Logic and Language: Exercise (Week 3)

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1 LGa

1.1

1. Unfolding the combinator f/g, we get the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{1_{A/B} : A/B \longrightarrow A/B}{\triangleright^{-1} 1_{A/B} : A/B \otimes B \longrightarrow A} \quad f : A \longrightarrow A'}{f \circ (\triangleright^{-1} 1_{A/B}) : A/B \otimes B \longrightarrow A'}}{\triangleright(f \circ (\triangleright^{-1} 1_{A/B})) : A/B \longrightarrow A'/B} \quad \dfrac{\dfrac{\dfrac{g : B \longrightarrow B' \quad \dfrac{\dfrac{1_{A/B'} : A/B' \longrightarrow A/B'}{\triangleright^{-1} 1_{A/B'} : (A/B') \otimes B' \longrightarrow A}}{\triangleleft\triangleright^{-1} 1_{A/B'} : B' \longrightarrow (A/B')\backslash A}}{(\triangleleft\triangleright^{-1} 1_{A/B'}) \circ g : B \longrightarrow (A/B')\backslash A}}{\triangleleft^{-1}((\triangleleft\triangleright^{-1} 1_{A/B'}) \circ g) : (A/B') \otimes B \longrightarrow A}}{\triangleright\triangleleft^{-1}((\triangleleft\triangleright^{-1} 1_{A/B'}) \circ g) : A/B' \longrightarrow A/B}}{(\triangleright(f \circ (\triangleright^{-1} 1_{A/B}))) \circ (\triangleright\triangleleft^{-1}((\triangleleft\triangleright^{-1} 1_{A/B'}) \circ g)) : A/B' \longrightarrow A'/B}$$

2. Applying the arrow reversal transformation $(\cdot)^\dagger$ to f/g, we get the reverse combinator:

$$\begin{aligned}
(f/g)^\dagger &= ((\triangleright(f \circ (\overset{-1}{\triangleright} 1_{A/B}))) \circ (\triangleright \overset{-1}{\triangleleft}((\triangleleft \overset{-1}{\triangleright} 1_{A/B'}) \circ g)))^\dagger \\
&= (\triangleright \overset{-1}{\triangleleft}((\triangleleft \overset{-1}{\triangleright} 1_{A/B'}) \circ g))^\dagger \circ (\triangleright(f \circ (\overset{-1}{\triangleright} 1_{A/B})))^\dagger \\
&= (\blacktriangleleft(\overset{-1}{\triangleleft}((\triangleleft \overset{-1}{\triangleright} 1_{A/B'}) \circ g))^\dagger) \circ (\blacktriangleleft(f \circ (\overset{-1}{\triangleright} 1_{A/B}))^\dagger) \\
&= (\blacktriangleleft \overset{-1}{\blacktriangleright}((\triangleleft \overset{-1}{\triangleright} 1_{A/B'}) \circ g)^\dagger) \circ (\blacktriangleleft((\overset{-1}{\triangleright} 1_{A/B})^\dagger \circ f^\dagger)) \\
&= (\blacktriangleleft \overset{-1}{\blacktriangleright}(g^\dagger \circ (\triangleleft \overset{-1}{\triangleright} 1_{A/B'})^\dagger)^\dagger) \circ (\blacktriangleleft((\overset{-1}{\blacktriangleleft} 1_{(A/B)^\dagger}) \circ f^\dagger)) \\
&= (\blacktriangleleft \overset{-1}{\blacktriangleright}(g^\dagger \circ (\blacktriangleright(\overset{-1}{\triangleright} 1_{A/B'})^\dagger))) \circ (\blacktriangleleft((\overset{-1}{\blacktriangleleft} 1_{B \oslash A}) \circ f^\dagger)) \\
&= (\blacktriangleleft \overset{-1}{\blacktriangleright}(g^\dagger \circ (\blacktriangleright \overset{-1}{\blacktriangleleft} 1_{(A/B')^\dagger}))) \circ (\blacktriangleleft((\overset{-1}{\blacktriangleleft} 1_{B \oslash A}) \circ f^\dagger)) \\
&= (\blacktriangleleft \overset{-1}{\blacktriangleright}(g^\dagger \circ (\blacktriangleright \overset{-1}{\blacktriangleleft} 1_{B' \oslash A}))) \circ (\blacktriangleleft((\overset{-1}{\blacktriangleleft} 1_{B \oslash A}) \circ f^\dagger))
\end{aligned}$$

Unfolding the combinator f/g, we get the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{1_{B'\obslash A} : B' \obslash A \longrightarrow B' \obslash A}{\blacktriangleleft^{-1} 1_{B'\obslash A} : A \longrightarrow B' \oplus (B' \obslash A)}}{\blacktriangleright \blacktriangleleft^{-1} 1_{B'\obslash A} : A \oslash (B' \obslash A) \longrightarrow B'} \quad g^\dagger : B' \longrightarrow B}{g^\dagger \circ (\blacktriangleright \blacktriangleleft^{-1} 1_{B'\obslash A}) : A \oslash (B' \obslash A) \longrightarrow B}}{\blacktriangleright^{-1}(g^\dagger \circ (\blacktriangleright \blacktriangleleft^{-1} 1_{B'\obslash A})) : A \longrightarrow B \oplus (B' \obslash A)}}{\blacktriangleleft \blacktriangleright^{-1}(g^\dagger \circ (\blacktriangleright \blacktriangleleft^{-1} 1_{B'\obslash A})) : B \obslash A \longrightarrow B' \obslash A} \quad \dfrac{\dfrac{f^\dagger : A' \longrightarrow A \quad \dfrac{1_{B\obslash A} : B \obslash A \longrightarrow B \obslash A}{\blacktriangleleft^{-1} 1_{B\obslash A} : A \longrightarrow B \oplus (B \obslash A)}}{(\blacktriangleleft^{-1} 1_{B\obslash A}) \circ f^\dagger : A' \longrightarrow B \oplus (B \obslash A)}}{\blacktriangleleft((\blacktriangleleft^{-1} 1_{B\obslash A}) \circ f^\dagger) : B \obslash A' \longrightarrow B \obslash A}}{(\blacktriangleleft \blacktriangleright^{-1}(g^\dagger \circ (\blacktriangleright \blacktriangleleft^{-1} 1_{B'\obslash A}))) \circ (\blacktriangleleft((\blacktriangleleft^{-1} 1_{B\obslash A}) \circ f^\dagger)) : B \obslash A' \longrightarrow B' \obslash A}$$

1.2

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \longrightarrow a}\,1_A \quad \dfrac{}{b \longrightarrow b}\,1_A}{a \oplus b \longrightarrow a \oplus b}\,(\oplus) \quad \dfrac{}{c \longrightarrow c}\,1_A}{(a \oplus b)/c \longrightarrow (a \oplus b)/c}\,(\backslash)}{((a \oplus b)/c) \otimes c \longrightarrow a \oplus b}\,\triangleright^{-1}}{c \oslash b \longrightarrow ((a \oplus b)/c)\backslash a}\,\underset{\cdot}{\mathbf{b}}}{((a \oplus b)/c) \otimes (c \oslash b) \longrightarrow a}\,\triangleleft^{-1}}{(a \oplus b)/c \longrightarrow a/(c \oslash b)}\,\triangleright$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{b \longrightarrow b}\,1_A \quad \dfrac{\dfrac{\dfrac{}{c \longrightarrow c}\,1_A \quad \dfrac{}{a \longrightarrow a}\,1_A}{c \otimes a \longrightarrow c \otimes a}\,(\otimes)}{c \longrightarrow (c \otimes a)/a}\,\triangleright}{b \oplus c \longrightarrow b \oplus ((c \otimes a)/a)}\,(\oplus)}{b \obslash (b \oplus c) \longrightarrow (c \otimes a)/a}\,\blacktriangleleft}{(b \oplus c) \otimes a \longrightarrow b \oplus (c \otimes a)}\,\underset{\cdot}{\mathbf{d}}^{-1}$$

2 LGf

2.1 Uniform *positive* bias

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                  \dfrac{}{np \vdash \boxed{np}}\;Ax
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                      \dfrac{}{s \vdash \boxed{s}}\;Ax
                    }{s \vdash s}\;\rightharpoonup
                  }{\boxed{s} \vdash s}\;\leftharpoondown
                }{np \cdot\otimes\cdot \boxed{np\backslash s} \vdash s}\;\backslash L
              }{np \cdot\otimes\cdot (np\backslash s) \vdash s}\;\leftharpoonup
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              \dfrac{}{s \vdash \boxed{s}}\;Ax
            }{np \cdot\otimes\cdot (\boxed{(np\backslash s)/s} \cdot\otimes\cdot s) \vdash s}\;/L
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        }{((np\backslash s)/s) \cdot\otimes\cdot s \vdash np \cdot\backslash\cdot s}\;rp
      }{s \vdash ((np\backslash s)/s) \cdot\backslash\cdot (np \cdot\backslash\cdot s)}\;rp
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            }{np \cdot\otimes\cdot (np\backslash s) \vdash ((np\backslash s)/s) \cdot\backslash\cdot (np \cdot\backslash\cdot s)}\;\leftharpoonup
          }{np \vdash (((np\backslash s)/s) \cdot\backslash\cdot (np \cdot\backslash\cdot s)) \cdot/\cdot (np\backslash s)}\;rp
        }{\boxed{np} \vdash (((np\backslash s)/s) \cdot\backslash\cdot (np \cdot\backslash\cdot s)) \cdot/\cdot (np\backslash s)}\;\leftharpoondown
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        \dfrac{}{n \vdash \boxed{n}}\;Ax
      }{\boxed{np/n} \cdot\otimes\cdot n \vdash (((np\backslash s)/s) \cdot\backslash\cdot (np \cdot\backslash\cdot s)) \cdot/\cdot (np\backslash s)}\;/L
    }{(np/n) \cdot\otimes\cdot n \vdash (((np\backslash s)/s) \cdot\backslash\cdot (np \cdot\backslash\cdot s)) \cdot/\cdot (np\backslash s)}\;\leftharpoonup
  }{((np/n) \cdot\otimes\cdot n) \cdot\otimes\cdot (np\backslash s) \vdash ((np\backslash s)/s) \cdot\backslash\cdot (np \cdot\backslash\cdot s)}\;rp
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  }{((np\backslash s)/s) \cdot\otimes\cdot (((np/n) \cdot\otimes\cdot n) \cdot\otimes\cdot (np\backslash s)) \vdash np \cdot\backslash\cdot s}\;rp
}{np \cdot\otimes\cdot (((np\backslash s)/s) \cdot\otimes\cdot (((np/n) \cdot\otimes\cdot n) \cdot\otimes\cdot (np\backslash s))) \vdash s}\;rp
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2.2 Uniform *negative* bias

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          \dfrac{}{\boxed{np} \vdash np}\;CoAx
        }{np \vdash np}\;\leftharpoonup
      }{np \vdash \boxed{np}}\;\rightharpoondown
      \quad
      \dfrac{}{\boxed{s} \vdash s}\;CoAx
    }{np \cdot\otimes\cdot \boxed{np\backslash s} \vdash s}\;\backslash L
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                \dfrac{}{\boxed{np} \vdash np}\;CoAx
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                    \dfrac{}{\boxed{n} \vdash n}\;CoAx
                  }{n \vdash n}\;\leftharpoonup
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              }{\boxed{np/n} \cdot\otimes\cdot n \vdash np}\;/L
            }{(np/n) \cdot\otimes\cdot n \vdash np}\;\leftharpoonup
          }{(np/n) \cdot\otimes\cdot n \vdash \boxed{np}}\;\rightharpoondown
          \quad
          \dfrac{}{\boxed{s} \vdash s}\;CoAx
        }{((np/n) \cdot\otimes\cdot n) \cdot\otimes\cdot \boxed{np\backslash s} \vdash s}\;\backslash L
      }{((np/n) \cdot\otimes\cdot n) \cdot\otimes\cdot (np\backslash s) \vdash s}\;\leftharpoonup
    }{((np/n) \cdot\otimes\cdot n) \cdot\otimes\cdot (np\backslash s) \vdash \boxed{s}}\;\rightharpoondown
  }{np \cdot\otimes\cdot (\boxed{(np\backslash s)/s} \cdot\otimes\cdot (((np/n) \cdot\otimes\cdot n) \cdot\otimes\cdot (np\backslash s))) \vdash s}\;/L
}{np \cdot\otimes\cdot (((np\backslash s)/s) \cdot\otimes\cdot (((np/n) \cdot\otimes\cdot n) \cdot\otimes\cdot (np\backslash s))) \vdash s}\;\leftharpoonup
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2.3 Bias with negative s and positive np, n

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\UnaryInfC{$np \vdash \boxed{np}$}
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\BinaryInfC{$np \cdot \otimes \cdot \boxed{np\backslash s} \vdash s$}
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\UnaryInfC{$np \vdash \boxed{np}$}
\AxiomC{}
\RightLabel{$CoAx$}
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\RightLabel{$\backslash L$}
\BinaryInfC{$np \cdot \otimes \cdot \boxed{np\backslash s} \vdash s$}
\RightLabel{$\leftharpoonup$}
\UnaryInfC{$np \cdot \otimes \cdot (np\backslash s) \vdash s$}
\RightLabel{rp}
\UnaryInfC{$np \vdash s \cdot / \cdot (np\backslash s)$}
\RightLabel{$\leftharpoondown$}
\UnaryInfC{$\boxed{np} \vdash s \cdot / \cdot (np\backslash s)$}
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\UnaryInfC{$((np/n) \cdot \otimes \cdot n) \vdash s \cdot / \cdot (np\backslash s)$}
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\RightLabel{Ax}
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\RightLabel{rp}
\BinaryInfC{$((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s) \vdash s$}
\RightLabel{$/L$}
\BinaryInfC{$np \cdot \otimes \cdot (\boxed{(np\backslash s)/s} \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s))) \vdash s$}
\RightLabel{$\leftharpoonup$}
\UnaryInfC{$np \cdot \otimes \cdot (((np\backslash s)/s) \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s))) \vdash s$}
\end{prooftree}
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