Logic and Language: Exercise (Week 5)

Orestis Melkonian [6176208], Konstantinos Kogkalidis [6230067]

1 LG: continuation semantics

1.1

For each term, we compute the ILL type by first observing the polarity of the sequent and then using the table to interpret complex types.

$$
\begin{aligned}
\lceil \textbf{some} \rceil &= \lceil np/n \rceil^{\perp} & \{\textit{Negative Hypothesis}\} \\
&= (\lceil np \rceil^{\perp} \otimes \lceil n \rceil)^{\perp} & \{\lceil A/B \rceil \textit{ with } A \textit{ and } B \textit{ positive}\} \\
&= (np^{\perp} \otimes n)^{\perp} & \{np \textit{ and } n \textit{ positive}\}
\end{aligned}
$$

$$
\begin{aligned}
\lceil \textbf{popular} \rceil &= \lceil n/n \rceil^{\perp} & \{\textit{Negative Hypothesis}\} \\
&= (\lceil n \rceil^{\perp} \otimes \lceil n \rceil)^{\perp} & \{\lceil A/B \rceil \textit{ with } A \textit{ and } B \textit{ positive}\} \\
&= (n^{\perp} \otimes n)^{\perp} & \{n \textit{ positive}\}
\end{aligned}
$$

$$
\begin{aligned}
\lceil \textbf{saint} \rceil &= \lceil n \rceil & \{\textit{Positive Hypothesis}\} \\
&= n & \{n \textit{ positive}\}
\end{aligned}
$$

$$
\begin{aligned}
\lceil \textbf{arrived} \rceil &= \lceil np \backslash s \rceil^{\perp} & \{\textit{Negative Hypothesis}\} \\
&= (\lceil np \rceil \otimes \lceil s \rceil)^{\perp} & \{\lceil B \backslash A \rceil \textit{ with } B \textit{ positive and } A \textit{ negative}\} \\
&= (np \otimes s^{\perp})^{\perp} & \{np \textit{ positive}, s \textit{ negative}\}
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= \lceil s \rceil \\
&= s^{\perp} & \{s \textit{ negative}\}
\end{aligned}
$$

$$
\begin{aligned}
z &= \lceil np \rceil \\
&= np & \{np \textit{ positive}\}
\end{aligned}
$$

$$
\begin{aligned}
y &= \lceil n \rceil \\
&= n & \{n \textit{ positive}\}
\end{aligned}
$$

The ILL types then are:

TERM	TYPE
$\lceil\textbf{some}\rceil$	$(np^\perp \otimes n)^\perp$
$\lceil\textbf{popular}\rceil$	$(n^\perp \otimes n)^\perp$
$\lceil\textbf{saint}\rceil$	n
$\lceil\textbf{arrived}\rceil$	$(np \otimes s^\perp)^\perp$
α	$s^\perp$
z	np
y	n

1.2

SOURCE TYPE	CONSTANT	$\lceil . \rceil^\ell$
n/n	popular	$\lambda\langle c, y\rangle.(c\ (\lambda z.\ \wedge (y\ z)\ (\text{POPULAR}\ z)))$

1.3

1. We compute the interpretation below:

$$\lceil\ddagger\rceil = \lambda a_0.(\lceil\textbf{arrived}\rceil\ \langle\lambda\beta_0.(\lceil\textbf{some}\rceil\ \langle\beta_0, \lambda\gamma_0.(\lceil\textbf{popular}\rceil\ \langle\gamma_0, \lambda a_1.(\lceil\textbf{saint}\rceil\ a_1)\rangle)\rangle), a_0\rangle)$$

2. The adjucted $\cdot^\ell$ translations are the following:

$$\lceil\textbf{some}\rceil^\ell = \lambda\langle x, k\rangle.(\exists\lambda z.(\wedge\ (k\ \lambda\theta.(\theta\ z))\ (x\ z)))$$
$$\lceil\textbf{popular}\rceil^\ell = \lambda\langle c, k\rangle.(c\ (\lambda z.(\wedge\ (\text{POPULAR}\ z)\ (k\ \lambda\theta.(\theta\ z)))))$$
$$\lceil\textbf{saint}\rceil^\ell = \lambda c.(c\ \text{SAINT})$$
$$\lceil\textbf{arrived}\rceil^\ell = \lambda\langle k, c\rangle.(k\ \lambda z.(c\ (\text{ARRIVED}\ z)))$$

We can now corroborate the α-equivalence of the two $\cdot^\ell$ translations:

$\lceil\ddagger\rceil^\ell = \lambda a_0.(\underline{\lambda\langle k, c\rangle.(k\ \lambda z.(c\ (\text{ARRIVED}\ z)))}$
$\qquad\langle\lambda\beta_0.(\lambda\langle x, k\rangle.(\exists\lambda z.(\wedge\ (k\ \lambda\theta.(\theta\ z))\ (x\ z)))\ \langle\beta_0, \lambda\gamma_0.($
$\qquad\quad\lambda\langle c, k\rangle.(c\ (\lambda z.(\wedge\ (\text{POP}\ z)\ (k\ \lambda\theta.(\theta\ z)))))\ \langle\gamma_0, \lambda a_1.(\lambda c.(c\ \text{SAINT})\ a_1)\rangle)\rangle), a_0\rangle)$
$\qquad\to^*_\beta \lambda a_0.(\underline{\lambda\langle x, k\rangle.(\exists\lambda z.(\wedge\ (k\ \lambda\theta.(\theta\ z))\ (x\ z)))}\ \langle\lambda z.(a_0\ (\text{ARRIVED}\ z)), \lambda\gamma_0.($
$\qquad\quad\lambda\langle c, k\rangle.(c\ (\lambda z.(\wedge\ (\text{POPULAR}\ z)\ (k\ \lambda\theta.(\theta\ z)))))\ \langle\gamma_0, \lambda a_1.(\lambda c.(c\ \text{SAINT})\ a_1)\rangle)\rangle)$
$\qquad\to^*_\beta \lambda a_0.(\exists\lambda z.(\wedge\ (\underline{\lambda\langle c, k\rangle.(c\ (\lambda z.(\wedge\ (\text{POPULAR}\ z)\ (k\ \lambda\theta.(\theta\ z)))))}$
$\qquad\quad\langle\lambda\theta.(\theta\ z), \lambda a_1.(\lambda c.(c\ \text{SAINT})\ a_1)\rangle)\ (a_0\ (\text{ARRIVED}\ z)))$
$\qquad\to^*_\beta \lambda a_0.(\exists\lambda z.(\wedge\ (\underline{\lambda\theta.(\theta\ z)}\ (\lambda z.(\wedge\ (\text{POPULAR}\ z)\ (\underline{\lambda c.(c\ \text{SAINT})}\ (\lambda\theta.(\theta\ z)))))))\ (a_0\ (\text{ARRIVED}\ z)))$
$\qquad\to^*_\beta \lambda \textcolor{red}{a_0}.(\exists\lambda z.(\wedge\ (\wedge\ (\text{POPULAR}\ z)\ (\text{SAINT}\ z))\ (\textcolor{red}{a_0}\ (\text{ARRIVED}\ z)))$
$\qquad\to_\alpha \lambda c.(\exists\lambda \textcolor{red}{z}.(\wedge\ (\wedge\ (\text{POPULAR}\ \textcolor{red}{z})\ (\text{SAINT}\ \textcolor{red}{z}))\ (c\ (\text{ARRIVED}\ \textcolor{red}{z})))$
$\qquad\to_\alpha \lambda c.(\exists\lambda x.(\wedge\ (\wedge\ (\text{POPULAR}\ x)\ (\text{SAINT}\ x))\ (c\ (\text{ARRIVED}\ x))))$
$\qquad= \lceil\dagger\rceil^\ell$

2 Pregroups

2.1

$$(\rightarrow) \qquad\qquad\qquad\qquad\qquad (\leftarrow)$$

(4)

$$\dfrac{\dfrac{}{1^l \to 1^l 1}\,(1) \quad \dfrac{}{1^l 1 \to 1}\,(2)}{1^l \to 1}\to \qquad\qquad \dfrac{\dfrac{}{1 \to 1^l 1}\,(2) \quad \dfrac{}{1^l 1 \to 1^l}\,(1)}{1 \to 1^l}\to$$

(5) $(\rightarrow)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A^{rl} \to A^{rl}1}\,(1) \quad \dfrac{}{1 \to A^r A}\,(2)}{A^{rl} \to A^{rl}(A^r A)}\to}{A^{rl} \to (A^{rl}A^r)A}\,(1) \quad \dfrac{}{A^{rl}A^r \to 1}\,(2)}{\dfrac{A^{rl} \to 1A}{A^{rl} \to A}}\to \quad \dfrac{}{1A \to A}\,(1)}{}$$

$(\leftarrow)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \to A1}\,(1) \quad \dfrac{}{1 \to A^r A^{rl}}\,(2)}{A \to A(A^r A^{rl})}\to}{A \to (AA^r)A^{rl}}\,(1) \quad \dfrac{}{AA^r \to 1}\,(3)}{\dfrac{A \to 1A^{rl}}{A \to A^{rl}}}\to \quad \dfrac{}{1A^{rl} \to A^{rl}}\,(1)}{}$$

(6) $(\rightarrow)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(AB)^l \to (1(AB))^l}\to \quad \dfrac{}{1 \to (B^l A^l)^r (B^l A^l)}\,(3)}{(AB)^l \to (((B^l A^l)^r (B^l A^l))(AB))^l}\to}{(AB)^l \to ((B^l A^l)^r ((B^l A^l)(AB)))^l}\,(1)}{(AB)^l \to ((B^l A^l)^r (B^l (A^l (AB))))^l}\,(1)}{(AB)^l \to ((B^l A^l)^r (B^l ((A^l A)B)))^l}\,(1) \quad \dfrac{}{A^l A \to 1}\,(2)}{(AB)^l \to ((B^l A^l)^r (B^l (1B)))^l}\to \quad \dfrac{}{1B \to B}\,(1)}{(AB)^l \to ((B^l A^l)^r (B^l B))^l}\to \quad \dfrac{}{B^l B \to 1}\,(2)}{\dfrac{(AB)^l \to ((B^l A^l)^r 1)^l}{\quad}\,todo \quad \dfrac{}{(B^l A^l)^r 1 \to (}}{\dfrac{(AB)^l \to ((B^l A^l)^r)^l}{(AB)^l \to B^l A^l}\,(5)}$$

2.2

We first calculate the pregroup translation of the given sequent:

$$\overline{(p/((q/q)/r))/r}$$
$$=\overline{(p/((q/q)/r))}\ r^\ell \qquad\qquad \{\overline{A/B} = \overline{A}(\overline{B})^\ell\}$$
$$=p\ \overline{((q/q)/r)}^\ell\ r^\ell \qquad\qquad \{\overline{A/B} = \overline{A}(\overline{B})^\ell\}$$
$$=p\ (\overline{(q/q)}\ r^\ell)^\ell\ r^\ell \qquad\qquad \{\overline{A/B} = \overline{A}(\overline{B})^\ell\}$$
$$=p\ ((q\ q^\ell)\ r^\ell)^\ell\ r^\ell \qquad\qquad \{\overline{A/B} = \overline{A}(\overline{B})^\ell\}$$
$$=p\ (q\ (q^\ell\ r^\ell))^\ell\ r^\ell \qquad\qquad \{rule\ (1)\ from\ 2.1\}$$
$$=p\ (q^\ell\ r^\ell)^\ell\ q^\ell\ r^\ell \qquad\qquad \{rule\ (6)\ from\ 2.1\}$$
$$=p\ r^{\ell\ell}\ q^{\ell\ell}\ q^\ell\ r^\ell \qquad\qquad \{rule\ (6)\ from\ 2.1\}$$

Finally, we prove the sequent by drawing a string diagram: