Logic and Language: Exercise (Week 5)

Orestis Melkonian [6176208], Konstantinos Kogkalidis [6230067]

1 LG: continuation semantics

1.1

For each term, we compute the ILL type by first observing the polarity of the sequent and then using the table to interpret complex types.

$$
\begin{aligned}
\lceil \textbf{some} \rceil &= \lceil np/n \rceil^{\perp} && \{\textit{Negative Hypothesis}\} \\
&= (\lceil np \rceil^{\perp} \otimes \lceil n \rceil)^{\perp} && \{\lceil A/B \rceil \textit{ with } A \textit{ and } B \textit{ positive}\} \\
&= (np^{\perp} \otimes n)^{\perp} && \{np \textit{ and } n \textit{ positive}\}
\end{aligned}
$$

$$
\begin{aligned}
\lceil \textbf{popular} \rceil &= \lceil n/n \rceil^{\perp} && \{\textit{Negative Hypothesis}\} \\
&= (\lceil n \rceil^{\perp} \otimes \lceil n \rceil)^{\perp} && \{\lceil A/B \rceil \textit{ with } A \textit{ and } B \textit{ positive}\} \\
&= (n^{\perp} \otimes n)^{\perp} && \{n \textit{ positive}\}
\end{aligned}
$$

$$
\begin{aligned}
\lceil \textbf{saint} \rceil &= \lceil n \rceil && \{\textit{Positive Hypothesis}\} \\
&= n && \{n \textit{ positive}\}
\end{aligned}
$$

$$
\begin{aligned}
\lceil \textbf{arrived} \rceil &= \lceil np \backslash s \rceil^{\perp} && \{\textit{Negative Hypothesis}\} \\
&= (\lceil np \rceil \otimes \lceil s \rceil)^{\perp} && \{\lceil B \backslash A \rceil \textit{ with } B \textit{ positive and } A \textit{ negative}\} \\
&= (np \otimes s^{\perp})^{\perp} && \{np \textit{ positive, } s \textit{ negative}\}
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= \lceil s \rceil \\
&= s^{\perp} && \{s \textit{ negative}\}
\end{aligned}
$$

$$
\begin{aligned}
z &= \lceil np \rceil \\
&= np && \{np \textit{ positive}\}
\end{aligned}
$$

$$
\begin{aligned}
y &= \lceil n \rceil \\
&= n && \{n \textit{ positive}\}
\end{aligned}
$$

The ILL types then are:

TERM	TYPE
$\lceil \mathbf{some} \rceil$	$(np^{\perp} \otimes n)^{\perp}$
$\lceil \mathbf{popular} \rceil$	$(n^{\perp} \otimes n)^{\perp}$
$\lceil \mathbf{saint} \rceil$	n
$\lceil \mathbf{arrived} \rceil$	$(np \otimes s^{\perp})^{\perp}$
α	$s^{\perp}$
z	np
y	n

1.2

SOURCE TYPE	CONSTANT	$\lceil . \rceil^{\ell}$
n/n	popular	$\lambda\langle c, y\rangle.(c\ (\lambda z.\ \wedge (y\ z)\ (\textsc{popular}\ z)))$

1.3

1. We compute the interpretation below:

$$
\begin{prooftree}
\AxiomC{β_0}
\RightLabel{$CoAx$}
\UnaryInfC{$\boxed{\beta_0 : np} \vdash np$}
\AxiomC{γ_0}
\RightLabel{$CoAx$}
\UnaryInfC{$\boxed{\gamma_0 : n} \vdash n$}
\AxiomC{α_1}
\RightLabel{$CoAx$}
\UnaryInfC{$\boxed{\alpha_1 : n} \vdash n$}
\RightLabel{$\leftharpoondown$}
\UnaryInfC{$\lceil saint \rceil\ \alpha_1 : n \vdash \alpha_1 : n$}
\RightLabel{$\rightharpoondown$}
\UnaryInfC{$n \vdash \boxed{\lambda\alpha_1.(\lceil \mathbf{saint} \rceil\ \alpha_1) : n}$}
\RightLabel{$/L$}
\BinaryInfC{$\boxed{\langle \gamma_0, \lambda\alpha_1.(\lceil \mathbf{saint} \rceil\ \alpha_1)\rangle : n/n} \vdash \gamma_0 : n \cdot / \cdot n$}
\RightLabel{$\leftharpoondown$}
\UnaryInfC{$(\lceil \mathbf{popular} \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil \mathbf{saint} \rceil\ \alpha_1)\rangle) : n/n \vdash \gamma_0 : n \cdot / \cdot n$}
\RightLabel{$\rightharpoondown$}
\UnaryInfC{$(n/n) \cdot \otimes \cdot n \vdash \boxed{\lambda\gamma_0.(\lceil \mathbf{popular} \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil \mathbf{saint} \rceil\ \alpha_1)\rangle) : n}$}
\RightLabel{$/L$}
\BinaryInfC{$\boxed{\langle \beta_0, \lambda\gamma_0.(\lceil \mathbf{popular} \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil \mathbf{saint} \rceil\ \alpha_1)\rangle)\rangle : np/n}$}
\RightLabel{$\leftharpoondown$}
\UnaryInfC{$(\lceil \mathbf{some} \rceil \langle \beta_0, \lambda\gamma_0.(\lceil \mathbf{popular} \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil \mathbf{saint} \rceil\ \alpha_1)\rangle)\rangle) : np/n \vdash \beta_0 : np \cdot / \cdot ((n/n) \cdot \otimes \cdot n)$}
\RightLabel{$\rightharpoondown$}
\UnaryInfC{$(np/n) \cdot \otimes \cdot ((n/n) \cdot \otimes \cdot n) \vdash \boxed{\lambda\beta_0.(\lceil \mathbf{some} \rceil \langle \beta_0, \lambda\gamma_0.(\lceil \mathbf{popular} \rceil \langle \gamma_0, \lambda\alpha_1(\lceil \mathbf{saint} \rceil\ \alpha_1)\rangle)\rangle) : np}$}
\AxiomC{α_0}
\RightLabel{$CoAx$}
\UnaryInfC{$\boxed{\alpha_0 : s} \vdash s$}
\RightLabel{$\backslash L$}
\BinaryInfC{$\boxed{(\lceil \mathbf{arrived} \rceil \langle \lambda\beta_0.(\lceil \mathbf{some} \rceil \langle \beta_0, \lambda\gamma_0.(\lceil \mathbf{popular} \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil \mathbf{saint} \rceil \alpha_1)\rangle)\rangle), \alpha_0\rangle) : np\backslash s} \vdash \alpha_0 : ((np/n) \cdot \otimes \cdot ((n/n) \cdot \otimes \cdot n)) \cdot \backslash \cdot s$}
\RightLabel{$\leftharpoondown$}
\UnaryInfC{$(\lceil \mathbf{arrived} \rceil \langle \lambda\beta_0.(\lceil some \rceil \langle \beta_0, \lambda\gamma_0.(\lceil \mathbf{popular} \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil \mathbf{saint} \rceil \alpha_1)\rangle)\rangle), \alpha_0\rangle) : np\backslash s \vdash \alpha_0 : ((np/n) \cdot \otimes \cdot ((n/n). \otimes .n)) \cdot \backslash \cdot s$}
\RightLabel{$\rightharpoondown$}
\UnaryInfC{$(np/n) \cdot \otimes \cdot ((n/n) \cdot \otimes \cdot n)) \cdot \otimes \cdot (np\backslash s) \vdash \boxed{\lambda\alpha_0.(\lceil \mathbf{arrived} \rceil \langle \lambda\beta_0.\lceil \mathbf{some} \rceil \langle \beta_0, \lambda\gamma_0.(\lceil \mathbf{popular} \rceil \langle \gamma_0, \lambda\alpha_1.(\lceil \mathbf{saint} \rceil\ \alpha_1)\rangle)\rangle), \alpha_0\rangle) : s}$}
\end{prooftree}
$$

Hence,

$$\lceil \ddagger \rceil = \lambda a_0.(\lceil \mathbf{arrived} \rceil\ \langle \lambda\beta_0.(\lceil \mathbf{some} \rceil\ \langle \beta_0, \lambda\gamma_0.(\lceil \mathbf{popular} \rceil\ \langle \gamma_0, \lambda a_1.(\lceil \mathbf{saint} \rceil\ a_1)\rangle)\rangle), a_0\rangle)$$

2. The adjucted $\cdot^{\ell}$ translations are the following:

$$
\begin{aligned}
\lceil \mathbf{some} \rceil^{\ell} &= \lambda\langle x, k\rangle.(\exists\lambda z.(\wedge\ (k\ \lambda\theta.(\theta\ z))\ (x\ z))) \\
\lceil \mathbf{popular} \rceil^{\ell} &= \lambda\langle c, k\rangle.(c\ (\lambda z.(\wedge\ (\text{POPULAR}\ z)\ (k\ \lambda\theta.(\theta\ z))))) \\
\lceil \mathbf{saint} \rceil^{\ell} &= \lambda c.(c\ \text{SAINT}) \\
\lceil \mathbf{arrived} \rceil^{\ell} &= \lambda\langle k, c\rangle.(k\ \lambda z.(c\ (\text{ARRIVED}\ z)))
\end{aligned}
$$

We can now corroborate the α-equivalence of the two $\cdot^{\ell}$ translations:

$$
\begin{aligned}
\lceil\ddagger\rceil^{\ell} = {} & \lambda a_0.(\underline{\lambda\langle k,c\rangle.(k\ \lambda z.(c\ (\text{ARRIVED}\ z)))} \\
& \quad \langle\lambda\beta_0.(\lambda\langle x,k\rangle.(\exists\lambda z.(\wedge\ (k\ \lambda\theta.(\theta\ z))\ (x\ z)))\ \langle\beta_0,\lambda\gamma_0.(\\
& \quad\quad \lambda\langle c,k\rangle.(c\ (\lambda z.(\wedge\ (\text{POP}\ z)\ (k\ \lambda\theta.(\theta\ z)))))\ \langle\gamma_0,\lambda a_1.(\lambda c.(c\ \text{SAINT})\ a_1)\rangle)\rangle),a_0\rangle) \\
\to^*_\beta {} & \lambda a_0.(\underline{\lambda\langle x,k\rangle.(\exists\lambda z.(\wedge\ (k\ \lambda\theta.(\theta\ z))\ (x\ z)))}\ \langle\lambda z.(a_0\ (\text{ARRIVED}\ z)),\lambda\gamma_0.(\\
& \quad\quad \lambda\langle c,k\rangle.(c\ (\lambda z.(\wedge\ (\text{POPULAR}\ z)\ (k\ \lambda\theta.(\theta\ z)))))\ \langle\gamma_0,\lambda a_1.(\lambda c.(c\ \text{SAINT})\ a_1)\rangle)\rangle) \\
\to^*_\beta {} & \lambda a_0.(\exists\lambda z.(\wedge\ (\underline{\lambda\langle c,k\rangle.(c\ (\lambda z.(\wedge\ (\text{POPULAR}\ z)\ (k\ \lambda\theta.(\theta\ z)))))} \\
& \quad\quad \langle\lambda\theta.(\theta\ z),\lambda a_1.(\lambda c.(c\ \text{SAINT})\ a_1)\rangle)\ (a_0\ (\text{ARRIVED}\ z))) \\
\to^*_\beta {} & \lambda a_0.(\exists\lambda z.(\wedge\ (\underline{\lambda\theta.(\theta\ z)}\ (\lambda z.(\wedge\ (\text{POPULAR}\ z)\ (\underline{\lambda c.(c\ \text{SAINT})}\ \underline{(\lambda\theta.(\theta\ z))})))))\ (a_0\ (\text{ARRIVED}\ z))) \\
\to^*_\beta {} & \lambda a_0.(\exists\lambda z.(\wedge\ (\wedge\ (\text{POPULAR}\ z)\ (\text{SAINT}\ z))\ (a_0\ (\text{ARRIVED}\ z))) \\
\to_\alpha {} & \lambda c.(\exists\lambda z.(\wedge\ (\wedge\ (\text{POPULAR}\ z)\ (\text{SAINT}\ z))\ (c\ (\text{ARRIVED}\ z))) \\
\to_\alpha {} & \lambda c.(\exists\lambda x.(\wedge\ (\wedge\ (\text{POPULAR}\ x)\ (\text{SAINT}\ x))\ (c\ (\text{ARRIVED}\ x))) \\
= {} & \lceil\dagger\rceil^{\ell}
\end{aligned}
$$

2 Pregroups

2.1

(4)

$$(\to)\qquad 1^{\ell} \xrightarrow{(1)} 1^{\ell}1 \xrightarrow{(2)} 1 \qquad\qquad (\leftarrow)\qquad 1 \xrightarrow{(2)} 11^{\ell} \xrightarrow{(1)} 1^{\ell}$$

(5)

$$(\to)$$
$$A^{rl} \xrightarrow{(1)} A^{rl}1 \xrightarrow{(3)} A^{rl}(A^{r}A) \xrightarrow{(1)} (A^{rl}A^{r})A \xrightarrow{(2)} 1A \xrightarrow{(1)} A$$

$$(\leftarrow)$$
$$A \xrightarrow{(1)} A1 \xrightarrow{(2)} A(A^{r}A^{rl}) \xrightarrow{(1)} (AA^{r})A^{rl} \xrightarrow{(3)} 1A^{rl} \xrightarrow{(1)} A^{rl}$$

(6)

$$(\to)$$
$$(AB)^{\ell} \xrightarrow{(1)} (1AB)^{\ell} \xrightarrow{(3)} (((B^{\ell}A^{\ell})^{r}(B^{\ell}A^{\ell}))AB)^{\ell} \xrightarrow{(1^*)} ((B^{\ell}A^{\ell})^{r}B^{\ell}(A^{\ell}A)B)^{\ell} \xrightarrow{(2+1)}$$
$$((B^{\ell}A^{\ell})^{r}B^{\ell}B)^{\ell} \xrightarrow{(2+1)} (B^{\ell}A^{\ell})^{rl} \xrightarrow{(5)} B^{\ell}A^{\ell}$$

$$(\leftarrow)$$
$$B^{\ell}A^{\ell} \xrightarrow{(1)} B^{\ell}A^{\ell}1 \xrightarrow{(2)} B^{\ell}A^{\ell}(AB)(AB)^{\ell} \xrightarrow{(1)} B^{\ell}(A^{\ell}A)B(AB)^{\ell} \xrightarrow{(2+1)} (B^{\ell}B)(AB)^{\ell} \xrightarrow{(2+1)} (AB)^{\ell}$$

(7)

$$B^{\ell} \xrightarrow{(1)} B^{\ell}1 \xrightarrow{(2)} B^{\ell}AA^{\ell} \xrightarrow{(A\to B)} B^{\ell}BA^{\ell} \xrightarrow{(2+1)} A^{\ell}$$

2.2

We first calculate the pregroup translation of the given sequent:

$$
\begin{aligned}
&\overline{(p/((q/q)/r))/r} && \\
=&\overline{(p/((q/q)/r))}\ r^{\ell} && \{\overline{A/B} = \overline{A}(\overline{B})^{\ell}\} \\
=&p\ \overline{((q/q)/r)}^{\ell}\ r^{\ell} && \{\overline{A/B} = \overline{A}(\overline{B})^{\ell}\} \\
=&p\ (\overline{(q/q)}\ r^{\ell})^{\ell}\ r^{\ell} && \{\overline{A/B} = \overline{A}(\overline{B})^{\ell}\} \\
=&p\ ((q\ q^{\ell})\ r^{\ell})^{\ell}\ r^{\ell} && \{\overline{A/B} = \overline{A}(\overline{B})^{\ell}\} \\
=&p\ (q\ (q^{\ell}\ r^{\ell}))^{\ell}\ r^{\ell} && \{\textit{rule (1) from 2.1}\} \\
=&p\ (q^{\ell}\ r^{\ell})^{\ell}\ q^{\ell}\ r^{\ell} && \{\textit{rule (6) from 2.1}\} \\
=&p\ r^{\ell^{\ell}}\ q^{\ell^{\ell}}\ q^{\ell}\ r^{\ell} && \{\textit{rule (6) from 2.1}\}
\end{aligned}
$$

Finally, we prove the sequent by drawing a string diagram: