Logic and Language: Exercise (Week 3)

Orestis Melkonian [6176208], Konstantinos Kogkalidis [6230067]

1 LGa

1.1

1. Unfolding the combinator f/g, we get the following derivation:

$$\cfrac{\cfrac{1_{A/B} : A/B \longrightarrow A/B}{\cfrac{\triangleright^{-1} 1_{A/B} : A/B \otimes B \longrightarrow A}{\cfrac{f \circ (\triangleright^{-1} 1_{A/B}) : A/B \otimes B \longrightarrow A'}{\triangleright(f \circ (\triangleright^{-1} 1_{A/B})) : A/B \longrightarrow A'/B}} \quad f : A \longrightarrow A'} \quad \cfrac{g : B \longrightarrow B' \quad \cfrac{\cfrac{\cfrac{1_{A/B'} : A/B' \longrightarrow A/B'}{\triangleright^{-1} 1_{A/B'} : (A/B') \otimes B' \longrightarrow A}}{\triangleleft\triangleright^{-1} 1_{A/B'} : B' \longrightarrow (A/B')\backslash A}}{\cfrac{(\triangleleft\triangleright^{-1} 1_{A/B'}) \circ g : B \longrightarrow (A/B')\backslash A}{\cfrac{\triangleleft^{-1}((\triangleleft\triangleright^{-1} 1_{A/B'}) \circ g) : (A/B') \otimes B \longrightarrow A}{\triangleright\triangleleft^{-1}((\triangleleft\triangleright^{-1} 1_{A/B'}) \circ g) : A/B' \longrightarrow A/B}}}}{(\triangleright(f \circ (\triangleright^{-1} 1_{A/B}))) \circ (\triangleright\triangleleft^{-1}((\triangleleft\triangleright^{-1} 1_{A/B'}) \circ g)) : A/B' \longrightarrow A'/B}$$

2. Applying the arrow reversal transformation $(\cdot)^\dagger$ to f/g, we get the reverse combinator:

$$(f/g)^\dagger = ((\triangleright(f \circ (\overset{-1}{\triangleright} 1_{A/B}))) \circ (\triangleright \overset{-1}{\triangleleft}((\triangleleft \overset{-1}{\triangleright} 1_{A/B'}) \circ g)))^\dagger$$

$$= (\triangleright \overset{-1}{\triangleleft}((\triangleleft \overset{-1}{\triangleright} 1_{A/B'}) \circ g))^\dagger \circ (\triangleright(f \circ (\overset{-1}{\triangleright} 1_{A/B})))^\dagger$$

$$= (\blacktriangleleft(\overset{-1}{\triangleleft}((\triangleleft \overset{-1}{\triangleright} 1_{A/B'}) \circ g))^\dagger) \circ (\blacktriangleleft(f \circ (\overset{-1}{\triangleright} 1_{A/B}))^\dagger)$$

$$= (\blacktriangleleft \overset{-1}{\blacktriangleright}((\triangleleft \overset{-1}{\triangleright} 1_{A/B'}) \circ g)^\dagger) \circ (\blacktriangleleft((\overset{-1}{\triangleright} 1_{A/B})^\dagger \circ f^\dagger))$$

$$= (\blacktriangleleft \overset{-1}{\blacktriangleright}(g^\dagger \circ (\triangleleft \overset{-1}{\triangleright} 1_{A/B'})^\dagger)^\dagger) \circ (\blacktriangleleft((\overset{-1}{\blacktriangleleft} 1_{(A/B)^\dagger}) \circ f^\dagger))$$

$$= (\blacktriangleleft \overset{-1}{\blacktriangleright}(g^\dagger \circ (\blacktriangleright(\overset{-1}{\triangleright} 1_{A/B'})^\dagger))) \circ (\blacktriangleleft((\overset{-1}{\blacktriangleleft} 1_{B \otimes A}) \circ f^\dagger))$$

$$= (\blacktriangleleft \overset{-1}{\blacktriangleright}(g^\dagger \circ (\blacktriangleright \overset{-1}{\blacktriangleleft} 1_{(A/B')^\dagger}))) \circ (\blacktriangleleft((\overset{-1}{\blacktriangleleft} 1_{B \otimes A}) \circ f^\dagger))$$

$$= (\blacktriangleleft \overset{-1}{\blacktriangleright}(g^\dagger \circ (\blacktriangleright \overset{-1}{\blacktriangleleft} 1_{B' \otimes A}))) \circ (\blacktriangleleft((\overset{-1}{\blacktriangleleft} 1_{B \otimes A}) \circ f^\dagger))$$

Unfolding the combinator f/g, we get the following derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{1_{B' \oslash A} : B' \oslash A \longrightarrow B' \oslash A}{\blacktriangleleft^{-1} 1_{B' \oslash A} : A \longrightarrow B' \oplus (B' \oslash A)}}{\blacktriangleright \blacktriangleleft^{-1} 1_{B' \oslash A} : A \oslash (B' \oslash A) \longrightarrow B' \quad g^\dagger : B' \longrightarrow B}}{g^\dagger \circ (\blacktriangleright \blacktriangleleft^{-1} 1_{B' \oslash A}) : A \oslash (B' \oslash A) \longrightarrow B}}{\blacktriangleright^{-1}(g^\dagger \circ (\blacktriangleright \blacktriangleleft^{-1} 1_{B' \oslash A})) : A \longrightarrow B \oplus (B' \oslash A)}}{\blacktriangleleft \blacktriangleright^{-1}(g^\dagger \circ (\blacktriangleright \blacktriangleleft^{-1} 1_{B' \oslash A})) : B \oslash A \longrightarrow B' \oslash A} \qquad \cfrac{\cfrac{\cfrac{f^\dagger : A' \longrightarrow A \quad \cfrac{1_{B \oslash A} : B \oslash A \longrightarrow B \oslash A}{\blacktriangleleft^{-1} 1_{B \oslash A} : A \longrightarrow B \oplus (B \oslash A)}}{(\blacktriangleleft^{-1} 1_{B \oslash A}) \circ f^\dagger : A' \longrightarrow B \oplus (B \oslash A)}}{\blacktriangleleft((\blacktriangleleft^{-1} 1_{B \oslash A}) \circ f^\dagger) : B \oslash A' \longrightarrow B \oslash A}}{}}{(\blacktriangleleft \blacktriangleright^{-1}(g^\dagger \circ (\blacktriangleright \blacktriangleleft^{-1} 1_{B' \oslash A}))) \circ (\blacktriangleleft((\blacktriangleleft^{-1} 1_{B \oslash A}) \circ f^\dagger)) : B \oslash A' \longrightarrow B' \oslash A}$$

1.2

1.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overline{a \longrightarrow a}\ 1_A \quad \overline{b \longrightarrow b}\ 1_A}{a \oplus b \longrightarrow a \oplus b}\ (\oplus) \quad \overline{c \longrightarrow c}\ 1_A}{(a \oplus b)/c \longrightarrow (a \oplus b)/c}\ (\backslash)}{((a \oplus b)/c) \otimes c \longrightarrow a \oplus b}\ \triangleright^{-1}}{c \oslash b \longrightarrow ((a \oplus b)/c)\backslash a}\ \mathbf{b}}{((a \oplus b)/c) \otimes (c \oslash b) \longrightarrow a}\ \triangleleft^{-1}}{(a \oplus b)/c \longrightarrow a/(c \oslash b)}\ \triangleright}$$

2.

$$\cfrac{\cfrac{\cfrac{\overline{b \longrightarrow b}\ 1_A \quad \cfrac{\cfrac{\overline{c \longrightarrow c}\ 1_A \quad \overline{a \longrightarrow a}\ 1_A}{c \otimes a \longrightarrow c \otimes a}\ (\otimes)}{c \longrightarrow (c \otimes a)/a}\ \triangleright}{b \oplus c \longrightarrow b \oplus ((c \otimes a)/a)}\ (\oplus)}{b \oslash (b \oplus c) \longrightarrow (c \otimes a)/a}\ \blacktriangleleft}{(b \oplus c) \otimes a \longrightarrow b \oplus (c \otimes a)}\ \mathbf{d}^{-1}$$

2 LGf

2.1 Uniform *positive* bias

$$
\dfrac{\dfrac{\dfrac{\dfrac{s \vdash \boxed{s}}{\dfrac{s \vdash s}{\boxed{s} \vdash s}\ {\scriptstyle\leftharpoonup}}\ {\scriptstyle\rightharpoonup}\ Ax}{\cdots}}{\cdots}}{\cdots}
$$

$$
\begin{array}{c}
\dfrac{s \vdash \boxed{s}}{\ }\ Ax \\[2pt]
\dfrac{s \vdash s}{\ }\ {\scriptstyle\rightharpoonup} \\[2pt]
\dfrac{\boxed{s} \vdash s}{\ }\ {\scriptstyle\leftharpoonup}
\end{array}
$$

$$
\dfrac{\ }{np \vdash \boxed{np}}\ Ax
$$

$$
\dfrac{np \cdot \otimes \cdot \boxed{np\backslash s} \vdash s}{\ }\ \backslash L
$$

$$
\dfrac{np \cdot \otimes \cdot (np\backslash s) \vdash s}{\ }\ {\scriptstyle\leftharpoonup}
$$

$$
\dfrac{\ }{s \vdash \boxed{s}}\ Ax
$$

$$
\dfrac{np \cdot \otimes \cdot \big(\boxed{(np\backslash s)/s} \cdot \otimes \cdot s\big) \vdash s}{\ }\ /L
$$

$$
\dfrac{np \cdot \otimes \cdot \big(((np\backslash s)/s) \cdot \otimes \cdot s\big) \vdash s}{\ }\ {\scriptstyle\leftharpoonup}
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$$
\dfrac{((np\backslash s)/s) \cdot \otimes \cdot s \vdash np \cdot \backslash \cdot s}{\ }\ rp
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$$
\dfrac{s \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}{\ }\ rp
$$

$$
\dfrac{\boxed{s} \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}{\ }\ {\scriptstyle\leftharpoonup}
$$

$$
\dfrac{\ }{np \vdash \boxed{np}}\ Ax
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$$
\dfrac{np \cdot \otimes \cdot \boxed{np\backslash s} \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}{\ }\ \backslash L
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$$
\dfrac{np \cdot \otimes \cdot (np\backslash s) \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}{\ }\ {\scriptstyle\leftharpoonup}
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$$
\dfrac{np \vdash (((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)) \cdot / \cdot (np\backslash s)}{\ }\ rp
$$

$$
\dfrac{\boxed{np} \vdash (((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)) \cdot / \cdot (np\backslash s)}{\ }\ {\scriptstyle\leftharpoonup}
$$

$$
\dfrac{\ }{n \vdash \boxed{n}}\ Ax
$$

$$
\dfrac{\boxed{np/n} \cdot \otimes \cdot n \vdash (((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)) \cdot / \cdot (np\backslash s)}{\ }\ /L
$$

$$
\dfrac{(np/n) \cdot \otimes \cdot n \vdash (((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)) \cdot / \cdot (np\backslash s)}{\ }\ {\scriptstyle\leftharpoonup}
$$

$$
\dfrac{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s) \vdash ((np\backslash s)/s) \cdot \backslash \cdot (np \cdot \backslash \cdot s)}{\ }\ rp
$$

$$
\dfrac{((np\backslash s)/s) \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s)) \vdash np \cdot \backslash \cdot s}{\ }\ rp
$$

$$
\dfrac{np \cdot \otimes \cdot (((np\backslash s)/s) \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s))) \vdash s}{\ }\ rp
$$

2.2 Uniform *negative* bias

$$
\dfrac{\ }{\boxed{n} \vdash n}\ CoAx
$$

$$
\dfrac{n \vdash n}{\ }\ {\scriptstyle\leftharpoonup}
$$

$$
\dfrac{n \vdash \boxed{n}}{\ }\ {\scriptstyle\rightharpoonup}
$$

$$
\dfrac{\ }{\boxed{np} \vdash np}\ CoAx
$$

$$
\dfrac{\boxed{np/n} \cdot \otimes \cdot n \vdash np}{\ }\ /L
$$

$$
\dfrac{(np/n) \cdot \otimes \cdot n \vdash np}{\ }\ {\scriptstyle\leftharpoonup}
$$

$$
\dfrac{(np/n) \cdot \otimes \cdot n \vdash \boxed{np}}{\ }\ {\scriptstyle\rightharpoonup}
$$

$$
\dfrac{\ }{\boxed{np} \vdash np}\ CoAx
$$

$$
\dfrac{np \vdash np}{\ }\ {\scriptstyle\leftharpoonup}
$$

$$
\dfrac{np \vdash \boxed{np}}{\ }\ {\scriptstyle\rightharpoonup}
$$

$$
\dfrac{\ }{\boxed{s} \vdash s}\ CoAx
$$

$$
\dfrac{np \cdot \otimes \cdot \boxed{np\backslash s} \vdash s}{\ }\ \backslash L
$$

$$
\dfrac{\ }{\boxed{s} \vdash s}\ CoAx
$$

$$
\dfrac{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot \boxed{np\backslash s} \vdash s}{\ }\ \backslash L
$$

$$
\dfrac{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s) \vdash s}{\ }\ {\scriptstyle\leftharpoonup}
$$

$$
\dfrac{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s) \vdash \boxed{s}}{\ }\ {\scriptstyle\rightharpoonup}
$$

$$
\dfrac{np \cdot \otimes \cdot \big(\boxed{(np\backslash s)/s} \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s))\big) \vdash s}{\ }\ /L
$$

$$
\dfrac{np \cdot \otimes \cdot \big(((np\backslash s)/s) \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s))\big) \vdash s}{\ }\ {\scriptstyle\leftharpoonup}
$$

2.3 Bias with negative s and positive np, n

$$
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  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          \cfrac{\cfrac{np \vdash \boxed{np} \;\; Ax \qquad \boxed{s} \vdash s \;\; CoAx}{np \cdot \otimes \cdot \boxed{np\backslash s} \vdash s} \backslash L}{np \cdot \otimes \cdot (np\backslash s) \vdash s} \llcorner
        }{np \vdash s \cdot / \cdot (np\backslash s)} rp
      }{\boxed{np} \vdash s \cdot / \cdot (np\backslash s)} \llcorner \qquad n \vdash \boxed{n} \;\; Ax
    }{\boxed{np/n} \cdot \otimes \cdot n \vdash s \cdot / \cdot (np\backslash s)} /L
  }{\cfrac{\cfrac{((np/n) \cdot \otimes \cdot n) \vdash s \cdot / \cdot (np\backslash s)}{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s) \vdash s} rp}{((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s) \vdash \boxed{s}} \urcorner} \llcorner
}{}
$$

$$
\cfrac{
  \cfrac{np \vdash \boxed{np} \;\; Ax \qquad \boxed{s} \vdash s \;\; CoAx}{np \cdot \otimes \cdot \boxed{np\backslash s} \vdash s} \backslash L
  \qquad
  \big[\,\text{(above)}\,\big]
}{
  \cfrac{np \cdot \otimes \cdot (\boxed{(np\backslash s)/s} \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s))) \vdash s}{np \cdot \otimes \cdot (((np\backslash s)/s) \cdot \otimes \cdot (((np/n) \cdot \otimes \cdot n) \cdot \otimes \cdot (np\backslash s))) \vdash s} \llcorner
} /L
$$