Dixit Stiglitz Aggregator and Imperfect Competition

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Consumers maximize:

$$
\begin{array}{cc}
\max_{\{q(\omega)\}_{\omega\in\Omega}} & \left[\int_{\omega\in\Omega} q(\omega)^{\frac{\sigma-1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \\
s.t. & \left[\int_{\omega\in\Omega} p(\omega)q(\omega)d\omega\right] \leq yL
\end{array}
$$

where $\sigma < 1$ is a parameter that controls the elasticity of substitution. Let $Q \equiv \left[\int_{\omega\in\Omega} q(\omega)^{\frac{\sigma-1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}}$ For each variety $\omega \in \Omega$, optimality satisfies:

$$
\begin{aligned}
\lambda p(\omega) &= \frac{\sigma}{\sigma-1}\left[\int_{\omega\in\Omega} q(\omega)^{\frac{\sigma-1}{\sigma}} d\omega\right]^{\frac{1}{\sigma-1}} \frac{\sigma-1}{\sigma} q(\omega)^{\frac{\sigma-1}{\sigma}} q(\omega)^{-1} \\
\iff p(\omega)q(\omega) &= \frac{1}{\lambda} Q^{\frac{1}{\sigma}} q(\omega)^{\frac{\sigma-1}{\sigma}}
\end{aligned}
$$

Replacing this in the budget constraint yields:

$$
\frac{1}{\lambda} Q^{\frac{1}{\sigma}} \left[\int_{\omega\in\Omega} q(\omega)^{\frac{\sigma-1}{\sigma}} d\omega\right] = yL \iff \lambda = \frac{Q}{yL}
$$

Replacing this in (2.1) results in:

$$
\begin{aligned}
\frac{Q}{yL} p(\omega)q(\omega) &= Q^{\frac{1}{\sigma}} q(\omega)^{\frac{\sigma-1}{\sigma}} \\
q(\omega) &= p(\omega)^{-\sigma} Q^{1-\sigma} (yL)^{\sigma} \qquad (1)
\end{aligned}
$$

Using the definition of Q:

$$
\begin{aligned}
Q &= \left[\int_{\omega\in\Omega}\left(p(\omega)^{-\sigma}Q^{1-\sigma}(yL)^{\sigma}\right)^{\frac{\sigma-1}{\sigma}}d\omega\right]^{\frac{\sigma}{\sigma-1}} \\
Q^{1-\sigma} &= Q(yL)^{-\sigma}\left[\int_{\omega\in\Omega}p(\omega)^{1-\sigma}d\omega\right]^{-\frac{\sigma}{\sigma-1}} \qquad (2)
\end{aligned}
$$

Which, replacing in (2.2) and rearranging, results in:

$$
q(\omega) = p(\omega)^{-\sigma}\left[\int_{\omega\in\Omega}p(\omega)^{1-\sigma}d\omega\right]^{\frac{\sigma}{1-\sigma}}Q \qquad (3)
$$

Note that the ideal price index for the aggregate good Q must satisfy:

$$
\begin{aligned}
PQ &= \left[\int_{\omega\in\Omega}p(\omega)q(\omega)d\omega\right] \\
PQ &= \left[\int_{\omega\in\Omega}p(\omega)^{1-\sigma}\left[\int_{\omega\in\Omega}p(\omega)^{1-\sigma}d\omega\right]^{\frac{\sigma}{1-\sigma}}Qd\omega\right] \\
P &= \left[\int_{\omega\in\Omega}p(\omega)^{1-\sigma}d\omega\right]^{\frac{1}{1-\sigma}}
\end{aligned}
$$

So we can write the demand function as:

$$
\begin{aligned}
q(\omega) &= p(\omega)^{-\sigma}\left[\int_{\omega\in\Omega}p(\omega)^{1-\sigma}d\omega\right]^{\frac{\sigma}{1-\sigma}}Q \\
&= \left(\frac{p(\omega)}{\left[\int_{\omega\in\Omega}p(\omega)^{1-\sigma}d\omega\right]^{\frac{1}{1-\sigma}}}\right)^{-\sigma}Q \\
&= \left(\frac{p(\omega)}{P}\right)^{-\sigma}Q
\end{aligned}
$$

Finally, realize that we can write the budget constraint as $PQ = \int_{\omega\in\Omega}p(\omega)q(\omega)d\omega = yL$. Solving for Q:

$$
Q = \frac{yL}{P} \qquad (4)
$$

Therefore,

$$
q(\omega) = \underbrace{\left(\frac{p(\omega)}{P}\right)^{-\sigma}}_{\text{decreasting in rel. price}} \cdot \underbrace{\frac{yL}{P}}_{\text{increasing in real income}}
$$

Firms hold a monopoly over variety ω. They take demand functions $q(\omega)$, aggregate price levels P, and marginal cost κ as given as choose prices to maximize profits:

$$\max_{p(\omega),q(\omega)} p(\omega)q(\omega) - \kappa q(\omega) = \max_{p(\omega)} \left(p(\omega) - \kappa\right)\left(\frac{p(\omega)}{P}\right)^{-\sigma}\frac{yL}{P}$$

Optimality satisfies:

$$\left(\frac{p(\omega)}{P}\right)^{-\sigma}\frac{yL}{P} - \frac{\sigma}{p(\omega)}\left(p(\omega) - \kappa\right)\left(\frac{p(\omega)}{P}\right)^{-\sigma}\frac{yL}{P} = 0 \iff p(\omega) = \frac{\sigma}{\sigma - 1}\kappa$$

So optimal prices are a mark-up $\frac{\sigma}{\sigma-1} > 1$ over marginal costs.