# Logic and Language: Exercise (Week 3)

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## 1 LGa

### 1.1

1. Unfolding the combinator $f/g$, we get the following derivation:

$$
\begin{prooftree}
\AxiomC{$1_{A/B} : A/B \longrightarrow A/B$}
\UnaryInfC{$\triangleright^{-1} 1_{A/B} : A/B \otimes B \longrightarrow A$}
\AxiomC{$f : A \longrightarrow A'$}
\BinaryInfC{$f \circ (\triangleright^{-1} 1_{A/B}) : A/B \otimes B \longrightarrow A'$}
\UnaryInfC{$\triangleright(f \circ (\triangleright^{-1} 1_{A/B})) : A/B \longrightarrow A'/B$}
\AxiomC{$g : B \longrightarrow B'$}
\AxiomC{$1_{A/B'} : A/B' \longrightarrow A/B'$}
\UnaryInfC{$\triangleright^{-1} 1_{A/B'} : (A/B') \otimes B' \longrightarrow A$}
\UnaryInfC{$\triangleleft\triangleright^{-1} 1_{A/B'} : B' \longrightarrow (A/B')\backslash A$}
\BinaryInfC{$(\triangleleft\triangleright^{-1} 1_{A/B'}) \circ g : B \longrightarrow (A/B')\backslash A$}
\UnaryInfC{$\triangleleft^{-1}((\triangleleft\triangleright^{-1} 1_{A/B'}) \circ g) : (A/B') \otimes B \longrightarrow A$}
\UnaryInfC{$\triangleright\triangleleft^{-1}((\triangleleft\triangleright^{-1} 1_{A/B'}) \circ g) : A/B' \longrightarrow A/B$}
\BinaryInfC{$(\triangleright(f \circ (\triangleright^{-1} 1_{A/B}))) \circ (\triangleright\triangleleft^{-1}((\triangleleft\triangleright^{-1} 1_{A/B'}) \circ g)) : A/B' \longrightarrow A'/B$}
\end{prooftree}
$$

2. Applying the arrow reversal transformation $(\cdot)^\dagger$ to $f/g$, we get the reverse combinator:

$$
\begin{aligned}
(f/g)^\dagger &= ((\triangleright(f \circ (\overset{-1}{\triangleright} 1_{A/B}))) \circ (\triangleright \overset{-1}{\triangleleft}((\triangleleft \overset{-1}{\triangleright} 1_{A/B'}) \circ g)))^\dagger \\
&= (\triangleright \overset{-1}{\triangleleft}((\triangleleft \overset{-1}{\triangleright} 1_{A/B'}) \circ g))^\dagger \circ (\triangleright(f \circ (\overset{-1}{\triangleright} 1_{A/B})))^\dagger \\
&= (\blacktriangleleft(\overset{-1}{\triangleleft}((\triangleleft \overset{-1}{\triangleright} 1_{A/B'}) \circ g))^\dagger) \circ (\blacktriangleleft(f \circ (\overset{-1}{\triangleright} 1_{A/B}))^\dagger) \\
&= (\blacktriangleleft \overset{-1}{\blacktriangleright}((\triangleleft \overset{-1}{\triangleright} 1_{A/B'}) \circ g)^\dagger) \circ (\blacktriangleleft((\overset{-1}{\triangleright} 1_{A/B})^\dagger \circ f^\dagger)) \\
&= (\blacktriangleleft \overset{-1}{\blacktriangleright}(g^\dagger \circ (\triangleleft \overset{-1}{\triangleright} 1_{A/B'})^\dagger)^\dagger) \circ (\blacktriangleleft((\overset{-1}{\blacktriangleleft} 1_{(A/B)^\dagger}) \circ f^\dagger)) \\
&= (\blacktriangleleft \overset{-1}{\blacktriangleright}(g^\dagger \circ (\blacktriangleright(\overset{-1}{\triangleright} 1_{A/B'})^\dagger))) \circ (\blacktriangleleft((\overset{-1}{\blacktriangleleft} 1_{B \obslash A}) \circ f^\dagger)) \\
&= (\blacktriangleleft \overset{-1}{\blacktriangleright}(g^\dagger \circ (\blacktriangleright \overset{-1}{\blacktriangleleft} 1_{(A/B')^\dagger}))) \circ (\blacktriangleleft((\overset{-1}{\blacktriangleleft} 1_{B \obslash A}) \circ f^\dagger)) \\
&= (\blacktriangleleft \overset{-1}{\blacktriangleright}(g^\dagger \circ (\blacktriangleright \overset{-1}{\blacktriangleleft} 1_{B' \obslash A}))) \circ (\blacktriangleleft((\overset{-1}{\blacktriangleleft} 1_{B \obslash A}) \circ f^\dagger))
\end{aligned}
$$

Unfolding the combinator $f/g$, we get the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{1_{B'\obslash A} : B' \obslash A \longrightarrow B' \obslash A}{\blacktriangleleft^{-1} 1_{B'\obslash A} : A \longrightarrow B' \oplus (B' \obslash A)}}{\blacktriangleright\blacktriangleleft^{-1} 1_{B'\obslash A} : A \oslash (B' \obslash A) \longrightarrow B'} \quad g^{\dagger} : B' \longrightarrow B}{g^{\dagger} \circ (\blacktriangleright\blacktriangleleft^{-1} 1_{B'\obslash A}) : A \oslash (B' \obslash A) \longrightarrow B}}{\blacktriangleright^{-1}(g^{\dagger} \circ (\blacktriangleright\blacktriangleleft^{-1} 1_{B'\obslash A})) : A \longrightarrow B \oplus (B' \obslash A)}}{\blacktriangleleft\blacktriangleright^{-1}(g^{\dagger} \circ (\blacktriangleright\blacktriangleleft^{-1} 1_{B'\obslash A})) : B \obslash A \longrightarrow B' \obslash A} \quad \dfrac{\dfrac{f^{\dagger} : A' \longrightarrow A \quad \dfrac{1_{B\obslash A} : B \obslash A \longrightarrow B \obslash A}{\blacktriangleleft^{-1} 1_{B\obslash A} : A \longrightarrow B \oplus (B \obslash A)}}{(\blacktriangleleft^{-1} 1_{B\obslash A}) \circ f^{\dagger} : A' \longrightarrow B \oplus (B \obslash A)}}{\blacktriangleleft((\blacktriangleleft^{-1} 1_{B\obslash A}) \circ f^{\dagger}) : B \obslash A' \longrightarrow B \obslash A}}{(\blacktriangleleft\blacktriangleright^{-1}(g^{\dagger} \circ (\blacktriangleright\blacktriangleleft^{-1} 1_{B'\obslash A}))) \circ (\blacktriangleleft((\blacktriangleleft^{-1} 1_{B\obslash A}) \circ f^{\dagger})) : B \obslash A' \longrightarrow B' \obslash A}$$

## 1.2

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \longrightarrow a}\,1_A \quad \dfrac{}{b \longrightarrow b}\,1_A}{a \oplus b \longrightarrow a \oplus b}\,(\oplus) \quad \dfrac{}{c \longrightarrow c}\,1_A}{(a \oplus b)/c \longrightarrow (a \oplus b)/c}\,(\backslash)}{((a \oplus b)/c) \otimes c \longrightarrow a \oplus b}\,\triangleright^{-1}}{c \oslash b \longrightarrow ((a \oplus b)/c)\backslash a}\,\underset{\cdot}{\mathbf{b}}}{((a \oplus b)/c) \otimes (c \oslash b) \longrightarrow a}\,\triangleleft^{-1}}{(a \oplus b)/c \longrightarrow a/(c \oslash b)}\,\triangleright$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{b \longrightarrow b}\,1_A \quad \dfrac{\dfrac{\dfrac{}{c \longrightarrow c}\,1_A \quad \dfrac{}{a \longrightarrow a}\,1_A}{c \otimes a \longrightarrow c \otimes a}\,(\otimes)}{c \longrightarrow (c \otimes a)/a}\,\triangleright}{b \oplus c \longrightarrow b \oplus ((c \otimes a)/a)}\,(\oplus)}{b \obslash (b \oplus c) \longrightarrow (c \otimes a)/a}\,\blacktriangleleft}{(b \oplus c) \otimes a \longrightarrow b \oplus (c \otimes a)}\,\underset{\cdot}{\mathbf{d}}^{-1}$$

# 2 LGf

## 2.1 Uniform *positive* bias

$$\dfrac{}{s \vdash \boxed{s}}\;Ax \qquad \dfrac{s \vdash \boxed{s}}{s \vdash s}\;\rightharpoonup \qquad \dfrac{s \vdash s}{\boxed{s} \vdash s}\;\leftharpoondown$$

$$\dfrac{\dfrac{}{np \vdash \boxed{np}}\;Ax \qquad \boxed{s} \vdash s}{np \cdot \otimes \cdot \boxed{np\backslash s} \vdash s}\;\backslash L \qquad \dfrac{np \cdot \otimes \cdot \boxed{np\backslash s} \vdash s}{np \cdot \otimes \cdot (np\backslash s) \vdash s}\;\leftharpoonup$$

$$\dfrac{np \cdot \otimes \cdot (np\backslash s) \vdash s \qquad \dfrac{}{s \vdash \boxed{s}}\;Ax}{np \cdot \otimes \cdot (\boxed{(np\backslash s)/s} \cdot \otimes \cdot s) \vdash s}\;/L$$

$$\dfrac{np \cdot \otimes \cdot (\boxed{(np\backslash s)/s} \cdot \otimes \cdot s) \vdash s}{np \cdot \otimes \cdot (((np\backslash s)/s) \cdot \otimes \cdot s) \vdash s}\;\leftharpoonup$$

$$\dfrac{np \cdot \otimes \cdot (((np\backslash s)/s) \cdot \otimes \cdot s) \vdash s}{((np\backslash s)/s) \cdot \otimes \cdot s \vdash np \cdot \backslash \cdot s}\;rp \qquad \dfrac{((np\backslash s)/s) \cdot \otimes \cdot s \vdash np \cdot \backslash \cdot s}{s \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}\;rp$$

$$\dfrac{s \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}{\boxed{s} \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}\;\leftharpoondown$$

$$\dfrac{\dfrac{}{np \vdash \boxed{np}}\;Ax \qquad \boxed{s} \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}{np \cdot \otimes \cdot \boxed{np\backslash s} \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}\;\backslash L$$

$$\dfrac{np \cdot \otimes \cdot \boxed{np\backslash s} \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}{np \cdot \otimes \cdot (np\backslash s) \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}\;\leftharpoonup$$

$$\dfrac{np \cdot \otimes \cdot (np\backslash s) \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}{np \vdash (((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)) \cdot / \cdot (np\backslash s)}\;rp$$

$$\dfrac{np \vdash (((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)) \cdot / \cdot (np\backslash s)}{\boxed{np} \vdash (((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)) \cdot / \cdot (np\backslash s)}\;\leftharpoondown$$

$$\dfrac{\boxed{np} \vdash (((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)) \cdot / \cdot (np\backslash s) \qquad \dfrac{}{n \vdash \boxed{n}}\;Ax}{\boxed{np/n} \cdot \otimes \cdot n \vdash (((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)) \cdot / \cdot (np\backslash s)}\;/L$$

$$\dfrac{\boxed{np/n} \cdot \otimes \cdot n \vdash (((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)) \cdot / \cdot (np\backslash s)}{(np/n) \cdot \otimes \cdot n \vdash (((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)) \cdot / \cdot (np\backslash s)}\;\leftharpoonup$$

$$\dfrac{(np/n) \cdot \otimes \cdot n \vdash (((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)) \cdot / \cdot (np\backslash s)}{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s) \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}\;rp$$

$$\dfrac{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s) \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}{((np\backslash s)/s) \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s)) \vdash np \cdot \backslash \cdot s}\;rp$$

$$\dfrac{((np\backslash s)/s) \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s)) \vdash np \cdot \backslash \cdot s}{np \cdot \otimes \cdot (((np\backslash s)/s) \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s))) \vdash s}\;rp$$

## 2.2 Uniform *negative* bias

$$\dfrac{}{\boxed{n} \vdash n}\;CoAx \qquad \dfrac{\boxed{n} \vdash n}{n \vdash n}\;\leftharpoonup \qquad \dfrac{n \vdash n}{n \vdash \boxed{n}}\;\rightharpoondown$$

$$\dfrac{\dfrac{}{\boxed{np} \vdash np}\;CoAx \qquad n \vdash \boxed{n}}{\boxed{np/n} \cdot \otimes \cdot n \vdash np}\;/L \qquad \dfrac{\boxed{np/n} \cdot \otimes \cdot n \vdash np}{(np/n) \cdot \otimes \cdot n \vdash np}\;\leftharpoonup \qquad \dfrac{(np/n) \cdot \otimes \cdot n \vdash np}{(np/n) \cdot \otimes \cdot n \vdash \boxed{np}}\;\rightharpoondown$$

$$\dfrac{(np/n) \cdot \otimes \cdot n \vdash \boxed{np} \qquad \dfrac{}{\boxed{s} \vdash s}\;CoAx}{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot \boxed{np\backslash s} \vdash s}\;\backslash L$$

$$\dfrac{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot \boxed{np\backslash s} \vdash s}{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s) \vdash s}\;\leftharpoonup \qquad \dfrac{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s) \vdash s}{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s) \vdash \boxed{s}}\;\rightharpoondown$$

$$\dfrac{}{\boxed{np} \vdash np}\;CoAx \qquad \dfrac{\boxed{np} \vdash np}{np \vdash np}\;\leftharpoonup \qquad \dfrac{np \vdash np}{np \vdash \boxed{np}}\;\rightharpoondown$$

$$\dfrac{np \vdash \boxed{np} \qquad \dfrac{}{\boxed{s} \vdash s}\;CoAx}{np \cdot \otimes \cdot \boxed{np\backslash s} \vdash s}\;\backslash L$$

$$\dfrac{np \cdot \otimes \cdot \boxed{np\backslash s} \vdash s \qquad ((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s) \vdash \boxed{s}}{np \cdot \otimes \cdot (\boxed{(np\backslash s)/s} \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s))) \vdash s}\;/L$$

$$\dfrac{np \cdot \otimes \cdot (\boxed{(np\backslash s)/s} \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s))) \vdash s}{np \cdot \otimes \cdot (((np\backslash s)/s) \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s))) \vdash s}\;\leftharpoonup$$

## 2.3 Bias with negative $s$ and positive $np, n$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{np \vdash \boxed{np}}\,Ax \qquad \dfrac{}{\boxed{s} \vdash s}\,CoAx}{\boxed{np\backslash s} \vdash np \cdot \backslash \cdot s}\,\backslash L \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{np \vdash \boxed{np}}\,Ax \qquad \dfrac{}{\boxed{s} \vdash s}\,CoAx}{np \cdot \otimes \cdot \boxed{np\backslash s} \vdash s}\,\backslash L}{np \cdot \otimes \cdot (np\backslash s) \vdash s}\,\leftharpoonup}{np \vdash s \cdot / \cdot (np\backslash s)}\,rp}{\boxed{np} \vdash s \cdot / \cdot (np\backslash s)}\,\leftharpoonup \qquad \dfrac{}{n \vdash \boxed{n}}\,Ax}{(np/n) \cdot \otimes \cdot n \vdash s \cdot / \cdot (np\backslash s)}\,/L}{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s) \vdash s}\,rp}{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s) \vdash \boxed{s}}\,\rightharpoonup}{\dfrac{\boxed{((np\backslash s)/s)} \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s)) \vdash np \cdot \backslash \cdot s}{\dfrac{((np\backslash s)/s) \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s)) \vdash np \cdot \backslash \cdot s}{np \cdot \otimes \cdot (((np\backslash s)/s) \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s))) \vdash s}\,rp}\,\leftharpoonup}\,/L}{}
$$