Logic and Language: Exercise (Week 5)

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1 LG: continuation semantics

1.1

For each term, we compute the ILL type by first observing the polarity of the sequent and then using the table to interpret complex types.

$$\begin{aligned}
\lceil \textbf{some} \rceil &= \lceil np/n \rceil^{\perp} && \{\textit{Negative Hypothesis}\}\\
&= (\lceil np \rceil^{\perp} \otimes \lceil n \rceil)^{\perp} && \{\lceil A/B \rceil \textit{ with } A \textit{ and } B \textit{ positive}\}\\
&= (np^{\perp} \otimes n)^{\perp} && \{np \textit{ and } n \textit{ positive}\}
\end{aligned}$$

$$\begin{aligned}
\lceil \textbf{popular} \rceil &= \lceil n/n \rceil^{\perp} && \{\textit{Negative Hypothesis}\}\\
&= (\lceil n \rceil^{\perp} \otimes \lceil n \rceil)^{\perp} && \{\lceil A/B \rceil \textit{ with } A \textit{ and } B \textit{ positive}\}\\
&= (n^{\perp} \otimes n)^{\perp} && \{n \textit{ positive}\}
\end{aligned}$$

$$\begin{aligned}
\lceil \textbf{saint} \rceil &= \lceil n \rceil && \{\textit{Positive Hypothesis}\}\\
&= n && \{n \textit{ positive}\}
\end{aligned}$$

$$\begin{aligned}
\lceil \textbf{arrived} \rceil &= \lceil np \backslash s \rceil^{\perp} && \{\textit{Negative Hypothesis}\}\\
&= (\lceil np \rceil \otimes \lceil s \rceil)^{\perp} && \{\lceil B \backslash A \rceil \textit{ with } B \textit{ positive and } A \textit{ negative}\}\\
&= (np \otimes s^{\perp})^{\perp} && \{np \textit{ positive, } s \textit{ negative}\}
\end{aligned}$$

$$\begin{aligned}
\alpha &= \lceil s \rceil\\
&= s^{\perp} && \{s \textit{ negative}\}
\end{aligned}$$

$$\begin{aligned}
z &= \lceil np \rceil\\
&= np && \{np \textit{ positive}\}
\end{aligned}$$

$$\begin{aligned}
y &= \lceil n \rceil\\
&= n && \{n \textit{ positive}\}
\end{aligned}$$

The ILL types then are:

TERM	TYPE
$\lceil \textbf{some} \rceil$	$(np^{\perp} \otimes n)^{\perp}$
$\lceil \textbf{popular} \rceil$	$(n^{\perp} \otimes n)^{\perp}$
$\lceil \textbf{saint} \rceil$	n
$\lceil \textbf{arrived} \rceil$	$(np \otimes s^{\perp})^{\perp}$
α	$s^{\perp}$
z	np
y	n

1.2

SOURCE TYPE	CONSTANT	$\lceil . \rceil^{\ell}$
n/n	popular	$\lambda\langle c, y\rangle.(c\ (\lambda z.\ \wedge (y\ z)\ (\text{POPULAR}\ z)))$

1.3

1.

$$\frac{\alpha_1}{\boxed{\alpha_1 : n} \vdash n}\ CoAx$$

$$\frac{\lceil saint \rceil\ \alpha_1 : n \vdash \alpha_1 : n}{n \vdash \boxed{\lambda\alpha_1.(\lceil saint \rceil\ \alpha_1) : n}}\ \rightharpoonup$$

$$\frac{\dfrac{\gamma_0}{\boxed{\gamma_0 : n} \vdash n}\ CoAx \qquad n \vdash \boxed{\lambda\alpha_1.(\lceil saint \rceil\ \alpha_1) : n}}{\boxed{\langle \gamma_0, \lambda\alpha_1.(\lceil saint \rceil\ \alpha_1)\rangle : n/n} \vdash \gamma_0 : n \cdot / \cdot n}\ /L$$

$$\frac{\dfrac{(\lceil popular \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil saint \rceil\ \alpha_1)\rangle) : n/n \vdash \gamma_0 : n \cdot / \cdot n}{(n/n) \cdot \otimes \cdot n \vdash \boxed{\lambda\gamma_0.(\lceil popular \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil saint \rceil\ \alpha_1)\rangle) : n}}\ \rightharpoonup}{}$$

$$\frac{\dfrac{\beta_0}{\boxed{\beta_0 : np} \vdash np}\ CoAx \qquad (n/n) \cdot \otimes \cdot n \vdash \boxed{\lambda\gamma_0.(\lceil popular \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil saint \rceil\ \alpha_1)\rangle) : n}}{\boxed{\langle \beta_0, \lambda\gamma_0.(\lceil popular \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil saint \rceil\ \alpha_1)\rangle)\rangle) : np/n}}\ /L$$

$$\frac{(\lceil some \rceil \langle \beta_0, \lambda\gamma_0.(\lceil popular \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil saint \rceil\ \alpha_1)\rangle)\rangle) : np/n \vdash \beta_0 : np \cdot / \cdot ((n/n) \cdot \otimes \cdot n)}{(np/n) \cdot \otimes \cdot ((n/n) \cdot \otimes \cdot n) \vdash \boxed{\lambda\beta_0.(\lceil some \rceil \langle \beta_0, \lambda\gamma_0.(\lceil popular \rceil \langle \gamma_0, \lambda\alpha_1(\lceil saint \rceil\ \alpha_1)\rangle)\rangle) : np}}\ \rightharpoonup \qquad \frac{\alpha_0}{\boxed{\alpha_0 : s} \vdash \ldots}$$

$$\boxed{(\lceil arrived \rceil \langle \beta_0.(\lceil some \rceil \langle \beta_0, \lambda\gamma_0.(\lceil popular \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil saint \rceil \alpha_1)\rangle)\rangle\rangle), \alpha_0\rangle) : np \backslash s \vdash \alpha_0 : ((np/n) \cdot \otimes \cdot ((n/n) \cdot \otimes \cdot \ldots}$$

$$\frac{(\lceil arrived \rceil \langle \beta_0.(\lceil some \rceil \langle \beta_0, \lambda\gamma_0.(\lceil popular \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil saint \rceil \alpha_1)\rangle)\rangle\rangle), \alpha_0\rangle) : np \backslash s \vdash \alpha_0 : ((np/n) \cdot \otimes \cdot ((n/n) \otimes \ldots}{(np/n) \cdot \otimes \cdot ((n/n) \cdot \otimes \cdot n)) \cdot \otimes \cdot (np \backslash s) \vdash \boxed{\lambda\alpha_0.(\lceil arrived \rceil \langle \lambda\beta_0.\lceil some \rceil \langle \beta_0, \lambda\gamma_0.(\lceil popular \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil saint \rceil\ \alpha_1))\ldots}}$$

2. We compute the interpretation below:

$$\lceil \ddagger \rceil = \lambda a_0.(\lceil \textbf{arrived} \rceil\ \langle \lambda\beta_0.(\lceil \textbf{some} \rceil\ \langle \beta_0, \lambda\gamma_0.(\lceil \textbf{popular} \rceil\ \langle \gamma_0, \lambda a_1.(\lceil \textbf{saint} \rceil\ a_1)\rangle)\rangle\rangle), a_0\rangle)$$

3. The adjucted $\cdot^{\ell}$ translations are the following:

$$\lceil \textbf{some} \rceil^{\ell} = \lambda\langle x, k\rangle.(\exists \lambda z.(\wedge\ (k\ \lambda\theta.(\theta\ z))\ (x\ z)))$$
$$\lceil \textbf{popular} \rceil^{\ell} = \lambda\langle c, k\rangle.(c\ (\lambda z.(\wedge\ (\text{POPULAR}\ z)\ (k\ \lambda\theta.(\theta\ z)))))$$
$$\lceil \textbf{saint} \rceil^{\ell} = \lambda c.(c\ \text{SAINT})$$
$$\lceil \textbf{arrived} \rceil^{\ell} = \lambda\langle k, c\rangle.(k\ \lambda z.(c\ (\text{ARRIVED}\ z)))$$

We can now corroborate the α-equivalence of the two $\cdot^\ell$ translations:

$$
\begin{aligned}
\lceil\ddagger\rceil^\ell = {} & \lambda a_0.(\underline{\lambda\langle k,c\rangle.(k\ \lambda z.(c\ (\mathrm{ARRIVED}\ z)))} \\
& \quad \langle\lambda\beta_0.(\lambda\langle x,k\rangle.(\exists\lambda z.(\wedge\ (k\ \lambda\theta.(\theta\ z))\ (x\ z)))\ \langle\beta_0,\lambda\gamma_0.(\\
& \qquad \lambda\langle c,k\rangle.(c\ (\lambda z.(\wedge\ (\mathrm{POP}\ z)\ (k\ \lambda\theta.(\theta\ z)))))\ \langle\gamma_0,\lambda a_1.(\lambda c.(c\ \mathrm{SAINT})\ a_1)\rangle)\rangle)),a_0\rangle) \\
\to_\beta^* {} & \lambda a_0.(\underline{\lambda\langle x,k\rangle.(\exists\lambda z.(\wedge\ (k\ \lambda\theta.(\theta\ z))\ (x\ z)))}\ \langle\lambda z.(a_0\ (\mathrm{ARRIVED}\ z)),\lambda\gamma_0.(\\
& \qquad \lambda\langle c,k\rangle.(c\ (\lambda z.(\wedge\ (\mathrm{POPULAR}\ z)\ (k\ \lambda\theta.(\theta\ z)))))\ \langle\gamma_0,\lambda a_1.(\lambda c.(c\ \mathrm{SAINT})\ a_1)\rangle)\rangle) \\
\to_\beta^* {} & \lambda a_0.(\exists\lambda z.(\wedge\ (\underline{\lambda\langle c,k\rangle.(c\ (\lambda z.(\wedge\ (\mathrm{POPULAR}\ z)\ (k\ \lambda\theta.(\theta\ z)))))} \\
& \qquad \langle\lambda\theta.(\theta\ z),\lambda a_1.(\lambda c.(c\ \mathrm{SAINT})\ a_1)\rangle)\ (a_0\ (\mathrm{ARRIVED}\ z))) \\
\to_\beta^* {} & \lambda a_0.(\exists\lambda z.(\wedge\ (\underline{\lambda\theta.(\theta\ z)}\ (\lambda z.(\wedge\ (\mathrm{POPULAR}\ z)\ (\underline{\lambda c.(c\ \mathrm{SAINT})}\ \underline{(\lambda\theta.(\theta\ z))}))))\ (a_0\ (\mathrm{ARRIVED}\ z))) \\
\to_\beta^* {} & \lambda a_0.(\exists\lambda z.(\wedge\ (\wedge\ (\mathrm{POPULAR}\ z)\ (\mathrm{SAINT}\ z))\ (a_0\ (\mathrm{ARRIVED}\ z))) \\
\to_\alpha {} & \lambda c.(\exists\lambda z.(\wedge\ (\wedge\ (\mathrm{POPULAR}\ z)\ (\mathrm{SAINT}\ z))\ (c\ (\mathrm{ARRIVED}\ z))) \\
\to_\alpha {} & \lambda c.(\exists\lambda x.(\wedge\ (\wedge\ (\mathrm{POPULAR}\ x)\ (\mathrm{SAINT}\ x))\ (c\ (\mathrm{ARRIVED}\ x))) \\
= {} & \lceil\dagger\rceil^\ell
\end{aligned}
$$

2 Pregroups

2.1

(4)

$(\to)$

$$
\dfrac{\dfrac{}{1^l \to 1^l 1}\,(1) \qquad \dfrac{}{1^l 1 \to 1}\,(2)}{1^l \to 1}\,\to
$$

$(\leftarrow)$

$$
\dfrac{\dfrac{}{1 \to 1^l 1}\,(2) \qquad \dfrac{}{1^l 1 \to 1^l}\,(1)}{1 \to 1^l}\,\to
$$

(5) $(\to)$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A^{rl} \to A^{rl}1}\,(1) \qquad \dfrac{}{1 \to A^r A}\,(2)}{A^{rl} \to A^{rl}(A^r A)}\,\to}{A^{rl} \to (A^{rl}A^r)A}\,(1) \qquad \dfrac{}{A^{rl}A^r \to 1}\,(2)}{A^{rl} \to 1A}\,\to \qquad \dfrac{}{1A \to A}\,(1)}{A^{rl} \to A}\,\to
$$

$(\leftarrow)$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \to A1}\,(1) \qquad \dfrac{}{1 \to A^r A^{rl}}\,(2)}{A \to A(A^r A^{rl})}\,\to}{A \to (AA^r)A^{rl}}\,(1) \qquad \dfrac{}{AA^r \to 1}\,(3)}{A \to 1A^{rl}}\,\to \qquad \dfrac{}{1A^{rl} \to A^{rl}}\,(1)}{A \to A^{rl}}\,\to
$$

(6) $(\to)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(AB)^l \to (1(AB))^l}\,\to \quad \dfrac{}{1 \to (B^lA^l)^r(B^lA^l)}\,(3)}{(AB)^l \to (((B^lA^l)^r(B^lA^l))(AB))^l}\,\to}{(AB)^l \to ((B^lA^l)^r((B^lA^l)(AB)))^l}\,(1)}{(AB)^l \to ((B^lA^l)^r(B^l(A^l(AB))))}\,(1)}{(AB)^l \to ((B^lA^l)^r(B^l((A^lA)B)))}\,(1) \quad \dfrac{}{A^lA \to 1}\,(2)}{(AB)^l \to ((B^lA^l)^r(B^l(1B)))^l}\,\to \quad \dfrac{}{1B \to B}\,(1)}{(AB)^l \to ((B^lA^l)^r(B^lB))^l}\,\to \quad \dfrac{}{B^lB \to 1}\,(2)}{(AB)^l \to ((B^lA^l)^r1)^l}\,\textit{todo} \quad \dfrac{}{(B^lA^l)^r1 \to (}}{(AB)^l \to ((B^lA^l)^r)^l}}{(AB)^l \to B^lA^l}\,(5)$$

2.2

We first calculate the pregroup translation of the given sequent:

$$\begin{array}{ll}
\overline{(p/((q/q)/r))/r} & \\
=\overline{(p/((q/q)/r))}\ r^\ell & \{\overline{A/B} = \overline{A}(\overline{B})^\ell\} \\
=p\ \overline{((q/q)/r)}^\ell\ r^\ell & \{\overline{A/B} = \overline{A}(\overline{B})^\ell\} \\
=p\ (\overline{(q/q)}\ r^\ell)^\ell\ r^\ell & \{\overline{A/B} = \overline{A}(\overline{B})^\ell\} \\
=p\ ((q\ q^\ell)\ r^\ell)^\ell\ r^\ell & \{\overline{A/B} = \overline{A}(\overline{B})^\ell\} \\
=p\ (q\ (q^\ell\ r^\ell))^\ell\ r^\ell & \{\textit{rule (1) from 2.1}\} \\
=p\ (q^\ell\ r^\ell)^\ell\ q^\ell\ r^\ell & \{\textit{rule (6) from 2.1}\} \\
=p\ r^{\ell^\ell}\ q^{\ell^\ell}\ q^\ell\ r^\ell & \{\textit{rule (6) from 2.1}\}
\end{array}$$

Finally, we prove the sequent by drawing a string diagram: