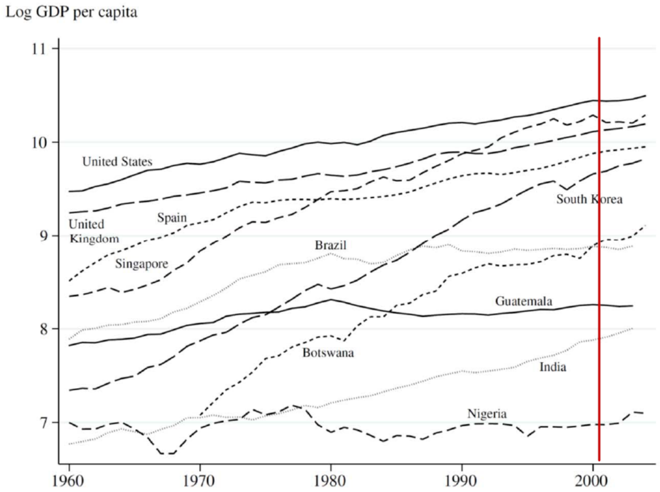
Econ 110A: Lecture 4

Carlos Góes[1]

[1]UC San Diego

UCSD

Why do countries have persistent differences in GDP per capita?



Reproduced from Ch. 1 of Daron Acemoglu, “Introduction to Modern Economic Growth,” 2014

Why do countries have persistent differences in GDP per capita?

- growth theories of the aggregate supply (production)
- investigate explanations of
  - differences in levels of GDP per capita (facts 3 and 6)
  - differences in growth experiences (fact 3)
  - sustained growth at the frontier (fact 2)

The Production Function

- Example: Ice Cream Factory
- Inputs:
  - Milk, Sugar, Salt
  - Chocolate/Strawberry
  - Cones/cups/containers
- Capital:
  - Freezer, Machines
  - Refrigerated trucks
  - Factory/building
- Intangibles:
  - License
  - Recipe
  - Business environment, Property rights

The Production Function

- we are interested in modeling the production of value added
- from the income approach to GDP, we know what factors are ultimately responsible for creation of value   e.g.: labor, management, capital, government
- this suggests a “factor-based” representation of the production function

$$\underbrace{Y}_{\substack{\text{output}\\ \text{value added}}} = F(\underbrace{A}_{\substack{\text{technology}\\ \text{institutions}\\ \text{ideas}}}, \underbrace{K}_{\text{capital}}, \underbrace{L}_{\text{labor}})$$

- $F(A, K, L)$ is your production function. If $F(\cdot,\cdot,\cdot)$ is Cobb-Douglas, then:

$$Y = A \cdot K^{\alpha} L^{1-\alpha}, \qquad 0 \leq \alpha \leq 1$$

Marginal Product: extra output produced by increasing one factor while keeping all the other factors fixed.

recall: economic agents make decision by “reasoning at the margin”

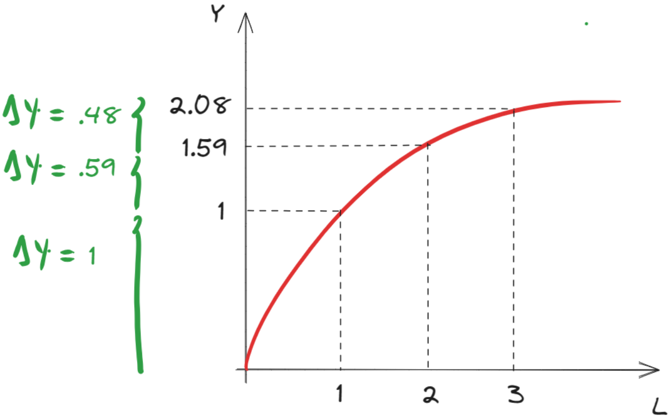
Production Function

Diminishing Marginal Product: Extra output produced by increasing one factor while keeping all the other fixed is decreasing in the one increasing factor.

With Cobb-Douglas Technology, the Marginal Product of Labor and Capital are:

$$MPL \equiv \frac{\partial Y}{\partial L} = \underbrace{(1-\alpha)\bar{A}\left(\frac{K}{L}\right)^{\alpha}}_{\text{decreasing in } L} \qquad MPK \equiv \frac{\partial Y}{\partial K} = \underbrace{\alpha\bar{A}\left(\frac{L}{K}\right)^{1-\alpha}}_{\text{decreasing in } K}$$

Example: Labor

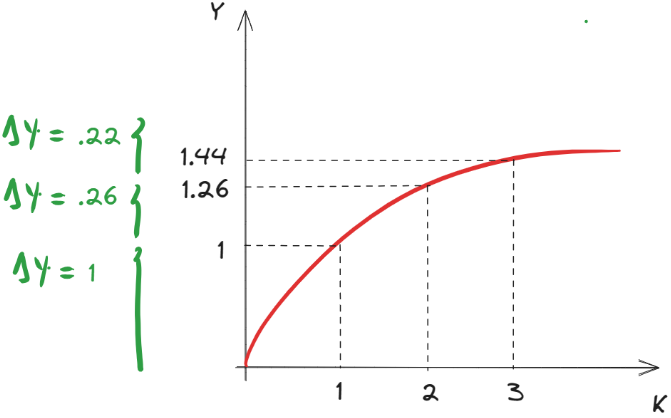


$$Y = \bar{A}K^{1/3}L^{2/3}$$

L and Y when $K = 1$ and $\bar{A} = 1$

L	Y
1	1
2	1.59
3	2.08
4	2.52

Example: Capital



$$Y = \bar{A}K^{1/3}L^{2/3}$$

L and Y when $L = 1$ and $\bar{A} = 1$

K	Y
1	1
2	1.26
3	1.44
4	1.59

Intuition

- Suppose you own a company with **50 employees** and **10 computers**
- An employee **does not** always need a computer...
- ...but **there are too few computers at this moment**, so sometimes **some employees stay idle**.
- If you start **buying computers, the first ones will be very productive** because they will be **matched with idle employees**.
- But at a certain point, new computers will start going idle for some time...
- And **if you buy too many computers** (say if your company has more computers than employees) **any additional computers will be useless** as they will be idle the whole time.
- **The production function of your firm exhibits diminishing returns in computers!**

Returns to Scale

Returns to Scale (RS): change in output when all factors are changed by the same proportion.

a function $F(\lambda K, \lambda L) = \lambda^s F(K, L)$ is

- **constant** returns to scale if $s = 1 \implies F(\lambda K, \lambda L) = \lambda F(K, L)$
- **increasing** returns to scale if $s > 1 \implies F(\lambda K, \lambda L) > \lambda F(K, L)$
- **decreasing** returns to scale if $s < 1 \implies F(\lambda K, \lambda L) < \lambda F(K, L)$

for some $\lambda > 0$.

Returns to Scale

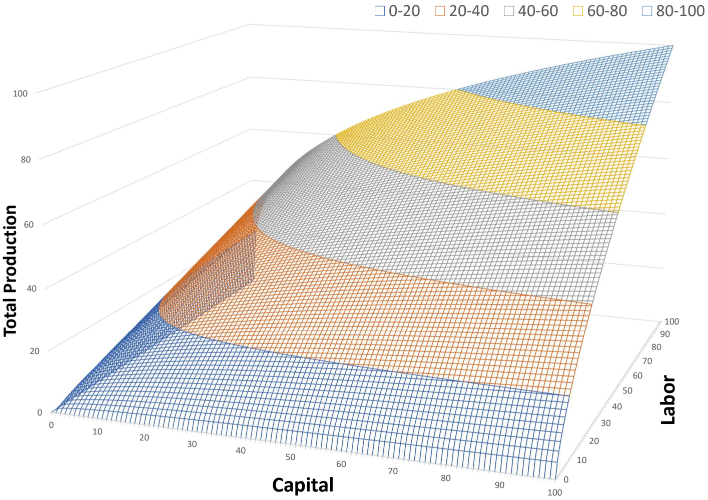
Claim: Cobb-Douglas is Constant Returns to Scale in (K,L)

Proof.

$$
\begin{aligned}
F(A, \lambda K, \lambda L) &= \bar{A}(\lambda K)^{\alpha}(\lambda L)^{1-\alpha} \\
&= \lambda \bar{A}(K)^{\alpha}(L)^{1-\alpha} \\
&= \lambda F(A, K, L)
\end{aligned}
$$

$\square$

Example: Capital and Labor jointly



$$\begin{aligned} Y &= \bar{A}K^{1/3}L^{2/3} \\ \bar{A} &= 1 \end{aligned}$$

Cobb-Douglas is Constant Returns to Scale in Capital and Labor jointly... but diminishing marginal returns while holding the other factor fixed...

Profit Maximization

$$\max_{\{K,L\}} \pi = \underbrace{P \cdot F(K,L)}_{\text{revenues}} - \left(\underbrace{w \cdot L + r \cdot K}_{\text{costs}}\right)$$

where

- P: price of the output good (if there is only one sector, we can normalize this $P = 1$, numéraire)
- $F(K,L)$: production function
- w: wages
- r: rental rate on capital

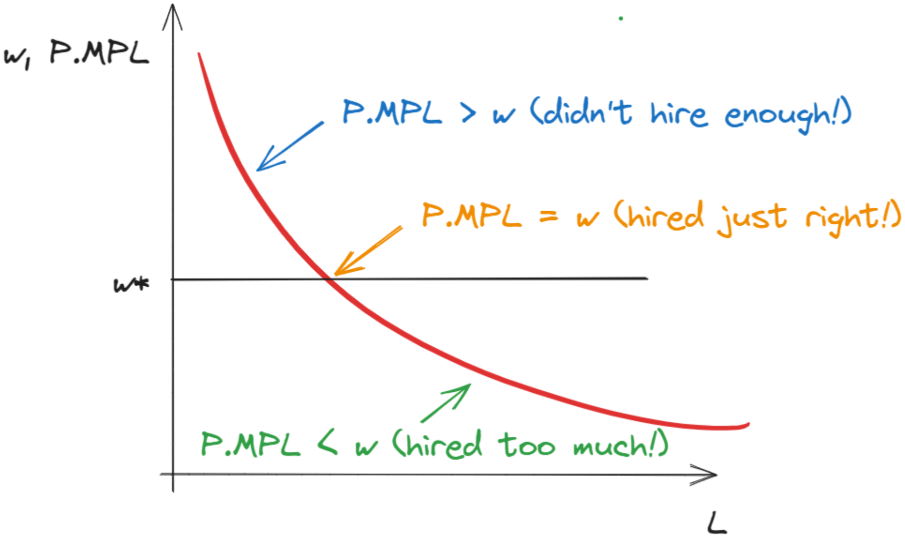
Profit Maximization

$$\max_{\{K,L\}} \pi = \underbrace{P \cdot F(K,L)}_{\text{revenues}} - \left(\underbrace{w \cdot L + r \cdot K}_{\text{costs}}\right)$$

First Order Conditions imply that, at the optimal:

$$\begin{aligned} \frac{\partial \pi}{\partial L} &= 0 \implies P \cdot \frac{\partial F(K,L)}{\partial L} = P \cdot MPL = w \\ \frac{\partial \pi}{\partial K} &= 0 \implies P \cdot \frac{\partial F(K,L)}{\partial K} = P \cdot MPK = r \end{aligned}$$

Intuition for optimality result



Optimality Conditions for Demand with Cobb-Douglas

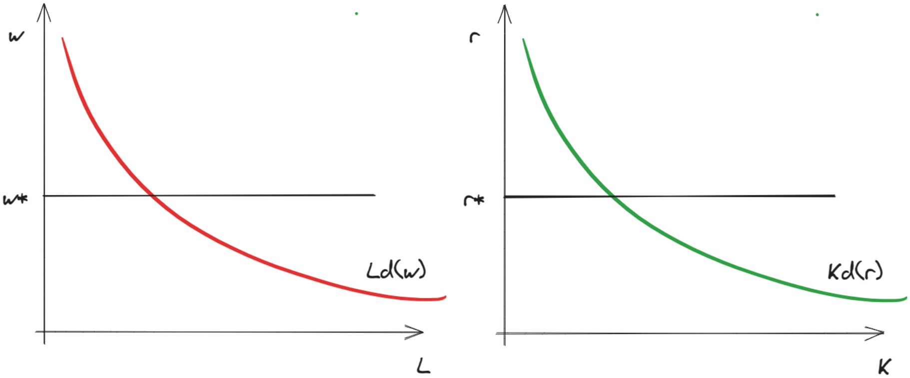
$$
\begin{aligned}
Y &= \bar{A}(K^d)^{\alpha}(L^d)^{1-\alpha} \\
P(1-\alpha)\bar{A}\left(\frac{K^d}{L^d}\right)^{\alpha} &= w \\
P\alpha\bar{A}\left(\frac{L^d}{K^d}\right)^{1-\alpha} &= r
\end{aligned}
$$

Optimality Conditions for Demand with Cobb-Douglas

Note that we can derive a labor demand and capital demand schedule from each of those, which are decreasing in factor prices:

$$
\begin{aligned}
L^d &= \left(\frac{(1-\alpha)\cdot A\cdot P}{w}\right)^{\frac{1}{\alpha}}\cdot K^d \\
K^d &= \left(\frac{\alpha\cdot A\cdot P}{r}\right)^{\frac{1}{1-\alpha}}\cdot L^d
\end{aligned}
$$

Demand

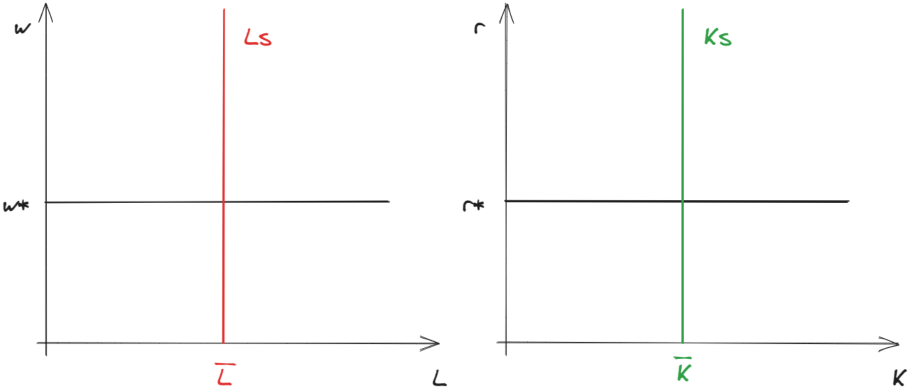


Supply side of the economy is simple

- Households supply labor and capital inelastically.
- Prices adjust to ensure that supply equals demand (market clearing condition)

$$
\begin{aligned}
L^d &= L^s = \bar{L} \qquad \text{(parameter)} \\
K^d &= K^s = \bar{K} \qquad \text{(parameter)}
\end{aligned}
$$

Supply



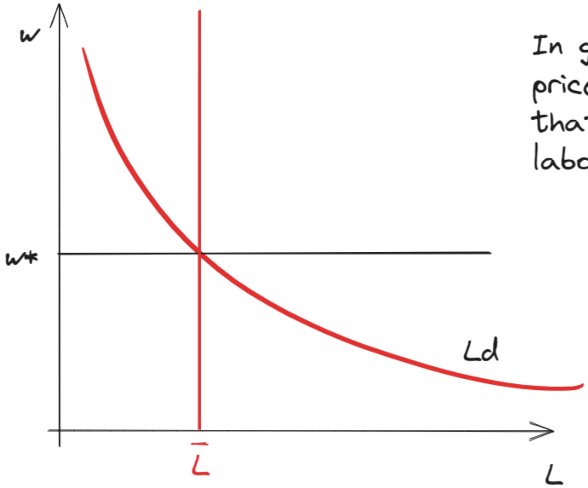
General Equilibrium

Endogeneous Variables: $Y, K, L, w, r,$ Numéraire: $P = 1$

Five equations for five unknowns

$$
\begin{aligned}
Y &= \bar{A}(K)^{\alpha}(L)^{1-\alpha} && (1)\\
P(1-\alpha)\bar{A}\left(\frac{K}{L}\right)^{\alpha} &= w && (2)\\
P\alpha\bar{A}\left(\frac{L}{K}\right)^{1-\alpha} &= r && (3)\\
L &= \bar{L} = L^{s} && (4)\\
K &= \bar{K} = K^{s} && (5)
\end{aligned}
$$

Supply markets



In general equilibrium, prices adjust to ensure that labor supply equals labor demand

Solution to the Production Model

Replacing the market clearing condition in and normalizing $P = 1$ to be the numéraire of this economy:

$$\begin{aligned}
Y^* &= \bar{A}(\bar{K})^{\alpha}(\bar{L})^{1-\alpha} && (6)\\
w^* &= (1-\alpha)\bar{A}\left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha} && (7)\\
r^* &= \alpha\bar{A}\left(\frac{\bar{L}}{\bar{K}}\right)^{1-\alpha} && (8)\\
L^* &= \bar{L} && (9)\\
K^* &= \bar{K} && (10)
\end{aligned}$$

Note that everything on the right-hand side of the equations is a parameter, so this is indeed an explicit solution!

Numerical Example

Suppose $\bar{K} = 20$, $\bar{L} = 160$, $\bar{A} = 1$, $\alpha = \frac{1}{3}$. What is the solution to the Production Model? Replacing in the set of equations before:

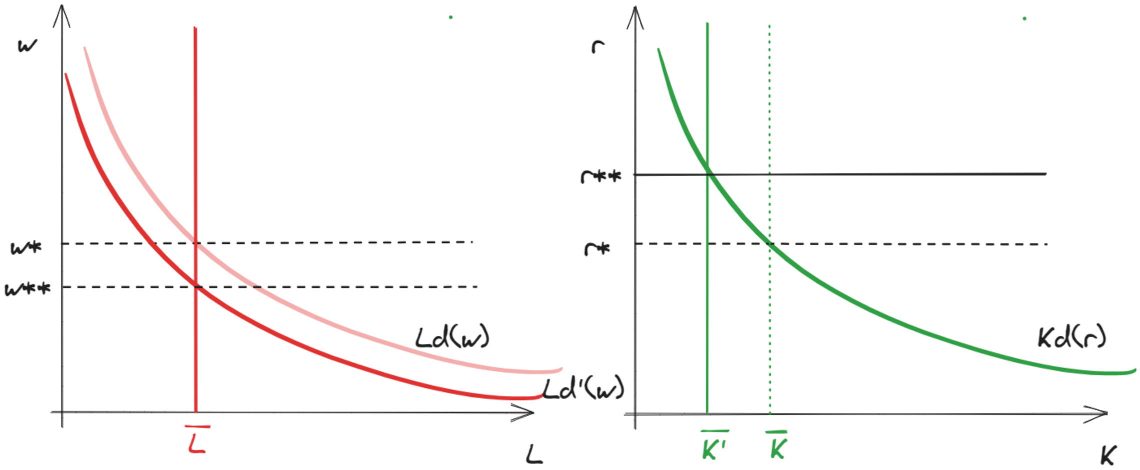
$$
\begin{aligned}
Y^* &= (20)^{\frac{1}{3}}(160)^{\frac{2}{3}} = 80 \\
w^* &= \frac{2}{3}\left(\frac{20}{160}\right)^{\frac{1}{3}} = \frac{1}{3} \\
r^* &= \frac{1}{3}\left(\frac{160}{20}\right)^{\frac{2}{3}} = \frac{4}{3} \\
L^* &= 160 \\
K^* &= 20
\end{aligned}
$$

Numerical Example

Now suppose the total available capital changes to $\bar{K}' = 10$. What happens?

$$\begin{aligned}
Y^{**} &= (10)^{\frac{1}{3}}(160)^{\frac{2}{3}} = 63.5 \\
w^{**} &= \frac{2}{3}\left(\frac{10}{160}\right)^{\frac{1}{3}} = 0.26 \\
r^{**} &= \frac{1}{3}\left(\frac{160}{10}\right)^{\frac{2}{3}} = 2.11 \\
L^{**} &= 160 \\
K^{**} &= 100
\end{aligned}$$

Numerical Example: Graphical Representation of Comparative Statics



Production Model: 4 Implications

- Production (value added) is equal to Income

$$Y^* = \bar{A}(\bar{K})^{\alpha}(\bar{L})^{1-\alpha} = r^* K^* + w^* L^*$$

Proof.

$\square$

Production Model: 4 Implications

- Production (value added) is equal to Income

$$Y^* = \bar{A}(\bar{K})^{\alpha}(\bar{L})^{1-\alpha} = r^* K^* + w^* L^*$$

Proof.

$$r^* K^* + w^* L^* \quad = \quad \alpha \bar{A} \left(\frac{\bar{L}}{\bar{K}} \right)^{1-\alpha} \bar{K} + (1-\alpha) \bar{A} \left(\frac{\bar{K}}{\bar{L}} \right)^{\alpha} \bar{L}$$

□

Production Model: 4 Implications

- Production (value added) is equal to Income

$$Y^* = \bar{A}(\bar{K})^{\alpha}(\bar{L})^{1-\alpha} = r^* K^* + w^* L^*$$

Proof.

$$\begin{aligned} r^* K^* + w^* L^* &= \alpha \bar{A} \left(\frac{\bar{L}}{\bar{K}}\right)^{1-\alpha} \bar{K} + (1-\alpha)\bar{A}\left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha} \bar{L} \\ &= \alpha \bar{A}(\bar{K})^{\alpha}(\bar{L})^{1-\alpha} + (1-\alpha)\bar{A}(\bar{K})^{\alpha}(\bar{L})^{1-\alpha} \end{aligned}$$

$\square$

Production Model: 4 Implications

- Production (value added) is equal to Income

$$Y^* = \bar{A}(\bar{K})^{\alpha}(\bar{L})^{1-\alpha} = r^* K^* + w^* L^*$$

Proof.

$$\begin{aligned}
r^* K^* + w^* L^* &= \alpha \bar{A}\left(\frac{\bar{L}}{\bar{K}}\right)^{1-\alpha} \bar{K} + (1-\alpha)\bar{A}\left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha} \bar{L} \\
&= \alpha \bar{A}(\bar{K})^{\alpha}(\bar{L})^{1-\alpha} + (1-\alpha)\bar{A}(\bar{K})^{\alpha}(\bar{L})^{1-\alpha} \\
&= (1-\alpha)Y^* + \alpha Y^* = Y^*
\end{aligned}$$

$\square$

Production Model: 4 Implications

- Production (value added) is equal to Income

$$Y^* = \bar{A}(\bar{K})^{\alpha}(\bar{L})^{1-\alpha} = r^* K^* + w^* L^*$$

Proof.

$$\begin{aligned}
r^* K^* + w^* L^* &= \alpha \bar{A} \left(\frac{\bar{L}}{\bar{K}}\right)^{1-\alpha} \bar{K} + (1-\alpha)\bar{A}\left(\frac{\bar{K}}{\bar{L}}\right)^{\alpha} \bar{L} \\
&= \alpha \bar{A}(\bar{K})^{\alpha}(\bar{L})^{1-\alpha} + (1-\alpha)\bar{A}(\bar{K})^{\alpha}(\bar{L})^{1-\alpha} \\
&= (1-\alpha) Y^* + \alpha Y^* = Y^*
\end{aligned}$$

□

- The profits of the representative firm are zero (follows from above)

$$\pi^* = Y^* - (r^* K^* + w^* L^*) = 0$$

Production Model: 4 Implications

- All available factors are utilized in equilibrium, so production depends on endowments of factors

$$Y^* = \bar{A}(\bar{K})^\alpha(\bar{L})^{1-\alpha}$$

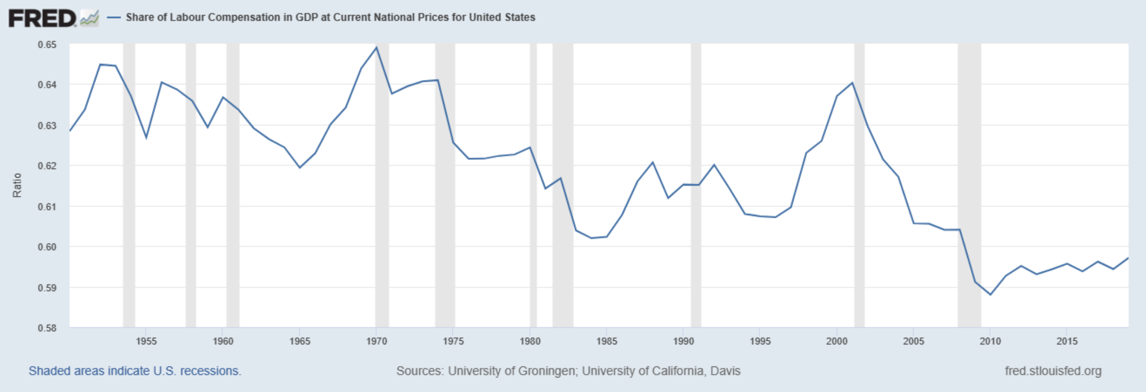
- Total payments to factors as share of output (factor shares) are determined by the production function

$$\frac{w^* L^*}{Y^*} = \frac{(1-\alpha)\bar{A}\left(\frac{\bar{K}}{\bar{L}}\right)^\alpha \bar{L}}{\bar{A}(\bar{K})^\alpha(\bar{L})^{1-\alpha}} = (1-\alpha)$$

$$\frac{r^* K^*}{Y^*} = \frac{\alpha\bar{A}\left(\frac{\bar{L}}{\bar{K}}\right)^{1-\alpha} \bar{K}}{\bar{A}(\bar{K})^\alpha(\bar{L})^{1-\alpha}} = \alpha$$

Production Model: 4 Implications

(Corollary) Under Cobb-Douglas, factor shares are constant.



Since 1950, labor share has been around 2/3 of GDP in the U.S., with some decline recently.

What have we learned?

- What are returns to scale
- What are diminishing marginal returns
- How to set up and solve a simple model
- In the production model, total output is proportional to technology/ideas/institutions A, total capital K and total labor force L, so countries with better technology, more wealth/capital stock, and larger populations will have larger economies