

Symmetry and the Standard Model

Chapter I Introduction

① Review of classical physics.

1.1. Classical mechanics

- Hamilton's principle
- Euler-Lagrange eqns.
- Noether's theorem ($\delta L = 0$ under $q \rightarrow q + \epsilon \delta q \Rightarrow$ current $j \stackrel{\text{def}}{=} \frac{\partial L}{\partial \dot{q}_i} \delta q_i : \frac{dj}{dt} = 0$)
i.e. conserved

1.2. Special Relativity

Lorentz tr. = x_0, x_1, x_2, x_3 rotations + t_0, t_1, t_2, t_3 rotations
(Euler trans.) (Lorentz boosts)

$$S = \int L dt = \int p_\mu dx^\mu \quad E^2 = p^2 + m^2$$

Physically allowable transformations:

$$\eta_{\mu\nu} = \Lambda_\mu^{M_1} \Lambda_\nu^{N_1} \eta_{M_1 N_1} \quad \eta_{\mu\nu} = \text{diag}\{-1, 1, 1, 1\}$$

(i.e. preserves inner product)

$$\eta_{00} = \Lambda_0^{M_1} \Lambda_0^{N_1} \eta_{M_1 N_1} \Rightarrow -1 = -(\Lambda_0^0)^2 + \sum_{i=1}^3 (\Lambda_0^i)^2 \Rightarrow (\Lambda_0^0)^2 \geq 1$$

proper tr. : $\det \Lambda = +1$

orthochronous tr. : $\Lambda_0^0 \geq 1$

improper tr. : $\det \Lambda = -1$ (changes parity)

non-orthochronous tr. : $\Lambda_0^0 \leq -1$ (e.g. time reverse)

Only proper and orthochronous tr. are physical, but other tr. are still required for Lorentz invariance.

1.3. Classical fields

$$S = \int dt L = \int d^4x \mathcal{L}, \text{ where } L = \int d^3x \mathcal{L}$$

$$H = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} = \Pi \dot{\phi} - \mathcal{L}$$

(Lagrangian and Hamiltonian)
densities

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i)} \right) = \frac{\partial \mathcal{L}}{\partial \phi_i} \quad i=1, \dots, n$$

Euler-Lagrange eqns. for ϕ_i :

$$\text{Noether's th: } \delta \mathcal{L} = 0 \text{ under } \phi \rightarrow \phi + \epsilon \delta \phi \Rightarrow j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi : \partial_\mu j^\mu = 0 \quad (j^\mu = \begin{pmatrix} j^0 \\ j \end{pmatrix} \begin{matrix} \leftarrow \text{charge density} \\ \leftarrow 3D \text{ current} \end{matrix})$$

1.4. Classical Electrodynamics

$$A^\mu = (\phi, \vec{A})^T \quad (4\text{-potential})$$

$$J^\mu = (g, \vec{j})^T \quad (4\text{-current})$$

$$F_{12} = \frac{q_1 q_2}{4\pi r^2}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

EM field Strength Tensor

$$\begin{cases} \partial_\mu F_{\nu\lambda} = 0 \\ \partial_\mu F^{\mu\nu} = J^\nu \end{cases}$$

$$\Rightarrow \begin{cases} \nabla \cdot \vec{E} = g \\ \nabla \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t} \end{cases} \quad \begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \end{cases}$$

Maxwell eqns.

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu$$

$$\text{Gauge tr.: } A^\mu \rightarrow A^\mu + \partial^\mu \chi : F^{\mu\nu} = \text{inv} \text{ and } A_\mu J^\mu = \text{inv} \Rightarrow \mathcal{L}_{EM} = \text{inv}$$

invariance: $A, B, C, \dots \rightarrow A', B', C', \dots \Rightarrow F(A, B, C, \dots) = F(A', B', C', \dots)$

symmetry: something remains the same under specific transformations

covariance: eqns. have the same form in all inertial frames of reference
(all laws of physics should be covariant)

② Open questions and road map

2.1. Open questions

1) 3 generations

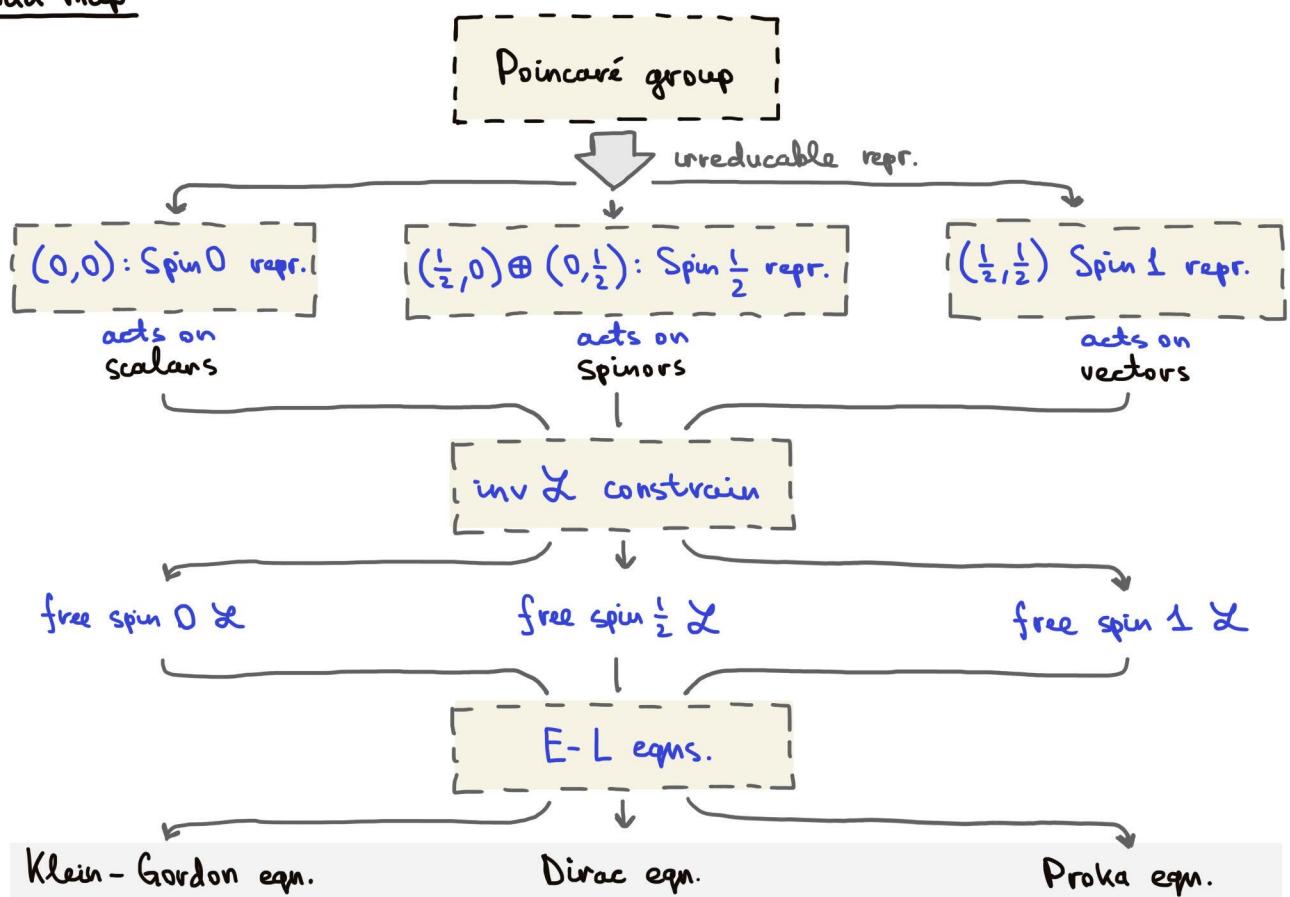
- 3 gauge theories \leftrightarrow 3 fundamental forces
($U(1) \times SU(2) \times SU(3)$) (em, weak, strong)
- 3 generations of quarks and leptons
- only include 3 lowest orders in Φ in the Lagrangian of the theory
(describes free fields and particles; higher order theories are unbound from below)
i.e. $E \rightarrow -\infty \Rightarrow$ not stable
- only use 3 lowest-dimensional representations of the double cover of the Poincaré group, which correspond to spins $0, \frac{1}{2}$ and 1 (no $\frac{3}{2}$ spin particles)

2) Values of constants

Coupling constants of interactions and masses of particles.

3) Quantum gravity

2.2. Road map



③ Brief overview of particle physics.

- elementary particles have mass, spin and charges
- elem. particles: bosons ($H, \gamma, W^\pm, Z^0, g_i : s=0, \pm$; forces)
fermions (quarks and leptons: $s=\frac{1}{2}$; matter)
- photons and gluons are massless ($m=0$); and W^\pm, Z^0 have mass due to spontaneous symmetry breaking (Higgs mechanism)
- difference between forces and matter because of Pauli's exclusion principle
- charges of fund. interactions: electric charge, isospin and color
- quarks and leptons have antiparticles
- quarks carry color charge, while leptons do not
- quarks can compose hadrons (composite particles that interact strongly):
mesons ($q\bar{q}$), baryon (qqq), tetraquarks, pentaquarks
(bosons)
(fermions)

| | | | e | isospin | color |
|---|---|---|------|---------|-------|
| u | c | t | +2/3 | +1/2 | v |
| d | s | b | -1/3 | -1/2 | v |

| | | | e | isospin | color |
|-----------|-----------|------------|----|---------|-------|
| ν_e | ν_μ | ν_τ | 0 | +1/2 | x |
| \bar{e} | μ^- | τ^- | -1 | -1/2 | x |

Brief history:

- | | |
|---------|---|
| 400 BCE | - Empedocles (earth, air, fire and water) and Democritus (atoms) |
| 1820 | - C. Ørsted (electricity is related to magnetism) |
| 1873 | - J.C. Maxwell |
| 1897 | - J.J. Thompson (electron) |
| 1911 | - E. Rutherford (structure of atoms) |
| 1950s | - S. Tomonaga, J. Schwinger and R. Feynman: QED |
| 1954 | - C.N. Yang, R. Mills: gauge invariance \Rightarrow 3 massless gauge bosons |
| 1961 | - S. Glashow: electro-weak theory with massless gauge bosons |
| 1963 | - H. Gell-Mann, G. Zweig: quarks |
| 1964 | - Guralnik, Hagen, Kibble; Higgs; Brout, Englert: Higgs mechanism |
| 1965 | - M-Y. Han, Y. Nambu, D. Greenberg: foundations of QCD |
| 1967 | - S. Weinberg, A. Salam: electro-weak theory |
| 19xx- | - experimental discoveries of particles (see Ch.2 in Robinson) |

Chapter II

Group Theory

① Basics

(G, \circ) is a **group**, $\circ: G \times G \rightarrow G$: 1. $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3 \quad \forall g_1, g_2, g_3 \in G$ (associativity)
 (closure) 2. $\exists e \in G: \forall g \in G \quad e \circ g = g \circ e = g$ (identity)
 3. $\forall g \in G \quad \exists \bar{g} \in G: \bar{g} \circ g = g \circ \bar{g} = e$ (inverse)

assoc. $\Rightarrow \exists! \bar{g}$ (let h_1, h_2 both be inverse to $g \Rightarrow h_1 = h_2 \circ e = h_1 \circ (g \circ h_2) = (h_1 \circ g) \circ h_2 = e \circ h_2 = h_2$)

1.1. Finite discrete groups

- a finite discrete group can be defined by a multiplication table
- each element appears only once in a row (and in a column)

let $g_i, g_j = g_i, g_k = g_k \Rightarrow g_{ii} = g_{ij}$ (because of $\exists! \bar{g}_j$)

$\text{ord } G = 2:$

| | | |
|-------|-------|-------|
| | e | g_1 |
| e | e | g_1 |
| g_1 | g_1 | e |

use property or:

$$\begin{aligned} \text{let } g_i \circ g_1 = g_1 &\Rightarrow \\ g_1 = g_i^{-1} \circ g_1 &= e \\ g_i \circ g_1 &= e \end{aligned}$$

$\text{ord } G = 3:$

| | | | |
|-------|-------|-------|-------|
| | e | g_1 | g_2 |
| e | e | g_1 | g_2 |
| g_1 | g_1 | g_2 | e |
| g_2 | g_2 | e | g_1 |

$\text{ord } G = n: \text{ finite group}$

$\text{ord } G = \infty: \text{ infinite group}$

(e.g. $\text{ord } S_n = n!$)

↑
symmetric group

Ex.: $(\mathbb{Z}_2, +), (\{\pm 1\}, \cdot)$, etc.

1.2. Representations

integers (residues) modulo n

Ex.: $(\mathbb{Z}_3, +):$ 1) $(\{[0]_3, [1]_3, [2]_3\}, +)$ 2) $(\{1, \exp(\frac{2}{3}\pi i), \exp(\frac{4}{3}\pi i)\}, \cdot)$

We will use linear representations, i.e. linear operators acting on some vector space:

$D(G)$: representation of (G, \circ) , and $D(g_i) \cdot D(g_k) = D(g_i \circ g_k)$

$|v\rangle = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \quad \langle v| = (v_1, \dots) \quad \langle u|v\rangle = u^T v$
 $\langle i|A|j\rangle$ - matrix elements of A if $\{|i\rangle\}$ - basis ($|A|j\rangle$ - new vector, $\langle i|A|j\rangle$ - its coordinates)
 (see linear algebra)

1. elements of $G \leftrightarrow$ orthonormal basis in some vector space

$$e \rightarrow |e\rangle = |\hat{e}_1\rangle \quad g_1 \rightarrow |g_1\rangle = |\hat{e}_2\rangle \quad g_2 \rightarrow |g_2\rangle = |\hat{e}_3\rangle \quad \dots$$

2. $D(G): D(g_i)|g_k\rangle \stackrel{\text{def}}{=} |g_i \circ g_k\rangle \quad (D(g_k)_{ij} = \langle g_i|D(g_k)|g_j\rangle)$

Examples:

1) $G = \{e, g_1\}$

| | | |
|-------|-------|-------|
| | e | g_1 |
| e | e | g_1 |
| g_1 | g_1 | e |

$$D(e)_{11} = \langle e|D(e)|e\rangle = \langle e|e\rangle = 1$$

$$D(e)_{12} = \langle e|D(e)|g_1\rangle = \langle e|g_1\rangle = 0$$

$$D(e)_{21} = \langle g_1|D(e)|e\rangle = \langle g_1|e\rangle = 0$$

$$D(e)_{22} = \langle g_1|D(e)|g_1\rangle = \langle g_1|g_1\rangle = 1$$

$$D(g_1)_{11} = \langle e|D(g_1)|e\rangle = \langle e|g_1\rangle = 0$$

$$D(g_1)_{12} = \langle e|D(g_1)|g_1\rangle = \langle e|e\rangle = 1$$

$$D(g_1)_{21} = \langle g_1|D(g_1)|e\rangle = \langle g_1|e\rangle = 1$$

$$D(g_1)_{22} = \langle g_1|D(g_1)|g_1\rangle = \langle g_1|g_1\rangle = 0$$

$$\Rightarrow D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow D(g_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then:

$$D(e) \cdot D(e) = D(e)$$

$$D(e) \cdot D(g_1) = D(g_1)$$

$$D(g_1) \cdot D(e) = D(g_1)$$

$$D(g_1) \cdot D(g_1) = D(e)$$

$$2) S_3: \quad e = (1\ 2\ 3) \quad g_3 = (2\ 1\ 3) \\ g_1 = (2\ 3\ 1) \quad g_4 = (3\ 2\ 1) \\ g_2 = (3\ 1\ 2) \quad g_5 = (1\ 3\ 2)$$

$$1. D(e) = id$$

$$2. D(g_1)_{11} = \langle e | D(g_1) | e \rangle = \langle e | g_1 \rangle = 0$$

$$D(g_1)_{12} = \langle e | D(g_1) | g_1 \rangle = \langle e | g_2 \rangle = 0$$

$$D(g_1)_{13} = \langle e | D(g_1) | g_2 \rangle = \langle e | e \rangle = 1$$

$$D(g_1)_{14} = \langle e | D(g_1) | g_3 \rangle = \langle e | g_5 \rangle = 0$$

$$D(g_1)_{15} = \langle e | D(g_1) | g_4 \rangle = \langle e | g_3 \rangle = 0$$

$$D(g_1)_{16} = \langle e | D(g_1) | g_5 \rangle = \langle e | g_4 \rangle = 0$$

| | e | g_1 | g_2 | g_3 | g_4 | g_5 |
|-------|-------|-------|-------|-------|-------|-------|
| e | e | g_1 | g_2 | g_3 | g_4 | g_5 |
| g_1 | g_1 | g_2 | e | g_5 | g_3 | g_4 |
| g_2 | g_2 | e | g_1 | g_4 | g_5 | g_3 |
| g_3 | g_3 | g_4 | g_5 | e | g_1 | g_2 |
| g_4 | g_4 | g_5 | g_3 | g_2 | e | g_1 |
| g_5 | g_5 | g_3 | g_4 | g_1 | g_2 | e |

$$\dots \Rightarrow D(g_1) = \begin{pmatrix} 0 & 0 & 1 & & & & \\ 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & & & & \\ 0 & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \dots D(S_3)$$

1.3. Reducibility and Irreducibility

Sometimes it's possible to find a repr. that is "smaller" than regular (therefore reducible & irreducible representations)

Regular repr.: $\dim V = n$, $n = \text{ord } G$

Example:

$$S_3: \quad D(g_1)|E\rangle = A_{3 \times 3} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \left| \Rightarrow \dots \right. \quad D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad D(g_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad D(g_2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$D(g_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad D(g_4) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad D(g_5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

3-dim repr. of S_3

2. Subgroups and Cosets.

2.1. Subgroups

$(H, \circ) \subset (G, \circ)$ is a subgroup of G : (H, \circ) is a group

Example:

- 1) $\{e, g_1, g_2\} \cong (\mathbb{Z}_3, +) \subset S_3$
- 2) $\{e, g_3\}, \{e, g_4\}, \{e, g_5\} \cong (\mathbb{Z}_2, +) \subset S_3$
- 3) $\{e\}, S_3 \subset S_3$ (trivial subgroups)

2.2. Cosets.

$gH \stackrel{\text{def}}{=} \{goh \mid h \in H\}$ left coset of H in G
 $Hg \stackrel{\text{def}}{=} \{hog \mid h \in H\}$ right coset of H in G

1. $H \subset G$ is normal: $\forall g \in G \quad gH = Hg$

2. G is abelian $\Rightarrow \forall H \subset G$ is normal

Example:

$$1) G = S_3$$

$$H = \{e, g_1, g_2\}$$

$$G_{012}$$

G_{012} is a normal subgroup of S_3

$$e G_{012} = \{e, g_1, g_2\}$$

$$g_1 G_{012} = \{g_1, g_2, e\}$$

$$g_2 G_{012} = \{g_2, e, g_1\}$$

$$g_3 G_{012} = \{g_3, g_4, g_5\}$$

$$g_4 G_{012} = \{g_4, g_5, g_3\}$$

$$g_5 G_{012} = \{g_5, g_3, g_4\}$$

$$G_{012} e = G_{012}$$

$$G_{012} g_1 = G_{012}$$

$$G_{012} g_2 = G_{012}$$

$$G_{012} g_3 = G_{345}$$

$$G_{012} g_4 = G_{345}$$

$$G_{012} g_5 = G_{345}$$

$$2) G = (\mathbb{Z}, +) \quad H = (2\mathbb{Z}, +)$$

H is a normal subgroup of G because G is abelian.

3. Factor groups.

$$g_1 = g_2 \text{ mod } H \subset G : g_2 \in g_1 H$$

$$1. g = g \text{ mod } H$$

$$2. g_1 = g_2 \text{ mod } H \Rightarrow g_2 = g_1 \text{ mod } H$$

$$3. g_1 = g_2 \text{ mod } H \quad g_2 = g_3 \text{ mod } H \Rightarrow g_1 = g_3 \text{ mod } H$$

"G mod H"

$G/H \stackrel{\text{def}}{=} \{gH \mid g \in G\}$

factor set of H in G

Th: $H \subset G$ is normal $\Leftrightarrow (G/H, \circ)$ is a group :
 factor group of $H \subset G$

1. assoc. is inherited
2. $g_1 H \circ g_2 H = (g_1 \circ g_2) H$
3. eH - identity
4. $(gH)^{-1} = g^{-1}H$

$$G \times \tilde{G} \stackrel{\text{def}}{=} \left(\{(g_i, \tilde{g}_k) \mid g_i \in G, \tilde{g}_k \in \tilde{G}\}, \circ \right), \text{ where } (g_i, \tilde{g}_k) \circ (g_j, \tilde{g}_l) \stackrel{\text{def}}{=} (g_i \circ g_j, \tilde{g}_k \circ \tilde{g}_l)$$

product group of G and \tilde{G}

Examples:

$$1) G = S_3 \quad H = G_{012} \quad S_3 / G_{012} = \{ \{e, g_1, g_2\}, \{g_3, g_4, g_5\} \} = \{ \tilde{E}, \tilde{G} \}$$

G_{012} is normal, therefore S_3 / G_{012} is a group:

$$\begin{array}{l} \tilde{E} \circ \tilde{G} = e \tilde{G} = \{g_3, g_4, g_5\} \\ \tilde{g}_1 \tilde{G} = \{g_5, g_3, g_4\} \\ \tilde{g}_2 \tilde{G} = \{g_4, g_5, g_3\} \\ \dots \\ \tilde{E} \circ \tilde{E} = \tilde{E} \\ \tilde{G} \circ \tilde{G} = \tilde{E} \end{array} \quad \Rightarrow \quad \begin{array}{|c|c|c|} \hline & \tilde{E} & \tilde{G} \\ \hline \tilde{E} & \tilde{E} & \tilde{G} \\ \hline \tilde{G} & \tilde{G} & \tilde{E} \\ \hline \end{array} \quad , \text{ i.e. } S_3 / G_{012} \cong (\mathbb{Z}_2, +)$$

(compare with S_3 mult. table)

$$2) \mathbb{Z} / 2\mathbb{Z} = \{0 + 2\mathbb{Z}, \pm 1 + 2\mathbb{Z}, \dots\} = \left[\begin{array}{l} 2k + 2\mathbb{Z} = \mathbb{Z}_{\text{even}} + k \\ (2k+1) + 2\mathbb{Z} = \mathbb{Z}_{\text{odd}} + k \end{array} \right] = \{\mathbb{Z}_{\text{even}}, \mathbb{Z}_{\text{odd}}\} \cong (\mathbb{Z}_2, +)$$

$$\mathbb{Z} / n\mathbb{Z} = \mathbb{Z}_n$$

$$3) G / G = \{eG, g_1 G, \dots\} = \{G, G, \dots\} = \{G\} \cong \{e\} \text{ (trivial group)} \quad , \text{ i.e. } G / G = e$$

$$G / e = \{e\{e\}, g_1 \{e\}, \dots\} = \{\{e\}, \{g_1\}, \dots\} \cong G \quad , \text{ i.e. } G / e = G$$

4. Reducibility

4.1. Algebras and Modules

$$\mathbb{R}[G] \stackrel{\text{def}}{=} \left\{ \sum_{i=0}^{n-1} a_i |g_i\rangle \mid a_i \in \mathbb{R}, g_i \in G \right\}, \text{ i.e. vector space, spanned by the elements of } G ;$$

each point corresponds to some lin. combination of $\{|g_i\rangle\}$

Algebra of a group G

$$D(g_i) \sum_{j=0}^{n-1} a_j |g_j\rangle = \sum_{j=0}^{n-1} a_j |g_i \circ g_j\rangle$$

Let $M = \{m_0, m_1, \dots, m_{q-1}\}$ be a set on which group G can act on. Next we will define a correspondence between M and an orthonormal basis in some q -dimensional vector space:
 (like with G)

$$m_0 \rightarrow |m_0\rangle \quad m_1 \rightarrow |m_1\rangle \dots \quad m_{q-1} \rightarrow |m_{q-1}\rangle$$

$$\mathbb{R}M \stackrel{\text{def}}{=} \left\{ \sum_{i=0}^{q-1} a_i |m_i\rangle \mid a_i \in \mathbb{R}, m_i \in M \right\}, \text{ i.e. vector space that repr. of } G \text{ acts on}$$

Module of a set M

Example:

$$G = S_3 \quad M = \{m_0, m_1, m_2\} \Rightarrow \mathbb{R}N = \mathbb{R}^3$$

$$g_i \sum_{j=0}^2 a_j |m_j\rangle = a_0 |g_i m_0\rangle + a_1 |g_i m_1\rangle + a_2 |g_i m_2\rangle \quad i=3: g_3 \sum_{j=0}^2 a_j |m_j\rangle = a_0 |m_1\rangle + a_1 |m_0\rangle + a_2 |m_2\rangle$$

$g_3 = (2 \ 1 \ 3) \quad \begin{pmatrix} \text{i.e. reflection in the } a_0 = a_1 \\ \text{plane in } \mathbb{R}^3, \text{ so } g \leftrightarrow \text{lin. trans} \\ \text{of } \mathbb{R}M \end{pmatrix}$

4.2. Invariant subspaces

$\mathbb{R}W \subset \mathbb{R}V$ is an **invariant subspace**: $\mathbb{R}W$ is closed under the action of G . ($\mathbb{R}V$ and \emptyset are trivial subspaces)

$D(G)$ acting on $\mathbb{R}N$ is **irreducible**: $\mathbb{R}N$ doesn't contain non-trivial inv. subspaces
(reducible otherwise)

Examples:

$$1) G = S_3 \quad M = \{m_0, m_1, m_2\} \Rightarrow \mathbb{R}N = \mathbb{R}^3$$

Then $\mathbb{R}W = \text{span}(|m_0\rangle + |m_1\rangle + |m_2\rangle)$ is an invariant subspace, i.e. $g_i \mathbb{R}W = \mathbb{R}W \quad \forall g_i \in G$
(line in \mathbb{R}^3)

$$2) G = S_3 \quad M = \{m_0, m_1, \dots, m_5\} \Rightarrow \mathbb{R}M = \mathbb{R}^6$$

$\forall g_i \in G \quad D(g_i) = \begin{pmatrix} \equiv & 0 \\ - & - \\ 0 & \equiv \end{pmatrix}$, i.e. the first 3 dimensions never mix with the last 3

\Downarrow
reducible $\Leftrightarrow \mathbb{R}M = \mathbb{R}V_1 \oplus \mathbb{R}V_2$, where $V_1 = \{m_0, m_1, m_2\}$
 $6 \quad 3 \quad 3$ $V_2 = \{m_3, m_4, m_5\}$

More generally, if $V = \bigoplus_{i=1}^k W_i$, then any matrix can be written as $X_n = \bigoplus_{i=1}^k A_{n_i}^{(i)}$, $X_n - n \times n$
($\dim V = n$, $\dim W_i = n_i$) $A_{n_i}^{(i)} - n_i \times n_i$ blocks
(block diagonal)

To find out if a repr. is reducible we can try to find a basis in which $\{D(g_i)\}$ have block diagonal form, i.e. find non-singular $S: D \rightarrow S^{-1}DS$:

$$D'(g_i) \cdot D'(g_j) = S^{-1}D(g_i)S S^{-1}D(g_j)S = S^{-1}D(g_i \circ g_j)S = D'(g_i \circ g_j), \text{ i.e. } D'(G) \text{ is also repr. of } G$$

(Jordan decomposition to find S ?)

Chapter III

Lie groups

1. Overview of groups.

From now on we will work with continuous groups ($g_i \rightarrow g(d_i)$).

$$1.1. GL(n, \mathbb{C}) : \mathbb{C}^n \xrightarrow{\text{repr.}} \mathbb{C}^n \cong \mathbb{R}^n \oplus \mathbb{R}^n \rightarrow \mathbb{R}^n \oplus \mathbb{R}^n$$

\uparrow lin. operators on V^n

, $\det A \neq 0 \Rightarrow$ space = inv.

$$GL(n, \mathbb{R}) : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$1.2. SL(n, \mathbb{C}) : \det A = +1 \Rightarrow \text{space + volume} = \text{inv.}$$

$$1.3. O(n) : A^T = A^{-1} \quad \det A = \pm 1, \quad \langle x, y \rangle = x^T y = x^T A^T A y = \langle x', y' \rangle \Rightarrow \text{space + volume + } \langle \cdot, \cdot \rangle = \text{inv.}$$

$$1.4. U(n) : A^T = A^{-1} \quad \det A = e^{i\theta}, \quad \langle x, y \rangle = x^T y = x^T A^T A y = \langle x', y' \rangle \Rightarrow \text{space + volume + } \langle \cdot, \cdot \rangle = \text{inv.}$$

$$1.5. SO(n) \subset O(n) :$$

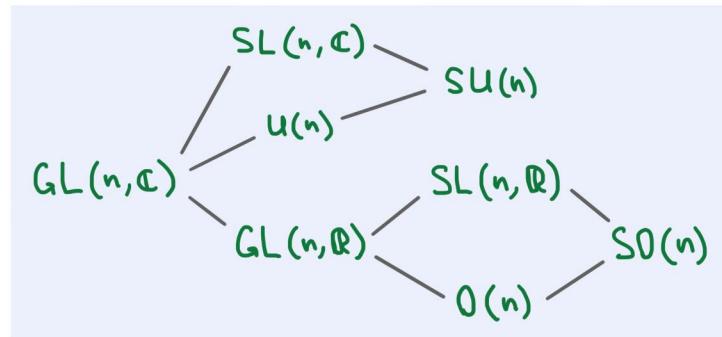
$SU(n) \subset U(n) : \det A = +1$ (i.e. only rotations) \Rightarrow space + volume + $\langle \cdot, \cdot \rangle$ + orientation = inv
(parity)

$$1.6. SO(m, n) \quad (SO(m), SO(n) \subset SO(m, n))$$

$$\langle x, y \rangle = - \sum_{i=1}^m x^i y^i + \sum_{i=m+1}^{m+n} x^i y^i$$

e.g. $SO(1, 3)$ – Lorentz group

($S = \text{inv. under } SO(1, 3)$, i.e. Lorentz invariance)



Examples:

1) $SO(2)$ repr. as rotations of a unit circle : $(\{\theta | \theta \in [0, 2\pi]\}, +)$

2) $SO(2)$ repr. as Euler matrices : $g(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mod 2\pi$

2. Generators.

Lie group is a group that is parameterized by a set of continuous parameters $d_i, i=1, \dots, n$.

$g(d_i)$: element of G

(We will choose parameters s.t. $g(d_i)|_{d_i=0} = e_J$
therefore $D_n(d_i)|_{d_i=0} = I$)

$D_n(g(d_i)) = D_n(d_i)$: representation of $g(d_i)$

1. Expansion for an element near e :

$$D_n(\delta d_i) = D_n(e) + \delta d_i \left. \frac{\partial D_n(d_i)}{\partial d_i} \right|_{d_i=0} + o(\delta d_i) I \approx I + i \delta d_i X_i,$$

2. Then $\forall g(d_i) \in G$:

$$D_n(d_i) = \lim_{N \rightarrow \infty} (D_n(\delta d_i))^N = \lim_{N \rightarrow \infty} (I + i \delta d_i X_i)^N = \lim_{N \rightarrow \infty} \left(I + \frac{i \delta d_i X_i}{N} \right)^N \Rightarrow$$

$$X_i \stackrel{\text{def}}{=} -i \left. \frac{\partial D_n(d_i)}{\partial d_i} \right|_{d_i=0}$$

generators of G

$$D_n(d_i) = \exp(i d_i X_i)$$

repr. of $g(d_i)$

There is an infinite amount of different sets of generators for a particular group, but they all are equivalent (like repr. of G).

dimension of G : number of generators / parameters of G . Ex: $SO(3)$: 3 generators

$$(\theta, \psi, \phi) \rightarrow (X_\theta, X_\psi, X_\phi)$$

3. Lie algebras

For a discrete group algebra was defined as V^n spanned by the elements of the group, but for Lie groups a different approach is needed (because groups are continuous): algebra is spanned by the generators of a Lie group instead.

Consider $g(\alpha_i), g(\beta_j) \in G$, then $g(\alpha_i) \circ g(\beta_j) \in G$; and for representations:

$$\exp(i\alpha_i X_i) \cdot \exp(i\beta_j X_j) = \exp(i\delta_k X_k)$$

$$1. i\delta_k X_k = \ln(\exp(i\delta_k X_k)) = \ln(I + \underbrace{\exp(i\alpha_i X_i) \cdot \exp(i\beta_j X_j) - I}_{x}) = \ln(I + x)$$

$$2. x \stackrel{\text{to 2nd order}}{\approx} (I + i\alpha_i X_i + \frac{1}{2}(i\alpha_i X_i)^2)(I + i\beta_j X_j + \frac{1}{2}(i\beta_j X_j)^2) - I \approx I + i\beta_j X_j - \frac{1}{2}(\beta_j X_j)^2 +$$

$$+ i\alpha_i X_i - \alpha_i X_i \beta_j X_j - \frac{1}{2}(\alpha_i X_i)^2 - I = i(\alpha_i X_i + \beta_j X_j) - \alpha_i X_i \beta_j X_j -$$

$$\boxed{\ln(I+x) = x - \frac{1}{2}x^2 + o(x^2)}$$

$$3. \ln(I+x) \approx x - \frac{1}{2}x^2 \approx i(\alpha_i X_i + \beta_j X_j) - \alpha_i X_i \beta_j X_j - \frac{1}{2}((\alpha_i X_i)^2 + (\beta_j X_j)^2) - \frac{1}{2}(-(\alpha_i X_i + \beta_j X_j) \cdot (\alpha_i X_i + \beta_j X_j)) = \dots + \frac{1}{2}((\alpha_i X_i)^2 + (\beta_j X_j)^2 + \alpha_i \beta_j (X_i X_j + X_j X_i)) = i(\alpha_i X_i + \beta_j X_j) + \frac{1}{2}\alpha_i \beta_j [X_j, X_i] = i(\alpha_i X_i + \beta_j X_j) - \frac{1}{2}[\alpha_i X_i, \beta_j X_j]$$

Then the result is:

$$\exp(i\alpha_i X_i) \cdot \exp(i\beta_j X_j) \approx \exp(i(\alpha_i X_i + \beta_j X_j) - \frac{1}{2}[\alpha_i X_i, \beta_j X_j])$$

Baker-Campbell-Hausdorff formula

$[X_i, X_j]$ is lin. combination of $\{X_k\}$ (because of closure), then

$$[X_i, X_j] = i f_{ijk} X_k, f_{ijk} = -f_{jik}$$

f_{ijk} - structure constants of G

- generators under the specific commutation relations form the Lie Algebra of a group
- structure constants f_{ijk} completely determine the structure of a group

The adjoint representation:

$$[X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0 \quad (\text{Jacobi Identity})$$

$$1. [X_i, [X_j, X_k]] = [X_i, i f_{jka} X_a] = (i f_{jka})(i f_{ial}) X_l = -f_{jka} f_{ial} X_l$$

$$2. f_{jka} f_{ial} X_l + f_{lia} f_{jal} X_l + f_{ija} f_{kal} X_l = 0$$

$$f_{jka} f_{ial} + f_{lia} f_{jal} + f_{ija} f_{kal} = 0$$

$$3. \text{Let } T_{bc}^a \stackrel{\text{def}}{=} -i f_{abc} \text{ (n nxn matrices, n = dim } G), \text{ then:}$$

$$f_{jka} f_{ial} = i^2 T_{ka}^j T_{al}^i = -(T^j \cdot T^i)_{ka}$$

$$f_{lia} f_{jal} = -f_{jka} f_{jal} = -i^2 T_{ia}^l T_{aj}^k = (T^l \cdot T^k)_{ia}$$

$$f_{ija} f_{kal} = -f_{ija} f_{kal} = -i f_{ija} T_{ka}^l$$

$$[T^a, T^b] = i f_{abc} T^c$$

\Rightarrow i.e. structure constants themselves form a repr. of G (adjoint representation)

4. Orthogonal and unitary groups.

4.1. $SO(2)$

Let's find a generator X and $D(G)$ from the property $\langle v, v \rangle = \langle v', v' \rangle$:

$$\theta \rightarrow R(\theta) = \exp(i\theta X) \text{ for some generator } X$$

$$\langle v', v' \rangle = v^T \exp(i\theta X^T) \exp(i\theta X) v \approx v^T (I + i\theta X^T + i\theta X) v = v^T v + v^T i\theta(X + X^T) v$$

↑ 1st order

$$X = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, i \text{ for } X = X^T$$

\Downarrow $X = -X^T$ (antisym.)

generator of $SO(2)$

Then if $g(\theta) \in G$ its representation will be:

$$\exp(i\theta X) = \exp(\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = I + \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{2}\theta^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{3!}\theta^3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \dots =$$

$$= \begin{pmatrix} 1 - \frac{1}{2}\theta^2 + \dots & \theta - \frac{1}{2!}\theta^3 + \dots \\ -(\theta - \frac{1}{3!}\theta^3 + \dots) & 1 - \frac{1}{2}\theta^2 + \dots \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

(i.e. Euler matrices)

$SO(2)$ is abelian and $[X, X] = 0 \Rightarrow f_{ijk} = 0$ (no adjoint repur.)

4.2. $SO(3)$

For $SO(3)$ let's go "backwards" and start with a representation:

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \quad R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix} \quad R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then for generators $X_i = -i \frac{\partial R_n(d_i)}{\partial d_i} \Big|_{d_i=0}$:

$$J_x = -i \frac{\partial R_x(\phi)}{\partial \phi} \Big|_{\phi=0} = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & \cos\phi \\ 0 & -\cos\phi & -\sin\phi \end{pmatrix} \Big|_{\phi=0} = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and similarly for } J_y \text{ and } J_z$$

$$J_x = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad J_y = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad J_z = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

generators of $SO(3)$

$[J_x, J_y] = i J_z$, and $[J_y, J_z] = i J_x$, $[J_z, J_x] = i J_y$ $\Rightarrow [J_i, J_j] = i \epsilon_{ijk} J_k$

\Downarrow

$f_{ijk} = \epsilon_{ijk}$

4.3. $SU(2)$

For $SU(2)$ let's start with $f_{ijk} = \epsilon_{ijk}$ (same as for $SO(3) \cong SU(2)/\mathbb{Z}_2$):

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad U^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \left| \Rightarrow \begin{cases} d = a^* \\ -b = c^* \\ -c = b^* \\ a = d^* \end{cases} \right. \Rightarrow U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \left| \Rightarrow \begin{array}{l} 3 \text{ real parameters} \\ (\text{just like } SO(3)) \\ \text{det } U = |a|^2 + |b|^2 = 1 \end{array} \right. \quad \text{and} \\ \det U = ad - bc \quad U^T = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

5. SU(2) and physical states.

5.1. Raising and Lowering operators

To understand the space Lie group acts on better, we will choose a basis from the eigenvectors of $D(G)$. For $SU(2)$ it's only possible to diagonalize one of the three generators at a time (because of $[,]$ relations): J^1, J^2, J^3 (J^3 is diagonal).

$$j, j' : \text{the biggest and smallest eigenvalues of } J^3 \\ |j; m\rangle : \text{an eigenvector of } J^3 \text{ with eigenvalue } m$$

$$J^3 |j; j\rangle = j |j; j\rangle \quad (\text{let's assume } |j; j\rangle \text{ is known}) \\ J^3 |j; m\rangle = m |j; m\rangle$$

$$J^\pm \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} (J^1 \pm i J^2) \\ \text{raising and lowering operators}$$

$$1. [J^3, J^\pm] = \frac{1}{\sqrt{2}} ([J^3, J^1] \pm i [J^3, J^2]) = \frac{1}{\sqrt{2}} (i \in_{31k} J^k \pm (-i) \in_{32k} J^k) = \\ = \frac{1}{\sqrt{2}} (i \in_{312} J^2 \mp \in_{321} J^1) = \frac{1}{\sqrt{2}} (\pm J^1 + i J^2) = \pm \frac{1}{\sqrt{2}} (J^1 \pm i J^2)$$

$$[J^3, J^\pm] = \pm J^\pm \\ [J^+, J^-] = J^3$$

$$2. [J^+, J^-] = \frac{1}{2} [J^1 + i J^2, J^1 - i J^2] = \frac{1}{2} (-i [J^1, J^2] + i [J^2, J^1]) = \\ = -\frac{1}{2} i ([J^1, J^2] + [J^1, J^2]) = -i [J^1, J^2] = J^3 \\ 3. J^i = (J^i)^\dagger \Rightarrow (J^\pm)^\dagger = J^\mp \quad (\text{i.e. hermitian adjoint operators})$$

5.2. Finding next eigenvector of J^3

$$J^3 |j; m\rangle = m |j; m\rangle$$

$$J^3 J^\pm |j; m\rangle = (\pm J^\pm + J^\pm J^3) |j; m\rangle = (m \pm 1) J^\pm |j; m\rangle$$

So $J^\pm |j; m\rangle$ is the eigenvector of J^3 with eigenvalue $(m \pm 1)$ if $|j; m\rangle$ is also the eigenvector of J^3 .

Hence the names of J^\pm ; also $J^+ |j; j\rangle = 0$ and $J^- |j; j'\rangle = 0$.

Therefore $J^- |j; j\rangle$ is the next eigenvector with eigenvalue $j-1$: $J^- |j; j\rangle = N_j |j; j-1\rangle$

$$1. |J^- |j; j\rangle|^2 = (J^- |j; j\rangle)^\dagger (J^- |j; j\rangle) = \langle j; j | J^+ J^- |j; j\rangle = \langle j; j | J^+ J^- - \underbrace{J^- J^+}_{=0} |j; j\rangle = \\ = \langle j; j | [J^+, J^-] |j; j\rangle = \langle j; j | J^3 |j; j\rangle = j \quad (|j; j\rangle \text{ is normalized})$$

$$2. |J^- |j; j\rangle|^2 = |N_j|^2 \langle j; j-1 | j; j-1 \rangle = |N_j|^2 \quad (|j; j-1\rangle \text{ is also normalized})$$

$$\text{Then } J^- |j; j\rangle = \sqrt{j} |j; j-1\rangle, \text{ or } |j; j-1\rangle = \frac{1}{\sqrt{j}} J^- |j; j\rangle.$$

5.3. Finding remaining eigenvectors of J^3

$$J^- |j; j-1\rangle = N_{j-1} |j; j-2\rangle$$

$$1. |J^- |j; j-1\rangle|^2 = \langle j; j-1 | J^+ J^- |j; j-1\rangle = \langle j; j | \frac{1}{\sqrt{j}} J^+ J^- \frac{1}{\sqrt{j}} J^- |j; j\rangle = \text{see [Rob, p.90]} = 2j-1$$

$$2. |J^- |j; j-1\rangle|^2 = |N_{j-1}|^2$$

$$\text{Then } J^- |j; j-1\rangle = \sqrt{2j-1} |j; j-2\rangle, \text{ or } |j; j-2\rangle = \frac{1}{\sqrt{2j-1}} J^- |j; j-1\rangle$$

We can continue this process and get the general expression for $|j; j-k\rangle$ and N_{j-k} :

$$|j; j-k\rangle = \frac{1}{N_{j-k}} (J^-)^k |j; j\rangle, N_{j-k} = \frac{1}{\sqrt{2}} \sqrt{(2j-k)(k+1)}$$

eigen vectors of J^3

$k=2j$: $N_{j-2j}=0$, therefore
 $|j; -j\rangle$ is the state with
the lowest eigenvalue.

So for a general repr. of $SU(2)$ we have $2j+1$ eigenstates of J^3 with eigenvalues $\{j, j-1, \dots, -j+1, -j\}$.

Therefore $j = \frac{n}{2}, n \in \mathbb{N}$ (i.e. $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$).

$$\langle j; m' | J^3 | j; m \rangle = m \delta_{m'm} \quad \downarrow N_{m-1}$$

$$\langle j; m' | J^+ | j; m \rangle = \frac{1}{\sqrt{2}} \sqrt{(j+m+1)(j-m)} \delta_{m'm+1}$$

$$\langle j; m' | J^- | j; m \rangle = \frac{1}{\sqrt{2}} \sqrt{(j+m)(j-m+1)} \delta_{m'm-1} \quad \uparrow N_m$$

⑥ $SU(2)$

6.1. $SU(2)$ for $j = \frac{1}{2}$

For $j = \frac{1}{2}$ repr. the eigenvalues for J^3 are $\{\frac{1}{2}, -\frac{1}{2}\}$, then $J_{1/2}^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $N_{1/2} = \frac{1}{\sqrt{2}}$.

$$\boxed{J^\pm = \frac{1}{2} (J^+ \pm i J^-) \Rightarrow \begin{aligned} J^1 &= \frac{1}{\sqrt{2}} (J^+ + J^-) \\ J^2 &= \frac{1}{\sqrt{2}} (J^- - J^+) \end{aligned} \quad [J_j^a]_{m'm} = \langle j; m' | J^a | j; m \rangle}$$

Let's find matrix elements of $J_{1/2}^1$ and $J_{1/2}^2$:

$$[J_{1/2}^1]_{11} = \langle \frac{1}{2}; -\frac{1}{2} | J_{1/2}^1 | \frac{1}{2}; -\frac{1}{2} \rangle = \langle \frac{1}{2}; -\frac{1}{2} | \frac{1}{\sqrt{2}} (J^+ + J^-) | \frac{1}{2}; -\frac{1}{2} \rangle = \frac{1}{\sqrt{2}} \langle \frac{1}{2}; -\frac{1}{2} | J^+ | \frac{1}{2}; -\frac{1}{2} \rangle = 0$$

$$[J_{1/2}^1]_{12} = \langle \frac{1}{2}; -\frac{1}{2} | J_{1/2}^1 | \frac{1}{2}; \frac{1}{2} \rangle = \langle \frac{1}{2}; -\frac{1}{2} | \frac{1}{\sqrt{2}} J^+ | \frac{1}{2}; \frac{1}{2} \rangle = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$[J_{1/2}^1]_{21} = \langle \frac{1}{2}; \frac{1}{2} | J_{1/2}^1 | \frac{1}{2}; -\frac{1}{2} \rangle = \langle \frac{1}{2}; \frac{1}{2} | \frac{1}{\sqrt{2}} J^+ | \frac{1}{2}; -\frac{1}{2} \rangle = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$[J_{1/2}^1]_{22} = \langle \frac{1}{2}; \frac{1}{2} | J_{1/2}^1 | \frac{1}{2}; \frac{1}{2} \rangle = \langle \frac{1}{2}; \frac{1}{2} | \frac{1}{\sqrt{2}} J^+ | \frac{1}{2}; \frac{1}{2} \rangle = 0$$

... (similarly for $J_{1/2}^2$)

$$J_{1/2}^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\sigma^1}{2} \quad J_{1/2}^2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\sigma^2}{2} \quad J_{1/2}^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\sigma^3}{2} \quad , \sigma^1, \sigma^2, \sigma^3: \text{Pauli spin matrices}$$

generators for $j = \frac{1}{2}$ repr. of $SU(2)$

$SU(2)$ will describe quantum spin for particles just like $SO(3)$ describes angular momentum.
(rotations in complex spinor space) vs. (rotations in \mathbb{R}^3)

6.2. $SU(2)$ for $j = 1$

For $j = 1$ the eigenvalues of J^3 are $\{1, 0, -1\}$, then $J_1^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and

...

$$N_1 = \sqrt{1} = 1 \\ N_0 = \sqrt{2-1} = 1$$

$$\boxed{J_1^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_1^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_1^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

generators for $j = 1$ repr. of $SU(2)$

6.3. $SU(2)$ for arbitrary j

So $SU(2)$ has 3 generators (J_j^1, J_j^2, J_j^3) for any $(2j+1)$ -dimensional representation and these representations are irreducible (for each j). Then eigenbasis (for J_j^3) is:

$$|j; j\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad |j; j-1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \dots \quad |j; -j\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad J_j^\pm \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} (J^+ \pm i J^-)$$

Example:

$$j = \frac{1}{2} : J_{\frac{1}{2}}^+ = \frac{1}{\sqrt{2}} \left(\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2} i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right) = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$J_{\frac{1}{2}}^- = \frac{1}{\sqrt{2}} \left(\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{1}{2} i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right) = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$J_{\frac{1}{2}}^+ | \frac{1}{2}; -\frac{1}{2} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = N_{\frac{1}{2}} | \frac{1}{2}; \frac{1}{2} \rangle$$

$$J_{\frac{1}{2}}^- | \frac{1}{2}; \frac{1}{2} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = N_{\frac{1}{2}} | \frac{1}{2}; -\frac{1}{2} \rangle$$

7. Root space.

7.1. Cartan subalgebra

- For $SU(2)$ we chose a basis in the space group is acting on from the eigenvectors of J_j^3 ; and linear combinations of non-diagonal J_j^1 and J_j^2 formed J_j^\pm , which transform eigenvectors to each other (changing eigenvalues by an amount defined by f_{ijk} , i.e. the commutation relations of the generators). Let's generalize this approach for an arbitrary Lie group.
- Generators span vector space with an inner product: $\langle T^i | T^j \rangle \stackrel{\text{def}}{=} \frac{1}{2\pi} \text{tr}(T^i T^j) = \delta^{ij}$. normalization constant
(i.e. $\{T^i\}$ orthogonal to each other)
- In the set $\{T^i\}$ there will be closed subalgebra (**Cartan subalgebra**) of (Cartan generators), which all commute with each other (and therefore can be simultaneously diagonalized).

$H^i, i=1, \dots, M$: Cartan generators

Ex: i) $SU(2)$ has 1 Cartan generator $\Rightarrow \text{rank } G = 1$

$E^i, i=1, \dots, N-M$: non-Cartan generators

$H^i = J_j^3 \quad E^i = J_j^1 \quad E^2 = J_j^2$

rank $G \leq M$, i.e. the number of Cartan generators 2) choosing x or p in QN because $[x, p] \neq 0$

We will choose a basis from the eigenvectors of $\{H^i\}$ s.t. each of them will have one eigenvalue with each of the M Cartan generators. So for every eigenvector $|l_j\rangle$ ($j=1, \dots, n$) we can get an M -dim. **weight vector** of its eigenvalues at $\{H^i\}$:

$$t_j \stackrel{\text{def}}{=} \begin{pmatrix} t_j^1 \\ t_j^2 \\ \vdots \\ t_j^M \end{pmatrix} \underset{1:1}{\leftrightarrow} |l_j\rangle$$

Ex.: $SU(2)$: 3 $(2j+1) \times (2j+1)$ operators acting on V^d , $d = 2j+1$
 $\Rightarrow d = 2j+1$ weight vectors ($N=1$: so they are scalars)
1) $j = \frac{1}{2}$: $t_1 = (\frac{1}{2}) \quad t_2 = (-\frac{1}{2})$
2) $j = 1$: $t_1 = (1) \quad t_2 = (0) \quad t_3 = (-1)$

eigenvalues
"Weights"
↓

Number n of $|D_n; t_j\rangle$ is determined by $\dim D_n(G)$ of a concrete repr.

Dimension M of each t_j is determined by the number of $\{H^i\}$, i.e. $\text{rank } G$.

$$H^i |D_n; t_j\rangle = t_j^i |D_n; t_j\rangle$$

7.2. Adjoint representation

We will use the adjoint repr. ($[T^a]_{bc} = -i f_{abc}$ $N \times N$ matrices, and root vectors \leftrightarrow weight vectors). (e.g. $j=1$ 3x3 repr. for $SU(2)$ is adjoint).

| | | | | |
|------------------------------------|---------------------|----------------------------------|-------------------|----------------------------------|
| N generators ($N \times N$) | \leftrightarrow | N eigenvectors (N -dim.) | \leftrightarrow | N root vectors (N -dim.) |
| $T^i \equiv T^{t_j}$ | $1:1$ | $ \text{Adj}; T^i\rangle$ | $1:1$ | t_j |
| H^{h_j} | $\swarrow \searrow$ | | | $h_j \quad e_j$ |
| E^{e_i} | | | | |

$$|\text{Adj}; H^{h_j}\rangle : j=1 \dots M \quad \text{Cartan}$$

$$|\text{Adj}; E^{e_i}\rangle : i=1 \dots N-M \quad \text{non-Cartan}$$

$$\langle \text{Adj}; T^{t_j} | \text{Adj}; T^{t_k} \rangle = \delta^{jk}$$

$$I = \sum_i |\text{Adj}_i; T^{t_i}\rangle \langle \text{Adj}_i; T^{t_i}| \quad -i f_{\text{alc}} = [T^{t_a}]_{bc} = \langle \text{Adj}_b; T^{t_b} | T^{t_a} | \text{Adj}_c; T^{t_c} \rangle$$

$$\begin{aligned} T^{t_a} |\text{Adj}_j; T^{t_b}\rangle &= \sum_c |\text{Adj}_j; T^{t_c}\rangle \langle \text{Adj}_j; T^{t_c}| T^{t_a} | \text{Adj}_j; T^{t_b}\rangle = \sum_c |\text{Adj}_j; T^{t_c}\rangle [T^a]_{cb} = \sum_c |\text{Adj}_j; T^{t_c}\rangle (-i f_{\text{alc}}) = \\ &= \sum_c i f_{\text{alc}} |\text{Adj}_j; T^{t_c}\rangle = |\text{Adj}_j; [T^{t_a}, T^{t_b}]\rangle \end{aligned}$$

$$T^{t_a} |\text{Adj}_j; T^{t_b}\rangle = |\text{Adj}_j; [T^{t_a}, T^{t_b}]\rangle$$

1. Cartan generators:

$$\begin{aligned} 1) H^{ha} |\text{Adj}_j; H^{hb}\rangle &= h_a^b |\text{Adj}_j; H^{hb}\rangle \\ H^{ha} |\text{Adj}_j; H^{hb}\rangle &= |\text{Adj}_j; [H^{ha}, H^{hb}]\rangle = 0 \quad \forall a, b \quad \left| \Rightarrow h_b = 0 \quad \forall b = 1, \dots, M \quad (H^{hj} \rightarrow H^j \text{ change of notation}) \right. \end{aligned}$$

$$\begin{aligned} 2) H^a |\text{Adj}_j; E^{ea}\rangle &= e_b^a |\text{Adj}_j; E^{ea}\rangle \\ H^a |\text{Adj}_j; E^{ea}\rangle &= |\text{Adj}_j; [H^a, E^{ea}]\rangle \quad \left| \Rightarrow [H^a, E^{ea}] = e_b^a E^{ea} \right. \end{aligned}$$

2. non-Cartan generators:

Consider a state $|\text{Adj}_j; T^{t_b}\rangle$ with H^c eigenvalue t_b^c . Let's act on it with E^{ea} and H^c :

$$\begin{aligned} H^c E^{ea} |\text{Adj}_j; T^{t_b}\rangle &= (H^c E^{ea} - E^{ea} H^c + E^{ea} H^c) |\text{Adj}_j; T^{t_b}\rangle = ([H^c, E^{ea}] + E^{ea} H^c) |\text{Adj}_j; T^{t_b}\rangle = \\ &= (e_a^c E^{ea} + E^{ea} t_b^c) |\text{Adj}_j; T^{t_b}\rangle = (t_b + e_a)^c E^{ea} |\text{Adj}_j; T^{t_b}\rangle \end{aligned}$$

So by acting on the eigenstate with non-Cartan E^{ea} we have shifted the H^c eigenvalue by one of the coordinates of the root vector (like raising and lowering operators in $\text{SU}(2)$).

Also $\nexists E^{-ea}$ should $\exists E^{-ea}$ which transforms the eigenvalue "back":

$$H^b E^{ea} |\text{Adj}_j; E^{-ea}\rangle = (-e_a + e_a)^b E^{ea} |\text{Adj}_j; E^{-ea}\rangle = 0$$

But only the states corresponding to Cartan generators have 0 eigenvalues $\Rightarrow E^{ea} |\text{Adj}_j; E^{-ea}\rangle = \sum_b N_a^b |\text{Adj}_j; H^b\rangle$

$$1) \langle \text{Adj}_j; H^c | E^{ea} |\text{Adj}_j; E^{-ea}\rangle = \sum_b N_a^b \langle \text{Adj}_j; H^c | \text{Adj}_j; H^b \rangle = \sum_b N_a^b \delta^{cb} = N_a^c$$

$$\begin{aligned} 2) \langle \text{Adj}_j; H^c | E^{ea} |\text{Adj}_j; E^{-ea}\rangle &= \langle \text{Adj}_j; H^c | \text{Adj}_j; [E^{ea}, E^{-ea}]\rangle = \frac{1}{2} \text{tr} (H^c [E^{ea}, E^{-ea}]) = \\ &= \frac{1}{2} \text{tr} (E^{-ea} [H^c, E^{ea}]) = \frac{1}{2} e_a^c \text{tr} (E^{-ea} E^{ea}) = e_a^c \delta^{aa} = e_a^c \end{aligned}$$

Therefore:

($\text{tr} = \text{inv under cyclic permutations}$)

$$E^{ea} |\text{Adj}_j; E^{-ea}\rangle = \sum_b e_a^b |\text{Adj}_j; H^b\rangle \quad \left| \Rightarrow [E^{ea}, E^{-ea}] = \sum_b e_a^b H^b \right.$$

$$E^{ea} |\text{Adj}_j; E^{-ea}\rangle = |\text{Adj}_j; [E^{ea}, E^{-ea}]\rangle \quad \left| \Rightarrow [E^{ea}, E^{-ea}] = \sum_b e_a^b H^b \right.$$

These results hold for any representation (but the adj. repr made them easy to obtain).

8. Adjoint representation of SU(2)

Adjoint repr. of $SU(2)$ is $j=1$ repr. (of 3×3 operators acting on V^3). Since $\dim SU(2) = 3$ and rank $SU(2) = 1$ there are 3 generators, one of which is Cartan generator:

$$\begin{aligned} E^1 &= J_1^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} & v_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & t_1 &= (+1) \\ H^1 &= J_1^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & v_2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & t_2 &= (0) \\ E^2 &= J_1^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} & v_3 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & t_3 &= (-1) \end{aligned}$$

↑ +1
• 0
↓ -1
root space

$[H^a, E^{e_b}] = e_b^a E^{e_b}$
 $[E^{e_a}, E^{-e_b}] = \sum_b e_b^b H^b$
 $\langle T^i, T^j \rangle = \frac{1}{2} \text{tr}(T^i T^j) = \delta^{ij}$
 (i.e. $\infty = 2$ for $j=1$ $SU(2)$ repr.)

8.1. Finding non-Cartan generators

$$\begin{aligned} [H^1, E^1] &= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} i E^2 & \Rightarrow E^1 \text{ and } E^2 \text{ don't satisfy the condition for non-Cartan generators,} \\ & \text{so let's take their linear combination instead:} \\ [H^1, E^2] &= -\frac{i}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = -\frac{i}{\sqrt{2}} E^1 & [H^1, \alpha E^1 \pm \beta i E^2] &= \frac{1}{\sqrt{2}} (\alpha i E^2 \pm \beta E^1), \text{ let's try } \alpha = \beta \\ \frac{\alpha}{\sqrt{2}} (E^1 + i E^2) &= \alpha \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & [H^1, \frac{\alpha}{\sqrt{2}} (E^1 + i E^2)] &= \alpha \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = t_1 \cdot \alpha \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & \Rightarrow \text{Satisfy condition} \\ \frac{\alpha}{\sqrt{2}} (E^1 - i E^2) &= \alpha \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & [H^1, \frac{\alpha}{\sqrt{2}} (E^1 - i E^2)] &= -\alpha \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = t_3 \cdot \alpha \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

Let's act with these operators on eigenbasis:

$$\begin{aligned} H^1 v_1 &= (+1) v_1 & \frac{\alpha}{\sqrt{2}} (E^1 + i E^2) v_1 &= 0 & \frac{\alpha}{\sqrt{2}} (E^1 - i E^2) v_1 &= \alpha v_2 & E^{+1} &= \frac{\alpha}{\sqrt{2}} (E^1 + i E^2) \\ H^1 v_2 &= (0) v_2 & \frac{\alpha}{\sqrt{2}} (E^1 + i E^2) v_2 &= \alpha v_1 & \frac{\alpha}{\sqrt{2}} (E^1 - i E^2) v_2 &= \alpha v_3 & , E^{-1} &= \frac{\alpha}{\sqrt{2}} (E^1 - i E^2) \\ H^1 v_3 &= (-1) v_3 & \frac{\alpha}{\sqrt{2}} (E^1 + i E^2) v_3 &= \alpha v_2 & \frac{\alpha}{\sqrt{2}} (E^1 - i E^2) v_3 &= 0 & \text{i.e. act like lowering} \\ &&&&&&&\text{and raising operators} \end{aligned}$$

8.2. Finding correspondence

So E^{+1} and E^{-1} correspond to t_1 and t_3 , therefore H^1 corresponds to $t_2 = (0)$ (as it should be since it's Cartan generator):

$$\begin{aligned} E^{+1} &\leftrightarrow t_1 = (+1) \leftrightarrow v_1 = |\text{Adj}_j; E^{+1}\rangle \\ H^1 &\leftrightarrow t_2 = (0) \leftrightarrow v_2 = |\text{Adj}_j; H^1\rangle \\ E^{-1} &\leftrightarrow t_3 = (-1) \leftrightarrow v_3 = |\text{Adj}_j; E^{-1}\rangle \end{aligned}$$

Finally let's find α :

$$\begin{aligned} [E^{+1}, E^{-1}] &= \alpha^2 H^1 & \Rightarrow E^{+1} &= \frac{1}{\sqrt{2}} (E^1 + i E^2) \\ [E^{+1}, E^{-1}] &= t_1 H^1 & \Rightarrow E^{-1} &= \frac{1}{\sqrt{2}} (E^1 - i E^2) \end{aligned}$$

raising and lowering operators for $SU(2)$

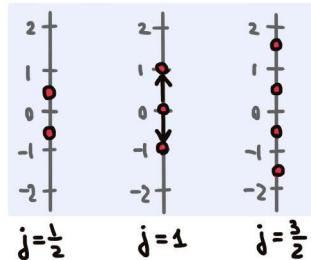
Physics perspective:

Eigenstates are particles with some charge (different values of charge = different points in root space) and generators are "tools" to change one state to another (i.e. things that describe how these particles interact).

8.3. Arbitrary representations

For any $SU(2)$ repr. there always will be 3 generators H^i, E^+, E^- (and they always act the same way: as raising and lowering operators, i.e. move \uparrow or \downarrow one position in the root space). Weight vectors are always 1-dimensional, but their number (as the number of eigenvectors) varies with the dimension of a representation: $n = 2j+1$.

$$\begin{aligned} j = \frac{1}{2} : n &= 2 \\ j = 1 : n &= 3 \\ j = \frac{3}{2} : n &= 4 \end{aligned}$$



So for any repr. of $SU(2)$ there are only 3 options: stay at the state, move up or move down.

⑨ $SU(3)$

We will study $SU(3)$ not in the adjoint repr., but in the fundamental repr. (3x3 operators on V^3). (for convenience)

$$\boxed{\begin{array}{l} \dim SU(n) = n^2 - 1 \\ \text{rank } SU(n) = n - 1 \Rightarrow \begin{array}{l} \text{Adj. repr.: } (n^2-1) \times (n^2-1) \\ \text{Fund. repr.: } n \times n \end{array} \\ \dim SO(n) = \frac{n(n-1)}{2} \\ (\text{because } A = -A^T) \end{array}} \quad \boxed{\begin{array}{l} \dim SU(3) = 8 \\ \text{rank } SU(3) = 2 \end{array}}$$

$$f_{123} = 1 \quad f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2} \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}$$

$$T^a = \frac{1}{2} \lambda^a, \quad \lambda^a - \text{Gell-Mann matrices} \quad a = 1, \dots, 8$$

SU(3) generators in fund. repr.

$$\begin{array}{llll} \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} & \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \end{array}$$

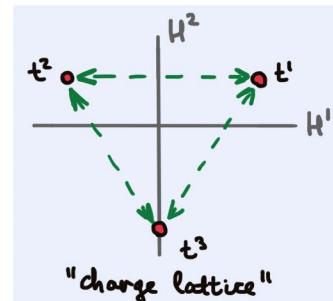
Cartan
(because diagonal)

non-Cartan

So generators are:

$$\begin{array}{ll} H^1 = T^3 & H^2 = T^8 \\ E^1 = T^1 & E^2 = T^2 \\ E^3 = T^4 & E^4 = T^5 \\ E^5 = T^6 & E^6 = T^7 \end{array}$$

$$\Rightarrow \begin{array}{l} t_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix} \quad t_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix} \quad t_3 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \\ \text{weight vectors} \end{array}$$



To understand what non-Cartan generators do one can use the general method (as for $SU(2)$), but we can take a shortcut. We are looking for raising and lowering operators, and $\{E^i\}$ doesn't fit for that, so we will try their lin. combinations (easy to see which one we need):

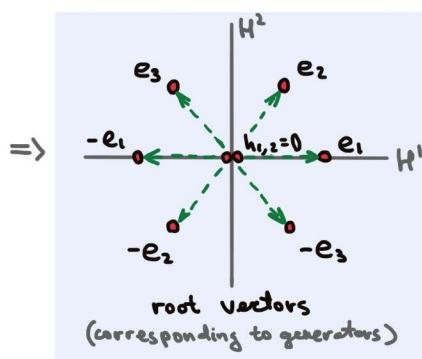
$$\begin{array}{ll} \text{A}^N: \quad \frac{1}{\sqrt{2}} (E^1 + iE^2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & V_2 \rightarrow V_1 \quad A^\dagger V_1 = 0 \quad A^\dagger V_2 = \frac{1}{\sqrt{2}} V_1 \quad A^\dagger V_3 = 0 \\ \frac{1}{\sqrt{2}} (E^1 - iE^2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & V_1 \rightarrow V_2 \quad A^\dagger V_1 = \frac{1}{\sqrt{2}} V_2 \quad A^\dagger V_2 = 0 \quad A^\dagger V_3 = 0 \end{array} \quad \left. \right\} \text{moves } \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{array}{ll} \text{B}^N: \quad \frac{1}{\sqrt{2}} (E^3 + iE^4) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & V_3 \rightarrow V_1 \quad B^\dagger V_1 = 0 \quad B^\dagger V_2 = 0 \quad B^\dagger V_3 = \frac{1}{\sqrt{2}} V_1 \\ \frac{1}{\sqrt{2}} (E^3 - iE^4) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & V_1 \rightarrow V_3 \quad B^\dagger V_1 = \frac{1}{\sqrt{2}} V_3 \quad B^\dagger V_2 = 0 \quad B^\dagger V_3 = 0 \end{array} \quad \left. \right\} \text{moves } \pm \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\begin{array}{ll} \text{C}^N: \quad \frac{1}{\sqrt{2}} (E^5 + iE^6) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & V_3 \rightarrow V_2 \quad C^\dagger V_1 = 0 \quad C^\dagger V_2 = 0 \quad C^\dagger V_3 = \frac{1}{\sqrt{2}} V_2 \\ \frac{1}{\sqrt{2}} (E^5 - iE^6) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & V_2 \rightarrow V_3 \quad C^\dagger V_1 = 0 \quad C^\dagger V_2 = \frac{1}{\sqrt{2}} V_3 \quad C^\dagger V_3 = 0 \end{array} \quad \left. \right\} \text{moves } \pm \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\begin{aligned} E^{\pm\left(\frac{1}{2}\right)} &= E^{\pm e_1} = \frac{1}{\sqrt{2}} (E^1 \pm i E^2) \\ E^{\pm\left(\frac{11}{2}\right)} &= E^{\pm e_2} = \frac{1}{\sqrt{2}} (E^3 \pm i E^4) \\ E^{\pm\left(\frac{-11}{2}\right)} &= E^{\pm e_3} = \frac{1}{\sqrt{2}} (E^5 \pm i E^6) \end{aligned}$$

$SU(3)$ non-Cartan generators



The root space diagram represents the way the state vectors in any repr. transform from one to another.

Because we didn't use the adjoint repr. there isn't 1:1 correspondence between the eigenstates, weight vectors and generators. That's why we have 2 diagrams: one represents the weight vectors, and another the root vectors (and the generators). In the adjoint repr. though it's the same space (e.g. e_2 will move $-e_1 \rightarrow e_3$ and $-e_3 \rightarrow e_1$).

Physics perspective:

Phys. interactions are described by some Lie group in a particular representation:

- particles are eigenstates of the Cartan generators of the group
- charges are eigenvalues of those eigenstates, i.e. weight vectors (N of charges = $\dim(\text{fund. } D(G))$)
- force-carrying particles are generators, i.e. describe how particles change:
 - 1) Cartan generators are force-carrying particles that interact with any particle charged under the same group by transforming energy and momentum, but not charge (δ and Z^0).
 - 2) non-Cartan generators are force-carrying particles that interact with any particle charged under the same group by transforming energy and momentum, and also charge (W^\pm, g_i).

Chapter IV

The Lorentz group

① The Lorentz Algebra.

$\Lambda \subset R$: rotations in \mathbb{R}^3

B : boosts (rotations in \mathbb{R}^4)

$$[R_x(\theta_x)]_v^M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_x & \sin\theta_x \\ 0 & 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} \quad [R_y(\theta_y)]_v^M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \quad [R_z(\theta_z)]_v^M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & \sin\theta_z & 0 \\ 0 & -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[B_x(\phi_x)]_v^M = \begin{pmatrix} \cosh\phi_x & -\sinh\phi_x & 0 & 0 \\ -\sinh\phi_x & \cosh\phi_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad [B_y(\theta_y)]_v^M = \begin{pmatrix} \cosh\phi_y & 0 & -\sinh\phi_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\phi_y & 0 & \cosh\phi_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad [B_z(\theta_z)]_v^M = \begin{pmatrix} \cosh\phi_z & 0 & 0 & -\sinh\phi_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi_z & 0 & 0 & \cosh\phi_z \end{pmatrix}$$

Then we can find generators for this repr.: $[J_i]_v^M = \left[-i \frac{dR_i(\theta_i)}{d\theta_i} \Big|_{\theta_i=0} \right]_v^M \quad [K_i]_v^M = \left[-i \frac{dB_i(\phi_i)}{d\phi_i} \Big|_{\phi_i=0} \right]_v^M$

$$\begin{array}{lll} [J_x]_v^M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} & [J_y]_v^M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} & [J_z]_v^M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ [K_x]_v^M = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & [K_y]_v^M = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & [K_z]_v^M = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\Rightarrow \begin{array}{l} [J_i, J_j] = i \epsilon_{ijk} J_k \\ [J_i, K_j] = i \epsilon_{ijk} K_k \\ [K_i, K_j] = -i \epsilon_{ijk} J_k \end{array}$$

interesting fact

Then the general Lorentz transformation: $L = \exp(i \vec{J} \cdot \vec{\theta} + i \vec{K} \cdot \vec{\phi})$, $\vec{J} = (J_x, J_y, J_z)$, $\vec{\theta} = (\theta_x, \theta_y, \theta_z)$, $\vec{K} = (K_x, K_y, K_z)$, $\vec{\phi} = (\phi_x, \phi_y, \phi_z)$

We can write down generators with lower indices:

$$\begin{array}{ll} [J_x]_{\mu\nu} = [J_x]_v^M & [K_x]_{\mu\nu} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ [J_y]_{\mu\nu} = [J_y]_v^M & [K_y]_{\mu\nu} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ [J_z]_{\mu\nu} = [J_z]_v^M & [K_z]_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \end{array}$$

All are hermitian,
but in this form
 $[J_i, K_j] = -i \epsilon_{ijk} K_k$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a & a & a & a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{pmatrix} = \begin{pmatrix} -a & -a & -a & -a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} a & a & a & a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{pmatrix} = \begin{pmatrix} a & a & a & a \\ -a & -a & -a & -a \\ -a & -a & -a & -a \\ -a & -a & -a & -a \end{pmatrix}$$

Structure of $SO(1,3)$:

$$N_i^\pm \stackrel{\text{def}}{=} \frac{1}{2}(J_i \pm i K_i)$$

, then

$$\begin{aligned} [N_i^+, N_j^+] &= i \epsilon_{ijk} N_k^+ \\ [N_i^-, N_j^-] &= i \epsilon_{ijk} N_k^- \\ [N_i^+, N_j^-] &= 0 \end{aligned}$$

So we have 2 copies of $SU(2)$ (for $\{N_i^+\}$ and $\{N_i^-\}$; this is true only for $SO(1,3)$). So representations will be specified by two values (j, j') and will be made up of matrices of $(2j+1)(2j'+1) \times (2j+1)(2j'+1)$ size.

$j \leftrightarrow \{N_i^+\}$ generators

$j' \leftrightarrow \{N_i^-\}$ generators

② Representations of the Lorentz group.

2.1. $(0,0)$ representation

1×1 matrices (i.e. scalars) acting on 1-dim vectors (i.e. scalars) (describes things that don't change)
under Lorentz trs.

2.2. $(\frac{1}{2}, 0)$ representation

Only N_i^+ are used: $N_i^- = \frac{1}{2}(J_i - iK_i) = 0 \Rightarrow J_i = iK_i \Rightarrow N_i^+ = \frac{1}{2}(J_i + iK_i) = iK_i$

$$\begin{aligned} N_i^+ &= iK_i \\ N_i^- &= \frac{1}{2}\sigma_i \end{aligned} \quad \Rightarrow \quad J_i = \frac{1}{2}\sigma_i \quad K_i = -\frac{i}{2}\sigma_i$$

$$\begin{aligned} R(\vec{\theta}) &= \exp\left(\frac{i}{2}\vec{\theta} \cdot \vec{\sigma}\right) \\ R(\vec{\phi}) &= \exp\left(\frac{1}{2}\vec{\phi} \cdot \vec{\sigma}\right) \end{aligned}$$

, $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$
Pauli vector

For example:

$$\begin{aligned} R_x(\theta_x) &= \exp\left(\frac{i}{2}\theta_x \sigma_x\right) = I + \frac{i}{2}\theta_x \sigma_x + \frac{1}{2}\left(\frac{i}{2}\theta_x \sigma_x\right)^2 + \dots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{i}{2}\theta_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{1}{2}\left(\frac{\theta_x}{2}\right)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \dots = \\ &= \begin{pmatrix} 1 - \frac{1}{2}\left(\frac{\theta_x}{2}\right)^2 + \dots & i\frac{\theta_x}{2} + \dots \\ i\frac{\theta_x}{2} + \dots & 1 - \frac{1}{2}\left(\frac{\theta_x}{2}\right)^2 + \dots \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta_x}{2} & i\sin\frac{\theta_x}{2} \\ i\sin\frac{\theta_x}{2} & \cos\frac{\theta_x}{2} \end{pmatrix} \quad \dots \end{aligned}$$

$$\begin{aligned} R_x(\theta_x) &= \begin{pmatrix} \cos\frac{\theta_x}{2} & i\sin\frac{\theta_x}{2} \\ i\sin\frac{\theta_x}{2} & \cos\frac{\theta_x}{2} \end{pmatrix} & R_y(\theta_y) &= \begin{pmatrix} \cos\frac{\theta_y}{2} & \sin\frac{\theta_y}{2} \\ -\sin\frac{\theta_y}{2} & \cos\frac{\theta_y}{2} \end{pmatrix} & R_z(\theta_z) &= \begin{pmatrix} \exp(i\frac{\theta_z}{2}) & 0 \\ 0 & \exp(-i\frac{\theta_z}{2}) \end{pmatrix} \\ B_x(\phi_x) &= \begin{pmatrix} \cosh\frac{\phi_x}{2} & \sinh\frac{\phi_x}{2} \\ \sinh\frac{\phi_x}{2} & \cosh\frac{\phi_x}{2} \end{pmatrix} & B_y(\phi_y) &= \begin{pmatrix} \cosh\frac{\phi_y}{2} & -i\sinh\frac{\phi_y}{2} \\ i\sinh\frac{\phi_y}{2} & \cosh\frac{\phi_y}{2} \end{pmatrix} & B_z(\phi_z) &= \begin{pmatrix} \exp(\frac{\phi_z}{2}) & 0 \\ 0 & \exp(-\frac{\phi_z}{2}) \end{pmatrix} \end{aligned}$$

2.3. $(0, \frac{1}{2})$ representation

Only N_i^- are used: $N_i^+ = \frac{1}{2}(J_i + iK_i) = 0 \Rightarrow J_i = -iK_i \Rightarrow N_i^- = \frac{1}{2}(J_i - iK_i) = -iK_i$

$$\begin{aligned} N_i^- &= -iK_i \\ N_i^+ &= \frac{1}{2}\sigma_i \end{aligned} \quad \Rightarrow \quad J_i = \frac{1}{2}\sigma_i \quad K_i = \frac{i}{2}\sigma_i$$

$$\begin{aligned} R(\vec{\theta}) &= \exp\left(\frac{i}{2}\vec{\theta} \cdot \vec{\sigma}\right) \\ R(\vec{\phi}) &= \exp\left(-\frac{1}{2}\vec{\phi} \cdot \vec{\sigma}\right) \end{aligned}$$

Lorentz tr. for $(0, \frac{1}{2})$

$$\begin{aligned} R_x(\theta_x) &= \begin{pmatrix} \cos\frac{\theta_x}{2} & i\sin\frac{\theta_x}{2} \\ i\sin\frac{\theta_x}{2} & \cos\frac{\theta_x}{2} \end{pmatrix} & R_y(\theta_y) &= \begin{pmatrix} \cos\frac{\theta_y}{2} & \sin\frac{\theta_y}{2} \\ -\sin\frac{\theta_y}{2} & \cos\frac{\theta_y}{2} \end{pmatrix} & R_z(\theta_z) &= \begin{pmatrix} \exp(i\frac{\theta_z}{2}) & 0 \\ 0 & \exp(-i\frac{\theta_z}{2}) \end{pmatrix} \\ B_x(\phi_x) &= \begin{pmatrix} \cosh\frac{\phi_x}{2} & -\sinh\frac{\phi_x}{2} \\ -\sinh\frac{\phi_x}{2} & \cosh\frac{\phi_x}{2} \end{pmatrix} & B_y(\phi_y) &= \begin{pmatrix} \cosh\frac{\phi_y}{2} & i\sinh\frac{\phi_y}{2} \\ -i\sinh\frac{\phi_y}{2} & \cosh\frac{\phi_y}{2} \end{pmatrix} & B_z(\phi_z) &= \begin{pmatrix} \exp(-\frac{\phi_z}{2}) & 0 \\ 0 & \exp(\frac{\phi_z}{2}) \end{pmatrix} \end{aligned}$$

2.4. Properties of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \det \sigma_i = -1 \quad \{1, -1\} : \text{eigenvalues} \\ \text{tr} \sigma_i = 0 \quad \sigma_i^\dagger = \sigma_i \text{ (hermitian)} \quad$$

$$1) \sigma_i^2 = \sigma_j^2 = \sigma_k^2 = -i\sigma_1\sigma_2\sigma_3 = I$$

$$2) \vec{\sigma} \triangleq \sigma_1 \hat{x} + \sigma_2 \hat{y} + \sigma_3 \hat{z} \quad (\text{Pauli vector}) \Rightarrow \vec{a} \cdot \vec{\sigma} = a_i \sigma_i \quad , \quad \vec{a} = (a_1, a_2, a_3)$$

$$3) [\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k \quad \{ \sigma_i, \sigma_j \} = 2\delta_{ij} I \quad \Rightarrow \sigma_i \sigma_j = \delta_{ij} I + i \epsilon_{ijk} \sigma_k \quad \begin{aligned} \sigma_1 \sigma_2 &= i \sigma_3 \\ \sigma_2 \sigma_3 &= i \sigma_1 \\ \sigma_3 \sigma_1 &= -i \sigma_2 \end{aligned}$$

$$4) (\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = a_i \sigma_i b_j \sigma_j = a_i b_j (\delta_{ij} I + i \epsilon_{ijk} \sigma_k) = a_i b_i I + i \sigma_k \epsilon_{ijk} a_i b_j = (\vec{a} \cdot \vec{b}) I + i \vec{a} \cdot (\vec{a} \times \vec{b})$$

$$5) (\hat{n} \cdot \vec{\sigma})^2 = n_1^2 \sigma_1^2 + n_2^2 \sigma_2^2 + n_3^2 \sigma_3^2 + n_1 n_2 (\underbrace{\sigma_1 \sigma_2 + \sigma_2 \sigma_1}_{=0}) + n_2 n_3 (\underbrace{\sigma_2 \sigma_3 + \sigma_3 \sigma_2}_{=0}) + n_1 n_3 (\underbrace{\sigma_1 \sigma_3 + \sigma_3 \sigma_1}_{=0}) = I$$

↑ unit vector ($n_1^2 + n_2^2 + n_3^2 = 1$)

$$6) \exp(i\alpha(\hat{n} \cdot \vec{\sigma})) = I \cos \alpha + i(\hat{n} \cdot \vec{\sigma}) \sin \alpha, \quad \hat{n} \text{-unit vector} \quad \begin{aligned} (\hat{n} \cdot \vec{\sigma})^m &= I \quad \forall m \in \mathbb{N} \\ (\hat{n} \cdot \vec{\sigma})^{m+1} &= \hat{n} \cdot \vec{\sigma} \quad \forall m \in \mathbb{N} \end{aligned}$$

$$*) (i\sigma_2) \vec{\sigma}^* (-i\sigma_2) = \sigma_2 \sigma_1^* \sigma_2 \hat{x} + \sigma_2 \sigma_2^* \sigma_2 \hat{y} + \sigma_2 \sigma_3^* \sigma_2 \hat{z} = \dots = -\vec{\sigma}$$

2.5. Relationship between $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$

$$(\frac{1}{2}, 0): \text{left-handed spinor repr. : } \psi_L \xrightarrow{\text{R}} \bar{\psi}_L = \exp(\frac{i}{2} \vec{\theta} \cdot \vec{\sigma}) \psi_L \quad \psi_L \xrightarrow{\text{B}} \bar{\psi}_L' = \exp(\frac{i}{2} \vec{\phi} \cdot \vec{\sigma}) \psi_L$$

$$(0, \frac{1}{2}): \text{right-handed spinor repr. : } \psi_R \xrightarrow{\text{R}} \bar{\psi}_R = \exp(\frac{i}{2} \vec{\theta} \cdot \vec{\sigma}) \psi_R \quad \psi_R \xrightarrow{\text{B}} \bar{\psi}_R' = \exp(-\frac{i}{2} \vec{\phi} \cdot \vec{\sigma}) \psi_R$$

Let's consider spinors $\bar{\psi}_L \stackrel{\text{def}}{=} i\sigma^2 \psi_L^*$ and $\bar{\psi}_R \stackrel{\text{def}}{=} i\sigma^2 \psi_R^*$ and their change under Lorentz tr.:

$$\text{R: } \bar{\psi}_L \rightarrow \bar{\psi}_L' = i\sigma^2 (\psi_L)^* = i\sigma^2 (\exp(\frac{i}{2} \vec{\theta} \cdot \vec{\sigma}) \psi_L)^* = i\sigma^2 \exp(-\frac{i}{2} \vec{\theta} \cdot \vec{\sigma}^*) \psi_L^* = i\sigma^2 \exp(-\frac{i}{2} \vec{\theta} \cdot \vec{\sigma}^*) (-i\sigma^2) (i\sigma^2) \psi_L^* =$$

$$= \exp(\frac{i}{2} \vec{\theta} \cdot \vec{\sigma}) i\sigma^2 \psi_L^* = \exp(\frac{i}{2} \vec{\theta} \cdot \vec{\sigma}) \bar{\psi}_L$$

$$\bar{\psi}_R \rightarrow \bar{\psi}_R' = i\sigma^2 (\psi_R)^* = \dots = \exp(\frac{i}{2} \vec{\theta} \cdot \vec{\sigma}) \bar{\psi}_R$$

$$\text{B: } \bar{\psi}_L \rightarrow \bar{\psi}_L' = i\sigma^2 (\psi_L')^* = i\sigma^2 (\exp(\frac{i}{2} \vec{\phi} \cdot \vec{\sigma}) \psi_L)^* = i\sigma^2 \exp(\frac{i}{2} \vec{\phi} \cdot \vec{\sigma}^*) \psi_L^* = i\sigma^2 \exp(\frac{i}{2} \vec{\phi} \cdot \vec{\sigma}^*) (-i\sigma^2) (i\sigma^2) \psi_L^* =$$

$$= \exp(-\frac{i}{2} \vec{\phi} \cdot \vec{\sigma}) i\sigma^2 \psi_L^* = \exp(-\frac{i}{2} \vec{\phi} \cdot \vec{\sigma}) \bar{\psi}_L$$

$$\bar{\psi}_R \rightarrow \bar{\psi}_R' = i\sigma^2 (\psi_R')^* = \dots = \exp(\frac{i}{2} \vec{\phi} \cdot \vec{\sigma}) \bar{\psi}_R$$

So $\bar{\psi}_L$ transforms as ψ_R , and $\bar{\psi}_R$ transforms as ψ_L :

$$\begin{aligned} \psi_L &= \bar{\psi}_R = i\sigma^2 \psi_R^* \\ i\sigma^2 \psi_L^* &= -i\sigma^2 (i\sigma^2 \psi_R^*)^* \\ \bar{\psi}_L &= \psi_R \end{aligned}$$

$$\begin{aligned} (i\sigma^2)^{-1} &= -i\sigma^2 \\ \Downarrow \\ L \rightarrow R: \psi &\rightarrow i\sigma^2 \psi^* \\ R \rightarrow L: \psi &\rightarrow -i\sigma^2 \psi^* \end{aligned}$$

Left-handed and right-handed repr. are conjugates of each other and any spinor in either repr. can be written as a spinor in the other repr.

2.6. $(\frac{1}{2}, \frac{1}{2})$ representation

Each state will have 2 indices: for left-handed $SU(2)$ and right-handed $SU(2)$ (and they won't interfere with each other, because $[N_i^+, N_i^-] = 0$).

$v^{\dot{a}\dot{b}}$: state $\begin{cases} \dot{a} - \text{right-handed indices } (N_i^+) \\ \dot{b} - \text{left-handed indices } (N_i^-) \end{cases} \Rightarrow 4$ components $\begin{cases} \xrightarrow{\text{2x2 matrix}} \\ \xrightarrow{4 \times 1 \text{ vector}} \end{cases}$

$$v^{\dot{a}\dot{b}} = \begin{pmatrix} \uparrow\uparrow & \uparrow\downarrow \\ \downarrow\uparrow & \downarrow\downarrow \end{pmatrix}, \quad |\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |\downarrow\uparrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |\downarrow\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\{N_i^+\}$ and $\{N_i^-\}$ are hermitian \Rightarrow consider $v^{\dot{a}\dot{b}}$ is hermitian:

$$v^{\dot{a}\dot{b}} = \begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix} = v^0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + v^1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + v^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + v^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (v^\mu \sigma_\mu)^{\dot{a}\dot{b}}$$

Therefore:

$$v^0 = \frac{1}{2}(v^{\dot{a}\dot{b}} + v^{\dot{b}\dot{a}}) \quad v^1 = \frac{1}{2}(v^{\dot{a}\dot{b}} - v^{\dot{b}\dot{a}}) \quad v^2 = \frac{i}{2}(v^{\dot{a}\dot{b}} - v^{\dot{b}\dot{a}}) \quad v^3 = \frac{i}{2}(v^{\dot{a}\dot{b}} + v^{\dot{b}\dot{a}})$$

General Lorentz transformation for $v^{\dot{a}\dot{b}}$:

$$v^{\dot{a}\dot{b}} \rightarrow v'^{\dot{a}\dot{b}} = [\exp(\frac{i}{2} \vec{\theta} \cdot \vec{\sigma} - \frac{i}{2} \vec{\phi} \cdot \vec{\sigma})]_{\dot{a}}^{\dot{c}} [\exp(\frac{i}{2} \vec{\theta} \cdot \vec{\sigma} + \frac{i}{2} \vec{\phi} \cdot \vec{\sigma})]_{\dot{b}}^{\dot{d}} v^{\dot{c}\dot{d}} \approx (I + \frac{i}{2} \vec{\theta} \cdot \vec{\sigma} - \frac{i}{2} \vec{\phi} \cdot \vec{\sigma})_{\dot{a}}^{\dot{c}} \cdot (I + \frac{i}{2} \vec{\theta} \cdot \vec{\sigma} + \frac{i}{2} \vec{\phi} \cdot \vec{\sigma})_{\dot{b}}^{\dot{d}} v^{\dot{c}\dot{d}} = \dots \approx \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\begin{aligned} v^0 &\rightarrow v'^0 = \frac{1}{2}(A+D) = v^0 + i(\theta_x v^1 + \theta_y v^2 + \theta_z v^3) \\ v^1 &\rightarrow v'^1 = \frac{1}{2}(B+C) = v^1 + i(\theta_x v^0 + \phi_z v^2 - \phi_y v^3) \\ v^2 &\rightarrow v'^2 = \frac{1}{2}(B-C) = v^2 + i(\theta_y v^0 - \phi_z v^1 + \phi_x v^3) \\ v^3 &\rightarrow v'^3 = \frac{1}{2}(A-D) = v^3 + i(\theta_z v^0 + \phi_y v^1 - \phi_x v^2) \end{aligned} \quad \Rightarrow \begin{pmatrix} v'^0 \\ v'^1 \\ v'^2 \\ v'^3 \end{pmatrix} = \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} + \begin{pmatrix} 0 & i\theta_x & i\theta_y & i\theta_z \\ i\theta_x & 0 & i\phi_z & -i\phi_y \\ i\theta_y & -i\phi_z & 0 & i\phi_x \\ i\theta_z & i\phi_y & -i\phi_x & 0 \end{pmatrix} \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}$$

i.e. Lorentz tr.
for vectors

So $(\frac{1}{2}, \frac{1}{2})$ repr. is a vector repr. of $SO(1,3)$.

$$\det(v^{\dot{a}\dot{b}}) = (v^0)^2 - (v^1)^2 - (v^2)^2 - (v^3)^2 = \text{inv} \quad \text{under } SO(1,3) \quad (\text{since } \det h = +1)$$

(as expected and demanded by SR)

③ Spinor indices.

Let's consider a spinor repr. of $SO(1,3)$: $\psi^{\dot{a}\dot{b}} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$, \dot{a} - right-handed indices, \dot{b} - left-handed indices

3.1 Dotted and undotted indices

$$(\psi^{\dot{a}\dot{b}})^T = \psi^{\dot{b}\dot{a}} = \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} = (\psi^{\dot{a}\dot{b}})^* \text{ as expected because } \psi^{\dot{a}\dot{b}} \text{ is hermitian}$$

So dotted and undotted indices can be interchanged with complex conjugation. However we know it's not all that's involved in switching between L and R spinor representations: we also need $i\sigma^2$.

3.2 Lowered and raised indices

$$L \rightarrow R: \psi \rightarrow i\sigma^2 \psi \quad (i\sigma^2)^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and let's act on it with } \{R_i\} \text{ and } \{\theta_i\}: \\ R \rightarrow L: \psi \rightarrow -i\sigma^2 \psi$$

$$R_x: (i\sigma^2)^{ac} = [R_x]_b^a [R_x]_d^c \quad (i\sigma^2)^{bd} = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \dots = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$B_x: (i\sigma^2)^{ac} = [B_x]_b^a [B_x]_d^c \quad (i\sigma^2)^{bd} = \begin{pmatrix} \operatorname{ch}\frac{\theta}{2} & -\operatorname{sh}\frac{\theta}{2} \\ -\operatorname{sh}\frac{\theta}{2} & \operatorname{ch}\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \operatorname{ch}\frac{\theta}{2} & -\operatorname{sh}\frac{\theta}{2} \\ -\operatorname{sh}\frac{\theta}{2} & \operatorname{ch}\frac{\theta}{2} \end{pmatrix} = \dots = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

...

So $i\sigma^2$ = inv under all Lorentz trs. $\Rightarrow (i\sigma^2)^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{ab}$: spinor metric (to raise and lower spinor indices)

$$(i\sigma^2)^{ab} = ((i\sigma^2)^{ab})^* \quad \Rightarrow (i\sigma^2)^{ab} = (i\sigma^2)^{\dot{a}\dot{b}} = -(i\sigma^2)_{ab} = -(i\sigma^2)_{\dot{a}\dot{b}} \leftarrow \text{convention} \\ (i\sigma^2)(-i\sigma^2) = I \quad \text{i.e. dotted and undotted indices are same for } i\sigma^2$$

If we follow this convention carefully we don't need to worry about remembering whether to use $i\sigma^2$ or $(-i\sigma^2)$ under $L \rightleftharpoons R$ change. Also we will consider ψ_L to have lowered and undotted indices (and ψ_R therefore will have raised and dotted):

$$\psi_L = \psi_a = \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \end{pmatrix} \quad \psi_R = \psi^{\dot{a}} = (i\sigma^2)^{\dot{a}\dot{b}} \psi_{\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_{L1}^* \\ \psi_{L2}^* \end{pmatrix} = \begin{pmatrix} \psi_{L2}^* \\ -\psi_{L1}^* \end{pmatrix} = \begin{pmatrix} \psi_{R1} \\ \psi_{R2} \end{pmatrix}$$

$$\psi_L^* = \psi_{\dot{a}} = \begin{pmatrix} \psi_{L1}^* \\ \psi_{L2}^* \end{pmatrix} \quad \psi_R^* = \psi^a = \begin{pmatrix} \psi_{L2} \\ -\psi_{L1} \end{pmatrix} = \begin{pmatrix} \psi_{R1}^* \\ \psi_{R2}^* \end{pmatrix}$$

$$\psi_L = \psi_a \xleftrightarrow{i\sigma^2} \psi^a = \psi_R^* \\ \psi_L^* = \psi_{\dot{a}} \xleftrightarrow{i\sigma^2} \psi^{\dot{a}} = \psi_R$$

dotted \leftrightarrow undotted : complex conjugation

raised \leftrightarrow lowered : using spinor metric

④ Lorentz Algebra for arbitrary dimensions.

4.1. Vector representations

The pattern for the form of R_i and B_i in higher dimensions is obvious (see the $SO(1,3)$ case), and after that the derivation of J_i and K_i is pretty straight-forward. Then if we lower indices for J_i and K_i we will get a basis set of 2-component hermitian matrices.

$$SO(1,n) : \frac{n(n+1)}{2} \text{ generators} = n \text{ boosts} + \frac{n(n-1)}{2} \text{ rotations}$$

Instead of using one index for counting generators, we will use two antisymmetric indices.

So $J^{MN} = -J^{NV}$ are generators (J^{M0} : boosts, J^{ij} : rotations) with matrix el.s $(J^{MN})^{ab} = -(J^{NV})^{ba}_{(n+1) \times (n+1)}$

$$(J^{MN})^{ab} = -i(\eta^{Ma}\eta^{Nb} - \eta^{Na}\eta^{Nb})$$

generators of $SO(1,n)$

$$[J^{MN}, J^{3\lambda}] = i(\eta^{\lambda M} J^{3\lambda} - \eta^{\lambda 3} J^{3\lambda} - \eta^{3\lambda} J^{3\lambda} + \eta^{3\lambda} J^{\lambda M})$$

Example:

Generators of rotations in \mathbb{R}^n : $L = \vec{r} \times \vec{p} = -i\vec{r} \times \frac{\partial}{\partial \vec{r}}$

$$\exp(i\alpha(-i\frac{\partial}{\partial x})) f(x) = \exp(\alpha\frac{\partial}{\partial x}) f(x) = (1 + \alpha\frac{\partial}{\partial x} + \frac{1}{2}\alpha^2\frac{\partial^2}{\partial x^2} + \dots) f(x) = f(x + \alpha)$$

Then $L^i = -i \epsilon_{ijk} x^j \partial^k$ or: i.e. $(-i\frac{\partial}{\partial x})$ - generator for spacial translations

$$L^{ij} = -i(x^i \partial j - x^j \partial i) \rightarrow L^{MN} = -i(x^M \partial^N - x^N \partial^M) \quad (\text{rotations in } \mathbb{R}^{1,n})$$

$$[L^{M0}, L^{3\lambda}] = \dots = i(\eta^{\lambda M} L^{3\lambda} - \eta^{\lambda 3} L^{3\lambda} - \eta^{3\lambda} L^{3\lambda} + \eta^{3\lambda} L^{\lambda M})$$

4.2. Spinor representations

A separation of Lorentz group into two copies of $SU(2)$ works only for $SO(1,3)$. For an arbitrary dimension there is a general approach for spinor repr. (using the framework of geometric algebra):

$$\text{Clifford Algebra: } \{\gamma^M | M=0, \dots, n-1\} : \{\gamma^M, \gamma^N\} = -2\eta^{MN} I$$

Now let's consider antisymmetric matrices $S^{MN} \stackrel{\text{def}}{=} \frac{i}{4} [\gamma^M, \gamma^N]$ with components $(S^{MN})^{ab}$:

$$1) S^{MN} = \frac{i}{4} (\gamma^M \gamma^N - \gamma^N \gamma^M) = \frac{i}{4} (\gamma^M \gamma^N - (-2\eta^{MN} I - \gamma^M \gamma^N)) = \frac{i}{2} (\gamma^M \gamma^N + \eta^{MN} I)$$

$$2) [S^{M0}, \gamma^3] = \dots = i(\gamma^3 \eta^{3M} - \gamma^M \eta^{33})$$

$$3) [S^{M0}, S^{3\lambda}] = \dots = -\frac{1}{2} (\gamma^3 \gamma^\lambda \eta^{3M} - \gamma^M \gamma^\lambda \eta^{33} - \gamma^3 \gamma^M \eta^{3M} - \gamma^3 \gamma^M \eta^{3\lambda}) = \\ = i(\eta^{\lambda M} S^{3\lambda} - \eta^{3\lambda} S^{3\lambda} - \eta^{3M} S^{3\lambda} - \eta^{3\lambda} S^{3M}) \quad (\text{i.e. define Lorentz algebra})$$

So as long as $\{\gamma^M\}$ satisfy the Clifford algebra, we can build the Lorentz algebra with $S^{MN} = \frac{i}{4} [\gamma^M, \gamma^N]$.