FALL 2023

CMPE371/CS571, Homework 1

Due date: 9.11.2023, 23:59

Q.1. For each of the following systems with input x(t) and output y(t), determine the system is (i) memoryless, (ii) stable, (iii) causal, (iv) linear and (v) time invariant. In each case give a short justification using the definitions of these properties.

(a) $y(t) = x(t)\sin(2t) + 1$

(b) $y(t) = x(t/2)$

(c) $y(t) = \int_{-\infty}^{t-1} 2x(\tau)d\tau$

(d) $y[n] = \sum_{k=-\infty}^{n} x[k+1]$

(e) $y[n] = x[n]\sum_{k=-\infty}^{\infty} \delta[n-3k]$

We have to analyze each of the given systems in terms of properties:

$y(t) = x(t)\sin(2t) + 1$

- This system is not memoryless. It depends on the current input value x(t).
- It is not stable in general. Because it has a sine function and sine functions may cause the output to become unbounded for certain functions.
- This system is causal. Since the output of t at any time only depends on past and current values of the input x(t).
- This system is linear. Since it is a linear combination of the input x(t), and a constant 1.
- This system is time-invariant. This equation doesn't specifically involve any time shifts.

$y(t) = x\left(\frac{t}{2}\right)$

- This system is memoryless. Since the output at any t time only depends on the current input value of x(t) and did not use previous input value of x(t), it is memoryless.

- This system is stable. It scales the input by a factor of 2. Because of that, it doesn't affect the boundedness of the output for a bounded input.

- This system is causal. Since the output of t at any time only depends on past and current values of the input x(t).

- This system is linear. Since it scales the value of x(t) by a constant factor 2, it is linear.

- This system is time-invariant. This equation doesn't specifically involve any time shifts.

$$y(t) = \int_{-\infty}^{t-1} 2x(\tau)\, d\tau$$

- This system is not memoryless. It contains an integral operation over a range of time.

- This system is not necessarily stable. A system is stable if every bounded input leads to a bounded output. In the given system, even if $x(t)$ is bounded, the integration over an infinite range (from $-\infty$ to $t-1$) can potentially lead to an unbounded output. Hence, without additional information about $x(t)$, we can not guarantee that the system is stable.

- This system is causal. If the output at any time t depends only on the values of the input at the present time and in the past, not on future inputs. In this system, the output $y(t)$ depends on $x(\tau)$ for τ up to $t-1$, which includes only past values.

- This system is linear. Since it performs a linear integration on the input $x(t)$.

- This system is time-invariant. Since it doesn't explicitly involve any time shifts.

$$y[n] = x[n] \sum_{k=-\infty}^{n} x[k+1]$$

- This system is memoryless. If the output at any time n depends only on the input at that same time n and not on past or future inputs. In this system, the output $y[n]$ at time n depends on past values of $x[n]$

- This system is not necessarily stable. Whether this system is stable depends on the properties of the input $x[n]$. If $x[n]$ is bounded, then the summation might also be bounded. However, if $x[n]$ is unbounded, the summation may not be bounded.

- This system is not causal. Since it depends on future values on the input $x[n+1]$ when calculating the output at n, it is not causal.

- This system is linear. Since it performs a linear operation as summation on the input $x[n+1]$, it is a linear system.

- This system is time-invariant. A system is time-invariant if a time shift in the input signal results in the same time shift in the output signal. If $x[n]$ is shifted to $x[n-\tau]$, the output becomes $\sum_{k=-\infty}^{n} x[k-\tau+1]$, which is equivalent to shifting $y[n]$ by τ.

$$y[n] = x[n] \sum_{k=-\infty}^{\infty} \delta[n-3k]$$

- This system is not memoryless. Since it relies on the current input value $x[n]$ and also involves a summation over an infinite range, it does not have any memory.
- This system is stable. Since $\delta[n-3k]$ is non-zero only for certain values of n (multiples of 3), and for those values, it equals 1, the output $y[n]$ will equal $x[n]$ at those points and zero otherwise. If $x[n]$ is bounded, so will be $y[n]$, therefore it is stable.
- This system is causal. Since $y[n]$ depends only on $x[n]$ at the same time instance and not on any future or past values of $x[n]$, it is casual.
- This system is linear. Since multiplying the input $x[n]$ by a constant or adding two inputs together will result in the output being scaled or added in the same way, it is linear.
- This system is not time-invariant. If $x[n]$ is shifted to $x[n-T]$, the output becomes $x[n-T] \sum_{k=-\infty}^{\infty} \delta[n-T-3k]$, which is not the same as shifting $y[n]$ by T due to the delta function. Since a time shift in the input signal does not have the same time shift in the output signal, it is not time-invariant.

Q.2. For an LTI system with impulse response h[n], prove that the system is causal if h[n] = 0 for negative values of n.

To prove that a linear time-invariant (LTI) system is causal when its impulse response h[n] is zero for negative values of n.

In an LTI system, the output y[n] for any input x[n] is given by the convolution of x[n] and the systems impulse response h[n]. The convolution sum is defined as:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

To the system to be causal, the output y[n] at time n must not be dependent on future values of x[n], which means that values of x[n] for n>k.

Given h[n] = 0 for n<0, let's analyze the convolution sum:

- When $n-k<0$, $h[n-k]=0$ by the given condition. Therefore, terms in the convolution sum where $n<k$ (indicating future values of the input) will not contribute to the output because $h[n-k]$ is zero for these terms.
- This leaves only terms where $n-k \geq 0$, or equivalently $k \leq n$, contributing to the sum. These terms correspond to the present and past values of the input x[n].

Therefore, the output y[n] at any given time n depends only on the values of the input up to that time and not on any future values. This satisfies the definition of casuality, proving that the system is causal under the given condition that $h[n] = 0$ for $n<0$.

Q.3. For the following system at initial rest,

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

$$h(t) = e^{-2t}u(t)$$

Determine whether this system is causal, memoryless and stable ?

Casuality: A system is said to be causal when its output at any given time t is solely dependent on inputs from the past and present. Because it is zero for negative and when we write the output as $y(t) = \int_0^t h(\tau) \cdot f(t-\tau) \cdot d\tau$, where $h(t)$ is the impulse response and $f(t)$ is the input action, $y(t)$ at any instant depends on the values of $f(t)$ for $t \leqslant \tau \leqslant t$, it satisfies the casuality property and is thus casual.

Memoryless: A system is said to be memoryless if its output is solely dependent on its current input and not on inputs from the past at any given time t. In this case, the derivative operation of the system is entirely dependent on the current input. As a result of it, it lacks memory.

Stability: A system is said to be memoryless if its output remains bounded for bounded inputs. The exponential term in $h(t)$ decays as t increases, stabilizing the system. We can see that when we are looking to the impulse response $h(t)$. From the given differential equation, we can find the impulse response as $h(t) = -2 \cdot e^{-2t} \cdot u(t)$. As t goes to infinity, $h(t)$ function goes to 0. Since the impulse response is absolutely integrable, this system is stable.

Q.4. Find the output y[n] of the system h[n] to the input x[n]. Use discrete-time convolution.

a. $x[n] = \{1,3,1\}$ where $n=\{0,1,2\}$ and
$h[n] = \{2,2,1\}$ where $n=\{0,1,2\}$

b. $x[n] = 8\delta[n+1] + 2\delta[n-1] + 2\delta[n-2] - \delta[n-3]$
$h[n] = \delta[n] + \delta[n-2]$

To find the output $y[n]$ of the system $h[n]$ to the input $x[n]$ using discrete-time convolution, we can use this formula:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

We should calculate the convolution for the given cases:

a-) $x[n] = \{1,3,1\}$ where $n = \{0,1,2\}$ and
$h[n] = \{2,2,1\}$ where $n = \{0,1,2\}$

$$y[n] = \sum_{k=-\infty}^{2} x[k] \cdot h[n-k]$$

For $(n=0)$:

$y[0] = x[0] \cdot h[0] = 1 \cdot 2 = 2$

For $(n=1)$:

$y[1] = x[0] \cdot h[1] + x[1] \cdot h[0] = 1 \cdot 2 + 3 \cdot 2 = 8$

For $(n=2)$:

$y[2] = x[0] \cdot h[2] + x[1] \cdot h[1] + x[2] \cdot h[0]$
$= 1 \cdot 1 + 3 \cdot 2 + 1 \cdot 2 = 9$

So, the output sequence $y[n]$ is $\{2, 8, 9\}$

b-) $x[n] = 8 \cdot \delta[n+1] + 2 \cdot \delta[n-1] + 2 \cdot \delta[n-2] - \delta[n-3]$ and

$h[n] = \delta[n] + \delta[n-2]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

For $(n=0)$:

$$y[0] = x[0] \cdot h[0] = 8 \cdot 1 = 8$$

For $(n=1)$:

$$y[1] = x[0] \cdot h[1] + x[1] \cdot h[0] = 8 \cdot 0 + 2 \cdot 1$$
$$= 2$$

For $(n=2)$:

$$y[2] = x[0] \cdot h[2] + x[1]\, h[1] + x[2] \cdot h[0]$$
$$= 8 \cdot 0 + 2 \cdot 0 + 2 \cdot 1$$
$$= 2$$

For $(n=3)$:

$$y[3] = x[0]\, h[3] + x[1]\, h[2] + x[2]\, h[1] + x[3]\, h[0].$$
$$= 8 \cdot 0 + 2 \cdot 0 + 2 \cdot 0 + (-1) \cdot 1$$
$$= -1$$

Therefore the output sequence of $y[n]$ is

$\{8, 2, 2, -1, 0, 0, \ldots\}$