Department of Computer Engineering

Signals and Systems for Computer Engineering

Fall 2023-2024, HW2

Due Date: 22.12.2023, 23:59

FULL NAME:

STUDENT ID:

SECTION:

Q1) For each of the signals given below;

(a) $\delta(t-8)$ **(b)** $e^{-at}u(t)$**, a is real and positive** **(c)** $e^{(-1+j2)t}u(t)$

I. **Find the Fourier transform**
II. **Sketch the magnitude as a function of frequency (include both positive and negative frequencies)**
III. **Sketch the phase as a function of frequency (include both positive and negative frequencies).**

a-) $\delta(t-8)$

I-) $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t-8) e^{-jwt} dt$

$= e^{-j8w} = \cos 8\omega - j\sin 8\omega$

through the sifting aspect of the unit impulse.

$|X(\omega)| = |e^{j8\omega}| = 1$ for all ω,

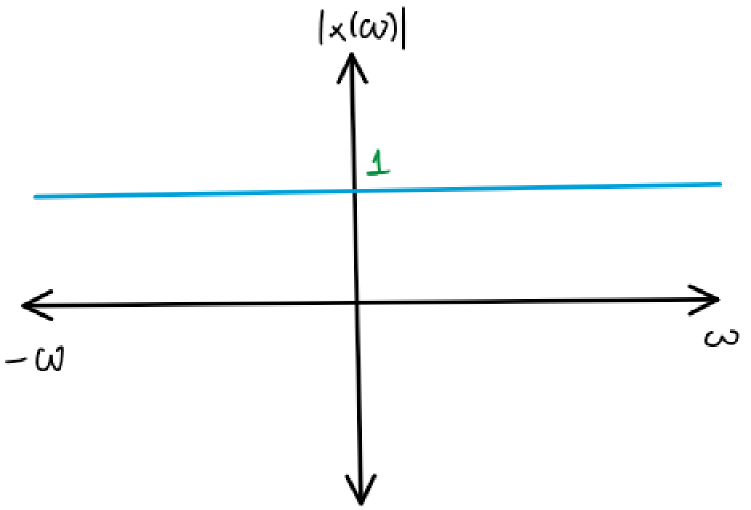
$\measuredangle X(\omega) = \tan^{-1}\left[\frac{Im\{X(\omega)\}}{Re\{X(\omega)\}}\right] = \tan^{-1}\left(\frac{-\sin 8\omega}{\cos 8\omega}\right) = -8\omega$

II-)

$$X(\omega) = Re\{X(\omega)\} + j\, Im\{X(\omega)\}$$
$$= e^{-j\omega \cdot 8} = \cos(-8\omega) + j\sin(-8\omega) = \cos(8\omega) - j\sin(8\omega)$$

$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \cos^2(8\omega) + \sin^2(8\omega)$

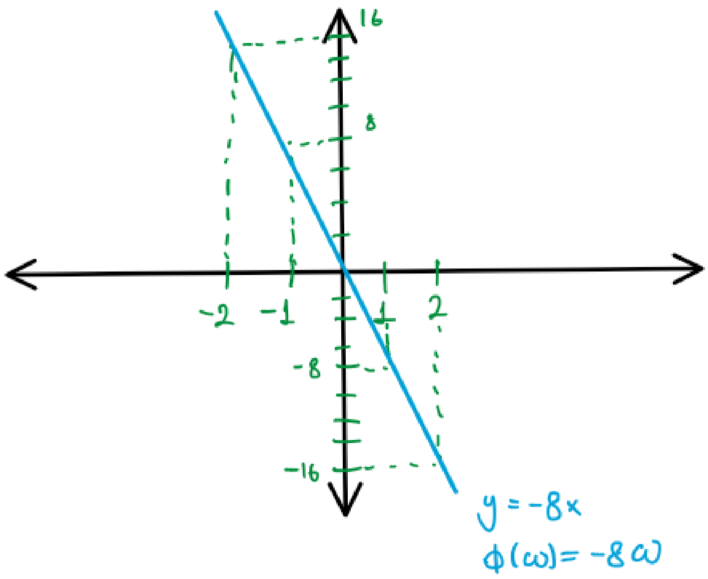
$$|X(\omega)| = \sqrt{\cos^2(8\omega) + \sin^2(8\omega)}$$
$$= 1$$



III-)

For the phase, we need the argument angle of the complex number $e^{8j\omega}$. Since Euler's formula puts $-e^{8j\omega}$ into the form $\cos(-8\omega) + j\sin(-8\omega)$, the angle between that point and the x-axis must be found.

This angle can be found by $\arctan\left(\frac{\sin(-8\omega)}{\cos(-8\omega)}\right) = \arctan\left(\tan(-8\omega)\right) = -8\omega$.



b-) $e^{-at}u(t)$, a is real and positive

I-) $$X(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt = \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

Since $Re\{a\} > 0$, e^{-at} goes to zero as t goes to infinity. As a result of it,

$$X(\omega) = \frac{-1}{a+j\omega}(0-1) = \frac{1}{a+j\omega},$$

$$|X(\omega)| = [X(\omega)X^*(\omega)]^{\frac{1}{2}} = \left[\frac{1}{a+j\omega}\left(\frac{1}{a-j\omega}\right)\right]^{\frac{1}{2}} \frac{1}{\sqrt{a^2+\omega^2}},$$

$$Re\{X(\omega)\} = \frac{X(\omega)+X^*(\omega)}{2} = \frac{a}{a^2+\omega^2},$$

$$Im\{X(\omega)\} = \frac{X(\omega)-X^*(\omega)}{2} = \frac{-\omega}{a^2+\omega^2},$$

$$\measuredangle X(\omega) = \tan^{-1}\left[\frac{Im\{X(\omega)\}}{Re\{X(\omega)\}}\right] = -\tan^{-1}\frac{\omega}{a}$$

II-)

$$X(\omega) = \frac{1}{a+j\omega} = \frac{1}{a+j\omega} \cdot \frac{(a-j\omega)}{(a-j\omega)} = \frac{a-j\omega}{a^2+\omega^2} - j\,\frac{\omega}{a^2+\omega^2}$$

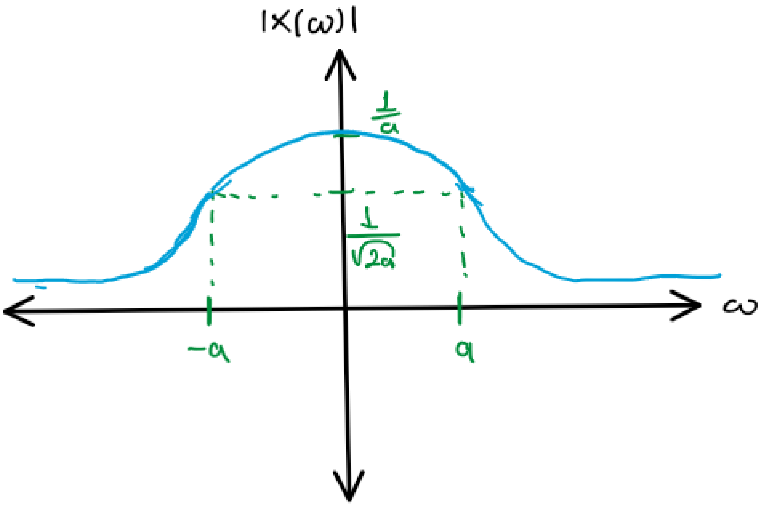
$$\mathrm{Re}\{X(\omega)\} = \frac{a}{a^2+\omega^2} \qquad \mathrm{Im}\{X(\omega)\} = \frac{-\omega}{a^2+\omega^2}$$

$$|X(\omega)| = \sqrt{\left(\frac{a}{a^2+\omega^2}\right)^2 + \left(\frac{-\omega}{a^2+\omega^2}\right)^2} = \sqrt{\frac{a^2}{a^4+2a^2\omega^2+\omega^4} + \frac{\omega^2}{a^4+2a^2\omega^2+\omega^4}}$$

$$= \sqrt{\frac{\cancel{a^2+\omega^2}}{(a^2+\omega^2)^{\cancel{2}}}} = \sqrt{\frac{1}{a^2+\omega^2}}$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

for any $a \in \mathbb{R}^+$

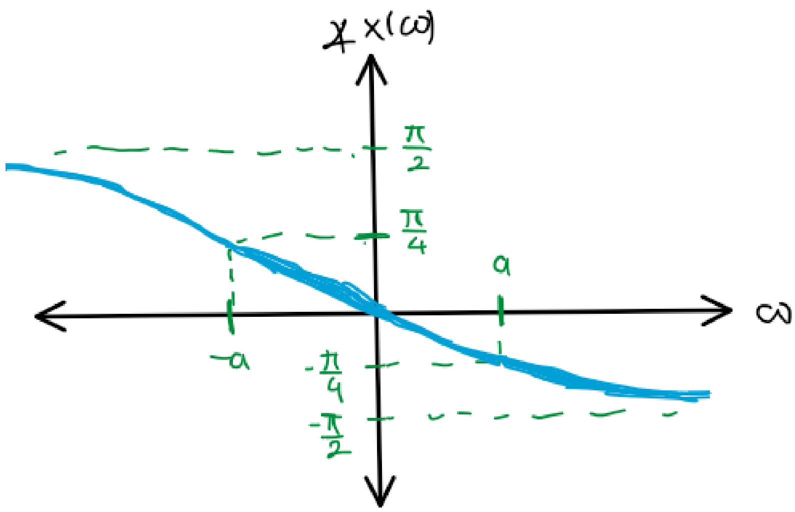


III-)

The Fourier transform of $e^{-at}u(t)$ (assuming $a > 0$) is $\frac{1}{a+i2\pi\omega}$.

- This can be represented as $\frac{1}{a+i2\pi\omega} = \frac{a}{a^2+(2\pi\omega)^2} - i\frac{2\pi\omega}{a^2+(2\pi\omega)^2}$.
- The phase is $\arg\left(\frac{1}{a+i2\pi\omega}\right) = \tan^{-1}\left(\frac{-2\pi\omega}{a}\right)$.

The phase is a function of ω and represents the phase shift introduced by the exponential decay in time.



c-) $e^{(-1+j2)t}u(t)$

I-) $$X(\omega) = \int_{-\infty}^{\infty} e^{(-1+j2)t}\, u(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{(-1+j2)t} e^{-j\omega t}\, dt$$

$$= \frac{1}{-1+j(2-\omega)}\, e^{(-1-j(2-\omega))t} \Big|_{0}^{\infty}$$

Since $Re\{-1+j(2-\omega)\} < 0$, $\lim_{t \to \infty} e^{[-1+j(2-\omega)]t} = 0$.

As a result of it,

$$X(\omega) = \frac{1}{1+j(\omega-2)}$$

$$|X(\omega)| = [X(\omega)X^*(\omega)]^{\frac{1}{2}} = \frac{1}{\sqrt{1+(\omega-2)^2}}$$

$$Re\{X(\omega)\} = \frac{X(\omega)+X^*(\omega)}{2} = \frac{1}{\sqrt{1+(\omega-2)^2}}$$

$$Im\{X(\omega)\} = \frac{X(\omega)-X^*(\omega)}{2} \cdot \frac{-(\omega-2)}{1+(\omega-2)^2}$$

$$\measuredangle X(\omega) = \tan^{-1}\left[\frac{Im\{X(\omega)\}}{Re\{X(\omega)\}}\right] = -\tan^{-1}(\omega-2)$$

II-)

$$X(\omega) = \frac{1}{1+j(-2+\omega)} = \frac{1}{1+j(-2+\omega)} \cdot \frac{1-j(-2+\omega)}{1-j(-2+\omega)} = \frac{1-j(-2+\omega)}{1+(-2+\omega)^2} = \frac{1+j2-j\omega}{\omega^2-4\omega+5}$$

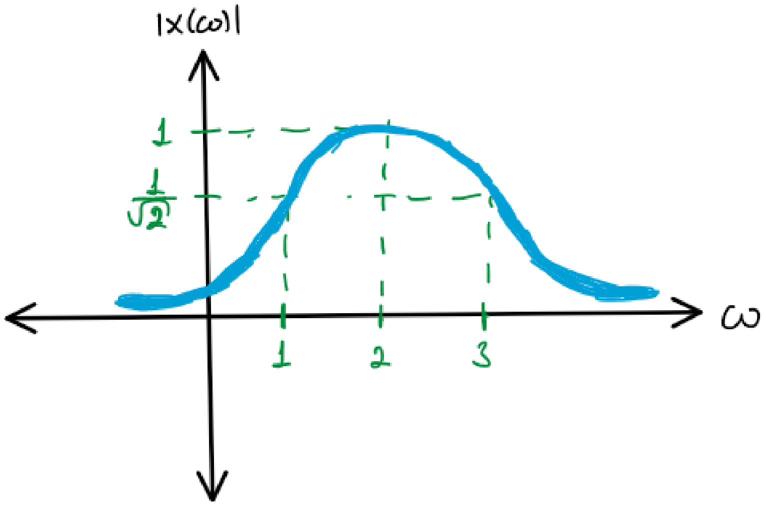
$$Re\{x(\omega)\} = \frac{1}{\omega^2-4\omega+5} \qquad Im\{x(\omega)\} = \frac{2-\omega}{\omega^2-4\omega+5}$$

$$|x(\omega)| = \sqrt{\frac{1}{(\omega^2-4\omega+5)^2} + \left(\frac{2-\omega}{\omega^2-4\omega+5}\right)^2} = \sqrt{\frac{\omega^2-4\omega+5}{(\omega^2-4\omega+5)^2}} = \frac{1}{\sqrt{\omega^2-4\omega+15}}$$

$$\sqrt{\omega^2-4\omega+5} \neq 0$$

$$\omega^2-4\omega+5 > 0$$

$$(\omega-2)^2+1$$



III-)

The Fourier transform of $e^{(-1+j2)t}u(t)$ is $\frac{1}{1-i(2-2\pi\omega)}$.

- This can be written as $\frac{1-i(2-2\pi\omega)}{1+(2-2\pi\omega)^2} = \frac{1}{1+(2-2\pi\omega)^2} - i\frac{2-2\pi\omega}{1+(2-2\pi\omega)^2}$.
- The phase is $\arg\left(\frac{1}{1-i(2-2\pi\omega)}\right) = \tan^{-1}\left(\frac{-(2-2\pi\omega)}{1}\right)$.

The phase in this case is also a function of ω, representing the complex exponential's effect in both decay and oscillation.