Stability in a many-to-one matching model with positive externalities among colleagues*

Omer Ali†

February 29, 2024

Abstract

I study a non-transferable utility, many-to-one matching model with externalities among workers at the same firm (colleagues). Firms have preferences over workers, while workers have preferences over the firm they join as well as the number of workers employed by that firm. I show that when workers prefer to join firms with more employees, and firms have *responsive* preferences with no capacity constraints, a stable matching exists only when there are two firms in the market. Relaxing any one of these conditions yields a counter-example with no stable matchings.

1 Introduction

Externalities can arise among agents in many two-sided settings including school choice and the labor market. A student's decision to attend a school, for example, may depend on the characteristics of the school itself as well as on the other students attending that school. A worker's decision to join a particular firm may depend on the firm itself as well as on their prospective colleagues. In this paper, I study a two-sided matching model in which agents on one side of the market exert externalities on one another, and investigate the conditions under which stable matchings exist.

I complement the existing literature on matching with externalities by analyzing a many-to-one setting in which firms match with many workers who prefer to join larger firms. I find that stable matchings exist but only under certain conditions: there can only be two firms in the market, and these firms must have responsive preferences.

The results in this paper are related to those in Dutta and Masso (1997), who also study a many-to-one matching model with externalities among workers. They derive restrictions on the preferences of agents that precipitate a non-empty core.[1] They find, for example, that core matchings exist when firms have substitutable preferences and workers care about firms first, then consider the workers they employ in a lexicographic manner. In contrast, I focus on stable matchings, and allow workers to have preferences that are not necessarily lexicographic. The main take-away from their analysis is that allowing for more general preferences may result in an empty core and, hence, no stable matching.

The results below are also related to those in Fisher and Hafalir (2016), who study a one-to-one matching model with externalities among all agents. In their set-up, both workers and firms prefer assignments with more matches, externalities affect all agents in a symmetric way, and agents do not take the effect of their actions on the level of externalities into account. They show that under these conditions, a stable matching exists. In contrast to this approach, I assume that externalities only affect workers, and allow them to take the effect of their actions on the level of externalities into account when they make their matching decisions. I use similar fixed point methods to show that a stable matching exists under these assumptions.

Many-to-one matching models with externalities are also analyzed by Bando (2012) and Salgado-Torres (2013) using an approach pioneered by Sasaki and Toda (1996) and refined by Hafalir (2008). In these papers, agents are endowed with 'estimation functions' that describe their beliefs about the outcomes that arise when they deviate from existing assignments. Pycia and Yenmez (2022) study a much more general many-to-many matching model with contracts in which agents on both sides of the market are affected by externalities. They derive restrictions on preferences that guarantee the existence of stable outcomes. A similar setting to the present model (but with transfers) has been studied by Lee (2014). The study investigates the existence of market tipping (in which only one platform prevails) or market splitting (in which two platforms coexist) equilibria in a model with firms that contract with two competing platforms.

There are additional studies that consider externalities in matching in environments that differ from the many-to-one setting here. Leshno (2022) studies a continuum economy where students have preferences over colleges that may depend on the assignment. To guarantee the existence of a stable matching, a restriction on preferences - diversity of preferences - is needed that prevents small changes in the assignment from precipitating a change in the college orderings of many students. Contreras and Torres-MartÃnez (2021); Imamura and Konishi (2023); Imamura et al. (2023), meanwhile, study one-to-one matching problems with externalities.

The existence of stable matchings in the presence of externalities turns out to be uncommon. For example, Dur and Wiseman (2019) consider a school choice problem, in which pairs

---

[1]All matchings in the core are stable.

of students (neighbors) have preferences that are interdependent such that joint assignment to some schools is strictly preferred to any alternative outcome in which they are separated. This makes the deferred acceptance mechanism no longer strategy-proof nor stable. Whereas agents in markets without externalities have preferences over agents on the other side of the market, in the presence of externalities, each agent may have preferences over all possible matching outcomes that can arise. As such, models with externalities often do not admit stable matchings for arbitrary preference profiles.[2] Motivated by this observation, and eschewing restrictions on preferences, Echenique and Yenmez (2007) propose an algorithm that finds core matchings (when they exist) in a model with externalities among colleagues. The approach I take in this paper is different and in line with studies that look for sufficient conditions for existence. Despite the simplicity of the setting I study in this paper, stable matchings only exist when there are two firms in the market. This result highlights the tradeoff between allowing for more complex interdependence between agents on the one hand, and finding stable matchings on the other.

The main result of the paper comes at the end of section 3. The proof uses Tarski's fixed point theorem to show that a fixed point, which corresponds to the externalities generated by a myopic-stable matching, exists. Roughly speaking, existence of myopic-stable matchings is driven by a monotonic relationship between the number of workers a firm starts out with, and the number of workers it ends up with at the end of a 'round' of matching. Assuming that firms have responsive preferences ensures that a firm always ends up with more workers than it would have had it begun the round with fewer workers. To reach a stable matching from this point requires verifying that workers move around in a way that does not result in any cycles.

Section 2 describes the model with two firms, and Section 3 outlines the existence proof. In Section 4, I show examples of cases in which either no stable matchings exist, or the existence proof fails to go through when the sufficient conditions I introduce in Section 3 are violated. I also describe an example in which even when these conditions hold, stable matchings fail to exist in general with more than two firms. Section 5 concludes and discusses directions for future research.

2   Model

Consider a model in which firms are matched with sets of workers. Let the set of firms be $\mathcal{F} = \{f_1, f_2\}$, the set of workers be $\mathcal{W} = \{w_1, \ldots, w_n\}$, and the set of all agents be $\mathcal{N} = \mathcal{F} \cup \mathcal{W}$. Each firm f_i has a quota, q_i, which determines the number of workers they are matched with

(empty slots are filled by multiple copies of the firm itself).[3] Workers have preferences over the firm they match with and the number of workers employed by that firm. Firms have preferences over sets of workers. A matching game, G, is a quintuple, $\{\mathcal{F}, \mathcal{W}, \mu, u_w, u_f\}$, of firms, workers, a matching (defined below), a utility function for workers, and a utility function for firms. Both firms and workers have strict preferences.

Definition 1. A mapping $\mu : \mathcal{N} \to \mathcal{N}$ is a **matching** if it satisfies the following conditions:

- $\mu(w) \in \mathcal{F} \cup \{w\}$ for each $w \in \mathcal{W}$

- $\mu(f_i) \subset \mathcal{W} \cup f_i : \ |\mu(f_i)| = q_i$ for each $f_i \in \mathcal{F}$

- For each w and f, $\mu(w) = f \iff w \in \mu(f)$

Let $\mathcal{M}$ be the set of all matchings. We define what we mean by an externality correspondence below.[4] When a firm employs a smaller number of workers than its capacity allows, the remaining slots are filled by multiple copies of itself, hence $\mu(f_i) \subset \mathcal{W} \cup f_i$.

Definition 2. An **externality correspondence** is a mapping $e : \mathcal{M} \to \mathbb{R}^2$ that associates a two-dimensional, real-valued vector, $e(\mu)$, to each matching, μ. Elements of this vector are indexed by the firm, so that $e_{f_i}(\mu)$ is the externality associated with firm f_i in matching μ.

The externality $e_{f_i}(\mu)$ is the number of workers matched with firm f_i in matching μ. Next, we define the payoffs of firms and workers at a given matching, μ. **Firm f's payoff** at matching μ is the utility they obtain from matching with the set of workers $\mu(f)$:

$$u_f(\mu) = u_f(\mu(f))$$

Worker w's payoff derives from the firm they are matched with as well as the level of externality associated with that firm in the going matching:

$$u_w(\mu) = u_w\left(\mu(w), e_{\mu(w)}(\mu)\right)$$

If worker w is single (i.e. $\mu(w) = w$), then $e_{\mu(w)} = 0$. The next set of definitions describe what we mean by stability in the model. The first definition describes individual rationality - one of the requirements that a stable matching must satisfy. A matching is individually rational if no worker prefers to be single than remain with their current match, and no firm prefers to relinquish any one of its workers. Since I assume that firms have responsive preferences, whenever they are willing to fire a set of workers, there must exist one worker they would prefer to exchange for an empty slot.

---

[3]Throughout the remainder of the paper, I assume that $q_i = n$ for both firms and show in Section 4 an example of a game with no stable matchings when this assumption is relaxed.

[4]The terminology we use is borrowed from Fisher and Hafalir (2016).

Definition 3. A matching is **individually rational** if $u_w(\mu) \geq u_w(w, 0)$ for every $w \in \mathcal{W}$, and $u_f(\mu(f)) \geq u_f(\{\mu(f)\backslash w\} \cup f)$ for every $f \in \mathcal{F}$ and any $w \in \mu(f)$.

Another way in which a matching may be unstable is when there is a worker who prefers to join another firm over the firm they are currently matched with, and that firm prefers to employ the worker.[5] The deviating firm and worker are said to block the going matching.

Definition 4. A matching, μ, is **blocked** by worker w and firm f if $u_w(f, e_f(\mu) + 1) > u_w(\mu(w), e_{\mu(w)}(\mu))$ and $u_f(\mu(f) \cup w) > u_f(\mu(f))$.

When the firm has already filled its quota, the blocking conditions for the worker and firm become $u_w(f, e_f(\mu)) > u_w(\mu(w), e_{\mu(w)}(\mu))$ and $u_f(\{\mu(f)\backslash w'\} \cup w) > u_f(\mu(f))$ for some $w' \in \mu(f)$, respectively.

Definition 5. A matching is **stable** if it is individually rational, and not blocked by any worker-firm pair.

This is the standard definition of stability, and the only difference between our setting and the classical notion is that workers now take the number of other workers employed at their target firm into account when contemplating whether or not to deviate.[6] Note that although the present setting is very similar to that in Fisher and Hafalir (2016), I allow workers to take their own deviation into account when blocking a matching. This endows workers with some rationality as they anticipate the effect their own deviation has on the size of a firm's workforce. The next section outlines the existence proof.

3   Existence of stable matchings

Let $E = \{e(\mu) : \mu \in \mathcal{M}\} \subset \mathbb{R}^{\mathcal{F}}$ be the set of possible vectors of externalities. For each $x \in E$, the auxiliary matching game, $G(x)$, is $\{\mathcal{F}, \mathcal{W}, \mu, u_w^x, u_f\}$, where the only change made to the ingredients of the matching game G is that workers' utilities are now indexed by the externality vector x and defined below.

Definition 6. For a given matching μ, and vector of externalities x, **worker w's payoff in the auxiliary game** $G(x)$ derives from the firm they are matched with as well as the level of externality associated with this firm according to the externality vector x:

---

[5]If the firm has already filled its quota of workers, then it must prefer to employ the deviating worker instead of one of its existing employees.

[6]A similar notion of stability is used in both Fisher and Hafalir (2016) and Pycia and Yenmez (2022) in which agents take other agents' matching decisions as fixed. This is an alternative approach to the one that models what agents believe will occur after they deviate through estimation functions (Sasaki and Toda (1996); Hafalir (2008); Salgado-Torres (2013)). Bando (2012) takes an intermediate approach and requires that group-wise deviations by a single firm and a set of workers only occur when they are credible. That is, when there are no deviations by a subset of the deviating workers and those already at the firm that make workers in this subset as well as the firm better off.

$$u_w^x(\mu) = u_w\left(\mu(w), x_{\mu(w)}\right)$$

Notice that each auxiliary matching game is now a potentially different matching game without externalities. Workers take the externalities associated with a firm as a fixed feature of that firm and, therefore, have a strict ranking of firms that is independent of the matching decisions of other agents. For a given auxiliary game, $G(x)$, we define what it means for a matching to be stable in the usual sense (i.e. in settings without externalities) in $G(x)$ and refer to this concept as *auxiliary-stability* to draw a distinction between stability in the game with externalities and stability in auxiliary games. As usual, stability requires individual rationality and no-blocking. The corresponding requirements for auxiliary games are defined below.

Definition 7. A matching μ is **auxiliary-rational in** $G(x)$ if $u_f(\mu(f)) \geq u_f\left(\{\mu(f)\backslash w\} \cup f\right)$ for every $f \in \mathcal{F}$ and any $w \in \mu(f)$, and $u_w(\mu(w), x_{\mu(w)}) = u_w^x(\mu(w)) \geq u_w^x(w) = u_w(w, 0)$ for all $w \in \mathcal{W}$.

Auxiliary-rationality is the same as individual rationality in the absence of externalities. In a matching that satisfies this condition, no firm prefers to relinquish any of its workers, and no worker prefers to be unmatched over remaining with the firm they are matched with.

Definition 8. A matching μ is **auxiliary-blocked in** $G(x)$ by worker w and firm f if $u_w(f, x_f) > u_w^x(\mu(w)) = u_w\left(\mu(w), x_{\mu(w)}\right)$ and $u_f\left(\{\mu(f)\backslash w'\} \cup w\right) > u_f\left(\mu(f)\right)$ for some $w' \in \mu(f)$.

A matching is auxiliary-blocked when, for a fixed vector of externalities, there is some firm-worker pair that would prefer to jointly deviate over remaining with their current matches. When no such firm-worker pair exists, and a matching is auxiliary-rational, it is auxiliary-stable.

Definition 9. A matching μ is **auxiliary-stable in** $G(x)$ if it is auxiliary-rational and not auxiliary-blocked in $G(x)$.

Each auxiliary game, $G(x)$, is a many-to-one matching game without externalities. As a result, an auxiliary-stable matching exists if firms' preferences are responsive (defined below). Responsiveness is a common restriction on preferences in the matching literature (see, for example, Roth and Sotomayor (1990)). Let $S(x)$ denote the set of stable matchings in the auxiliary game $G(x)$: $S(x) = \{\mu : \mu$ *is auxiliary stable in* $G(x)\}$. Below is the definition of responsive preferences, and a result guaranteeing that auxiliary games always admit an auxiliary-stable matching.

Definition 10. (Definition 5.2 in [Roth and Sotomayor (1990)]) A firm's preference relation, $\succ_f$, over sets of workers is **responsive** (to preference relation $\tilde{\succ}$ over individual workers) if for any set of workers S, and any $w \notin S$ and $w' \in S$, $S \succ \{S \setminus w'\} \cup \{w\} \iff w \tilde{\succ} w'$.

Lemma 1. *For each vector of externalities $x \in E$, if firms' preferences are responsive, then $S(x)$, the set of stable matchings of auxiliary game $G(x)$, is non-empty.*

Proof. See the appendix. □

Definition 11. A worker's preferences, $\succ_w$, are **increasing in the number of colleagues** if for any two auxiliary games, $G(e_1)$ and $G(e_2)$, such that $e_1 \geq e_2$,

$$f_1 \succ_w f \ in \ G(e_2) \implies f_1 \succ_w f \ in \ G(e_1) \ for \ f \in \{f_2, \emptyset\}$$

$$f_2 \succ_w f \ in \ G(e_1) \implies f_2 \succ_w f \ in \ G(e_2) \ for \ f \in \{f_1, \emptyset\}$$

Preferences are increasing in the number of colleagues if, whenever a worker prefers some firm, say firm 1, over either firm 2 or remaining unmatched when firm one is matched with some number of workers, then they also prefer firm 1 to either firm 2 or being unmatched when firm 1 is matched with more workers. Before stating the next result, we introduce the concept of *myopic-stability*. This form of stability requires that matchings are auxiliary-stable with respect to the vector of externalities they generate. Formally, let $x_\mu = e(\mu)$ be the set of externalities generated by matching μ. Matching μ is **myopic-stable** if it is auxiliary stable in $G(x_\mu)$.

While we will be able to relax the assumption of myopia, we use it in an intermediate step in the existence proof. The next step in showing existence of stable matchings in G is to define a correspondence, $T : E \to E$, in the following way:

$$T(x) = \{e(\mu) : \mu \in S(x)\}$$

In words, $T(x)$ is the set of possible externalities associated with auxiliary-stable matchings of the game $G(x)$. The fixed points of T correspond to myopic-stable matchings, as the next result shows.

Proposition 1. *There is a myopic-stable matching μ^* of the matching game G iff there is an x^* such that $x^* \in T(x^*)$, where $x^* = e(\mu^*)$.*

Proof. See the appendix. □

Having shown that fixed points of T correspond to externalities generated by myopic-stable matchings, we now show that fixed points of T do exist. Our strategy is to apply Tarski's

fixed point theorem (Tarski (1955)), which means that we need to guarantee that there is an increasing selection of T (since T is a correspondence) and that E is a partially ordered complete lattice.[7] A selection of the correspondence T is a function $f : E \to E$ such that for any $x \in E$, $f(x) \in T(x)$. E endowed with a partial order is a complete lattice if any subset of E has a greatest lower bound and least upper bound that exist in E.

Proposition 2. *If there exists an increasing selection f of T, then a myopic-stable matching, μ^*, exists.*

Proof. By Tarski's fixed point theorem, an increasing selection of T has a fixed point, which is also a fixed point of T. By Proposition 1, this fixed point corresponds to the externalities generated by a myopic-stable matching, μ^*. □

We are now ready to show that myopic-stable matchings exist, as long as three conditions hold: (i) firms have responsive preferences, (ii) workers have preferences that are increasing in the number of colleagues, and (iii) firms do not have capacity constraints. First, we define an appropriate partial order on the set of externalities, E, and show that the resulting partially ordered set is a complete lattice. Then we check that T is increasing with respect to the partial order defined. Once these two conditions are satisfied, Tarski's fixed point theorem applies. The partial order is described in lemma 2 in the appendix, which also shows that E equipped with this partial order is a complete lattice. We obtain the existence of myopic-stable matchings in the next proposition.

Proposition 3. *For workers with preferences that are increasing in the number of colleagues, and two firms with responsive preferences and $q_i = n$, a myopic-stable matching exists.*

Proof. See the appendix. □

Showing that stable matchings exist once we have a myopic stable matching is relatively straightforward, and we present the argument in the penultimate result in this section.

Proposition 4. *For firms with responsive preferences, workers with preferences that are increasing in the number of colleagues and $q_i = n$, if a myopic-stable matching exists, then so does a stable matching.*

Proof. See the appendix. □

Putting these results together yields the main result of the paper: a stable matching exists whenever the preferences of firms and workers satisfy the restrictions above.

---

[7]Tarski's fixed point theorem states the following: let $F : X \to X$ be a map from a complete lattice, $(X, \leq)$, onto itself. If F is increasing $(x \geq y \implies F(x) \geq F(y))$, the set of fixed points of F, $\{x \in X : x = F(x)\}$, is a non-empty complete lattice.

Theorem 1. *For two firms with responsive preferences, workers with preferences that are increasing in the number of colleagues and $q_i = n$, a stable matching exists.*

Proof. This follows from propositions 1-4. □

The assumptions we imposed on preferences to ensure that a stable matching exists allow for the application of Tarski's fixed point theorem in Proposition 3. The three restrictions we require, (1) that there are only two firms, (2) that firms have responsive preferences, and (3) that firms have no capacity constraints, are all necessary, as will be shown in the next section.

4 Examples

This section contains examples of games without stable matchings that arise when the assumptions on firms' capacities, their preferences and their number are relaxed. In the first example, I show that when there are capacity constraints, it is possible to find a preference profile in which firms' preferences are responsive, workers' preferences are increasing in the number of colleagues, and yet no stable matching exists.

Responsive preferences are a special case of preferences that satisfy 'substitutability'.[8] In the standard many-to-one matching model without externalities, a stable matching exists whenever firms have substitutable preferences. However, example 2 shows that substitutable preferences are not sufficient to guarantee that stable matchings exist - it describes a game with two firms, workers whose preferences are increasing in the number of colleagues and no stable matching.

Finally, example 3 shows that when there are three firms (with responsive preferences) and three workers with preferences that are increasing in the number of colleagues, it is possible to construct preferences for workers that preclude the existence of stable matchings.

Example 1. An example of a game with firms that have capacity constraints in which no stable matching exists:[9] Consider a game with two firms and two workers. Let both firms have responsive preferences with the following underlying ranking of workers: $w_1 \succ w_2 \succ \emptyset$. Let worker i's preferences ($\succ_i$) be increasing in the number of colleagues and order firm-employee pairs in the following way:

$$(f_1, 2) \succ_1 (f_2, 1) \succ_1 (f_1, 1) \succ_1 \emptyset$$

$$(f_2, 1) \succ_2 (f_1, 2) \succ_2 (f_1, 1) \succ_2 \emptyset$$

---

[8] A preference relation $\succ$ satisfies substitutability if for any $w, w' \in W$, if $w \in S \subseteq W$ and $S \succ S'$, $\forall S' \subseteq W$, then $w \in T \subseteq W \backslash w'$ where $T \succ T'$, $\forall T' \subseteq W \backslash w'$

[9] I would like to thank Ichiro Obara for suggesting this example.

Namely, worker 1 is preferred by both firms to worker 2, and worker 1 prefers firm 1 to firm 2 only when firm 1 employs one other worker. Meanwhile, worker 2 always prefers firm 2 to firm 1. Finally, suppose that firm 2's capacity is 1: it can only employ one worker, while firm 1 can employ 2 workers. To see that there is no stable matching, consider the following cycle, beginning at matching $\mu_1 = \{\{f_1, w_2\}, \{f_2, w_1\}\}$:

$\mu_1 = \{\{f_1, w_2\}, \{f_2, w_1\}\}$ is blocked by (f_1, w_1), leading to μ_2.

$\mu_2 = \{\{f_1, w_1, w_2\}, \{f_2, \emptyset\}\}$ is blocked by (f_2, w_2), leading to μ_3.

$\mu_3 = \{\{f_1, w_1\}, \{f_2, w_2\}\}$ is blocked by (f_2, w_1), leading to μ_4.

$\mu_4 = \{\{f_1, \emptyset\}, \{f_2, w_1\}, \{\emptyset, w_2\}\}$ is blocked by (f_1, w_2), leading back to μ_1.

Notice that starting at any matching of this game leads to some point in this cycle through either an individual or a pair-wise block. Since all agents prefer to be matched, it is clear that no matching in which there is an unmatched agent is stable, so we only need to consider matchings in which all agents are matched. μ_1, μ_2 and μ_3 are, therefore, the only possible candidates for stable matchings. But these matchings are part of the cycle above, hence no stable matchings exist.

Example 2. An example of a game with firms that have substitutable preferences in which no stable matching exists: Let $F = \{f_1, f_2\}$, $W = \{w_1, w_2\}$, and suppose that firms' preferences are such that $\{w_1\} \succ_f \{w_1, w_2\} \succ_f \{w_2\} \succ_f \emptyset$. First, notice that although these preferences are substitutable, they are not responsive. If they were responsive to some underlying preference relation, $\succ$, then $w_2 \succ \emptyset$ would imply that $\{w_1, w_2\} \succ_f \{w_1, \emptyset\} = w_1$. Let workers' preferences be such that $(f_1, 2) \succ_w (f_2, 2) \succ_w (f_1, 1) \succ_w (f_2, 1)$, and notice that there are no stable matchings in which any worker is unmatched. This is because any unmatched worker would prefer to match with some firm. Moreover, if one worker is unmatched, then there must be a vacant firm, which would prefer to employ them than remain unmatched. However, any matching in which all workers are matched is not stable either:

$\{\{f_1, w_1, w_2\}, f_2\}$ is blocked by f_1

$\{\{f_1, w_1\}, \{f_2, w_2\}\}$ is blocked by (f_2, w_1)

$\{\{f_1\}, \{f_2, w_1, w_2\}\}$ is blocked by f_2

$\{\{f_1, w_2\}, \{f_2, w_1\}\}$ is blocked by (f_1, w_1)

If there is no stable matching in which some workers are unmatched, and none in which all are matched, then stable matchings don't exist in this example.

In the final example, I show that whenever there are 3 firms and 3 workers, it is possible to construct preferences for workers such that no stable matching exists even when firms have responsive preferences and no capacity constraints.

Example 3. An example of a game with 3 firms in which no stable matching exists:

suppose there are 3 firms and 3 workers.[10] Firms have responsive preferences and all workers are acceptable. The following describes the preferences of workers.

$$w_1 : (f_3, 2) \succ_1 (f_2, 2) \succ_1 (f_2, 1) \succ_1 (f_1, 3) \succ_1 (f_3, 1) \succ_1 \emptyset$$
$$w_2 : (f_2, 2) \succ_2 (f_1, 2) \succ_2 (f_1, 1) \succ_2 (f_2, 1) \succ_2 \emptyset$$
$$w_3 : (f_1, 2) \succ_3 (f_3, 2) \succ_3 (f_3, 1) \succ_3 (f_1, 1) \succ_3 \emptyset$$

Workers' preferences are increasing in the number of colleagues, hence satisfy the condition in Definition 11. However, in every matching between workers and firms, there exists a series of deviations that lead to the cycle described below. The starting matching is $\mu_1 = \{\{f_1, w_2, w_3\} \{f_2, w_1\}, f_3\}$.

μ_1 is blocked by (f_2, w_2), leading to $\mu_2 = \{\{f_1, w_3\} \{f_2, w_1, w_2\} f_3\}$.
μ_2 is blocked by (f_3, w_3), leading to $\mu_3 = \{f_1, \{f_2, w_1, w_2\}, \{f_3, w_3\}\}$.
μ_3 is blocked by (f_3, w_1), leading to $\mu_4 = \{f_1, \{f_2, w_2\}, \{f_3, w_1, w_3\}\}$.
μ_4 is blocked by (f_1, w_2), leading to $\mu_5 = \{\{f_1, w_2\}, f_2, \{f_3, w_1, w_3\}\}$.
μ_5 is blocked by (f_1, w_3), leading to $\mu_6 = \{\{f_1, w_2, w_3\}, f_2, \{f_3, w_1\}\}$.
μ_6 is blocked by (f_2, w_1), leading back to μ_1.

The existence of a cycle alone does not prove that there is no stable matching. It can be shown that starting at any potentially stable matching leads, through a sequence of blocks, to the cycle above. The remainder of this example is relegated to the appendix.

5   Conclusion

In this paper, I described a matching model in which workers exert externalities on their colleagues. I showed that a stable matching exists when workers' preferences are increasing in the number of colleagues, and two firms have responsive preferences with no capacity constraints. The results contribute to the literature on matching models with externalities among colleagues by providing sufficient conditions for the existence of stable matchings and examples in which stable matchings no longer exist when these conditions are relaxed. The framework is reminiscent of an oligopolistic labor market with two firms in which workers prefer to join larger firms, or a school choice problem with two schools and students who prefer larger cohorts.

---

[10]When there are more than 3 firms and 3 workers, a preference profile can be constructed so that the resulting game has no stable matching by assuming that firms only find 3 workers acceptable, and these 3 workers only find three firms acceptable.

There are some open questions that remain. While the examples in section 4 show that stable matchings may not exist, it may be the case that stronger assumptions on the preferences of firms and workers that rule out the type of cycles presented above would allow for stable matchings to exist. One natural question to ask is whether stable matchings emerge when preferences of workers are such that they only care about the relative difference in the number of workers between firms. This would shrink the number of deviations workers make, and could reasonably result in more stable matchings.

Finally, agents see ahead up to the effect of their own deviation. This is a departure from Fisher and Hafalir (2016), in which agents are myopic. However, it would be reasonable to assume that agents can also anticipate the responses of others. In which case, a suitable notion of stability (for example, farsighted stability as in Mauleon et al. (2011) and Ray and Vohra (2015)) may yield a different conclusion.

References

Bando, Keisuke, "Many-to-one matching markets with externalities among firms," *Journal of Mathematical Economics*, 2012, *48* (1), 14–20.

Contreras, José Luis and Juan Pablo Torres-Martĩnez, "The roommate problem with externalities," *International Journal of Game Theory*, March 2021, *50* (1), 149–165.

Dur, Umut Mert and Thomas Wiseman, "School choice with neighbors," *Journal of Mathematical Economics*, August 2019, *83*, 101–109.

Dutta, Bhaskar and Jordi Masso, "Stability of Matchings When Individuals Have Preferences over Colleagues," *Journal of Economic Theory*, 1997, *75* (2), 464–475.

Echenique, Federico and Jorge Oviedo, "A theory of stability in many-to-many matching markets," *Theoretical Economics*, 2006, *1* (2), 233–273.

_ **and M. Bumin Yenmez**, "A solution to matching with preferences over colleagues," *Games and Economic Behavior*, 2007, *59* (1), 46 – 71.

Fisher, James C. D. and Isa E. Hafalir, "Matching with aggregate externalities," *Mathematical Social Sciences*, May 2016, *81*, 1–7.

Gale, D. and L. S. Shapley, "College Admissions and the Stability of Marriage," *The American Mathematical Monthly*, 1962, *69* (1), pp. 9–15.

Hafalir, Isa E., "Stability of marriage with externalities," *International Journal of Game Theory*, March 2008, *37* (3), 353–369.

Imamura, Kenzo and Hideo Konishi, "Assortative Matching with Externalities and Farsighted Agents," *Dynamic Games and Applications*, June 2023, *13* (2), 497–509.

_ , _ , **and Chen-Yu Pan**, "Stability in matching with externalities: Pairs competition and oligopolistic joint ventures," *Journal of Economic Behavior & Organization*, January 2023, *205*, 270–286.

Lee, Robin S., "Competing Platforms," *Journal of Economics & Management Strategy*, September 2014, *23* (3), 507–526.

Leshno, Jacob, "Stable Matching with Peer-Dependent Preferences in Large Markets: Existence and Cutoff Characterization," August 2022.

Mauleon, Ana, Vincent J. Vannetelbosch, and Wouter Vergote, "von Neumann-Morgenstern farsightedly stable sets in two-sided matching," *Theoretical Economics*, September 2011, *6* (3), 499–521.

Pycia, Marek and M Bumin Yenmez, "Matching with Externalities," *The Review of Economic Studies*, 06 2022, *90* (2), 948–974.

Ray, Debraj and Rajiv Vohra, "The Farsighted Stable Set," *Econometrica*, May 2015, *83* (3), 977–1011.

Roth, Alvin E and Marilda A. Oliveira Sotomayor, "Two-sided Matching: A Study in Game-theoretic Modeling and Analysis," *Econometric Society Monographs*, 1990, (18).

Salgado-Torres, Alfredo, "Many to one matching: Externalities and stability," April 2013.

Sasaki, Hiroo and Manabu Toda, "Two-Sided Matching Problems with Externalities," *Journal of Economic Theory*, 1996, *70* (1), 93–108.

Tarski, Alfred, "A lattice-theoretical fixpoint theorem and its applications," *Pacific journal of Mathematics*, 1955, *5* (2), 285–309.

Appendix

Proof. (of lemma 1)

Responsive preferences satisfy substitutability. Any many-to-one matching game without externalities in which firms' preferences satisfy substitutability admits a stable matching by theorem 6.5 in Roth and Sotomayor (1990). □

Proof. (of proposition 1)

First, we will show that if μ^* is stable, then $e\left(\mu^*\right) \in T(x^*)$.

Suppose, to the contrary, that $e(\mu^*) \notin T(x^*)$. Notice that this implies that $\mu^* \notin S(x^*)$, i.e. that μ^* is not auxiliary stable in $G(x^*)$. This means that either (i) μ^* is not auxiliary-rational, or (ii) it is auxiliary-blocked in $G(x^*)$.

If (i) is true, then either there exists some $w \in \mathcal{W}$ such that $u_w\left(\mu^*(w), x^*_{\mu^*(w)}\right) < u_w(w, 0)$, or there exists some $f \in \mathcal{F}$ with $u_f(\mu^*) < u_f(f)$. If the latter is true, then μ^* cannot be stable (a contradiction). Moreover, μ^* cannot be stable if the former is true, since $u_w\left(\mu^*(w), x^*_{\mu^*(w)}\right) = u_w\left(\mu^*(w), e(\mu^*)_{\mu^*(w)}\right) = u_w(\mu^*)$ and $u_w(\mu^*) < u_w(w, 0)$ implies that μ^* is not individually rational.

If (ii) is the case, then there exists $w \in \mathcal{W}$ and f$\in \mathcal{F}$ such that $u_w(f, x^*_f) > u_w\left(\mu^*(w), x^*_{\mu^*(w)}\right)$ and $u_f\left(\mu^*(f) \cup w\right) > u_f\left(\mu^*(f)\right)$. Since $x^* = e(\mu^*)$, this violates the stability of μ^*. It must, therefore be the case that a stable μ^* is a fixed point of T.

Next, we show that a fixed point of T must be generated by a stable matching. Let $x^* \in T(x^*)$ such that $x^* = e(\mu^*)$ - such a matching exists by non-emptiness of T. Now suppose that μ^* is not stable. Either (i) μ^* is not individually rational, or else (ii) it is blocked by some $w \in \mathcal{W}$ and $f \in \mathcal{F}$.

(i) in the first case, the existence of some $f \in \mathcal{F}$ such that $u_f(\mu^*) < u_f(f)$ violates the auxiliary stability of μ^* in $G(x^*)$, which is guaranteed by the definition of μ^* as a matching such that $e(\mu^*) \in T(x^*)$. If, instead, individual rationality is violated by the existence of some $w \in \mathcal{W}$ with $u_w(\mu^*) < u_w(w, 0)$, then since $u_w(\mu^*) = u_w(\mu^*(w), e(\mu^*)_{\mu^*(w)}) = u_w(\mu^*(w), x^*_{\mu^*(w)})$, this violates the auxiliary stability of μ^* in $G(x^*)$.

(ii) in the second case, the existence of some $w \in \mathcal{W}$ and $f \in \mathcal{F}$ such that $u_f(\mu^*(f) \cup w) > u_f(\mu^*(f))$ and$u_w(f, e(\mu^*)_{\mu^*(w)})) > u_w(\mu^*(f))$ violates auxiliary stability of μ^* in $G(x^*)$, since $e(\mu^*) = x^*$ by definition. □

Lemma 2. $E = \left\{(x, y, n - x - y) \in \mathbb{R}^3 : x, y \geq 0, x + y \leq n\right\}$ *endowed with the partial order* $(x_1, y_1, n - x_1 - y_1) \geq (x_2, y_2, n - x_2 - y_2) \iff x_1 \geq x_2$ *and* $y_1 \leq y_2$ *is a complete lattice.*

Proof. We want to show that for any $A \subseteq E$, there exists a supremum and infimum in E. Let $\bar{x} = \max_{(x,y,n-x-y) \in A}\{x\}$, and $\underline{y} = \min_{(x,y,n-x-y) \in A}\{y\}$ and define $s = (\bar{x}, \underline{y}, n - \bar{x} - \underline{y})$. Then s is the supremum of A, and is in E. To see that s is the supremum of A, notice that $\bar{x} \geq x$ for any $(x, y, n - x - y) \in A$ and $\underline{y} \leq y$ for any $(x, y, n - x - y) \in A$, by definition. To see that

$s \in E$, notice that since $\bar{x} = x$ for some $(x, \tilde{y}, n - x - \tilde{y}) \in A \subseteq E$, it must be the case that $\bar{x} + \tilde{y} \leq n$. Moreover, since $y \leq \tilde{y}$, it follows that $\bar{x} + y \leq \bar{x} + \tilde{y} \leq n$.

Similarly, let $\underline{x} = \min_{(x,y,n-x-y)\in A}\{x\}$, $\bar{y} = \max_{(x,y,n-x-y)\in A}\{y\}$, and $n = (\underline{x}, \bar{y}, n - \underline{x} - \bar{y})$. Then n is an infimum of A since $\underline{x} \leq x$ and $\bar{y} \geq y$ for any $(x, y, n - x - y) \in A$. To see that $n \in E$, notice that $\bar{y} = y$ for some $(\tilde{x}, y, n - \tilde{x} - y) \in A$, and so $\tilde{x} + \bar{y} \leq n$. Since $\underline{x} \leq \tilde{x}$, it must be the case that $\underline{x} + \bar{y} \leq \tilde{x} + \bar{y} \leq n$. $\qquad\square$

Proof. (of proposition 3) We will show that there exists an increasing (with respect to the partial order defined in lemma 2) selection of T. Let $(x_1, y_1, n - x_1 - y_1), (x_2, y_2, n - x_2 - y_2) \in E$ be such that $(x_1, y_1, n - x_1 - y_1) \geq (x_2, y_2, n - x_2 - y_2)$. We will show that for any $\mu_1 \in T(x_1, y_1, n - x_1 - y_1)$ and $\mu_2 \in T(x_2, y_2, n - x_2 - y_2)$, $e(\mu_1) \geq e(\mu_2)$. This implies that any selection of T is increasing.

Let $G(1) := G(x_1, y_1, n - x_1 - y_1)$ and $G(2) := G(x_2, y_2, n - x_2 - y_2)$, and notice that if $W_1(1) = \{w \in W : f_1 \succ_w f_2 \text{ in } G(1)\}$ and $W_1(2) = \{w \in W : f_1 \succ_w f_2 \text{ in } G(2)\}$, then $W_1(2) \subseteq W_1(1)$.

To proceed, we make use of the following chain of implications: $e(\mu_1) \geq e(\mu_2) \iff |\mu_1(f_1)| \geq |\mu_2(f_1)|$ and $|\mu_1(f_2)| \leq |\mu_2(f_1)| \impliedby \mu_1(f_1) \supseteq \mu_2(f_1)$ and $\mu_1(f_2) \subseteq \mu_2(f_2)$.

Claim 1. $\mu_1(f_1) \supseteq \mu_2(f_1)$

Proof. Suppose that $\mu_1(f_1) \not\supseteq \mu_2(f_1)$. Then there exists some $w \in \mu_2(f_1)\backslash\mu_1(f_1)$. $w \in \mu_2(f_1)$ means that $w \succ_{f_1} \emptyset$ and that $f_1 \succ_w \emptyset$ in $G(2)$. In addition it means that either $f_1 \succ_w f_2$, or that $f_2 \succ_w f_1$ and $\emptyset \succ_{f_2} w$ in $G(2)$. First, notice that $f_1 \succ_w \emptyset$ in $G(2)$ implies that $f_1 \succ_w \emptyset$ in $G(1)$. If it is the case that $f_1 \succ_w f_2$ in $G(2)$, then it follows that $f_1 \succ_w f_2$ in $G(1)$, and we arrive at a contradiction to the statement that $w \notin \mu_1(f_1)$. Suppose, instead, that $f_2 \succ_w f_1$ and $\emptyset \succ_{f_2} w$ in $G(2)$. Since preferences of firms are unchanged across games, this means that $\emptyset \succ_{f_2} w$ in $G(1)$ as well, hence $w \notin \mu_1(f_2)$. But since $f_1 \succ_w \emptyset$ in $G(1)$ and $w \succ_{f_1} \emptyset$, it must be the case that $w \in \mu_1(f_1)$ - a contradiction. $\qquad\square$

Claim 2. $\mu_1(f_2) \subseteq \mu_2(f_2)$

Proof. Suppose that $\mu_1(f_2) \not\subseteq \mu_2(f_2)$. Then there exists some $w \in \mu_1(f_2)\backslash\mu_2(f_2)$. $w \in \mu_1(f_2) \implies w \succ_{f_2} \emptyset$ and $f_2 \succ_w \emptyset$ in $G(1)$. In addition, $w \in \mu_1(f_2) \implies$ either (i) $f_2 \succ_w f_1$ in $G(1)$, or (ii) $f_1 \succ_w f_2$ and $\emptyset \succ_{f_1} w$ in $G(1)$.

Case (i): $f_2 \succ_w \emptyset$ in $G(1) \implies f_2 \succ_w \emptyset$ in $G(2)$. $f_2 \succ_w f_1$ in $G(1) \implies f_2 \succ_w f_1$ in $G(2)$. Since $w \succ_{f_2} \emptyset$ and $\{f_2 \succ_w f_1, f_2 \succ_w \emptyset\}$ in $G(2)$, w and f_2 are a blocking pair in $G(2)$ - a contradiction. $\qquad\square$

We have shown that $e(\mu_1) \geq e(\mu_2)$ for any $\mu_1 \in T(x_1, y_1, n - x_1 - y_1)$ and $\mu_2 \in T(x_2, y_2, n - x_2 - y_2)$. Pick any selection f of T - by the above, it must be an increasing selection. $\qquad\square$

Proof. (of proposition 4)

Start at some myopic-stable matching μ with some workers employed at the two firms while some are unemployed. Define the following sets of workers:

$$W^0(\mu) := \{w \in \mu(f_1) \cup \mu(f_2) : \emptyset \succ_w (\mu(w), |\mu(\mu(w))|)\}$$

$$W^1(\mu) := \{w \in \mu(f_2) : (f_1, |\mu(f_1)| + 1) \succ_w (f_2, |\mu(f_2)|) \ and \ w \succ_{f_1} \emptyset\}$$

$$W^2(\mu) := \{w \in \mu(f_1) : (f_2, |\mu(f_2)| + 1) \succ_w (f_1, |\mu(f_1)|) \ and \ w \succ_{f_2} \emptyset\}$$

Notice that since μ is myopic-stable, $W^0(\mu) = \emptyset$. There are 4 possible cases we need to consider:

(1) $W^1(\mu) = W^2(\mu) = \emptyset$
(2) $W^1(\mu) = \emptyset$, $W^2(\mu) \neq \emptyset$
(3) $W^1(\mu) \neq \emptyset$, $W^2(\mu) = \emptyset$
(4) $W^1(\mu) \neq \emptyset$, $W^2(\mu) \neq \emptyset$

Case 1:

Suppose that case (1) is true. Workers in the unemployment pool may prefer to join one of the two firms when they take the effect of their deviation on the number of the firm's employees into account. Define the following sets of workers:

$$U^1(\mu) := \{w : \mu(w) = \emptyset, \ (f_1, |\mu(f_1)| + 1) \succ_w \emptyset, \ and \ w \succ_{f_1} \emptyset\}$$

$$U^2(\mu) := \{w : \mu(w) = \emptyset, \ (f_2, |\mu(f_2)| + 1) \succ_w \emptyset, \ and \ w \succ_{f_2} \emptyset\}$$

There are 4 possible cases:

(i) $U^1(\mu) = U^2(\mu) = \emptyset$
(ii) $U^1(\mu) = \emptyset$, $U^2(\mu) \neq \emptyset$
(iii) $U^1(\mu) \neq \emptyset$, $U^2(\mu) = \emptyset$
(iv) $U^1(\mu) \neq \emptyset$, $U^2(\mu) \neq \emptyset$

If (i) is true, then μ is stable.

Cases 1-ii (and, by symmetry, case 1-iii):

Suppose (ii) is true. Let all workers in $U^2(\mu)$ move to f_2. This may cause other workers to want to move from the unemployment pool to f_2. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of unemployed workers is finite. Call the new matching $\tilde{\mu}$. Since f_2 now has more workers, $W^2(\tilde{\mu})$ may be

17

non-empty, while $W^1(\tilde{\mu})$ must be empty because $W^1(\mu)$ is empty, and f_2 now has more workers relative to f_1. Notice that for any matching μ^*, if $W^1(\mu^*) = W^2(\mu^*) = U^1(\mu^*) = U^2(\mu^*) = \emptyset$, then μ^* is stable.

- Step 1:

    - Allow all workers in $W^0(\tilde{\mu}) \subseteq \tilde{\mu}(f_1)$ (if any) to move to the unemployment pool. This may cause other workers to want to move from f_1 to the unemployment pool. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of workers matched with f_1 is finite; call the resulting new matching μ^1.

    - If $W^2(\mu^1)$ is empty, consider the sets $U^1(\mu^1)$ and $U^2(\mu^1)$. Since $U^1(\mu)$ is empty, so is $U^1(\mu^1)$. It may be the case, however, that $U^2(\mu^1)$ is non-empty. If so, allow workers to move to f_2. This may cause other workers to want to move from the unemployment pool to f_2. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of unemployed workers is finite. Call the new matching $\tilde{\mu}^1$.

    - If $W^2(\mu^1)$ is non-empty, allow workers to move to f_2. This may cause other workers to want to move from f_1 to f_2. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of workers employed by f_1 is finite, and call the new matching $\tilde{\mu}^1$. Consider the sets $U^1(\tilde{\mu}^1)$ and $U^2(\tilde{\mu}^1)$. Since $U^1(\mu)$ is empty, so is $U^1(\tilde{\mu}^1)$. It may be the case, however, that $U^2(\tilde{\mu}^1)$ is non-empty. If so, allow workers to move to f_2. This may cause other workers to want to move from the unemployment pool to f_2. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of unemployed workers is finite. Call the new matching $\tilde{\mu}^1$.

- Step k:

    - Allow all workers in $W^0(\tilde{\mu}^{k-1}) \subseteq \tilde{\mu}^{k-1}(f_1)$ (if any) to move to the unemployment pool. This may cause other workers to want to move from f_1 to the unemployment pool. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of workers employed by f_1 is finite. Call the new matching μ^{k-1}.

    - If $W^2(\mu^{k-1})$ is empty, consider the sets $U^1(\mu^{k-1})$ and $U^2(\mu^{k-1})$. Since $U^1(\mu)$ is empty, so is $U^1(\mu^{k-1})$. If $U^2(\mu^{k-1})$ is non-empty, allow workers to move to f_2. This may cause other workers to want to move from f_1 to f_2. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of workers employed by f_1 is finite, and call the new matching $\tilde{\mu}^k$.

– If $W^2(\mu^{k-1})$ is non-empty, allow workers to move to f_2. This may cause other workers to want to move from f_1 to f_2. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of workers employed by f_1 is finite. Call the new matching $\tilde{\mu}^{k-1}$. Consider the sets $U^1(\tilde{\mu}^{k-1})$ and $U^2(\tilde{\mu}^{k-1})$. Since $U^1(\mu)$ is empty, so is $U^1(\tilde{\mu}^{k-1})$. It may be the case, however, that $U^2(\tilde{\mu}^{k-1})$ is non-empty. If so, allow workers to move to f_2. This may cause other workers to want to move from the unemployment pool to f_2. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of unemployed workers is finite. Call the new matching $\tilde{\mu}^k$.

This process must stop at some finite k^*, since the number of agents who are either matched with firm 1 or unemployed is finite: it must be the case that $\tilde{\mu}^{k^*}$ is stable. Notice that the same process can be implemented for case (iii) when $U^1(\mu) \neq \emptyset$.

Case 1-iv:

Without loss of generality, let workers in $U^1(\mu)$ move to firm 1 and call the resulting matching $\tilde{\mu}$. $W^2(\tilde{\mu})$ is empty since $W^2(\mu)$ is empty and there are more workers in firm 1 relative to firm 2 in $\tilde{\mu}$ than in μ.

- Step 1:

  – Allow all workers in $W^0(\tilde{\mu}) \subseteq \tilde{\mu}(f_2)$ (if any) to move to the unemployment pool. This may cause other workers to want to move from f_2 to the unemployment pool. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of workers matched with f_2 is finite; call the resulting new matching μ^1.

  – If both $W^1(\mu^1)$ and $U^1(\mu^1)$ are empty, follow the steps for case **1-ii**.

  – If $W^1(\mu^1)$ is empty, and $U^1(\mu^1)$ is non-empty, allow workers to move to firm 1. This may cause other workers to want to move from the unemployment pool to f_1. If so, allow them and subsequent sets of similar workers to move. The process ends at some point since the number of workers in the unemployment pool is finite; call the new matching $\tilde{\mu}^1$.

  – If $W^1(\mu^1)$ is non-empty, allow workers to move to firm 1. This may cause other workers to want to move from f_2 to f_1. If so, allow them and subsequent sets of similar workers to move. The process ends at some point since the number of workers matched with f_2 is finite; call the new matching $\hat{\mu}^1$.

    * If $U^1(\hat{\mu}^1)$ is non-empty, allow workers to move to firm 1. This may cause other workers to want to move from the unemployment pool to f_1. If so, allow them and subsequent sets of similar workers to move. The process ends at some

point since the number of workers in the unemployment pool is finite; and call the new matching $\tilde{\mu}^1$.

* If $U^1(\hat{\mu}^1)$ is empty, follow the steps for case **1-ii**.

- Step k:

  - Allow all workers in $W^0(\tilde{\mu}^{k-1}) \subseteq \tilde{\mu}^{k-1}(f_2)$ (if any) to move to the unemployment pool. This may cause other workers to want to move from f_2 to the unemployment pool. If so, allow them and subsequent sets of similar workers to move. This process ends at some point since the number of workers matched with f_2 is finite; call the resulting new matching μ^k.

  - If both $W^1(\mu^k)$ and $U^1(\mu^k)$ are empty, follow the steps for case **1-ii**.

  - If $W^1(\mu^k)$ is empty, and $U^1(\mu^k)$ is non-empty, allow workers to move to firm 1. This may cause other workers to want to move from the unemployment pool to f_1. If so, allow them and subsequent sets of similar workers to move. The process ends at some point since the number of workers in the unemployment pool is finite; call the new matching $\tilde{\mu}^k$.

  - If $W^1(\mu^k)$ is non-empty, allow workers to move to firm 1. This may cause other workers to want to move from f_2 to f_1. If so, allow them and subsequent sets of similar workers to move. The process ends at some point since the number of workers matched with f_2 is finite; call the new matching $\hat{\mu}^k$.

    * If $U^1(\hat{\mu}^k)$ is non-empty, allow workers to move to firm 1. This may cause other workers to want to move from the unemployment pool to f_1. If so, allow them and subsequent sets of similar workers to move. The process ends at some point since the number of workers in the unemployment pool is finite; and call the new matching $\tilde{\mu}^k$.

    * If $U^1(\hat{\mu}^k)$ is empty, follow the steps for case **1-ii**.

The process leads to case **1-ii** at some k^* since the number of workers that are either matched with firm 2 or unemployed is finite.

Note that if some workers moved from firm 2 to firm 1 during the process, then $U^2(\mu^{k^*})$ is empty, since μ is myopic-stable. Namely, for all workers with $\mu(w) = \emptyset$, it must be the case that one of the following statements is true:

- $\emptyset \succ_w (f_1, |\mu(f_1)|)$ and $\emptyset \succ_w (f_2, |\mu(f_2)|)$

- $\emptyset \succ_w (f_1, |\mu(f_1)|)$, $(f_2, |\mu(f_2)|) \succ_w \emptyset$, and $\emptyset \succ_{f_2} w$

- $\emptyset \succ_w (f_2, |\mu(f_2)|)$, $(f_1, |\mu(f_1)|) \succ_w \emptyset$, and $\emptyset \succ_{f_1} w$

20

- $(f_1, |\mu(f_1)|) \succ_w \emptyset$, $\emptyset \succ_{f_1} w$, $(f_2, |\mu(f_2)|) \succ_w \emptyset$, and $\emptyset \succ_{f_2} w$

If at least one worker moves from firm 2 to firm 1, $|\mu^{k^*}(f_2)| < |\mu(f_2)| - 1$, and workers in $U^2(\mu)$ now prefer to remain unemployed even after taking their own deviations into account.

Had we allowed workers to move from $U^2(\mu)$ instead of $U^1(\mu)$, we would return to case **1-iii** instead of **1-ii**.

Case 2:

Suppose $W^1(\mu) = \emptyset$ and $W^2(\mu)$ is non-empty. Allow workers in $W^2(\mu)$ to move to firm 2 and call the new matching μ^1. Since $W^1(\mu)$ is empty, so is $W^1(\mu^1)$. If $W^2(\mu^1)$ is non-empty, allow workers to move to firm 2 and call the resulting matching μ^2. Continue the process until $W^2(\mu^{k^*})$ is empty - such a k^* must exist since the number of workers matched with firm 1 is finite. Since $W^2(\mu)$ was non-empty and μ is a myopic-stable matching, $W^1(\mu^{k^*}) = \emptyset$, and we can follow the steps in **case 1**.

Case 3:

This is the same as **case 2** but with workers moving to firm 1 instead of firm 2.

Case 4:

When both $W^1(\mu)$ and $W^2(\mu)$ are non-empty, allowing any one of the sets of workers to move to its preferred firm (and allowing subsequent sets of similar workers to move in the same direction) takes us back to **case 1**, since μ is myopic-stable. To see this, suppose we let workers in $W^1(\mu)$ move to firm 1 and call the new matching μ^1. Since μ is myopic-stable, $W^2(\mu^1) = \emptyset$.

- Step 1:

    - Suppose $W^1(\mu^1) \neq \emptyset$. Allow workers in $W^1(\mu^1)$ to move to firm 1 and call the new matching μ^2.

- Step $k - 1$:

    - Suppose $W^1(\mu^{k-1}) \neq \emptyset$. Allow workers in $W^1(\mu^{k-1})$ to move to firm 1, and call the new matching μ^k.

The process must end at some point, and there must exist a k^*: $W^1(\mu^{k^*}) = \emptyset$, since the number of workers matched with firm 2 is finite. Moreover, $W^2(\mu^{k^*}) = \emptyset$ by myopic-stability of μ. This takes us back to **case 1**.

$\square$

Example 3 (continued) Since all the workers prefer to be matched with some firm than remain unemployed, and firms find all workers acceptable, there will be no unemployed

workers at any stable matching. It suffices, therefore, to consider matchings in which all workers are employed. There are 27 such matchings (including the 6 shown in the main text). In 6 of these 27 matchings, one worker is matched with each firm. For example, $\mu = \{\{f_1, w_1\}, \{f_2, w_2\}, \{f_3, w_3\}\}$ is such a matching. None of these matchings can be stable, since workers always prefer to join another firm with an existing employee than remain at a firm that only employs one worker. Next, consider matchings in which two workers are matched with one firm and the remaining worker is matched with another firm. There are 18 such matchings, none of which are stable. To see this, notice that in 6 of these matchings, worker 1 is the only employee in the firm they are matched with. If that firm is either firm 1 or 3, they would prefer to move to firm 2 (even when it doesn't employ the other two workers). When that firm is firm 2, and the other two workers are employed by firm 3, then worker 1 prefers to deviate and join them. Finally, when worker 1 is matched with firm 2 and the other two workers are matched with firm 1, we are back to μ_1, which we know is not stable.

In 6 of the matchings in which two workers are matched with one firm and one worker is matched with another, worker 2 is the only worker employed by some firm. If they are employed by either firm 2 or 3, they block the going matching and move to firm 1. If they are employed by firm 1 and the remaining workers are employed by firm 2, they block the matching by moving to firm 2. Finally, if they are employed by firm 1 and the rest are employed by firm 3, we are back to μ_5, which is not stable.

In the remaining 6 matchings, worker 3 is the sole employee of some firm. This firm cannot be firm 1 in a stable matching, since worker 3 strictly prefers to join firm 3 with any number of employees than remain at firm 1. The worker can also never be employed by firm 2 since it individually blocks such a matching. If the worker is employed by firm 3 and the remaining workers are employed by firm 1, then it prefers to deviate and join them. If they are employed by firm 2, then worker 1 deviates and joins firm 3.

Finally, there are 3 matchings in which all 3 workers are matched with the same firm. Call them μ^1, μ^2, and μ^3, where μ^j is the matching in which all workers are matched with firm j. μ^2 is not stable because worker 3 prefers to be single. μ^1 is not stable because worker 1 prefers to join firm 2. μ^3 is not stable because worker 2 prefers to be single.

This shows that the game presented in example 3 has no stable matching.