Discussion Session 4: Describing and Characterizing Competitive Equilibria

Carlos Góes

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1 Describing a Sequence-of-Markets Equilibrium

This subsection describes the Neoclassical Growth Model. For more details, refer to Chapter 1 of Stokey & Lucas' textbook. It builds on the foundation of the Solow-Swan model presented in undergraduate macroeconomics —with the difference that all decisions, including savings, are endogenous. Like in the Solow model, given assumptions below, there will be a unique **steady state** for this economy —that is, a dynamic equilibrium where all allocations are unchanged from one period to the next.

Assumptions

1. Time is discrete: $t = [0, \ldots, T]'$
2. A complete, transitive, and reflexive preference schedule $\succcurlyeq$ has a numerical representation $u(\cdot)$.
3. We assume $u : \mathbb{R}^+ \to \mathbb{R}$ is continuously differentiable, stricly increasing, and strictly concave, such that $\lim_{c \to 0} u'(c_t) \to -\infty$
4. Preferences are given consumption streams: $\{c_t\}_t^\infty = \{c_0, c_1, \ldots\}$
5. Future period consumption is discounted by discount rate $\beta \in (0,1)$.
6. Lifetime utility is represented by a sum of discounted consumption streams: $\sum_{t=0}^\infty \beta^t u(c_t)$.
7. Individuals are endowed with initial capital k_0 and, every period t, one unit of time that can be split between work (n) or leisure (l), such that $n + l = 1$.
8. Firms have a technology that transforms inputs into outputs: $Y_t = F(K_t, N_t)$.
9. We assume $F : \mathbb{R}^+ \to \mathbb{R}^+$ is continuously differentiable, constant returns to scale (homogeneous of degree one), strictly-quasi-concave, with $F(0, n) = 0 \quad \forall n$; $F_K(K, N) > 0, F_N(K, N) > 0 \quad \forall N, K$; $\lim_{K \to 0} F'(K, 1) \to \infty$; $\lim_{K \to \infty} F'(K, 1) \to 0$.

10. Households exist within measure one. By "measure one households," we mean that there are uncountably many households and their measure is normalized to one, i.e.: $i \sim Uniform(0,1)$. This abstraction helps with the assumption that each household individually has no aggregate impact —they operate in a competitive market as price takers. Mathematically, since each i has measure zero, they can have no impact over the aggregate economy. Under these assumptions, we can characterize the aggregate stocks $K_t = \int_0^1 k_i di, N_t = \int_0^1 n_i di$.

11. Small letters represent individual household/firm variables or parameters while capital letters represent aggregate variables.

1.1 Defining the Sequence of Markets Equilibrium

Household preferences take the form:

$$\sum_{t=0}^{\infty} \beta^t [\log(c_t) + \alpha \log(1 - n_t)] \tag{1}$$

i.e., the household derives utility from consumption (c_t) and leisure ($l_t = 1 - n_t$). The preference for leisure relative to consumption is controlled by α.

Households face a budget constraint:

$$\underbrace{c_i + i_t}_{expenditures} \leq \underbrace{w_t n_t + r_t k_t + \pi_t}_{income} \tag{2}$$

They supply units of labor (n_t) and are paid a market wage (w_t) thereby earning labor income ($w_t n_t$). Similarly, households supply units of capital (k_t) and are paid a market rental rate of capital (r_t) thereby earning capital income ($k_t r_t$). Households own the firms, thereby earning (if any) profits π_t. Households can choose to consume c_t or invest i_t their income.

Investments (i_t) add to capital stock (k_t) the household owns, which depreciates every period at a fixed rate (δ). The law of motion for capital is given by:

$$k_{t+1} = (1 - \delta) k_t + i_t \tag{3}$$

Putting those together, the constrained maximization problem households face is, therefore:

$$\begin{aligned} \max_{\{c_t, k_{t+1}, i_t, n_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t [\log(c_t) + \alpha \log(1 - n_t)] \\ s.t. \quad & c_i + k_{t+1} \leq w_t n_t + r_t k_t + (1 - \delta) k_t + \pi_t, \qquad k_0 > 0 \text{ given} \end{aligned} \tag{4}$$

Firms have access to the following technology:

$$Y_t = A K_t^{\theta} N_t^{1-\theta} \tag{5}$$

where Y_t is total output and A is an exogenous parameter for technology or productivity.

Firms face an unconstrained static maximization problem, which tries to maximize profit as the difference between output and inputs:

$$\max_{K_t^d, N_t^d} A(K_t^d)^\theta (N_t^d)^{1-\theta} - w_t N_t^d - r_t K_t^d \tag{6}$$

Definition 1 (Sequence-of-markets equilibrium)**.** A sequence-of-markets equilibrium consists of:

i) household choice streams for consumption, work, and savings: $\{c_t, n_t, k_{t+1}\}_{t=0}^\infty$.

ii) firms' choice streams for capital and labor demand: $\{K_t^d, N_t^d\}_{t=0}^\infty$.

iii) prices streams: $\{w_t, r_t\}_{t=0}^\infty$.

Such that:

a) Given (iii), (i) solves the households' problem.

b) Given (iii), (ii) solves the firms' problem.

c) Markets clear:

- $N_{d,t} = N_t$
- $K_{d,t} = K_t$
- $C_t + K_{t+1} = A(K_t^d)^\theta (N_t^d)^{1-\theta} + (1-\delta)K_t$
- Due to free entry, there are zero profits: $\pi_t = 0$.

1.2 Characterizing the SoM Equilibrium

Since preferences are strictly concave, the Karush–Kuhn–Tucker conditions are satisfied and we can rule out corner solutions, and write the inequality constraint with an equality. We can also rewrite the second constraint in terms of $i_t = k_{t+1} - (1-\delta)k_t$ and plug it in the first constraint to reduce the maximization problem.

The Lagrangian for the household maximization problem is:

$$\begin{aligned} \mathscr{L} &= \sum_{t=0}^\infty \beta^t [\log(c_t) + \alpha \log(1-n_t)] + \\ & \sum_{t=1}^\infty \lambda_t [w_t n_t + r_t k_t + \pi_t + (1-\delta)k_t - c_i - k_{t+1}] \end{aligned} \tag{7}$$

with the following first order conditions

$$
\begin{aligned}
c_t &: \quad \beta^t \frac{1}{c_t} = \lambda_t & (8) \\
k_{t+1} &: \quad \lambda_t = \lambda_{t+1}[r_{t+1} + (1-\delta)] & (9) \\
n_t &: \quad \beta^t \frac{\alpha}{1-n_t} = \lambda_t w_t & (10)
\end{aligned}
$$

Combining (8) and (9) yields the **Euler Equation**:

$$
\begin{aligned}
\beta^t \frac{1}{c_t} &= \beta^{t+1} \frac{1}{c_{t+1}}[r_{t+1} + (1-\delta)] \\
\frac{1}{c_t} &= \beta \frac{1}{c_{t+1}}[r_{t+1} + (1-\delta)] & (11)
\end{aligned}
$$

which intuitively states that, at the optimal point, the marginal utility of present consuming one unit of output (the left hand side of equation 12) must equal the discounted marginal utility of saving one unit of output today and consuming it tomorrow. Note that the right hand side of the equation incorporates how much the saved unit will yield in capital income and how much it will depreciate.

Combining (8) and (10) yields the **Labor Leisure Optimality Condition**:

$$
\begin{aligned}
\beta^t \frac{\alpha}{n_t} &= \lambda_t w_t \\
\frac{1}{c_t} w_t &= \frac{\alpha}{1-n_t} & (12)
\end{aligned}
$$

which states that, at the optimal, the marginal utility of consuming the earnings of one additional unit of work must equate the marginal disutility of working an additional unit.

The optimality conditions for the firm's problem are:

$$
\begin{aligned}
K_t^d &: \quad \theta A (K_t^d)^{\theta-1} (N_t^d)^{1-\theta} = r_t & (13) \\
N_t^d &: \quad (1-\theta) A (K_t^d)^{\theta} (N_t^d)^{-\theta} = w_t & (14)
\end{aligned}
$$

Equation (13) states that the optimal quantity of capital demanded will be such that the marginal product of capital equals the marginal cost of renting an additional unit of capital. Equation (14) states that the optimal quantity of labor demanded will be such that the marginal product of labor equals the marginal cost of hiring an additional unit of labor.

2 Describing the Recursive Competitive Equilibrium

As we have seen in class, we can write the dynamic problem we have seen in the previous section with a recursive formulation:

$$\begin{aligned} V(K,k) = \quad & \max_{c,n,k'} \quad \log(c) + \alpha \log(1-n) + \beta V(K',k') \\ & s.t. \quad w(K)n + r(K)k + (1-\delta)k = k' + c \\ & \quad K' = \hat{G}(K) \end{aligned} \tag{15}$$

where c, n, k are individual consumption, labor and capital allocations, K, N are aggregate capital and labor allocations, $w(K), r(K)$ are price functions, α, δ are parameters, and $\hat{G}(K)$ is a perceived law of motion for capital.

Assume further there exists a representative firm that produces the output good using technology $F(K, L) = AK^{\theta}N^{1-\theta}$ and solving subsequent problem:

$$\max_{K_d,N_d} \quad AK_d^{\theta}N_d^{1-\theta} - r(K)K_d - w(K)N_d \tag{16}$$

Definition 2 (Recursive competitive equilibrium)**.** A Recursive Competitive Equilibrium consists of:

i) a value function $V(K,k)$ and policy functions $c(K,k), k'(K,k), n(K,k)$

ii) a perceived law of motion for aggregate capital $\hat{G}(K)$

iii) price functions $w(K), r(K)$

iv) firms policy functions $K_d(K), N_d(K)$

such that:

a) Given (ii) and (iii), (i) solves the household problem

b) Given (iii), (iv) solves the firm's problem

c) Markets clear:

- $K_d(K) = K$
- $N_d(K) = n(K,k)$
- $c(K,K) + k'(K,K) = F(K_d(K), N_d(K)) + (1-\delta)K$

d) Expectations are correct: $\hat{G}(K) = k'(K,K)$.

2.1 Characterizing the Recursive Competitive Equilibrium

The Lagrangian for the household problem is:

$$\mathscr{L} = \log(c) + \alpha \log(1-n) + \beta V(K', k') + \lambda[k' + c - w(K)n - r(K)k - (1-\delta)k] \quad (17)$$

with the following first order conditions:

$$\begin{aligned} c &: \quad \frac{1}{c} = \lambda \\ n &: \quad \frac{\alpha}{1-n} = \lambda w(K) \\ k' &: \quad \lambda = \beta V_{k'}(K', k') \end{aligned}$$

which can be combined into a **labor-leisure optimality condition**:

$$\frac{\alpha}{1-n} = \frac{w(K)}{c} \quad (18)$$

To derive the envelope condition, first write the value function with the optimal values for $c(K,k), n(K,k), k'(K,k)$ and replacing for $c(K,k)$ using the budget constraint:

$$V(K,k) = \log\Big(w(K)n(K,k) + r(K)k + (1-\delta)k - k'(K,k)\Big) + \alpha \log(1 - n(K,k)) + \beta V(K', k')$$

then take the derivative of the value function with respect to k, ignoring any indirect effects through the policy functions:

$$\frac{\partial V(K,k)}{\partial k}\Big|_{c(K,k)=c, n(K,k)=n, k'(K,k)=k'} = \frac{r(K) + (1+\delta)}{c}$$

Shifting this one period forward and using the FOC for k' yields the **euler equation**:

$$\frac{1}{c} = \beta \frac{1}{c'}[r(K') + (1-\delta)] \quad (19)$$

The firm has the following optimality conditions:

$$\begin{aligned} \theta A K_d^{\theta-1} N_d^{1-\theta} &= r(K) \quad (20) \\ (1-\theta) A K_d^{\theta} N_d^{-\theta} &= w(K) \quad (21) \end{aligned}$$

By market clearing:

$$
\begin{aligned}
K_d &= K = \int_0^1 k_i di && (22)\\
N_d &= N = \int_0^1 n_i di && (23)\\
AK^{\theta}N^{1-\theta} + (1-\delta)K &= K' + C && (24)
\end{aligned}
$$

and that expectations are correct:

$$\hat{G}(K) = K(K, K) \tag{25}$$

The optimality coinditions above characterize the RCE.

3 Solving for the steady state

By comparing the optimality conditions in either version, you can see that they are identical. Therefore, we can use either of them to derive the steady-state of this economy.

In order to characterize the **steady-state** for this economy, we simply assume quantities are identical across time and arrive at the following set of equations. The goal is to solve for every variable in terms of parameters.

We start from the Euler Equation, which yields:

$$\frac{1}{c} = \beta \frac{1}{c}[r + (1-\delta)] \iff r = \beta^{-1} - (1-\delta)$$

which is in terms of parameters. From there, we can solve for the capital to output ratio in terms of parameters, using one of the FOCs of the firm:

$$\theta \frac{Y}{K} = r \implies \frac{K}{Y} = \frac{\theta}{r}$$

which is in terms of parameters. Additionally, using the goods market clearance condition in the steady state, we can write:

$$Y + (1-\delta)K = K + C \implies \frac{C}{Y} = 1 - \delta \frac{K}{Y}$$

which is in terms of parameters. We then move to labor leisure condition, which, combined with the optimality condition for the firm, yields:

$$\frac{1}{C}(1-\theta)\frac{Y}{N} = \frac{\alpha}{1-N} \iff N = \frac{\left[\frac{1}{\alpha}\left(\frac{C}{Y}\right)^{-1}(1-\theta)\right]}{\left[1 + \frac{1}{\alpha}\left(\frac{C}{Y}\right)^{-1}(1-\theta)\right]}$$

which is in terms of parameters. Moving to the production function, note that we can rewrite:

$$Y = A\left(\frac{K}{Y}\right)^{\theta} N^{1-\theta} Y^{\theta} \iff Y = \left[A\left(\frac{K}{Y}\right)^{\theta} N^{1-\theta}\right]^{\frac{1}{1-\theta}}$$

where all terms on the RHS are in terms of parameters. Now that we have steady state output and labor, we can express wages in terms of parameters:

$$w = (1-\theta)\frac{Y}{N}$$

Therefore, our steady state can be fully characterized by the following equations:

$$\begin{aligned}
r &= \beta^{-1} - (1-\delta) \\
\frac{K}{Y} &= \frac{\theta}{r} \\
\frac{C}{Y} &= 1 - \delta\frac{K}{Y} \\
N &= \frac{\left[\frac{1}{\alpha}\left(\frac{C}{Y}\right)^{-1}(1-\theta)\right]}{\left[1 + \frac{1}{\alpha}\left(\frac{C}{Y}\right)^{-1}(1-\theta)\right]} \\
Y &= \left[A\left(\frac{K}{Y}\right)^{\theta} N^{1-\theta}\right]^{\frac{1}{1-\theta}} \\
w &= (1-\theta)\frac{Y}{N}
\end{aligned}$$