# Basic maths

## 1. Logic symbols.

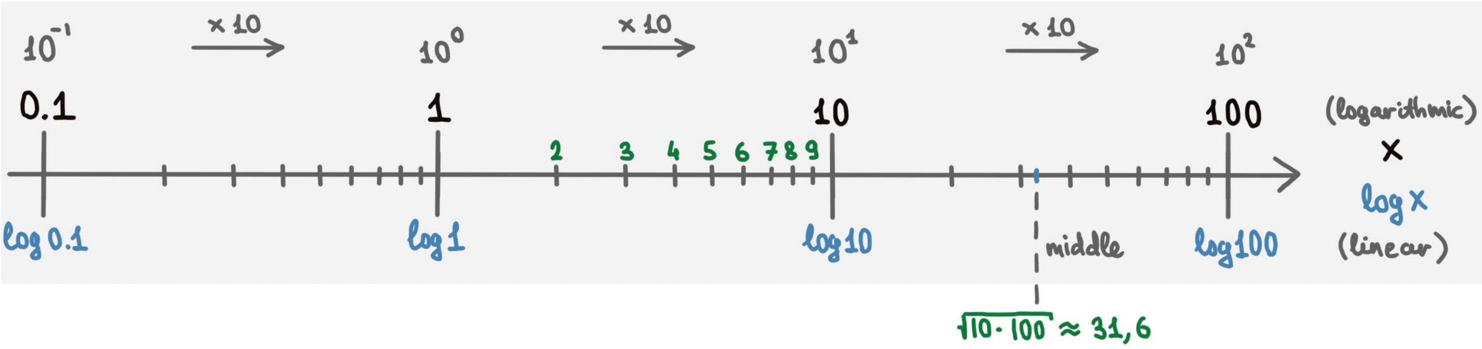
| | | | |
|---|---|---|---|
| $\Rightarrow$ | material implication | $\top$ | tautology (truth) |
| $\Leftrightarrow$ | material equivalence | $\bot$ | contradiction (falsity) |
| $\neg$ | negation (not) | $\forall$ | universal quantification |
| $\mathbb{D}$ | domain of discourse (universal set) | $\exists$ | existential quantification |
| $\wedge$ | conjunction (and) | $\exists!$ | uniqueness quantification |
| $\vee$ | disjunction (or) | $\vdash$ | syntactically entails (proves) |
| $\oplus$ | exclusive disjunction (xor) | $\vDash$ | semantically entails (models) |

## 2. Logarithms.

$$a^{\log_a x} = x \qquad \log_a(xy) = \log_a x + \log_a y \qquad \log_x y = (\log_y x)^{-1} \qquad x^{\log_a y} = y^{\log_a x}$$

$$\log_a a^x = x \qquad \log_a(x/y) = \log_a x - \log_a y \qquad \log_{x^n}(y^m) = \frac{m}{n}\log_x y \qquad \log_a b = \frac{\log_c b}{\log_c a}$$

Log scale:




$\log 2 \approx 0.301$

$\log 3 \approx 0.477$

$\log 4 = 2\log 2 \approx 0.602$

$\log 5 = \log 10 - \log 2 \approx 0.699$

$\log 6 = \log 2 + \log 3 \approx 0.778$

$\log 7 \approx 0.845$

$\log 8 = 3 \cdot \log 2 \approx 0.903$

$\log 9 = 2 \cdot \log 3 \approx 0.954$

## 3. Trigonometry.

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad e^{ix} = \cos x + i \sin x$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \qquad \cosh x = \frac{e^{x} + e^{-x}}{2} \qquad e^{x} = \sinh x + \cosh x$$

$$\sin x = -i \sinh(ix)$$
$$\sinh x = -i \sin(ix)$$
$$\cos x = \cosh(ix)$$
$$\cosh x = \cos(ix)$$

$$\cos^2 x + \sin^2 x = 1 \qquad \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$1 + \tan^2 x = \frac{1}{\cos^2 x} \qquad \sin^2 x = \frac{1 - \cos 2x}{2} \qquad \sin 2x = 2 \sin x \cos x$$

$$1 + \cot^2 x = \frac{1}{\sin^2 x} \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \qquad \cos 2x = \cos^2 x - \sin^2 x$$

# 4. Sequences.

## 4.1. Arithmetic progression.

$\boxed{a_n = a + (n-1)\,d}$ , where $d$ is a common difference

- $a_n = a_m + (n-m)\,d$
- sum of a progression:

$$S_n = \sum_{k=1}^{n} (a + (k-1)d) = a + (a+d) + \ldots + (a + (n-1)d)$$

$$S_n = \sum_{k=1}^{n} (a_n - (n-k)d) = (a_n - (n-1)d) + \ldots + (a_n - d) + a_n$$

$$\Rightarrow \quad 2S_n = n\,(a_1 + a_n)$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

- $S_{mn} = \sum_{k=m}^{n} (a + (k-1)d) = \frac{n-m}{2}(a_m + a_n)$
- $P_n = \prod_{k=0}^{n-1} (a + kd) = \prod_{k=1}^{n-1} d\left(\frac{a}{d} + k\right) = d^n \frac{\Gamma\left(\frac{a}{d} + n\right)}{\Gamma\left(\frac{a}{d}\right)}$
- arithmetic mean: $A(a_1, \ldots, a_n) = \frac{S_n}{n} = \frac{1}{2}(a_1 + a_n)$ $\qquad a_n = \frac{1}{2}(a_{n-1} + a_{n+1})$

## 4.2. Geometric progression.

$\boxed{b_n = b\,r^{n-1}}$ , where $r$ is a common ratio

- $b_n = b_m\, r^{m-n}$
- sum of a progression:

$$S_n = \sum_{k=1}^{n} b\,r^{k-1} = b + br + br^2 + \ldots + br^{n-1}$$

$$(1-r)S_n = (1-r)(1 + r + r^2 + \ldots + r^{n-1})\,b$$

$$(1-r)S_n = (1 + r + r^2 + \ldots + r^{n-1} - r - r^2 - \ldots - r^n)\,b \quad \Rightarrow \quad S_n = \frac{b(1-r^n)}{1-r} = \frac{b_1 - b_{n+1}}{1-r}$$

- $S_{mn} = \sum_{k=m}^{n} b\,r^{k-1} = \frac{b(r^{m-1} - r^n)}{1-r}$
- $S_n^{(even)} = \sum_{k=0}^{n} b\,r^{2k} = \sum_{k=0}^{n} b\,(r^2)^k = \frac{b(1 - r^{2n+2})}{1-r^2}$
- $S_n^{(odd)} = \sum_{k=0}^{n} b\,r^{2k+1} = \sum_{k=0}^{n} (br)(r^2)^k = \frac{br(1 - r^{2n+2})}{1-r^2}$
- geometric series:

$$\sum_{k=0}^{\infty} b\,r^k = \frac{b}{1-r}, \quad |r| < 1$$

$$\sum_{k=0}^{\infty} b\,r^{2k} = \frac{b}{1-r^2}, \quad |r| < 1$$

$$\sum_{k=0}^{\infty} b\,r^{2k+1} = \frac{br}{1-r^2}, \quad |r| < 1$$

- $P_n = \prod_{k=0}^{n-1} b\,r^k = b^n r^{1+2+\ldots+(n-1)} = b^n r^{\frac{1}{2}(n-1)n} = \left(b\sqrt{r^{n-1}}\right)^n = \left(\sqrt{b_1 \cdot b_n}\right)^n$
- geometric mean: $G(b_1, \ldots, b_n) = \sqrt[n]{P_n} = \sqrt{b_1 \cdot b_n}$ $\qquad b_n = \sqrt{b_{n-1} \cdot b_{n+1}}$

## 4.3. Arithmetico-geometric sequence.

$$\boxed{t_n = a_n \cdot b_n}, \text{ where } \begin{array}{l} a_n = a + (n-1)d \\ b_n = b r^{n-1} \end{array}$$

- sum of a progression:

$$S_n = \sum_{k=1}^{n} (a+(k-1)d) b r^{k-1} = ab + (a+d)br + (a+2d)br^2 + \ldots + (a+(n-1)d)br^{n-1}$$

$$(1-r)S_n = ab + (a+d)br + (a+2d)br^2 + \ldots + (a+(n-1)d)br^{n-1} -$$

$$- abr - (a+d)br^2 - (a+2d)br^3 - \ldots - (a+(n-1)d)br^n =$$

$$= ab + db(r + r^2 + \ldots + r^{n-1}) - (a+(n-1)d)br^n =$$

$$= ab + db(r + r^2 + \ldots + r^{n-1} + r^n) - (dbr^n + (a+(n-1)d)br^n) =$$

$$= ab + dbr(1 + r + \ldots + r^{n-1}) - (a+nd)br^n$$

$$S_n = \frac{ab - (a+nd)br^n}{1-r} + \frac{dbr(1-r^n)}{(1-r)^2} = \frac{a_1 b_1 - a_{n+1}b_{n+1}}{1-r} + \frac{dr(b_1 - b_{n+1})}{(1-r)^2}$$

- $S = \sum_{k=1}^{\infty} t_n = \frac{ab}{1-r} + \frac{drb}{(1-r)^2}$, $|r|<1$

- Examples:

1) $t_n = (n-1)2^{-(n-1)}$ $\quad \left\{\frac{0}{1}, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \ldots\right\}$

   $a = 0 \quad d = 1$
   $b = 1 \quad r = 1/2$

2) $t_n = (n-1)r^{n-1}$, $|r|<1$ $\quad$ Gabriel's staircase

   $a = 0 \quad d = 1$
   $b = 1 \quad |r|<1$

   $$\sum_{k=1}^{\infty} k r^k = \frac{r}{(1-r)^2}$$

## 4.4. Algebraic identities.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} 2k = n(n+1)$$

$$\sum_{k=1}^{n} (2k-1) = n^2$$

$$(a+b)^n = \sum_{k=1}^{n} \binom{n}{k} a^{n-k} b^k$$

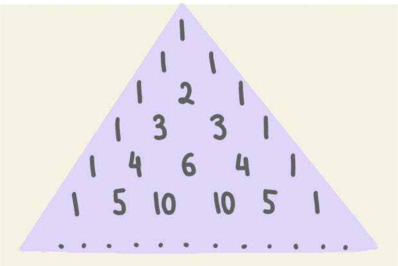
$$2^n = \sum_{k=1}^{n} \binom{n}{k}$$

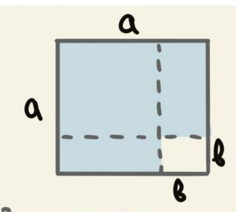
$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$$

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \ldots - ab^{n-2} + b^{n-1}), \; n = 2k+1$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$$




$$a^2 - b^2 = (a-b)a + (a-b)b$$

## 5. Differentiation.

$$(x^k)' = k x^{k-1}$$

$$(\sin x)' = \cos x \qquad (\tan x)' = \frac{1}{\cos^2 x} \qquad \left(\arcsin \frac{x}{a}\right)' = \frac{1}{\sqrt{a^2 - x^2}} \qquad \left(\arctan \frac{x}{a}\right)' = \frac{a}{a^2 + x^2}$$

$$(\cos x)' = -\sin x \qquad (\cot x)' = -\frac{1}{\sin^2 x} \qquad \left(\arccos \frac{x}{a}\right)' = -\frac{1}{\sqrt{a^2 - x^2}} \qquad \left(\operatorname{arccot} \frac{x}{a}\right)' = -\frac{a}{a^2 + x^2}$$

$$(\sinh x)' = \cosh x \qquad (\tanh x)' = \frac{1}{\cosh^2 x} \qquad \left(\operatorname{arcsinh} \frac{x}{a}\right)' = \frac{1}{\sqrt{x^2 + a^2}} \qquad \left(\operatorname{arctanh} \frac{x}{a}\right)' = \frac{a}{a^2 - x^2}$$

$$(\cosh x)' = \sinh x \qquad (\coth x)' = -\frac{1}{\sinh^2 x} \qquad \left(\operatorname{arccosh} \frac{x}{a}\right)' = \frac{1}{\sqrt{x^2 - a^2}} \qquad \left(\operatorname{arccoth} \frac{x}{a}\right)' = \frac{a}{a^2 - x^2}$$

$$(a^x)' = a^x \ln a \qquad (\log_a x)' = \frac{1}{x \ln a} \qquad (u^v)' = (e^{v \ln u})'$$

$$(e^x)' = e^x \qquad (\ln x)' = \frac{1}{x}$$

## 6. Integration.

$$\int x^n dx = \frac{x^{n+1}}{n+1},\ n \neq -1 \qquad \int a^x dx = \frac{a^x}{\ln a} \qquad \int \log_a x \, dx = \frac{x \ln x - x}{\ln a}$$

$$\int \frac{dx}{x} = \ln|x| \qquad \int e^x dx = e^x \qquad \int \ln x \, dx = x \ln x - x$$

$$\int \sin x \, dx = -\cos x \qquad \int \tan x \, dx = -\ln|\cos x| \qquad \int \sinh x \, dx = \cosh x \qquad \int \tanh x \, dx = \ln|\cosh x|$$

$$\int \cos x \, dx = \sin x \qquad \int \cot x \, dx = \ln|\sin x| \qquad \int \cosh x \, dx = \sinh x \qquad \int \coth x \, dx = \ln|\sinh x|$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} \qquad \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} \qquad \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right|$$

# 7. Taylor series.

$$(1+x)^{\alpha} = \sum_{k=1}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-(k-1))}{k!} x^k + \dots, \quad \alpha \in \mathbb{R},\ |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \qquad \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \qquad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots \qquad \frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots$$

$$e^x = \sum_{k=1}^{\infty} \frac{x^k}{k!} \qquad a^x = \sum_{k=1}^{\infty} \frac{(x \ln a)^k}{k!}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \dots$$

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \dots$$

# Linear Algebra (from 3Blue1Brown and wiki)

## 1. Linear span of vectors.

$$\boxed{\operatorname{span}(X) \overset{\text{def}}{=} \underbrace{\left\{ \sum_{i=1}^{k} \lambda_i x_i \,\middle|\, k \in \mathbb{N},\ \lambda_i \in K,\ x_i \in X \right\}}_{\text{span of vectors}}}$$

, where $\underset{\text{(set of vectors)}}{X = \{x_1, \dots, x_n\}} \in \underset{\text{(vector space)}}{V(K)}$

Properties:

1. $\operatorname{span}(X)$ is the smallest linear subspace of $V$ that contains $X$. (can take this as a definition of span)
2. Basis is the smallest subspace of $V$ that spans $V$.

## 2. Linear transformations.

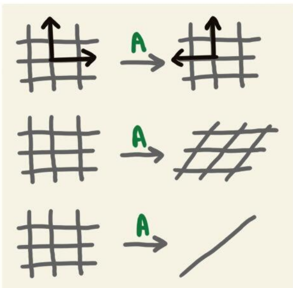
$$A(\vec{x}) = A\left(\sum_{i=1}^{n} x^i \vec{e}_i\right) = \sum_{i=1}^{n} x^i A(\vec{e}_i) = \begin{pmatrix} \sum_{i=1}^{n} x^i A^1(\vec{e}_i) \\ \cdots \\ \sum_{i=1}^{n} x^i A^m(\vec{e}_i) \end{pmatrix} = \begin{array}{c} \begin{array}{ccc} A(\vec{e}_1) & & A(\vec{e}_n) \end{array} \\ \begin{pmatrix} A^1(\vec{e}_1) & \dots & A^1(\vec{e}_n) \\ \dots & \dots & \dots \\ A^m(\vec{e}_1) & \dots & A^m(\vec{e}_n) \end{pmatrix} \end{array} \begin{pmatrix} x^1 \\ \dots \\ x^n \end{pmatrix}$$

Properties:

1. $A : \mathbb{R}^n \to \mathbb{R}^m$ : 1.1. $A(\vec{0}) = \vec{0}$
   1.2. grid lines remain parallel and evenly spaced
2. Columns of $A$ are a new basis $\{A(\vec{e}_i)\}$ in old coordinates.
3. If two columns or rows of $A$ are linearly dependent (i.e. $A$ is not full-ranked), then $A$ squashes $V$ to a space that is 1 dimension smaller.

Examples:

1) Rotations of $\mathbb{R}^2$ ($\pi/2$) $\quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
2) Shear of $\mathbb{R}^2$ $\quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
3) Squashing of $\mathbb{R}^2$ $\quad A = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$



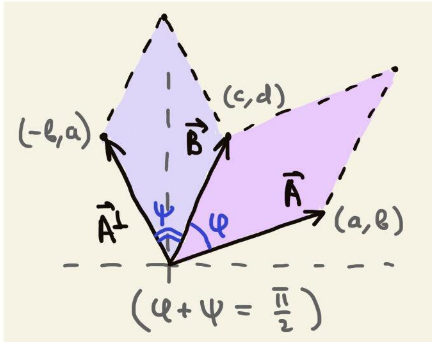

## 3. Determinant

Consider an oriented volume in $V$ and $A : V \to W$. Then $\det A$ represents a scale factor when transforming this volume under $A$.

Properties:

1. $\det A = 0$ : $A$ squshes space
2. $\det A \neq 0 \iff \exists A^{-1}$
3. $\operatorname{rank} A \overset{\text{def}}{=} \dim(\operatorname{im} A)$, where $\operatorname{im} A$ is a span of column vectors of $A$ (column space)
   (rank helps to tell when a solution to a system of linear eqns. exists)

Determinant formula for $\mathbb{R}^2$:

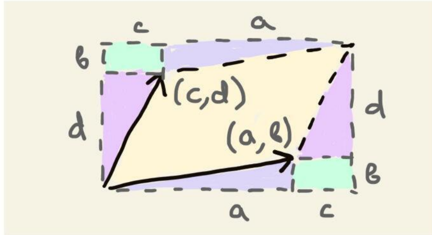



1) $S_{AB} = S_{A^\perp B}$

$S_{AB} = |\vec{A}|\cdot|\vec{B}|\cdot\sin\varphi = |\vec{A}||\vec{B}|\cos\left(\frac{\pi}{2}-\varphi\right) =$

$= |\vec{A}^\perp||\vec{B}|\cos\psi = \langle \vec{A}^\perp, \vec{B}\rangle = (-b \;\; a)\begin{pmatrix} c \\ d\end{pmatrix} =$

$= ad - bc$




2) $S_{AB} = (a+c)(b+d) - ab - dc - 2bc = ad - bc$

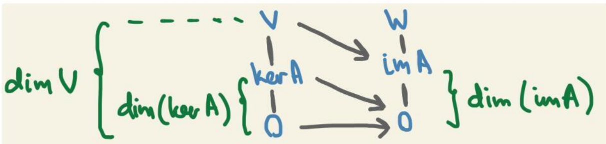
## 4. 4 fundamental subspaces.

Consider $A: \mathbb{R}^n \to \mathbb{R}^m$, then:

| | |
|---|---|
| $\operatorname{im} A$ : $\dim(\operatorname{im} A) = r$ (rank) <br> (column space) | $\operatorname{coim} A \stackrel{\text{def}}{=} \operatorname{im} A^T$ : $\dim(\operatorname{coim} A) = r$ (rank) <br> (row space) |
| $\ker A$ : $\dim(\ker A) = n - r$ (nullity) <br> (null space) | $\operatorname{coker} A \stackrel{\text{def}}{=} \ker A^T$ : $\dim(\operatorname{coker} A) = m - r$ (corank) <br> (left nullspace) |

$$\boxed{\dim(\operatorname{im} A) + \dim(\ker A) = \dim V}$$

rank-nullity theorem




$\ker A = (\operatorname{im} A^T)^\perp \qquad \operatorname{coker} A = (\operatorname{im} A)^\perp$

## 5. Bonus memes. (from 3B1B)

- cross product det formula derivation
- eigenvectors and eigenvalues (remain on a span after trans.)
- geometric interpretation of Kramer's rule
- derivative matrix for polynomials

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} = \frac{d}{dx} \quad \text{for } p_n(x) = \sum_{k=0}^{n} c_k x^k \quad (\{1, x, x^2, \ldots\} \text{ is a basis})$$

$$\boxed{(\vec{a}\cdot\vec{b})^2 + \|\vec{a}\times\vec{b}\|^2 = \|\vec{a}\|^2\cdot\|\vec{b}\|^2}$$

Lagrange identity

# Infinite series (from Zorich b.I §3.1.4)

## 1) Numerical series.

$(\{a_n\},\{S_n\})$ is a **series**: $S_n = \sum_{k=1}^{n} a_k \ \forall n \in \mathbb{N}$, where $\{a_n\}$ : terms of a series
$\{S_n\}$ : partial sums
$S = \lim_{n\to\infty} S_n$ : sum of a series

$\sum_{n=1}^{\infty} a_n$ is **convergent** : $\{S_n\}$ converges
$\sum_{n=1}^{\infty} a_n$ is **divergent** : $\{S_n\}$ diverges (or $\nexists \lim_{n\to\infty} S_n$)

**Th.1.1 (Cauchy conv. cr.):**
$\sum_{n=1}^{\infty} a_n$ converges $\iff \forall \varepsilon>0\ \exists N\in\mathbb{N} : \forall k \geq \ell > N \ \left|\sum_{m=\ell}^{k} a_m\right| < \varepsilon$

**Proof:**

$\{S_n\}$ conv. $\overset{\text{Cauchy}}{\iff} \forall \varepsilon>0\ \exists N\in\mathbb{N}: \forall k \geq \ell > N \ |S_k - S_\ell| < \varepsilon$

$|S_k - S_\ell| = \left|\sum_{m=1}^{k} a_m - \sum_{m=1}^{\ell} a_m\right| = \left|\sum_{m=\ell+1}^{k} a_m\right|$

**Corollaries:**

- if only a finite number of terms of a series are changed, the resulting new series will converge/diverge if the original series did.
- $\sum_{n=1}^{\infty} a_n$ conv. $\Rightarrow \lim_{n\to\infty} a_n = 0$ (necessary cond. for conv.)
- $\sum_{n=1}^{\infty} a_n$ conv. $\Rightarrow r_m = \sum_{n=m+1}^{\infty} a_n$ is s.t. $\lim_{m\to\infty} r_m = 0$ (m-th remainder of a series)
- $\sum_{n=1}^{\infty} Ca_n$ conv. $\Rightarrow \sum_{n=1}^{\infty} a_n$ conv., where $C \neq 0$

**Examples:**

1) $\sum_{n=0}^{\infty} q^n$ geometric series

$|q| \geq 1$: $|q^n| = |q|^n \geq 1 \Rightarrow \lim_{n\to\infty} q^n \neq 0 \Rightarrow$ diverges

$|q| < 1$: $S_n = 1 + q + \ldots + q^n = \frac{1-q^n}{1-q} \Rightarrow S = \lim_{n\to\infty} S_n = \frac{1}{1-q}$ converges

2) $\sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series

$\lim_{n\to\infty} \frac{1}{n} = 0$

$S_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ $\quad |S_{2n} - S_n| = \frac{1}{n+1} + \ldots + \frac{1}{n+n} > n \cdot \frac{1}{2n} = \frac{1}{2} \Rightarrow$ diverges

## 2) Absolute and conditional convergence.

Consider $\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - \ldots$ It diverges since $\lim_{n\to\infty} a_n \neq 0$. But we can make it converge by unserting parantheses or moving the terms around:

$$(1-1) + (1-1) + \ldots \rightarrow 0$$
$$1 + (-1+1) + (-1+1) + \ldots \rightarrow 1$$
$$1 + 1 - 1 + 1 - \ldots = 1 + (1-1) + (1-1) + \ldots \rightarrow 2$$

So the usual laws for dealing with finite sums generally can't be extended to series. There is nevertheless a type of series that can be handled like finite sums, these are the so-called absolutely convergent series.

$\sum_{n=1}^{\infty} a_n$ is absolutely conv. : $\sum_{n=1}^{\infty} |a_n|$ converges

$\sum_{n=1}^{\infty} a_n$ is conditionaly conv. : $\sum_{n=1}^{\infty} |a_n|$ diverges, but $\sum_{n=1}^{\infty} a_n$ conv.

Properties:

- $\sum_{n=1}^{\infty} a_n$ conv. abs. $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ conv. (since $|a_\ell + \ldots + a_k| \leq |a_k| + \ldots + |a_\ell| = ||a_k| + \ldots + |a_\ell||$)
- $\sum_{n=1}^{\infty} a_n$ conv. abs. $\Rightarrow$ $\forall$ permutation $\sigma$ $\sum_{n=1}^{\infty} a_n = S$ (Cauchy)
- $\sum_{n=1}^{\infty} a_n$ conv. cond. $\Rightarrow$ $\forall S$ $\exists$ permutation $\sigma$ : $\sum_{n=1}^{\infty} a_n = S$ (Riemann)
- $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ conv. abs. $\Rightarrow$ $\sum_{n=1}^{\infty} a_n b_n$ and $\sum_{n=1}^{\infty} (a_n + b_n)$ conv. abs. and $S^+ = A + B$, $S^* = A \cdot B$

To better understand abs. convergence, let's investigate series $\sum_{n=1}^{\infty} p_n$ with only non-negative terms:

- $\{S_n\}$ for $\sum_{n=1}^{\infty} p_n$ is non-decresing
- $\sum_{n=1}^{\infty} p_n$ conv. $\Leftrightarrow$ $\{S_n\}$ is bounded above
- If $\exists N \in \mathbb{N}$ : $p_n \leq \tilde{p}_n$ $\forall n > N$, then
  1. $\sum_{n=1}^{\infty} \tilde{p}_n$ conv. $\Rightarrow$ $\sum_{n=1}^{\infty} p_n$ conv.
  2. $\sum_{n=1}^{\infty} p_n$ diver. $\Rightarrow$ $\sum_{n=1}^{\infty} p_n$ diver.

  direct comparison test
- If $\exists N \in \mathbb{N}$ : $\frac{p_{n+1}}{p_n} \leq \frac{\tilde{p}_{n+1}}{\tilde{p}_n}$ $\forall n > N$, then
  1. $\sum_{n=1}^{\infty} \tilde{p}_n$ conv. $\Rightarrow$ $\sum_{n=1}^{\infty} p_n$ conv.
  2. $\sum_{n=1}^{\infty} p_n$ diver. $\Rightarrow$ $\sum_{n=1}^{\infty} p_n$ diver.

  ratio comparison test
- If $\exists \lim_{n\to\infty} \frac{\tilde{p}_n}{p_n} = c$ s.t. $c < \infty$, then
  1. $\sum_{n=1}^{\infty} \tilde{p}_n$ conv. $\Leftrightarrow$ $\sum_{n=1}^{\infty} p_n$ conv.
  2. $\sum_{n=1}^{\infty} p_n$ diver. $\Leftrightarrow$ $\sum_{n=1}^{\infty} p_n$ diver.

  limit comparison test

**Th. 2.1 (Weierstrass M-test):**
Consider $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$. If $\exists N \in \mathbb{N}$ : $|a_n| \leq b_n$ $\forall n > N$, then $\sum_{n=1} b_n$ conv. $\Rightarrow$ $\sum_{n=1} a_n$ abs. conv.
(i.e. $\Sigma a_n$ is majorized by $\Sigma b_n$)

**Th. 2.2** (Cauchy's root test):

Consider $\sum_{n=1}^{\infty} a_n$ and $\alpha = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$ (if exists), then:
1. $\alpha < 1$ : conv. abs.
2. $\alpha > 1$ : diverges
3. $\alpha = 1$ : ?

**Th. 2.3** (d'Alembert's ratio test):

Consider $\sum_{n=1}^{\infty} a_n$ and $\alpha = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ (if exists), then:
1. $\alpha < 1$ : conv. abs.
2. $\alpha > 1$ : diverges
3. $\alpha = 1$ : ?

**Th. 2.4** (Maclaurin's integral test):

Consider non-increasing $f: [m, +\infty) \to \mathbb{R}$ s.t. $f(x) \geq 0$. Then $\sum_{n=m}^{\infty} f(n)$ conv. $\Leftrightarrow \exists \int_m^{\infty} f(x)\,dx$.

**Th. 2.5** (Leibniz test):

Consider $\sum_{n=1}^{\infty} (-1)^{n+1} p_n$. Then if $\{p_n\}$ is non-increasing and $\lim_{n\to\infty} p_n = 0$, then $\sum_{n=1}^{\infty} (-1)^{n+1} p_n$ conv.

**Th. 2.6** (Dirichlet-Abel test):

Consider non-increasing $\{b_n\}$ s.t. $\lim_{n\to\infty} b_n = 0$ and $\{a_n\}$ that is bounded above, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

**Th. 2.7** (Cauchy condensation test):

Consider non-increasing $\{p_n\}$. Then $\sum_{n=1}^{\infty} p_n$ conv. $\Leftrightarrow \sum_{n=0}^{\infty} 2^n p_{2^n}$ conv.

**Examples:**

1) $\sum_{n=1}^{\infty} \frac{1}{n^\lambda}$ hyperharmonic series

- $f(x) = \frac{1}{x^\lambda}$ is non-increasing on $[1, +\infty)$ and $f(x) > 0$, then:

$$\int_1^{\infty} \frac{dx}{x^\lambda} = \begin{cases} \left.\frac{x^{1-\lambda}}{1-\lambda}\right|_1^{\infty}, & \lambda \neq 1 \\ \left.\ln x\right|_1^{\infty}, & \lambda = 1 \end{cases}, \text{ then } \begin{array}{l} \lambda < 1: \text{ diverges} \\ \lambda = 1: \text{ diverges} \\ \lambda > 1: \text{ conv. and } \sum_{n=1}^{\infty} \frac{1}{n^\lambda} = \frac{1}{\lambda - 1} \end{array}$$

- $\{p_n\}$ is non-increasing for $\lambda \geq 0 \overset{\text{Cauchy}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{1}{n^\lambda}$ conv./diver. whenever $\sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^\lambda}$ does

$\sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^\lambda} = \sum_{n=0}^{\infty} (2^{1-\lambda})^n$, then $\alpha = \overline{\lim_{n\to\infty}} \sqrt[n]{(2^{1-\lambda})^n} = 2^{1-\lambda}$

Series converges whenever $\alpha < 1$ (Cauchy test) $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^\lambda}$ conv. if $\lambda > 1$

2) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^\lambda} = 1 - \frac{1}{2^\lambda} + \frac{1}{3^\lambda} - \dots$ , $\lambda > 1$

$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n^\lambda}\right| = \sum_{n=1}^{\infty} \frac{1}{n^\lambda} \overset{\text{M-test}}{\Longrightarrow}$ conv. abs.

3) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

$\{p_n\}$ is non-increasing and $\lim_{n\to\infty} p_n = 0 \overset{\text{Leibniz}}{\Longrightarrow}$ conv.

# Fields on manifolds (Zorich b. II § 15.3.6.)

## 1. Commutator of vector fields

$$\boxed{[X,Y]f \overset{df}{=} X(Yf) - Y(Xf)}, \quad X,Y: C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R}),\ f \in C^\infty(M,\mathbb{R})$$

$$[X,Y] = X^i\partial_i(Y^j\partial_j) - Y^i\partial_i(X^j\partial_j) = X^i\partial_i Y^j\partial_j + \cancel{X^iY^j}\partial_{ij} - Y^i\partial_i X^j\partial_j -$$
$$- \cancel{Y^iX^j}\partial_{ij} = (X^i\partial_i Y^j - Y^i\partial_i X^j)\partial_j$$

## 2. Exterior derivative

$$\boxed{d\omega(X,Y) = X(\omega Y) - Y(\omega X) - \omega[X,Y]}, \quad \omega: TM \to \mathbb{R}$$

Proof:

$\omega = \omega_i dx^i \qquad X = X^i\partial_i \qquad \omega(X) = \omega_i X^i$

$d\omega = \partial_i\omega_j\, dx^i \wedge dx^j \qquad Y = Y^i\partial_i \qquad \omega(Y) = \omega_i Y^i$

1) $X(\omega Y) - Y(\omega X) = X^i\partial_i(\omega_j Y^j) - Y^i\partial_i(\omega_j X^j) = X^i(Y^j\partial_i\omega_j + \omega_j\partial_i Y^j) -$
$- Y^i(X^j\partial_i\omega_j + \omega_j\partial_i X^j)$

$\omega[X,Y] = \omega_j(X^i\partial_i Y^j - Y^i\partial_i X^j)$

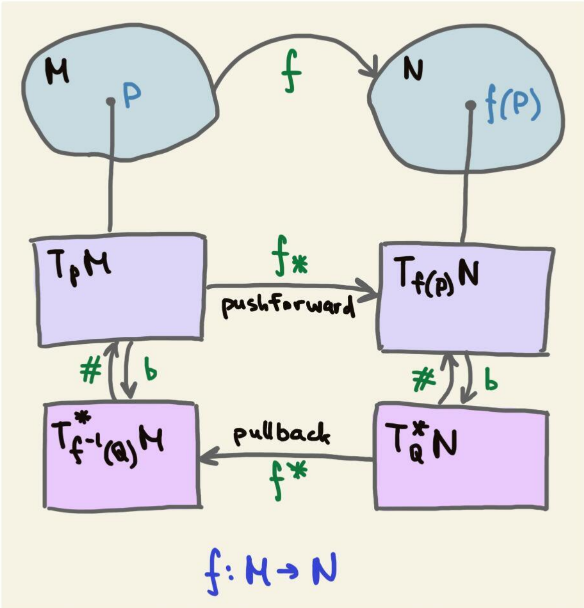
2) $X(\omega Y) - Y(\omega X) - \omega[X,Y] = \ldots = \partial_i\omega_j(X^iY^j - X^jY^i) = \partial_i\omega_j \begin{vmatrix} X^i & X^j \\ Y^i & Y^j \end{vmatrix}$

3) $d\omega(X,Y) = \partial_i\omega_j\, dx^i\wedge dx^j(X,Y) = \partial_i\omega_j \begin{vmatrix} X^i & X^j \\ Y^i & Y^j \end{vmatrix}$

$$\boxed{d\omega(X_1,\ldots,X_{m+1}) = \sum_{i=1}^{m+1}(-1)^{i+1}X_i\,\omega(X_1,\ldots,\hat{X}_i,X_{m+1}) + \sum_{1\le i<j\le m+1}(-1)^{i+j}\omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{m+1})}$$

Proof: (analogous)

# 3. Pullback and pushforward




$$\boxed{f_*\vec{v}(\phi) \overset{\text{def}}{=} \vec{v}(\phi \circ f)}\ , \quad f_*: T_pM \to T_{f(p)}N$$

pushforward

$\phi: N \to \mathbb{R}$

$\vec{v} \in T_pM$

$f_*\vec{v} \in T_{f(p)}N$

$$\boxed{f^*\omega(\vec{v}) \overset{\text{def}}{=} \omega(f_*\vec{v})}\ , \quad f^*: T^*_QN \to T^*_{f^{-1}(Q)}M$$

pullback

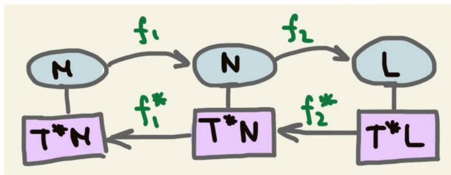
$\vec{v} \in T_{f^{-1}(Q)}M$

$\omega \in T^*_QN$

$f^*\omega \in T^*_{f^{-1}(Q)}M$

## Properties:

1. $f_*\vec{v}\,(\phi) = \vec{v}\,(\phi \circ f) = v^i \partial_i (\phi \circ f) = v^i \dfrac{\partial \phi}{\partial f^j} \dfrac{\partial f^j}{\partial x^i} = \left(v^i \dfrac{\partial f^j}{\partial x^i} \dfrac{\partial}{\partial f^j}\right) \phi$
2. $f_*[X, Y] = [f_*X, f_*Y]$
3. $f^*(\alpha_1 \omega_1 + \alpha_2 \omega_2) = \alpha_1 f^*\omega_1 + \alpha_2 f^*\omega_2$
4. $f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$
5. $f^* \circ d = d \circ f^*$
6. $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$



## 4. Lie derivative

$\vec{v} \rightarrow \{\varphi_t \,|\, t \in \mathbb{R}\}$ 1-parametric group of diffeomorphisms on $M$

$$D_{v(x)} f \equiv v(f)(x) = \lim_{t\to 0} \frac{f(\varphi_t(x)) - f(x)}{t} = \lim_{t\to 0} \frac{\varphi_t^* f - f}{t}(x)$$

$$\boxed{\mathcal{L}_X \omega \overset{\text{def}}{=} \lim_{t\to 0} \frac{\varphi_t^*\omega - \omega}{t}(x)}\ , \ \mathcal{L}_X : \Omega^k(M) \to \Omega^k(M) \quad \text{linear}$$

1. $\mathcal{L}_X(\omega_1 \wedge \omega_2) = \mathcal{L}_X \omega_1 \wedge \omega_2 + \omega_1 \wedge \mathcal{L}_X \omega_2$

2. $f \in \Omega^0(M)$ : $\mathcal{L}_X f = Xf = df(X)$

   $\mathcal{L}_X df = d(Xf)$ ?

## 5. Interior derivative (product)

$$\boxed{i_X\omega(X_1,\dots,X_{k-1}) \overset{\text{def}}{=} \omega(X, X_1, \dots, X_{k-1})}\ ,$$

$i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ linear
$\omega \in \Omega^k(M)$
$i_X\omega \in \Omega^{k-1}(M)$

1. $f \in \Omega^0(M)$ : $i_X f = 0 \ \forall f$

   $i_X df = df(X) = X^i \partial_i f = Xf$

   $\omega \in \Omega^1(M)$: $i_X\omega = \omega(X)$

2. $i_X\omega = \frac{1}{(k-1)!} X^i \omega_{i i_2 \dots i_k} dx^{i_2} \wedge \dots \wedge dx^{i_k}$

3. $i_X(\omega_1 \wedge \omega_2) = i_X\omega_1 \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge i_X \omega_2$

$$\boxed{\mathcal{L}_X = i_X d + d\, i_X}$$

homotopy identity

4. $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$

   $\mathcal{L}_X \circ i_X = i_X \circ \mathcal{L}_X$

5. $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$

   $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$

*. $\nabla_X Y - \nabla_Y X = [X,Y]$ , if $\Gamma^i_{jk} = \Gamma^i_{kj}$

$$\nabla_X Y - \nabla_Y X = X^i \nabla_i (Y^j \partial_j) - Y^i \nabla_i (X^j \partial_j) = X^i (\nabla_i Y^j \partial_j + Y^j \nabla_i \partial_j) -$$
$$- Y^i (\nabla_i X^j \partial_j + X^j \nabla_i \partial_j) = X^i (\partial_i Y^j + \Gamma^j_{ik} Y^k)\partial_j + X^i Y^j \Gamma^k_{ji} \partial_k -$$
$$- Y^i (\partial_i X^j + \Gamma^j_{ik} X^k)\partial_j - Y^i X^j \Gamma^k_{ji} \partial_k = (X^i \partial_i Y^j - Y^i \partial_i X^j)\partial_j$$

# 6. Geometry of k-vectors and k-forms

## 6.1. k-vectors (a.k.a. k-blades, multivectors, Clifford numbers)

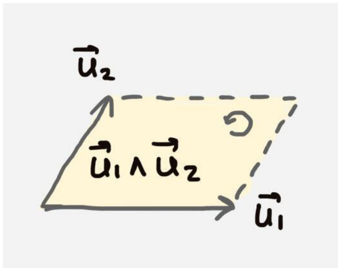
$u \in \Lambda^k V$, $\dim V = n$

Components of $u = u_1 \wedge \ldots \wedge u_k$ are oriented $(k-1)$-volumes, that are obtained from projecting a k-dimensional parallelepiped on the hyperplanes defined by the coordinate axes. And $\|u\|$ is the volume of this parallelepiped.

Examples:

1) $M = \mathbb{R}^2$

$$\left.\begin{aligned} \vec{u}_1 &= u_1^1 \vec{e}_1 + u_1^2 \vec{e}_2 \\ \vec{u}_2 &= u_2^1 \vec{e}_1 + u_2^2 \vec{e}_2 \end{aligned}\right| \Rightarrow \vec{u}_1 \wedge \vec{u}_2 = \underbrace{\begin{vmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \end{vmatrix}}_{\pm S} \vec{e}_1 \wedge \vec{e}_2$$



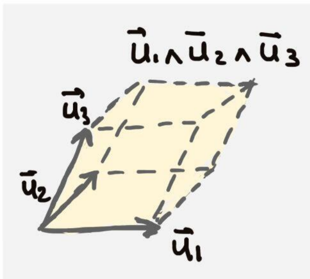

$$\|\vec{u}_1 \wedge \vec{u}_2\| = S_{(\vec{u}_1, \vec{u}_2)}$$

2) $M = \mathbb{R}^3$ $\quad \vec{u}_j = u_j^i \vec{e}_i$

$$\vec{u}_1 \wedge \vec{u}_2 = \underbrace{\begin{vmatrix} u_1^2 & u_2^2 \\ u_1^3 & u_2^3 \end{vmatrix}}_{\pm S_{yz}} \vec{e}_2 \wedge \vec{e}_3 + \underbrace{\begin{vmatrix} u_1^1 & u_2^1 \\ u_1^3 & u_2^3 \end{vmatrix}}_{\pm S_{xz}} \vec{e}_1 \wedge \vec{e}_3 + \underbrace{\begin{vmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \end{vmatrix}}_{\pm S_{xy}} \vec{e}_1 \wedge \vec{e}_2 \quad \left(\vec{u}_1 \wedge \vec{u}_2 \overset{*}{\leftrightarrow} \vec{u}_1 \times \vec{u}_2\right)$$

$$\|\vec{u}_1 \wedge \vec{u}_2\| = S_{(\vec{u}_1, \vec{u}_2)} \quad \left(= \sqrt{S_{yz}^2 + S_{xz}^2 + S_{xy}^2}\right)$$

$$\vec{u}_1 \wedge \vec{u}_2 \wedge \vec{u}_3 = \underbrace{\begin{vmatrix} u_1^1 & u_2^1 & u_3^1 \\ u_1^2 & u_2^2 & u_3^2 \\ u_1^3 & u_2^3 & u_3^3 \end{vmatrix}}_{\pm V} \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3$$

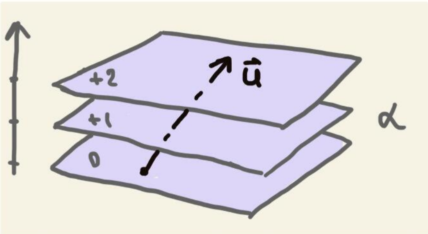



$$\|\vec{u}_1 \wedge \vec{u}_2 \wedge \vec{u}_3\| = V_{(\vec{u}_1, \vec{u}_2, \vec{u}_3)}$$

## 6.2. k-forms

### 1-forms:

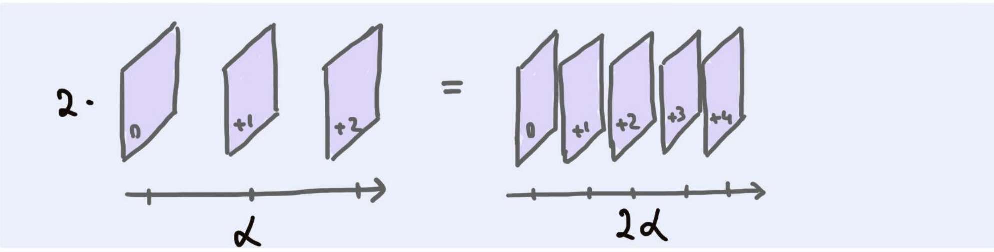
Geometrically 1-form can be viewed as a family of parallel hyperplanes with a chosen orientation. Then the result of a 1-form acting on a vector is a number of hyperplanes pierced by this vector (taking orientation into account).



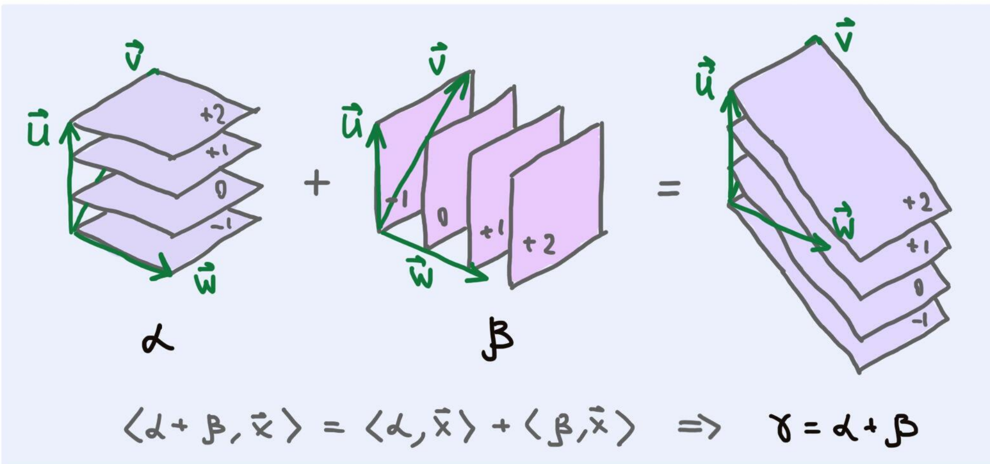

$\langle \alpha, \bar{u} \rangle$ number of intersections between $\bar{u}$ and $\alpha$

Geometrical interpretation of linear operations:

1. Scalar multiplication.




2. Addition.

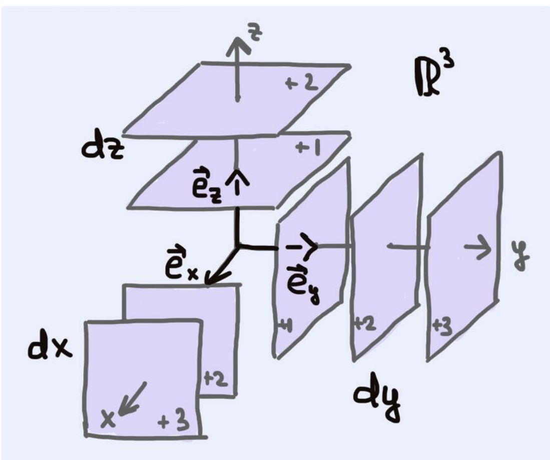


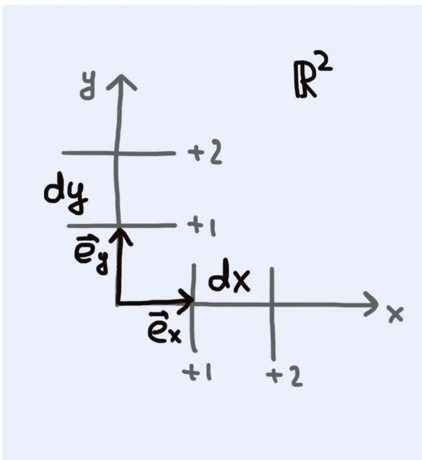

$\alpha \begin{cases} \langle \alpha, \vec{u} \rangle = \langle \alpha, \vec{v} \rangle = 3 \\ \langle \alpha, \vec{w} \rangle = 0 \end{cases}$

$\beta \begin{cases} \langle \beta, \vec{u} \rangle = \langle \beta, \vec{v} \rangle = 0 \\ \langle \beta, \vec{w} \rangle = 2{,}5 \end{cases}$

$\gamma \begin{cases} \langle \gamma, \vec{u} \rangle = \langle \gamma, \vec{v} \rangle = 3 \\ \langle \gamma, \vec{w} \rangle = 2{,}5 \end{cases}$

Coordinate 1-forms: $\left( \langle dx^\alpha, \partial_\beta \rangle = \delta^\alpha_\beta \right)$



# 1-forms in $\mathbb{R}^{1,3}$:

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

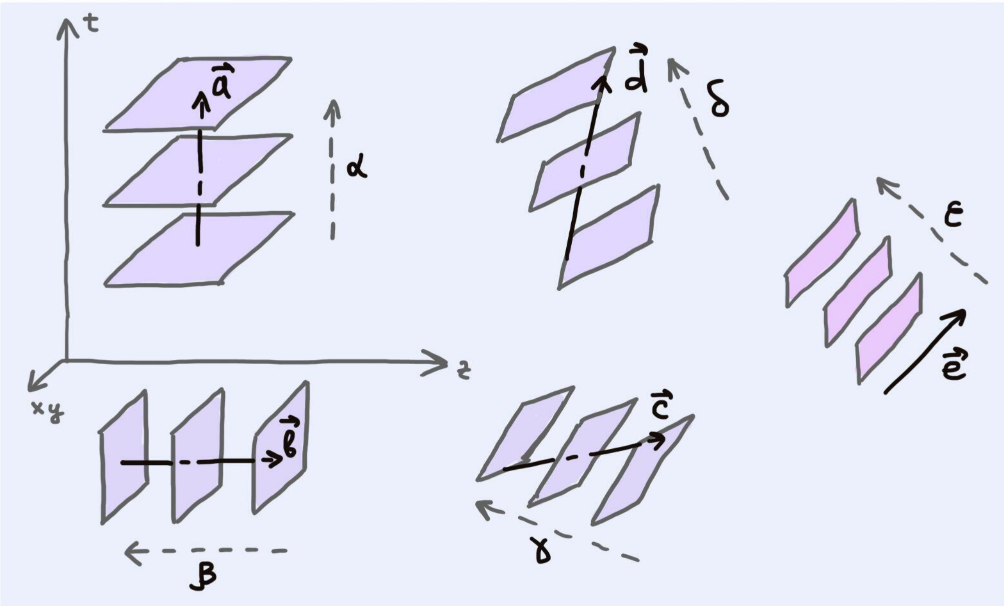
Let's find vectors corresponding to 1-forms:

$$\underset{\text{covector}}{\omega}(\vec{u}) = \langle \underset{\text{vector}}{\vec{\omega}}, \vec{u} \rangle \quad \forall\, \vec{u} \in TM$$

covector $\simeq$ vector

$Pr_{\vec{\omega}}(\vec{u}) = \omega(\vec{u})$. Then hypersurfaces of $\omega$ can be formed as spans of vectors perpendicular to $\vec{\omega}$. Consequently, $\omega(\vec{\omega}) = \langle \vec{\omega}, \vec{\omega} \rangle$ (number of intersections between $\omega$ and $\vec{\omega}$).

## Examples:

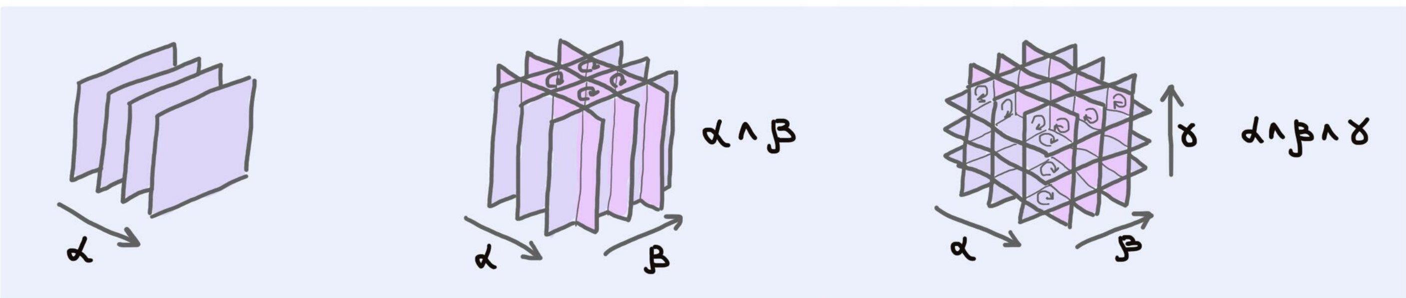



here hyperplanes (k=3) are depicted as planes (k=2)

$(\langle \vec{e}, \vec{e} \rangle = 0)$

# k-forms $(k>1)$:

k-forms can be viewed as collections of intersecting 1-forms (taking orientation into account). Then the result of a k-form $\omega = \omega^1 \wedge \ldots \wedge \omega^k$ acting on a k-vector $\vec{u}_1 \wedge \ldots \wedge \vec{u}_k$ is a number of k-form structures intersected by a k-parallelipiped defined by the k-vector $\vec{u}_1 \wedge \ldots \wedge \vec{u}_k$. For example, in the case of $\omega \in \Omega^2$ in $\mathbb{R}^3$ 2-form is a honeycomb-like structure and $\langle \omega, \vec{u}_1 \wedge \vec{u}_2 \rangle$ is a number of tubes cut by an oriented parallelogram $\vec{u}_1 \wedge \vec{u}_2$.



# $\boxed{\text{Hodge Duality}}$ (Tevian Dray, 1999)

+ Flanders "Diff. forms"

Basic operations on diff. forms $\to$ $\wedge$ (exterior product), $d$ (exterior derivative), $*$ (Hodge dual) $\to$ $\delta$ (coderivative), $\Delta$ (Laplace operator)

$V$ : vector space $(\dim V = n)$

$g$ : metric $\left( g^{ij} = g(\sigma^i, \sigma^j) = \pm \delta^{ij}\right.$, where $\{\sigma^i \in V, i = 1, \dots, n\}$ orthonormal basis$)$

$s$ : signature of $g$ (# of "$-$" signs in $g$, i.e. nonstandard definition)

$$\Lambda^0 V = \mathbb{R} \qquad \Lambda^1 V = V \quad \dots \quad \Lambda^p V = \emptyset,\ p>n \ ; \quad \dim \Lambda^p V = \binom{n}{p}$$

$$\{\sigma^I \mid \sigma^I = \sigma^{i_1} \wedge \dots \wedge \sigma^{i_p} \ ; \ I = \{i_k \mid k = 1, \dots, p \ ; \ 1 \leq i_1 < \dots < i_p \leq n\}\}$$ basis in $\Lambda^p V$

$\alpha \in \Lambda^p V$ decomposable : $\exists\, \alpha^i \in V : \alpha = \alpha^1 \wedge \dots \wedge \alpha^p$

$$\boxed{\langle \alpha, \beta \rangle \overset{\text{def}}{=} \det \langle \alpha^i, \beta^j \rangle}$$ , where $\alpha, \beta \in \Lambda^p V$; $\alpha, \beta$ – decomposable with $\{\alpha^i\}$ and $\{\beta^j\}$

inner product of p-vectors

1) $\langle \sigma^I, \sigma^J \rangle = \left( \prod_{k=1}^{p} \langle \sigma^{i_k}, \sigma^{i_k} \rangle \right) \delta^{IJ} = \pm \delta^{IJ}$, i.e. $\{\sigma^I\}$ – orthonorm basis in $\Lambda^p V$

(Note: $\langle \cdot, \cdot \rangle$ same notation for inner product in different $\Lambda^p V$)

2) $\langle \omega, \omega \rangle = \prod_{k=1}^{n} \langle \sigma^{i_k}, \sigma^{i_k} \rangle = (-1)^s$, where $\omega = \sigma^1 \wedge \dots \wedge \sigma^n \in \Lambda^n V$

(n-vector which defines orientation)

<u>Th.</u>: $\forall f \in V^* \ \exists\, \beta \in V : \ f(\alpha) = \langle \alpha, \beta \rangle \ \forall \alpha \in V$ ($\langle \cdot, \cdot \rangle$ defined by metric in $V$)

<u>Col.</u>: $\forall \lambda \in \Lambda^p V \ \exists\, f_\lambda \in (\Lambda^{n-p} V)^* : \ \lambda \wedge \theta = f_\lambda(\theta)\, \omega \ \forall \theta \in \Lambda^{n-p} V ; \ \omega \in \Lambda^n V$

$\Updownarrow$ $*\lambda \in \Lambda^{n-p} V$ $\qquad$ $\Updownarrow$ $\lambda \wedge \theta = (-1)^s \langle \theta, *\lambda \rangle\, \omega$

Therefore:

$$\boxed{\lambda \wedge \theta = (-1)^s \langle \theta, *\lambda \rangle\, \omega}$$ , $*: \Lambda^p V \to \Lambda^{n-p} V$ (or $\Omega^p \to \Omega^{n-p}$), $\omega = e_1 \wedge \dots \wedge e_n \in \Lambda^n V$

$\left[ \Lambda^p V \simeq \Lambda^{n-p} V \text{ because } \binom{n}{p} = \binom{n}{n-p} \right]$

e.g. $\Lambda^2(\mathbb{R}^3) \overset{*}{\leftrightarrow} \Lambda^1(\mathbb{R}^3)$ : $\quad u \wedge v \overset{*}{\leftrightarrow} u \times v$

($\wedge$-product $\leftrightarrow$ $\times$-product) $\qquad$ (bivector $\leftrightarrow$ pseudovector)

# Practical definition:

Consider $\lambda = \sigma^1 \wedge \ldots \wedge \sigma^p$, then $\forall$ basis vector (orthonormal) $\sigma^I \in \Lambda^{n-p} V$ $\quad \lambda \wedge \sigma^I = (-1)^s \langle \sigma^I, *\lambda \rangle \omega$

$\lambda \wedge \sigma^I = 0$ if $I \neq \{p+1, \ldots, n\}$
$\langle \sigma^I, *\lambda \rangle = (*\lambda)_I$ (I-component of $*\lambda$)
$\Rightarrow$ $*\lambda = c\, \sigma^{p+1} \wedge \ldots \wedge \sigma^n$, $c \in \mathbb{R}$

Therefore:

$$\omega = (-1)^s \langle \sigma^{I'}, c\,\sigma^{I'} \rangle \omega \quad , \quad I' = \{p+1, \ldots, n\}$$

$$c = \frac{(-1)^s}{\langle \sigma^{I'}, \sigma^{I'} \rangle} = \frac{\langle \omega, \omega \rangle}{\langle \sigma^{I'}, \sigma^{I'} \rangle} = \langle \lambda, \lambda \rangle = \langle \sigma^1, \sigma^1 \rangle \cdot \ldots \cdot \langle \sigma^p, \sigma^p \rangle \quad \text{, then:}$$

$$\boxed{*(\sigma^1 \wedge \ldots \wedge \sigma^p) = \langle \sigma^1, \sigma^1 \rangle \ldots \langle \sigma^p, \sigma^p \rangle\, \sigma^{p+1} \wedge \ldots \wedge \sigma^n}$$

, or $\{i_1, \ldots, i_n\}$ any permutation, but:
LHS = RHS for even perm.
LHS = − RHS for odd perm.

## Examples:

1) $V = \mathbb{R}^2$ $\quad ds^2 = dx^2 + dy^2$ $\quad \omega = dx \wedge dy$ $\quad (s = 0)$

$*dx = dy$ $\qquad *1 = dx \wedge dy$

$*dy = -dx$ $\qquad *(dx \wedge dy) = 1$

2) $V = \mathbb{R}^{1,2}$ $\quad ds^2 = -dt^2 + dx^2 + dy^2$ $\quad \omega = dt \wedge dx \wedge dy$ $\quad (s = 1)$

$*1 = dt \wedge dx \wedge dy$ $\qquad *(dt \wedge dx) = -dy$

$*dt = -dx \wedge dy$ $\qquad *(dx \wedge dy) = dt$

$*dx = dy \wedge dt$ $\qquad *(dy \wedge dt) = -dx$

$*dy = dt \wedge dx$ $\qquad *(dt \wedge dx \wedge dy) = -1$

3) $V = \mathbb{R}^{1,3}$ $\quad ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ $\quad \omega = dt \wedge dx \wedge dy \wedge dz$ $\quad (s = 3)$

$*dt = dx \wedge dy \wedge dz$ $\qquad *(dt \wedge dx) = -dy \wedge dz$

$*dx = dy \wedge dz \wedge dt$ $\qquad *(dt \wedge dy) = -dz \wedge dx$

$*dy = -dz \wedge dt \wedge dx$ $\qquad *(dt \wedge dz) = -dx \wedge dy$

$*dz = dt \wedge dx \wedge dy$ $\qquad *(dx \wedge dy) = -dz \wedge dt$

$*1 = dt \wedge dx \wedge dy \wedge dz$ $\qquad *(dy \wedge dz) = -dx \wedge dt$

$*(dt \wedge dx \wedge dy \wedge dz) = -1$ $\qquad *(dz \wedge dx) = -dy \wedge dt$

e.g. $*(dx \wedge dy) = \eta^{11} \cdot \eta^{22} \cdot (-1)^I dz \wedge dt$, where $(-1)^I = \text{sign}\{\overbrace{1\ 2}^{dx\,dy}\ \overbrace{3\ 0}^{dz\,dt}\} = -1$ (odd perm.)

$*(dx \wedge dy) = (-1)^3 dz \wedge dt$

$*(dx \wedge dy) = -dz \wedge dt$

# Properties of Hodge $*$-operator:

1) $\boxed{*1 = \omega \qquad *\omega = (-1)^s}$ , $1 \in \Lambda^0 V$, $\omega \in \Lambda^n V$

**Proof:**

$$\left.\begin{array}{l} \exists\, \sigma^I = \sigma^1 \wedge \dots \wedge \sigma^p \\ \exists\, \sigma^J = \sigma^{p+1} \wedge \dots \wedge \sigma^n \end{array}\right| \Rightarrow *\sigma^I = \langle \sigma^I, \sigma^I \rangle \sigma^J \quad \text{and} \quad \begin{array}{l} \langle 1, 1 \rangle = 1 \\ \langle \omega, \omega \rangle = (-1)^s \end{array}, \ s: \text{signature of } g$$

$$\Updownarrow$$

$$*\alpha = \|\alpha\|^2 \beta, \quad \begin{array}{l} \alpha \in \Lambda^p \\ \beta \in \Lambda^{n-p} \end{array}$$

2) $\boxed{**\alpha = (-1)^{p(n-p)+s} \alpha}$ , $\alpha \in \Lambda^p V$

formally should write $*^{-1} *$

$n = 2k+1$: $p(n-p)$ even $\forall p$

$n = 2k$ : $p(n-p)$ even for even $p$
$p(n-p)$ odd for odd $p$

**Proof:**

$$\sigma^J \wedge \sigma^I = (-1)^{p(n-p)} \sigma^I \wedge \sigma^J = (-1)^{p(n-p)} \omega \quad \Rightarrow \quad \sigma^J \wedge \left( (-1)^{p(n-p)} \sigma^I \right) = \omega$$

Therefore:

$$*\sigma^J = \langle \sigma^J, \sigma^J \rangle (-1)^{p(n-p)} \sigma^I$$

$$**\sigma^I = *\left( \langle \sigma^I, \sigma^I \rangle \sigma^J \right) = \underbrace{\langle \sigma^I, \sigma^I \rangle \langle \sigma^J, \sigma^J \rangle}_{\langle \omega, \omega \rangle = (-1)^s} (-1)^{p(n-p)} \sigma^I = (-1)^{n(n-p)+s} \sigma^I$$

3) $\boxed{\alpha \wedge *\beta = \langle \alpha, \beta \rangle \omega}$ , $\alpha, \beta \in \Lambda^p V$

one can take this as a definition of $*$
(but be carefull with signs)

**Proof:**

$$\lambda \wedge \theta \overset{\text{def}}{=} (-1)^s \langle \theta, *\lambda \rangle \omega$$

$$\left.\begin{array}{l} \exists\ \lambda = *\beta \in \Lambda^{n-p} V \\ \quad \theta = \alpha \in \Lambda^p V \end{array}\right| \Rightarrow \quad *\beta \wedge \alpha = (-1)^s \langle \alpha, **\beta \rangle \omega$$

$$(-1)^{p(n-p)} \alpha \wedge *\beta = (-1)^s (-1)^{p(n-p)+s} \langle \alpha, \beta \rangle \omega$$

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \omega$$

4) $\boxed{\langle \alpha, \beta \rangle = (-1)^s * (\alpha \wedge *\beta)}$ , $\alpha, \beta \in \Lambda^p V$

one can take this as a definition of $\langle \cdot, \cdot \rangle$

**Proof:**

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \omega \quad \Rightarrow \quad \langle \alpha, \beta \rangle = (-1)^s * (\alpha \wedge *\beta)$$

## Examples:

1) $M = \mathbb{R}^3$ $\quad ds^2 = dx^2 + dy^2 + dz^2$ $\quad (s=0)$

$f \in \Omega^0$ : $**f = f$ $\qquad \beta \in \Omega^2$ : $**\beta = \beta$

$\alpha \in \Omega^1$ : $**\alpha = \alpha$ $\qquad \omega \in \Omega^3$ : $**\omega = \omega$

$$\boxed{**\alpha = (-1)^{p(n-p)+s}\alpha, \ \alpha \in \Omega^p(M)}$$

2) $M = \mathbb{R}^{1,3}$ $\quad ds^2 = dt^2 - d\vec{r}^{\,2}$ $\quad (s=3)$

$J \in \Omega^1$ : $**J = J$ $\qquad \omega \in \Omega^4$ : $**\omega = -\omega$

$F \in \Omega^2$ : $**F = -F$ $\qquad f \in \Omega^0$ : $**f = -f$

$B \in \Omega^3$ : $**B = B$

3) $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = E_x \, dt\wedge dx + E_y \, dt \wedge dy + E_z \, dt\wedge dz - B_x \, dy \wedge dz -$
$- B_y \, dz \wedge dx - B_z \, dx \wedge dy$ (Faraday 2-form)

$*F = -B_x \, dt\wedge dx - B_y \, dt\wedge dy - B_z \, dt \wedge dz - E_x \, dy\wedge dz - E_y \, dz \wedge dx - E_z \, dx \wedge dz$

$j = \rho\, dt - j_x dx - j_y dy - j_z dz$ (current 3-form, or vice versa: current 1-form could use any definition ; $*J = j$)

$J \overset{\text{def}}{=} *j = \rho \, dx\wedge dy \wedge dz - j_x \, dt\wedge dy \wedge dz - j_y \, dt \wedge dz \wedge dx - j_z \, dt \wedge dx \wedge dy$

Then:

$$\boxed{\begin{aligned} dF &= 0 \\ d*F &= 4\pi J \end{aligned}} \Rightarrow dJ = 0 \text{ (charge conservation)}$$

Maxwell eqn-ns

4) $M = \mathbb{R}^{1,3}$ (or $M = \mathbb{R}^4$)

$\omega \in \Omega^0 \cup \Omega^1 \cup \Omega^3 \cup \Omega^4$ : $\forall \omega$ decomosable

$\omega \in \Omega^2$ : $\omega$ decomposable $\Leftrightarrow$ $\omega \wedge \omega = 0$ (proof on stack exchange; just direct computation)

$$\boxed{\delta \overset{\text{def}}{=} (-1)^{n(k-1)+s+1} * d *}\ , \quad \delta : \Omega^{k+1} \to \Omega^{k}, \quad d : \Omega^{k} \to \Omega^{k+1}$$

codifferential (wikipedia)

**Properties:**

1. $(\eta, \delta\xi) = (d\eta, \xi)$, $\quad \eta \in \Omega^k(M)$, $\xi \in \Omega^{k+1}(M)$ $\quad \left[ (\alpha, \beta) \overset{\text{def}}{=} \int_M \alpha \wedge *\beta \,, \ \alpha, \beta \in \Omega^k(M) \right]$

(i.e. $\delta$ and $d$ are adjoint operators)

**Proof:**

$$\int_M d(\eta \wedge *\xi) = \int_M (d\eta \wedge *\xi + (-1)^k \eta \wedge d*\xi) = \int_M (d\eta \wedge *\xi + (-1)^k \eta \wedge * *^{-1} d * \xi) =$$

$$= \int_M (d\eta \wedge *\xi - (-1)^{k+1} \eta \wedge *\delta\xi) = (d\eta, \xi) - (\eta, \delta\xi) \underset{\uparrow}{=} 0$$

from Stokes' theorem

2. $\delta^2 = 0$ (from def. and $d^2 = 0$)

$$\boxed{\Delta \overset{\text{def}}{=} (\delta + d)^2 = \delta d + d\delta}$$

Laplace-de Rham operator

1. $(\Delta\eta, \xi) = (\eta, \Delta\xi)$
2. $(\Delta\eta, \eta) \geqslant 0$
3. $V = \mathbb{R}^3$, then:

| | | | | | |
|---|---|---|---|---|---|
| $df$ | $\leftrightarrow$ | $\nabla f$ | $*d\vec{A}$ | $\leftrightarrow$ | $\nabla \times \vec{A}$ |
| $\delta d f$ | $\leftrightarrow$ | $\Delta f$ | $\delta \vec{A}$ | $\leftrightarrow$ | $\nabla \cdot \vec{A}$ |