi - z_0)^{-(n+1)}\, d\xi\right] (z - z_0)^n = \sum_{n=-1}^{-\infty} c_n (z - z_0)^n$$

$$\sum_{-\infty}^{\infty} c_n (z - z_0)^n = \underbrace{\sum_{n=0}^{\infty} c_n (z - z_0)^n}_{\text{regular part}} + \underbrace{\sum_{n=-1}^{-\infty} c_n (z - z_0)^n}_{\text{principal part}} = \Sigma_1 + \Sigma_2$$

Σ_1:
- power series in $(z - z_0)$: $\sum_{n=0}^{\infty} c_n (z - z_0)^n$
- domain of convergence: $\{|z - z_0| < R\}$, $R = \left(\limsup_{n \to \infty} \sqrt[n]{|c_n|}\right)^{-1}$

Σ_2:
- power series in $\frac{1}{z - z_0}$: $\sum_{n=1}^{\infty} c_{-n} \left(\frac{1}{z - z_0}\right)^n$
- domain of convergence: $\{|z - z_0| > r\}$, $r = \limsup_{n \to \infty} \sqrt[n]{|c_{-n}|}$

So the domain of convergence for the Laurent series is $V = \{r < |z - z_0| < R\}$ (if $r < R$).

Th. 1.2 (Cauchy inequalities):
If $f: V \to \mathbb{C}$, $V = \{r < |z - z_0| < R\}$, is hol., then $|c_n| \leq \frac{\|f\|_{\gamma_s}}{s^n}$, $\gamma_s = \{|z - z_0| = s\}$.

Expressions for c_n are rarely used in practice since they require computation of integrals.

Example:

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} \quad \text{is hol. in the annuli} \quad \begin{aligned} V_1 &= \{0<|z|<1\} \\ V_2 &= \{1<|z|<2\} \\ V_3 &= \{2<|z|<\infty\} \end{aligned}$$

V_1: $\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$ conv. for $|z|<2$

$-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ conv. for $|z|<1$

$$f(z) = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right)z^n$$

V_2: $\frac{1}{z-2} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$ conv. for $|z|<2$

$-\frac{1}{z-1} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=-1}^{-\infty} z^n$ conv. for $|z|>1$

$$f(z) = -\left(\sum_{n=-1}^{-\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right)$$

V_3: $\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \frac{1}{2}\cdot\frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \frac{1}{2}\sum_{n=-1}^{-\infty}\left(\frac{z}{2}\right)^n$ conv. for $|z|>2$

$-\frac{1}{z-1} = -\sum_{n=-1}^{-\infty} z^n$ conv. for $|z|>1$

$$f(z) = \sum_{n=-1}^{-\infty}\left(\frac{1}{2^{n+1}} - 1\right)z^n$$

1.2. Relation between the Laurent and Fourier series.

Consider $\phi: \mathbb{R}\to\mathbb{R}$ that is integrable on $[0,2\pi]$, then:

$$\phi(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt), \quad a_n = \frac{1}{\pi}\int_0^{2\pi}\phi(t)\cos nt\,dt \quad b_n = \frac{1}{\pi}\int_0^{2\pi}\phi(t)\sin nt\,dt$$

$$\phi(t) = \sum_{-\infty}^{\infty} c_n \exp(int), \quad \begin{aligned} c_n &= \frac{a_n - i b_n}{2} = \frac{1}{2\pi}\int_0^{2\pi}\phi(t)\exp(-int)\,dt, \quad n = 0,1,2,\dots \\ c_n &= \frac{a_{-n} + i b_{-n}}{2} = \frac{1}{2\pi}\int_0^{2\pi}\phi(t)\exp(-int)\,dt, \quad n = -1,-2,\dots \end{aligned}$$

Now let's set $e^{it} = z$ and $\phi(t) = f(e^{it}) = f(z)$, then:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}\,dt = \frac{1}{2\pi i}\oint_{|z|=1}\frac{f(z)\,dz}{z^{n+1}}.$$

So the Fourier series of $\phi(t)$, $t\in[0,2\pi]$, is the Laurent series of $f(z)$, $z = e^{it}$, on the unit circle $|z|=1$. Also if the Fourier series converges, the corresponding Laurent series may have an empty domain of convergence.

Example:

$$\phi(t) = \frac{a\sin t}{a^2 - 2a\cos t + 1} \qquad z = e^{it} \qquad \sin t = \frac{e^{it} - e^{-it}}{2i} \qquad \cos t = \frac{e^{it} + e^{-it}}{2}$$

$$f(z) = \frac{a\left(z - \frac{1}{z}\right)}{2i\left(a^2 - a\left(z + \frac{1}{z}\right) + 1\right)} = \frac{1}{2i}\frac{(z^2-1)\frac{1}{z}}{\left(a - \frac{z^2+1}{z} + \frac{1}{a}\right)} = \frac{1}{2i}\frac{1 - z^2}{-\left(-z^2 - 1 + \left(a + \frac{1}{a}\right)z\right)} =$$

$$= \frac{1}{2i}\frac{1-z^2}{z^2 - \left(a + \frac{1}{a}\right)z + 1} = \frac{1}{2i}\frac{1-z^2}{(z-a)\left(z - \frac{1}{a}\right)} = \frac{1}{2i}\left(\frac{A}{z-a} + \frac{B}{z - \frac{1}{a}}\right) =$$

$$1 - z^2 = A\left(z - \frac{1}{a}\right) + B(z-a) \qquad \begin{aligned} z = a:&\quad 1 - a^2 = A\frac{a^2-1}{a} \Rightarrow A = -a \\ z = \frac{1}{a}:&\quad 1 - \frac{1}{a^2} = B\frac{1-a^2}{a} \Rightarrow B = -\frac{1}{a} \end{aligned}$$

$$= \frac{1}{2i}\left(-\frac{a}{z-a} - \frac{1/a}{z - 1/a}\right) = \frac{1}{2i}\left(\frac{1}{1 - z/a} + \frac{1}{1 - az}\right) = \frac{1}{2i}\sum_{n=1}^{\infty} a^n\left(z^n - \frac{1}{z^n}\right) = \sum_{n=1}^{\infty} a^n \sin(nt)$$

$$V = \left\{|a| < z < \frac{1}{|a|}\right\}$$

② Singularities.

2.1. Types of singularities.

The most interesting points in the study of hol. functions are those where functions cease being holomorphic — the singular points. So hol. functions are characterized by their singularities (can be isolated or non-isolated).

Consider $f: \Omega \to \mathbb{C}$ hol. on $\Omega \setminus \{z_0\}$, i.e. $z = z_0$ is an isolated singularity, then:

$z = z_0$ is a removable sing.: $\exists \lim\limits_{z \to z_0} f(z) = A$

$z = z_0$ is a pole : $\exists \lim\limits_{z \to z_0} f(z) = \infty$

$z = z_0$ is an essential sing. : $\bar{\exists} \lim\limits_{z \to z_0} f(z)$

Examples:

1) $f(z) = \frac{\sin z}{z}$ $\quad z = 0$: $f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$

$\lim\limits_{z \to 0} f(z) = 1 \implies z = 0$ is a rem. sing.

2) $f(z) = \frac{1}{z^n}$, $n \in \mathbb{N}$ $\quad z = 0$: $\lim\limits_{z \to 0} f(z) = \infty \implies z = 0$ is a pole

3) $f(z) = \exp\left(\frac{1}{z}\right)$ $\quad z = 0$: $\lim\limits_{z \to +0} f(z) = \infty$ $\quad \lim\limits_{z \to -0} f(z) = 0 \implies z = 0$ is an essential sing.

Singularities can be non-isolated:

4) $f(z) = \frac{1}{\sin(\pi / z)}$ $\quad z = \frac{1}{n}$, $n \in \mathbb{Z}$ are poles

$z = 0$ is a limit point of poles (non-isolated)

5) $f(z) = \sum\limits_{n=0}^{\infty} z^{2^n} = 1 + z^2 + z^4 + z^8 + \dots$, conv. for $|z| < 1$ and $|z| = 1$ is a circle of sing-es

2.2. Zeros of a function.

Since singularities often appear because the denominator of a fraction vanishes, it's useful to study zeros of hol. functions.

$z = z_0$ is a zero of order n of f: $\quad f(z_0) = f'(z_0) = \dots = f^{(n-1)}(z_0) = 0$.
(multiplicity) $\quad$ (the order describes the rate at which f vanishes near z_0)

$z = z_0$ is a pole of order n of f: $\quad z = z_0$ is a zero of order n of $1/f$
(multiplicity) $\quad$ (the order describes the rate at which f grows near z_0)

Analytic continuation (Ch. II § 2.3.) shows that zeros of a non-trivial hol. function must be isolated, i.e. if $z = z_0$ is a zero of f, then $\forall z \in \mathring{U}(z_0)$ $f(z) \neq 0$.

Th. 2.1:

If $f: \mathbb{C} \to \mathbb{C}$, Ω is open and connected, is hol., $f \not\equiv 0$, and $z = z_0$ is a zero of order n of f, then $\exists U(z_0)$ s.t. $\forall z \in U(z_0)$ $f(z) = (z - z_0)^n g(z)$, where g is hol. and $g \neq 0$.

Proof:

$\forall z \in U(z_0)$ $f(z) = \sum\limits_{k=0}^{\infty} a_k (z - z_0)^k$ $\qquad f \not\equiv 0 \implies \exists n \in \mathbb{N}$ s.t. $a_n \neq 0$, then:

$f(z) = (z - z_0)^n (a_n + a_{n+1}(z - z_0) + \dots) = (z - z_0)^n g(z)$, i.e. g is hol. and $g \neq 0$ (since $a_n \neq 0$)

one can easily show that n is uniquely determined and it coincides with the order of zero $z = z_0$

2.3. Poles of a function.

Now using Th.2.1. we can describe the type of a singularity possesed by $1/f$ at z_0:

Th. 2.2:
If $f:\Omega\to\mathbb{C}$, Ω is open and connected, is hol., $f\neq 0$, and $z=z_0$ is pole of order n of f, then $\exists\, \mathring{U}(z_0)$ s.t. $\forall z\in\mathring{U}(z_0)$ $f(z)=(z-z_0)^{-n}h(z)$, where h is hol. and $h\neq 0$.

Proof:

$$\text{Th.2.1.} \Rightarrow \frac{1}{f(z)}=(z-z_0)^n g(z) \Rightarrow f(z)=(z-z_0)^{-n}h(z), \text{ where } h(z)=\frac{1}{g(z)}.$$

Cor. 2.3:
If $z=z_0$ is a pole of order n, then $\forall z\in U(z_0)$ $f(z)=\frac{a_{-n}}{(z-z_0)^n}+\ldots+\frac{a_{-1}}{z-z_0}+G(z)$, G is hol.

Proof:

$$f(z)=(z-z_0)^{-n}h(z)=(z-z_0)^{-n}(A_0+A_1(z-z_0)+\ldots)=\frac{a_{-n}}{(z-z_0)^n}+\ldots+\frac{a_{-1}}{(z-z_0)}+G(z).$$

All the terms of the principal part of f have primitives except the a_{-1} term (see Ch.I §7). So $\frac{1}{2\pi i}\oint_\gamma P(z)\,dz=a_{-1}$, where $P(z)$ is the principal part of f at $z=z_0$. This number is called the residue of f at the pole $z=z_0$ (this will be discussed in §4).

2.4. Properties of singular points.

The type of an isolated singularity $z=z_0$ is closely related to the Laurent expansion of f in $\mathring{U}(z_0)$:

$z=z_0$ is a removable sing.	$\Leftrightarrow$	1. $f(z)=\sum_{n=0}^{\infty}c_n(z-z_0)^n$ $\forall z\in\mathring{U}(z_0)$ 2. f is bounded in $\mathring{U}(z_0)$	i.e. no principal part
$z=z_0$ is a pole of order N	$\Leftrightarrow$	1. $f(z)=\sum_{n=-N}^{\infty}c_n(z-z_0)^n$ $\forall z\in\mathring{U}(z_0)$ 2. $z=z_0$ is a zero of order N of $1/f$	i.e. finite number of terms for the principal part
$z=z_0$ is an essential sing.	$\Leftrightarrow$ $\Rightarrow$	1. $f(z)=\sum_{-\infty}^{\infty}c_n(z-z_0)^n$ $\forall z\in\mathring{U}(z_0)$ 2. $\forall A\in\overline{\mathbb{C}}$ $\exists\{z_n: z_n\to z_0\}: \lim_{n\to\infty}f(z_n)=A$	i.e. inf. number of terms for the principal part

Claims above are also true for the isolated singularities at $z=\infty$ with a caveat that in this case the part of the Laurent expansion with the negative powers is the regular part and the part with the positive powers is the principal part. It's easy to see after the substitution $z=\frac{1}{\omega}$:

$$f(z)=f(\tfrac{1}{\omega})=\phi(\omega) \Rightarrow \lim_{z\to\infty}f(z)=\lim_{\omega\to 0}\phi(\omega)$$

ϕ has the same type of singularity at $\omega=0$ as f at $z=\infty$

Th. 2.4 (Casorati-Weierstrass):
If $f:\Omega\to\mathbb{C}$, Ω is open, is hol. on $\Omega\setminus\{z_0\}$, where $z=z_0$ is an essential singularity, then $f(\Omega\setminus\{z_0\})$ is dense in $\mathbb{C}$, i.e. $\overline{f(\Omega\setminus\{z_0\})}=\mathbb{C}$.

(A is **dense** in X: $\forall x\in X$ $\forall U(x)$ is s.t. $A\cap U(x)\neq\emptyset$, i.e. every neighbourhood of x contains at least one point from A, e.g. $\mathbb{Q}$ is dense in $\mathbb{R}$)

3. Classification of holomorphic functions.

Using Liouville's th. we can classify hol. functions according to their singularities:

3.1. Constant functions.

no singularities on $\overline{\mathbb{C}}$

3.2. Entire functions.

singularity at $z=\infty$

1) $z=\infty$ is a rem. sing.: $f = \text{const}$ (from Liouville's th.)
2) $z=\infty$ is a pole : $g(z) = c_1 z + \ldots + c_N z^N$ principal part of f at $z=\infty$

   Then $f-g$ is entire and has a rem. sing at $z=\infty$, so $f-g = \text{const}$, therefore,

   $$f(z) = \sum_{n=0}^{N} c_n (z-z_0)^n$$
3) $z=\infty$ is an ess. sing. : entire transcendental functions, e.g. $\exp(z)$, $\sin(z)$, $\cos(z)$

3.3. Meromorphic functions.

all singularities in $\mathbb{C}$ are poles (countably many)

Th. 3.1:
If f is meromorphic and $z=\infty$ is a pole or rem. sing., then f is a rational function.

Proof:

$g_\nu(z) = \dfrac{c_{-N_\nu}^{(\nu)}}{(z-a_\nu)^{N_\nu}} + \ldots + \dfrac{c_{-1}^{(\nu)}}{(z-a_\nu)}$ principal part of f near a pole a_ν $(\nu = 1,\ldots,n)$

$g(z) = c_1 z + \ldots + c_N z^N$ principal part of f near $z=\infty$ (if rem. sing., then $g=0$)

Now consider $\phi(z) = f(z) - g(z) - \sum_{\nu=1}^{n} g_\nu(z)$, it has no singularities and hence $\phi = c_0$.
Then we can conclude that $f(z) = c_0 + g(z) + \sum_{\nu=1}^{n} g_\nu(z)$, i.e. a rational function.

So a rational function is determined by its zeros and poles (up to a multiplicative const).

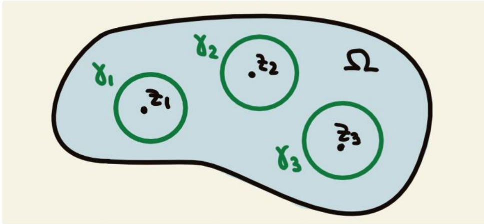
④ Residues.

4.1. Residue theorem.

Consider $f:\Omega\to\mathbb{C}$, Ω is a domain, hol. on $\Omega\setminus\{z_1,\dots,z_n\}$. Then Cauchy th. implies:

$\int_{\partial\Omega} f(z)\,dz = \sum_{\nu=1}^{n} \oint_{\gamma_\nu} f(z)\,dz$, where $\gamma_\nu = \{|z-z_\nu| = r\}$

doesn't depend on r



$$\boxed{\int_{\partial\Omega} f(z)\,dz = 2\pi i \sum_{\nu=1}^{n} \operatorname*{res}_{z_\nu} f}\ , \quad \operatorname*{res}_{z_\nu} f \overset{\text{def}}{=} \frac{1}{2\pi i}\oint_{\gamma_\nu} f(z)\,dz$$

residue formula

So the computation of the integral of a hol. function along the boundary of a domain is reduced to the computation of the integrals over small contours around its singularities (evaluation of residues).

Properties:

1. $\operatorname*{res}_{z_0} f = c_{-1}$ (see §2.3.)
2. $z=z_0$ is a removable sing.: $\operatorname*{res}_{z_0} f = 0$ (since $c_{-1}=0$)
3. $z=z_0$ is a pole of order 1 (simple pole): $\operatorname*{res}_{z_0} f = \lim_{z\to z_0}(z-z_0)f(z)$

   $z=z_0$ is a pole of order n: $\operatorname*{res}_{z_0} f = \lim_{z\to z_0}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]$ (see §2.3.)
4. $z=z_0$ is an essential sing.: compute the principal part of the Laurent expansion (no easier formulas)
5. $f(z) = \frac{\phi(z)}{\psi(z)}$, ϕ and ψ are hol at $z=z_0$

   $z=z_0$ is a simple pole: $\psi(z_0)=0$ $\psi'(z_0)\neq 0$ $\phi(z_0)\neq 0$ $\Rightarrow \operatorname*{res}_{z_0} f = \lim_{z\to z_0}(z-z_0)\frac{\phi(z)}{\psi(z)} = \frac{\phi(z_0)}{\psi'(z_0)}$

One can also define residues at $z=\infty$:

$$\operatorname*{res}_{\infty} f \overset{\text{def}}{=} \frac{1}{2\pi i}\oint_{\gamma_R^-} f(z)\,dz = -c_{-1}$$, i.e. γ_R^- has a neg. orientation.

The terms with the neg. powers constitute the regular part of the Laurent expansion at $z=\infty$. Therefore, unlike at finite singular points the residue at infinity may be non-zero even if $z=\infty$ is not a singularity.

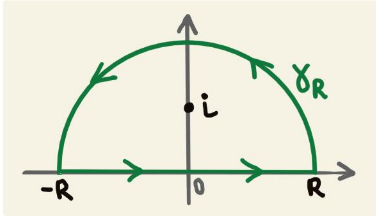
Th. 4.1:
If $f:\mathbb{C}\to\mathbb{C}$ is hol. on $\mathbb{C}\setminus\{z_1,\dots,z_n\}$, then $\sum_{\nu=1}^{n}\operatorname*{res}_{z_\nu} f + \operatorname*{res}_{\infty} f = 0$.

4.2. The evaluation of integrals.

The calculus of residues provides a powerful technique to compute integrals. The main idea is to extend f to $\mathbb{C}$ and choose an appropriate toy contour such that $\lim_{R\to\infty}\oint_{\gamma_R} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$. The choice of γ_R is motivated by the decay behaviour of f.

Examples:

1) $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ $\qquad f(z) = \frac{1}{1+z^2}$



$\gamma_R = \gamma_R^+ \cup [-R,R]$

pole $z=i$: $\operatorname{res}_i f = \lim_{z\to i} (z-i)\frac{1}{1+z^2} = \frac{1}{2i}$

pole $z=-i$: $\operatorname{res}_{-i} f = \lim_{z\to -i} (z+i)\frac{1}{1+z^2} = -\frac{1}{2i}$

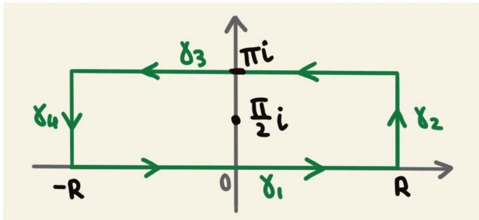
$$\oint_{\gamma_R} f(z)\,dz = 2\pi i \cdot \frac{1}{2i} = \pi$$

$$\left|\int_{\gamma_R^+} f(z)\,dz\right| \leq \pi R \cdot \frac{B}{R^2} = \frac{\pi B}{R} \Rightarrow \lim_{R\to\infty} \int_{\gamma_R^+} f(z)\,dz = 0 \Rightarrow I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

2) $I = \int_{-\infty}^{\infty} \frac{\cos x}{\exp(x)+\exp(-x)}\,dx$

$$\sin(x) = \sin(-x) \Rightarrow \int_{-\infty}^{\infty} \frac{\sin(x)}{\exp(x)+\exp(-x)}\,dx = 0 \Rightarrow f(z) = \frac{\exp(iz)}{\exp(z)+\exp(-z)}$$

1. Singularities:



$$e^z + e^{-z} = 0 \Rightarrow e^{2z} = -1 \Rightarrow e^{2x} e^{2iy} = -1$$

$$\left.\begin{array}{l} e^{2x} = 1 \Rightarrow x = 0 \\ e^{2iy} = -1 \Rightarrow y = \frac{\pi}{2} + \pi k,\ k\in\mathbb{Z} \end{array}\right| \Rightarrow \underbrace{z = i\left(\tfrac{\pi}{2} + \pi k\right)}_{\text{poles}}$$

$$\frac{d}{dz}\left(e^z + e^{-z}\right)\Big|_{z=i\frac{\pi}{2}} = \left(e^z - e^{-z}\right)\Big|_{z=i\frac{\pi}{2}} = 2i \neq 0 \Rightarrow z = i\tfrac{\pi}{2} \text{ is a simple pole}$$

$$\operatorname{res}_{i\pi/2} f = \operatorname{res}_{i\pi/2} \frac{\phi}{\psi} = \frac{\phi(i\pi/2)}{\psi'(i\pi/2)} = \frac{\exp(i\cdot i\pi/2)}{\exp(i\pi/2)-\exp(-i\pi/2)} = \frac{e^{-\pi/2}}{2i} \Rightarrow \oint_{\gamma(R)} f(z)\,dz = 2\pi i \cdot \frac{e^{-\pi/2}}{2i} = \pi e^{-\frac{\pi}{2}}$$

2. Evaluation of integrals:

$\gamma_2(R)$: $$\left|\int_{\gamma_2(R)} f(z)\,dz\right| \leq \int_{\gamma_2(R)} |f(z)|\,dz \leq \pi \cdot \sup_{y\in[0,\pi]} \left|\frac{e^{i(R+iy)}}{e^R e^{iy} + e^{-R}e^{-iy}}\right|$$

$$\left|\frac{e^{i(R+iy)}}{e^R\left(e^{iy} + e^{-2R}e^{-iy}\right)}\right| \leq \frac{e^{-y}}{e^R\left|e^{iy} + e^{-2R}e^{-iy}\right|} \leq \frac{e^{-y}}{e^R\left(|e^{iy}| + e^{-2R}|e^{-iy}|\right)} \overset{y\in[0,\pi]}{\leq} \frac{1}{e^R\left(1-e^{-2R}\right)}$$

$$\lim_{R\to\infty} \int_{\gamma_2(R)} f(z)\,dz = 0$$

$\gamma_4(R)$: analogous to $\gamma_2(R)$, so $\lim_{R\to\infty} \int_{\gamma_4(R)} f(z)\,dz = 0$.

$\gamma_3(R)$: $$\int_{\gamma_3(R)} f(z)\,dz = \int_R^{-R} \frac{e^{i(x+i\pi)}}{e^{x+\pi i} + e^{-x-\pi i}}\,dx = e^{-\pi}\int_R^{-R} \frac{e^{ix}}{-e^x - e^{-x}}\,dx = e^{-\pi}\int_{-R}^{R} \frac{e^{ix}\,dx}{e^x + e^{-x}} = e^{-\pi}\int_{\gamma_1(R)} f\,dz$$

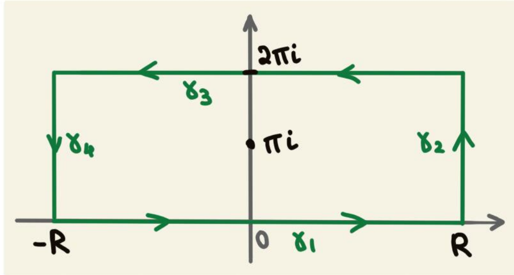
Then we have:

$$I + e^{-\pi} I = \pi e^{-\pi/2}$$

$$I = \frac{\pi e^{-\pi/2}}{1+e^{-\pi}} = \frac{\pi}{e^{\pi/2} + e^{-\pi/2}}$$

3) $I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx\ ,\quad 0<a<1 \qquad f(z) = \frac{e^{az}}{1+e^z}$

1. Singularities:



$z = \pi i : \quad \operatorname*{res}_{\pi i} f = \operatorname*{res}_{\pi i} \frac{\phi}{\psi} = \frac{\phi(\pi i)}{\psi'(\pi i)} = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i}$

$z = -\pi i : \quad \operatorname*{res}_{-\pi i} f = -e^{-a\pi i}$

Then $\oint_{\gamma(R)} f(z)\,dz = -2\pi i\, e^{a\pi i}$.

2. Evaluating integrals:

$\gamma_2(R): \left|\int_{\gamma_2(R)} f(z)\,dz\right| \leqslant \int_0^{2\pi} \left|\frac{e^{a(R+iy)}}{1+e^{R+iy}}\right| dy \leqslant C\, e^{(a-1)R} \quad \Rightarrow \quad \lim_{R\to\infty} \int_{\gamma_2(R)} f(z)\,dz = 0 \quad (\text{since } a<1)$

$\sup_{\gamma_2(R)} \left|\frac{e^{aR}e^{iay}}{1+e^R e^{iy}}\right| \leqslant \frac{e^{aR}}{e^R-1} = e^{(a-1)R}\frac{1}{1-e^{-R}}$

$\gamma_4(R): \left|\int_{\gamma_4(R)} f(z)\,dz\right| \leqslant \int_{2\pi}^{0} \left|\frac{e^{a(-R+iy)}}{1+e^{-R+iy}}\right| dy \leqslant C\, e^{-aR} \quad \Rightarrow \quad \lim_{R\to\infty} \int_{\gamma_4(R)} f(z)\,dz = 0 \quad (\text{since } a>0)$

$\sup_{\gamma_4(R)} \left|\frac{e^{-aR}e^{iay}}{1+e^{-R} e^{iy}}\right| \leqslant \frac{e^{-aR}}{1-e^{-R}}$

$\gamma_3(R): \int_{\gamma_3(R)} f(z)\,dz = \int_R^{-R} \frac{e^{ax}\cdot e^{ai2\pi}}{1+e^x e^{i2\pi}}\,dx = e^{i2\pi a}\int_R^{-R} \frac{e^{ax}}{1+e^x}\,dx = -e^{i2\pi a}\int_{\gamma_1(R)} f(z)\,dz$

Then we have:

$I - e^{i2\pi a} I = -2\pi i\, e^{i\pi a} \quad \Rightarrow \quad I = \frac{-2\pi i\, e^{i\pi a}}{1-e^{i2\pi a}} = \frac{-2\pi i}{e^{-i\pi a}-e^{i\pi a}} = \frac{\pi}{\sin(\pi a)}$

4) $I = \oint_{|z|=2} \frac{dz}{(z^8+1)^2} \qquad f(z) = \frac{1}{(z^8+1)^2}$

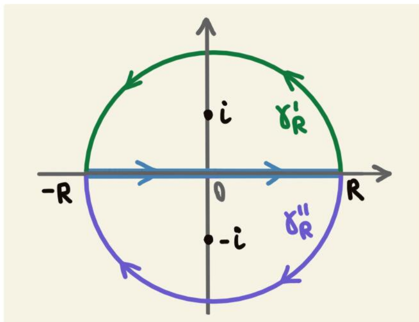
$z=\infty$: zero of order 16 $\Rightarrow$ Laurent expansion starts at z^{-16} $\Rightarrow$ $c_{-1} = \operatorname*{res}_{\infty} f = 0$

$I = 2\pi i \sum_{\nu=1}^{8} \operatorname*{res}_{z_\nu} f = -2\pi i \operatorname*{res}_{\infty} f = 0$

5) $\phi(t) = \int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\,dx\ ,\quad t\in\mathbb{R} \qquad f(z) = \frac{e^{itz}}{1+z^2}$

$\phi(t)$ is majorized by $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$ $\Rightarrow$ conv. abs.

1. Singularites:



$z = i: \quad \operatorname*{res}_{i} f = \frac{e^{it\cdot i}}{2\cdot i} = \frac{e^{-t}}{2i} \quad \Rightarrow \quad \oint_{\gamma'} f(z)\,dz = \pi e^{-t}$

$z = -i: \quad \operatorname*{res}_{-i} f = \frac{e^{it(-i)}}{2(-i)} = -\frac{e^{t}}{2i} \quad \Rightarrow \quad \oint_{\gamma''} f(z)\,dz = -\pi e^{t}$

2. Evaluating integrals:

$t\geqslant 0: \left|\int_{\gamma'_R} \frac{e^{itz}}{1+z^2}\,dz\right| \leqslant \frac{\pi R}{R^2-1} \quad \Rightarrow \quad \lim_{R\to\infty}\int_{\gamma'_R} f(z)\,dz = 0 \quad \Rightarrow \quad \int_{-\infty}^{\infty} f(x)\,dx = \pi e^{-t}$

$t< 0: \left|\int_{\gamma''_R} \frac{e^{itz}}{1+z^2}\,dz\right| \leqslant \frac{\pi R}{R^2-1} \quad \Rightarrow \quad \lim_{R\to\infty}\int_{\gamma''_R} f(z)\,dz = 0 \quad \Rightarrow \quad \int_{-\infty}^{\infty} f(x)\,dx = \pi e^{t}$

$\phi(t) = \int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\,dx = \pi e^{-|t|}$

5. The complex logarithm.

We want the logarithm to be the inverse of the exponential function, then it's natural to set $\log z = \ln|z| + i \arg z$, $z \neq 0$ (log: complex natural log., ln: real natural log.).

Globally $\log z$ is multi-valued (since $\arg z$ is defined uniquely up to $2\pi k$, $k \in \mathbb{Z}$), so to make it single-valued one should restrict the domain of $\log z$ (choose a branch/sheet). But locally one can always give unambiguous definition of the logarithm.

Th.5.1:
If Ω is simly-connected and $1 \in \Omega$ $0 \notin \Omega$, then $\exists$ a branch $F(z) = \log z|_{\Omega}$ s.t. it is hol. on Ω, $\exp(F(z)) = z$ $\forall z \in \Omega$, and $F(r) = \ln r$ whenever $r \in \mathbb{R}$ and $r \in U(1)$.

Proof:

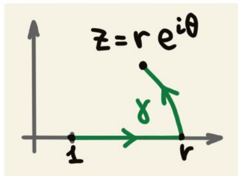
$F(z) = \int_1^z \frac{dw}{w}$ (doesn't depends on the chosen path) ... $F(z)$ is hol. and $F'(z) = \frac{1}{z}$.

$$\frac{d}{dz}\left(z e^{-F(z)}\right) = e^{-F(z)} - z F'(z) e^{-F(z)} = \left(1 - z \frac{1}{z}\right) e^{-F(z)} = 0 \Rightarrow e^{F(z)} = z$$

Properties:

- In the slit plane $\Omega = \mathbb{C} \setminus (-\infty; 0]$ we have the principal branch of $\log z$, $\theta \in (-\pi, \pi)$:

$$\log z = \int_\gamma \frac{dw}{w} = \int_1^r \frac{dx}{x} + \int_0^\theta \frac{ire^{it}}{re^{it}} dt = \ln r + i\theta$$



- In general $\log(z_1 z_2) \neq \log z_1 + \log z_2$ on a particular branch.
- Taylor expansion for the principal branch:

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}, \quad |z| < 1$$

- Using the complex logarithm (the principal branch) we can define the powers z^α $\forall \alpha \in \mathbb{C}$:

$z^\alpha \stackrel{def}{=} e^{\alpha \log z}$ (usual power properties are preserved, e.g. $1^\alpha = 1$, $(z^{1/\alpha})^\alpha = z$)

Every $\omega \neq 0$ can be rewritten as $\omega = e^z$. A generalization of this fact is given in:

Th.5.2:
If $f: \Omega \to \mathbb{C}$, Ω is a domain, is hol. and $f \neq 0$, then $\exists g: \Omega \to \mathbb{C}$ s.t. $f(z) = e^{g(z)}$, i.e. $g(z)$ is a particular branch of $\log f(z)$.

Proof:

$$g(z) = \int_{z_0}^{z} \frac{f'(w)}{f(w)} dw + c_0, \quad e^{c_0} = f(z_0) \qquad g'(z) = \frac{f'(z)}{f(z)}$$

$$\frac{d}{dz}\left(f(z) e^{-g(z)}\right) = f'(z) e^{-g(z)} - f(z) g'(z) e^{-g(z)} = \left(f'(z) - f(z)\frac{f'(z)}{f(z)}\right) e^{-g(z)} = 0 \Rightarrow f(z) = e^{g(z)}$$

⑥ The argument principle and its applications.

6.1. The argument principle.

Consider $f:\Omega\to\mathbb{C}$ that is hol. and $\log f(z) = \ln|f(z)| + i\arg f(z)$. Then $\frac{f'(z)}{f(z)} = \frac{d}{dz}\log f(z)$.

Observe that:

$$\frac{\left(\prod_{k=1}^{N} f_k\right)'}{\prod_{k=1}^{N} f_k} = \sum_{k=1}^{N}\frac{f_k'}{f_k}, \text{ e.g. } \frac{(f_1f_2)'}{f_1f_2} = \frac{f_1'f_2 + f_1f_2'}{f_1f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$$

Now let's investigate the behaviour of $\frac{f'}{f}$ near zeros and poles of f:

1. $z = z_0$ is a zero of order n of f:

$f(z) = (z-z_0)^n g(z)$, where g is hol. and $g(z)\neq 0\ \forall z\in U(z_0)$

$$\frac{f'(z)}{f(z)} = \frac{n(z-z_0)^{n-1}g(z) + (z-z_0)^n g'(z)}{(z-z_0)^n g(z)} = \frac{n}{z-z_0} + \frac{g'(z)}{g(z)}$$

$z=z_0$ is a simple pole of $\frac{f'}{f}$ and $\operatorname{res}_{z_0}\frac{f'}{f} = n$

2. $z=z_0$ is a pole of order n of f:

$f(z) = (z-z_0)^{-n}h(z)$, where h is hol. and $h(z)\neq 0\ \forall z\in U(z_0)$

$$\frac{f'(z)}{f(z)} = \frac{-n(z-z_0)^{-n-1}h(z) + (z-z_0)^{-n}h'(z)}{(z-z_0)^{-n}h(z)} = \frac{-n}{z-z_0} + \frac{h'(z)}{h(z)}$$

$z=z_0$ is a simple pole of $\frac{f'}{f}$ and $\operatorname{res}_{z_0}\frac{f'}{f} = -n$

So if $f:\mathbb{C}\to\mathbb{C}$ is meromorphic, then $\forall$ domain Ω s.t. $\partial\Omega$ has no zeros and poles:

$$\oint_{\partial\Omega}\frac{f'(z)}{f(z)}dz = \oint_{\partial\Omega} d(\log f(z)) = \left(\ln|f(r)| + i\arg f(z)\right)\Big|_{\alpha}^{\beta} = i\left[\arg f(z(\beta)) - \arg f(z(\alpha))\right] = i\,\underbrace{\Delta_{\partial\Omega}\arg f(z)}_{\text{increment of }\arg f(z)\text{ along }\partial\Omega}$$

$$\oint_{\partial\Omega}\frac{f'(z)}{f(z)}dz = 2\pi i\sum_{\nu=1}^{M}\operatorname{res}_{z_\nu}\frac{f'}{f}, \text{ where } \{z_1,\dots,z_M\} \text{ are zeros and poles of } f$$

Now if we define the winding number of $f(\partial\Omega)$ around $z=0$ as $\operatorname{ind}_0 f(\partial\Omega) \overset{\text{def}}{=} \frac{1}{2\pi}\Delta_{\partial\Omega}\arg f(z)$, (i.e. the number of turns the vector $f(z)$ makes around the origin as z varies along $\partial\Omega$), then we can conclude that:

$$\boxed{\operatorname{ind}_0 f(\partial\Omega) = N - P}$$

, N is the number of zeros of f inside Ω, P is the number of poles of f inside Ω (counted with their orders)

Argument principle

- If we define z_0-point of f as the solution to $f(z) = z_0$, then:

$$N_{z_0} - P = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f'(z)\,dz}{f(z)-z_0} = \frac{1}{2\pi}\Delta_{\partial\Omega}\arg(f(z)-z_0) = \operatorname{ind}_{z_0} f(\partial\Omega)$$

- The argument principle can be generalized as the following:

$$\frac{1}{2\pi i}\int_{\partial\Omega}F(z)\frac{f'(z)}{f(z)}dz = \underbrace{\sum_{\nu}F(z_\nu)}_{\text{zeros of } f} - \underbrace{\sum_{\mu}F(z_\mu)}_{\text{poles of } f}, \text{ where } F \text{ is hol.}$$

6.2. Applications of the argument principle.

Th. 6.1 (Rouche):
Consider $f:\Omega\to\mathbb{C}$ and $g:\Omega\to\mathbb{C}$ that are hol., where Ω is open, $0\in\Omega$, and $D\subset\Omega$. If $|f(z)|>|g(z)|$ $\forall z\in\partial D$, then f and $f+g$ have the same number of zeros inside D.

- So hol. functions can be perturbed slightly without changing the number of its zeros.
- Also the Rouche's th. can be used to proove the fundamental th. of algebra.

Th. 6.2 (Open mapping th.):
If $f:\Omega\to\mathbb{C}$, Ω is a domain, is hol. and $f\neq$ const, then $f(\Omega)$ is a domain.

- So hol. functions map open sets to open sets.
- Also the Open mapping th. helps to solve the problem of local inversion of a hol. function. If $z=z_0$ is non-critical ($f'(z_0)\neq 0$), then f is locally invertable near $z=z_0$:

$$f^{-1}(\omega)=\sum_{n=0}^{\infty} d_n(\omega-\omega_0)^n,\quad \omega_0=f(z_0)\quad d_0=z_0\quad d_n=\frac{1}{n!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^n}\left(\frac{z-z_0}{f(z)-f(z_0)}\right)^n,\ n=1,2,\dots$$

Example:

$f(z)=z\,e^{-az}$ $\qquad z_0=0$

$d_0=f(z_0)=0$

$$d_n=\frac{1}{n!}\lim_{z\to 0}\frac{d^{n-1}}{dz^n}\left(\frac{z}{f(z)}\right)^n=\frac{1}{n!}\lim_{z\to 0}\frac{d^{n-1}}{dz^n}\left(e^{anz}\right)=\frac{1}{n!}\lim_{z\to 0}\left((an)^{n-1}e^{anz}\right)=\frac{(an)^{n-1}}{n!}$$

Th. 6.3 (Maximum modulus principle):
If $f:\Omega\to\mathbb{C}$, Ω is a domain, is hol. and $f\neq$ const, then f can't attain a max. in Ω.

Cor. 6.4:
Consider a domain Ω s.t. $\overline{\Omega}$ is compact. If $f:\Omega\to\mathbb{C}$ is hol. on Ω and cont. on $\overline{\Omega}$, then $\sup\limits_{z\in\Omega}|f(z)|\leqslant\sup\limits_{z\in\partial\Omega}|f(z)|$.

- So non-constant hol. functions can attain max. only on the boundaries of compact domains.
- The maximum modulus principle is reminiscant of the same property for harmonic functions:

Consider hol. $f:\Omega\to\mathbb{C}$, Ω is open, then $\forall D_r(z_0)\subset\Omega$ we have $f(z)=\sum\limits_{n=0}^{\infty}a_n(z-z_0)^n$.

$$a_n=\frac{f^{(n)}(z_0)}{n!}=\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}}=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}}\,ire^{i\theta}\,d\theta=\frac{1}{2\pi r^n}\int_0^{2\pi}f(z_0+re^{i\theta})\,e^{-in\theta}\,d\theta$$

So the coefficients of the power expansion of a hol. function are equal (up to r^n) to the Fourie coefficients of this function. Also since $f(z)=\frac{1}{2\pi}\int_0^{2\pi}f(z_0+re^{i\theta})\,d\theta$, by taking the real parts of both sides, we obtain:

$$\mathrm{Re}(f(z_0))=u(z_0)=\frac{1}{2\pi}\int_0^{2\pi}u(z_0+re^{i\theta})\,dt$$, i.e. mean-value property

Recall that u is harmonic whenever f is hol., so the mean-value property is true for both hol. and harmonic functions. Therefore, for such functions max. value can only be attained on the boundaries of the sets where these functions are defined.