Discussion Session 3: Inefficient Equilibria: Economy with Externalities

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September 1, 2023

In this section, we will introduce market failures to illustrate how the competitive equilibrium allocation can be Pareto inefficient. We define a flow of pollution P_t, which decreases household welfare. There is a measure one of households in this economy, each with preferences that take the following form:

$$\sum_{t=0}^{\infty} \beta^t \left[\log c_t + \chi \log(1 - n_t) - D(P_t) \right] \tag{1}$$

where c_t is consumption; n_t is labor; and $D(\cdot)$ is a strictly increasing, strictly convex differentiable function. We assume pollution is an increasing function of output: $P_t = F(Y_t)$. We will first characterize the recursive competitive equilibrium, then compare it to the benevolent social planner's solution.

1 Recursive Competitive Equilibrium

We define the households recursive problem as:

$$\begin{aligned} v(K,k) \quad &= \quad \max_{k',c,n} \log c + \chi \log(1-n) - D(P(K)) + \beta v(K',k') \\ s.t. \quad & k' + c \leq nw(K) + kr(K) + (1-\delta)k \\ & K' = \hat{G}(K) \quad \text{(perceived law of motion for capital)} \end{aligned} \tag{2}$$

with Lagrangian and FOCs:

$$\begin{aligned} \mathscr{L} \quad &= \quad \log c + \chi \log(1-n) - D(P(K)) + \beta v(K',k') + \\ & \quad\quad \lambda[nw(K) + kr(K) + (1-\delta)k - k' - c] \\ c \quad &: \quad \frac{1}{c} = \lambda \\ n \quad &: \quad \frac{\chi}{1-n} = \lambda w(k) \\ k' \quad &: \quad \lambda = \beta v_{k'}(K',k') \end{aligned} \tag{3}$$

To derive $v_k(k, K)$, we substitute for $c(k, K)$ in the value function, evaluate $n(K, k), k'(k, K)$ at their optimal points, and take the derivative with respect to k, capturing only the direct effect:

$$
\begin{aligned}
v(K,k) &= \log(n(K,k)w(K) + kr(K) + (1-\delta)k - k'(K,k) \\
&+ \chi \log(1 - n(K,k)) - D(P(K)) + \beta v(K', k') \\
\implies v_k(K,k) &= \frac{r(K) + (1-\delta)}{c(K,k)}
\end{aligned}
$$

Combining the FOCs and the envelope condition result in our standard Euler equation and labor-leisure condition (dropping the parenthesis of the choice variables for ease of notation):

$$
\underbrace{\frac{1}{c(K,k)}}_{\text{marginal utility of consumption}} = \underbrace{\beta \frac{[r(\hat{G}(K)) + (1-\delta)]}{c(\hat{G}(K), k'(K,k))}}_{\substack{\text{discounted marginal utility} \\ \text{of consuming savings tomorrow}}} \tag{4}
$$

$$
\underbrace{\frac{\chi}{1 - n(K,k)}}_{\text{marginal disutility of work}} = \underbrace{\frac{w(K)}{c(K,k)}}_{\substack{\text{marginal utility of} \\ \text{consuming marginal wage}}} \tag{5}
$$

The firms problem is:

$$
\max_{K_d, L_d} AK_d^{\theta} N_d^{1-\theta} - K_d r(K) - L_d w(K) \tag{6}
$$

with optimality conditions:

$$
r(K) = \theta A [K_d(K)]^{\theta - 1} [N_d(K)]^{1-\theta} w(K) = (1-\theta) A [K_d(K)]^{\theta} [N_d(K)]^{-\theta}
$$

In equilibrium, it must be the case that the factor market clears, i.e.:

$$
K_d(K) = K, \quad N_d(K) = N
$$

Definition 1 (Recursive Competitive Equilibrium)**.** A Recursive Competitive Equilibrium is define as:

1. A household value function $v(K, k)$ and household policy functions $k'(K, k), n(K, k), c(K, k)$;
2. Firm demand functions $K_d(K), N_d(K)$;
3. Price functions $w(K), r(K)$; and a
4. A perceived law of motion for capital for capital $K' = \hat{G}(K)$;

such that

a) Given 3 and 4, 1 solves the consumer's problem;

b) Given 3, 2 solves the firm's problem;

c) Factor markets clear, i.e. $K \equiv \int_0^i k(i)di = K^d(K)$ and $N \equiv \int_0^i n(K,k(i))di = N^d(K)$;

d) Goods markets clear, i.e. $c(K,K) + k'(K,K) = AK^\theta N^{1-\theta} + (1-\delta)K$; and

e) Expectations are correct, i.e. $K' = k'(K,K) = \hat{G}(K)$.

In equilibrium, we know $K' = \hat{G}(K)$. Therefore, in equilibrium, the following must hold:

$$\frac{1}{c(K,k)} = \beta \frac{[\theta A[K_d(K')]^{\theta-1}[N_d(K')]^{1-\theta} + (1-\delta)]}{c(K',k'(K))} \tag{7}$$

$$\frac{\chi}{1-n(K,k)} = \frac{(1-\theta)A[K_d(K)]^\theta [N_d(K)]^{-\theta}}{c(K,k)} \tag{8}$$

2 Planner's Problem

We now turn to the social planner's recursive problem. The benevolent planner tries to maximize social welfare, so they do not care about individual variables, only about aggregates. The problem is described as such:

$$V(K) = \max_{K',C,N,P,Y} \log C + \chi \log(1-N) - D(P) + \beta V(K') \tag{9}$$

$$\begin{aligned} s.t. \quad & K' + C \leq Y + (1-\delta)K \\ & P = F(Y) \\ & Y = AK^\theta N^{1-\theta} \end{aligned} \tag{10}$$

with Lagrangian and FOCS:

$$\begin{aligned}
\mathscr{L} &= \log C + \chi \log(1-N) - D(F(AK^\theta N^{1-\theta})) + \beta V(K') + \\
&\quad \Lambda[AK^\theta [N(K)]^{1-\theta} + (1-\delta)K - K'(K) - C(K)] \\
C &: \Lambda = \frac{1}{C(K)} \\
N &: \frac{\chi}{1-N(K)} = [\Lambda - D'(F(Y))F'(Y)](1-\theta)AK^\theta [N(K)]^{-\theta} \\
K' &: \Lambda = \beta v'(K')
\end{aligned}$$

To derive $v'(K')$, we can use the envelope condition, writing out the value function explicitly as functions of K, then ignoring any indirect reoptimization effect:

$$\begin{aligned} v(K) &= \log(AK^\theta N(K)^{1-\theta} + (1-\delta)K - K'(K)) + \chi\log(1-N(K)) - D(F(AK^\theta N(K)^{1-\theta})) + \beta V(K') \\ \implies v'(K) &= \frac{\theta AK^{\theta-1}[N(K)]^{1-\theta} + (1+\delta)}{C(K)} - D'(P(K))F'(Y(K))\theta AK^{\theta-1}[N(K)]^{1-\theta} \end{aligned}$$

We can now combine the FOCs to derive the planner's version of the euler equation and the labor leisure condition.

$$\begin{aligned} \underbrace{\frac{1}{C(K)}}_{\text{marginal utility of consumption}} &= \underbrace{\beta\left(\left[\frac{1}{C(K'(K))} - D'(P(K'))F'(Y(K'))\right]\theta A[K'(K)]^{\theta-1}[N(K')]^{1-\theta} + \frac{1}{C(K')}(1-\delta)\right)}_{\substack{\text{discounted marginal utility}\\ \text{of consuming savings tomorrow}}} \\ \underbrace{\frac{\chi}{1-N(K)}}_{\text{marginal disutility of work}} &= \underbrace{\left[\frac{1}{C(K)} - D'(F(Y(K)))F'(Y(K))\right](1-\theta)AK^\theta[N(K)]^{-\theta}}_{\substack{\text{marginal utility of}\\ \text{consuming marginal wage}}} \end{aligned}$$

Definition 2 (Solution to the Recursive Planner's Problem)**.** A solution to the recursive planner's problem is defined as a social value function $v(K)$ and social policy functions $K'(K), N(K), C(K), Y(K), P(K)$ such that the value function and the policy functions maximize lifetime social utility, the allocation is feasible, and resources in society are exhausted, i.e. $C(K) + K'(K) = Y(K) + (1-\delta)K$, with $Y(K) = AK^\theta[N(K)]^{1-\theta}$.

3 Comparing the solutions

Now let us compare the Euler Equation from the Planner's Problem to the Competitive Equilbrium (dropping the function notation for simplicity):

$$\begin{aligned} \frac{1}{C} &= \beta\left(\left[\frac{1}{C'} - D'(P')F'(Y')\right]\theta A(K')^{\theta-1}(N')^{1-\theta} + \frac{1}{C'}(1-\delta)\right) \quad \text{(Planner's EE)} \\ \frac{1}{c} &= \beta\frac{[\theta AK^{\theta-1}N^{1-\theta} + (1-\delta)]}{c'} \quad \text{(CE's EE)} \end{aligned}$$

Note that they differ by the term: $-\beta D'(F(Y'))F'(Y')\theta AK'^{\theta-1}N'^{1-\theta}$, which denotes the discounted disutility of having to experience higher pollution in the next period due to higher production as a function of higher savings. The planner internalizes the disutility of extra production that happens through pollution, while households ignore them in the competitive equilibrium. Therefore, in the competitive equilibrium households oversave and overconsume compared to the social optimum.

We can observe a similar pattern from comparing the labor leisure conditions:

$$\frac{\chi}{1-N} = \left[\frac{1}{C} - D'(F(Y))F'(Y)\right](1-\theta)AK^{\theta}N^{-\theta} \quad \text{(Planner's LL)}$$
$$\frac{\chi}{1-n} = \frac{(1-\theta)AK^{\theta}N^{-\theta}}{c} \quad \text{(CE's LL)}$$

Again, consumer overwork compared to the social optimum. You can see that by realizing that the marginal disutility of work is higher in the competitive equilibrium ($D'(F(Y))F'(Y) > 0$) and recalling that the utility function is concave in leisure (that is, convex in hours).