Discussion Session 1: Fundamentals of Value Function Iteration

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1 Mathematical Foundations

This section states definitions and theorems that are essential for understanding Value Function Iteration. Proofs will be omitted throughout the section. The two main results are the Contraction Mapping Theorem and Blackwell's sufficient conditions for a contraction. The other definitions and results are necessary foundations for those two main results. Useful references with complete proofs are Alexis Toda's Mathematics for Economists Lecture Notes (Chapter 7), Daron Acemoglu's MIT Macro Lecture Notes, and Stokey and Lucas textbook (Chapters 3 and 4).

Definition 1 (Metric)**.** Let $\mathscr{X}$ be some nonempty set. Then a function $\rho : \mathscr{X} \times \mathscr{X} \to \mathbb{R}$ is a **metric** if $\forall x, y, z \in \mathscr{X}$:

1. (positivity) $\rho(x, y) \geq 0 \quad \forall x, y \in \mathscr{X}$ and $\rho(x, y) = 0$ if and only if $x = y$.
2. (symmetry) $\rho(x, y) = \rho(y, x) \quad \forall x, y \in \mathscr{X}$.
3. (triangle inequality) $\rho(x, y) + \rho(y, z) \geq \rho(x, z) \quad \forall x, y, z \in \mathscr{X}$

Example 1 (Euclidian distance on R^N). *Define $\mathscr{X} := \mathbb{R}^N$ and, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$,*

$$\rho(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{n=1}^{N} (x_n - y_n)^2}$$

Is this a metric?

One can immediately see that positivity is satisfied by the fact that $\rho(\cdot, \cdot)$ is a sum of squares. Symmetry is also satisfied since $\boldsymbol{x} = \boldsymbol{y}$ if and only if $x_i = y_i$ for every $i \leq N$. Finally, the fact $\rho(\cdot, \cdot)$ satisfies the triangle inequality, follows from the Cauchy-Schwarz inequality.

Definition 2 (Metric space)**.** The pair $(\rho, \mathscr{X})$ is called a **metric space** when ρ is a metric on $\mathscr{X}$.

Definition 3 (Cauchy sequence)**.** A sequence $\{x_n\}_{n=1}^{\infty}$ on set $\mathscr{X}$ is called *Cauchy* if

$$(\forall \varepsilon > 0)(\exists N)(m, n > N) \quad s.t. \quad |x_m - x_n| < \varepsilon$$

Example 2. *The sequence $x_n = \frac{1}{2^n}$ is Cauchy. For any $\varepsilon > 0$, we can choose N such that $\frac{1}{2^{N-1}} < \varepsilon$. Then, for $n, m > N$:*

$$|x_n - x_m| = \left| \frac{1}{2^n} - \frac{1}{2^m} \right| \leq \frac{1}{2^n} + \frac{1}{2^m} \leq \frac{1}{2^N} + \frac{1}{2^N} = \frac{1}{2^{N-1}} < \varepsilon$$

Definition 4 (Complete metric space)**.** A metric space $(\rho, \mathscr{X})$ is called *complete* if, any *Cauchy* sequence on $\mathscr{X}$ is convergent, that is, $x = \lim_{n\to\infty}\{x_n\}_{n\in\mathbb{Z}}^{\infty}$ is in $\mathscr{X}$ whenever

$$(\forall \varepsilon > 0)(\exists N)(m, n > N) \quad s.t. \quad \rho(x_m, x_n) < \varepsilon$$

Example 3. *Let ρ be the absolute value metric, such that $\rho(x, y) = |x - y|$. The metric space $(\rho, (0, 1))$ is not complete, since the sequence $x_n = \frac{1}{n}$ is defined over $(0, 1)$, but its limit lies outside of $(0, 1)$. However, the metric space $(\rho, [0, 1])$ is complete.*

Definition 5 (Operator)**.** Let $(\rho, \mathscr{X})$ be a metric space. An operator is any function $T : \mathscr{X} \to \mathscr{X}$ mapping $\mathscr{X}$ onto itself.

Definition 6 (Contraction Mapping)**.** Let $(\rho, \mathscr{X})$ be a metric space and $T : \mathscr{X} \to \mathscr{X}$ be an operator. If, for some $\beta \in (0, 1)$,

$$\rho(T(x), T(y)) \leq \beta\rho(x, y) \quad \forall x, y \in \mathscr{X}$$

then T is a contraction mapping with modulus β.

Definition 7 (Fixed Point)**.** Let $(\rho, \mathscr{X})$ be a metric space and $T : \mathscr{X} \to \mathscr{X}$ be an operator. A fixed point is any element of $\mathscr{X}$ satisfying $T(x) = x$.

Theorem 1 (Contraction Mapping Theorem)**.** *Let $(\rho, \mathscr{X})$ be a complete metric space and $T : \mathscr{X} \to \mathscr{X}$ be a contraction with modulus β. Then:*

1. *T has a unique fixed $x^* \in \mathscr{X}$ such that $T(x^*) = x^*$;*
2. *for any $x_0 \in \mathscr{X}$, we have $x^* = \lim_{n\to\infty} T^n(x_0)$; and*
3. *for any $x_0 \in \mathscr{X}$ we have that $\rho(T^n(x_0), x^*) \leq \beta^n \rho(x_0, x^*)$.*

Proof. Omitted □

Definition 8 (Sup norm)**.** Let $f : \mathscr{X} \to \mathbb{R}$ and $g : \mathscr{X} \to \mathbb{R}$ be continuous functions mapping $\mathscr{X}$ to $\mathbb{R}$. We say that $f \leq g$ if $f(x) \leq g(x) \forall x \in \mathscr{X}$ and define the distance of between two functions f, g as:

$$||f - g|| = \sup_{x\in\mathscr{X}} |f(x) - g(x)|$$

Theorem 2 (Blackwell's Sufficient Condition for a Contraction)**.** *Let $\mathscr{X}$ be a topological space and $C_b(\mathscr{X})$ be the set of all continuous bounded functions that map $\mathscr{X}$ to $\mathbb{R}$. If $T : C_b(\mathscr{X}) \to C_b(\mathscr{X})$ satisfies:*

1. *(monotonicity)* $f \leq g \implies Tf \leq Tg \quad \forall f, g \in C_b(\mathscr{X})$.
2. *(discounting)* $\exists \beta \in [0, 1)$ *s.t.* $T(f + a) \leq Tf + \beta a \quad \forall f, g \in C_b(\mathscr{X}), \quad a \in \mathbb{R}$.

then T is a contraction.

Proof. Omitted □

2 Fundamentals of Value Function Iteration

In class, we saw that the social planner solves a sequence problem of the following kind:

$$\begin{aligned} \max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t \log(C_t) \\ & C_t + K_{t+1} \leq K_t^{\theta} \quad \forall t \geq 0 \\ & C_0, K_0 \geq 0, \qquad K_0 \text{ given.} \end{aligned}$$

The solution for this problem is a sequence of consumption and capital allocation streams $\{C_t, K_{t+1}\}_{t=0}^{\infty}$, such that the optimality conditions are satisfied and the maximum possible utility is attained.

Suppose that this problem had been solved for all possible values of K_0. In that case, we could define a function $v(K_0) : \mathbb{R}_+ \to \mathbb{R}$, that returns the value of maximized discounted lifetime utility for any possible K_0. We call this a **value function**. Formally:

$$\begin{aligned} v(K_0) \quad = \quad & \max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(C_t) \\ & C_t + K_{t+1} \leq K_t^{\theta} \quad \forall t \geq 0, \quad C_0, K_0 \geq 0, \quad K_0 \text{ given.} \end{aligned}$$

which can be rewritten as:

$$\begin{aligned} v(K_0) \quad = \quad & \max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \log(C_0) + \beta \sum_{t=0}^{\infty} \beta^t \log(C_{t+1}) \\ & C_t + K_{t+1} \leq K_t^{\theta} \quad \forall t \geq 0, \quad C_0, K_0 \geq 0, \quad K_0 \text{ given.} \end{aligned}$$

Define $x = x_t, \quad x' = x_{t+1}$ for any variable x at period t. We can then rewrite the value function recursively. We call this recursive form the **Bellman Equation** (or Bellman Operator):

$$v(K)_{\text{state variables}} = max_{C,K'}{}_{\text{choice variables}} \quad \log(C) + \beta v(K') \tag{1}$$
$$C + K' \leq K^\theta$$

which becomes an unconstrained value function if we rewrite the constraint as $C = K^\theta - K'$ and substitute it into the value function:

$$v(K) = \max_{K'} \log(K^\theta - K') + \beta V(K') \tag{2}$$

We can now try to solve this problem through **value function iteration** (VFI). The intuition behind VFI is that if solve the maximization problem for each period and iterate through many periods of our recursive problem, we can reach a fixed point that will show the optimal choice for the infinite utility maximization problem.

The intuition here is that, since the discount rate $\beta > 0$, for a large enough N, the marginal discounted utility added at t by solving the problem at $t + N$ is near zero. Therefore, if we iterate the problem back enough we can find the limit function that optimizes consumption in a infinite horizon.

Mathematically, we know, from the **contraction mapping theorem** (Theorem 1.1) that contractions over complete metric spaces achieve a unique fixed point. Therefore, to ensure that our value function has a fixed point, we need to show that it is a contraction.

Define $[Tv](K) = \max_{K'} \log(K^\theta - K') + \beta v(K')$. Let $C_b(\mathbb{R}_+)$ be the space of all continuous bounded functions mapping $\mathbb{R}_+$ to $\mathbb{R}$ and $(C_b(\mathbb{R}_+), ||\cdot||)$ be a complete metric space, where $||\cdot||$ is the sup norm, and $T : C_b(\mathbb{R}_+) \to C_b(\mathbb{R}_+)$ be an operator.

To show that there is a unique value function that maximizes lifetime utility for $K \in \mathbb{R}_+$, we need to show that T is a contraction. For that, it is sufficient to show that T satisfies monotonicity and discounting (*Blackwell's Sufficient Conditions*).

- **Monotonicity:** If $v(K') \geq w(K') \forall K' \in \mathbb{R}_+$, then $\log(K^\theta - K') + \beta v(K') \geq \log(K^\theta - K') + \beta w(K'), \forall K' \in \mathbb{R}_+$. In particular $\max_{K'} \log(K^\theta - K') + \beta v(K') \geq \max_{K'} \log(K^\theta - K') + \beta w(K')$, or $[Tv](K) \geq [Tw](K)$. Therefore, T is monotonic.

- **Discounting** For scalar a, $T[v + a](K) = \max_{K'} \log(K^\theta - K') + \beta(v(K') + a) = [Tv](K) + \beta a$. Therefore, T satisfies discounting.

Since our value function operator is a contraction over a complete metric space, it reaches a fixed point, which is **limit function** mapping $\mathbb{R}_+$ to $\mathbb{R}$ which reflects the maximized lifetime utility whenever the state variable is K.