# Discussion Session 3: Inefficient Equilibria: Economy with Externalities

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In this section, we will introduce market failures to illustrate how the competitive equilibrium allocation can be Pareto inefficient. We define a flow of pollution $P_t$, which decreases household welfare. Preferences take the following form:

$$\sum_{t=0}^{\infty} \beta^t \left[ \log c_t + \chi \log(1-n_t) - D(P_t) \right] \tag{1}$$

where $c_t$ is consumption; $n_t$ is labor; and $D(\cdot)$ is a strictly increasing, strictly convex differentiable function. We assume pollution is an increasing function of output: $P_t = F(Y_t)$. We will first characterize the recursive competitive equilibrium, then compare it to the benevolent social planner's solution. We will not focus on the producting side.

## 1 Recursive Competitive Equilibrium

We define the households recursive problem as:

$$\begin{aligned} v(K,k) &= \max_{k',c,n} \log c + \chi \log(1-n) - D(P(K)) + \beta v(K',k') \\ s.t. &\quad k' + c \leq nw(K) + kr(K) + (1-\delta)k \\ &\quad K' = \hat{G}(K) \quad \text{(perceived law of motion for capital)} \\ &\quad P' = \hat{P}(K) \quad \text{(perceived law of motion for polution)} \end{aligned} \tag{2}$$

with Lagrangian and FOCs:

$$
\begin{aligned}
\mathscr{L} &= \log c + \chi \log(1-n) - D(P(K)) + \beta v(K',k') + \\
&\quad \lambda[nw(K) + kr(K) + (1-\delta)k - k' - c] \\
c &: \frac{1}{c} = \lambda \\
n &: \frac{\chi}{1-n} = \lambda w(k) \\
k' &: \lambda = \beta v_{k'}(K',k')
\end{aligned} \tag{3}
$$

To derive $v_k(k,K)$, we substitute for $c(k,K)$ in the value function, evaluate $n(K,k), k'(k,K)$ at their optimal points, and take the derivative with respect to $k$, capturing only the direct effect:

$$
\begin{aligned}
v(K,k) &= \log c(n(K,k)w(K) + kr(K) + (1-\delta)k - k'(K,k) \\
&+ \chi \log(1-n(K,k)) - D(P(K)) + \beta v(K',k') \\
\implies v_k(K,k) &= \frac{r(K) + (1-\delta)}{c(K,k)} \implies v_k(K,k) = \frac{r'(K') + (1-\delta)}{c'(K',k')}
\end{aligned}
$$

Combining the FOCs and the envelope condition result in our standard Euler equation and labor-leisure condition (dropping the parenthesis of the choice variables for ease of notation):

$$
\underbrace{\frac{1}{c}}_{\text{marginal utility of consumption}} = \underbrace{\beta \frac{[r'(K') + (1-\delta)]}{c'}}_{\substack{\text{discounted marginal utility} \\ \text{of consuming savings tomorrow}}} \tag{4}
$$

$$
\underbrace{\frac{\chi}{1-n}}_{\text{marginal disutility of work}} = \underbrace{\frac{w(K)}{c}}_{\substack{\text{marginal utility of} \\ \text{consuming marginal wage}}} \tag{5}
$$

The firms problem is:

$$
\max_{K_d, L_d} AK_d^{\theta} N_d^{1-\theta} - K_d r(K) - L_d w(K) \tag{6}
$$

with optimality conditions:

$$
r(K) = \theta A K_d^{\theta-1} N_d^{1-\theta} w(K) = (1-\theta) A K_d^{\theta} N_d^{-\theta}
$$

In equilibrium, it must be the case that the factor market clears, i.e.:

$$
K_d = K, \quad N_d = N
$$

Combining the above results with the optimality conditions for the consumer yields:

$$
\begin{aligned}
\frac{1}{c} &= \beta \frac{[\theta A K'^{\theta-1} N'^{1-\theta} + (1-\delta)]}{c'} &\quad (7)\\
\frac{\chi}{1-n} &= \frac{(1-\theta) A K^{\theta} N^{-\theta}}{c} &\quad (8)
\end{aligned}
$$

## 2 Planner's Problem

The planner's recursive problem is:

$$
\begin{aligned}
V(K) &= \max_{K',C,N,P,Y} \log C + \chi \log(1-N) - D(P) + \beta V(K') &\quad (9)\\
s.t. &\quad K' + C \leq Y + (1-\delta)K \\
&\quad P = F(Y) \\
&\quad Y = A K^{\theta} N^{1-\theta} &\quad (10)
\end{aligned}
$$

with Lagrangian and FOCS:

$$
\begin{aligned}
\mathscr{L} &= \log C + \chi \log(1-N) - D(F(AK^{\theta}N^{1-\theta})) + \beta V(K') + \\
&\quad \Lambda[AK^{\theta}N^{1-\theta} + (1-\delta)K - K' - C] \\
C &: \Lambda = \frac{1}{C} \\
N &: \frac{\chi}{1-N} = [\Lambda - D'(F(Y))F'(Y)](1-\theta)AK^{\theta}N^{-\theta} \\
K' &: \Lambda = \beta v'(K')
\end{aligned}
$$

To derive $v'(K')$, we can use the envelope condition:

$$
\begin{aligned}
v(K) &= \log(AK^{\theta}N(K)^{1-\theta} + (1-\delta)K - K'(K)) + \chi \log(1-N(K)) - D(F(AK^{\theta}N(K)^{1-\theta})) + \beta V(K') \\
\implies v'(K) &= \frac{\theta A K^{\theta-1} N(K)^{1-\theta} + (1+\delta)}{C(K)} - D'(P(K))F'(Y(K))\theta A K^{\theta-1} N(K)^{1-\theta}
\end{aligned}
$$

Now let us compare the Euler Equation from the Planner's Problem to the Competitive Equilbrium:

$$
\begin{aligned}
\frac{1}{C} &= \beta\left(\left[\frac{1}{C'} - D'(P')F'(Y')\right]\theta A (K')^{\theta-1}(N')^{1-\theta} + \frac{1}{C'}(1-\delta)\right) &\quad \text{(Planner's EE)}\\
\frac{1}{c} &= \beta \frac{[\theta A K^{\theta-1} N^{1-\theta} + (1-\delta)]}{c'} &\quad \text{(CE's EE)}
\end{aligned}
$$

Note that they differ by the term: $-\beta D'(F(Y'))F'(Y')\theta AK'^{\theta-1}N'^{1-\theta}$, which denotes the discounted disutility of having to experience higher pollution in the next period due to higher production as a function of higher savings. The planner internalizes the disutility of extra production that happens through pollution, while households ignore them in the competitive equilibrium. Therefore, in the competitive equilibrium households oversave and overconsume compared to the social optimum.

We can observe a similar pattern from comparing the labor leisure conditions:

$$\begin{aligned}
\frac{\chi}{1-N} &= \left[\frac{1}{C} - D'(F(Y))F'(Y)\right](1-\theta)AK^{\theta}N^{-\theta} \quad \text{(Planner's LL)} \\
\frac{\chi}{1-n} &= \frac{(1-\theta)AK^{\theta}N^{-\theta}}{c} \quad \text{(CE's LL)}
\end{aligned}$$

Again, consumer overwork compared to the social optimum. You can see that by realizing that the marginal disutility of work is higher in the competitive equilibrium ($D'(F(Y))F'(Y) > 0$) and recalling that the utility function is concave in leisure (that is, convex in hours).