Lecture 9: Introduction to Heterogeneous Agents – the Aiyagari Model

Carlos Góes

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1 Environment

Time is discrete. There are infinitely many ex ante identical households distributed along the unit interval. Each household maximizes the present discounted value of lifetime utility:

$$\begin{aligned} v(a_0, z_0) \equiv \quad & \max_{\{c_t, a_{t+1}\}_{t\geq 0}} \quad \sum_{t=0}^{\infty} \beta^t \mathbb{E}_0 \left[u(c_t) \right] \\ & s.t. \quad c_t + a_{t+1} = (1 + r_t) a_t + w_t z_t \\ & \qquad z_t = \Pi(z_{t-1}), \quad z_t \in \{\bar{z}_1, \cdots, \bar{z}_N\} \quad N < \infty, \\ & \qquad c_t \geq 0, \quad a_t \geq -b \end{aligned}$$

where c_t is consumption; a_t are asset holdings, which must respect a borrowing constraint such that net assets are never smaller than $-b$; r_t are rental rates; and w_t are wages.

Households are endowed with one unit of hours and supply labor inelastically. Households are hit with an efficiency shock z_t. You can interpret z_t as productivity shocks or unemployment risk shocks, which are independently and identically distributed across households.

Note that, while individuals are *ex ante* identical, the labor supply draw z_t makes them *ex post* heterogeneous. The productivity shock follows the **Markov process** $\Pi(z_{t-1})$ with potential values potential values $\mathcal{Z} = \{\bar{z}_1, \cdots, \bar{z}_N\}$. A Markov Process is also called a "memoryless process," i.e. one in which the probability distribution of z_t only depends on z_{t-1}). Formally, in a Markov process: $\mathbb{P}\left(z_{t+1} = \bar{z}_i | z_t\right) = \mathbb{P}\left(z_{t+1} = \bar{z}_i | z_t, z_{t-1}, \cdots\right)$. The elements matrix Π describe the transition probabilities for labor supply, i.e.: $\pi(\bar{z}_i|\bar{z}_{i'}) \equiv \mathbb{P}\left(z_{t+1} = \bar{z}_i | z_t = \bar{z}_{i'}\right)$.

The first important remark about this model is that it is a **model of incomplete markets**. Individuals face idiosyncratic shocks z_t but they can only invest in a risk free asset a_t. There is no market for state-contingent assets, which would allow household to insure themselves against risk.

Furthermore, each period, this economy is described by a joint distribution of households over states and assets $\Phi(a_t, z_t)$. Aggregate variables, such as prices, are a complicated function of this distribution. In general, this is a very complicated problem to solve: it is a highly dimensional problem that involves converging over a fixed point for every part of this distribution.

As we will see later, we restrict the analysis to understanding the **stationary distribution** of this economy. A stationary distribution in one in which aggregate variables are constant and the share of households in each state is also constant. It does not mean, however, that the assets of every household will be constant. In fact, households will be facing idiosyncratic shocks and changing states, but will do so in such a fashion that the share of individuals moving in and out of each states is such that they exactly cancel out.

The production side of this economy is simple, since we focus primarily in heterogeneity across households. There is a continuum of identical firms with Cobb-Douglas technology $Y_t = A_t K_t^{\theta} N_t^{1-\theta}$ that hire labor and capital at the spot market. To accommodate depreciation, we define the gross interest rate $r_t^{gross} = r_t + \delta$, which will be the prevailing rate at the rental market for capital (ask yourself why).

2 Define and characterize the recursive equilibrium

Household problem. The problem above can be written recursively as such:

$$
\begin{aligned}
v(a,z,\Phi) &= \max_{c,a'} u(c) + \beta \mathbb{E}\left[v(a',z',\Phi')\middle| z\right] \\
&= \max_{c,a'} u(c) + \beta \sum_{z'} \pi(z'|z) v(a',z',\Phi') \\
s.t. &\quad c + a' = (1+r(\Phi))a + w(\Phi)z \\
&\quad a' \geq -b \\
&\quad \Phi' = \hat{G}(\Phi)
\end{aligned}
$$

To solve for the optimality conditions, set up a Lagrangian:

$$\mathcal{L} = u(c) + \beta \sum_{z'} \pi(z'|z) v(a',z',\Phi') + \lambda[(1+r(\Phi))a + w(\Phi)z - c - a'] + \mu[a' + b]$$

Households optimality conditions satisfy:

$$
\begin{aligned}
c &: \quad u'(c) = \lambda \\
a &: \quad \lambda = \beta \sum_{z'} \pi(z'|z) v_{a'}(a',z',\Phi') + \mu
\end{aligned}
$$

To derive $v_a(a,z,\Phi)$, write the value function replacing for consumption, with the choice variable evaluated at its policy function $a'(a,z,\Phi)$:

$$v(a,z,\Phi) = u((1+r(\Phi))a + w(\Phi)z - a'(a,z,\Phi)) + \beta \sum_{z'} \pi(z'|z) v(a',z',\Phi')$$

and, using the envelope theorem, take derivatives ignoring the reoptimization effect:

$$v_a(a,z,\Phi) = u'(c)(1+r(\Phi))$$

Evaluating this derivative one period forward and combining the two first order conditions yield the euler equation:

$$u'(c) = \beta \sum_{z'} \pi(z'|z) u'(c')(1+r(\Phi')) + \mu$$

Whenever the borrowing constraint does not bind, $\mu = 0$ and the Euler Equation becomes more familiar. If the budget constraint binds, however, then $\mu > 0$ and

$$u'(c) > \beta \sum_{z'} \pi(z'|z) u'(c')(1+r(\Phi'))$$

which implies that the household would rather have a higher level of c.

Firms problem. The static firms problem is:

$$\max_{K_d,N_d} A(K_d)^{\theta}(N_d)^{1-\theta} - w(\Phi)N_d - r^{gross}(\Phi)K_d$$

with optimality conditions:

$$r(\Phi) = \theta A \left(\frac{K_d}{N_d}\right)^{-(1-\theta)} - \delta, \qquad w(\Phi) = (1-\theta)A\left(\frac{K_d}{N_d}\right)^{\theta}$$

Solving for K_d/N_d in the first equation, we can express wages as:

$$w(\Phi) = (1-\theta)A\left(\frac{\theta A}{r(\Phi)+\delta}\right)^{\frac{\theta}{1-\theta}}$$

Definition 1 (Recursive Competitive Equilibrium)**.** A Recursive Competitive Equilibrium consists of:

(a) a household value function $v(a, z, \Phi)$ and policy functions $c(a, z, \Phi)$, $a'(a, z, \Phi)$;

(b) firms factor demand functions $K_d(\Phi)$, $N_d(\Phi)$;

(c) prices functions $r(\Phi)$, $w(\Phi)$;

(d) a joint distribution over assets and states $\Phi(a, z)$;

(e) a perceived law of motion for the aggregate state $\Phi(a', z') = H(\Phi(a, z))$.
such that:

(i) Given (c), (d) and (e), (a) solves the household problem;

(ii) Given (c) and (d), (b) solves the firms problem;

(iii) Factor markets clear:

$$N(\Phi) = N_d(\Phi) = 1, \qquad K(\Phi) = K_d(\Phi) = \int a d\Phi(a, z)$$

(iv) Goods markets clear:

$$\int c(a, z, \Phi) d\Phi(a, z) + \int a'(a, z, \Phi) d\Phi(a, z) = A[K(\Phi)]^{\theta}[N(\Phi)]^{1-\theta} + (1-\delta)K(\Phi)$$

(v) Expectations are correct:

$$\Phi'(a', z') = \hat{G}(\Phi)$$

Definition 2 (Stationary Equilibrium with Heterogeneous Agents)**.** A Stationary Equilibrium with Heterogeneous Agents consists of a Recursive Competitive Equilibrium that satisfies:

(a) Aggregate prices are fixed for periods t, $t+1$;

(b) Aggregate stocks $K(\Phi), N$ are fixed for periods t, $t+1$;

(c) The joint distribution of households in the assets, states space is fixed for periods t, $t+1$.

3 Solving for the Stationary Distribution

Here I will describe the intuition behind the model in the stationary distribution. In a companion Jupyter Notebook, I will provide a code in Julia that solves for the model. You can use that model to understand what is going on under the hood. However, instead of writing the solver algorithm ourselves, as we have done previously, we will take advantage of the QuantEcon library. The code presented here is largely based on the QuantEcon Julia Notebooks and on Florian Oswald's lecture notes.

The solution to this problem works in two steps. First, taking prices as given, we calculate what is the solution to the dynamic problem households face. Then, we solve for the prices that will make factor markets clear.

Solution to the dynamic problem Note that, given prices, the structure of the problem is such that there are N value functions in this problem:

$$\begin{aligned} v(a,\bar{z}_1,\Phi) &= \max_{a'} u((1+r(\Phi))a + w(\Phi)\bar{z}_1 - a') + \beta \sum_{\bar{z}_n \in \mathcal{Z}} \pi(\bar{z}_n|z_1) v(a',\bar{z}_n,\Phi') \\ &\vdots \\ v(a,\bar{z}_N,\Phi) &= \max_{a'} u((1+r(\Phi))a + w(\Phi)\bar{z}_N - a') + \beta \sum_{\bar{z}_n \in \mathcal{Z}} \pi(\bar{z}_n|\bar{z}_N) v(a',\bar{z}_n,\Phi') \end{aligned}$$

We can solve this problem using discrete dynamic programming. First, we create a discretized version of the asset space $\mathcal{A} = \{\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_K\}$, with, for all k, $\bar{a}_k - \bar{a}_{k-1} = c$. Since both a and z are state variables, the state-space in this problem is $\mathcal{A} \times \mathcal{Z} = \{(\bar{a}_1,\bar{z}_1),(\bar{a}_1,\bar{z}_2),\cdots,(\bar{a}_1,\bar{z}_N),(\bar{a}_2,\bar{z}_1),\cdots,(\bar{a}_K,\bar{z}_1),\cdots,(\bar{a}_K,\bar{z}_N)\}$.

We could use our own solver, as we have done in Lecture 2. As mentioned, we will instead resort to QuantEcon's DDP solver. The solver requires three objects:

1. A utility matrix $U[s,a']$, which lives in the $\mathcal{A} \times \mathcal{Z} \times \mathcal{Z}$ space and states the payoff under action $a' = \bar{a}_j$ and states $(\bar{a}_i, \bar{z}_n)$:

$$U = \begin{bmatrix} u((1+r(\Phi))\bar{a}_1 + w(\Phi)\bar{z}_1 - \bar{a}_1) & u((1+r(\Phi))\bar{a}_1 + w(\Phi)\bar{z}_1 - \bar{a}_2) & \cdots & u((1+r(\Phi))\bar{a}_1 + w(\Phi)\bar{z}_1 - \bar{a}_N) \\ u((1+r(\Phi))\bar{a}_1 + w(\Phi)\bar{z}_2 - \bar{a}_1) & u((1+r(\Phi))\bar{a}_1 + w(\Phi)\bar{z}_2 - \bar{a}_2) & \cdots & u((1+r(\Phi))\bar{a}_1 + w(\Phi)\bar{z}_2 - \bar{a}_N) \\ \vdots & \ddots & \ddots & \vdots \\ u((1+r(\Phi))\bar{a}_1 + w(\Phi)\bar{z}_N - \bar{a}_1) & u((1+r(\Phi))\bar{a}_1 + w(\Phi)\bar{z}_N - \bar{a}_2) & \cdots & u((1+r(\Phi))\bar{a}_1 + w(\Phi)\bar{z}_N - \bar{a}_N) \\ u((1+r(\Phi))\bar{a}_2 + w(\Phi)\bar{z}_1 - \bar{a}_1) & u((1+r(\Phi))\bar{a}_2 + w(\Phi)\bar{z}_1 - \bar{a}_2) & \cdots & u((1+r(\Phi))\bar{a}_2 + w(\Phi)\bar{z}_1 - \bar{a}_N) \\ \vdots & \ddots & \ddots & \vdots \\ u((1+r(\Phi))\bar{a}_K + w(\Phi)\bar{z}_N - \bar{a}_1) & u((1+r(\Phi))\bar{a}_K + w(\Phi)\bar{z}_N - \bar{a}_2) & \cdots & u((1+r(\Phi))\bar{a}_K + w(\Phi)\bar{z}_N - \bar{a}_N) \end{bmatrix}$$

2. A transition matrix $Q[s,a',s']$, which lives in the $\mathcal{A} \times \mathcal{Z} \times \mathcal{Z} \times \mathcal{A} \times \mathcal{Z}$ space and states the probability of moving from state s to state s' if the individual chooses action a today. Since in our model the choice a does not affect the probability of moving across states, even though we have to create a matrix with those dimensions, it will be the case that:

$$Q_{(\bar{a}_i,\bar{z}_j),\bar{a}_q,(\bar{a}_m,\bar{z}_n)} = \pi(\bar{z}_n|\bar{z}_j) = \mathbb{P}(\bar{z}_n|\bar{z}_j)$$

3. A intertemporal discount rate β.

For given guesses of $r(\Phi), w(\Phi)$, and the objects U, P, β defined above, we can then retrieve a N value functions using QuantEcon's solver. In the companion Jupyter Notebook, we solve the model with $N = 2$ productivity states. The value functions and policy functions look like the following:

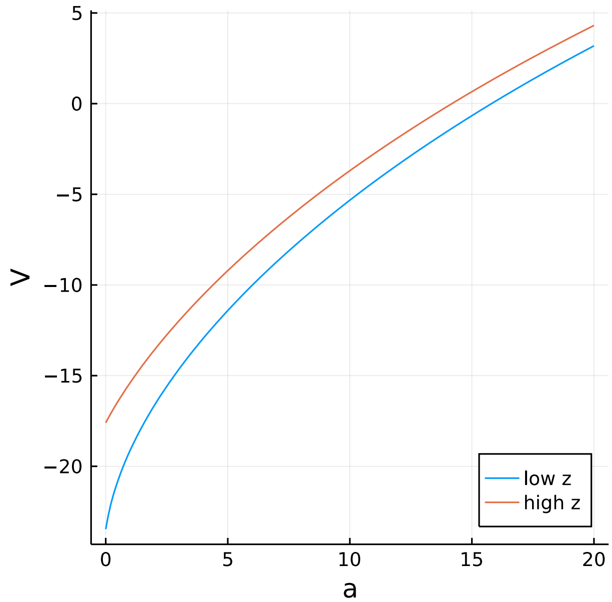

Figure 1: Value Functions

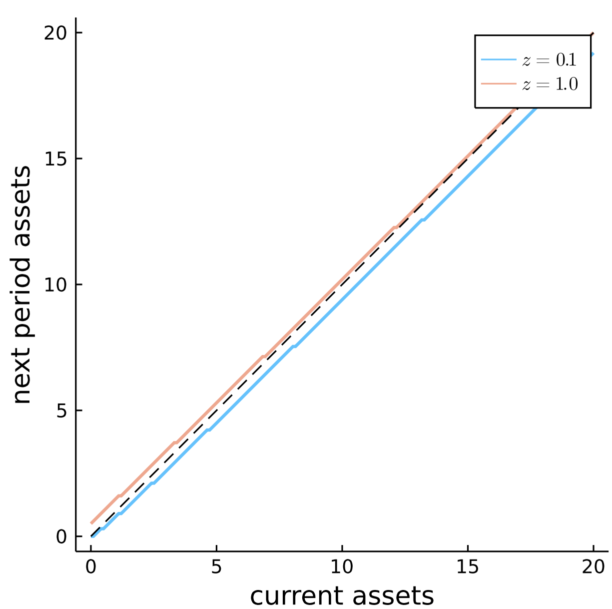

Figure 2: Policy Functions

Stationary Distribution. One of the resulting objects of this simulation is a **stationary distribution**. A stationary distribution is one satisfying:

$$\tilde{\Phi} = \tilde{\Phi}\mathbb{P}(a', z'|a, z)$$

where $\mathbb{P}(a', z'|a, z)$ is a matrix of transition probabilities from state (a, z) to state (a', z'). Note that the distribution is *stationary* because it remains unchanged after applying the transition matrix to it. This stationary property is induced by the Markov process. Given our guesses of prices, this is the stationary distribution of households consistent with the model (which might not be the true one):

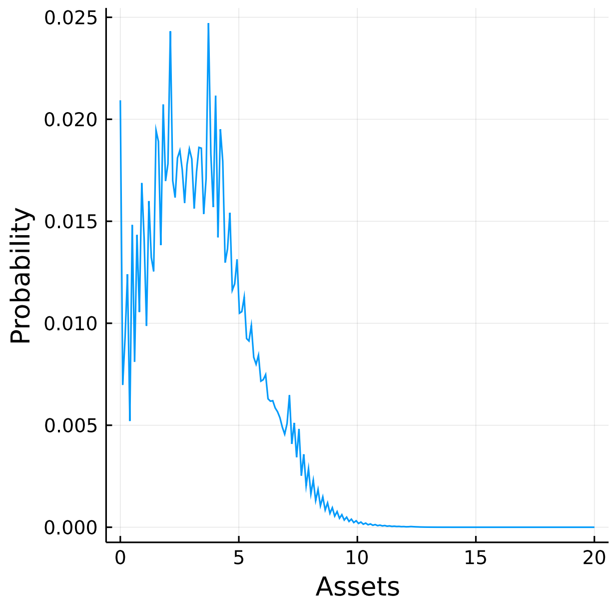

Figure 3: Candidate Stationary Distribution given guess of prices

General equilibrium. For now, we have simply guessed prices $r(\Phi)$ and $w(\Phi)$ and solved for the optimal consumer behavior given those prices. Since households treat prices as given, we can compute their optimal behavior for any guess of prices. However, in general equilibrium, as stated in the definition of the RCE, factor markets must clear, such that:

$$N(\Phi) = N_d(\Phi) = 1, \qquad K(\Phi) = K_d(\Phi) = \int a d\Phi(a, z)$$

In the equation above, $K_d(\Phi)$ must be equal the total (or average, since we have a measure one of households in this economy) assets saved, which will be the total capital supplied in this economy. For this to happen, prices must adjust to make sure that capital supply equals capital demand. Remember that, by optimality, the following conditions must hold:

$$r(\Phi) = \theta A \left(K_d(\Phi)\right)^{-(1-\theta)} - \delta, \qquad w(r(\Phi)) = (1-\theta)A\left(\frac{\theta A}{r(\Phi)+\delta}\right)^{\frac{\theta}{1-\theta}}$$

For each guess $r(\Phi)$, we can calculate the amounts of capital optimally supplied and demanded. This values are shown in the figure below.

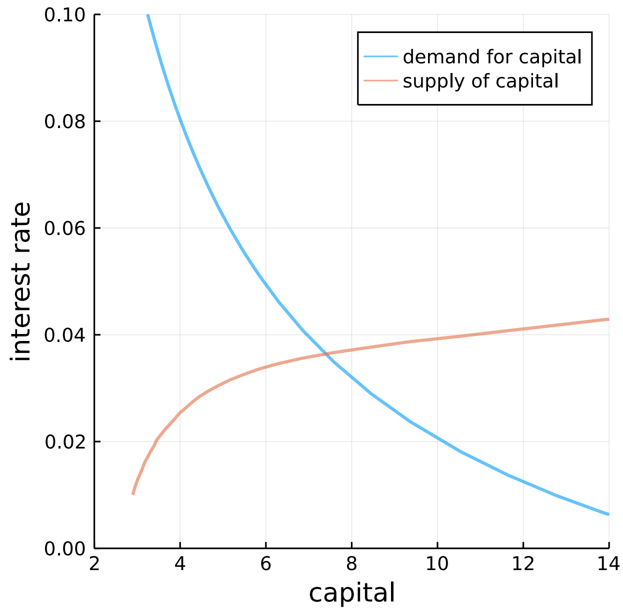

Figure 4: Supply and Demand for Capital at different Prices

To find the equilibrium price numerically, we can:

1. Choose some search interval $\mathcal{R} = \{\underline{r}, \cdots, \bar{r}\}$ and some tolerance level ε;
2. For each $r \in \mathcal{R}$:
   (a) Calculate capital demand $K_d(r) = \left(\frac{r+\delta}{A\theta}\right)^{-\frac{1}{1-\theta}}$;
   (b) Solve for the stationary distribution $\tilde{\Gamma}^r$ using the steps mentioned above and calculate capital supply as $K_s(r) = \int a d\Phi^r(a, z)$.
   (c) Calculate the excess capital supply function $\eta(r) = K_s(r) - K_d(r)$;
3. If, for some $r^* \in \mathcal{R}$, $|\eta(r^*)| < \varepsilon$, stop and choose r^* as your equilibrium interest rate; otherwise, choose another grid and tolerance and restart the algorithm.

I plot the excess supply function below. Naturally the equilibrium price will be whenever supply equals demand and excess supply is zero (up to numerical approximation error). Once we have our equilibrium prices, we can calculate $w(r^*)$ and compute the final stationary distribution repeating the solution for the dynamic problem. I also plot it below.

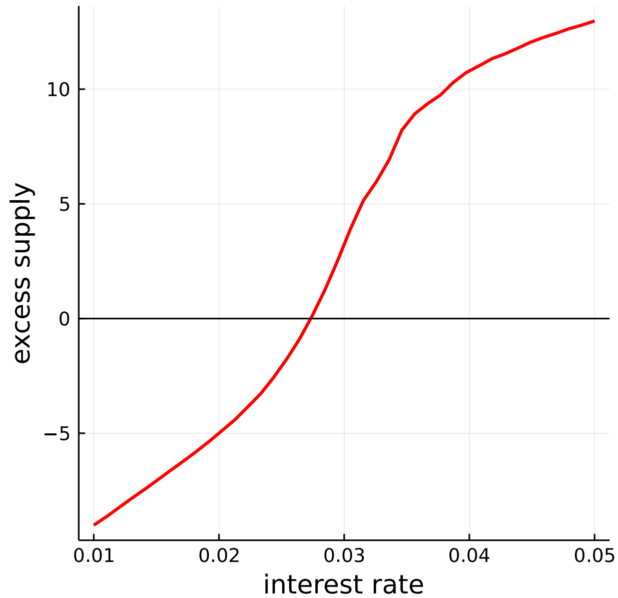

Figure 5: The excess supply function

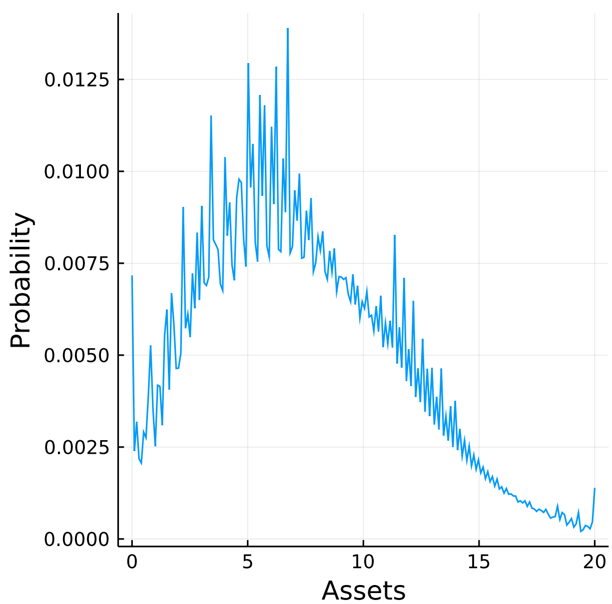

Figure 6: Final stationary distribution