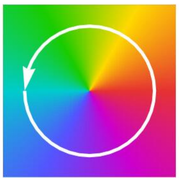
Complex Analysis (Exercises)

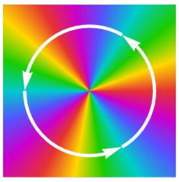
[Sh.]: R. Shakarchi, "Problems and Solutions for Complex Analysis".
[Cah.]: K. Cahill, "Physical Mathematics".
[Has.]: S. Hassani, "Mathematical Physics".
[AWH]: Arfken, Weber, Harris, "Mathematical Methods".

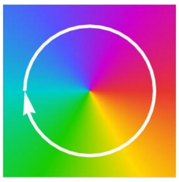
1. Complex functions.

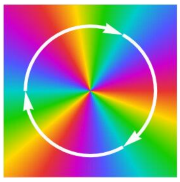
Domain coloring (or a color wheel graph) is a technique for visualizing complex functions by assigning a color to each point of $\mathbb{C}$: hue represents $\arg(z)$ and height represents $|z|$.

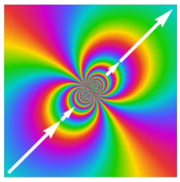
$z = -1$:	$\arg z = -\pi$	cyan
$z = -i$:	$\arg z = -\pi/2$	magenta
$z = 1$:	$\arg z = 0$	red
$z = i$:	$\arg z = \pi/2$	olive


zero


multiple zero


pole


multiple pole


essential singularity

$f(z) = \frac{1}{z} = \frac{1}{r} e^{-i\theta}$ inversion through the unit circle

$D = \{z \mid |z| < 1\}$

$C = \{z \mid |z| = 1\}$

$D \setminus \{0\} \mapsto \mathbb{C} \setminus \bar{D}$

$\mathbb{C} \setminus \bar{D} \mapsto D \setminus \{0\}$

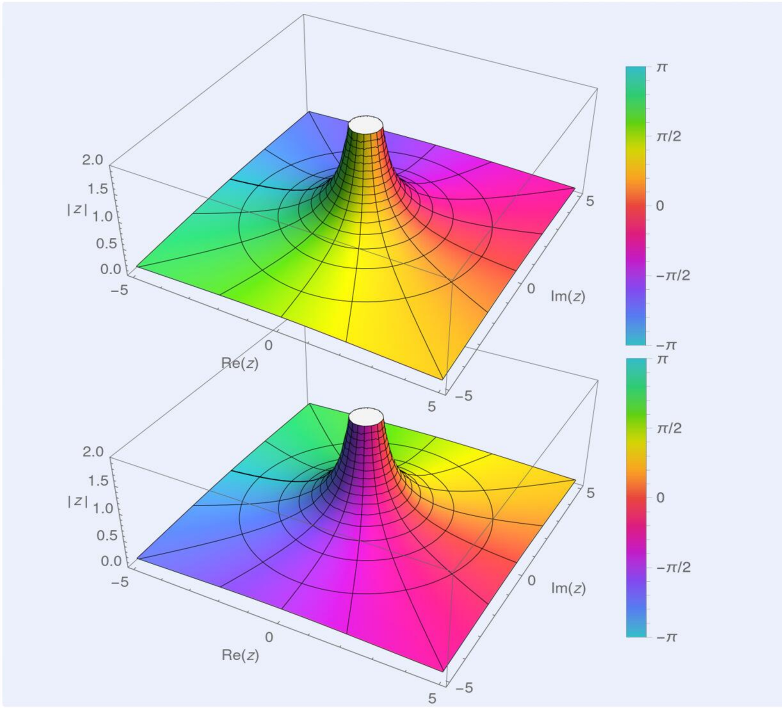
$C \mapsto C$

$f(z) = \frac{1}{\bar{z}} = \frac{1}{r} e^{i\theta}$ reflection through the unit circle

$D \setminus \{0\} \mapsto \mathbb{C} \setminus \bar{D}$

$\mathbb{C} \setminus \bar{D} \mapsto D \setminus \{0\}$

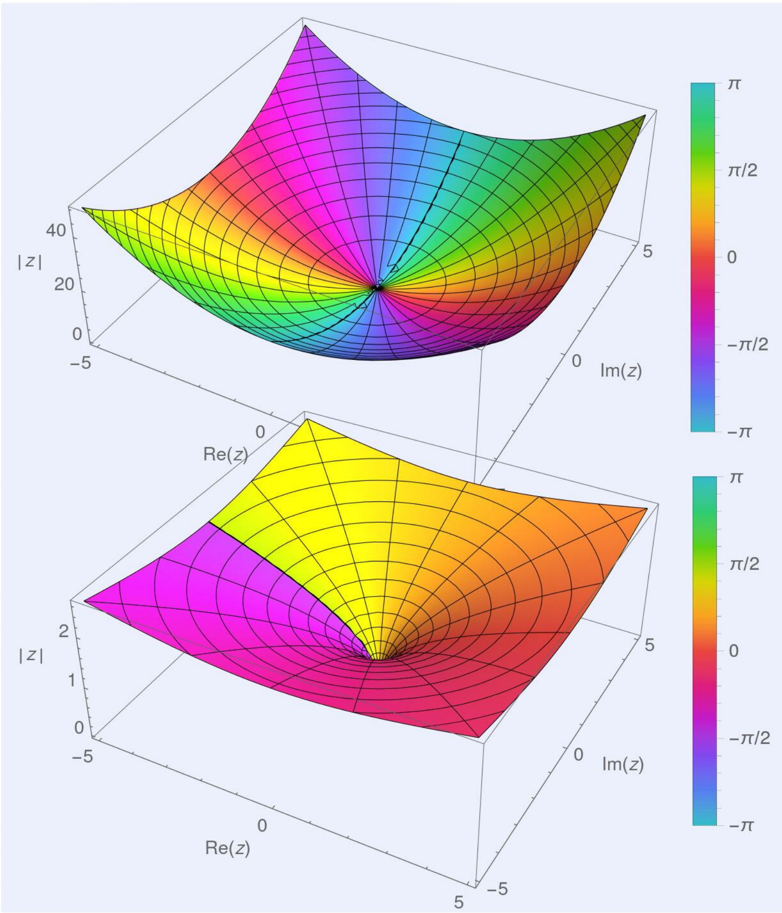
$C \mapsto C$



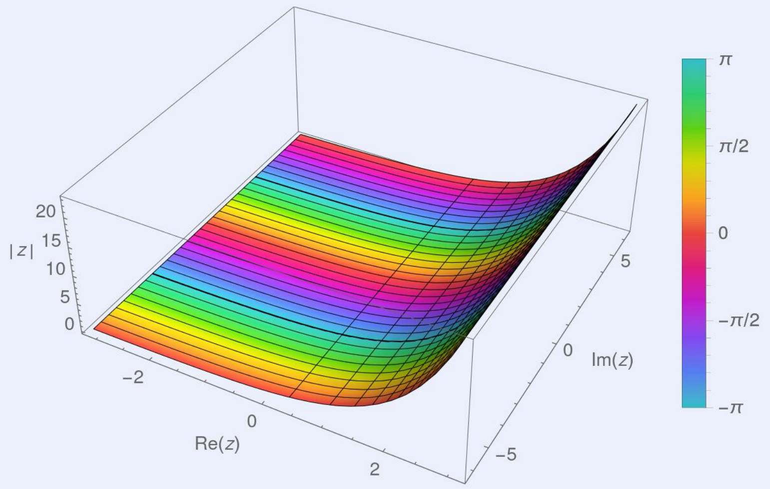
$f(z) = z^2$

$f(z) = \sqrt{z}$

multivalued

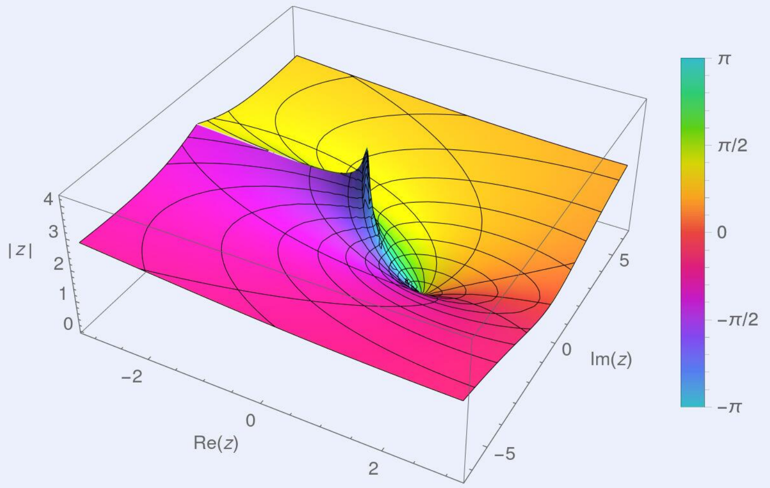


$f(z) = e^z$



$f(z) = \log(z)$

multivalued



$f(z) = \sin(z)$

Finding zeros:

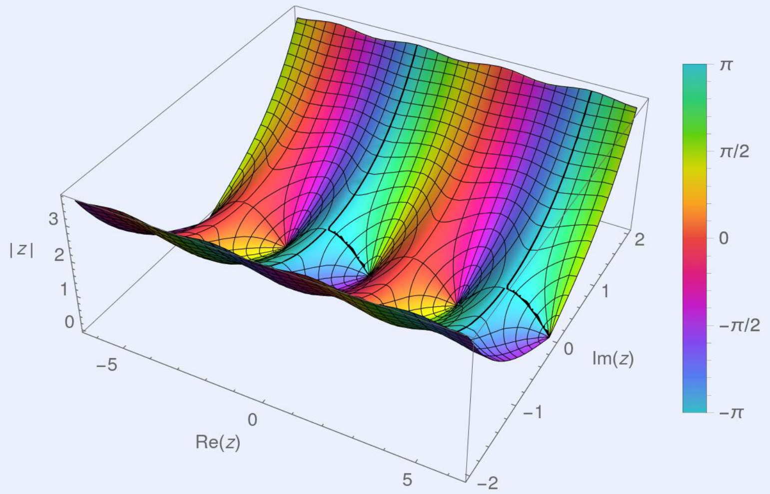
$$\frac{e^{iz} - e^{-iz}}{2i} = 0 \qquad e^{2iz} = 1 \qquad e^{-2y} e^{2ix} = 1$$

$$\begin{cases} -2y = 1 \\ 2x = 2\pi k, k \in \mathbb{Z} \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = \pi k, k \in \mathbb{Z} \end{cases}$$

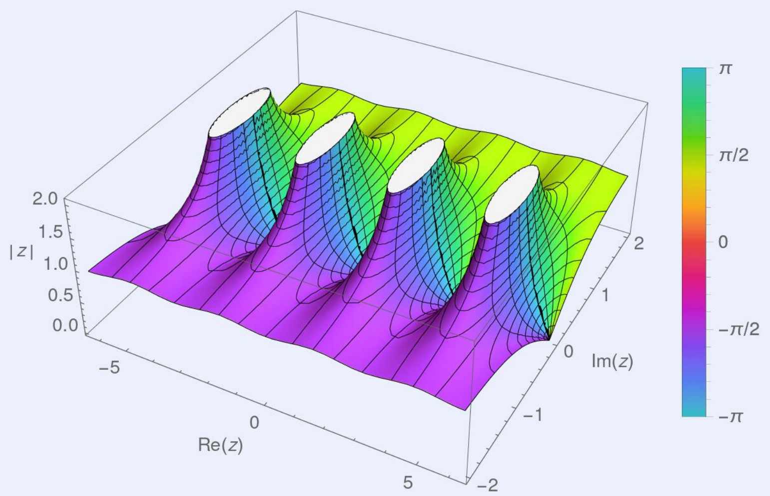
$$\frac{d}{dz}(\sin z) = \frac{i e^{iz} + i e^{-iz}}{2i} = \cos(z)$$

$$\left.\frac{d}{dz}(\sin z)\right|_{z=\pi k} = \begin{cases} 1, & k = 2n \\ -1, & k = 2n+1 \end{cases}, n \in \mathbb{Z} \neq 0$$

So $z = \pi k$ $(k \in \mathbb{Z})$ are simple zeros of $\sin(z)$.



$f(z) = \tan(z)$



② Limits and power series.

Ex. 2.4.1. (Sh.)

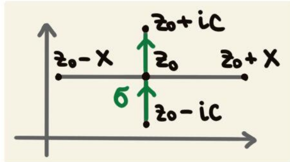
$f(z) = \frac{z^2}{z-2}$ Expand up to $n=3$ at $z_0 = 1$.

Solution:

$$f(z) = \frac{((z-1)+1)^2}{-(1-(z-1))} = -((z-1)^2 + 2(z-1) + 1)\sum_{k=0}^{\infty}(z-1)^k = -1-(2+1)(z-1)-(1+2+1)(z-1)^2 -$$
$$-(1+2+1)(z-1)^3 + o(z^3) = -1 - 3(z-1) - 4(z-1)^2 - 4(z-1)^3 + o(z^3)$$

Ex. 3.2.7. (Sh.)

$\lim\limits_{x\to 0} \int_\sigma \left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha'}\right) dz$, where $\sigma(t) = z_0 + itc$, $t\in[-1,1]$ $c>0$; $\alpha = z_0 + x$; $\alpha' = z_0 - x$, $x>0$



Solution:

$$\int_\sigma\left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha'}\right)dz = \int_{-1}^{1}\left(\frac{1}{z_0+itc-\alpha} - \frac{1}{z_0+itc-\alpha'}\right)ic\,dt = ic\int_{-1}^{1}\left(\frac{1}{itc-x} + \frac{1}{itc+x}\right)dt = \int_{-1}^{1}\frac{ic\cdot 2x\,dt}{-t^2c^2-x^2} =$$
$$= -\frac{2ic}{x}\int_{-1}^{1}\frac{dt}{1+(c/x)^2t^2} = -\frac{2ic}{x}\cdot\frac{x}{c}\arctan\left(\frac{tc}{x}\right)\Big|_{-1}^{1} = -4i\arctan\left(\frac{c}{x}\right)$$

$$\lim_{x\to 0}\int_\sigma\left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha'}\right)dz = -4i\cdot\frac{\pi}{2} = -2\pi i$$

Ex. 3.2.8. (Sh.)

$\lim\limits_{B\to\infty}\int_{-B}^{B}\left(\frac{1}{t+ix} - \frac{1}{t-ix}\right)dt$, $x>0$

Solution:

$$\int_{-B}^{B}\left(\frac{1}{t+ix} - \frac{1}{t-ix}\right)dt = \int_{-B}^{B}\frac{-2ix}{t^2+x^2}dt = -\frac{2i}{x}\int_{-B}^{B}\frac{dt}{1+(t/x)^2} = -\frac{2i}{x}\cdot x\arctan\left(\frac{t}{x}\right)\Big|_{-B}^{B} = -4i\arctan\left(\frac{B}{x}\right)$$

$$\lim_{B\to\infty}\int_{-B}^{B}\left(\frac{1}{t+ix} - \frac{1}{t-ix}\right)dt = -4i\cdot\frac{\pi}{2} = -2\pi i$$

Ex. 3.6.3. (Sh.)

$\lim\limits_{y\to 0}(\log(a+iy) - \log(a-iy))$, $y>0$

Solution:

1. $(-\pi,\pi)$ branch of $\log z$

   $a>0$: $\log(a+iy) - \log(a-iy) = \log(\sqrt{a^2+y^2}\,e^{i\varepsilon_y}) - \log(\sqrt{a^2+y^2}\,e^{-i\varepsilon_y}) =$
   $= \ln\sqrt{a^2+y^2} + i\varepsilon_y - \ln\sqrt{a^2+y^2} - (-i\varepsilon_y) = 2i\varepsilon_y$

   $\lim\limits_{y\to 0}(\log(a+iy) - \log(a-iy)) = 0$

   $a<0$: $\log(a+iy) - \log(a-iy) = \log(\sqrt{a^2+y^2}\,e^{i(\pi-\varepsilon_y)}) - \log(\sqrt{a^2+y^2}\,e^{i(-\pi+\varepsilon_y)}) =$
   $= i(\pi-\varepsilon_y) - i(\varepsilon_y-\pi) = 2i(\pi-\varepsilon_y)$

   $\lim\limits_{y\to 0}(\log(a+iy) - \log(a-iy)) = 2\pi i$

2. $(0,2\pi)$ branch of $\log z$

   $a>0$: $\lim\limits_{y\to 0}(\log(a+iy) - \log(a-iy)) = -2\pi i$

   $a<0$: $\lim\limits_{y\to 0}(\log(a+iy) - \log(a-iy)) = 0$

Ex. 3.6.1. (Sh.)

a) $\log(i) = \log e^{i\pi/2} = i\frac{\pi}{2}$

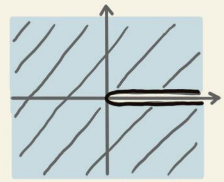
b) $\log(-i) = \log e^{i3\pi/2} = i\frac{3\pi}{2}$

c) $\log(i-1) = \log(\sqrt{2}\, e^{i3\pi/4}) = \frac{1}{2}\ln 2 + i\frac{3\pi}{4}$

d) $i^i = (e^{i\pi/2})^i = e^{-\pi/2}$

e) $(-i)^i = (e^{i3\pi/2})^i = e^{-3\pi/2}$

f) $(-1)^i = (e^{i\pi})^i = e^{-\pi}$



$(0,2\pi)$ branch of $\log z$

③ The Laurent expansion.

Ex. 5.2.4. (Sh.)

$f(z) = \frac{z}{1+z^3}$ Expand f in a series of positive and negative powers of z

Solution:

$$|z|<1: \quad f(z) = z\,\frac{1}{1-(-z^3)} = z\left(1 + (-z^3) + (-z^3)^2 + \dots\right) = z - z^4 + z^7 - \dots = \boxed{\sum_{n=0}^{\infty} (-1)^n z^{3n+1}}$$

$$|z|>1: \quad f(z) = \frac{1}{z^2}\,\frac{1}{1-(-z^{-3})} = \frac{1}{z^2}\left(1 + \left(-\frac{1}{z^3}\right) + \left(-\frac{1}{z^3}\right)^2 + \dots\right) = \frac{1}{z^2} - \frac{1}{z^5} + \frac{1}{z^8} - \dots = \boxed{\sum_{n=0}^{\infty} (-1)^n z^{-(3n+2)}}$$

Ex. 5.2.5 (Sh.)

Give the Laurent expansion:

a) $f(z) = \frac{z}{z+2}$ for $|z|>2$

$$f(z) = \frac{1}{1-(-2/z)} = \sum_{n=0}^{\infty}\left(-\frac{2}{z}\right)^n = \boxed{\sum_{n=0}^{-\infty}\left(-\frac{z}{2}\right)^n}$$

b) $f(z) = \sin\left(\frac{1}{z}\right)$ for $|z|>0$

$$f(z) = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{z^{-(2n+1)}}{(2n+1)!}}$$

c) $f(z) = \cos\left(\frac{1}{z}\right)$ for $|z|>0$

$$f(z) = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{z^{-2n}}{(2n)!}}$$

d) $f(z) = \frac{1}{z-3}$ for $|z|>3$

$$f(z) = \frac{1}{z}\,\frac{1}{1-3/z} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n = \sum_{n=1}^{\infty}\frac{3^{n-1}}{z^n} = \boxed{\sum_{n=-1}^{-\infty}\frac{z^n}{3^{n+1}}}$$

Ex. 5.2.8. (Sh.)

$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$ is hol. in the annuli
$V_1 = \{0<|z|<1\}$
$V_2 = \{1<|z|<2\}$
$V_3 = \{2<|z|<\infty\}$

V_1:

$$\frac{1}{z-2} = -\frac{1}{2}\,\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \quad \text{conv. for } |z|<2$$

$$-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{conv. for } |z|<1$$

$$\boxed{f(z) = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right)z^n}$$

V_2:

$$\frac{1}{z-2} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \quad \text{conv. for } |z|<2$$

$$-\frac{1}{z-1} = -\frac{1}{z}\,\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=-1}^{-\infty} z^n \quad \text{conv. for } |z|>1$$

$$\boxed{f(z) = -\left(\sum_{n=-1}^{-\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right)}$$

V_3:

$$\frac{1}{z-2} = \frac{1}{z}\,\frac{1}{1-\frac{2}{z}} = \frac{1}{2}\cdot\frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \frac{1}{2}\sum_{n=-1}^{-\infty}\left(\frac{z}{2}\right)^n \quad \text{conv. for } |z|>2$$

$$-\frac{1}{z-1} = -\sum_{n=-1}^{-\infty} z^n \quad \text{conv. for } |z|>1$$

$$\boxed{f(z) = \sum_{n=-1}^{-\infty}\left(\frac{1}{2^{n+1}} - 1\right)z^n}$$

④ Residues.

Ex. 6.1.1-16.

Calculate the residue of f at $z=0$:

1) $f(z) = \dfrac{z^2+1}{z} = z + \dfrac{1}{z} \Rightarrow \boxed{\underset{z=0}{\operatorname{res}} f(z) = 1}$

2) $f(z) = \dfrac{z^2+3z-5}{z^3} = \dfrac{1}{z} + 3\dfrac{1}{z^2} - 5\dfrac{1}{z^3} \Rightarrow \boxed{\underset{z=0}{\operatorname{res}} f(z) = 1}$

3) $f(z) = \dfrac{z^3}{(z-1)(z^4+2)}$ $\quad z=0$ is not a sing. $\Rightarrow \boxed{\underset{z=0}{\operatorname{res}} f(z) = 0}$

5) $f(z) = \dfrac{\sin(z)}{z^4} = \dfrac{1}{z^4}\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \dots\right) = \dfrac{1}{z^3} - \dfrac{1}{3!}\dfrac{1}{z} + \dfrac{1}{5!}z - \dfrac{1}{7!}z^3 + \dots \Rightarrow \boxed{\underset{z=0}{\operatorname{res}} f(z) = -\dfrac{1}{6}}$

6) $f(z) = \dfrac{\sin(z)}{z^5} = \dfrac{1}{z^5}\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \dots\right) = \dfrac{1}{z^4} - \dfrac{1}{3!}\dfrac{1}{z^2} + \dfrac{1}{5!} - \dfrac{1}{7!}z^2 + \dots \Rightarrow \boxed{\underset{z=0}{\operatorname{res}} f(z) = 0}$

9) $f(z) = \dfrac{e^z}{z} = \dfrac{1}{z}\left(1 + z + \dfrac{z^2}{2!} + \dots\right) = \dfrac{1}{z} + 1 + \dfrac{z}{2!} + \dots \Rightarrow \boxed{\underset{z=0}{\operatorname{res}} f(z) = 1}$

13) $f(z) = \dfrac{\log(1+z)}{z^2} = \dfrac{1}{z^2}\left(z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \dots\right) = \dfrac{1}{z} - \dfrac{1}{2} + \dfrac{z}{3} - \dots \Rightarrow \boxed{\underset{z=0}{\operatorname{res}} f(z) = 1}$

14) $f(z) = \dfrac{e^z}{\sin(z)} = e^z\left(z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dots\right)^{-1} = \dfrac{e^z}{z}\left(1 - \left(\dfrac{z^2}{3!} - \dfrac{z^4}{5!} + \dots\right)\right)^{-1} = \dfrac{1 + z + \frac{z^2}{2} + \dots}{z} \cdot \Big(1 +$

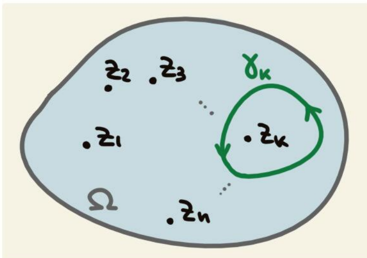
$+ \left(\dfrac{z^2}{3!} - \dfrac{z^4}{5!} + \dots\right) + \left(\dfrac{z^2}{3!} - \dfrac{z^4}{5!} + \dots\right)^2 + \dots\Big) = \left(\dfrac{1}{z} + 1 + \dfrac{z}{2} + \dots\right)\cdot\left(1 + \left(\dfrac{z^2}{3!} - \dfrac{z^4}{5!} + \dots\right) + \dots\right)$

$\Rightarrow \boxed{\underset{z=0}{\operatorname{res}} f(z) = 1}$

Ex. 6.1.17. (Sh.)

$$\underset{z=1}{\operatorname{res}} \frac{1}{z^n-1} = \lim_{z\to 1} \frac{z-1}{z^n-1} = \lim_{z\to 1} \left(1 + z + \dots + z^{n-1}\right)^{-1} = \boxed{\frac{1}{n}}$$

Ex. 6.1.18. (Sh.)



$$f(z) = \frac{1}{(z-z_1)\cdot\ldots\cdot(z-z_n)}$$

$$\oint_{\gamma_k} f(z)\,dz = 2\pi i \underset{z=z_k}{\operatorname{res}} f(z) = 2\pi i \lim_{z\to z_k} \frac{z-z_k}{(z-z_1)\cdot\ldots\cdot(z-z_n)} = \boxed{2\pi i \prod_{j\neq k} \frac{1}{z_k-z_j}}$$

Ex. 6.1.19. (Sh.)

$f(z) = \dfrac{1}{z^4-1}$ $\qquad z_0 = i$ $\qquad \gamma(t) = i + \frac{1}{2}e^{it},\ t\in[0,2\pi]$

a) $\underset{z=i}{\operatorname{res}} f(z) = \dfrac{1}{(i-1)}\cdot\dfrac{1}{(i+1)}\cdot\dfrac{1}{(i+i)} = \dfrac{1}{(i^2-1)2i} = -\dfrac{1}{4i} = \boxed{\dfrac{i}{4}}$

b) $\oint_{\gamma} f(z)\,dz = 2\pi i \underset{z=i}{\operatorname{res}} f(z) = 2\pi i \cdot \dfrac{i}{4} = \boxed{-\dfrac{\pi}{2}}$

5. Complex integration.

Ex. 5.1.

1) $f(z) = (z - z_0)^n \qquad \gamma(t) = z_0 + re^{it}, \quad t \in [0, 2\pi]$

$$\left.\begin{array}{l} \gamma'(t) = ire^{it} \\ f(\gamma(t)) = r^n e^{int} \end{array}\right| \Rightarrow \oint_\gamma f(z)\,dz = ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt = \begin{cases} n \neq -1: \dfrac{r^{n+1}}{n+1}\left(e^{i(n+1)2\pi} - 1\right) = \boxed{0} \\ n = -1: i\,(2\pi - 0) = \boxed{2\pi i} \end{cases}$$

$\oint_\gamma \frac{dz}{z} = 2\pi i \neq 0 \Rightarrow f(z) = \frac{1}{z}$ doesn't have a primitive.

2) $f(z) = z^n, \; n \neq -1 \qquad \gamma(t)$ is arbitrary, but $\gamma(t) \neq 0$ for $n < 0$.

$\frac{d}{dt}\gamma^{n+1}(t) = (n+1)\gamma^n(t)\gamma'(t)$, then:

$\int_\gamma f(z)\,dz = \int_a^b \gamma^n(t)\gamma'(t)\,dt = \boxed{\frac{1}{n+1}\left(\gamma^{n+1}(b) - \gamma^{n+1}(a)\right)}$, i.e. is independent of γ

$a = b: \oint_\gamma z^n\,dz = 0 \quad \forall$ closed γ

3) $f(z) = \frac{z^m}{z^n}, \quad m, n \in \mathbb{Z} \qquad \gamma(t) = re^{it}, \quad t \in [0, 2\pi]$

$$\oint_\gamma f(z)\,dz = \int_0^{2\pi} \frac{r^m}{r^n} e^{i(m-n)t}\, ire^{it}\,dt = \frac{r^{m+1}}{r^n}\, i \int_0^{2\pi} e^{i(m+1-n)t}\,dt = \begin{cases} i\,\dfrac{r^{m+1}}{r^n}\,\dfrac{e^{2\pi i(m+1-n)} - 1}{m+1-n} = 0, & m+1-n \neq 0 \\ 2\pi i \cdot r^{m+1-n}, & m+1-n = 0 \end{cases}$$

$$\boxed{\frac{1}{2\pi i}\,\frac{1}{r^{m+1-n}} \oint_\gamma \frac{z^m}{z^n}\,dz = \delta_{m+1,n}}$$

4) $f(z) = \frac{1}{(z^8 + 1)^2} \qquad \gamma(t) = 2e^{it}, \quad t \in [0, 2\pi]$

$z = \infty$: zero of order 16 $\Rightarrow$ Laurent expansion starts at z^{-16} $\Rightarrow c_{-1} = \operatorname{res}_\infty f = 0$

$$\oint_\gamma \frac{dz}{(z^8+1)^2} = 2\pi i \sum_{\nu=1}^{8} \operatorname{res}_{z_\nu} f = -2\pi i \operatorname{res}_\infty f = \boxed{0}$$

Ex. 6.1.26. (Sh.)

Calculate $\oint_\gamma f(z)dz$, where $\gamma(t) = 8e^{it}$, $t \in [0,2\pi]$

b) $f(z) = \frac{1}{1-\cos(z)}$

Solution:

$\cos(z) = 1 : \quad z = 2\pi k,\ k \in \mathbb{Z}$ poles of f

$$f(z) = \left(1 - 1 + \frac{1}{2!}z^2 - \frac{1}{4!}z^3 + \ldots\right)^{-1} = \frac{2}{z^2}\left(1 - \frac{2z^2}{4!} + \frac{2z^4}{6!} - \ldots\right)^{-1} = \frac{2}{z^2}\left(1 + \left(\frac{2z^2}{4!} - \frac{2z^4}{6!} + \ldots\right) + \ldots\right)$$

$$\left.\begin{array}{l} \Rightarrow \operatorname*{res}_{z=0} f(z) = 0 \\ f(z+2\pi) = f(z) \end{array}\right| \Rightarrow \operatorname*{res}_{z=2\pi k} f(z) = 0 \Rightarrow \boxed{\oint_\gamma f(z)dz = 0}$$

d) $f(z) = \tan(z)$

Solution:

$$\oint_\gamma f(z)dz = \oint_\gamma \frac{\sin(z)}{\cos(z)}dz = -\oint_\gamma \frac{d(\cos(z))}{\cos(z)} = -2\pi i \underbrace{\sum_k n_k}_{\text{number of zeros of } \cos(z) \text{ inside } \gamma} \quad \text{(argument principle)}$$

$\cos(z) = 0 : \quad z = \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z} \quad \Rightarrow \left\{\frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \frac{5\pi}{2}, -\frac{5\pi}{2}\right\}$ are inside γ

$$\oint_\gamma f(z)dz = -2\pi i \cdot 6 = \boxed{-12\pi i}$$

Ex. 6.19. (Cah.)

$f(z) = \sum_{n=1}^{\infty} \frac{z}{z - n^{-s}}$ Calculate $\oint_\gamma f(z)dz$, where $\gamma(t) = 2e^{it}$, $t \in [0,2\pi]$.

Solution:

$$\oint_\gamma f(z)dz = 2\pi i \sum_k \operatorname*{res}_{z=z_k} f(z) = 2\pi i \cdot \sum_{n=1}^{\infty} \lim_{z \to n^{-s}} (z - n^{-s}) f(z) = 2\pi i \cdot \sum_{n=1}^{\infty} \frac{1}{n^s} = \boxed{2\pi i \cdot \zeta(s)}$$

Riemann zeta function

Ex. 6.21. (Cah.)

$f(z) = \frac{1}{(z-1)(z-2)^2}$ Calculate $\oint_\gamma f(z)dz$, where $\gamma(t) = 4e^{it}$, $t \in [0,2\pi]$.

Solution:

$$\oint_\gamma f(z)dz = 2\pi i \left(\operatorname*{res}_{z=1} f(z) + \operatorname*{res}_{z=2} f(z)\right)$$

Method 1: $\operatorname*{res}_{z=1} f(z) = \lim_{z \to 1} (z-1) f(z) = 1$

$a_{-2}(2) = \lim_{z \to 2} (z-2)^2 f(z) = 1$

$a_{-1}(2) = \lim_{z \to 2} (z-2)\left(f(z) - \frac{a_{-2}(2)}{(z-2)^2}\right) = -1 = \operatorname*{res}_{z=2} f(z)$

Method 2: $\operatorname*{res}_{z=1} f(z) = a_{-1}(1) = \frac{1}{2\pi i} \oint_{\gamma_\varepsilon} \frac{dz}{(z-1)(z-2)^2} \underset{\substack{\text{Cauchy's} \\ \text{int. formulas}}}{=} \left.\left(\frac{1}{(z-2)^2}\right)\right|_{z=1} = 1$

$\operatorname*{res}_{z=2} f(z) = a_{-1}(2) = \frac{1}{2\pi i} \oint_{\gamma_\varepsilon} \frac{dz}{(z-1)(z-2)^2} = \left.\frac{d}{dz}\left(\frac{1}{z-1}\right)\right|_{z=2} = -\left.\frac{1}{(z-1)^2}\right|_{z=2} = -1$

Method 3: $\gamma \to \gamma_R$ s.t. $\gamma \sim \gamma_R$

$\left|\oint_{\gamma_R} f(z)dz\right| \approx \frac{2\pi R}{R^3} = \frac{2\pi}{R^2} \qquad R \to \infty : \oint_{\gamma_R} f(z)dz \to 0$

$$\boxed{\oint_\gamma f(z)dz = 0}$$

6. Evaluation of definite integrals.

6.1. Real integrals.

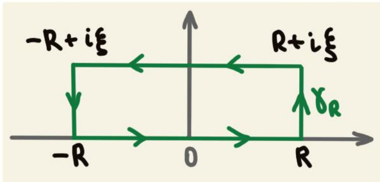
Ex. 6.1.

$$\int_{-\infty}^{\infty} \exp(-\pi x^2)\exp(-2\pi i x\xi)\,dx \qquad x,\xi \in \mathbb{R}$$

Solution:

$\xi = 0$: $\displaystyle\int_{-\infty}^{\infty} \exp(-\pi x^2)\,dx = 1$ (from real analysis)

$\xi > 0$: Consider $f(z) = \exp(-\pi z^2)$, it's entire. Let's choose a rectangular contour γ_R and apply Cauchy's th. to it:



$$\oint_{\gamma_R} f(z)\,dz = 0$$

1) $\displaystyle I_0(R) = \int_{-R}^{R} \exp(-\pi x^2) \qquad \lim_{R\to\infty} I_0(R) = 1$

2) $\displaystyle I_1(R) = \int_0^{\xi} f(R+iy)\,i\,dy = \int_0^{\xi} \exp(-\pi(R^2+2iRy-y^2))\,i\,dy$

$|I_1(R)| \leq C\cdot\exp(-\pi R^2) \implies \displaystyle\lim_{R\to\infty} I_1(R) = 0$

$\displaystyle I_2(R) = \int_{\xi}^{0} f(-R+iy)\,i\,dy = \int_{\xi}^{0} \exp(-\pi(R^2-2iRy-y^2))\,i\,dy$

$|I_2(R)| \leq C\cdot\exp(-\pi R^2) \implies \displaystyle\lim_{R\to\infty} I_2(R) = 0$

3) $\displaystyle I_3 = \int_{-\infty}^{\infty} f(x+i\xi)\,dx = \int_{-\infty}^{\infty} \exp(-\pi(x^2+2ix\xi-\xi^2))\,dx = -\exp(\pi\xi^2)\int_{-\infty}^{\infty} \exp(-\pi x^2)\exp(-2\pi i x\xi)\,dx$

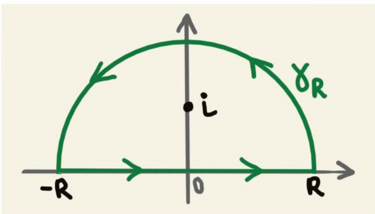
$I_0 + I_1 + I_2 + I_3 = 0 \implies \boxed{\displaystyle\int_{-\infty}^{\infty} \exp(-\pi x^2)\exp(-2\pi i x\xi)\,dx = \exp(-\pi\xi^2)}$

$\xi < 0$: analogous

Ex. 6.2.

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \qquad f(z) = \frac{1}{1+z^2}$$

Solution:



$\gamma_R = \gamma_R^+ \cup [-R,R]$

pole $z = i$: $\displaystyle\operatorname{res}_i f = \lim_{z\to i}(z-i)\frac{1}{1+z^2} = \frac{1}{2i}$

pole $z = -i$: $\displaystyle\operatorname{res}_{-i} f = \lim_{z\to -i}(z+i)\frac{1}{1+z^2} = -\frac{1}{2i}$

$$\oint_{\gamma_R} f(z)\,dz = 2\pi i\cdot\frac{1}{2i} = \pi$$

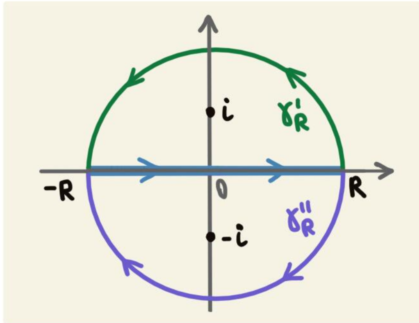
$\displaystyle\left|\int_{\gamma_R^+} f(z)\,dz\right| \leq \pi R\cdot\frac{B}{R^2} = \frac{\pi B}{R} \implies \lim_{R\to\infty}\int_{\gamma_R^+} f(z)\,dz = 0 \implies \boxed{I = \int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \pi}$

Ex. 6.2.5. (Sh.)

$$\phi(t) = \int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\,dx\ ,\quad t\in\mathbb{R} \qquad f(z) = \frac{e^{itz}}{1+z^2}$$

Solution:

$\phi(t)$ is majorized by $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ $\Rightarrow$ conv. abs.



1. Singularites:

$$z = i:\ \operatorname*{res}_{i} f = \frac{e^{it\cdot i}}{2\cdot i} = \frac{e^{-t}}{2i} \Rightarrow \oint_{\gamma'} f(z)\,dz = \pi e^{-t}$$

$$z=-i:\ \operatorname*{res}_{-i} f = \frac{e^{it(-i)}}{2(-i)} = -\frac{e^{t}}{2i} \Rightarrow \oint_{\gamma''} f(z)\,dz = -\pi e^{t}$$

2. Evaluating integrals:

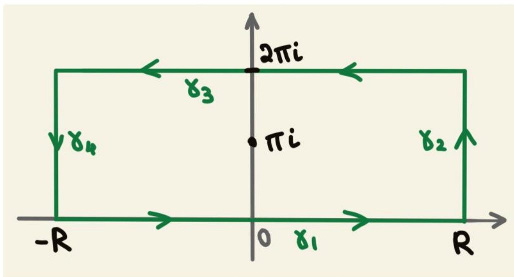
$$t>0:\ \left|\int_{\gamma_R'} \frac{e^{itz}}{1+z^2}\,dz\right| \leqslant \frac{\pi R}{R^2-1} \Rightarrow \lim_{R\to\infty}\int_{\gamma_R'} f(z)\,dz = 0 \Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \pi e^{-t}$$

$$t<0:\ \left|\int_{\gamma_R''} \frac{e^{itz}}{1+z^2}\,dz\right| \leqslant \frac{\pi R}{R^2-1} \Rightarrow \lim_{R\to\infty}\int_{\gamma_R''} f(z)\,dz = 0 \Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \pi e^{t}$$

$$\boxed{\phi(t) = \int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\,dx = \pi e^{-|t|}}$$

Ex. 6.2.13. (Sh.)

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx\ ,\quad 0<a<1 \qquad f(z) = \frac{e^{az}}{1+e^z}$$



1. Singularities:

$$z=\pi i:\ \operatorname*{res}_{\pi i} f = \operatorname*{res}_{\pi i}\frac{\Phi}{\Psi} = \frac{\Phi(\pi i)}{\Psi'(\pi i)} = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i}$$

$$z=-\pi i:\ \operatorname*{res}_{-\pi i} f = -e^{-a\pi i}$$

Then $\oint_{\gamma(R)} f(z)\,dz = -2\pi i\, e^{a\pi i}$.

2. Evaluating integrals:

$$\gamma_2(R):\ \left|\int_{\gamma_2(R)} f(z)\,dz\right| \leqslant \int_0^{2\pi}\left|\frac{e^{a(R+iy)}}{1+e^{R+iy}}\right|dy \leqslant C\,e^{(a-1)R} \Rightarrow \lim_{R\to\infty}\int_{\gamma_2(R)} f(z)\,dz = 0 \quad (\text{since } a<1)$$

$$\sup_{\gamma_2(R)}\left|\frac{e^{aR}e^{iay}}{1+e^R e^{iy}}\right| \leqslant \frac{e^{aR}}{e^R-1} = e^{(a-1)R}\,\frac{1}{1-e^{-R}}$$

$$\gamma_4(R):\ \left|\int_{\gamma_4(R)} f(z)\,dz\right| \leqslant \int_{2\pi}^{0}\left|\frac{e^{a(-R+iy)}}{1+e^{-R+iy}}\right|dy \leqslant C\,e^{-aR} \Rightarrow \lim_{R\to\infty}\int_{\gamma_4(R)} f(z)\,dz = 0 \quad (\text{since } a>0)$$

$$\sup_{\gamma_4(R)}\left|\frac{e^{-aR}e^{iay}}{1+e^{-R}e^{iy}}\right| \leqslant \frac{e^{-aR}}{1-e^{-R}}$$

$$\gamma_3(R):\ \int_{\gamma_3(R)} f(z)\,dz = \int_R^{-R}\frac{e^{ax}\cdot e^{ai2\pi}}{1+e^x e^{i2\pi}}\,dx = e^{i2\pi a}\int_R^{-R}\frac{e^{ax}}{1+e^x}\,dx = -e^{i2\pi a}\int_{\gamma_1(R)} f(z)\,dz$$

Then we have:

$$I - e^{i2\pi a} I = -2\pi i\, e^{i\pi a} \Rightarrow I = \frac{-2\pi i\, e^{i\pi a}}{1-e^{i2\pi a}} = \frac{-2\pi i}{e^{-i\pi a}-e^{i\pi a}} = \boxed{\frac{\pi}{\sin(\pi a)}}$$

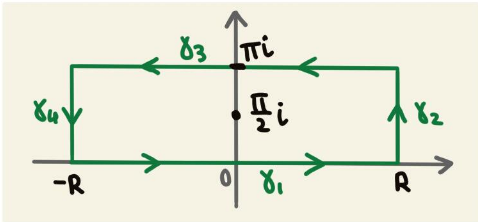
Ex. 6.2.11. (Sh.)

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{\exp(x) + \exp(-x)}\, dx$$

Solution:

$$\sin(x) = \sin(-x) \Rightarrow \int_{-\infty}^{\infty} \frac{\sin(x)}{\exp(x)+\exp(-x)}\, dx = 0 \Rightarrow f(z) = \frac{\exp(iz)}{\exp(z)+\exp(-z)}$$

1. Singularities:



$$e^{z} + e^{-z} = 0 \Rightarrow e^{2z} = -1 \Rightarrow e^{2x} e^{2iy} = -1$$

$$\left.\begin{array}{lll} e^{2x} = 1 & \Rightarrow & x = 0 \\ e^{2iy} = -1 & \Rightarrow & y = \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z} \end{array}\right| \Rightarrow \underset{\text{poles}}{z = i\left(\tfrac{\pi}{2} + \pi k\right)}$$

$$\frac{d}{dz}\left(e^{z} + e^{-z}\right)\Big|_{z = i\frac{\pi}{2}} = \left(e^{z} - e^{-z}\right)\Big|_{z = i\frac{\pi}{2}} = 2i \neq 0 \Rightarrow z = i\frac{\pi}{2} \text{ is a simple pole}$$

$$\operatorname*{res}_{i\pi/2} f = \operatorname*{res}_{i\pi/2} \frac{\phi}{\psi} = \frac{\phi(i\pi/2)}{\psi'(i\pi/2)} = \frac{\exp(i \cdot i\pi/2)}{\exp(i\pi/2) - \exp(-i\pi/2)} = \frac{e^{-\pi/2}}{2i} \Rightarrow \oint_{\gamma(R)} f(z)\,dz = 2\pi i \cdot \frac{e^{-\pi/2}}{2i} = \pi e^{-\frac{\pi}{2}}$$

2. Evaluation of integrals:

$\gamma_2(R)$: $$\left|\int_{\gamma_2(R)} f(z)\,dz\right| \leqslant \int_{\gamma_2(R)} |f(z)|\,dz \leqslant \pi \cdot \sup_{y \in [0,\pi]} \left|\frac{e^{i(R+iy)}}{e^{R}e^{iy} + e^{-R}e^{-iy}}\right|$$

$$\left|\frac{e^{i(R+iy)}}{e^{R}\left(e^{iy} + e^{-2R}e^{-iy}\right)}\right| \leqslant \frac{e^{-y}}{e^{R}\left|e^{iy} + e^{-2R}e^{-iy}\right|} \leqslant \frac{e^{-y}}{e^{R}\left(|e^{iy}| + e^{-2R}|e^{-iy}|\right)} \underset{y\in[0,\pi]}{\leqslant} \frac{1}{e^{R}\left(1 - e^{-2R}\right)}$$

$$\lim_{R\to\infty} \int_{\gamma_2(R)} f(z)\,dz = 0$$

$\gamma_4(R)$: analogous to $\gamma_2(R)$, so $\lim_{R\to\infty} \int_{\gamma_4(R)} f(z)\,dz = 0$.

$\gamma_3(R)$: $$\int_{\gamma_3(R)} f(z)\,dz = \int_{R}^{-R} \frac{e^{i(x+i\pi)}}{e^{x+\pi i} + e^{-x-\pi i}}\,dx = e^{-\pi}\int_{R}^{-R} \frac{e^{ix}}{-e^{x} - e^{-x}}\,dx = e^{-\pi}\int_{-R}^{R} \frac{e^{ix}\,dx}{e^{x} + e^{-x}} = e^{-\pi}\int_{\gamma_1(R)} f\,dz$$

Then we have:

$$I + e^{-\pi} I = \pi e^{-\pi/2}$$

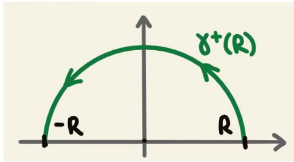
$$I = \frac{\pi e^{-\pi/2}}{1 + e^{-\pi}} = \frac{\pi}{e^{\pi/2} + e^{-\pi/2}}$$

6.2. Jordan's lemma.

Th. 6.1. (Jordan's lemma):
Consider a semicircle $\gamma^+(R) = \{Re^{i\theta} \mid \theta \in [0,\pi]\}$ in the upper half-plane. If $f(z) = e^{iaz} g(z)$, $a > 0$, is meromorphic, then $\left| \int_{\gamma^+(R)} f(z)dz \right| \leq \frac{\pi}{a} \|g\|_{\gamma^+(R)}$.

- If $\lim_{R\to\infty} |g(z)| = 0$, then $\lim_{R\to\infty} \int_{\gamma^+(R)} f(z)dz = 0$.
- Jordan's lemma yields a simple way to calculate improper integrals along the real axis (using the residue th.):

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_k \operatorname*{res}_{z=z_k} f(z)$$



- If $a < 0$, then consider a semicircle $\gamma^-(R)$ in the lower half-plane.

Proof:

$$\int_{\gamma^+(R)} f(z)dz = \int_0^{\pi} g(Re^{i\theta}) e^{iaR(\cos\theta + i\sin\theta)} iRe^{i\theta} d\theta = iR \int_0^{\pi} g(Re^{i\theta}) e^{-aR\sin\theta} e^{i(aR\cos\theta + \theta)} d\theta$$

$$\left| \int_{\gamma^+(R)} f(z)dz \right| \leq R \int_0^{\pi} |g(Re^{i\theta})| e^{-aR\sin\theta} d\theta \leq R \|g\|_{\gamma^+(R)} \cdot 2\int_0^{\pi/2} e^{-aR\sin\theta} d\theta \leq \left[\theta \in [0, \tfrac{\pi}{2}] : \overset{\text{concave}}{\sin\theta \geq \tfrac{2}{\pi}\theta} \right] \leq$$

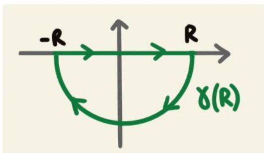
$$\leq 2R \|g\|_{\gamma^+(R)} \int_0^{\pi/2} e^{-aR\frac{2}{\pi}\theta} d\theta = 2R \|g\|_{\gamma^+(R)} \left(-\frac{\pi}{2aR}\right) e^{-aR\frac{2}{\pi}\theta} \Big|_0^{\pi/2} = \frac{\pi}{a} \|g\|_{\gamma^+(R)} (1 - e^{-aR}) \leq \frac{\pi}{a} \|g\|_{\gamma^+(R)}$$

Ex. 6.22. (Cah.)

$$\int_{-\infty}^{\infty} \frac{dx}{(x-i)(x-2i)(x-3i)}$$

Solution:

Consider $\gamma^-(t) = Re^{-it}$, $t \in [0,\pi]$. All the poles of $f(z) = \frac{1}{(z-i)(z-2i)(z-3i)}$ are in the upper half-plane, so $\oint_{\gamma} f(z)dz = \int_{-R}^{R} f(x)dx + \int_{\gamma^-(R)} f(z)dz = 0$.



$$\left| \int_{\gamma^-(R)} f(z)dz \right| \leq \pi R \cdot \frac{1}{R^3} \Rightarrow \lim_{R\to\infty} \int_{\gamma^-(R)} f(z)dz = 0 \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x-i)(x-2i)(x-3i)} = 0$$

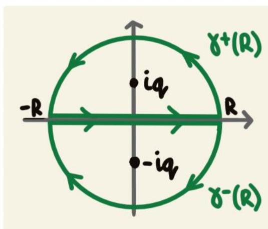
Ex. 6.24. (Cah.)

$$\int_0^{\infty} \frac{\cos x}{q^2 + x^2} dx, \quad q > 0$$

Solution:

$$\int_0^{\infty} \frac{\cos x}{q^2+x^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos x}{q^2+x^2} dx = \frac{1}{4} \int_{-\infty}^{\infty} \frac{e^{ix} dx}{(x-iq)(x+iq)} + \frac{1}{4} \int_{-\infty}^{\infty} \frac{e^{-ix} dx}{(x-iq)(x+iq)} = \frac{1}{4}(I_1 + I_2)$$

Consider $\gamma^+(R) = Re^{it}$ for I_1 and $\gamma^-(R) = Re^{-it}$ for I_2 $(t \in [0,\pi])$.



$$\left.\begin{array}{l} \lim_{R\to\infty} \int_{\gamma^+(R)} f_1(z)dz \underset{\text{Jordan's lemma}}{=} 0 \\ \lim_{R\to\infty} \int_{\gamma^-(R)} f_2(z)dz = 0 \end{array}\right| \Rightarrow \begin{array}{l} I_1 = 2\pi i \cdot \operatorname*{res}_{z=iq} f_1(z) \\ I_2 = \underset{\uparrow\ \gamma^- \text{ has neg. orient.}}{-} 2\pi i \cdot \operatorname*{res}_{z=-iq} f_2(z) \end{array}$$

$$I = \frac{1}{4} \cdot 2\pi i \left(\lim_{z\to iq} \frac{e^{iz}}{z+iq} - \lim_{z\to -iq} \frac{e^{-iz}}{z-iq} \right) = \frac{\pi i}{2} \left(\frac{e^{-q}}{2iq} + \frac{e^{-q}}{2iq} \right) = \frac{\pi}{2} \frac{e^{-q}}{q}$$

Ex. 6.26. (Cah.)

$$I(x)=\int_{-\infty}^{\infty}\frac{e^{ikx}}{k^2+m^2}\,dk \qquad \text{i.e. Fourie transform of } f(k)=\frac{1}{k^2+m^2}$$

Solution:

$f(z)=\frac{e^{izx}}{z^2+m^2}$ As before let's consider γ^+ and γ^- half-circle contours.

1. $x>0$: $\lim\limits_{R\to\infty}\int_{\gamma^+(R)} f(z)dz=0$ (Jordan's lemma)

$$I(x)=\oint_{\gamma^+}\frac{e^{izx}\,dk}{(z+im)(z-im)}=2\pi i\cdot\operatorname*{res}_{z=im} f(z)=2\pi i\cdot\lim_{z\to im}\frac{e^{izx}}{z+im}=2\pi i\cdot\frac{e^{-mx}}{2im}=\frac{\pi}{m}e^{-mx}$$

2. $x<0$: $\lim\limits_{R\to\infty}\int_{\gamma^-(R)} f(z)dz=0$ (Jordan's lemma)

$$I(x)=\oint_{\gamma^-}\frac{e^{izx}\,dk}{(z+im)(z-im)}=-2\pi i\cdot\operatorname*{res}_{z=-im} f(z)=-2\pi i\cdot\lim_{z\to -im}\frac{e^{izx}}{z-im}=-2\pi i\cdot\frac{e^{mx}}{-2im}=\frac{\pi}{m}e^{mx}$$

$$\boxed{I(x)=\frac{\pi}{m}e^{-m|x|}}$$

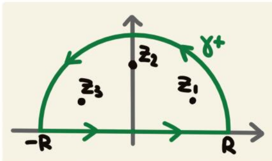
Ex. 6.2.1. (Sh.)

a) $\int_{-\infty}^{\infty}\frac{dx}{1+x^6}$ $\qquad f(z)=\frac{1}{1+z^6}$

Solution:

Consider $\gamma^+(t)=Re^{it}$, $t\in[0,\pi]$. Since $\left|\int_{\gamma^+(R)} f(z)dz\right|\leq \pi R\cdot\frac{B}{R^6}$ we have $\lim\limits_{R\to\infty}\int_{\gamma^+(R)} f(z)dz=0$.

Next let's find the poles in the upper half-plane:



$z^6=-1 \Rightarrow z=e^{i\frac{\pi+2\pi k}{6}}$, $k=0,1,\dots,5$

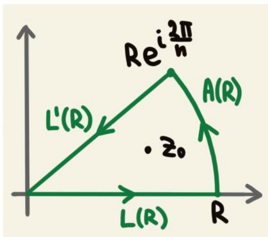
$$\int_{-\infty}^{\infty}\frac{dx}{1+x^6}=\oint_{\gamma}\frac{dz}{1+z^6}=2\pi i\sum_{k=1}^{3}\operatorname*{res}_{z=z_k} f(z)=2\pi i\sum_{k=1}^{3}\left(\frac{1}{6z^5}\right)\Big|_{z=z_k}=$$

$$=\frac{\pi i}{3}\left(e^{-i\frac{5\pi}{6}}+e^{-i\frac{5\pi}{2}}+e^{-i\frac{25\pi}{6}}\right)=\frac{\pi i}{3}\left(-\frac{\sqrt3}{2}-\frac12 i-i+\frac{\sqrt3}{2}-\frac12 i\right)=\boxed{\frac{2\pi}{3}}$$

b) $\int_{0}^{\infty}\frac{dx}{1+x^n}$, $n>1$ $\qquad f(z)=\frac{1}{1+z^n}$

Solution:

Consider $\gamma(R)=L(R)+A(R)+L'(R)$ s.t. it has only one pole inside it: $z_0=e^{i\frac{\pi}{n}}$.



1. Since $\left|\int_{A(R)} f(z)dz\right|\leq\frac{2\pi}{n}R\cdot\frac{B}{R^n}$ we have $\lim\limits_{R\to\infty}\int_{A(R)} f(z)dz=0$.

2. $$\int_{L'(R)} f(z)dz=\int_R^0\frac{e^{i\frac{2\pi}{n}}\,dt}{1+(te^{i\frac{2\pi}{n}})^n}=-e^{i\frac{2\pi}{n}}\int_0^R\frac{dt}{1+(te^{i\frac{2\pi}{n}})^n}=-e^{i\frac{2\pi}{n}}\int_{L(R)} f(z)dz$$

Then:

$$\oint_{\gamma} f(z)dz=2\pi i\cdot\operatorname{res}(f,e^{i\frac{\pi}{n}})=2\pi i\cdot\left(\frac{1}{nz^{n-1}}\right)\Big|_{z=e^{i\frac{\pi}{n}}}=2\pi i\cdot\left(-\frac1n\right)\cdot e^{i\frac{\pi}{n}}$$

$$\oint_{\gamma} f(z)dz=\left(1-e^{i\frac{2\pi}{n}}\right)\int_0^{\infty}\frac{dx}{1+x^n} \quad\Rightarrow\quad \frac{e^{i\frac{\pi}{n}}-e^{-i\frac{\pi}{n}}}{2i}\int_0^{\infty}\frac{dx}{1+x^n}=\frac{\pi}{n}$$

$$\boxed{\int_0^{\infty}\frac{dx}{1+x^n}=\frac{\pi/n}{\sin(\pi/n)}}$$

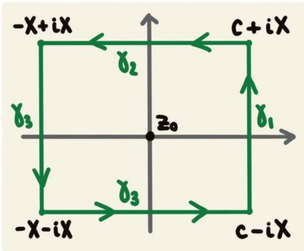
Ex. 6.2.7. (Sh.)

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a^z}{z}\,dz \quad , \quad c>0 \quad a>0 \qquad\qquad f(z)=\frac{a^z}{z}=\frac{e^{\ln a\cdot z}}{z}$$

Solution:

1. $a=1$:
$$\int_{c-i\infty}^{c+i\infty}\frac{dz}{z}=\int_{-\infty}^{\infty}\frac{i\,dt}{c+it}=i\int_{-\infty}^{\infty}\frac{c-it}{c^2+t^2}\,dt=i\int_{-\infty}^{\infty}\frac{c\,dt}{c^2+t^2}+\overbrace{\int_{-\infty}^{\infty}\frac{t\,dt}{c^2+t^2}}^{=0}=i\arctan\frac{t}{c}\Big|_{-\infty}^{\infty}=i\pi$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{dz}{z}=\frac{1}{2}$$



2. $a>1$: consider $\gamma=\gamma_1+\gamma_2+\gamma_3+\gamma_4$

$$\int_{\gamma_2(X)}f(z)\,dz=\int_{c}^{-X}\frac{e^{\ln a(t+iX)}}{t+iX}\,dt$$

$$\left|\int_{\gamma_2(X)}f(z)\,dz\right|\leqslant\int_{-X}^{c}\frac{e^{\ln a\cdot t}}{|t+iX|}\,dt\leqslant\frac{1}{X}\int_{-X}^{c}e^{\ln a\cdot t}\,dt=\frac{e^{c\ln a}-e^{X\ln a}}{X\ln a}\quad\Rightarrow\quad\lim_{X\to\infty}\int_{\gamma_2(X)}f(z)\,dz=0$$

$$\int_{\gamma_3(X)}f(z)\,dz=\int_{X}^{-X}\frac{e^{\ln a(-X+it)}}{-X+it}\,dt$$

$$\left|\int_{\gamma_3(X)}f(z)\,dz\right|\leqslant\int_{-X}^{X}\frac{e^{-X\ln a}}{|-X+it|}\,dt\leqslant\frac{e^{-X\ln a}}{X}\int_{-X}^{X}dt=2e^{-X\ln a}\quad\Rightarrow\quad\lim_{X\to\infty}\int_{\gamma_3(X)}f(z)\,dz=0$$

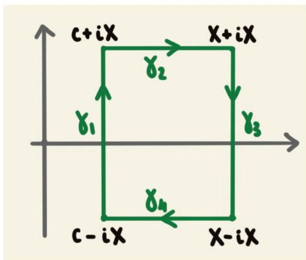
$$\int_{\gamma_4(X)}f(z)\,dz=\int_{-X}^{c}\frac{e^{\ln a(t-iX)}}{t-iX}\,dt$$

$$\left|\int_{\gamma_4(X)}f(z)\,dz\right|\leqslant\int_{-X}^{c}\frac{e^{\ln a\cdot t}}{|t-iX|}\,dt\leqslant\frac{1}{X}\int_{-X}^{c}e^{\ln a\cdot t}\,dt=\frac{e^{c\ln a}-e^{X\ln a}}{X\ln a}\quad\Rightarrow\quad\lim_{X\to\infty}\int_{\gamma_4(X)}f(z)\,dz=0$$

Then we have:

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{a^z}{z}\,dz=\frac{1}{2\pi i}\int_{\gamma_1}\frac{a^z}{z}\,dz=\operatorname*{res}_{z=0}f(z)=a^z\Big|_{z=0}=1$$

3. $a<1$: consider $\gamma=\gamma_1+\gamma_2+\gamma_3+\gamma_4$



$$\int_{\gamma_2}f(z)\,dz=\int_{\gamma_3}f(z)\,dz=\int_{\gamma_4}f(z)\,dz=0 \quad \text{(arguing as before)}$$

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{a^z}{z}\,dz=\frac{1}{2\pi i}\int_{\gamma_1}\frac{a^z}{z}\,dz=0$$

So we have:

$$\boxed{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a^z}{z}\,dz=\begin{cases}0\ , & 0<a<1\\ 1/2\ , & a=1\\ 1\ , & a>1\end{cases}}$$

6.3. Singularities on a contour.

If there is a singularity on a contour of integration, it will cause the integral to diverge. To resolve this issue we:

1. Bypass the point of singularity by going around it.
2. Use the Cauchy principal value to compute the integral along the real axis.

Consider $f:\mathbb{R}\to\mathbb{R}$ that has singularities $\{x_1,\dots,x_n\}$, then:

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx \overset{\text{def}}{=} \lim_{\delta\to 0^+}\left(\int_{-\infty}^{x_1-\delta} f(x)dx + \int_{x_1+\delta}^{x_2-\delta} f(x)dx + \dots + \int_{x_n+\delta}^{\infty} f(x)\,dx\right)$$

This procedure doesn't make the integral to converge. A convergent integral would be uniquely determined and would exist even when all the deltas approach zero independently of each other. On the hand, we can make a divergent integral to have any value by appropriately choosing the relations between $\delta_1,\dots,\delta_n$. For example:

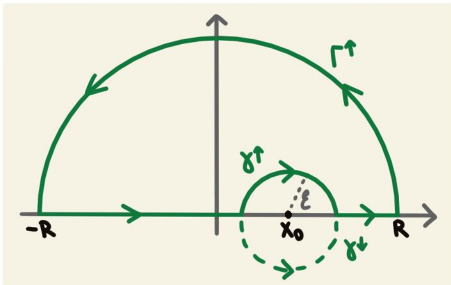
$$I = \int_{-a}^{b}\frac{dx}{x} \qquad \delta=\delta_1=\delta_2:\ \lim_{\delta_1,\delta_2\to 0}\left(\int_{-a}^{-\delta_1}\frac{dx}{x}+\int_{\delta_2}^{b}\frac{dx}{x}\right) = \ln\delta - \ln a + \ln b - \ln\delta = \ln\left(\frac{b}{a}\right)$$

$$\delta=\delta_1=2\delta_2:\ \lim_{\delta_1,\delta_2\to 0}\left(\int_{-a}^{-\delta_1}\frac{dx}{x}+\int_{\delta_2}^{b}\frac{dx}{x}\right) = \ln\delta - \ln a + \ln b - (\ln\delta+\ln 2) = \ln\left(2\frac{b}{a}\right)$$

Now let's consider a meromorphic function $f:\mathbb{C}\to\mathbb{C}$ that is hol. at $z=x_0$ and let's evaluate the integral of $\frac{f(z)}{z-x_0}$ along the real axis:

$$\oint_{\gamma(R)}\frac{f(z)}{z-x_0}dz = \text{P.V.}\int_{-R}^{R}\frac{f(x)}{x-x_0}dx + \int_{\gamma^\uparrow(\varepsilon)}\frac{f(z)}{z-x_0}dz + \int_{\Gamma^\uparrow(R)}\frac{f(z)}{z-x_0}dz$$

$$\oint_{\gamma(R)}\frac{f(z)}{z-x_0}dz = \text{P.V.}\int_{-R}^{R}\frac{f(x)}{x-x_0}dx + \int_{\gamma^\downarrow(\varepsilon)}\frac{f(z)}{z-x_0}dz + \int_{\Gamma^\uparrow(R)}\frac{f(z)}{z-x_0}dz$$



1. $\lim_{R\to\infty}\int_{\Gamma^\uparrow(R)}\frac{f(z)}{z-x_0}dz = 0$ (Jordan's lemma)

2. $$\lim_{\varepsilon\to 0}\int_{\gamma^\uparrow(\varepsilon)}\frac{f(z)}{z-x_0}dz = \lim_{\varepsilon\to 0}\int_{\gamma^\uparrow(\varepsilon)}\frac{f(x_0)+f'(x_0)(z-x_0)+\frac{1}{2}f''(x_0)(z-x_0)^2+\dots}{z-x_0}dz =$$
$$= \lim_{\varepsilon\to 0}\int_{\pi}^{0}\frac{f(x_0)+f'(x_0)\varepsilon e^{it}+\frac{1}{2}f''(x_0)\varepsilon^2 e^{2it}+\dots}{\varepsilon e^{it}}\, i\varepsilon e^{it}dt =$$
$$= \lim_{\varepsilon\to 0} i\int_{\pi}^{0}\left(f(x_0)+f'(x_0)e^{it}\varepsilon+\frac{1}{2}f''(x_0)e^{2it}\varepsilon^2+\dots\right)dt =$$
$$= \lim_{\varepsilon\to 0}\left(-i\pi f(x_0)+C_1\varepsilon+C_2\varepsilon^2+\dots\right) = -i\pi f(x_0)$$

3. $\lim_{\varepsilon\to 0}\int_{\gamma^\downarrow(\varepsilon)}\frac{f(z)}{z-x_0}dz = \dots = i\pi f(x_0)$

$\gamma^\uparrow$: $\oint_{\gamma}\frac{f(z)}{z-x_0}dz = 2\pi i\sum_k \operatorname*{res}_{z=z_k}\left(\frac{f(z)}{z-x_0}\right)$

$\gamma^\downarrow$: $\oint_{\gamma}\frac{f(z)}{z-x_0}dz = 2\pi i\left(\underbrace{\sum_k \operatorname*{res}_{z=z_k}\left(\frac{f(z)}{z-x_0}\right)}_{\text{residues in the upper half-plane}} + \underbrace{f(x_0)}_{\text{residue at } z=x_0}\right)$

$$\Rightarrow \boxed{\text{P.V.}\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx = i\pi f(x_0) + 2\pi i\sum_k \operatorname*{res}_{z=z_k}\left(\frac{f(z)}{z-x_0}\right)}$$

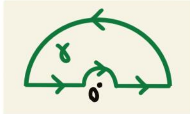
(i.e. same result for both contours)

Ex. 11.3.10. (Has.)

$$I = \int_0^\infty \frac{\sin x}{x}\,dx$$

Solution:

$x = 0$: removable singularities $\left(\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right)$



$$I = \lim_{\varepsilon\to 0}\int_\varepsilon^\infty \frac{\sin x}{x}\,dx = \lim_{\varepsilon\to 0}\left(\int_\varepsilon^\infty \frac{e^{ix}}{2ix}\,dx - \int_\varepsilon^\infty \frac{e^{-ix}}{2ix}\,dx\right) = \lim_{\varepsilon\to 0}\left(\int_\varepsilon^\infty \frac{e^{ix}}{2ix}\,dx + \int_{-\infty}^{-\varepsilon} \frac{e^{ix}}{2ix}\,dx\right) =$$

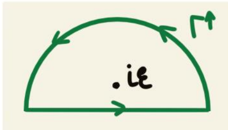
$$= \text{P.V.}\int_{-\infty}^{\infty} \frac{e^{ix}}{2ix}\,dx = \frac{1}{2i}\cdot i\pi e^0 = \boxed{\frac{\pi}{2}}$$

Ex. 11.3.12. (Has.)

$$f(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{ikx}}{x - i\varepsilon}\,dx\,,\quad \varepsilon > 0$$

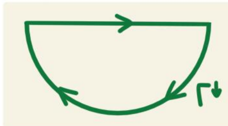
Solution:

$k > 0$: using contour $\Gamma^\uparrow$



$$f(k) = \frac{1}{2\pi i}\oint_\gamma \frac{e^{ikz}}{z - i\varepsilon}\,dz = \operatorname*{res}_{z=i\varepsilon}\left(\frac{e^{ikz}}{z-i\varepsilon}\right) = e^{ik\cdot i\varepsilon} = e^{-k\varepsilon}$$

$k < 0$: using contour $\Gamma^\downarrow$



$$f(k) = \frac{1}{2\pi i}\oint_\gamma \frac{e^{ikz}}{z - i\varepsilon}\,dz = 0$$ (no poles in the lower half-plane)

$\boxed{f(k) \overset{\varepsilon\to 0}{=\!=} \begin{cases} 1, & k > 0 \\ 0, & k < 0 \end{cases}}$, i.e. θ step function $\boxed{\theta(x) \overset{\varepsilon\to 0}{=\!=} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{ixt}}{t - i\varepsilon}\,dt}$

6.4. Multiple singularities on a contour.

Suppose there are two singular points on the real axis at $x = x_1$ and $x = x_2$.

$$\oint_\gamma \frac{f(z)\,dz}{(z-x_1)(z-x_2)} = \text{P.V.}\int_{-R}^{R} \dots + \int_{\gamma_1^\uparrow(\varepsilon)} \dots + \int_{\gamma_2^\uparrow(\varepsilon)} \dots + \int_{\Gamma^\uparrow(R)} \dots$$

1. $$\int_{\gamma_1^\uparrow(\varepsilon)} \frac{f(z)\,dz}{(z-x_1)(z-x_2)} = \int_{\gamma_1^\uparrow(\varepsilon)} \frac{f(x_1) + f'(x_1)(z-x_1) + \frac{1}{2} f''(x_1)(z-x_1)^2 + \dots}{(z-x_1)(z-x_2)}\,dz =$$

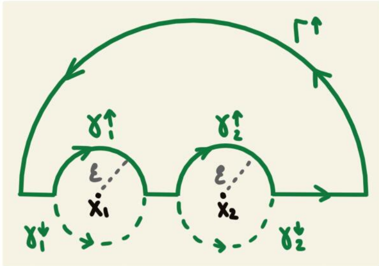
$$= \int_\pi^0 \frac{f(x_1) + f'(x_1)\varepsilon e^{it} + \frac{1}{2} f''(x_0)\varepsilon^2 e^{2it} + \dots}{\varepsilon e^{it}(x_1 + \varepsilon e^{it} - x_2)}\, i\varepsilon e^{it}\,dt =$$

$$= \int_\pi^0 \frac{f(x_1) + f'(x_1)\varepsilon e^{it} + \frac{1}{2} f''(x_0)\varepsilon^2 e^{2it} + \dots}{x_1 - x_2 + \varepsilon e^{it}}\, i\,dt \overset{\varepsilon\to 0}{=\!=} -i\pi \frac{f(x_1)}{x_1 - x_2}$$

2. $$\int_{\gamma_2^\uparrow(\varepsilon)} \frac{f(z)\,dz}{(z-x_1)(z-x_2)} = \dots \overset{\varepsilon\to 0}{=\!=} -i\pi \frac{f(x_2)}{x_2 - x_1}$$

3. $$\int_{\gamma_1^\downarrow(\varepsilon)} \frac{f(z)\,dz}{(z-x_1)(z-x_2)} = \dots \overset{\varepsilon\to 0}{=\!=} -i\pi \frac{f(x_1)}{x_1 - x_2}$$

4. $$\int_{\gamma_2^\downarrow(\varepsilon)} \frac{f(z)\,dz}{(z-x_1)(z-x_2)} = \dots \overset{\varepsilon\to 0}{=\!=} i\pi \frac{f(x_2)}{x_2 - x_1}$$



Then when $R \to \infty$ and $\varepsilon \to 0$ for $\gamma_1^\uparrow$ and $\gamma_2^\uparrow$ we get:

residues in the upper half-plane

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{(x-x_1)(x-x_2)} - i\pi\left(\frac{f(x_1)}{x_1 - x_2} + \frac{f(x_2)}{x_2 - x_1}\right) = 2\pi i \cdot \sum_k \operatorname*{res}_{z=z_k}\left(\frac{f(z)}{(z-x_1)(z-x_2)}\right)$$

$$\boxed{\text{P.V.}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{(x-x_1)(x-x_2)} = i\pi \frac{f(x_2) - f(x_1)}{x_2 - x_1} + 2\pi i \cdot \sum_k \operatorname*{res}_{z=z_k}\left(\frac{f(z)}{(z-x_1)(z-x_2)}\right)}$$

- we get the same result if we choose $\gamma_1^\downarrow$ and $\gamma_2^\downarrow$ instead (as expected)
- $x_1 \to x_2$: $\text{P.V.}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{(x-x_0)^2} = i\pi f'(x_0) + 2\pi i \sum_k \operatorname*{res}_{z=z_k}\left(\frac{f(z)}{(z-x_0)^2}\right)$

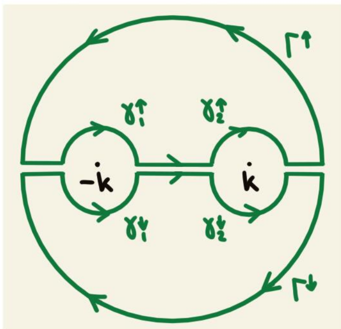
Ex. 11.3.13. (Has.)

$$I = \int_{-\infty}^{\infty} \frac{e^{itx}}{x^2 - k^2}\,dx \quad , \quad k, t \in \mathbb{R}$$

Solution:

1. $t > 0$: using a contour in the upper half-plane

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{e^{itx}\,dx}{(x-k)(x+k)} = i\pi \frac{e^{ikt} - e^{-ikt}}{2k} = -\frac{\pi}{k}\sin(kt)$$



2. $t < 0$: using a contour in the lower half-plane

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{e^{itx}\,dx}{(x-k)(x+k)} = -i\pi \frac{e^{ikt} - e^{-ikt}}{2k} = \frac{\pi}{k}\sin(kt)$$

because $\gamma_1^\downarrow$ and $\gamma_2^\downarrow$

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2 - k^2}\,dx = -\frac{\pi}{k}\sin(k|t|)$$

6.5. Avoiding branch cuts.

When integrating multi-valued functions:

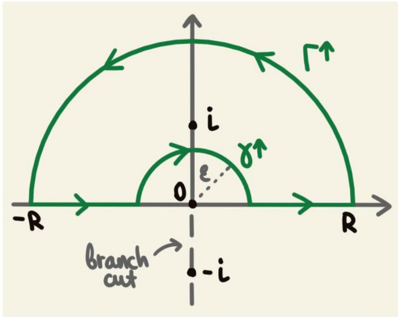
- choose a branch to make f single-valued
- choose a contour that avoids branch cuts and singularities (f is not defined there)

> **Branch cut**: a line that separates different branches of a multi-valued function.
> **Branch point**: the point common to all the branch cuts.

Ex. 6.3.

$$I = \int_0^\infty \frac{\ln x}{1+x^2}\,dx \qquad f(z) = \frac{\ln z}{1+z^2}$$

Solution:



Consider $\arg z \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ branch of $\ln z$ and a contour γ

$$\oint_{\gamma(R)} f(z)\,dz = \int_{-R}^{-\varepsilon} f(x)\,dx + \int_{\gamma^{\uparrow}(\varepsilon)} f(z)\,dz + \int_{\varepsilon}^{R} f(x)\,dx + \int_{\Gamma^{\uparrow}(R)} f(z)\,dz$$

1. $$\left|\int_{\Gamma^{\uparrow}(R)} f(z)\,dz\right| \leqslant \int_{\Gamma^{\uparrow}(R)} |f(z)|\,dz = \int_0^{\pi} \frac{|\ln R + it|}{|1+R^2e^{2it}|}\,|iRe^{it}|\,dt \leqslant \pi R \cdot \frac{\ln R + \pi}{R^2-1}$$

$$\lim_{R\to\infty} \frac{R(\ln R + \pi)}{R^2-1} = \lim_{R\to\infty} \frac{1+\ln R + \pi}{2R} = 0$$

2. $$\left|\int_{\gamma^{\uparrow}(\varepsilon)} f(z)\,dz\right| \leqslant \ldots \leqslant \pi\varepsilon \cdot \frac{\ln\varepsilon + \pi}{1-\varepsilon^2}$$

$$\lim_{\varepsilon\to 0} \frac{\varepsilon(\ln\varepsilon + \pi)}{1-\varepsilon^2} = \lim_{\varepsilon\to 0} \frac{\ln\varepsilon}{1/\varepsilon - \varepsilon} = \lim_{\varepsilon\to 0} \frac{1/\varepsilon}{-1/\varepsilon^2 - 1} = \lim_{\varepsilon\to 0} \frac{-\varepsilon}{1+\varepsilon^2} = 0$$

3. $$\int_{-\infty}^{0} f(z)\,dz = \int_{-\infty}^{0} \frac{\ln|z| + i\arg z}{1+z^2}\,dz = \int_{-\infty}^{0} \frac{\ln|x|}{1+x^2}\,dx + i\pi\int_{-\infty}^{0} \frac{dx}{1+x^2} = \int_0^{\infty} \frac{\ln x}{1+x^2}\,dx + i\pi \cdot \operatorname{artan} x\Big|_{-\infty}^{0} =$$

$$= \int_0^{\infty} \frac{\ln x}{1+x^2}\,dx + i\pi \cdot \frac{\pi}{2}$$

Then we have:

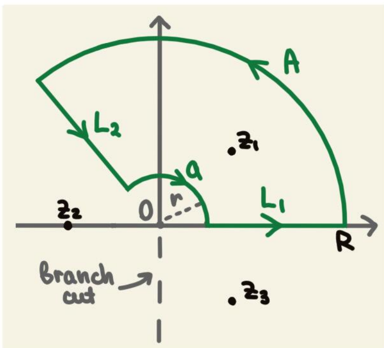
$$2\int_0^{\infty} \frac{\ln x}{1+x^2}\,dx + i\frac{\pi^2}{2} = 2\pi i \cdot \operatorname*{res}_{z=i} f(z) = 2\pi i \cdot \frac{\ln i}{2i} = \pi \cdot \ln e^{i\frac{\pi}{2}} = i\frac{\pi^2}{2}$$

$$\boxed{\int_0^{\infty} \frac{\ln x}{1+x^2}\,dx = 0}$$

Ex. 11.8.7. (AHW)

$$I = \int_0^\infty \frac{\ln x}{1+x^3}\,dx \qquad f(z) = \frac{\ln z}{1+z^3}$$

Solution:



Consider $\arg z \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ branch of $\ln z$ and a contour γ

$$\oint_{\gamma(R)} f(z)\,dz = \int_{L_1(R)} f(z)\,dz + \underbrace{\int_{A(R)} f(z)\,dz}_{\to 0} + \underbrace{\int_{L_2(R)} f(z)\,dz}_{\sim I} + \underbrace{\int_{a(r)} f(z)\,dz}_{\to 0}$$

...

$$\boxed{I = -\frac{2\pi^2}{27}}$$