D^3 as a 2-MCFL

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1.3 Challenge

[This is not a hand-in exercise. If you can solve it by Dec 5, there will be a present for you!]

Let D^n be the language over an n-symbol alphabet, lexicographically ordered $a_1 < \cdots < a_n$, where words satisfy the following conditions:

- 1. each word contains an equal number of the n alphabet symbols
- 2. for every prefix p of a word, the number of a_i in $p \ge$ the number of a_{i+1} $(1 \le i \le n-1)$

 D^n generalizes the familiar language of balanced brackets, in which case you have an alphabet of size 2, say $\{a,b\}$, with 'opening bracket' a preceding 'closing bracket' b in the lexicographic ordering.

The conjecture (Makoto Kanazawa, p.c.) is that for $n \ge 2$, D^n is the language of a non-wellnested (n-1)-MCFG.

Give a 2-MCFG for D^3 , i.e. words over a 3-letter alphabet $\{a, b, c\}$ (with the usual lexicographic order) satisfying conditions (1) and (2) above. Give the ACG encoding of your MCFG for D^3 .

Reference M. Moortgat (2014), A note on multidimensional Dyck languages.

DYCK WORDS

- abc
- aabbcc
- abcabcabacbc

Dyck words

- abc
- aabbcc
- abcabcabacbc

Non-dyck words

aabb

Dyck words

- abc
- aabbcc
- abcabcabacbc

Non-dyck words

- aabb
- aabbbcc

Dyck words

- abc
- aabbcc
- abcabcabacbc

Non-dyck words

- aabb
- aabbbcc
- abcacb

Dyck words

- abc
- aabbcc
- abcabcabacbc

Non-dyck words

- aabb
- aabbbcc
- abcacb

ababacbcabcc

Dyck words

- abc
- aabbcc
- abcabcabacbc

Non-dyck words

- aabb
- aabbbcc
- abcacb



Dyck words

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- aabbcc
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Non-dyck words

- aabb
- aabbbcc
- abcacb



Dyck words

- abc
- aabbcc
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Non-dyck words

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- aabbbcc
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Dyck words

- abc
- aabbcc
- abcabcabacbc

Non-dyck words

- aabb
- aabbbcc
- abcacb



$$S(xy) \leftarrow W(x, y). \tag{1}$$

$$W(\epsilon, xyabc) \leftarrow W(x, y). \tag{2}$$

$$W(\epsilon, xaybc) \leftarrow W(x, y). \tag{3}$$

$$\dots \dots$$

$$W(abxcy, \epsilon) \leftarrow W(x, y). \tag{60}$$

$$W(abcxy, \epsilon) \leftarrow W(x, y). \tag{61}$$

$$W(\epsilon, abc). \tag{62}$$

$$W(a, bc). \tag{63}$$

$$W(ab, c). \tag{64}$$

$$W(abc, \epsilon). \tag{65}$$

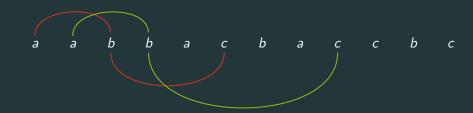




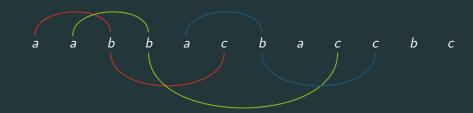
Straddling counter-example



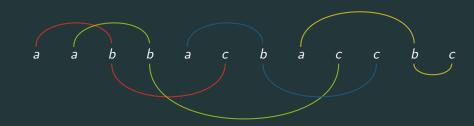
Straddling counter-example



Straddling counter-example



Straddling counter-example



Straddling counter-example

Meta-grammars: Introduction

NOTATION

 $\mathcal{O}_m[\![$ conclusion \leftarrow premises | $\{$ partial orderings of inserted elements $\}]\![$.

Meta-grammars: Introduction

NOTATION

 $\mathcal{O}_m[$ conclusion \leftarrow premises $| \{ partial \ orderings \ of \ inserted \ elements \}].$

Meta-grammar G₁

$$\mathcal{O}_{2}[W \leftarrow \epsilon \mid \{a < b < c\}]].$$

$$\mathcal{O}_{2}[W \leftarrow W_{xy} \mid \{x < y, \ a < b < c\}]].$$
TRIPLE
INSERTION

Meta-grammars: Introduction

NOTATION

 $\mathcal{O}_m[[conclusion \leftarrow premises \mid \{partial \ orderings \ of \ inserted \ elements \}]].$

Meta-grammar G₁

$$\begin{array}{c} \mathcal{O}_2 \llbracket W \leftarrow \epsilon \mid \{a < b < c\} \rrbracket. \\ \mathcal{O}_2 \llbracket W \leftarrow W_{xy} \mid \{x < y, \ a < b < c\} \rrbracket. \end{array} \right\} \begin{array}{c} \text{TRIPLE} \\ \text{INSERTION} \\ + \\ \mathcal{O}_2 \llbracket W \leftarrow W_{xy}, W_{zw} \mid \{x < y, \ z < w\} \rrbracket. \end{array} \right\} \begin{array}{c} \text{INTERLEAVING} \\ \text{WORDS} \end{array}$$

G₂: Adding states

$$\left. \begin{array}{l}
\mathcal{O}_{2}[A^{+} \leftarrow \epsilon \mid \{a\}]]. \\
\mathcal{O}_{2}[B^{+} \leftarrow \epsilon \mid \{b\}]]. \\
\mathcal{O}_{2}[C^{+} \leftarrow \epsilon \mid \{c\}]].
\end{array} \right\} \text{BASE CASES}$$

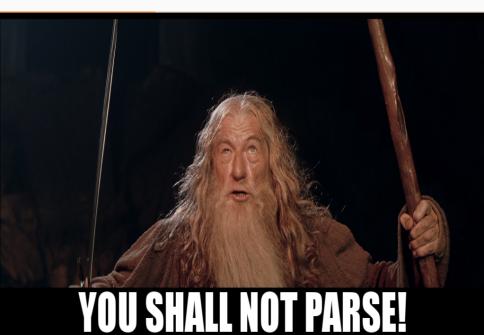
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\begin{array}{c}
\mathcal{O}_{2}[\![C^{-} \leftarrow A^{+}, B^{+} \mid \{x < y < z < w\}]\!]. \\
\mathcal{O}_{2}[\![B^{-} \leftarrow A^{+}, C^{+} \mid \{x < y < z < w\}]\!]. \\
\mathcal{O}_{2}[\![A^{-} \leftarrow B^{+}, C^{+} \mid \{x < y < z < w\}]\!]. \\
\mathcal{O}_{2}[\![A^{+} \leftarrow C^{-}, B^{-} \mid \{x < y < z < w\}]\!]. \\
\mathcal{O}_{2}[\![B^{+} \leftarrow C^{-}, A^{-} \mid \{x < y < z < w\}]\!]. \\
\mathcal{O}_{2}[\![C^{+} \leftarrow B^{-}, A^{-} \mid \{x < y < z < w\}]\!]. \\
\forall K \in \mathcal{S} \setminus W: \mathcal{O}_{2}[\![K \leftarrow K_{xy}, W_{zw} \mid \{x < y, z < w\}]\!].
\end{array}
```

$$\mathcal{O}_2[W \leftarrow A^+, A^- \mid \{x < y < z < w\}]].$$

$$\mathcal{O}_2[W \leftarrow C^-, C^+ \mid \{x < y < z < w\}]].$$
CLOSURES

G_3 : G_2 + Universal triple insertion

$$\begin{split} \mathsf{G}_3 &= \mathsf{G}_2 \, + \, \forall \,\, \mathrm{K} \in \mathcal{S} \setminus \mathrm{W} : \\ & \mathcal{O}_2 \llbracket \mathrm{K} \leftarrow \mathrm{K}_{xy} \mid \{x < y, \,\, a < b < c\} \rrbracket. \end{split}$$

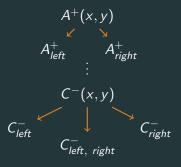


EXAMPLE

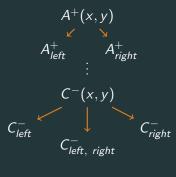
$$A^{+}(x,y)$$

$$A^{+}_{left} A^{+}_{right}$$

EXAMPLE

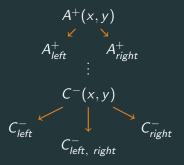


EXAMPLE



WHY?

EXAMPLE

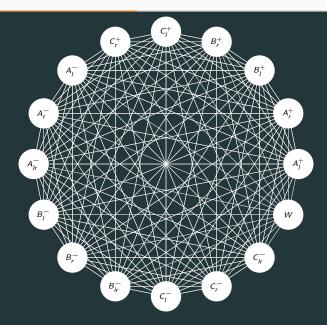


WHY?

New orders in interactions

$$C^-(xz, wy) \leftarrow A^+_{left}(x, y), B^+(z, w).$$

Refining states: Interactions



G₄: Automatic Rule Inference

State descriptors \mathcal{D}

$$\begin{array}{lll} \mathrm{W} \mapsto (\epsilon, \epsilon) & \mathrm{A}_r^- \mapsto (\epsilon, bc) \\ \mathrm{A}_l^+ \mapsto (a, \epsilon) & \mathrm{A}_{lr}^- \mapsto (b, c) \\ \mathrm{A}_r^+ \mapsto (\epsilon, a) & \mathrm{B}_l^- \mapsto (ac, \epsilon) \\ \mathrm{B}_l^+ \mapsto (b, \epsilon) & \mathrm{B}_r^- \mapsto (\epsilon, ac) \\ \mathrm{B}_r^+ \mapsto (\epsilon, b) & \mathrm{B}_{lr}^- \mapsto (a, c) \\ \mathrm{C}_l^+ \mapsto (c, \epsilon) & \mathrm{C}_l^- \mapsto (ab, \epsilon) \\ \mathrm{C}_r^+ \mapsto (\epsilon, c) & \mathrm{C}_r^- \mapsto (\epsilon, ab) \\ \mathrm{A}_l^- \mapsto (bc, \epsilon) & \mathrm{C}_{lr}^- \mapsto (a, b) \end{array}$$

Automatic Rule Inference: Example

Case of
$$A_{lr}^-(x_b, y_c) + B_{lr}^-(z_a, w_c)$$

PERMUTATION

:

(zxw, y)

÷

(xzw, y)

:

Automatic Rule Inference: Example

Case of
$$A_{lr}^-(x_b, y_c) + B_{lr}^-(z_a, w_c)$$

PERMUTATIONDESCRIPTOR
$$\vdots$$
 \vdots (zxw, y) \longrightarrow (abc, c) \vdots \vdots (xzw, y) \longrightarrow (bac, c) \vdots \vdots

Automatic Rule Inference: Example

Case of
$$A_{lr}^-(x_b, y_c) + B_{lr}^-(z_a, w_c)$$

PERMUTATION
 DESCRIPTOR
 ELIMINATED

$$\vdots$$

$$\vdots$$

$$(zxw, y)$$

$$(c, \epsilon) \leftrightarrow C_I^+$$

$$\vdots$$

$$\vdots$$

$$(xzw, y)$$

$$(bac, c)$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

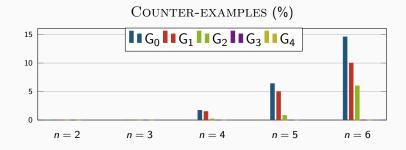
$$\vdots$$

$$\vdots$$

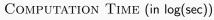
$$\vdots$$

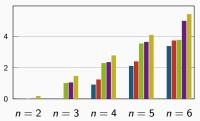
$$\vdots$$

Results



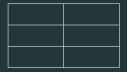






Correspondences: Young Tableau

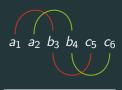
 $a_1 \ a_2 \ b_3 \ b_4 \ c_5 \ c_6$



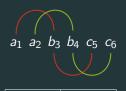
Correspondences: Young Tableau



Correspondences: Young Tableau

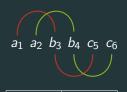


1	2
3	4
5	6



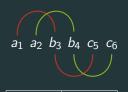
1	2
3	4
5	6

	1
2	3
4	5



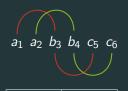
1	2
3	4
5	6

1	
2	3
4	5



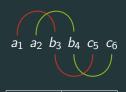
1	2
3	4
5	6

1	3
2	
4	5



1	2
3	4
5	6

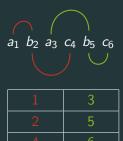
1	3
2	5
4	



1	2
3	4
5	6

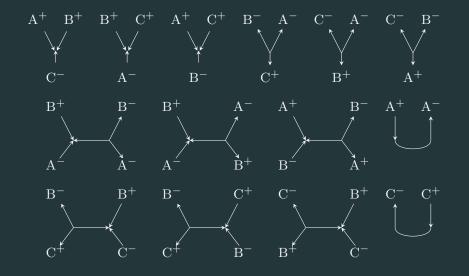
1	3
2	5
4	6





Correspondences: Spider Webs

Growth rules



full of adventure, full of knowledge.

When you depart for Ithaca, wish for the road to be long,

C. P. Cavafy Ithaca