Discussion Session 6: The Romer Model

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In this section, we will present a groundbreaking model by Nobel Laureate Paul Romer [**?**]. The model's main insight is that innovation through the introduction of new ideas and products have spillovers. That is, society as a whole benefits from innovation, even if they did not innovate individually. This happens because **ideas are not rival goods** and **ideas are only partially excludable**. For instance, my use of the Pythagorean theorem does not prevent others from making use of it and I cannot exclude others from using it if they want to.

The implication for macroeconomics is that if a technology (production function) includes a nonrival input, it cannot be constant returns to scale in all its inputs. In fact, nonrivalry of ideas give rise to **increasing returns to scale** in a context of monopolistic competition.

The model has three sectors: a research and development sector, an intermediate goods sector, and a final goods sector. Households rent their human capital and buy goods from the final goods firms.

0.1 Characterizing the competitive equilibrium

There is measure one of households who solve the following problem:

$$\max_{a_{t+1}, c_t} \sum_{t=0}^{\infty} \beta^t \log c_t \qquad s.t. \quad c_t + a_{t+1} \le (1 + r_{t+1})a_t + w_t n \tag{1}$$

where c_t are units of the final good. Households supply inelastically n units of labor to the market. Households can accumulate assets a_t to smooth consumption over time. The intertemporal euler equation of this simple household problem is:

$$\frac{c_{t+1}}{c_t} = \beta(1 + r_{t+1})$$

There are many perfectly competitive **final goods producers** (FGP), all endowed with the same technology:

$$Y_t = n^{1-\alpha} \int_0^{M_t} x_t(\omega)^\alpha d\omega$$

Each period, a final goods producer sources labor n_t and infinitely many intermediate inputs $x_t(\omega)$ of each variety $\omega \in [0, M_t]$. Note that there is a time-dependent measure of varieties M_t. As we will see below, this measure increases as the research and development sector produces new ideas. They maximize their profits through the following problem:

$$\max_{n_t^d, \{x_t^d(\omega)\}_{\omega \in [0, M_t]}} (n_t^d)^{1-\alpha} \int_0^{M_t} (x_t^d(\omega))^\alpha d\omega - w_t n_t^d - \int_0^{M_t} p_t(\omega) x_t^d(\omega) d\omega \qquad (2)$$

Differentiating under the integral sign in gives the demand functions:

$$(1-\alpha)(n_t^d)^{-\alpha} \int_0^{M_t} (x_t^d(\omega))^\alpha d\omega = w_t \qquad (3)$$

$$\alpha (n_t^d)^{1-\alpha} (x_t^d(\omega))^{\alpha-1} = p_t(\omega) \text{ for each } \omega \in (0, M_t) \qquad (4)$$

Rearranging, optimal demand functions for variety ω are decreasing in prices $p_t(\omega)$ and increasing in total use of labor n_t^d:

$$x_t^d(\omega) = p(\omega)^{-\frac{1}{1-\alpha}} \cdot n_t^d \cdot \alpha^{\frac{1}{1-\alpha}} \text{ for each } \omega \in [0, M_t]$$

There is an **intermediate goods producer** that has perpetual rights over the production of each variety $\omega \in [0, M_t]$. Each of them is endowed with a linear technology that transforms one unit of the final good into one unit of the intermediate good: $x(\omega) = F(y) = y$. They take marginal costs κ and demand curves $x_t^d(\omega)$ and choose optimal prices to maximize profits:

$$\begin{aligned}
\max_{p_t(\omega)} \pi_t(\omega) &= p_t(\omega) x_t^d(\omega) - \kappa x_t^d(\omega) \\
&= p_t(\omega) p(\omega)^{-\frac{1}{1-\alpha}} \cdot n_t^d \cdot \alpha^{\frac{1}{1-\alpha}} - \kappa p(\omega)^{-\frac{1}{1-\alpha}} \cdot n_t^d \cdot \alpha^{\frac{1}{1-\alpha}} \\
&= \left(p(\omega)^{-\frac{\alpha}{1-\alpha}} - \kappa p(\omega)^{-\frac{1}{1-\alpha}} \right) \cdot n_t^d \cdot \alpha^{\frac{1}{1-\alpha}}
\end{aligned}$$

whose optimality condition satisfy:

$$-\frac{\alpha}{1-\alpha} p_t(\omega)^{-\frac{1}{1-\alpha}} + \frac{\kappa}{1-\alpha} p_t(\omega)^{-\frac{2-\alpha}{1-\alpha}} = 0 \implies p_t(\omega) = \frac{1}{\alpha}\kappa \qquad \forall \omega \in [0, M_t]$$

Importantly, we note that the price $p_t(\omega) = \frac{1}{\alpha}\kappa$, which is a simple markup over marginal cost, does not depend on ω. They are the same for every variety. The implication is that demanded inputs $x_t(\omega)$ and profits $\pi_t(\omega)$ will also be equal for every variety. Since labor supply n is constant in every period and is inelastically supplied at the market, then $n_t^d = n \forall t$ and in equilibrium both profits and demanded inputs for each variety ω will be constant over time.

$$
\begin{aligned}
x_t^d(\omega) &= \bar{x} = \kappa^{-\frac{1}{1-\alpha}} \cdot n \cdot \alpha^{\frac{2}{1-\alpha}} & \forall \omega \in [0, M_t] \\
\pi_t(\omega) &= \bar{\pi} = \frac{1}{\alpha}\kappa\bar{x} - \kappa\bar{x} = \frac{1-\alpha}{\alpha} \cdot \kappa^{-\frac{\alpha}{1-\alpha}} \cdot n \cdot \alpha^{\frac{2}{1-\alpha}} & \forall \omega \in [0, M_t]
\end{aligned}
$$

We can also calculate total output:

$$
\begin{aligned}
Y_t &= n^{1-\alpha} \int_0^{M_t} x_t(\omega)^\alpha d\omega \\
&= n^{1-\alpha} \int_0^{M_t} \bar{x}^\alpha d\omega \\
&= n^{1-\alpha} \left[\kappa^{-\frac{1}{1-\alpha}} \cdot n \cdot \alpha^{\frac{2}{1-\alpha}} \right]^\alpha \cdot \int_0^{M_t} 1 d\omega \\
&= \left(\frac{\alpha^2}{\kappa} \right)^{\frac{\alpha}{1-\alpha}} \cdot n \cdot M_t
\end{aligned}
$$

and wages:

$$
w_t n = (1-\alpha)Y_t = (1-\alpha) \cdot \left(\frac{\alpha^2}{\kappa} \right)^{\frac{\alpha}{1-\alpha}} \cdot n \cdot M_t
$$

There is a **research sector in this economy**, which creates new varieties according to the following law of motion:

$$
M_{t+1} = M_t + \eta Z_t
$$

where Z_t is investment in RD in units of the final good. Once a new variety exists, its owners have a perpetual patent over its design. The economic value of a new variety net present value of producing the new varieties and selling them as intermediate inputs to final goods producers:

$$
V_t = \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \frac{1}{1+r_{t+k}} \pi_{t+j}(\omega) \right) = \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \frac{1}{1+r_{t+k}} \bar{\pi} \right)
$$

Note that entrepreneurs invest Z_t at period t, but the additional varieties $M_{t+1} - M_t$ only materialize in the following period. They fund themselves by borrowing Z_t at the assets markets at period t and repay $(1+r_{t+1})Z_t$ at $t+1$. Therefore, entrepreneurs will only invest at t if the present discounted value of varieties at $t+1$ exceeds its marginal cost:

$$
\underbrace{\eta V_{t+1} Z_t}_{\text{benefit}} - \underbrace{(1+r_{t+1})Z_t}_{\text{cost}} \geq 0 \iff \underbrace{\frac{1}{1+r_{t+1}} V_{t+1}}_{\text{PV of } V_{t+1} \text{ at } t} \geq \frac{1}{\eta}
$$

1 Balanced Growth Path

Along the Balanced Growth Path, we are looking for constant growth rages of consumption g_c and output g_y. From the euler equation, it is clear that the interest rate must be constant for every period $t \geq \underline{o}$, where $\underline{o}$ is the first period in which the economy is at a balanced growth path:

$$\frac{c_{t+1}}{c_t} = \underbrace{g_c}_{\text{constant}} = \underbrace{\beta(1 + r_{t+1})}_{\text{constant}} \qquad \forall t \geq \underline{o}$$

$$\implies r_t = r_{t+1} = r_{t+2} = \bar{r} \qquad \forall t \geq \underline{o}$$

Therefore, along the Balanced Growth Path, the value of new varieties satisfies:

$$V_t = \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \frac{1}{1+r_{t+k}} \bar{\pi} \right) = \sum_{j=0}^{\infty} \left(\frac{1}{1+\bar{r}} \right)^j \bar{\pi} = \frac{1+\bar{r}}{\bar{r}} \bar{\pi} \qquad \forall t \geq \underline{o}$$

By free-entry, varieties will be created up to the point in which:

$$\frac{1}{1+\bar{r}} V_t = \frac{1}{\eta} \implies \bar{r} = \eta \cdot \bar{\pi}$$

We can now go back to the Euler Equation and characterize g_c:

$$\frac{c_{t+1}}{c_t} = g_c = \beta(1 + \eta \cdot \bar{\pi}) \qquad \forall t \geq \underline{o}$$

$$= \beta \left(1 + \eta \cdot \frac{1-\alpha}{\alpha} \cdot \kappa^{\frac{\alpha}{1-\alpha}} \cdot n \cdot \alpha^{\frac{2}{1-\alpha}} \right)$$

We can show that $g_y = g_w = g_m$. For g_y, calculate:

$$\frac{Y_{t+1}}{Y_t} = g_y = \frac{\left(\frac{\alpha^2}{\kappa} \right)^{\frac{\alpha}{1-\alpha}} \cdot n \cdot M_{t+1}}{\left(\frac{\alpha^2}{\kappa} \right)^{\frac{\alpha}{1-\alpha}} \cdot n \cdot M_t} = \frac{M_{t+1}}{M_t} = g_m$$

Likewise, for wages:

$$\frac{w_{t+1}}{w_t} = g_w = \frac{(1-\alpha) \cdot \left(\frac{\alpha^2}{\kappa} \right)^{\frac{\alpha}{1-\alpha}} \cdot M_{t+1}}{(1-\alpha) \cdot \left(\frac{\alpha^2}{\kappa} \right)^{\frac{\alpha}{1-\alpha}} \cdot M_t} = \frac{M_{t+1}}{M_t} = g_m$$

Now turn to the law of motion for varieties. We can write it as:

$$\frac{M_{t+1}}{M_t} = g_m = 1 + \eta \frac{Z_t}{M_t} \qquad \forall t \geq \underline{o}$$

which implies that the ratio $\frac{Z_t}{M_t}$ is a constant ($\forall t \geq \underline{o}$). This can only be the case if Z_t and M_t are growing at the same rate. Therefore: $g_z = g_m$.

Now turn to the resource constraint:

$$
\begin{aligned}
C_t &= Y_t - Z_t - \int_0^{M_t} x_t(\omega) d\omega \\
g_c^{t-\underline{o}} C_0 &= g_y^{t-\underline{o}} Y_0 - g_z^{t-\underline{o}} Z_0 - \bar{x} g_m^{t-\underline{o}} M_0 \\
g_c^{t-\underline{o}} C_0 &= g_m^{t-\underline{o}} Y_0 - g_m^{t-\underline{o}} Z_0 - \bar{x} g_m^{t-\underline{o}} M_0 \qquad (\because g_y = g_z = g_m) \\
g_c^{t-\underline{o}} C_0 &= g_m^{t-\underline{o}} (Y_0 - Z_0 - \bar{x} M_0) \\
g_c^{t-\underline{o}} &= g_m^{t-\underline{o}} \qquad (\because C_0 = Y_0 - Z_0 - \bar{x} M_0) \\
g_c &= g_m
\end{aligned}
$$

Finally, since by the asset market clearance $a_t = Z_t$, it is trivial that $g_a = g_z = g_m$. Previously, we have concluded that profits per variety ($g_\pi = 1$) and demand per variety ($g_x = 1$) are constant along the balanced growth path.

Therefore, along the BGP:

$$
\begin{aligned}
g_m &= g_y = g_c = g_w = g_z = g_a \\
g_x &= g_\pi = 1
\end{aligned}
$$