

Logic and Language: Exercise (Week 3)

Orestis Melkonian [6176208], Konstantinos Kogkalidis [6230067]

1 LGa

1.1

1. Unfolding the combinator f/g , we get the following derivation:

$$\begin{array}{c}
 \frac{1_{A/B} : A/B \longrightarrow A/B}{\Downarrow^{-1} 1_{A/B} : A/B \otimes B \longrightarrow A} \quad f : A \longrightarrow A' \quad \frac{g : B \longrightarrow B' \quad \frac{1_{A/B'} : A/B \longrightarrow A/B'}{\Downarrow^{-1} 1_{A/B'} : (A/B') \otimes B' \longrightarrow A}}{\Downarrow^{-1} 1_{A/B'} : B' \longrightarrow (A/B') \setminus A}} \\
 \frac{f \circ (\Downarrow^{-1} 1_{A/B}) : A/B \otimes B \longrightarrow A'}{\Downarrow(f \circ (\Downarrow^{-1} 1_{A/B})) : A/B \longrightarrow A'/B} \quad \frac{(\Downarrow \Downarrow^{-1} 1_{A/B'}) \circ g : B \longrightarrow (A/B') \setminus A}{\Downarrow^{-1}((\Downarrow \Downarrow^{-1} 1_{A/B'}) \circ g) : (A/B') \otimes B \longrightarrow A}} \\
 \frac{\Downarrow(f \circ (\Downarrow^{-1} 1_{A/B})) : A/B \longrightarrow A'/B \quad \Downarrow \Downarrow^{-1}((\Downarrow \Downarrow^{-1} 1_{A/B'}) \circ g) : A/B' \longrightarrow A/B}{(\Downarrow(f \circ (\Downarrow^{-1} 1_{A/B}))) \circ (\Downarrow \Downarrow^{-1}((\Downarrow \Downarrow^{-1} 1_{A/B'}) \circ g)) : A/B' \longrightarrow A'/B}
 \end{array}$$

2. Applying the arrow reversal transformation $(\cdot)^\dagger$ to f/g , we get the reverse combinator:

$$\begin{aligned}
 (f/g)^\dagger &= ((\Downarrow(f \circ (\Downarrow^{-1} 1_{A/B}))) \circ (\Downarrow \Downarrow^{-1}((\Downarrow \Downarrow^{-1} 1_{A/B'}) \circ g)))^\dagger \\
 &= (\Downarrow \Downarrow^{-1}((\Downarrow \Downarrow^{-1} 1_{A/B'}) \circ g))^\dagger \circ (\Downarrow(f \circ (\Downarrow^{-1} 1_{A/B})))^\dagger \\
 &= (\Downarrow \Downarrow^{-1}((\Downarrow \Downarrow^{-1} 1_{A/B'}) \circ g))^\dagger \circ (\Downarrow(f \circ (\Downarrow^{-1} 1_{A/B})))^\dagger \\
 &= (\Downarrow \Downarrow^{-1}((\Downarrow \Downarrow^{-1} 1_{A/B'}) \circ g))^\dagger \circ (\Downarrow((\Downarrow^{-1} 1_{A/B})^\dagger \circ f^\dagger)) \\
 &= (\Downarrow \Downarrow^{-1}(g^\dagger \circ (\Downarrow \Downarrow^{-1} 1_{A/B'})^\dagger))^\dagger \circ (\Downarrow((\Downarrow^{-1} 1_{A/B})^\dagger \circ f^\dagger)) \\
 &= (\Downarrow \Downarrow^{-1}(g^\dagger \circ (\Downarrow \Downarrow^{-1} 1_{A/B'})^\dagger))^\dagger \circ (\Downarrow((\Downarrow^{-1} 1_{B \otimes A}) \circ f^\dagger)) \\
 &= (\Downarrow \Downarrow^{-1}(g^\dagger \circ (\Downarrow \Downarrow^{-1} 1_{(A/B')^\dagger}))^\dagger) \circ (\Downarrow((\Downarrow^{-1} 1_{B \otimes A}) \circ f^\dagger)) \\
 &= (\Downarrow \Downarrow^{-1}(g^\dagger \circ (\Downarrow \Downarrow^{-1} 1_{B' \otimes A})))^\dagger \circ (\Downarrow((\Downarrow^{-1} 1_{B \otimes A}) \circ f^\dagger))
 \end{aligned}$$

Unfolding the combinator f/g , we get the following derivation:

$$\begin{array}{c}
\frac{1_{B' \otimes A} : B' \otimes A \longrightarrow B' \otimes A}{\blacktriangleleft^{-1} 1_{B' \otimes A} : A \longrightarrow B' \oplus (B' \otimes A)} \\
\frac{\blacktriangleright \blacktriangleleft^{-1} 1_{B' \otimes A} : A \otimes (B' \otimes A) \longrightarrow B' \quad g^\dagger : B' \longrightarrow B}{g^\dagger \circ (\blacktriangleright \blacktriangleleft^{-1} 1_{B' \otimes A}) : A \otimes (B' \otimes A) \longrightarrow B} \\
\frac{\blacktriangleright^{-1} (g^\dagger \circ (\blacktriangleright \blacktriangleleft^{-1} 1_{B' \otimes A})) : A \longrightarrow B \oplus (B' \otimes A)}{\blacktriangleleft \blacktriangleright^{-1} (g^\dagger \circ (\blacktriangleright \blacktriangleleft^{-1} 1_{B' \otimes A})) : B \otimes A \longrightarrow B' \otimes A} \\
\frac{(\blacktriangleleft \blacktriangleright^{-1} (g^\dagger \circ (\blacktriangleright \blacktriangleleft^{-1} 1_{B' \otimes A}))) \circ (\blacktriangleleft ((\blacktriangleleft^{-1} 1_{B \otimes A}) \circ f^\dagger)) : B \otimes A' \longrightarrow B' \otimes A}{\quad}
\end{array}
\quad
\begin{array}{c}
\frac{1_{B \otimes A} : B \otimes A \longrightarrow B \otimes A}{\blacktriangleleft^{-1} 1_{B \otimes A} : A \longrightarrow B \oplus (B \otimes A)} \\
\frac{f^\dagger : A' \longrightarrow A \quad \blacktriangleleft^{-1} 1_{B \otimes A} : A \longrightarrow B \oplus (B \otimes A)}{(\blacktriangleleft^{-1} 1_{B \otimes A}) \circ f^\dagger : A' \longrightarrow B \oplus (B \otimes A)} \\
\frac{\blacktriangleleft ((\blacktriangleleft^{-1} 1_{B \otimes A}) \circ f^\dagger) : B \otimes A' \longrightarrow B \otimes A}{\quad}
\end{array}$$

1.2

1.

$$\begin{array}{c}
\frac{\frac{\frac{}{a \longrightarrow a} 1_A \quad \frac{}{b \longrightarrow b} 1_A}{a \oplus b \longrightarrow a \oplus b} (\oplus) \quad \frac{}{c \longrightarrow c} 1_A}{\frac{(a \oplus b)/c \longrightarrow (a \oplus b)/c}{((a \oplus b)/c) \otimes c \longrightarrow a \oplus b} (\backslash)} \\
\frac{\frac{(a \oplus b)/c \longrightarrow (a \oplus b)/c}{((a \oplus b)/c) \otimes c \longrightarrow a \oplus b} \blacktriangleright^{-1}}{c \otimes b \longrightarrow ((a \oplus b)/c) \backslash a} \mathfrak{b} \\
\frac{c \otimes b \longrightarrow ((a \oplus b)/c) \backslash a}{((a \oplus b)/c) \otimes (c \otimes b) \longrightarrow a} \blacktriangleleft^{-1} \\
\frac{((a \oplus b)/c) \otimes (c \otimes b) \longrightarrow a}{(a \oplus b)/c \longrightarrow a/(c \otimes b)} \blacktriangleright
\end{array}$$

2.

$$\begin{array}{c}
\frac{\frac{\frac{}{c \longrightarrow c} 1_A \quad \frac{}{a \longrightarrow a} 1_A}{c \otimes a \longrightarrow c \otimes a} (\otimes)}{\frac{c \otimes a \longrightarrow c \otimes a}{c \longrightarrow (c \otimes a)/a} \blacktriangleright} \\
\frac{\frac{}{b \longrightarrow b} 1_A \quad \frac{c \otimes a \longrightarrow c \otimes a}{c \longrightarrow (c \otimes a)/a} \blacktriangleright}{b \oplus c \longrightarrow b \oplus ((c \otimes a)/a)} (\oplus) \\
\frac{b \oplus c \longrightarrow b \oplus ((c \otimes a)/a)}{b \otimes (b \oplus c) \longrightarrow (c \otimes a)/a} \blacktriangleleft \\
\frac{b \otimes (b \oplus c) \longrightarrow (c \otimes a)/a}{(b \oplus c) \otimes a \longrightarrow b \oplus (c \otimes a)} \mathfrak{d}^{-1}
\end{array}$$