Logic and Language: Exercise (Week 5)

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1 LG: continuation semantics

1.1

For each term, we compute the ILL type by first observing the polarity of the sequent and then using the table to interpret complex types.

$$
\begin{aligned}
\lceil \textbf{some} \rceil &= \lceil np/n \rceil^\perp && \{\textit{Negative Hypothesis}\} \\
&= (\lceil np \rceil^\perp \otimes \lceil n \rceil)^\perp && \{\lceil A/B \rceil \textit{ with } A \textit{ and } B \textit{ positive}\} \\
&= (np^\perp \otimes n)^\perp && \{np \textit{ and } n \textit{ positive}\}
\end{aligned}
$$

$$
\begin{aligned}
\lceil \textbf{popular} \rceil &= \lceil n/n \rceil^\perp && \{\textit{Negative Hypothesis}\} \\
&= (\lceil n \rceil^\perp \otimes \lceil n \rceil)^\perp && \{\lceil A/B \rceil \textit{ with } A \textit{ and } B \textit{ positive}\} \\
&= (n^\perp \otimes n)^\perp && \{n \textit{ positive}\}
\end{aligned}
$$

$$
\begin{aligned}
\lceil \textbf{saint} \rceil &= \lceil n \rceil && \{\textit{Positive Hypothesis}\} \\
&= n && \{n \textit{ positive}\}
\end{aligned}
$$

$$
\begin{aligned}
\lceil \textbf{arrived} \rceil &= \lceil np \backslash s \rceil^\perp && \{\textit{Negative Hypothesis}\} \\
&= (\lceil np \rceil \otimes \lceil s \rceil)^\perp && \{\lceil B \backslash A \rceil \textit{ with } B \textit{ positive and } A \textit{ negative}\} \\
&= (np \otimes s^\perp)^\perp && \{np \textit{ positive, } s \textit{ negative}\}
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= \lceil s \rceil \\
&= s^\perp && \{s \textit{ negative}\}
\end{aligned}
$$

$$
\begin{aligned}
z &= \lceil np \rceil \\
&= np && \{np \textit{ positive}\}
\end{aligned}
$$

$$
\begin{aligned}
y &= \lceil n \rceil \\
&= n && \{n \textit{ positive}\}
\end{aligned}
$$

The ILL types then are:

TERM	TYPE
$\lceil\mathbf{some}\rceil$	$(np^{\perp} \otimes n)^{\perp}$
$\lceil\mathbf{popular}\rceil$	$(n^{\perp} \otimes n)^{\perp}$
$\lceil\mathbf{saint}\rceil$	n
$\lceil\mathbf{arrived}\rceil$	$(np \otimes s^{\perp})^{\perp}$
α	$s^{\perp}$
z	np
y	n

1.2

SOURCE TYPE	CONSTANT	$\lceil . \rceil^{\ell}$
n/n	popular	$\lambda\langle c, y\rangle.(c\ (\lambda z.\ \wedge (y\ z)\ (\textsc{popular}\ z)))$

1.3

1. We compute the interpretation below:

$$
\dfrac{
\dfrac{\beta_0}{\boxed{\beta_0 : np} \vdash np}\ \mathit{CoAx}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\gamma_0}{\boxed{\gamma_0 : n} \vdash n}\ \mathit{CoAx}
\qquad
\dfrac{
\dfrac{
\dfrac{\alpha_1}{\boxed{\alpha_1 : n} \vdash n}\ \mathit{CoAx}
}{\lceil saint\rceil\ \alpha_1 : n \vdash \alpha_1 : n}\leftharpoonup
}{n \vdash \boxed{\lambda\alpha_1.(\lceil saint\rceil\ \alpha_1) : n}}\rightharpoonup
}{\boxed{\langle\gamma_0, \lambda\alpha_1.(\lceil saint\rceil\ \alpha_1)\rangle : n/n} \vdash \gamma_0 : n \cdot / \cdot n}\ /L
}{(\lceil popular\rceil\langle\gamma_0, \lambda\alpha_1.(\lceil saint\rceil\ \alpha_1)\rangle) : n/n \vdash \gamma_0 : n \cdot / \cdot n}\leftharpoonup
}{(n/n) \cdot \otimes \cdot n \vdash \boxed{\lambda\gamma_0.(\lceil popular\rceil\langle\gamma_0, \lambda\alpha_1.(\lceil saint\rceil\ \alpha_1)\rangle) : n}}\rightharpoonup
}{\boxed{\langle\beta_0, \lambda\gamma_0.(\lceil popular\rceil\langle\gamma_0, \lambda\alpha_1.(\lceil saint\rceil\ \alpha_1)\rangle)\rangle : np/n}}\ /L
$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\vdots}{\boxed{\langle\beta_0, \lambda\gamma_0.(\lceil popular\rceil\langle\gamma_0, \lambda\alpha_1.(\lceil saint\rceil\ \alpha_1)\rangle)\rangle : np/n}}
}{(\lceil some\rceil\langle\beta_0, \lambda\gamma_0.(\lceil popular\rceil\langle\gamma_0, \lambda\alpha_1.(\lceil saint\rceil\ \alpha_1)\rangle)\rangle) : np/n \vdash \beta_0 : np \cdot / \cdot ((n/n) \cdot \otimes \cdot n)}\leftharpoonup
}{(np/n) \cdot \otimes \cdot ((n/n) \cdot \otimes \cdot n) \vdash \boxed{\lambda\beta_0.(\lceil some\rceil\langle\beta_0, \lambda\gamma_0.(\lceil popular\rceil\langle\gamma_0, \lambda\alpha_1(\lceil saint\rceil\ \alpha_1)\rangle)\rangle) : np}}\rightharpoonup
\qquad
\dfrac{\alpha_0}{\boxed{\alpha_0 : s \vdash s}}\ \mathit{CoAx}
}{\boxed{(\lceil arrived\rceil\langle\beta_0.(\lceil some\rceil\langle\beta_0, \lambda\gamma_0.(\lceil popular\rceil\langle\gamma_0, \lambda\alpha_1.(\lceil saint\rceil\alpha_1)\rangle)\rangle), \alpha_0\rangle) : np\backslash s \vdash \alpha_0 : ((np/n) \cdot \otimes \cdot ((n/n) \cdot \otimes \cdot n)) \cdot \backslash \cdot s}}\ \backslash L
$$

$$
\dfrac{
\dfrac{\vdots}{(\lceil arrived\rceil\langle\beta_0.(\lceil some\rceil\langle\beta_0, \lambda\gamma_0.(\lceil popular\rceil\langle\gamma_0, \lambda\alpha_1.(\lceil saint\rceil\alpha_1)\rangle)\rangle), \alpha_0\rangle) : np\backslash s \vdash \alpha_0 : ((np/n) \cdot \otimes \cdot ((n/n) \otimes n)) \cdot \backslash \cdot s}\leftharpoonup
}{(np/n) \cdot \otimes \cdot ((n/n) \cdot \otimes \cdot n)) \cdot \otimes \cdot (np\backslash s) \vdash \boxed{\lambda\alpha_0.(\lceil arrived\rceil\langle\lambda\beta_0.\lceil some\rceil\langle\beta_0, \lambda\gamma_0.(\lceil popular\rceil\langle\gamma_0, \lambda\alpha_1.(\lceil saint\rceil\ \alpha_1)\rangle)\rangle), \alpha_0\rangle) : s}}\rightharpoonup
$$

2. The adjucted $\cdot^{\ell}$ translations are the following:

$$
\begin{aligned}
\lceil\mathbf{some}\rceil^{\ell} &= \lambda\langle x, k\rangle.(\exists\lambda z.(\wedge\ (k\ \lambda\theta.(\theta\ z))\ (x\ z))) \\
\lceil\mathbf{popular}\rceil^{\ell} &= \lambda\langle c, k\rangle.(c\ (\lambda z.(\wedge\ (\mathrm{POPULAR}\ z)\ (k\ \lambda\theta.(\theta\ z))))) \\
\lceil\mathbf{saint}\rceil^{\ell} &= \lambda c.(c\ \mathrm{SAINT}) \\
\lceil\mathbf{arrived}\rceil^{\ell} &= \lambda\langle k, c\rangle.(k\ \lambda z.(c\ (\mathrm{ARRIVED}\ z)))
\end{aligned}
$$

We can now corroborate the α-equivalence of the two $\cdot^\ell$ translations:

$$
\begin{aligned}
\lceil\ddagger\rceil^\ell = {} & \lambda a_0.(\underline{\lambda\langle k,c\rangle.(k\ \lambda z.(c\ (\text{ARRIVED}\ z)))} \\
& \quad \langle\lambda\beta_0.(\lambda\langle x,k\rangle.(\exists\lambda z.(\wedge\ (k\ \lambda\theta.(\theta\ z))\ (x\ z)))\ \langle\beta_0,\lambda\gamma_0.(\\
& \quad\quad \lambda\langle c,k\rangle.(c\ (\lambda z.(\wedge\ (\text{POP}\ z)\ (k\ \lambda\theta.(\theta\ z)))))\ \langle\gamma_0,\lambda a_1.(\lambda c.(c\ \text{SAINT})\ a_1)\rangle)\rangle), a_0\rangle) \\
\to_\beta^* {} & \lambda a_0.(\underline{\lambda\langle x,k\rangle.(\exists\lambda z.(\wedge\ (k\ \lambda\theta.(\theta\ z))\ (x\ z)))}\ \langle\lambda z.(a_0\ (\text{ARRIVED}\ z)),\lambda\gamma_0.(\\
& \quad \lambda\langle c,k\rangle.(c\ (\lambda z.(\wedge\ (\text{POPULAR}\ z)\ (k\ \lambda\theta.(\theta\ z)))))\ \langle\gamma_0,\lambda a_1.(\lambda c.(c\ \text{SAINT})\ a_1)\rangle)\rangle) \\
\to_\beta^* {} & \lambda a_0.(\exists\lambda z.(\wedge\ (\underline{\lambda\langle c,k\rangle.(c\ (\lambda z.(\wedge\ (\text{POPULAR}\ z)\ (k\ \lambda\theta.(\theta\ z)))))} \\
& \quad \langle\lambda\theta.(\theta\ z),\lambda a_1.(\lambda c.(c\ \text{SAINT})\ a_1)\rangle)\ (a_0\ (\text{ARRIVED}\ z))) \\
\to_\beta^* {} & \lambda a_0.(\exists\lambda z.(\wedge\ (\underline{\lambda\theta.(\theta\ z)}\ (\lambda z.(\wedge\ (\text{POPULAR}\ z)\ (\underline{\lambda c.(c\ \text{SAINT})}\ \underline{(\lambda\theta.(\theta\ z)}))))))\ (a_0\ (\text{ARRIVED}\ z))) \\
\to_\beta^* {} & \lambda a_0.(\exists\lambda z.(\wedge\ (\wedge\ (\text{POPULAR}\ z)\ (\text{SAINT}\ z))\ (a_0\ (\text{ARRIVED}\ z))) \\
\to_\alpha {} & \lambda c.(\exists\lambda z.(\wedge\ (\wedge\ (\text{POPULAR}\ z)\ (\text{SAINT}\ z))\ (c\ (\text{ARRIVED}\ z))) \\
\to_\alpha {} & \lambda c.(\exists\lambda x.(\wedge\ (\wedge\ (\text{POPULAR}\ x)\ (\text{SAINT}\ x))\ (c\ (\text{ARRIVED}\ x))) \\
= {} & \lceil\dagger\rceil^\ell
\end{aligned}
$$

2 Pregroups

2.1

(4)

$$(\rightarrow) \qquad 1^\ell \xrightarrow{(1)} 1^\ell 1 \xrightarrow{(2)} 1 \qquad\qquad (\leftarrow) \qquad 1 \xrightarrow{(2)} 11^\ell \xrightarrow{(1)} 1^\ell$$

(5)

$$(\rightarrow)$$

$$A^{rl} \xrightarrow{(1)} A^{rl}1 \xrightarrow{(3)} A^{rl}(A^r A) \xrightarrow{(1)} (A^{rl}A^r)A \xrightarrow{(2)} 1A \xrightarrow{(1)} A$$

$$(\leftarrow)$$

$$A \xrightarrow{(1)} A1 \xrightarrow{(2)} A(A^r A^{rl}) \xrightarrow{(1)} (AA^r)A^{rl} \xrightarrow{(3)} 1A^{rl} \xrightarrow{(1)} A^{rl}$$

(6)

$$(\rightarrow)$$

$$(AB)^\ell \xrightarrow{(1)} (1AB)^\ell \xrightarrow{(3)} (((B^\ell A^\ell)^r(B^\ell A^\ell))AB)^\ell \xrightarrow{(1^*)} ((B^\ell A^\ell)^r B^\ell (A^\ell A)B)^\ell \xrightarrow{(2+1)}$$
$$((B^\ell A^\ell)^r B^\ell B)^\ell \xrightarrow{(2+1)} (B^\ell A^\ell)^{rl} \xrightarrow{(5)} B^\ell A^\ell$$

$$(\leftarrow)$$

$$B^\ell A^\ell \xrightarrow{(1)} B^\ell A^\ell 1 \xrightarrow{(2)} B^\ell A^\ell (AB)(AB)^\ell \xrightarrow{(1)} B^\ell (A^\ell A)B(AB)^\ell \xrightarrow{(2+1)} (B^\ell B)(AB)^\ell \xrightarrow{(2+1)} (AB)^\ell$$

(7)

$$B^\ell \xrightarrow{(1)} B^\ell 1 \xrightarrow{(2)} B^\ell AA^\ell \xrightarrow{(A\to B)} B^\ell BA^\ell \xrightarrow{(2+1)} A^\ell$$

2.2

We first calculate the pregroup translation of the given sequent:

$$
\begin{aligned}
&\overline{(p/((q/q)/r))/r} && \\
=&\overline{(p/((q/q)/r))}\ r^{\ell} && \{\overline{A/B} = \overline{A}(\overline{B})^{\ell}\} \\
=&p\ \overline{((q/q)/r)}^{\ell}\ r^{\ell} && \{\overline{A/B} = \overline{A}(\overline{B})^{\ell}\} \\
=&p\ (\overline{(q/q)}\ r^{\ell})^{\ell}\ r^{\ell} && \{\overline{A/B} = \overline{A}(\overline{B})^{\ell}\} \\
=&p\ ((q\ q^{\ell})\ r^{\ell})^{\ell}\ r^{\ell} && \{\overline{A/B} = \overline{A}(\overline{B})^{\ell}\} \\
=&p\ (q\ (q^{\ell}\ r^{\ell}))^{\ell}\ r^{\ell} && \{\textit{rule (1) from 2.1}\} \\
=&p\ (q^{\ell}\ r^{\ell})^{\ell}\ q^{\ell}\ r^{\ell} && \{\textit{rule (6) from 2.1}\} \\
=&p\ r^{\ell^{\ell}}\ q^{\ell^{\ell}}\ q^{\ell}\ r^{\ell} && \{\textit{rule (6) from 2.1}\}
\end{aligned}
$$

Finally, we prove the sequent by drawing a string diagram: