Logic and Language: Exercise (Week 6)

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1 Syntax

1.1

First, we define the rules of *rightward extraction* $\widehat{\alpha}^r_\diamond$, $\widehat{\sigma}^r_\diamond$:

$$\frac{f : A \otimes (B \otimes \Diamond C) \to D}{\widehat{\alpha}^r_\diamond f : (A \otimes B) \otimes \Diamond C \to D} \qquad \frac{f : (A \otimes \Diamond C) \otimes B \to D}{\widehat{\sigma}^r_\diamond f : (A \otimes B) \otimes \Diamond C \to D}$$

We can now proceed with the derivation of

$$n \otimes ((n\backslash n)/(s/\Diamond\Box np)) \otimes ((np/n) \otimes n) \otimes ((np\backslash s)/np)) \to n$$

as follows:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1_n}
\UnaryInfC{$n \vdash n$}
\AxiomC{}
\RightLabel{1_n}
\UnaryInfC{$n \vdash n$}
\RightLabel{$\backslash$}
\BinaryInfC{$n\backslash n \vdash n\backslash n$}
\AxiomC{}
\RightLabel{1_{np}}
\UnaryInfC{$np \vdash np$}
\AxiomC{}
\RightLabel{1_n}
\UnaryInfC{$n \vdash n$}
\RightLabel{$/$}
\BinaryInfC{$np/n \vdash np/n$}
\RightLabel{$\triangleright^{-1}$}
\UnaryInfC{$(np/n) \otimes n \vdash np$}
\AxiomC{}
\RightLabel{1_s}
\UnaryInfC{$s \vdash s$}
\RightLabel{$\backslash$}
\BinaryInfC{$np\backslash s \vdash ((np/n) \otimes n)\backslash s$}
\AxiomC{}
\RightLabel{1_{np}}
\UnaryInfC{$np \vdash np$}
\RightLabel{$\Box$}
\UnaryInfC{$\Box np \vdash \Box np$}
\RightLabel{$\triangledown^{-1}$}
\UnaryInfC{$\Diamond\Box np \vdash np$}
\RightLabel{$/$}
\BinaryInfC{$(np\backslash s)/np \vdash (((np/n) \otimes n)\backslash s)/\Diamond\Box np$}
\RightLabel{$\triangleright^{-1}$}
\UnaryInfC{$(np\backslash s)/np \otimes \Diamond\Box np \vdash ((np/n) \otimes n)\backslash s$}
\RightLabel{$\triangleleft^{-1}$}
\UnaryInfC{$((np/n) \otimes n) \otimes ((np\backslash s)/np \otimes \Diamond\Box np) \vdash s$}
\RightLabel{$\widehat{\alpha}^r_\diamond$}
\UnaryInfC{$(((np/n) \otimes n) \otimes ((np\backslash s)/np)) \otimes \Diamond\Box np \vdash s$}
\RightLabel{$\triangleright$}
\UnaryInfC{$((np/n) \otimes n) \otimes ((np\backslash s)/np) \vdash s/\Diamond\Box np$}
\RightLabel{$/$}
\BinaryInfC{$(n\backslash n)/(s/\Diamond\Box np) \vdash (n\backslash n)/(((np/n) \otimes n) \otimes (np\backslash s)/np)$}
\RightLabel{$\triangleright^{-1}$}
\UnaryInfC{$(n\backslash n)/(s/\Diamond\Box np) \otimes (((np/n) \otimes (np\backslash s)/np) \vdash n\backslash n$}
\RightLabel{$\triangleleft^{-1}$}
\UnaryInfC{$n \otimes ((n\backslash n)/(s/\Diamond\Box np) \otimes (((np/n) \otimes n) \otimes (np\backslash s)/np)) \vdash n$}
\end{prooftree}
$$

2 Interpretation

2.1

We start by assigning a temporary variable at each rule application in the proof tree:

$$\dfrac{\dfrac{}{np \vdash np}\,1_{np} \qquad \dfrac{}{n \vdash n}\,1_{n}}{f : np/n \vdash np/n}\,/$$

$$\dfrac{f : np/n \vdash np/n}{g : (np/n) \otimes n \vdash np}\,\triangleright^{-1}$$

$$\dfrac{g : (np/n) \otimes n \vdash np \qquad \dfrac{}{s \vdash s}\,1_{s}}{h : np\backslash s \vdash ((np/n) \otimes n)\backslash s}\,\backslash$$

$$\dfrac{\dfrac{\dfrac{}{np \vdash np}\,1_{np}}{\Box np \vdash \Box np}\,\Box}{d : \Diamond\Box np \vdash np}\,\triangledown^{-1}$$

$$\dfrac{h : np\backslash s \vdash ((np/n) \otimes n)\backslash s \qquad d : \Diamond\Box np \vdash np}{i : (np\backslash s)/np \vdash (((np/n) \otimes n)\backslash s)/\Diamond\Box np}\,/$$

$$\dfrac{i : (np\backslash s)/np \vdash (((np/n) \otimes n)\backslash s)/\Diamond\Box np}{j : (np\backslash s)/np \otimes \Diamond\Box np \vdash ((np/n) \otimes n)\backslash s}\,\triangleright^{-1}$$

$$\dfrac{j : (np\backslash s)/np \otimes \Diamond\Box np \vdash ((np/n) \otimes n)\backslash s}{k : ((np/n) \otimes n) \otimes ((np\backslash s)/np \otimes \Diamond\Box np) \vdash s}\,\triangleleft^{-1}$$

$$\dfrac{k : ((np/n) \otimes n) \otimes ((np\backslash s)/np \otimes \Diamond\Box np) \vdash s}{l : (((np/n) \otimes n) \otimes ((np\backslash s)/np)) \otimes \Diamond\Box np \vdash s}\,\widehat{\alpha}^{r}_{\diamond}$$

$$\dfrac{l : (((np/n) \otimes n) \otimes ((np\backslash s)/np)) \otimes \Diamond\Box np \vdash s}{n : ((np/n) \otimes n) \otimes ((np\backslash s)/np) \vdash s/\Diamond\Box np}\,\triangleright$$

$$\dfrac{\dfrac{}{n \vdash n}\,1_{n} \qquad \dfrac{}{n \vdash n}\,1_{n}}{m : n\backslash n \vdash n\backslash n}\,\backslash$$

$$\dfrac{m : n\backslash n \vdash n\backslash n \qquad n : ((np/n) \otimes n) \otimes ((np\backslash s)/np) \vdash s/\Diamond\Box np}{o : (n\backslash n)/(s/\Diamond\Box np) \vdash (n\backslash n)/(((np/n) \otimes n) \otimes (np\backslash s)/np)}\,/$$

$$\dfrac{o : (n\backslash n)/(s/\Diamond\Box np) \vdash (n\backslash n)/(((np/n) \otimes n) \otimes (np\backslash s)/np)}{p : (n\backslash n)/(s/\Diamond\Box np) \otimes (((np/n) \otimes (np\backslash s)/np) \vdash n\backslash n}\,\triangleright^{-1}$$

$$\dfrac{p : (n\backslash n)/(s/\Diamond\Box np) \otimes (((np/n) \otimes (np\backslash s)/np) \vdash n\backslash n}{q : n \otimes ((n\backslash n)/(s/\Diamond\Box np) \otimes (((np/n) \otimes n) \otimes (np\backslash s)/np)) \vdash n}\,\triangleleft^{-1}$$

We now work our way top-down through the proof-tree, writing the interpretation of each formula using the rules of 3.1.

$$
\begin{aligned}
\lceil f \rceil &: (1_{\mathsf{N}} \otimes \eta_{\mathsf{N}} \otimes 1_{\mathsf{N}}) \circ (1_{\mathsf{N}\otimes\mathsf{N}} \otimes 1_{\mathsf{N}} \otimes 1_{\mathsf{N}}) \circ (1_{\mathsf{N}\otimes\mathsf{N}} \otimes \epsilon_{\mathsf{N}}) \\
\lceil g \rceil \equiv \lceil \triangleright^{-1} f \rceil &: (\lceil f \rceil \otimes 1_{\mathsf{N}}) \circ (1_{\mathsf{N}} \otimes \epsilon_{\mathsf{N}}) \\
\lceil h \rceil &: (1_{\mathsf{N}} \otimes \eta_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}} \otimes 1_{\mathsf{S}}) \circ (1_{\mathsf{N}} \otimes \lceil g \rceil \otimes 1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}}) \circ (\epsilon_{\mathsf{N}} \otimes 1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}}) \\
\lceil d \rceil &: 1_{\mathsf{N}} \\
\lceil i \rceil &: (\lceil h \rceil \otimes \eta_{\mathsf{N}} \otimes 1_{\mathsf{N}}) \circ (1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{S}} \otimes \lceil d \rceil \otimes 1_{\mathsf{N}}) \otimes (1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{S}} \otimes \epsilon_{\mathsf{S}}) \\
\lceil j \rceil \equiv \lceil \triangleright^{-1} i \rceil &: (\lceil i \rceil \otimes 1_{\mathsf{N}}) \circ (1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}} \otimes \epsilon_{\mathsf{N}}) \\
\lceil k \rceil \equiv \lceil \triangleleft^{-1} j \rceil &: (1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}} \otimes \lceil j \rceil) \circ (\epsilon_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}} \otimes 1_{\mathsf{S}}) \\
\lceil l \rceil \equiv \widehat{\alpha}^{r}_{\diamond} k &: \alpha \circ \lceil k \rceil \\
\lceil m \rceil &: (1_{\mathsf{N}} \otimes \eta_{\mathsf{N}} \otimes 1_{\mathsf{N}}) \circ (1_{\mathsf{N}} \otimes 1_{\mathsf{N}} \otimes 1_{\mathsf{N}\otimes\mathsf{N}}) \circ (\epsilon_{\mathsf{N}} \otimes 1_{\mathsf{N}\otimes\mathsf{N}}) \\
\lceil n \rceil \equiv \lceil \triangleright l \rceil &: (1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}} \otimes \eta_{\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{N}}) \circ (\lceil l \rceil \otimes 1_{\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{N}}) \\
\lceil o \rceil &: (\lceil m \rceil \otimes \eta_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{N}} \otimes 1_{\mathsf{S}\otimes\mathsf{N}}) \circ (1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{N}} \otimes \lceil n \rceil \otimes 1_{\mathsf{S}\otimes\mathsf{N}}) \\
&\quad \circ (1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{N}} \otimes \epsilon_{\mathsf{S}\otimes\mathsf{N}}) \\
\lceil p \rceil \equiv \lceil \triangleright^{-1} o \rceil &: (\lceil o \rceil \otimes 1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{N}}) \circ (1_{\mathsf{N}\otimes\mathsf{N}} \otimes \epsilon_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{N}}) \\
\lceil q \rceil \equiv \lceil \triangleleft^{-1} p \rceil &: (1_{\mathsf{N}} \otimes \lceil p \rceil) \circ (\epsilon_{\mathsf{N}} \otimes 1_{\mathsf{N}})
\end{aligned}
$$

Recursively unwrapping the above, we obtain the (unarguably humongous) final interpretation:

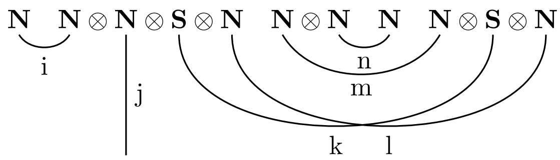
$$
\begin{aligned}
&(1_{\mathsf{N}} \otimes ((((((1_{\mathsf{N}} \otimes \eta_{\mathsf{N}} \otimes 1_{\mathsf{N}}) \circ (1_{\mathsf{N}} \otimes 1_{\mathsf{N}} \otimes 1_{\mathsf{N}\otimes\mathsf{N}}) \circ (\epsilon_{\mathsf{N}} \otimes 1_{\mathsf{N}\otimes\mathsf{N}})) \otimes \eta_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{N}} \otimes 1_{\mathsf{S}\otimes\mathsf{N}}) \circ (1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{N}} \\
&\otimes ((1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}} \otimes \eta_{\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{N}}) \circ ((\alpha \circ ((1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}} \otimes ((((((1_{\mathsf{N}} \otimes \eta_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}} \otimes 1_{\mathsf{S}}) \circ (1_{\mathsf{N}} \otimes ((((1_{\mathsf{N}} \otimes \eta_{\mathsf{N}} \otimes 1_{\mathsf{N}}) \circ (1_{\mathsf{N}\otimes\mathsf{N}} \\
&\otimes 1_{\mathsf{N}} \otimes 1_{\mathsf{N}}) \circ (1_{\mathsf{N}\otimes\mathsf{N}} \otimes \epsilon_{\mathsf{N}})) \otimes 1_{\mathsf{N}}) \circ (1_{\mathsf{N}} \otimes \epsilon_{\mathsf{N}})) \otimes 1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}}) \circ (\epsilon_{\mathsf{N}} \otimes 1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}})) \otimes \eta_{\mathsf{N}} \otimes 1_{\mathsf{N}}) \circ (1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{S}} \\
&\otimes 1_{\mathsf{N}} \otimes 1_{\mathsf{N}}) \otimes (1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{S}} \otimes \epsilon_{\mathsf{S}})) \otimes 1_{\mathsf{N}}) \circ (1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}} \otimes \epsilon_{\mathsf{N}}))) \circ (\epsilon_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}} \otimes 1_{\mathsf{S}}))) \otimes 1_{\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{N}})) \otimes 1_{\mathsf{S}\otimes\mathsf{N}})) \\
&\otimes 1_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{N}}) \circ (1_{\mathsf{N}\otimes\mathsf{N}} \otimes \epsilon_{\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{N}\otimes\mathsf{S}\otimes\mathsf{N}}))) \circ (\epsilon_{\mathsf{N}} \otimes 1_{\mathsf{N}})
\end{aligned}
$$

2.2

By working our way from the leaves of the proof tree, we get the following generalized Kronecker delta:

$$
\begin{aligned}
&\mathbf{island}_i \otimes \mathbf{that}_{j,k,l,m} \otimes \mathbf{the}_{n,o} \otimes \mathbf{hurricane}_p \otimes \mathbf{destroyed}_{q,r,s} \xrightarrow{\delta^{i,k,l,m,n,o}_{j,t,r,s,q,p}} \mathbf{v}_r^{obj} \in \mathrm{N} \\
\mathbf{v}_r^{obj} =&\mathbf{island}_i \otimes \mathbf{that}_{i,j,k,l} \otimes \mathbf{the}_{m,n} \otimes \mathbf{hurricane}_n \otimes \mathbf{destroyed}_{m,k,l} \quad \text{(relabeled)}
\end{aligned}
$$

We give the matching diagram in the figure below:



2.3

In order to calculate the semantic value for the relative clause body 'the hurricane destroyed', we first apply $\mathbf{the}_{MN}$ to $\mathbf{hurricane}_N$. The operation yields the noun-phrase $\mathbf{the\ hurricane}_M$, represented by a row-vector equal to that of **hurricane**. The verb $\mathbf{destroyed}_{MKL}$ is then applied to the resulting vector, thereupon we obtain the final result $\mathbf{the\ hurricane\ destroyed}_{KL}$. Concretely, $\mathbf{the\ hurricane\ destroyed}_{KL} = \mathbf{destroyed}_{MKL}(\mathbf{the}_{MN}\mathbf{hurricane}_N)$ is a 2 by 3 matrix, the elements of which are:

$$
\begin{pmatrix} 12 & -19 & 3 \\ 5 & 10 & 1 \end{pmatrix}
$$

given by:

$$
\mathbf{the\ hurricane\ destroyed}(k,l) = \sum_{m \epsilon M} \mathbf{hurricane}(m) \times \mathbf{destroyed}(m,k,l), \ \forall\ k \ \epsilon\ K,\ l \ \epsilon\ L
$$

The corresponding Python code is given below:

```
import numpy as np
hurricane, island, the = np.array([3,-5,5]), np.array([-5,4,0]), np.eye(3)
destroyed = np.array([[[4,-3,1],[5,5,2]], [[-1,-2,2],[2,-3,0]], [[-1,-4,2],[0,-4,-1]]])
the_hurricane = np.matmul(the, hurricane)  # == hurricane
the_hurricane_destroyed = np.tensordot(the_hurricane, destroyed, axes=1)
```

2.4

The interpreted type for the relative pronoun is:

$$\lceil (n\backslash n)/(s/\Diamond\Box np)\rceil = \lceil n\backslash n\rceil \otimes \lceil s/\Diamond\Box np\rceil = \lceil n\rceil \otimes \lceil n\rceil \otimes \lceil s\rceil \otimes \lceil \Diamond\Box np\rceil = N \otimes N \otimes S \otimes N$$

We can now give the following Frobenius recipe for **that**:

$$I \cong I \otimes I \xrightarrow{\eta_N \otimes \eta_N} N \otimes N \otimes N \otimes N \cong N \otimes N \otimes N \otimes I \otimes N \xrightarrow{1_N \otimes \mu_N \otimes \zeta_S \otimes 1_N} N \otimes N \otimes S \otimes N$$

In order to obtain the final interpretation, we do the following (dictated from the above recipe):

1. Reduce the rank of the transitive verb by summing over the S component, thus obtaining the following matrix:

$$\textbf{collapsed\_destroyed} = \begin{pmatrix} \begin{pmatrix} 9 & 2 & 3 \end{pmatrix} \\ \begin{pmatrix} 1 & -5 & 2 \end{pmatrix} \\ \begin{pmatrix} -1 & -8 & 1 \end{pmatrix} \end{pmatrix}$$

2. Apply **collapsed_destroyed** to **the_hurricane** in subject position:

$$\textbf{the\_hurricane\_destroyed} = \begin{pmatrix} 17 & -9 & 4 \end{pmatrix}$$

3. Multiply **the_hurricane_destroyed** element-wise with **island**:

$$\textbf{island\_that\_the\_hurricane\_destroyed} = \begin{pmatrix} -85 & -36 & 0 \end{pmatrix}$$

The corresponding Python code is given below:

```python
import numpy as np
hurricane, island, the = np.array([3,-5,5]), np.array([-5,4,0]), np.eye(3)
destroyed = np.array([[[4,-3,1],[5,5,2]], [[-1,-2,2],[2,-3,0]], [[-1,-4,2],[0,-4,-1]]])
the_hurricane = np.matmul(the, hurricane)  # == hurricane
collapsed_destroyed = np.sum(destroyed, axis=1) # sum over the S dimension
the_hurricane_destroyed = np.matmul(hurricane, collapsed_destroyed)
island_that_the_hurricane_destroyed = island * hurricane_destroyed # element-wise
```