Econ 110A: Lecture 6

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So far we talked about growth without actually talking about growth...

Growth without growth?

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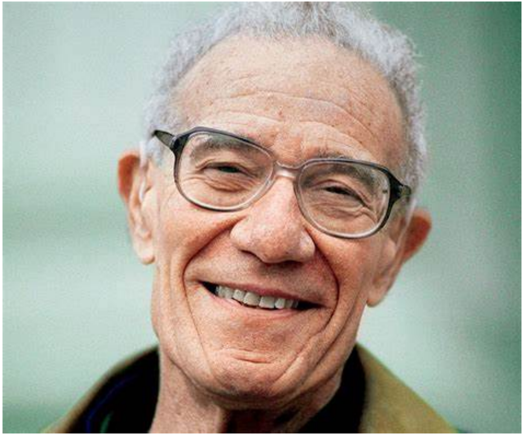
We tried to explain income levels across differences at a given moment, not growth rates, which only indirectly speaks about growth.

Robert (Bob) Solow
1924-



Nobel Prize in Economics, 1987

An introduction to Growth Dynamics

Questions we asked:

- Why some countries are so much richer than others?

- Capital matters but only partially. TFP plays a much bigger role.

An introduction to Growth Dynamics

Questions we asked:

- Why some countries are so much richer than others?
- Capital matters but only partially. TFP plays a much bigger role.
- Why do some countries grow faster than others?
- Can the answer to this question help understand the role of TFP?

Two Pictures from 1960

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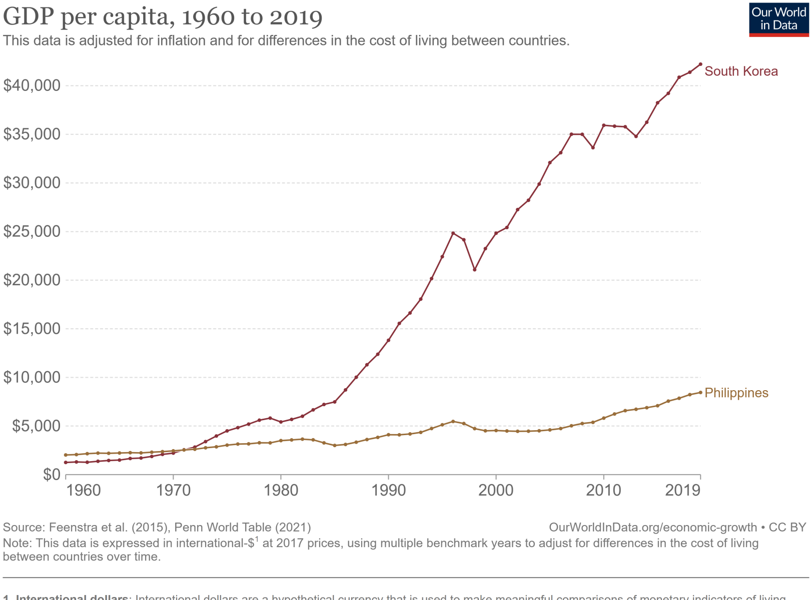
	South Korea	Philippines
Per Capita GDP	$1,500	$1,500
Population	25M	25M
Working Age Population	50%	50%
Attending College at 20	5%	13%

Two Pictures from 1960



More specific questions:

- Can differences in capital accumulation explain differences in growth of GDP per capita across countries?

An introduction to Growth Dynamics

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- Can differences in capital accumulation explain differences in growth of GDP per capita across countries?

- Is capital accumulation the ultimate source of sustained growth in GDP per capita?

The Solow Growth Model

We add the time dimension!

- Production:

$$Y_t = \bar{A} K_t^{\alpha} L_t^{1-\alpha}, \qquad \alpha \in (0,1), \quad t \in \{0,1,2,\cdots\}$$

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- Resource constraint:

$$Y_t = \underbrace{C_t}_{\text{consumption}} + \underbrace{S_t}_{\text{savings}}, \qquad t \in \{0, 1, 2, \cdots\}$$

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- Capital Accumulation:

$$\begin{aligned} K_{t+1} &= K_t + \underbrace{I_t}_{\text{investment}} - \underbrace{\bar{d} \cdot K_t}_{\text{depreciation}}, \qquad \bar{d} \in (0,1), \quad t \in \{0,1,2,\cdots\} \\ K_{t+1} - K_t &\equiv \Delta K_{t+1} = I_t - \bar{d} \cdot K_t \end{aligned}$$

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- Production:

$$L_t = \bar{L}, \quad t \in \{0, 1, 2, \cdots\}$$

- Investment:

$$I_t = S_t = \bar{s} Y_t, \quad \bar{s} \in (0, 1), \quad t \in \{0, 1, 2, \cdots\}$$

The Solow Growth Model: Taking Stock

Normalizing the price of the output good $P_t = 1$ each period, for each period $t \in \{0, 1, 2, \cdots\}$, given parameters $\bar{d}$, $\bar{s}$, $\bar{A}$, $\bar{L}$, α and the initial value of capital K_0 there are five unknowns Y_t, K_{t+1}, L_t, C_t, I_t and five equations:

$$
\begin{aligned}
Y_t &= \bar{A} K_t^{\alpha} L_t^{1-\alpha} && (1)\\
Y_t &= C_t + I_t && (2)\\
\Delta K_{t+1} &= I_t - \bar{d} \cdot K_t && (3)\\
L_t &= \bar{L} && (4)\\
I_t &= \bar{s} Y_t && (5)
\end{aligned}
$$

that characterize the solution to this model.

The Solow Growth Model: Factor Markets?

What about factor markets?

- We can add factor markets, satisfying:

$$
\begin{aligned}
w_t &= MPL_t = (1-\alpha)\frac{Y_t}{L_t} \\
r_t &= MPK_t = \alpha\frac{Y_t}{K_t}
\end{aligned}
$$

- But nothing else would change in the model.

- So to simplify, we keep these two equations and unknowns out!

Solving the Solow Growth Model

Solving fully the model with equations is very hard. We usually do that numerically, using a computer. But here are some things we can do with pen and paper to simplify the problem:

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- Show solution on a diagram (Solow Diagram)

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Solving fully the model with equations is very hard. We usually do that numerically, using a computer. But here are some things we can do with pen and paper to simplify the problem:

- Reduce equations to strictly necessary
- Show solution on a diagram (Solow Diagram)
- Solve for the “Long Run” of the model (Steady State)

Solving the Solow Growth Model

Strategy: Reduce system of equations from five to two

- Plug-in (2) and (5) are redundant, not independent: if $I_t = \bar{s} Y_t$, then $\implies C_t = (1 - \bar{s}) Y_t$ ("Walras' Law"; reduces the system to 4)

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- Plug-in (4) into (1), becomes $Y_t = \bar{A}K_t^{\alpha}\bar{L}^{1-\alpha}$ (reduces the system to 3)

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- Plug-in (5) into (3), becomes $\Delta K_{t+1} = \bar{s}Y_t - \bar{d} \cdot K_t$. Using above, $\Delta K_{t+1} = \bar{s}\bar{A}K_t^{\alpha}\bar{L}^{1-\alpha} - \bar{d} \cdot K_t$ (reduces the system to 2)

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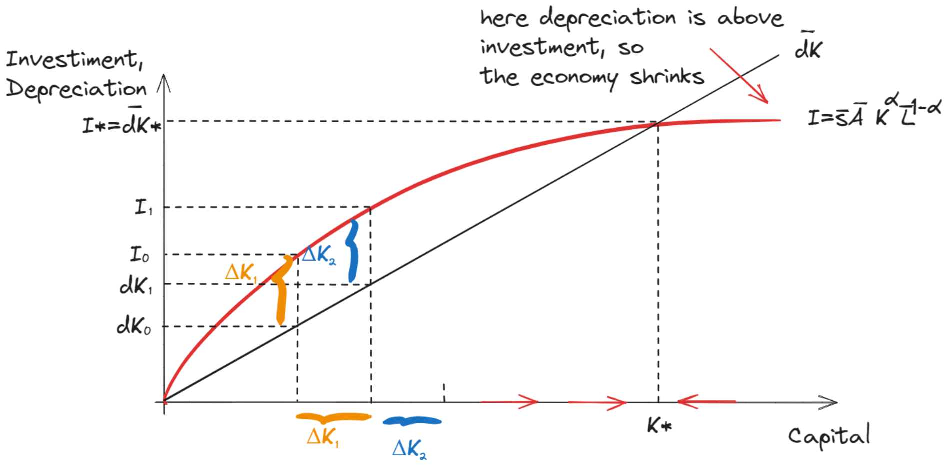
Final system:

$$\begin{aligned} Y_t &= \bar{A}K_t^{\alpha}\bar{L}^{1-\alpha} \\ \Delta K_{t+1} &= \bar{s}\bar{A}K_t^{\alpha}\bar{L}^{1-\alpha} - \bar{d} \cdot K_t \end{aligned}$$

Capital Dynamics (second equation)



Solow Model: The Steady State

$$
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At the Steady-State (SS), $\Delta K_{t+1} = 0$ and $K_{t+1} = K_t$, so we might as well call it K^*. The same is true for Y, so we call it Y^*. Let us look for K^*, Y^* that satisfy the definition of a SS in the system above:

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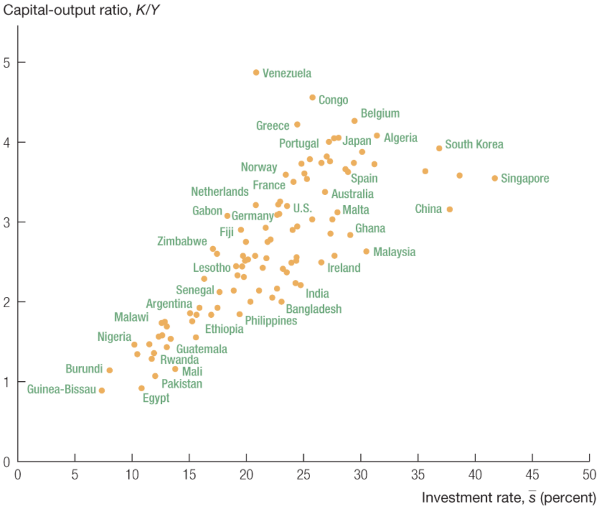
$$
\begin{aligned}
\bar{s}\bar{A}(K^*)^{\alpha}\bar{L}^{1-\alpha} - \bar{d} \cdot K^* = 0 \iff K^* &= \left(\frac{\bar{s}\bar{A}}{\bar{d}}\right)^{\frac{1}{1-\alpha}} \bar{L} \\
\implies Y^* &= \bar{A}^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}}{\bar{d}}\right)^{\frac{\alpha}{1-\alpha}} \bar{L}
\end{aligned}
$$

Solow Model: The Steady State

Note that the model predicts that the capital-to-output ratio is increasing in the investment rate:

$$\frac{K^*}{Y^*} = \frac{\bar{s}}{\bar{d}}$$

In the data, there is indeed a positive correlation between those variables.



Source: Penn World Tables, Version 9.1. The capital-output ratio is measured in the year 2017, while the investment rate is averaged over the period 1990 to 2017.

Solow Model: The Steady State

Other predictions of the model do not have a great fit...

$$y^* \equiv \frac{Y^*}{\bar{L}} = \bar{A}^{\frac{1}{1-\alpha}} \left(\frac{\bar{s}}{\bar{d}}\right)^{\frac{\alpha}{1-\alpha}}$$

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Now assume $\bar{d}_{rich} = \bar{d}_{poor}$ and let us make a similar decomposition as we did with the production model:

$$\underbrace{\frac{y^*_{rich}}{y^*_{poor}}}_{64} = \underbrace{\left(\frac{\bar{A}_{rich}}{\bar{A}_{poor}}\right)^{\frac{3}{2}}}_{32} \times \underbrace{\left(\frac{\bar{s}_{rich}}{\bar{s}_{poor}}\right)^{\frac{1}{2}}}_{2}$$

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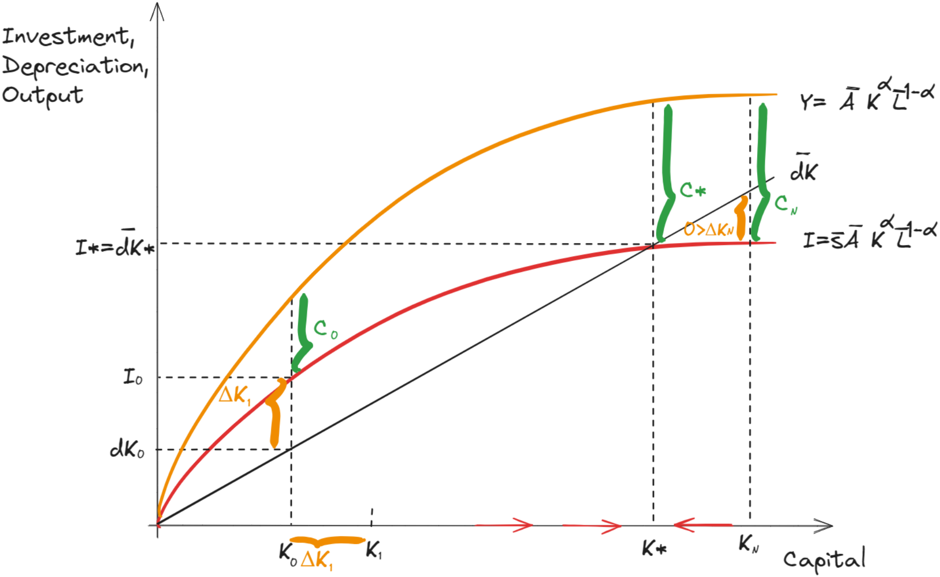
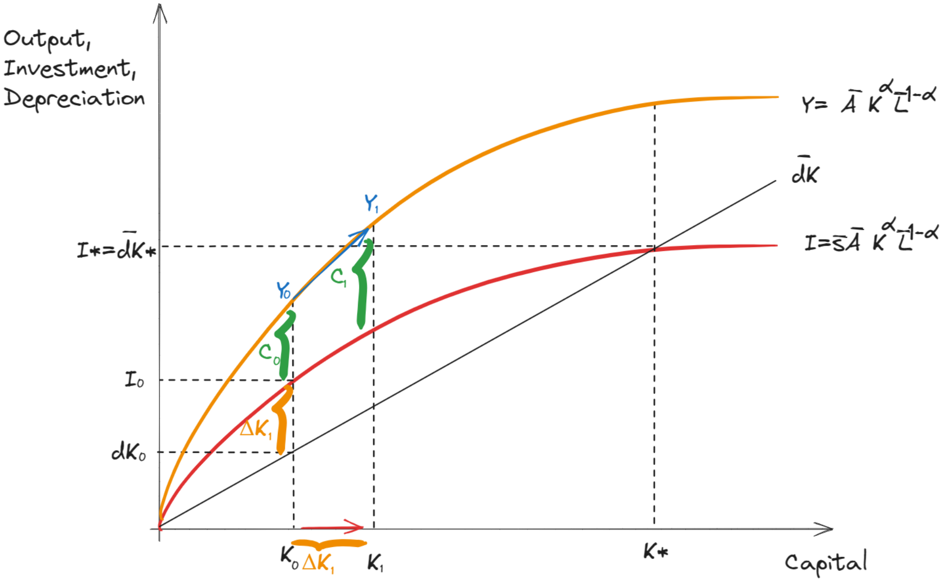
while in the production model

$$\underbrace{\frac{y^*_{rich}}{y^*_{poor}}}_{64} = \underbrace{\frac{\bar{A}_{rich}}{\bar{A}_{poor}}}_{13} \times \underbrace{\left(\frac{\bar{k}^*_{rich}}{\bar{k}^*_{poor}}\right)^{\frac{1}{3}}}_{5}$$

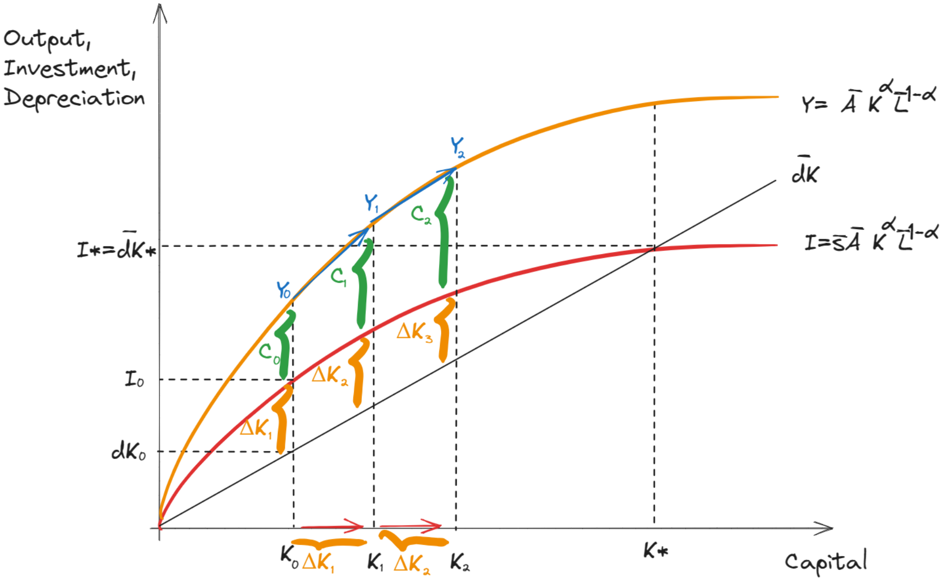
Solow Model: The Complete Diagram



Solow Model: The Complete Diagram, tracing out the dynamics



Solow Model: The Complete Diagram, tracing out the dynamics



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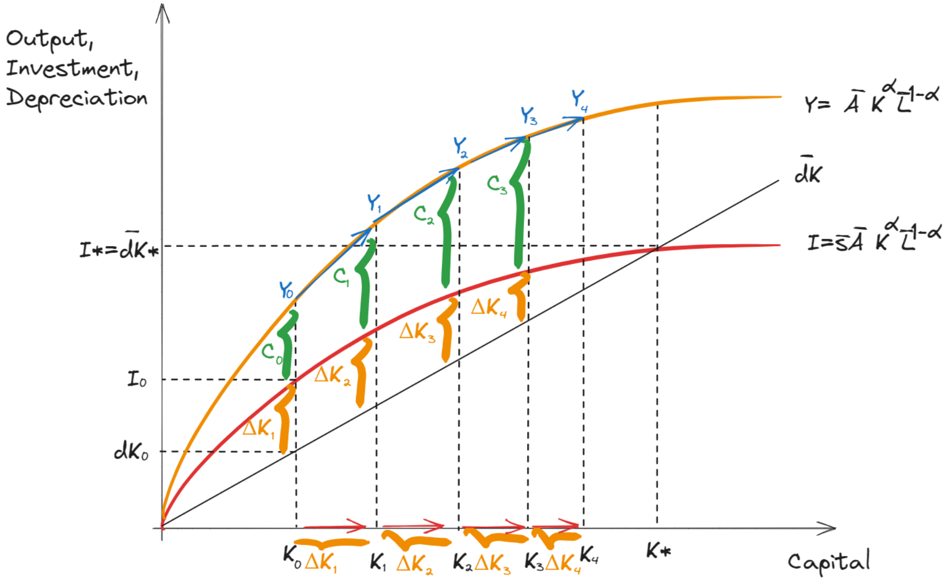
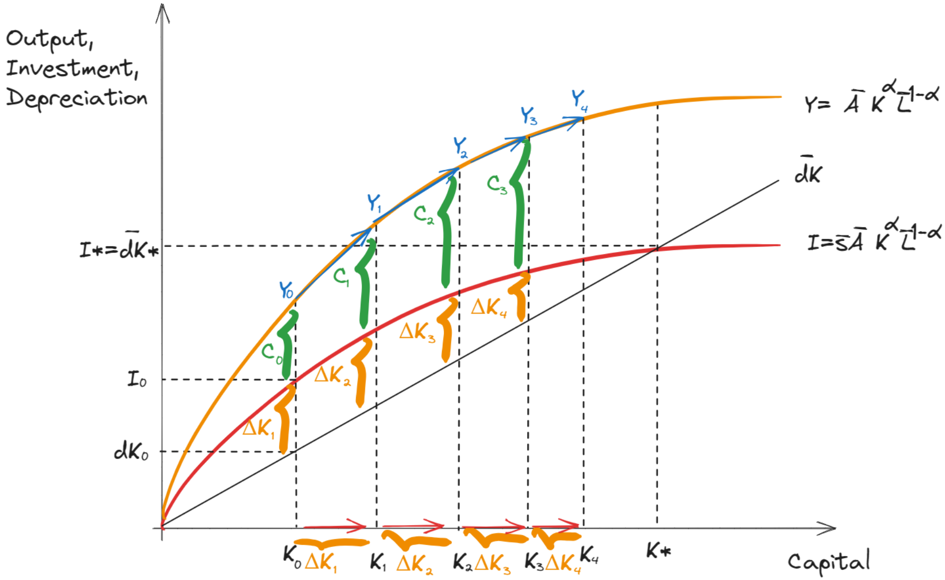


Solow Model: tracing out the dynamics

Assume $\alpha = 1/3$, $\bar{L} = 100$, $\bar{s} = .2$, $\bar{d} = .1$.

t	K_t	$\bar{d}K_t$	Y_t	I_t	ΔK_{t+1}
0	**250**	$250 \cdot 0.1 =$ **25**	$250^{\frac{1}{3}} 100^{\frac{2}{3}} =$ **135.7**	$135.7 \cdot 0.2 =$ **27.1**	$(27.1 - 25) =$ **2.1**
1	$250 + 2.1 =$ **252.1**	$252.1 \cdot 0.1 =$ **25.2**	$252.1^{\frac{1}{3}} 100^{\frac{2}{3}} =$ **136.1**	$136.1 \cdot 0.2 =$ **27.2**	$(27.2 - 25.2) =$ **2**
2	$252.1 + 2 =$ **254.1**	$254.1 \cdot 0.1 =$ **25.4**	$254.1^{\frac{1}{3}} 100^{\frac{2}{3}} =$ **136.5**	$136.5 \cdot 0.2 =$ **27.3**	$(27.3 - 25.4) =$ **1.9**
3	$254.1 + 1.9 =$ **256**	$256 \cdot 0.1 =$ **25.6**	$256^{\frac{1}{3}} 100^{\frac{2}{3}} =$ **136.8**	$136.8 \cdot 0.2 =$ **27.4**	$(27.4 - 25.6) =$ **1.8**

Solow Model: The Complete Diagram, tracing out the dynamics



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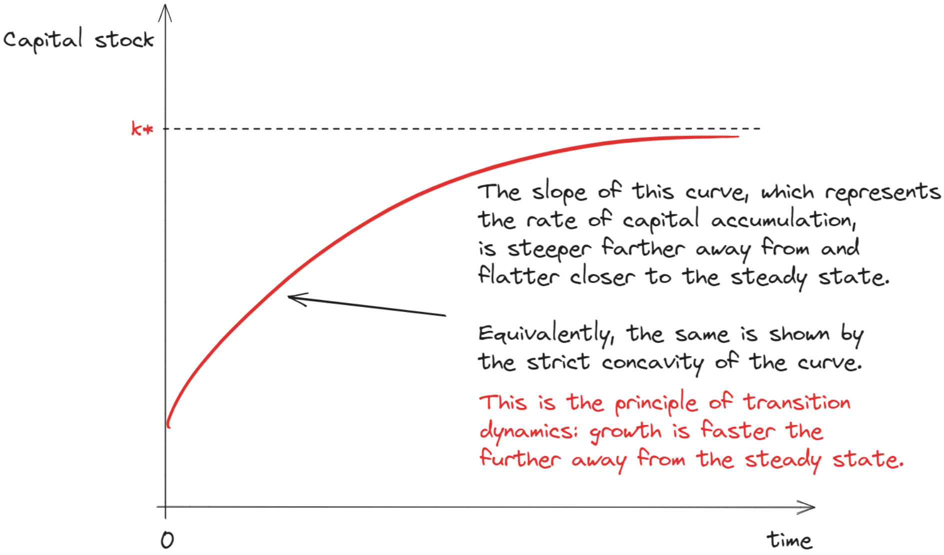
t	K_t	$\bar{d}K_t$	Y_t	I_t	ΔK_{t+1}
0	100.0	10.0	100.0	20.0	10.0
1	110.0	11.0	103.2	20.6	9.6
2	119.6	12.0	106.2	21.2	9.3
3	128.9	12.9	108.8	21.8	8.9

Solow Model: principle of transition dynamics



Solow Model: principle of transition dynamics
The mathematics behind the principle

$$\Delta K_{t+1} = \bar{s}Y_t - \bar{d}K_t$$

Solow Model: principle of transition dynamics

The mathematics behind the principle

$$\begin{aligned}
\Delta K_{t+1} &= \bar{s}Y_t - \bar{d}K_t \\
\Delta \frac{K_{t+1}}{K_t} &= \bar{s}\frac{Y_t}{K_t} - \bar{d}
\end{aligned}$$

Solow Model: principle of transition dynamics

The mathematics behind the principle

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\Delta \frac{K_{t+1}}{K_t} &= \bar{s} \frac{Y_t}{K_t} - \bar{s} \frac{Y^*}{K^*} \qquad \left(\because \frac{K^*}{Y^*} = \frac{\bar{s}}{\bar{d}} \implies \bar{d} = \bar{s} \frac{Y^*}{K^*} \right)
\end{aligned}$$

Solow Model: principle of transition dynamics
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\Delta \frac{K_{t+1}}{K_t} &= \bar{s} \times \left(\frac{Y_t}{K_t} - \frac{Y^*}{K^*} \right)
\end{aligned}
$$

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\Delta \frac{K_{t+1}}{K_t} &= \bar{s} \frac{Y^*}{K^*} \times \left(\frac{Y_t / K_t}{Y^* / K^*} - 1 \right)
\end{aligned}
$$

The mathematics behind the principle

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Solow Model: principle of transition dynamics

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\end{aligned}
$$

So $\Delta \frac{K_{t+1}}{K_t}$ will be large if the gap $\frac{K^*}{K_t}$ is large.

Important aside: there are no explicit banks, but there is still a real interest rate

Real interest rate: the amount of output a person can earn by saving one unit of output for a year, or the amount of output a person must pay to borrow one unit of output for a year.

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- **Production view**: save 1 unit at time $t \rightarrow$ invest 1 unit $I_t \rightarrow$ get 1 unit of K_{t+1}: rented at $r = MPK$, but depreciates at $(1 - \bar{d})$.

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If markets are fully integrated, by nonarbitrage, the following must hold:

$$1 + R \quad = \quad r + (1 - \bar{d})$$

$$\implies \underbrace{r}_{\text{return of financial capital/assets}} = \underbrace{R + \bar{d}}_{\text{return on physical capital/assets}}$$

The forces behind the Solow Model

- If a society has some endowment of capital, it can save and invest to grow, accumulate capital stock and grow richer.
- However, with a fixed population and diminishing marginal returns to capital, growth cannot go on forever in this mode.
- In fact, in this “growth model” there is no long-run growth: is a unique steady-state in which the economy **does not grow**!
- The Solow model does a good job of explaining differences in capital accumulation across countries.

It has limits but it was a breakthrough

> *All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive"*

Bob Solow (1956)