Symmetry and the Standard Model

Chapter I

Introduction

1. Review of classical physics.

1.1. Classical mechanics

- Hamilton's principle
- Euler-Lagrange eqns.
- Noether's theorem $\left(\delta L = 0 \text{ under } q \to q + \epsilon \delta q \implies \text{current } j \stackrel{\text{def}}{=} \frac{\partial L}{\partial \dot{q}} \delta q \;:\; \frac{dj}{dt} = 0 \right)$
  i.e. conserved

1.2. Special Relativity

Lorentz tr. = xy, xz, yz rotations + tx, ty, tz rotations
(Euler trans.) (Lorentz Boosts)

$$S = \int L\, dt = \int p_\mu dx^\mu \qquad E^2 = p^2 + m^2$$

Physically allowable transformations:

$$\eta_{\mu\nu} = \Lambda^{\mu'}_{\ \mu} \Lambda^{\nu'}_{\ \nu} \eta_{\mu'\nu'}$$

$\eta_{\mu\nu} = \mathrm{diag}\{-1, 1, 1, 1\}$
(i.e. preserves inner product)

$$\eta_{00} = \Lambda^{\mu'}_{\ 0} \Lambda^{\nu'}_{\ 0} \eta_{\mu'\nu'} \implies -1 = -\left(\Lambda^{0'}_{\ 0}\right)^2 + \sum_{i=3}^{3} \left(\Lambda^{i'}_{\ 0}\right)^2 \implies \left(\Lambda^{0'}_{\ 0}\right)^2 \geq 1$$

proper tr.: $\det \Lambda = +1$
improper tr.: $\det \Lambda = -1$ (changes parity)

ortochronuos tr.: $\Lambda^{0'}_{0} \geq 1$
non-ortochronous tr.: $\Lambda^{0'}_{0} \leq -1$ (e.g. time reverse)

Only proper and ortochronous tr. are physical, but other tr. are still required for Lorentz invariance.

1.3. Classical fields

$$S = \int dt\, L = \int d^4x\, \mathcal{L}, \text{ where } \quad L = \int d^3x\, \mathcal{L}, \quad \mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} = \Pi \dot{\phi} - \mathcal{L}$$

(Lagrangian and Hamiltonian densities)

$$\boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = \frac{\partial \mathcal{L}}{\partial \phi_i}}, \quad i = 1, \dots, n$$

Euler-Lagrange eqns. for ϕ_i

Noether's th: $\delta \mathcal{L} = 0$ under $\phi \to \phi + \epsilon \delta \phi \implies j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \phi \;:\; \partial_\mu j^\mu = 0 \quad \left(j^\mu = \begin{pmatrix} j^0 \\ \vec{j} \end{pmatrix} \begin{matrix} \leftarrow \text{charge density} \\ \leftarrow \text{3D current} \end{matrix} \right)$

1.4. Classical Electrodynamics

$A^\mu = (\phi, \vec{A})^T$ (4-potential)
$J^\mu = (\rho, \vec{j})^T$ (4-current)

$$F_{12} = \frac{q_1 q_2}{4\pi r^2} \qquad \vec{B} = \nabla \times \vec{A}, \quad \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

EM field Strength Tensor

$$\boxed{\begin{aligned} \partial_{[\mu} F_{\nu\lambda]} &= 0 \\ \partial_\mu F^{\mu\nu} &= J^\nu \end{aligned}} \implies \begin{aligned} \nabla \cdot \vec{E} &= \rho & \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{B} &= \vec{j} + \frac{\partial \vec{E}}{\partial t} & \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned}$$

Maxwell eqns.

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu$$

Gauge tr.: $A^\mu \to A^\mu + \partial^\mu \chi$: $F^{\mu\nu} = \text{inv}$ and $A_\mu J^\mu = \text{inv} \implies \mathcal{L}_{EM} = \text{inv}$

invariance: $A, B, C, \dots \to A', B', C', \dots \implies F(A, B, C, \dots) = F(A', B', C', \dots)$
symmetry: something remains the same under specific transformations
covariance: eqns. have the same form in all inertial frames of reference
(all laws of physics should be covariant)

2. Open questions and road map

2.1. Open questions

1) 3 generations

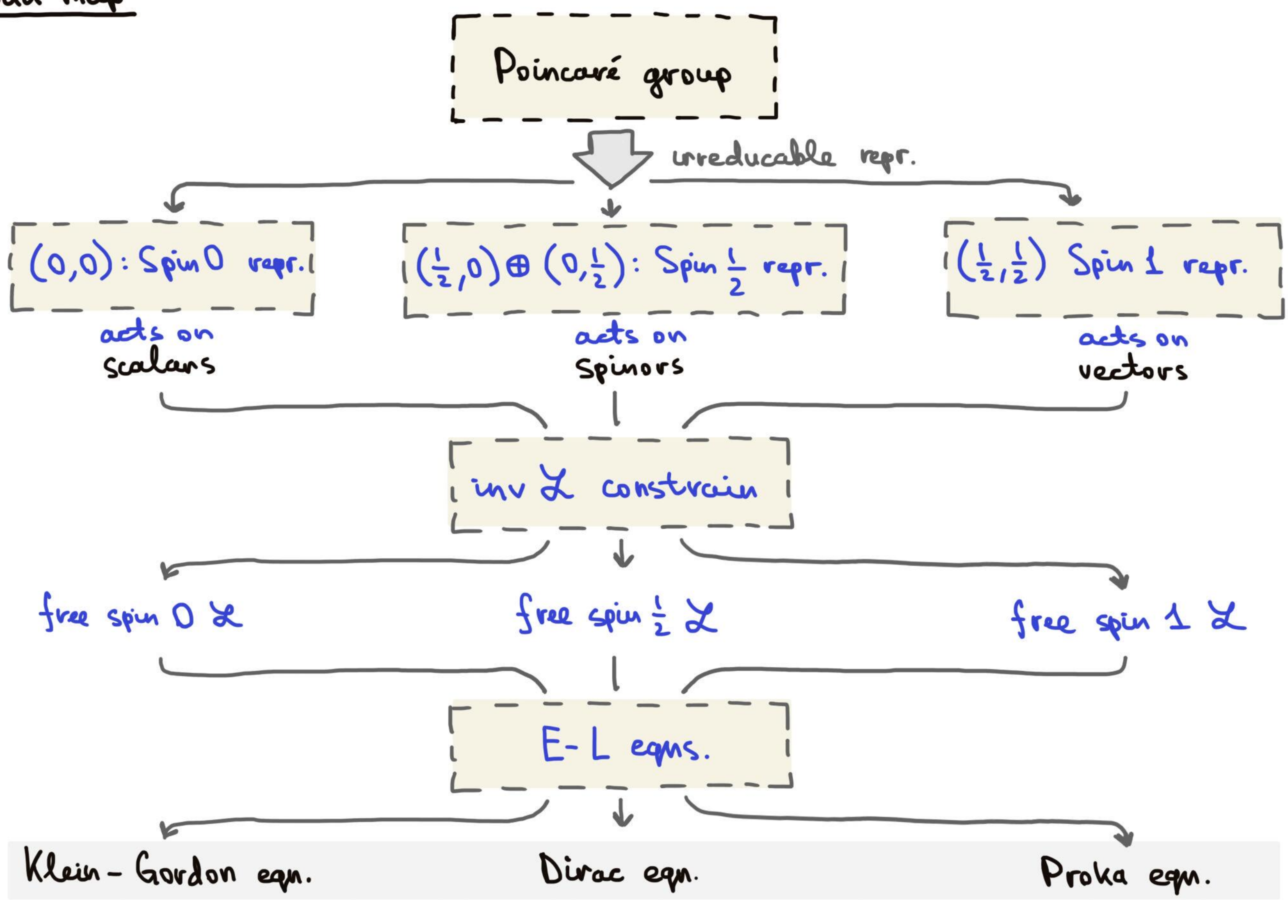
- 3 gauge theories $\leftrightarrow$ 3 fundamental forces
  ($U(1)\times SU(2)\times SU(3)$) (em, weak, strong)
- 3 generations of quarks and leptons
- only include 3 lowest orders in Φ in the Lagrangian of the theory
  (describes free fields and particles; higher order theories are unbound from below)
  i.e. $E \to -\infty \Rightarrow$ not stable
- only use 3 lowest-dimensional representations of the double cover of the Poincaré group, which correspond to spins $0, \frac{1}{2}$ and 1 (no $\frac{3}{2}$ spin particles)

2) Values of constants

Coupling constants of interactions and masses of particles.

3) Quantum gravity

2.2. Road map



3.) Brief overview of particle physics.

- elementary particles have mass, spin and charges
- elem. particles: bosons ($H, \gamma, W^{\pm}, Z^0, g_i$: $s=0,1$; forces)
  fermions (quarks and leptons: $s=\frac{1}{2}$; matter)
- photons and gluons are massless ($m=0$); and $W^{\pm}, Z^0$ have mass due to spontaneous symmetry breaking (Higgs mechanism)
- difference between forces and matter because of Pauli's exclusion principle
- charges of fund. interactions: electric charge, isospin and color
- quarks and leptons have antiparticles
- quarks carry color charge, while leptons do not
- quarks can compose hadrons (composite particles that interact strongly):
  mesons ($q\bar{q}$) (bosons), baryon (qqq) (fermions), tetraquarks, pentaquarks

			e	isospin	color
u	c	t	+2/3	+1/2	✓
d	s	b	−1/3	−1/2	✓

			e	isospin	color
ν_e	ν_μ	ν_τ	0	+1/2	×
e^-	μ^-	τ^-	−1	−1/2	×

Brief history:

400 BCE – Empedocles (earth, air, fire and water) and Democritus (atoms)

1820 – C. Ørsted (electricity is related to magnetism)

1873 – J.C. Maxwell

1897 – J.J. Thompson (electron)

1911 – E. Rutherford (structure of atoms)

1950s – S. Tomonaga, J. Schwinger and R. Feynman: QED

1954 – C.N. Yang, R. Mills : gauge invariance => 3 massless gauge bosons

1961 – S. Glashow : electro-weak theory with massless gauge bosons

1963 – M. Gell-Mann, G. Zweig : quarks

1964 – Guralnik, Hagen, Kibble; Higgs; Brout, Englert : Higgs mechanism

1965 – M-Y. Han, Y. Nambu, O. Greenberg : foundations of QCD

1967 – S. Weinberg, A. Salam : electro-weak theory

19xx– – experimental discoveries of particles (see Ch.2 in Robinson)

Chapter II
Group Theory

① Basics

$(G,\circ)$ is a **group**, $\circ: G\times G \to G$ (closure): 1. $g_1\circ(g_2\circ g_3) = (g_1\circ g_2)\circ g_3 \quad \forall g_1,g_2,g_3 \in G$ (associativity)
2. $\exists e\in G: \forall g\in G \quad e\circ g = g\circ e = g$ (identity)
3. $\forall g\in G \ \exists g^{-1}\in G: \ g^{-1}\circ g = g\circ g^{-1} = e$ (inverse)

assoc. $\Rightarrow \exists!\, g^{-1}$ (let h_1, h_2 both be inverse to g $\Rightarrow h_1 = h_1 e = h_1(gh_2) = (h_1 g)h_2 = eh_2 = h_2$)

1.1. Finite discrete groups

- a finite discrete group can be defined by a multiplication table
- each element appears only once in a row (and in a column)

let $g_{i_1} g_j = g_{i_2} g_j = g_k \Rightarrow g_{i_1} = g_{i_2}$ (because of $\exists!\, g_j^{-1}$)

ord $G = n$: finite group
ord $G = \infty$: infinite group

(e.g. ord $S_n = n!$)
symmetric group

ord $G = 2$:

	e	g_1
e	e	g_1
g_1	g_1	e

use property or:
let $g_1 g_1 = g_1 \Rightarrow$
$g_1 = g_1^{-1} g_1 = e \Rightarrow$
$g_1 g_1 = e$

ord $G = 3$:

	e	g_1	g_2
e	e	g_1	g_2
g_1	g_1	g_2	e
g_2	g_2	e	g_1

Ex.: $(\mathbb{Z}_2,+)$, $(\{1,-1\},\cdot)$, etc.

1.2. Representations

Ex: $(\mathbb{Z}_3,+)$: 1) $(\{[0]_3,[1]_3,[2]_3\},+)$ — integers (residues) modulo n — 2) $(\{1, \exp(\tfrac{2}{3}\pi i), \exp(\tfrac{4}{3}\pi i)\},\cdot)$

We will use linear representations, i.e. linear operators acting on some vector space:

$D(G)$: **representation** of $(G,\circ)$, and $D(g_i)\cdot D(g_k) = D(g_i\circ g_k)$

$|v\rangle = \begin{pmatrix} v^1 \\ \vdots \end{pmatrix}$ $\quad \langle v| = (v_1 \dots)$ $\quad \langle u|v\rangle = u^T v$

$\langle i|A|j\rangle$ – matrix elements of A if $\{|i\rangle\}$ – basis ($A|j\rangle$ – new vector, $\langle i|A|j\rangle$ – its coodinates)
(see linear algebra)

1. elements of G $\leftrightarrow$ orthonormal basis in some vector space

$e \to |e\rangle = |\hat{e}_1\rangle \quad g_1 \to |g_1\rangle = |\hat{e}_2\rangle \quad g_2 \to |g_2\rangle = |\hat{e}_3\rangle \quad \dots$

2. $D(G)$: $D(g_i)|g_k\rangle \overset{\text{def}}{=} |g_i\circ g_k\rangle \quad (D(g_k)_{ij} = \langle g_i|D(g_k)|g_j\rangle)$

Examples:

1) $G = \{e, g_1\}$

	e	g_1
e	e	g_1
g_1	g_1	e

$D(e)_{11} = \langle e|D(e)|e\rangle = \langle e|e\rangle = 1$
$D(e)_{12} = \langle e|D(e)|g_1\rangle = \langle e|g_1\rangle = 0$
$D(e)_{21} = \langle g_1|D(e)|e\rangle = \langle g_1|e\rangle = 0$
$D(e)_{22} = \langle g_1|D(e)|g_1\rangle = \langle g_1|g_1\rangle = 1$

$\Rightarrow D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$D(g_1)_{11} = \langle e|D(g_1)|e\rangle = \langle e|g_1\rangle = 0$
$D(g_1)_{12} = \langle e|D(g_1)|g_1\rangle = \langle e|e\rangle = 1$
$D(g_1)_{21} = \langle g_1|D(g_1)|e\rangle = \langle g_1|g_1\rangle = 1$
$D(g_1)_{22} = \langle g_1|D(g_1)|g_1\rangle = \langle g_1|e\rangle = 0$

$\Rightarrow D(g_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Then:
$D(e)\cdot D(e) = D(e)$
$D(e)\cdot D(g_1) = D(g_1)$
$D(g_1)\cdot D(e) = D(g_1)$
$D(g_1)\cdot D(g_1) = D(e)$

2) S_3: $e = (1\ 2\ 3)$ $\quad g_3 = (2\ 1\ 3)$
$g_1 = (2\ 3\ 1)$ $\quad g_4 = (3\ 2\ 1)$
$g_2 = (3\ 1\ 2)$ $\quad g_5 = (1\ 3\ 2)$

	e	g_1	g_2	g_3	g_4	g_5
e	e	g_1	g_2	g_3	g_4	g_5
g_1	g_1	g_2	e	g_5	g_3	g_4
g_2	g_2	e	g_1	g_4	g_5	g_3
g_3	g_3	g_4	g_5	e	g_1	g_2
g_4	g_4	g_5	g_3	g_2	e	g_1
g_5	g_5	g_3	g_4	g_1	g_2	e

1. $D(e) = id$

2. $D(g_1)_{11} = \langle e | D(g_1) | e \rangle = \langle e | g_1 \rangle = 0$
$D(g_1)_{12} = \langle e | D(g_1) | g_1 \rangle = \langle e | g_2 \rangle = 0$
$D(g_1)_{13} = \langle e | D(g_1) | g_2 \rangle = \langle e | e \rangle = 1$
$D(g_1)_{14} = \langle e | D(g_1) | g_3 \rangle = \langle e | g_5 \rangle = 0$
$D(g_1)_{15} = \langle e | D(g_1) | g_4 \rangle = \langle e | g_3 \rangle = 0$
$D(g_1)_{16} = \langle e | D(g_1) | g_5 \rangle = \langle e | g_4 \rangle = 0$

$$\ldots \Rightarrow D(g_1) = \begin{pmatrix} \begin{smallmatrix} 0&0&1\\1&0&0\\0&1&0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0&1&0\\0&0&1\\1&0&0 \end{smallmatrix} \end{pmatrix} \ldots D(S_3)$$

1.3. Reducibility and Irreducibility

Sometimes it's possible to find a repr. that is "smaller" than regular (therefore reducible & irreducible representations)

Regular repr.: $\dim V = n$, $n = \operatorname{ord} G$

Example:

S_3: $D(g_i)|E\rangle = A_{3\times3}\begin{pmatrix}1\\2\\3\end{pmatrix}$
$D(g_i)|E\rangle = |g_i\rangle$

$$\Rightarrow \ldots \overbrace{\begin{array}{lll} D(e) = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} & D(g_1) = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} & D(g_2) = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} \\ D(g_3) = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} & D(g_4) = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} & D(g_5) = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} \end{array}}^{\text{3-dim repr. of } S_3}$$

2. Subgroups and Cosets.

2.1. Subgroups

$(H,\circ) \subset (G,\circ)$ is a subgroup of G : $(H,\circ)$ is a group

Example:

S_3: 1) $\{e, g_1, g_2\} \simeq (\mathbb{Z}_3, +) \subset S_3$
2) $\{e, g_3\}, \{e, g_4\}, \{e, g_5\} \simeq (\mathbb{Z}_2, +) \subset S_3$
3) $\{e\}, S_3 \subset S_3$ (trivial subgroups)

2.2. Cosets.

$gH \stackrel{\text{def}}{=} \{g \circ h \mid h \in H\}$ left coset of H in G
$Hg \stackrel{\text{def}}{=} \{h \circ g \mid h \in H\}$ right coset of H in G

1. $H \subset G$ is normal: $\forall g \in G \quad gH = Hg$
2. G is abelian $\Rightarrow \forall H \subset G$ is normal

Example:

1) $G = S_3$
$H = \{e, g_1, g_2\}$
G_{012}

G_{012} is a normal subgroup of S_3

$e\, G_{012} = \{e, g_1, g_2\}$ same $G_{012}\, e = G_{012}$
$g_1 G_{012} = \{g_1, g_2, e\}$ $\quad G_{012}\, g_1 = G_{012}$
$g_2 G_{012} = \{g_2, e, g_1\}$ $\quad G_{012}\, g_2 = G_{012}$
$g_3 G_{012} = \{g_3, g_4, g_5\}$ $\quad G_{012}\, g_3 = G_{345}$
$g_4 G_{012} = \{g_4, g_5, g_3\}$ $\quad G_{012}\, g_4 = G_{345}$
$g_5 G_{012} = \{g_5, g_3, g_4\}$ same $G_{012}\, g_5 = G_{345}$

2) $G = (\mathbb{Z}, +)$ $\quad H = (2\mathbb{Z}, +)$

H is a normal subgroup of G because G is abelian.

3. Factor groups.

$g_1 = g_2 \mod H \subset G :\quad g_2 \in g_1 H$

1. $g = g \mod H$
2. $g_1 = g_2 \mod H \Rightarrow g_2 = g_1 \mod H$
3. $g_1 = g_2 \mod H \quad g_2 = g_3 \mod H \Rightarrow g_1 = g_3 \mod H$

} equivalence relation $\Rightarrow$

"G mod H"

$$\boxed{G/H \overset{\text{def}}{=} \{gH \mid g \in G\}}$$

factor set of H in G

Th: $H \subset G$ is normal $\Leftrightarrow$ $(G/H, \circ)$ is a group :
1. assoc. is inherited
2. $g_1 H \circ g_2 H = (g_1 \circ g_2) H$
3. eH – identity
4. $(gH)^{-1} = g^{-1} H$

factor group of $H \subset G$

$$\boxed{G \times \tilde{G} \overset{\text{def}}{=} \left(\{(g_i, \tilde{g}_k) \mid g_i \in G, g_k \in \tilde{G}\}, \bullet\right)}$$, where $(g_i, \tilde{g}_k) \bullet (g_j, \tilde{g}_\ell) \overset{\text{def}}{=} (g_i \circ g_j, \tilde{g}_k \tilde{\circ} \tilde{g}_\ell)$

product group of G and $\tilde{G}$

Examples:

1) $G = S_3 \quad H = G_{012} \qquad S_3 / G_{012} = \{\{e, g_1, g_2\}, \{g_3, g_4, g_5\}\} = \{\tilde{E}, \tilde{G}\}$

G_{012} is normal, therefore S_3/G_{012} is a group:

$$\tilde{E} \circ \tilde{G} = \begin{matrix} e\tilde{G} \\ g_1\tilde{G} \\ g_2\tilde{G} \end{matrix} = \begin{matrix} \{g_3, g_4, g_5\} \\ \{g_5, g_3, g_4\} \\ \{g_4, g_5, g_3\} \end{matrix} = \tilde{G} = \tilde{G} \circ \tilde{E}$$

$\dots$

$\tilde{E} \circ \tilde{E} = \tilde{E}$

$\tilde{G} \circ \tilde{G} = \tilde{E}$

$\Rightarrow$

	$\tilde{E}$	$\tilde{G}$
$\tilde{E}$	$\tilde{E}$	$\tilde{G}$
$\tilde{G}$	$\tilde{G}$	$\tilde{E}$

(compare with S_3 mult. table)

, i.e. $S_3 / G_{012} \simeq (\mathbb{Z}_2, +)$

2) $\mathbb{Z}/2\mathbb{Z} = \{0 + 2\mathbb{Z}, \pm 1 + 2\mathbb{Z}, \dots\} = \begin{bmatrix} 2k + 2\mathbb{Z} = \mathbb{Z}_{\text{even}} & \forall k \\ (2k+1) + 2\mathbb{Z} = \mathbb{Z}_{\text{odd}} & \forall k \end{bmatrix} = \{\mathbb{Z}_{\text{even}}, \mathbb{Z}_{\text{odd}}\} \simeq (\mathbb{Z}_2, +)$

$\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$

3) $G/G = \{eG, g_1 G, \dots\} = \{G, G, \dots\} = \{G\} \simeq \{e\}$ (trivial group)

$G/e = \{e\{e\}, g_1\{e\}, \dots\} = \{\{e\}, \{g_1\}, \dots\} \simeq G$

, i.e. $G/G = e$, $G/e = G$

4. Reducibility

4.1. Algebras and Modules

$$\boxed{\mathbb{R}[G] \overset{\text{def}}{=} \left\{\sum_{i=0}^{n-1} a_i |g_i\rangle \;\middle|\; a_i \in \mathbb{R}, g_i \in G\right\}}$$, i.e. vector space, spanned by the elements of G ; each point corresponds to some lin. combination of $\{|g_i\rangle\}$

Algebra of a group G

$$D(g_i) \sum_{j=0}^{n-1} a_j |g_j\rangle = \sum_{j=0}^{n-1} a_j |g_i \circ g_j\rangle$$

Let $M = \{m_0, m_1, \dots, m_{q-1}\}$ be a set on which group G can act on. Next we will define a correspondence between M and an orthonormal basis in some q-dimensional vector space : (like with G)

$m_0 \to |m_0\rangle \quad m_1 \to |m_1\rangle \quad \dots \quad m_{q-1} \to |m_{q-1}\rangle$

$$\boxed{\mathbb{R}M \overset{\text{def}}{=} \left\{\sum_{i=0}^{q-1} a_i |m_i\rangle \;\middle|\; a_i \in \mathbb{R}, m_i \in M\right\}}$$, i.e. vector space that repr. of G acts on

Module of a set M

Example:

$G = S_3 \qquad M = \{m_0, m_1, m_2\} \implies \mathbb{R}M = \mathbb{R}^3$

$g_i \sum_{j=0}^{2} a_j |m_j\rangle = a_0 |g_i m_0\rangle + a_1 |g_i m_1\rangle + a_1 |g_i m_2\rangle \qquad i=3: \; g_3 \sum_{j=0}^{2} a_j |m_j\rangle = a_0 |m_1\rangle + a_1 |m_0\rangle + a_2 |m_2\rangle$

$g_3 = (2\,1\,3)$

(i.e. reflection in the $a_0 = a_1$ plane in $\mathbb{R}^3$, so $g \leftrightarrow$ lin. trans of $\mathbb{R}M$)

4.2. Invariant subspaces

$\mathbb{R}W \subset \mathbb{R}V$ is an invariant subspace of $D(G)$: $\mathbb{R}W$ is closed under the action of G. ($\mathbb{R}V$ and $\emptyset$ are trivial subspaces)

$D(G)$ acting on $\mathbb{R}M$ is irreducible : $\mathbb{R}M$ doesn't contain non-trivial inv. subspaces (reducible otherwise)

Examples:

1) $G = S_3 \qquad M = \{m_0, m_1, m_2\} \implies \mathbb{R}M = \mathbb{R}^3$

Then $\mathbb{R}W = \text{span}(|m_0\rangle + |m_1\rangle + |m_2\rangle)$ (line in $\mathbb{R}^3$) is an invariant subspace, i.e. $g_i \mathbb{R}W = \mathbb{R}W \;\; \forall g_i \in G$

2) $G = S_3 \qquad M = \{m_0, m_1, \dots, m_5\} \implies \mathbb{R}M = \mathbb{R}^6$

$\forall g_i \in G \quad D(g_i) = \left(\begin{array}{c|c} \equiv & 0 \\ \hline 0 & \equiv \end{array}\right)$, i.e. the first 3 dimensions never mix with the last 3

$\Downarrow$

reducible repr. $\Leftarrow \underset{6}{\mathbb{R}M} = \underset{3}{\mathbb{R}V_1} \oplus \underset{3}{\mathbb{R}V_2}$, where $V_1 = \{m_0, m_1, m_2\}$, $V_2 = \{m_3, m_4, m_5\}$

More generally, if $V = \bigoplus_{i=1}^{k} W_i$ ($\dim V = n$, $\dim W_i = n_i$), then any matrix can be written as $X_n = \bigoplus_{i=1}^{k} A^{(i)}_{n_i}$ (block diagonal), X_n — $n \times n$, $A^{(i)}_{n_i}$ — $n_i \times n_i$ blocks

To find out if a repr. is reducible we can try to find a basis in which $\{D(g_i)\}$ have block diagonal form, i.e. find non-singular ($\det S \neq 0$) $S: D \to S^{-1} D S$:

$D'(g_i) \cdot D'(g_j) = S^{-1} D(g_i) S S^{-1} D(g_j) S = S^{-1} D(g_i \circ g_j) S = D'(g_i \circ g_j)$, i.e. $D'(G)$ is also repr. of G

(Jordan decomposition to find S ?)

Chapter III
Lie groups

1. Overview of groups.

From now on we will work with continuous groups ($g_i \to g(\alpha_i)$).

1.1. $GL(n,\mathbb{C})$: $\mathbb{C}^n \to \mathbb{C}^n \simeq \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^n$
repr. = lin. operators on V^n
$\cup$, $\det A \neq 0 \Rightarrow$ space = inv.
$GL(n,\mathbb{R})$: $\mathbb{R}^n \to \mathbb{R}^n$

1.2. $SL(n,\mathbb{C})$: $\det A = +1 \Rightarrow$ space + volume = inv.

1.3. $O(n)$: $A^T = A^{-1}$ $\det A = \pm 1$, $\langle x,y\rangle = x^T y = x'^T A^T A y' = \langle x', y'\rangle \Rightarrow$ space + volume + $\langle\cdot,\cdot\rangle$ = inv.

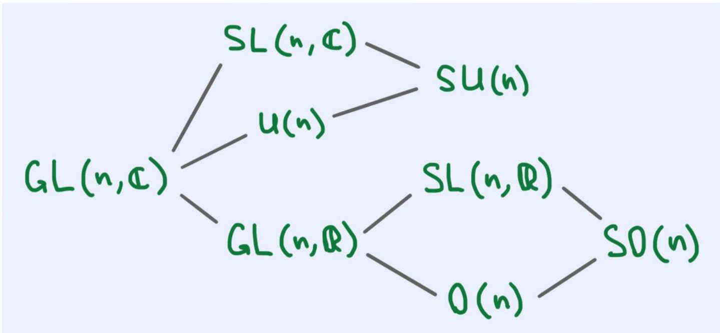
1.4. $U(n)$: $A^\dagger = A^{-1}$ $\det A = e^{i\varphi}$, $\langle x,y\rangle = x^\dagger y = x'^\dagger A^\dagger A y' = \langle x', y'\rangle \Rightarrow$ space + volume + $\langle\cdot,\cdot\rangle$ = inv.

1.5. $SO(n) \subset O(n)$:
$SU(n) \subset U(n)$: $\det A = +1$ (i.e. only rotations) $\Rightarrow$ space + volume + $\langle\cdot,\cdot\rangle$ + orientation = inv
(parity)

1.6. $SO(m,n)$ ($SO(m), SO(n) \subset SO(m,n)$)

$$\langle x,y\rangle = -\sum_{i=1}^{m} x^i y^i + \sum_{i=m+1}^{m+n} x^i y^i$$

e.g. $SO(1,3)$ — Lorentz group
(S = inv. under $SO(1,3)$, i.e. Lorentz invariance)



Examples:

1) $SO(2)$ repr. as rotations of a unit circle : $(\{\theta \mid \theta \in [0,2\pi)\}, +)$
mod 2π

2) $SO(2)$ repr. as Euler matrices : $g(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

2. Generators.

Lie group is a group that is parameterized by a set of continuous parameters α_i, $i = 1,\dots,n$.

$g(\alpha_i)$: element of G

$D_n(g(\alpha_i)) = D_n(\alpha_i)$: representation of $g(\alpha_i)$

(We will choose parameters s.t. $g(\alpha_i)|_{\alpha_i=0} = e$, therefore $D_n(\alpha_i)|_{\alpha_i=0} = I$)

1. Expansion for an element near e:

$$D_n(\delta\alpha_i) = D_n(e) + \delta\alpha_i \frac{\partial D_n(\alpha_i)}{\partial \alpha_i}\Big|_{\alpha_i=0} + o(\delta\alpha_i) I \approx I + i\,\delta\alpha_i X_i ,$$

$$\boxed{X_i \overset{\text{def}}{=} -i \frac{\partial D_n(\alpha_i)}{\partial \alpha_i}\Big|_{\alpha_i=0}}$$
generators of G

2. Then $\forall\, g(\alpha_i) \in G$:

$$D_n(\alpha_i) = \lim_{N\to\infty} (D_n(\delta\alpha_i))^N = \lim_{N\to\infty} (I + i\delta\alpha_i X_i)^N = \lim_{N\to\infty}\left(I + \frac{i\alpha_i X_i}{N}\right)^N \Rightarrow \boxed{D_n(\alpha_i) = \exp(i\alpha_i X_i)}$$
repr. of $g(\alpha_i)$

There is an infinite amount of different sets of generators for a particular group, but they all are equivalent (like repr. of G).

dimension of G : number of generators / parameters of G.
Ex: $SO(3)$: 3 generators
$(\theta, \varphi, \psi) \to (X_\theta, X_\varphi, X_\psi)$

3. Lie algebras

For a discrete group algebra was defined as V^n spanned by the elements of the group, but for Lie groups a different approach is needed (because groups are continuous): algebra is spanned by the generators of a Lie group instead.

Consider $g(\alpha_i), g(\beta_j) \in G$, then $g(\alpha_i)\circ g(\beta_j) \in G$; and for representations:

$$\exp(i\alpha_i X_i)\cdot \exp(i\beta_j X_j) = \exp(i\delta_k X_k)$$

1. $i\delta_k X_k = \ln(\exp(i\delta_k X_k)) = \ln\big(I + \overbrace{\exp(i\alpha_i X_i)\cdot\exp(i\beta_j X_j) - I}^{x}\big) = \ln(I + x)$

2. $x \overset{\text{to 2nd order}}{\approx} \left(I + i\alpha_i X_i + \frac{1}{2}(i\alpha_i X_i)^2\right)\left(I + i\beta_j X_j + \frac{1}{2}(i\beta_j X_j)^2\right) - I \approx I + i\beta_j X_j - \frac{1}{2}(\beta_j X_j)^2 +$
$$+ i\alpha_i X_i - \alpha_i X_i \beta_j X_j - \frac{1}{2}(\alpha_i X_i)^2 - I = i(\alpha_i X_i + \beta_j X_j) - \alpha_i X_i \beta_j X_j - \frac{1}{2}\left((\alpha_i X_i)^2 + (\beta_j X_j)^2\right)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + o(x^2)$$

3. $\ln(I+x) \approx x - \frac{1}{2}x^2 \approx i(\alpha_i X_i + \beta_j X_j) - \alpha_i X_i \beta_j X_j - \frac{1}{2}\left((\alpha_i X_i)^2 + (\beta_j X_j)^2\right) - \frac{1}{2}\left(-(\alpha_i X_i + \beta_j X_j)\cdot\right.$
$$\left.\cdot(\alpha_i X_i + \beta_j X_j)\right) = \dots + \frac{1}{2}\left((\alpha_i X_i)^2 + (\beta_j X_j)^2 + \alpha_i\beta_j(X_iX_j + X_jX_i)\right) = i(\alpha_i X_i + \beta_j X_j) +$$
$$+ \frac{1}{2}\alpha_i\beta_j[X_j, X_i] = i(\alpha_i X_i + \beta_j X_j) - \frac{1}{2}[\alpha_i X_i, \beta_j X_j]$$

Then the result is:

$$\boxed{\exp(i\alpha_i X_i)\cdot\exp(i\beta_j X_j) \approx \exp\left(i(\alpha_i X_i + \beta_j X_j) - \frac{1}{2}[\alpha_i X_i, \beta_j X_j]\right)}$$

Baker-Campbell-Hausdorff formula

$[X_i, X_j] \propto$ lin. combination of $\{X_k\}$ (because of closure), then $\boxed{[X_i, X_j] = i f_{ijk} X_k}$, $f_{ijk} = -f_{jik}$

f_{ijk} – structure constants of G

- generators under the specific commutation relations form the Lie Algebra of a group
- structure constants f_{ijk} completely determine the structure of a group

The adjoint representation:

$$[X_i,[X_j,X_k]] + [X_j,[X_k,X_i]] + [X_k,[X_i,X_j]] = 0 \quad \text{(Jacobi Identity)}$$

1. $[X_i,[X_j,X_k]] = [X_i, i f_{jka} X_a] = (i f_{jka})(i f_{iab}) X_b = -f_{jka} f_{iab} X_b$

2. $f_{jka} f_{iab} X_b + f_{kia} f_{jab} X_b + f_{ija} f_{kab} X_b = 0$

   $f_{jka} f_{iab} + f_{kia} f_{jab} + f_{ija} f_{kab} = 0$

3. Let $T^a_{bc} \overset{def}{=} -i f_{abc}$ (n $n\times n$ matrices, $n = \dim G$), then:

$$f_{jka} f_{iab} = i^2 T^j_{ka} T^i_{ab} = -(T^j \cdot T^i)_{kb}$$
$$f_{kia} f_{jab} = -f_{ika} f_{jab} = -i^2 T^i_{ka} T^j_{ab} = (T^i\cdot T^j)_{kb}$$
$$f_{ija} f_{kab} = -f_{ija} f_{akb} = -i f_{ija} T^a_{kb}$$

$\Longrightarrow$ $\boxed{[T^a, T^b] = i f_{abc} T^c}$

i.e. structure constants themselves form a repr. of G (adjoint representation)

4. Orthogonal and unitary groups.

4.1. SO(2)

Let's find a generator X and D(G) from the property $\langle v, v\rangle = \langle v', v'\rangle$:

$\theta \to R(\theta) = \exp(i\theta X)$ for some generator X

$$\langle v', v'\rangle = v^T \exp(i\theta X^T)\exp(i\theta X)\, v \underset{\text{1st order}}{\approx} v^T(I + i\theta X^T + i\theta X)\, v = v^T v + v^T i\theta(X + X^T)\, v$$

$$\Downarrow$$

$$X = -X^T \text{ (antisym.)}$$

$$\boxed{X = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}, \; i \text{ for } X = X^\dagger$$

generator of SO(2)

Then $\forall g(\theta) \in G$ its representation will be:

$$\exp(i\theta X) = \exp\left(\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = I + \theta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{2}\theta^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{3!}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \ldots =$$

$$= \begin{pmatrix} 1 - \frac{1}{2}\theta^2 + \ldots & \theta - \frac{1}{3!}\theta^3 + \ldots \\ -(\theta - \frac{1}{3!}\theta^3 + \ldots) & 1 - \frac{1}{2}\theta^2 + \ldots \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

(i.e. Euler matrices)

SO(2) is abelian and $[X, X] = 0 \Rightarrow f_{ijk} = 0$ (no adjoint repr.)

4.2. SO(3)

For SO(3) let's go "backwards" and start with a representation:

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \quad R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix} \quad R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then for generators $X_i = -i \left.\frac{\partial D_n(\alpha_i)}{\partial \alpha_i}\right|_{\alpha_i = 0}$:

$$J_x = -i\left.\frac{\partial R_x(\phi)}{\partial\phi}\right|_{\phi=0} = -i\left.\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & \cos\phi \\ 0 & -\cos\phi & -\sin\phi \end{pmatrix}\right|_{\phi=0} = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$ and similarly for J_y and J_z

$$\boxed{J_x = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad J_y = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad J_z = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

generators of SO(3)

, and $[J_x, J_y] = iJ_z$, $[J_y, J_z] = iJ_x$, $[J_z, J_x] = iJ_y$ $\Rightarrow$ $\boxed{[J_i, J_j] = i\epsilon_{ijk}J_k}$

$$\Downarrow$$

$$f_{ijk} = \epsilon_{ijk}$$

4.3. SU(2)

For SU(2) let's start with $f_{ijk} = \epsilon_{ijk}$ (same as for $SO(3) \simeq SU(2)/\mathbb{Z}_2$):

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad U^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\det U = ad - bc \quad U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

$$\Rightarrow \begin{cases} d = a^* \\ -b = c^* \\ -c = b^* \\ a = d^* \end{cases} \Rightarrow U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

$$\det U = |a|^2 + |b|^2 = 1$$

$\Rightarrow$ 3 real parameters (just like SO(3)) and 3 generators

5. SU(2) and physical states.

5.1. Raising and Lowering operators

To understand the space Lie group acts on better, we will choose a basis from the eigenvectors of $D(G)$. For $SU(2)$ it's only possible to diagonalize one of the three generators at a time (because of $[\,,\,]$ relations): J^1, J^2, J^3 (J^3 is diagonal).

j, j' : the biggest and smallest eigenvalues of J^3

$|j;m\rangle$: an eigenvector of J^3 with eigenvalue m

$$J^3|j;j\rangle = j\,|j;j\rangle \qquad J^3|j;m\rangle = m\,|j;m\rangle$$

(let's assume $|j;j\rangle$ is known)

$$\boxed{J^{\pm} \overset{def}{=} \frac{1}{\sqrt{2}}\left(J^1 \pm iJ^2\right)}$$

raising and lowering operators

1. $[J^3, J^{\pm}] = \frac{1}{\sqrt{2}}\left([J^3,J^1] \pm i[J^3,J^2]\right) = \frac{1}{\sqrt{2}}\left(i\epsilon_{31k}J^k \pm (-1)\epsilon_{32k}J^k\right) =$
$= \frac{1}{\sqrt{2}}\left(i\epsilon_{312}J^2 \mp \epsilon_{321}J^1\right) = \frac{1}{\sqrt{2}}\left(\pm J^1 + iJ^2\right) = \pm\frac{1}{\sqrt{2}}\left(J^1 \pm iJ^2\right)$

2. $[J^+, J^-] = \frac{1}{2}[J^1+iJ^2, J^1-iJ^2] = \frac{1}{2}\left(-i[J^1,J^2] + i[J^2,J^1]\right) =$
$= -\frac{1}{2}i\left([J^1,J^2]+[J^1,J^2]\right) = -i[J^1,J^2] = J^3$

3. $J^i = (J^i)^\dagger \Rightarrow (J^{\pm})^\dagger = J^{\mp}$ (i.e. hermitian adjoint operators)

$$\boxed{\begin{aligned}[J^3, J^{\pm}] &= \pm J^{\pm}\\ [J^+, J^-] &= J^3\end{aligned}}$$

5.2. Finding next eigenvector of J^3

$$J^3|j;m\rangle = m\,|j;m\rangle$$

$$J^3J^{\pm}|j;m\rangle = (\pm J^{\pm} + J^{\pm}J^3)|j;m\rangle = (m\pm1)\,J^{\pm}|j;m\rangle$$

So $J^{\pm}|j;m\rangle$ is the eigenvector of J^3 with eigenvalue $(m\pm1)$ if $|j;m\rangle$ is also the eigenvector of J^3. Hence the names of $J^{\pm}$; also $J^+|j;j\rangle = 0$ and $J^-|j;j'\rangle = 0$.

Therefore $J^-|j;j\rangle$ is the next eigenvector with eigenvalue $j-1$: $J^-|j;j\rangle = N_j|j;j-1\rangle$

1. $\left|J^-|j;j\rangle\right|^2 = (J^-|j;j\rangle)^\dagger(J^-|j;j\rangle) = \langle j;j|J^+J^-|j;j\rangle = \langle j;j|J^+J^- - \underbrace{J^-J^+|j;j\rangle}_{=0} =$
$= \langle j;j|[J^+,J^-]|j;j\rangle = \langle j;j|J^3|j;j\rangle = j$ ($|j;j\rangle$ is normalized)

2. $\left|J^-|j;j\rangle\right|^2 = |N_j|^2\langle j;j-1|j;j-1\rangle = |N_j|^2$ ($|j;j-1\rangle$ is also normalized)

Then $J^-|j;j\rangle = \sqrt{j}\,|j;j-1\rangle$, or $|j;j-1\rangle = \frac{1}{\sqrt{j}}J^-|j;j\rangle$.

5.3. Finding remaining eigenvectors of J^3

$$J^-|j;j-1\rangle = N_{j-1}|j;j-2\rangle$$

1. $\left|J^-|j;j-1\rangle\right|^2 = \langle j;j-1|J^+J^-|j;j-1\rangle = \langle j;j|\frac{1}{\sqrt{j}}J^+J^+J^-\frac{1}{\sqrt{j}}J^-|j;j\rangle =$ see [Rob., p.90] $= 2j-1$

2. $\left|J^-|j;j-1\rangle\right|^2 = |N_{j-1}|^2$

Then $J^-|j;j-1\rangle = \sqrt{2j-1}\,|j;j-2\rangle$, or $|j;j-2\rangle = \frac{1}{\sqrt{2j-1}}J^-|j;j-1\rangle$

We can continue this process and get the general expression for $|j;j-k\rangle$ and N_{j-k} :

$$\boxed{|j;j-k\rangle = \frac{1}{N_{j-k}}(J^-)^k\,|j;j\rangle}\,, \quad N_{j-k} = \frac{1}{\sqrt{2}}\sqrt{(2j-k)(k+1)}$$

eigen vectors of J^3

$k=2j$: $N_{j-2j}=0$, therefore $|j;-j\rangle$ is the state with the lowest eigenvalue.

So for a general repr. of SU(2) we have $2j+1$ eigenstates of J^3 with eigenvalues $\{j, j-1, \dots, -j+1, -j\}$. Therefore $j=\frac{n}{2}$, $n\in\mathbb{N}$ (i.e. $j=0,\frac{1}{2},1,\frac{3}{2},\dots$).

$$\langle j;m'|J^3|j;m\rangle = m\,\delta_{m'm}$$
$$\langle j;m'|J^+|j;m\rangle = \underbrace{\tfrac{1}{\sqrt{2}}\sqrt{(j+m+1)(j-m)}}_{N_{m-1}}\,\delta_{m'\,m+1}$$
$$\langle j;m'|J^-|j;m\rangle = \underbrace{\tfrac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}}_{N_m}\,\delta_{m'\,m-1}$$

6. SU(2)

6.1. SU(2) for $j=\frac{1}{2}$

For $j=\frac{1}{2}$ repr. the eigenvalues for J^3 are $\{\frac{1}{2},-\frac{1}{2}\}$, then $J^3_{1/2} = \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $N_{1/2}=\frac{1}{\sqrt{2}}$.

$$J^\pm = \frac{1}{2}(J^1 \pm iJ^2) \Rightarrow \begin{array}{l} J^1 = \frac{1}{\sqrt{2}}(J^- + J^+) \\ J^2 = \frac{1}{\sqrt{2}}(J^- - J^+)\end{array} \qquad [J^a_j]_{m'm} = \langle j;m'|J^a|j;m\rangle$$

Let's find matrix elements of $J^1_{1/2}$ and $J^2_{1/2}$:

$$[J^1_{1/2}]_{11} = \langle\tfrac{1}{2};-\tfrac{1}{2}|J^1_{1/2}|\tfrac{1}{2};-\tfrac{1}{2}\rangle = \langle\tfrac{1}{2};-\tfrac{1}{2}|\tfrac{1}{\sqrt{2}}(J^-+J^+)|\tfrac{1}{2};-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}\langle\tfrac{1}{2};-\tfrac{1}{2}|J^+|\tfrac{1}{2};-\tfrac{1}{2}\rangle = 0$$
$$[J^1_{1/2}]_{12} = \langle\tfrac{1}{2};-\tfrac{1}{2}|J^1_{1/2}|\tfrac{1}{2};\tfrac{1}{2}\rangle = \langle\tfrac{1}{2};-\tfrac{1}{2}|\tfrac{1}{\sqrt{2}}J^-|\tfrac{1}{2};\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}\cdot\tfrac{1}{\sqrt{2}} = \tfrac{1}{2}$$
$$[J^1_{1/2}]_{21} = \langle\tfrac{1}{2};\tfrac{1}{2}|J^1_{1/2}|\tfrac{1}{2};-\tfrac{1}{2}\rangle = \langle\tfrac{1}{2};\tfrac{1}{2}|\tfrac{1}{\sqrt{2}}J^+|\tfrac{1}{2};-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}\cdot\tfrac{1}{\sqrt{2}} = \tfrac{1}{2}$$
$$[J^1_{1/2}]_{22} = \langle\tfrac{1}{2};\tfrac{1}{2}|J^1_{1/2}|\tfrac{1}{2};\tfrac{1}{2}\rangle = \langle\tfrac{1}{2};\tfrac{1}{2}|\tfrac{1}{\sqrt{2}}J^-|\tfrac{1}{2};\tfrac{1}{2}\rangle = 0$$

... (similarly for $J^2_{1/2}$)

$$\boxed{J^1_{1/2} = \frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \frac{\sigma^1}{2} \qquad J^2_{1/2} = \frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \frac{\sigma^2}{2} \qquad J^3_{1/2} = \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \frac{\sigma^3}{2}}$$

$\sigma^1, \sigma^2, \sigma^3$: Pauli spin matrices

generators for $j=\frac{1}{2}$ repr. of SU(2)

SU(2) will describe quantum spin for particles just like SO(3) describes angular momentum.
(rotations in complex spinor space) vs. (rotations in $\mathbb{R}^3$)

6.2. SU(2) for $j=1$

For $j=1$ the eigenvalues of J^3 are $\{1,0,-1\}$, then $J^3_1 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$ and

$N_1 = \sqrt{1} = 1$

$N_0 = \sqrt{2-1} = 1$

...

$$\boxed{J^1_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \qquad J^2_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix} \qquad J^3_1 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}}$$

generators for $j=1$ repr. of SU(2)

6.3. SU(2) for arbitrary j

So SU(2) has 3 generators (J^1_j, J^2_j, J^3_j) for any $(2j+1)$-dimensional representation and these representations are irreducible (for each j). Then eigenbasis (for J^3_j) is:

$$|j;j\rangle = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \quad |j;j-1\rangle = \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix} \quad \dots \quad |j;-j\rangle = \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}$$

and $J^\pm_j \overset{\text{def}}{=} \frac{1}{\sqrt{2}}(J^1 \pm iJ^2)$

Example:

$$j=\tfrac12:\quad J^+_{1/2} = \tfrac{1}{\sqrt2}\left(\tfrac12\begin{pmatrix}0&1\\1&0\end{pmatrix} + \tfrac12 i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right) = \tfrac{1}{2\sqrt2}\begin{pmatrix}0&2\\0&0\end{pmatrix} = \tfrac{1}{\sqrt2}\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$J^-_{1/2} = \tfrac{1}{\sqrt2}\left(\tfrac12\begin{pmatrix}0&1\\1&0\end{pmatrix} - \tfrac12 i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right) = \tfrac{1}{2\sqrt2}\begin{pmatrix}0&0\\2&0\end{pmatrix} = \tfrac{1}{\sqrt2}\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

$$J^+_{1/2}\,|\tfrac12;-\tfrac12\rangle = \tfrac{1}{\sqrt2}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \tfrac{1}{\sqrt2}\begin{pmatrix}1\\0\end{pmatrix} = N_{1/2}\,|\tfrac12;\tfrac12\rangle$$

$$J^-_{1/2}\,|\tfrac12;\tfrac12\rangle = \tfrac{1}{\sqrt2}\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \tfrac{1}{\sqrt2}\begin{pmatrix}0\\1\end{pmatrix} = N_{1/2}\,|\tfrac12;-\tfrac12\rangle$$

7. Root space.

7.1. Cartan subalgebra

- For SU(2) we chose a basis in the space group is acting on from the eigenvectors of J^3_j; and linear combinations of non-diagonal J^1_j and J^2_j formed $J^\pm$, which transform eigenvectors to each other (changing eigenvalues by an amount defined by f_{ijk}, i.e. the commutation relations of the generators). Let's generalize this approach for an arbitrary Lie group.

normalisation constant

- Generators span vector space with an inner product: $\langle T^i | T^j\rangle \overset{\text{def}}{=} \frac{1}{\varkappa}\,\mathrm{tr}(T^i T^j) = \delta^{ij}$.

(i.e. $\{T^i\}$ orthogonal to each other)

- In the set $\{T^i\}$ there will be closed subalgebra (Cartan subalgebra) of Cartan generators, which all commute with each other (and therefore can be simultaneasly diagonalised).

$H^i,\ i=1,\dots,M$: Cartan generators

$E^i,\ i=1,\dots,N-M$: non-Cartan generators

$\text{rank}\,G \overset{\text{def}}{=} M$, i.e. the number of Cartan generators

Ex: 1) SU(2) has 1 Cartan generator $\Rightarrow$ rank $G = 1$

$H^1 = J^3_j \quad E^1 = J^1_j \quad E^2 = J^2_j$

2) choosing x or p in QM because $[x,p]\neq 0$

We will choose a basis from the eigenvectors of $\{H^i\}$ s.t. each of them will have one eigenvalue with each of the M Cartan generators. So for every eigenvector $|j\rangle$ $(j=1,\dots,n)$ we can get an M-dim. weight vector of its eigenvalues of $\{H^i\}$:

$$t_j \overset{\text{def}}{=} \begin{pmatrix} t^1_j \\ t^2_j \\ \vdots \\ t^M_j \end{pmatrix} \underset{1:1}{\leftrightarrow} |j\rangle$$

Ex.: SU(2): 3 $(2j+1)\times(2j+1)$ operators acting on V^d, $d = 2j+1$

$\Rightarrow d = 2j+1$ weight vectors ($M=1$: so they are scalars)

1) $j=\frac12$: $\quad t_1 = (\frac12) \quad t_2 = (-\frac12)$

2) $j = 1$: $\quad t_1 = (1) \quad t_2 = (0) \quad t_3 = (-1)$

eigenvalues
‖
weights

$$H^i\,|D_n; t_j\rangle = t^i_j\,|D_n; t_j\rangle$$

Number n of $|D_n; t_j\rangle$ is determined by $\dim D_n(G)$ of a concrete repr.
Dimension M of each t_j is determined by the number of $\{H^i\}$, i.e. rank G.

7.2. Adjoint representation

We will use the adjoint repr. ($[T^a]_{bc} = -i f_{abc}$ N $N\times N$ matrices, and root vectors $\leftrightarrow$ weight vectors). (e.g. $j=1$ 3×3 repr. for SU(2) is adjoint).

N generators (N×N)		N eigenvectors (N-dim.)		N root vectors (M-dim.)
$T^j \equiv T^{t_j}$	$\underset{1:1}{\leftrightarrow}$	$\lvert Adj; T^{t_j}\rangle$	$\underset{1:1}{\leftrightarrow}$	t_j
H^{h_j} E^{e_j}				h_j e_j

$|Adj; H^{h_j}\rangle : j = 1\dots M$ Cartan

$|Adj; E^{e_j}\rangle : j = 1\dots N-M$ non-Cartan

$$\langle Adj; T^{t_j} | Adj; T^{t_k}\rangle = \delta^{jk}$$

$$I = \sum_i |Adj; T^{ti}\rangle\langle Adj; T^{ti}| \qquad -i f_{abc} = [T^{ta}]_{bc} = \langle Adj; T^{tb}| T^{ta} |Adj; T^{tc}\rangle$$

$$T^{ta}|Adj; T^{tb}\rangle = \sum_c |Adj; T^{tc}\rangle\langle Adj; T^{tc}| T^{ta}|Adj; T^{tb}\rangle = \sum_c |Adj; T^{tc}\rangle [T^a]_{cb} = \sum_c |Adj; T^{tc}\rangle(-i f_{acb}) =$$

$$= \sum_c i f_{abc} |Adj; T^{tc}\rangle = |Adj; [T^{ta}, T^{tb}]\rangle$$

$$\boxed{T^{ta}|Adj; T^{tb}\rangle = |Adj; [T^{ta}, T^{tb}]\rangle}$$

1. Cartan generators:

1) $H^{ha}|Adj; H^{hb}\rangle = h^a_b |Adj; H^{hb}\rangle$

$H^{ha}|Adj; H^{hb}\rangle = |Adj; [H^{ha}, H^{hb}]\rangle = 0 \quad \forall a, b$

$\Rightarrow h_b = 0 \quad \forall b = 1, \dots, M \qquad (H^{hj} \to H^j$ change of notation$)$

2) $H^a|Adj; E^{eb}\rangle = e^a_b |Adj; E^{eb}\rangle$

$H^a|Adj; E^{eb}\rangle = |Adj; [H^a, E^{eb}]\rangle$

$\Rightarrow \boxed{[H^a, E^{eb}] = e^a_b E^{eb}}$

2. non-Cartan generators:

Consider a state $|Adj; T^{tb}\rangle$ with H^c eigenvalue t^c_b. Let's act on it with E^{ea} and H^c:

$$H^c E^{ea}|Adj; T^{tb}\rangle = (H^c E^{ea} - E^{ea}H^c + E^{ea}H^c)|Adj; T^{tb}\rangle = ([H^c, E^{ea}] + E^{ea}H^c)|Adj; T^{tb}\rangle =$$

$$= (e^c_a E^{ea} + E^{ea} t^c_b)|Adj; T^{tb}\rangle = (t_b + e_a)^c E^{ea}|Adj; T^{tb}\rangle$$

So by acting on the eigenstate with non-Cartan E^{ea} we have shifted the H^c eigenvalue by one of the coordinates of the root vector (like raising and lowering operators in su(2)).

Also $\forall E^{ea}$ should $\exists E^{-ea}$ which transforms the eigenvalue "back":

$$H^b E^{ea}|Adj; E^{-ea}\rangle = (-e_a + e_a)^b E^{ea}|Adj; E^{-ea}\rangle = 0$$

But only the states corresponding to Cartan generators have 0 eigenvalues $\Rightarrow E^{ea}|Adj; E^{-ea}\rangle = \sum_b N^b_a |Adj; H^b\rangle$

1) $\langle Adj; H^c| E^{ea}|Adj; E^{-ea}\rangle = \sum_b N^b_a \langle Adj; H^c|Adj; H^b\rangle = \sum_b N^b_a \delta^{cb} = N^c_a$

2) $\langle Adj; H^c| E^{ea}|Adj; E^{-ea}\rangle = \langle Adj; H^c|Adj; [E^{ea}, E^{-ea}]\rangle = \frac{1}{\varkappa} \operatorname{tr}(H^c [E^{ea}, E^{-ea}]) =$

$$= \frac{1}{\varkappa} \operatorname{tr}(E^{-ea}[H^c, E^{ea}]) = \frac{1}{\varkappa} e^c_a \operatorname{tr}(E^{-ea} E^{ea}) = e^c_a \delta^{aa} = e^c_a$$

(tr = inv under cyclic permutations)

Therefore:

$E^{ea}|Adj; E^{-ea}\rangle = \sum_b e^b_a |Adj; H^b\rangle$

$E^{ea}|Adj; E^{-ea}\rangle = |Adj; [E^{ea}, E^{-ea}]\rangle$

$\Rightarrow \boxed{[E^{ea}, E^{-ea}] = \sum_b e^b_a H^b}$

These results hold for any representation (but the adj. repr made them easy to obtain).

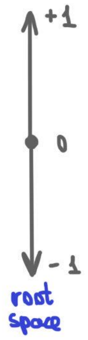
8. Adjoint representation of SU(2)

Adjoint repr. of SU(2) is $j=1$ repr. (of 3×3 operators acting on V^3). Since $\dim SU(2)=3$ and rank $SU(2)=1$ there are 3 generators, one of which is Cartan generator:

$$E^1 = J^1_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \qquad v_1=\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad t_1=(+1)$$

$$H^1 = J^3_1 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix} \qquad v_2=\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad t_2=(0)$$

$$E^2 = J^2_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix} \qquad v_3=\begin{pmatrix}0\\0\\1\end{pmatrix} \qquad t_3=(-1)$$



$$[H^a, E^{e_B}] = e^a_B E^{e_B}$$
$$[E^{e_a}, E^{-e_a}] = \sum_B e^B_a H^B$$
$$\langle T^i, T^j\rangle = \frac{1}{2}\mathrm{tr}(T^iT^j) = \delta^{ij}$$
(i.e. $\varkappa = 2$ for $j=1$ SU(2) repr.)

8.1. Finding non-Cartan generators

$$[H^1, E^1] = \frac{1}{2}\begin{pmatrix}0&1&0\\-1&0&1\\0&-1&0\end{pmatrix} = \frac{1}{\sqrt{2}}iE^2$$

$$[H^1, E^2] = -\frac{i}{2}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} = -\frac{i}{\sqrt{2}}E^1$$

$\Rightarrow$ E^1 and E^2 don't satisfy the condition for non-Cartan generators, so let's take their linear combination instead:

$$[H^1, \alpha E^1 \pm \beta i E^2] = \frac{1}{\sqrt{2}}(\alpha i E^2 \pm \beta E^1), \text{ let's try } \alpha=\beta$$

$$\frac{\alpha}{\sqrt{2}}(E^1+iE^2) = \alpha\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix} \qquad [H^1, \tfrac{\alpha}{\sqrt{2}}(E^1+iE^2)] = \alpha\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix} = t_1\cdot\alpha\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$$

$$\frac{\alpha}{\sqrt{2}}(E^1-iE^2) = \alpha\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix} \qquad [H^1, \tfrac{\alpha}{\sqrt{2}}(E^1-iE^2)] = -\alpha\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix} = t_3\cdot\alpha\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$$

$\Rightarrow$ Satisfy condition

Let's act with these operators on eigenbasis:

$$H^1v_1 = (1)v_1 \qquad \frac{\alpha}{\sqrt{2}}(E^1+iE^2)v_1 = 0 \qquad \frac{\alpha}{\sqrt{2}}(E^1-iE^2)v_1 = \alpha v_2$$

$$H^1v_2 = (0)v_2 \qquad \frac{\alpha}{\sqrt{2}}(E^1+iE^2)v_2 = \alpha v_1 \qquad \frac{\alpha}{\sqrt{2}}(E^1-iE^2)v_2 = \alpha v_3$$

$$H^1v_3 = (-1)v_3 \qquad \frac{\alpha}{\sqrt{2}}(E^1+iE^2)v_3 = \alpha v_2 \qquad \frac{\alpha}{\sqrt{2}}(E^1-iE^2)v_3 = 0$$

$$E^{+1} = \frac{\alpha}{\sqrt{2}}(E^1+iE^2), \quad E^{-1} = \frac{\alpha}{\sqrt{2}}(E^1-iE^2)$$

i.e. act like lowering and raising operators

8.2. Finding correspondence

So E^{+1} and E^{-1} correspond to t_1 and t_3, therefore H^1 corresponds to $t_2=(0)$ (as it should be since it's Cartan generator):

$$E^{+1} \leftrightarrow t_1=(+1) \leftrightarrow v_1 = |Adj; E^{+1}\rangle$$
$$H^1 \leftrightarrow t_2=(0) \leftrightarrow v_2 = |Adj; H^1\rangle$$
$$E^{-1} \leftrightarrow t_3=(-1) \leftrightarrow v_3 = |Adj; E^{-1}\rangle$$

Physics perspective:

Eigenstates are particles with some charge (different values of charge = different points in root space) and generators are "tools" to change one state to another (i.e. things that describe how these particles interact).

Finally let's find α:

$$[E^{+1}, E^{-1}] = \alpha^2 H^1$$
$$[E^{+1}, E^{-1}] = t_1 H^1$$

$$\Rightarrow \boxed{E^{+1} = \frac{1}{\sqrt{2}}(E^1+iE^2), \quad E^{-1} = \frac{1}{\sqrt{2}}(E^1-iE^2)}$$

raising and lowering operators for SU(2)

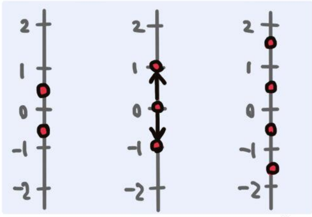
8.3. Arbitrary representations

For any SU(2) repr. there always will be 3 generators H^1, E^+, E^- (and they always act the same way: as raising and lowering operators, i.e. move ↑ or ↓ one position in the root space). Weight vectors are always 1-dimensional, but their number (as the number of eigenvectors) varies with the dimension of a representation: $n = 2j+1$.

$j = \frac{1}{2}$: $n = 2$

$j = 1$: $n = 3$

$j = \frac{3}{2}$: $n = 4$



$j=\frac{1}{2}$ $j=1$ $j=\frac{3}{2}$

So for any repr. of SU(2) there are only 3 options: stay at the state, move up or move down.

9. SU(3)

We will study SU(3) not in the adjoint repr., but in the fundamental repr. (3×3 operators on V^3).
(for convenience)

$\dim SU(n) = n^2 - 1$

$\text{rank}\, SU(n) = n-1$ $\Rightarrow$ Adj. repr.: $(n^2-1)\times(n^2-1)$, Fund. repr.: $n \times n$

$\dim SO(n) = \frac{n(n-1)}{2}$ (because $A = -A^T$)

$\dim SU(3) = 8$

$\text{rank}\, SU(3) = 2$

$f_{123} = 1 \qquad f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2} \qquad f_{458} = f_{678} = \frac{\sqrt{3}}{2}$

$\boxed{T^a = \frac{1}{2}\lambda^a}$, λ^a - Gell-Mann matrices, $a = 1, \dots, 8$

SU(3) generators in fund. repr.

$\lambda^3 = \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix}$ $\lambda^1 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix}$ $\lambda^4 = \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix}$ $\lambda^6 = \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix}$

$\lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&-2 \end{pmatrix}$ $\lambda^2 = \begin{pmatrix} 0&-i&0\\i&0&0\\0&0&0 \end{pmatrix}$ $\lambda^5 = \begin{pmatrix} 0&0&-i\\0&0&0\\i&0&0 \end{pmatrix}$ $\lambda^7 = \begin{pmatrix} 0&0&0\\0&0&-i\\0&i&0 \end{pmatrix}$

Cartan (because diagonal) non-Cartan

So generators are:

$H^1 = T^3 \quad H^2 = T^8$

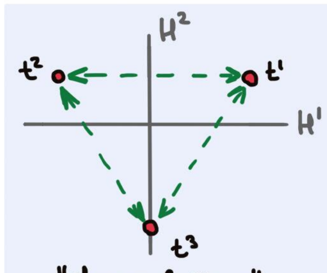
$E^1 = T^1 \quad E^2 = T^2$

$E^3 = T^4 \quad E^4 = T^5$

$E^5 = T^6 \quad E^6 = T^7$

$\Rightarrow \quad t_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix} \quad t_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2\sqrt{3}} \end{pmatrix} \quad t_3 = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$

weight vectors



"charge lattice"

To understand what non-Cartan generators do one can use the general method (as for SU(2)), but we can take a shortcut. We are looking for raising and lowering operators, and $\{E^i\}$ doesn't fit for that, so we will try their lin. combinations (easy to see which one we need):

$A^{\uparrow\downarrow}$:

$\frac{1}{\sqrt{2}}(E^1 + iE^2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix}$ $v_2 \to v_1$ $\quad A^\uparrow v_1 = 0 \quad A^\uparrow v_2 = \frac{1}{\sqrt{2}} v_1 \quad A^\uparrow v_3 = 0$

$\frac{1}{\sqrt{2}}(E^1 - iE^2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&0&0\\1&0&0\\0&0&0 \end{pmatrix}$ $v_1 \to v_2$ $\quad A^\downarrow v_1 = \frac{1}{\sqrt{2}} v_2 \quad A^\downarrow v_2 = 0 \quad A^\downarrow v_3 = 0$

moves $\pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$B^{\uparrow\downarrow}$:

$\frac{1}{\sqrt{2}}(E^3 + iE^4) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix}$ $v_3 \to v_1$ $\quad B^\uparrow v_1 = 0 \quad B^\uparrow v_2 = 0 \quad B^\uparrow v_3 = \frac{1}{\sqrt{2}} v_1$

$\frac{1}{\sqrt{2}}(E^3 - iE^4) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&0&0\\0&0&0\\1&0&0 \end{pmatrix}$ $v_1 \to v_3$ $\quad B^\downarrow v_1 = \frac{1}{\sqrt{2}} v_3 \quad B^\downarrow v_2 = 0 \quad B^\downarrow v_3 = 0$

moves $\pm \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$

$C^{\uparrow\downarrow}$:

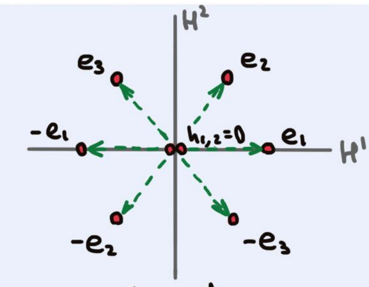
$\frac{1}{\sqrt{2}}(E^5 + iE^6) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix}$ $v_3 \to v_2$ $\quad C^\uparrow v_1 = 0 \quad C^\uparrow v_2 = 0 \quad C^\uparrow v_3 = \frac{1}{\sqrt{2}} v_2$

$\frac{1}{\sqrt{2}}(E^5 - iE^6) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0&0&0\\0&0&0\\0&1&0 \end{pmatrix}$ $v_2 \to v_3$ $\quad C^\downarrow v_1 = 0 \quad C^\downarrow v_2 = \frac{1}{\sqrt{2}} v_3 \quad C^\downarrow v_3 = 0$

moves $\pm \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$

$$\boxed{\begin{aligned} E^{\pm\binom{1}{0}} &= E^{\pm e_1} = \tfrac{1}{\sqrt{2}}\left(E^1 \pm i E^2\right) \\ E^{\pm\binom{1/2}{\sqrt{3}/2}} &= E^{\pm e_2} = \tfrac{1}{\sqrt{2}}\left(E^3 \pm i E^4\right) \\ E^{\pm\binom{-1/2}{\sqrt{3}/2}} &= E^{\pm e_3} = \tfrac{1}{\sqrt{2}}\left(E^5 \pm i E^6\right) \end{aligned}} \implies$$

SU(3) non-Cartan generators



root vectors
(corresponding to generators)

The root space diagram represents the way the state vectors in any repr. transform from one to another.

Because we didn't use the adjoint repr. there isn't 1:1 correspondence between the eigenstates, weight vectors and generators. That's why we have 2 diagrams: one represents the weight vectors, and another the root vectors (and the generators). In the adjoint repr. though it's the same space (e.g. e_2 will move $-e_1 \to e_3$ and $-e_3 \to e_1$).

Physics perspective:

Phys. interactions are described by some Lie group in a particular representation:

- particles are eigenstates of the Cartan generators of the group
- charges are eigenvalues of those eigenstates, i.e. weight vectors (N of charges $= \dim(\text{fund. } D(G))$)
- force-carrying particles are generators, i.e. describe how particles change:
  1) Cartan generators are force-carrying particles that interact with any particle charged under the same group by transforming energy and momentum, but not charge (γ and Z^0).
  2) non-Cartan generators are force-carrying particles that interact with any particle charged under the same group by transforming energy and momentum, and also charge ($W^\pm$, g_i).

Chapter IV
The Lorentz group

1. The Lorentz Algebra.

$$\Lambda < \begin{array}{l} R: \text{rotations in } \mathbb{R}^3 \\ B: \text{boosts (rotations in } \mathbb{R}^{1,1}) \end{array}$$

$$[R_x(\theta_x)]^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_x & \sin\theta_x \\ 0 & 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} \quad [R_y(\theta_y)]^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \quad [R_z(\theta_z)]^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & \sin\theta_z & 0 \\ 0 & -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[B_x(\phi_x)]^\mu{}_\nu = \begin{pmatrix} \mathrm{ch}\phi_x & -\mathrm{sh}\phi_x & 0 & 0 \\ -\mathrm{sh}\phi_x & \mathrm{ch}\phi_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad [B_y(\theta_y)]^\mu{}_\nu = \begin{pmatrix} \mathrm{ch}\phi_y & 0 & -\mathrm{sh}\phi_y & 0 \\ 0 & 1 & 0 & 0 \\ -\mathrm{sh}\phi_y & 0 & \mathrm{ch}\phi_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad [B_z(\theta_z)]^\mu{}_\nu = \begin{pmatrix} \mathrm{ch}\phi_z & 0 & 0 & -\mathrm{sh}\phi_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{sh}\phi_z & 0 & 0 & \mathrm{ch}\phi_z \end{pmatrix}$$

Then we can find generators for this repr.: $[J_i]^\mu{}_\nu = \left[-i \frac{dR_i(\theta_i)}{d\theta_i}\Big|_{\theta_i=0}\right]^\mu{}_\nu$ $\quad [K_i]^\mu{}_\nu = \left[-i \frac{dB_i(\phi_i)}{d\phi_i}\Big|_{\phi_i=0}\right]^\mu{}_\nu$

$$[J_x]^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad [J_y]^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad [J_z]^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[K_x]^\mu{}_\nu = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad [K_y]^\mu{}_\nu = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad [K_z]^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k \\ [J_i, K_j] &= i\epsilon_{ijk} K_k \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k \end{aligned}}$$

interesting fact

Then the general Lorentz transformation: $L = \exp(i\vec{J}\cdot\vec{\theta} + i\vec{K}\cdot\vec{\phi})$, $\vec{J} = (J_x, J_y, J_z)$ $\vec{\theta} = (\theta_x, \theta_y, \theta_z)$, $\vec{K} = (K_x, K_y, K_z)$ $\vec{\phi} = (\phi_x, \phi_y, \phi_z)$

We can write down generators with lower indices:

$$[J_x]_{\mu\nu} = [J_x]^\mu{}_\nu \qquad [K_x]_{\mu\nu} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[J_y]_{\mu\nu} = [J_y]^\mu{}_\nu \qquad [K_y]_{\mu\nu} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[J_z]_{\mu\nu} = [J_z]^\mu{}_\nu \qquad [K_z]_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

All are hermitian, but in this form $[J_i, K_j] = -i\epsilon_{ijk} K_k$

$$\begin{bmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & a & a & a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{pmatrix} = \begin{pmatrix} -a & -a & -a & -a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a & a & a & a \\ a & a & a & a \\ a & a & a & a \\ a & a & a & a \end{pmatrix} = \begin{pmatrix} a & a & a & a \\ -a & -a & -a & -a \\ -a & -a & -a & -a \\ -a & -a & -a & -a \end{pmatrix} \end{bmatrix}$$

Structure of SO(1,3):

$$\boxed{N_i^\pm \stackrel{\text{def}}{=} \frac{1}{2}(J_i \pm i K_i)}, \text{ then } \quad \begin{aligned} [N_i^+, N_j^+] &= i\epsilon_{ijk} N_k^+ \\ [N_i^-, N_j^-] &= i\epsilon_{ijk} N_k^- \\ [N_i^+, N_j^-] &= 0 \end{aligned}$$

So we have 2 copies of SU(2) (for $\{N_i^+\}$ and $\{N_i^-\}$; this is true only for SO(1,3)). So representations will be specified by two values (j, j') and will be made up of matrices of $(2j+1)(2j'+1) \times (2j+1)(2j'+1)$ size.

$j \leftrightarrow \{N_i^+\}$ generators

$j' \leftrightarrow \{N_i^-\}$ generators

② Representations of the Lorentz group.

2.1. $(0,0)$ representation

1×1 matrices (i.e. scalars) acting on 1-dim vectors (i.e. scalars) (describes things that don't change under Lorentz trs.)

2.2. $(\frac{1}{2},0)$ representation

Only N_i^+ are used: $N_i^- = \frac{1}{2}(J_i - iK_i) = 0 \Rightarrow J_i = iK_i \Rightarrow N_i^+ = \frac{1}{2}(J_i + iK_i) = iK_i$

$$\left.\begin{matrix} N_i^+ = iK_i \\ N_i^+ = \frac{1}{2}\sigma_i \end{matrix}\right| \Rightarrow \boxed{\begin{matrix} J_i = \frac{1}{2}\sigma_i \\ K_i = -\frac{i}{2}\sigma_i \end{matrix}} \qquad \boxed{\begin{matrix} R(\vec{\theta}) = \exp\left(\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}\right) \\ R(\vec{\phi}) = \exp\left(\frac{1}{2}\vec{\phi}\cdot\vec{\sigma}\right) \end{matrix}} \, , \ \vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$$

Lorentz tr. for $(\frac{1}{2},0)$ — Pauli vector

For example:

$$R_x(\theta_x) = \exp\left(\frac{i}{2}\theta_x\sigma_x\right) = I + \frac{i}{2}\theta_x\sigma_x + \frac{1}{2}\left(\frac{i}{2}\theta_x\sigma_x\right)^2 + \ldots = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{i}{2}\theta_x\begin{pmatrix}0&1\\1&0\end{pmatrix} - \frac{1}{2}\left(\frac{\theta_x}{2}\right)^2\begin{pmatrix}1&0\\0&1\end{pmatrix} + \ldots =$$

$$= \begin{pmatrix} 1-\frac{1}{2}\left(\frac{\theta_x}{2}\right)^2+\ldots & i\frac{\theta_x}{2}+\ldots \\ i\frac{\theta_x}{2}+\ldots & 1-\frac{1}{2}\left(\frac{\theta_x}{2}\right)^2+\ldots \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta_x}{2} & i\sin\frac{\theta_x}{2} \\ i\sin\frac{\theta_x}{2} & \cos\frac{\theta_x}{2} \end{pmatrix} \quad \ldots$$

$$R_x(\theta_x) = \begin{pmatrix} \cos\frac{\theta_x}{2} & i\sin\frac{\theta_x}{2} \\ i\sin\frac{\theta_x}{2} & \cos\frac{\theta_x}{2} \end{pmatrix} \quad R_y(\theta_y) = \begin{pmatrix} \cos\frac{\theta_y}{2} & \sin\frac{\theta_y}{2} \\ -\sin\frac{\theta_y}{2} & \cos\frac{\theta_y}{2} \end{pmatrix} \quad R_z(\theta_z) = \begin{pmatrix} \exp(i\frac{\theta_z}{2}) & 0 \\ 0 & \exp(i\frac{\theta_z}{2}) \end{pmatrix}$$

$$B_x(\phi_x) = \begin{pmatrix} ch\frac{\phi_x}{2} & sh\frac{\phi_x}{2} \\ sh\frac{\phi_x}{2} & ch\frac{\phi_x}{2} \end{pmatrix} \quad B_y(\phi_y) = \begin{pmatrix} ch\frac{\phi_y}{2} & -i\,sh\frac{\phi_y}{2} \\ i\,sh\frac{\phi_y}{2} & ch\frac{\phi_y}{2} \end{pmatrix} \quad B_z(\theta_z) = \begin{pmatrix} \exp(\frac{\phi_z}{2}) & 0 \\ 0 & \exp(\frac{\phi_z}{2}) \end{pmatrix}$$

2.3. $(0,\frac{1}{2})$ representation

Only N_i^- are used: $N_i^+ = \frac{1}{2}(J_i + iK_i) = 0 \Rightarrow J_i = -iK_i \Rightarrow N_i^- = \frac{1}{2}(J_i - iK_i) = -iK_i$

$$\left.\begin{matrix} N_i^- = -iK_i \\ N_i^- = \frac{1}{2}\sigma_i \end{matrix}\right| \Rightarrow \boxed{\begin{matrix} J_i = \frac{1}{2}\sigma_i \\ K_i = \frac{i}{2}\sigma_i \end{matrix}} \qquad \boxed{\begin{matrix} R(\vec{\theta}) = \exp\left(\frac{i}{2}\vec{\theta}\cdot\vec{\sigma}\right) \\ R(\vec{\phi}) = \exp\left(-\frac{1}{2}\vec{\phi}\cdot\vec{\sigma}\right) \end{matrix}}$$

Lorentz tr. for $(0,\frac{1}{2})$

$$R_x(\theta_x) = \begin{pmatrix} \cos\frac{\theta_x}{2} & i\sin\frac{\theta_x}{2} \\ i\sin\frac{\theta_x}{2} & \cos\frac{\theta_x}{2} \end{pmatrix} \quad R_y(\theta_y) = \begin{pmatrix} \cos\frac{\theta_y}{2} & \sin\frac{\theta_y}{2} \\ -\sin\frac{\theta_y}{2} & \cos\frac{\theta_y}{2} \end{pmatrix} \quad R_z(\theta_z) = \begin{pmatrix} \exp(i\frac{\theta_z}{2}) & 0 \\ 0 & \exp(i\frac{\theta_z}{2}) \end{pmatrix}$$

$$B_x(\phi_x) = \begin{pmatrix} ch\frac{\phi_x}{2} & -sh\frac{\phi_x}{2} \\ -sh\frac{\phi_x}{2} & ch\frac{\phi_x}{2} \end{pmatrix} \quad B_y(\phi_y) = \begin{pmatrix} ch\frac{\phi_y}{2} & i\,sh\frac{\phi_y}{2} \\ -i\,sh\frac{\phi_y}{2} & ch\frac{\phi_y}{2} \end{pmatrix} \quad B_z(\theta_z) = \begin{pmatrix} \exp(-\frac{\phi_z}{2}) & 0 \\ 0 & \exp(-\frac{\phi_z}{2}) \end{pmatrix}$$

2.4. Properties of Pauli matrices

$$\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix} \quad \sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix} , \quad \begin{matrix}\det\sigma_i = -1 \\ \operatorname{tr}\sigma_i = 0\end{matrix} \quad \begin{matrix}\{1,-1\}: \text{eigenvalues} \\ \sigma_i^\dagger = \sigma_i \ (\text{hermitian})\end{matrix}$$

1) $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -i\,\sigma_1\sigma_2\sigma_3 = I$

2) $\vec{\sigma} \overset{\text{def}}{=} \sigma_1\hat{x} + \sigma_2\hat{y} + \sigma_3\hat{z}$ (Pauli vector) $\Rightarrow \vec{a}\cdot\vec{\sigma} = a_i\sigma_i$, $\vec{a} = (a_1, a_2, a_3)$

3) $$\left.\begin{matrix} [\sigma_i,\sigma_j] = 2i\,\epsilon_{ijk}\sigma_k \\ \{\sigma_i,\sigma_j\} = 2\,\delta_{ij} I \end{matrix}\right| \Rightarrow \sigma_i\sigma_j = \delta_{ij} I + i\,\epsilon_{ijk}\sigma_k \Rightarrow \begin{matrix} \sigma_1\sigma_2 = i\sigma_3 \\ \sigma_2\sigma_3 = i\sigma_1 \\ \sigma_3\sigma_1 = -i\sigma_2 \end{matrix}$$

4) $(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = a_i\sigma_i\, b_j\sigma_j = a_i b_j(\delta_{ij}I + i\epsilon_{ijk}\sigma_k) = a_i b_i I + i\sigma_k\,\epsilon_{ijk}a_i b_j = (\vec{a}\cdot\vec{b})I + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b})$

5) $(\hat{n}\cdot\vec{\sigma})^2 = n_1^2\sigma_1^2 + n_2^2\sigma_2^2 + n_3^2\sigma_3^2 + n_1 n_2\underbrace{(\sigma_1\sigma_2+\sigma_2\sigma_1)}_{=0} + n_2 n_3\underbrace{(\sigma_2\sigma_3+\sigma_3\sigma_2)}_{=0} + n_1 n_3\underbrace{(\sigma_1\sigma_3+\sigma_3\sigma_1)}_{=0} = I$

unit vector $(n_1^2+n_2^2+n_3^2 = 1)$

$\Downarrow$

$(\hat{n}\cdot\vec{\sigma})^{2m} = I \ \forall m\in\mathbb{N}$

$(\hat{n}\cdot\vec{\sigma})^{2m+1} = \hat{n}\cdot\vec{\sigma} \ \forall m\in\mathbb{N}$

6) $\exp(i\,a(\hat{n}\cdot\vec{\sigma})) = I\cos a + i(\hat{n}\cdot\vec{\sigma})\sin a$, $a\in\mathbb{R}$, $\hat{n}$ – unit vector (from previous properties)

*) $(i\sigma_2)\,\vec{\sigma}^*(-i\sigma_2) = \sigma_2\sigma_1^*\sigma_2\,\hat{x} + \sigma_2\sigma_2^*\sigma_2\,\hat{y} + \sigma_2\sigma_3^*\sigma_2\,\hat{z} = \ldots = -\vec{\sigma}$

2.5. Relationship between $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$

$(\frac{1}{2},0)$: left-handed spinor repr. : $\psi_L \xrightarrow{R} \psi'_L = \exp(\frac{i}{2}\vec{\theta}\cdot\vec{\sigma})\psi_L$ $\qquad \psi_L \xrightarrow{B} \psi'_L = \exp(\frac{1}{2}\vec{\phi}\cdot\vec{\sigma})\psi_L$

$(0,\frac{1}{2})$: right-handed spinor repr. : $\psi_R \xrightarrow{R} \psi'_R = \exp(\frac{i}{2}\vec{\theta}\cdot\vec{\sigma})\psi_R$ $\qquad \psi_R \xrightarrow{B} \psi'_R = \exp(-\frac{1}{2}\vec{\phi}\cdot\vec{\sigma})\psi_R$

Let's consider spinors $\bar{\psi}_L \overset{def}{=} i\sigma^2\psi_L^*$ and $\bar{\psi}_R \overset{def}{=} i\sigma^2\psi_R^*$ and their change under Lorentz tr. :

index, not square!

$$R:\ \bar{\psi}_L \to \bar{\psi}'_L = i\sigma^2(\psi'_L)^* = i\sigma^2\left(\exp(\tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma})\psi_L\right)^* = i\sigma^2\exp(-\tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma}^*)\psi_L^* = i\sigma^2\exp(-\tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma}^*)(-i\sigma^2)(i\sigma^2)\psi_L^* =$$

$$= \exp(\tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma})\, i\sigma^2\psi_L^* = \exp(\tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma})\bar{\psi}_L$$

$$\bar{\psi}_R \to \bar{\psi}'_R = i\sigma^2(\psi'_R)^* = \dots = \exp(\tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma})\bar{\psi}_R$$

$$B:\ \bar{\psi}_L \to \bar{\psi}'_L = i\sigma^2(\psi'_L)^* = i\sigma^2\left(\exp(\tfrac{1}{2}\vec{\phi}\cdot\vec{\sigma})\psi_L\right)^* = i\sigma^2\exp(\tfrac{1}{2}\vec{\phi}\cdot\vec{\sigma}^*)\psi_L^* = i\sigma^2\exp(\tfrac{1}{2}\vec{\phi}\cdot\vec{\sigma}^*)(-i\sigma^2)(i\sigma^2)\psi_L^* =$$

$$= \exp(-\tfrac{1}{2}\vec{\phi}\cdot\vec{\sigma})\, i\sigma^2\psi_L^* = \exp(-\tfrac{1}{2}\vec{\phi}\cdot\vec{\sigma})\bar{\psi}_L$$

$$\bar{\psi}_R \to \bar{\psi}'_R = i\sigma^2(\psi'_R)^* = \dots = \exp(\tfrac{1}{2}\vec{\phi}\cdot\vec{\sigma})\bar{\psi}_R$$

So $\bar{\psi}_L$ transforms as ψ_R, and $\bar{\psi}_R$ transforms as ψ_L :

$$\psi_L = \bar{\psi}_R = i\sigma^2\psi_R^*$$
$$i\sigma^2\psi_L^* = -i\sigma^2(i\sigma^2\psi_R^*)^*$$
$$\bar{\psi}_L = \psi_R$$

$(i\sigma^2)^{-1} = -i\sigma^2$

$\Downarrow$

$L\to R:\ \psi \to i\sigma^2\psi^*$

$R\to L:\ \psi \to -i\sigma^2\psi^*$

$\Rightarrow$ Left-handed and right-handed repr. are conjugates of each other and any spinor in either repr. can be written as a spinor in the other repr.

2.6. $(\frac{1}{2},\frac{1}{2})$ representation

Each state will have 2 indices: for left-handed SU(2) and right-handed SU(2) (and they won't interfere with each other, because $[N_i^+, N_i^-]=0$).

$v^{\dot{a}b}$: state $\quad \dot{a}$ – right-handed indices (N_i^-) $\quad b$ – left-handed indices (N_i^+) $\Rightarrow$ 4 components $\nearrow$ 2x2 matrix $\searrow$ 4x1 vector

$$v^{\dot{a}b} = \begin{pmatrix} \uparrow\uparrow & \uparrow\downarrow \\ \downarrow\uparrow & \downarrow\downarrow \end{pmatrix},\quad |\uparrow\uparrow\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\quad |\uparrow\downarrow\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\quad |\downarrow\uparrow\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\quad |\downarrow\downarrow\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$\{N_i^+\}$ and $\{N_i^-\}$ are hermitian $\Rightarrow$ consider $v^{\dot{a}b}$ is hermitian :

$$v^{\dot{a}b} = \begin{pmatrix} v^0+v^3 & v^1-iv^2 \\ v^1+iv^2 & v^0-v^3 \end{pmatrix} = v^0\begin{pmatrix}1&0\\0&1\end{pmatrix} + v^1\begin{pmatrix}0&1\\1&0\end{pmatrix} + v^2\begin{pmatrix}0&-i\\i&0\end{pmatrix} + v^3\begin{pmatrix}1&0\\0&-1\end{pmatrix} = (v^\mu\sigma_\mu)^{\dot{a}b}$$

Therefore:

$$v^0 = \tfrac{1}{2}(v^{\uparrow\uparrow}+v^{\downarrow\downarrow})\qquad v^1 = \tfrac{1}{2}(v^{\uparrow\downarrow}+v^{\downarrow\uparrow})\qquad v^2 = \tfrac{i}{2}(v^{\uparrow\downarrow}-v^{\downarrow\uparrow})\qquad v^3 = \tfrac{1}{2}(v^{\uparrow\uparrow}-v^{\downarrow\downarrow})$$

General Lorentz transformation for $v^{\dot{a}b}$:

$$v^{\dot{a}b} \to v'^{\dot{c}d} = \left[\exp(\tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} - \tfrac{1}{2}\vec{\phi}\cdot\vec{\sigma})\right]^{\dot{c}}_{\dot{a}}\left[\exp(\tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \tfrac{1}{2}\vec{\phi}\cdot\vec{\sigma})\right]^{d}_{b} v^{\dot{a}b} \approx (I + \tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} - \tfrac{1}{2}\vec{\phi}\cdot\vec{\sigma})^{\dot{c}}_{\dot{a}}\,\cdot$$

$$\cdot(I + \tfrac{i}{2}\vec{\theta}\cdot\vec{\sigma} + \tfrac{1}{2}\vec{\phi}\cdot\vec{\sigma})^{d}_{b} v^{\dot{a}b} = \dots \approx \begin{pmatrix}A & B\\ C & D\end{pmatrix}$$

$$v^0 \to v^{0'} = \tfrac{1}{2}(A+D) = v^0 + i(\theta_x v^1 + \theta_y v^2 + \theta_z v^3)$$
$$v^1 \to v^{1'} = \tfrac{1}{2}(B+C) = v^1 + i(\theta_x v^0 + \phi_z v^2 - \phi_y v^3)$$
$$v^2 \to v^{2'} = \tfrac{i}{2}(B-C) = v^2 + i(\theta_y v^0 - \phi_z v^1 + \phi_x v^3)$$
$$v^3 \to v^{3'} = \tfrac{1}{2}(A-D) = v^3 + i(\theta_z v^0 + \phi_y v^1 - \phi_x v^2)$$

$$\Rightarrow \begin{pmatrix}v^{0'}\\v^{1'}\\v^{2'}\\v^{3'}\end{pmatrix} = \begin{pmatrix}v^0\\v^1\\v^2\\v^3\end{pmatrix} + \begin{pmatrix}0 & i\theta_x & i\theta_y & i\theta_z\\ i\theta_x & 0 & i\phi_z & -i\phi_y\\ i\theta_y & -i\phi_z & 0 & i\phi_x\\ i\theta_z & i\phi_y & -i\phi_x & 0\end{pmatrix}\begin{pmatrix}v^0\\v^1\\v^2\\v^3\end{pmatrix}$$

i.e Lorentz tr. for vectors

So $(\frac{1}{2},\frac{1}{2})$ repr. is a vector repr. of SO(1,3).

$\det(v^{\dot{a}b}) = (v^0)^2 - (v^1)^2 - (v^2)^2 - (v^3)^2 = inv$ under SO(1,3) (since $\det\Lambda = +1$)
(as expected and demanded by SR)

③ Spinor indices.

Let's consider a spinor repr. of SO(1,3): $V^{\dot{a}b} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$, $\dot{a}$ – right-handed indices, b – left-handed indices

3.1. Dotted and undotted indices

$$(V^{\dot{a}b})^T = V^{b\dot{a}} = \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} = (V^{\dot{a}b})^*$$ as expected because $V^{\dot{a}b}$ is hermitian

So dotted and undotted indices can be interchanged with complex conjugation. However we know it's not all that's involved in switching between L and R spinor representations: we also need $i\sigma^2$.

3.2. Lowered and raised indices

$L \to R: \psi \to i\sigma^2\psi$

$R \to L: \psi \to -i\sigma^2\psi$

$(i\sigma^2)^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and let's act on it with $\{R_i\}$ and $\{B_i\}$:

$$R_x: (i\sigma^2)'^{ac} = [R_x]^a_b [R_x]^c_d (i\sigma^2)^{bd} = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \ldots = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$B_x: (i\sigma^2)'^{ac} = [B_x]^a_b [B_x]^c_d (i\sigma^2)^{bd} = \begin{pmatrix} ch\frac{\varphi}{2} & -sh\frac{\varphi}{2} \\ -sh\frac{\varphi}{2} & ch\frac{\varphi}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} ch\frac{\varphi}{2} & -sh\frac{\varphi}{2} \\ -sh\frac{\varphi}{2} & ch\frac{\varphi}{2} \end{pmatrix} = \ldots = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

...

So $i\sigma^2$ = inv under all Lorentz trs. => $\boxed{(i\sigma^2)^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{ab}}$: spinor metric (to raise and lower spinor indices)

$(i\sigma^2)^{ab} = \left((i\sigma^2)^{ab}\right)^*$

$(i\sigma^2)(-i\sigma^2) = I$

$\Rightarrow (i\sigma^2)^{ab} = (i\sigma^2)^{\dot{a}\dot{b}} = -(i\sigma^2)_{ab} = -(i\sigma^2)_{\dot{a}\dot{b}}$ ← convention

i.e. dotted and undotted indices are same for $i\sigma^2$

If we follow this convention carefully we don't need to worry about remembering whether to use $i\sigma^2$ or $(-i\sigma^2)$ under $L \rightleftarrows R$ change. Also we will consider ψ_L to have lowered and undotted indices (and ψ_R therefore will have raised and dotted):

$$\psi_L = \psi_a = \begin{pmatrix} \psi_{L1} \\ \psi_{L2} \end{pmatrix} \qquad \psi_R = \psi^{\dot{a}} = (i\sigma^2)^{\dot{a}\dot{b}}\psi_{\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \psi^*_{L1} \\ \psi^*_{L2} \end{pmatrix} = \begin{pmatrix} \psi^*_{L2} \\ -\psi^*_{L1} \end{pmatrix} = \begin{pmatrix} \psi_{R1} \\ \psi_{R2} \end{pmatrix}$$

$$\psi^*_L = \psi_{\dot{a}} = \begin{pmatrix} \psi^*_{L1} \\ \psi^*_{L2} \end{pmatrix} \qquad \psi^*_R = \psi^a = \begin{pmatrix} \psi_{L2} \\ -\psi_{L1} \end{pmatrix} = \begin{pmatrix} \psi^*_{R1} \\ \psi^*_{R2} \end{pmatrix}$$

$$\begin{array}{ccc} \psi_L = \psi_a & \xleftrightarrow{i\sigma^2} & \psi^a = \psi^*_R \\ {*}\updownarrow & & \updownarrow{*} \\ \psi^*_L = \psi_{\dot{a}} & \xleftrightarrow{i\sigma^2} & \psi^{\dot{a}} = \psi_R \end{array}$$

dotted ↔ undotted : complex conjugation

raised ↔ lowered : using spinor metric

4. Lorentz Algebra for arbitrary dimensions.

4.1. Vector representations

The pattern for the form of R_i and B_i in higher dimensions is obvious (see the $SO(1,3)$ case), and after that the derivation of J_i and K_i is pretty straight-forward. Then if we lower indices for J_i and K_i we will get a basis set of 2-component hermitian matrices.

$SO(1,n)$: $\frac{n(n+1)}{2}$ generators $=$ n boosts $+$ $\frac{n(n-1)}{2}$ rotations

Instead of using one index for counting generators, we will use two antisymmetric indices.
So $J^{\mu\nu} = -J^{\nu\mu}$ are generators ($J^{\mu 0}$: boosts, J^{ij}: rotations) with matrix el.s $(J^{\mu\nu})^{ab} = -(J^{\mu\nu})^{ba}$
$(n+1)\times(n+1)$

$$\boxed{(J^{\mu\nu})^{ab} = -i\left(\eta^{\mu a}\eta^{\nu b} - \eta^{\nu a}\eta^{\mu b}\right)} \qquad \boxed{[J^{\mu\nu}, J^{\rho\lambda}] = i\left(\eta^{\lambda\mu}J^{\rho\nu} - \eta^{\lambda\nu}J^{\rho\mu} - \eta^{\rho\mu}J^{\lambda\nu} + \eta^{\rho\nu}J^{\lambda\mu}\right)}$$

generators of $SO(1,n)$

Example:

Generators of rotations in $\mathbb{R}^n$: $\vec{L} = \vec{r}\times\vec{p} = -i\vec{r}\times\frac{\partial}{\partial\vec{r}}$

$$\exp\left(ia\left(-i\frac{\partial}{\partial x}\right)\right)f(x) = \exp\left(a\frac{\partial}{\partial x}\right)f(x) = \left(1 + a\frac{\partial}{\partial x} + \frac{1}{2}a^2\frac{\partial^2}{\partial x^2} + \dots\right)f(x) = f(x+a)$$

i.e. $\left(-i\frac{\partial}{\partial x}\right)$ – generator for spacial translations

Then $L^i = -i\,\epsilon_{ijk}x^j\partial^k$ or:

$$L^{ij} = -i\left(x^i\partial^j - x^j\partial^i\right) \rightarrow L^{\mu\nu} = -i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right) \quad (\text{rotations in } \mathbb{R}^{1,n})$$

$$[L^{\mu\nu}, L^{\rho\lambda}] = \dots = i\left(\eta^{\lambda\mu}L^{\rho\nu} - \eta^{\lambda\nu}L^{\rho\mu} - \eta^{\rho\mu}L^{\lambda\nu} + \eta^{\rho\nu}L^{\lambda\mu}\right)$$

4.2. Spinor representations

A separation of Lorentz group into two copies of $SU(2)$ works only for $SO(1,3)$. For an arbitrary dimension there is a general approach for spinor repr. (using the framework of geometric algebra):

Clifford Algebra: $\{\gamma^\mu \mid \mu = 0,\dots,n-1\}$: $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}I$

Now let's consider antisymmetric matrices $\boxed{S^{\mu\nu} \stackrel{def}{=} \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$ with components $(S^{\mu\nu})^{ab}$:

1) $S^{\mu\nu} = \frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{i}{4}(\gamma^\mu\gamma^\nu - (-2\eta^{\mu\nu}I - \gamma^\mu\gamma^\nu)) = \frac{i}{2}(\gamma^\mu\gamma^\nu + \eta^{\mu\nu}I)$

2) $[S^{\mu\nu}, \gamma^\rho] = \dots = i\left(\gamma^\nu\eta^{\rho\mu} - \gamma^\mu\eta^{\nu\rho}\right)$

3) $[S^{\mu\nu}, S^{\rho\lambda}] = \dots = -\frac{i}{2}\left(\gamma^\nu\gamma^\lambda\eta^{\rho\mu} - \gamma^\mu\gamma^\lambda\eta^{\nu\rho} - \gamma^\rho\gamma^\nu\eta^{\lambda\mu} - \gamma^\rho\gamma^\mu\eta^{\nu\lambda}\right) =$

$= i\left(\eta^{\lambda\mu}S^{\rho\nu} - \eta^{\nu\lambda}S^{\rho\mu} - \eta^{\rho\mu}S^{\lambda\nu} - \eta^{\nu\rho}S^{\lambda\mu}\right)$ (i.e. define Lorentz algebra)

So as long as $\{\gamma^\mu\}$ satisfy the Clifford algebra, we can build the Lorentz algebra with $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.