Discussion Session 5: Neoclassical Growth with Endogenous Labor Supply

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Time is discrete and there are infinitely many periods indexed by $t \in \{0, 1, 2, \cdots\}$. Households are indexed by i and uniformly distributed over the unit interval: $i \sim \text{Uniform}(0, 1)$. Each of them has a continuous and strictly concave utility function $u(c, \ell)$:

$$u_i(c, \ell) = \frac{c^{1-\gamma} - 1}{1 - \gamma} - \psi \frac{\ell^{1+\phi} - 1}{1 + \phi}$$

representing continuous quasi-concave preferences over consumption and labor.

Households supply ℓ_t labor units to the market at price w_t per unit. Each household can also accumulate assets, which we call k_t. They can borrow or lend out capital k_t for price r_t. Every period, assets depreciate at rate $\delta \in [0, 1]$.

Both the goods and the factor markets are perfectly competitive. There are infinitely many identical firms endowed with a constant returns to scale technology $Y = F(K, L) = AK^\theta L^{1-\theta}$. Each firm sources capital and labor from factor markets and chooses its input bundles to maximize profits (π_t), which they pay out as dividends to the households.

1. [5 points] Taking k_0 as given, define the Sequence of Markets Equilibrium.

- Households' Problem

$$\begin{aligned} v(K_0, k_0) \equiv \quad & \max_{\{c_t, \ell_t, k_{t+1}\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma} - 1}{1 - \gamma} - \psi \frac{\ell_t^{1+\phi} - 1}{1 + \phi} \right] \\ & s.t. \quad c_t + k_{t+1} \leq r_t k_t + w_t \ell_t + (1 - \delta) k_t + \pi_t \\ & \qquad k_0 \text{ given} \end{aligned}$$

- Firms' Problem

$$(\forall t) \quad \pi_t(K) \equiv \max_{\{K_t^d, L_t^d\}} A(K_t^d)^\theta (L_t^d)^{1-\theta} - r_t K_t^d - w_t N_t^d$$

Definition 1 (Sequence-of-markets equilibrium)**.** A sequence-of-markets equilibrium consists of:

(i) household choice streams for consumption, work, and savings: $\{c_t, \ell_t, k_{t+1}\}_{t=0}^{\infty}$.
(ii) firms' choice streams for capital and labor demand: $\{K_t^d, L_t^d\}_{t=0}^{\infty}$.
(iii) prices streams: $\{w_t, r_t\}_{t=0}^{\infty}$.

Such that:

(a) Given (iii), (i) solves the households' problem.
(b) Given (iii), (ii) solves the firms' problem.
(c) Markets clear:
  - $L_{d,t} = L_t = \int_0^1 \ell_t(i)di$
  - $K_{d,t} = K_t = \int_0^1 k_t(i)di$
  - $C_t + K_{t+1} = A(K_t^d)^\theta (L_t^d)^{1-\theta} + (1-\delta)K_t$
  - Due to free entry, there are zero profits: $\pi_t = 0$.

- [5 points] Express the household problem recursively and define the Recursive Competitive Equilibrium.

$$\begin{aligned}
v(K_0, k_0) &\equiv \max_{\{c_t,\ell_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}-1}{1-\gamma} - \psi\frac{\ell_t^{1+\phi}-1}{1+\phi}\right] \\
&= \max_{\{c_t,\ell_t,k_{t+1}\}_{t=0}^{\infty}} \left\{\left[\frac{c_0^{1-\gamma}-1}{1-\gamma} - \psi\frac{\ell_0^{1+\phi}-1}{1+\phi}\right] + \sum_{t=1}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}-1}{1-\gamma} - \psi\frac{\ell_t^{1+\phi}-1}{1+\phi}\right]\right\} \\
&= \max_{\{c_t,\ell_t,k_{t+1}\}_{t=0}^{\infty}} \left\{\left[\frac{c_0^{1-\gamma}-1}{1-\gamma} - \psi\frac{\ell_0^{1+\phi}-1}{1+\phi}\right] + \sum_{j=1}^{\infty} \beta^j \left[\frac{c_j^{1-\gamma}-1}{1-\gamma} - \psi\frac{\ell_j^{1+\phi}-1}{1+\phi}\right]\right\} \\
&= \max_{\{c_t,\ell_t,k_{t+1}\}_{t=0}^{\infty}} \left\{\left[\frac{c_0^{1-\gamma}-1}{1-\gamma} - \psi\frac{\ell_0^{1+\phi}-1}{1+\phi}\right] + \beta\sum_{j=0}^{\infty} \beta^j \left[\frac{c_{j+1}^{1-\gamma}-1}{1-\gamma} - \psi\frac{\ell_{j+1}^{1+\phi}-1}{1+\phi}\right]\right\} \\
&= \max_{\{c_0,\ell_0,k_1\}} \left[\frac{c_0^{1-\gamma}-1}{1-\gamma} - \psi\frac{\ell_0^{1+\phi}-1}{1+\phi}\right] + \beta v(K_1, k_1)
\end{aligned}$$

Noting that the starting period is arbitrary this transformation is valid not only for $t = 0, 1$, but for any $t, t+1 \in \{0, 1, \cdots\}$, we can drop the time subscripts and denote:

$$v(K, k) = \max_{\{c,\ell,k'\}} \left[\frac{c^{1-\gamma}-1}{1-\gamma} - \psi\frac{\ell^{1+\phi}-1}{1+\phi}\right] + \beta v(K', k')$$

- Households' Problem:

$$v(K,k) = \max_{\{c,\ell,k'\}} \left[\frac{c^{1-\gamma}-1}{1-\gamma} - \psi\frac{\ell^{1+\phi}-1}{1+\phi}\right] + \beta v(K',k')$$
$$s.t. \quad c + k' \leq w(K)\ell + r(K)k + (1-\delta)k + \pi(K), \qquad K' = \hat{G}(K)$$

- Firms' Problem:

$$\pi(K) = \max_{\{K^d,L^d\}} A(K^d)^\theta (L^d)^{1-\theta} - r(K)K^d - w(K)N^d$$

Definition 2 (Recursive competitive equilibrium)**.** A Recursive Competitive Equilibrium consists of:

i) a value function $v(K,k)$ and policy functions $c(K,k), k'(K,k), \ell(K,k)$
ii) a perceived law of motion for aggregate capital $\hat{G}(K)$
iii) price functions $w(K), r(K)$
iv) firms policy functions $K_d(K), N_d(K), \pi(K)$

such that:

a) Given (ii) and (iii), (i) solves the household problem
b) Given (iii), (iv) solves the firm's problem
c) Markets clear:
   - $K^d(K) = K \equiv \int_0^1 k(i)di$
   - $L^d(K) = \ell(K,K) = \int_0^1 \ell(K,k)(i)di$
   - $c(K,K) + k'(K,K) = A[K^d(K)]^\theta[L^d(K)]^{1-\theta} + (1-\delta)K$
d) Expectations are correct: $\hat{G}(K) = k'(K,K)$.
e) Due to free entry, profits are zero $\pi(K) = 0$.

2. [10 points] Using either characterization, solve for the competitive markets equilibrium. Show that household allocations can be characterized by two conditions: the Euler Equation and a Labor-Leisure trade-off. Provide an economic interpretation of each condition

   [OPTION 1: Characterize the SoM Equilibrium]

   - Households' Problem

     The household Lagrangian can be expressed as such:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}-1}{1-\gamma} - \psi\frac{\ell_t^{1+\phi}-1}{1+\phi} + \lambda_t[r_t k_t + w_t\ell_t + (1-\delta)k_t + \pi_t - c_t - k_{t+1}]\right]$$

There are infinitely many first order conditions satisfying, for all $t \geq 0$:

$$\begin{aligned} c_t &: \quad c_t^{-\gamma} - \lambda_t = 0 \\ \ell_t &: \quad -\psi \ell_t^{\phi} + \lambda_t w_t = 0 \\ k_{t+1} &: \quad -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1}[r_{t+1} + (1-\delta)] = 0 \end{aligned}$$

which can be combined into an Euler Equation [EE] and a labor-leisure [LL] conditions:

$$\begin{aligned} EE &: \quad \underbrace{c_t^{-\gamma}}_{\text{marginal utility of consumption today}} = \underbrace{\beta[r_{t+1} + (1-\delta)]c_{t+1}^{-\gamma}}_{\text{marginal utility of consuming savings tomorrow}} \\ LL &: \quad \underbrace{\psi \ell_t^{\phi}}_{\text{marginal disutility of work}} = \underbrace{c_t^{-\gamma} w_t}_{\text{marginal benefit of work}} \end{aligned}$$

- Firms' Problem

  The solution for the firms' static problem satisfies:

$$K_t^d = \begin{cases} 0 \text{ if } \theta \frac{Y_t}{K_t^d} < r_t \\ K_t^d = \left(\frac{\theta A}{r_t}\right)^{\frac{1}{1-\theta}} \cdot L_t^d \text{ if } \theta \frac{Y_t}{K_t^d} = r_t \\ \infty \text{ if } \theta \frac{Y_t}{K_t^d} > r_t \end{cases} \qquad L_t^d = \begin{cases} 0 \text{ if } (1-\theta) \frac{Y_t}{L_t^d} < w_t \\ L_t^d = \left(\frac{(1-\theta) A}{w_t}\right)^{\frac{1}{1-\theta}} \cdot K_t^d \text{ if } (1-\theta) \frac{Y_t}{L_t^d} = w_t \\ \infty \text{ if } (1-\theta) \frac{Y_t}{L_t^d} > w_t \end{cases}$$

  Clearly, if an equilibrium exists, it must be the case that:

$$\frac{Y_t}{K_t^d} = r_t, \qquad (1-\theta)\frac{Y_t}{L_t^d} = w_t$$

[OPTION 2: Characterize the Recursive Competitive Equilibrium]

- Households' Problem

  The household Lagrangian can be expressed as such:

$$\mathcal{L} = \left[\frac{c^{1-\gamma} - 1}{1-\gamma} - \psi \frac{\ell^{1+\phi} - 1}{1+\phi}\right] + \beta v(K', k') + \lambda[r(K)k + w(K)\ell + (1-\delta)k + \pi(K) - c - k']$$

  There are infinitely many first order conditions satisfying, for all $t \geq 0$:

$$\begin{aligned} c &: \quad c^{-\gamma} - \lambda = 0 \\ \ell &: \quad -\psi \ell^{\phi} + \lambda w(K) = 0 \\ k' &: \quad -\lambda + \beta v_{k'}(K, K') = 0 \end{aligned}$$

In order to derive $v_{k'}(K,K')$ we substitute for c in the value function, and for the optimal choices $k'(K,k), \ell(K,k)$ take the derivative with respect to the state variable k ignoring the reoptimization effect (Envelope Condition):

$$\begin{aligned}
v(K,k) &= \left[\frac{[r(K)k + w(K)\ell(K,k) + (1-\delta)k + \pi - k'(K,k)]^{1-\gamma} - 1}{1-\gamma} \right. \\
&- \left. \psi \frac{\ell(K,k)^{1+\phi} - 1}{1+\phi} \right] + \beta v(K', k'(K,k)) \\
\implies v_k(K,k) &= [c(K,k)]^{-\gamma}[r(K) + (1-\delta)]
\end{aligned}$$

Combining the FOCs and the envelope condition, we can derive an Euler Equation [EE] and a labor-leisure [LL] conditions:

$$\begin{aligned}
EE &: \underbrace{c^{-\gamma}}_{\text{marginal utility of consumption today}} = \underbrace{\beta[r(K') + (1-\delta)](c')^{-\gamma}}_{\text{marginal utility of consuming savings tomorrow}} \\
LL &: \underbrace{\psi \ell^{\phi}}_{\text{marginal disutility of work}} = \underbrace{c^{-\gamma} w(K)}_{\text{marginal benefit of work}}
\end{aligned}$$

- Firms' Problem
  The solution for the firms' static problem satisfies:

$$K^d = \begin{cases} 0 \text{ if } \theta \frac{Y}{K^d} < r(K) \\ K^d = \left(\frac{\theta A}{r(K)}\right)^{\frac{1}{1-\theta}} \cdot L^d \text{ if } \theta \frac{Y}{K^d} = r(K) \\ \infty \text{ if } \theta \frac{Y}{K^d} > r(K) \end{cases} \qquad L^d = \begin{cases} 0 \text{ if } (1-\theta) \frac{Y}{L^d} < w(K) \\ L^d = \left(\frac{(1-\theta)A}{w}\right)^{\frac{1}{1-\theta}} \cdot K \text{ if } (1-\theta) \frac{Y}{L^d} = w(K) \\ \infty \text{ if } (1-\theta) \frac{Y}{L^d} > w(K) \end{cases}$$

  Clearly, if an equilibrium exists, it must be the case that:

$$\theta \frac{Y}{K^d} = r(K), \qquad (1-\theta) \frac{Y}{L^d} = w(K)$$

3. [5 points] Characterize the model's steady-state. Does the economy exhibit sustained economic growth in steady state? Why or why not.

   We drop subscripts and primes and state that every variable is expressed at the steady-state. From the Euler Equation, we know that:

$$1 = \beta[r + (1-\delta)] \implies r = \frac{1}{\beta} - (1-\delta)$$

   which is in terms of parameters. From there, we can solve for the capital to output ratio in terms of parameters, using one of the FOCs of the firm:

$$\theta\frac{Y}{K} = r \implies \frac{K}{Y} = \frac{\theta}{r}$$

such that K/Y is in terms of parameters.

From above, we know that $r = F_K(K, L), w = F_L(K, L)$. Furthermore, since $F(\cdot, \cdot)$ is homogeneous of degree one, by Euler's Theorem, $F_K(K, L)K + F_L(K, L)L = Y$. Therefore, $rK + wL = Y$. We also know that, since there is a measure one of households uniformly distributed over the unit interval, $k = K, \ell = L, c = C$.

We can then rewrite the budget constraint as:

$$C_t + K_{t+1} = r_t K_t + w_t L_t + (1-\delta)K_t = Y_t + (1-\delta)K_t$$

In the steady state, we can write:

$$Y + (1-\delta)K = K + C \implies \frac{C}{Y} = 1 - \delta\frac{K}{Y}$$

which is in terms of parameters. From the labor-leisure condition and the optimality condition for the firm, we know:

$$(1-\theta)\frac{Y}{L} = w, \qquad \psi L^{\phi} = C^{-\gamma} w(K) \implies \psi L^{\phi} = C^{-\gamma}(1-\theta)\frac{Y}{L}$$

$$\begin{aligned}
\psi L^{\phi} &= C^{-\gamma}(1-\theta)\frac{Y}{L} \\
L &= \left[\frac{1}{\psi}(C/Y)^{-\gamma}(1-\theta)\right]^{\frac{1}{1+\phi}} Y^{\frac{1-\gamma}{1+\phi}}
\end{aligned}$$

Now replace this into the production function:

$$\begin{aligned}
Y &= AK^{\theta}L^{1-\theta} \\
Y &= A\left(\frac{K}{Y}\right)^{\theta} L^{1-\theta} Y^{\theta} \\
Y &= A\left(\frac{K}{Y}\right)^{\theta} \left[\frac{1}{\psi}(C/Y)^{-\gamma}(1-\theta)\right]^{\frac{1-\theta}{1+\phi}} Y^{\frac{(1-\gamma)(1-\theta)+\theta(1+\phi)}{1+\phi}} \\
Y^{\frac{(1-\theta)(\phi+\gamma)}{1+\phi}} &= A\left(\frac{K}{Y}\right)^{\theta} \left[\frac{1}{\psi}(C/Y)^{-\gamma}(1-\theta)\right]^{\frac{1-\theta}{1+\phi}} \\
Y &= \left[A\left(\frac{K}{Y}\right)^{\theta}\right]^{\frac{1+\phi}{(1-\theta)(\phi+\gamma)}} \left[\frac{1}{\psi}(C/Y)^{-\gamma}(1-\theta)\right]^{\frac{1}{\phi+\gamma}}
\end{aligned}$$

which is in terms of parameters. We can now solve for L:

$$L = \left[\frac{1}{\psi}(C/Y)^{-\gamma}(1-\theta)\right]^{\frac{1}{\phi+\gamma}} \left[A\left(\frac{K}{Y}\right)^{\theta}\right]^{\frac{1-\gamma}{(1-\theta)(\phi+\gamma)}}$$

and w:

$$w = (1-\theta)\frac{Y}{L}$$

We have now fully characterized the steady state in terms of parameters:

$$\begin{aligned}
r &= \frac{1}{\beta} - (1-\delta) \\
\frac{K}{Y} &= \frac{\theta}{r} \\
\frac{C}{Y} &= 1 - \delta\frac{K}{Y} \\
Y &= \left[A\left(\frac{K}{Y}\right)^{\theta}\right]^{\frac{1+\phi}{(1-\theta)(\phi+\gamma)}} \left[\frac{1}{\psi}(C/Y)^{-\gamma}(1-\theta)\right]^{\frac{1}{\phi+\gamma}} \\
L &= \left[\frac{1}{\psi}(C/Y)^{-\gamma}(1-\theta)\right]^{\frac{1}{\phi+\gamma}} \left[A\left(\frac{K}{Y}\right)^{\theta}\right]^{\frac{1-\gamma}{(1-\theta)(\phi+\gamma)}} \\
w &= (1-\theta)\frac{Y}{L}
\end{aligned}$$

4. [5 points] Suppose TFP grows at a constant rate $\mu > 0$ so that $A_{t+1} = A_t \cdot \exp(\mu)$. Does the model admit a balanced growth path? Explain how your answer depends the value of γ.

   In this case, the problem no longer satisfies the $S\&L$ assumptions since $\Gamma(k)$ is now unbounded. Using a change of variables, denote $A_t = Z_t^{1-\theta}$. You will realize that, in that case, the problem is identical to the Solow model presented in class. For full points, it is enough to state the fact above and state the growth rate $\exp(\mu/(1-\theta))$ and to state that the BGP only exists if $\gamma = 1$ (e.g. Uzawa (1961)).

   For completeness, I show below the full derivation of this growth rate (this was not necessary, but might be useful for practice).

$$\frac{A_{t+1}}{A_t} = \left(\frac{Z_{t+1}}{Z_t}\right)^{1-\theta} \implies \frac{Z_{t+1}}{Z_t} = \exp(\mu)^{\frac{1}{1-\theta}} = \exp\left(\frac{\mu}{1-\theta}\right) \implies Z_t = Z_0 \exp\left(\frac{\mu}{1-\theta}t\right)$$

Denote $\tilde{x}_t \equiv x_t/Z_t$ for any variable x for all t. We can then re-state the problem in the following way:

$$
\begin{aligned}
v(K_0, k_0) \equiv \quad \max_{\{\tilde{c}_t, \ell_t, \tilde{k}_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \hat{\beta}^t \left\{ \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} - z_t^{-(1-\gamma)} \psi \frac{\ell_t^{1+\phi}}{1+\phi} \right\} \\
s.t. \qquad & \tilde{c}_t + k_{t+1} \exp(\mu/[1-\alpha]) \leq r_t \tilde{k}_t + w_t \ell_t \frac{1}{z_t} + (1-\delta)\tilde{k}_t \\
& \tilde{k}_0 \text{ given}
\end{aligned}
$$

where $\hat{\beta} \equiv \beta \exp((1-\gamma) \cdot \mu/[1-\theta] \cdot t) z_0$. Without loss of generality, we will normalize $z_0 = 1$. Also note that we dropped constant part of the utility function for ease of exposition. Nonetheless, this is a monotonic transformation of the utility function that does not change the ordering of the underlying preference relations. This transformed problem is guaranteed to have a steady-state under the S-L assumptions if $\hat{\beta} < 1$. We will derive it below.

The Lagrangian for this problem is:

$$
\mathcal{L} = \sum_{t=0}^{\infty} \hat{\beta}^t \left\{ \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} - z_t^{-(1-\gamma)} \psi \frac{\ell_t^{1+\phi}}{1+\phi} + \lambda_t [r_t \tilde{k}_t + w_t \frac{\ell_t}{z_t} + (1-\delta)\tilde{k}_t - \tilde{c}_t - \tilde{k}_{t+1} \exp(\mu/[1-\theta])] \right\}
$$

with FOCs:

$$
\begin{aligned}
\tilde{c}_t &: \quad \tilde{c}_t^{-\gamma} = \lambda_t \\
\ell_t &: \quad z_t^{-(1-\gamma)} \psi \ell_t^{\phi} = \lambda_t \frac{w_t}{z_t} \\
\tilde{k}_{t+1} &: \quad \lambda_t \exp(\mu/[1-\theta]) = \lambda_{t+1}[r_{t+1} + (1-\delta)]
\end{aligned}
$$

To derive the euler equation condition, combine the first and third FOCs:

$$
\tilde{c}_t^{-\gamma} \exp(\mu/[1-\theta]) = \hat{\beta}[r_{t+1} + (1-\delta)]\tilde{c}_{t+1}^{-\gamma}
$$

To derive the labor-leisure condition, combine the two first FOCs:

$$
\begin{aligned}
z_t^{-(1-\gamma)} \psi \ell_t^{\phi} &= \frac{w_t}{z_t} \tilde{c}_t^{-\gamma} \\
\psi \ell_t^{\phi} &= \frac{w_t}{z_t} z_t \left(\frac{c_t}{z_t} \right)^{-\gamma} z_t^{-\gamma} \\
\psi \ell_t^{\phi} &= w_t c_t^{-\gamma}
\end{aligned}
$$

so this model delivers a standard labor-leisure condition. Since the firms problem is static, the optimality conditions do not change. Note that:

$$\tilde{Y}_t = \frac{Y_t}{z_t} = (K_t/z_t)^\theta N_t^{1-\theta} = \tilde{K}_t^\theta N_t^{1-\theta}$$

Clearly, then the FOCs of the firm problem satisfy:

$$\theta \frac{\tilde{Y}_t}{\tilde{K}_t^d} = r_t, \qquad (1-\theta)\frac{\tilde{Y}_t}{L_t^d} = \frac{w_t}{z_t}$$

Note that we can write $\tilde{Y}_t = r_t \tilde{K}_t^d + \frac{w_t}{z_t} L_t^d$. Also, by market clearing, $\tilde{K}_t^d = \tilde{k}_t$ and $L_t^d = \ell_t$. Therefore, we can re-write the budget constraint as:

$$\tilde{c}_t + \tilde{k}_{t+1} = \tilde{Y}_t + (1-\delta)\tilde{k}_t$$

From evaluating the euler equation of transformed problem at its steady state point, we know that:

$$\hat{\beta}[r + (1-\delta)] = \exp(\mu/[1-\theta]) \implies r = \hat{\beta}\exp(\mu/[1-\theta]) - (1-\delta) = \beta^{-1}\exp(-(1-\gamma)) - (1-\delta)$$

which is terms of parameters. From the first order condition of the firm:

$$\frac{\tilde{K}}{\tilde{Y}} = \frac{r}{\theta}$$

which is in terms of parameters. From the resource constraint, we can write:

$$\frac{\tilde{C}}{\tilde{Y}} = 1 - \delta\frac{\tilde{K}}{\tilde{Y}}$$

which is in terms of parameters. Now turn to the labor leisure condition combined with the optimality condition for the firm. After solving for L, we can write:

$$\begin{aligned}
\psi L^\phi &= wC^{-\gamma} \\
\psi L^{1+\phi} &= (1-\theta) z \tilde{Y} C^{-\gamma} \qquad \text{(from firm's FOC)} \\
\psi L^{1+\phi} &= (1-\theta) Y C^{-\gamma} \\
\iff L &= \left[\frac{1}{\psi}(1-\theta)\frac{C}{Y}^{-\gamma} Y^{1-\gamma}\right]^{\frac{1}{1+\phi}} \\
L &= \left[\frac{1}{\psi}(1-\theta)\left(\frac{\tilde{C}}{\tilde{Y}}\right)^{-\gamma}(\tilde{Y}z)^{1-\gamma}\right]^{\frac{1}{1+\phi}}
\end{aligned}$$

Now use the production function:

$$\begin{aligned} \tilde{Y} &= \tilde{K}^{\theta} L^{1-\theta} \\ \iff \tilde{Y} &= \left(\frac{\tilde{K}}{\tilde{Y}}\right)^{\theta} \left[\frac{1}{\psi}(1-\theta)\left(\frac{\tilde{C}}{\tilde{Y}}\right)^{-\gamma} (\tilde{Y}z)^{1-\gamma}\right]^{\frac{1-\theta}{1+\phi}} \end{aligned}$$

We now realize that L, Y depend on z whenever $\gamma \neq 1$, so the model is not consistent with a BGP with a fixed labor supply. If $\gamma = 1$, however, there exists a unique BGP with a fixed labor supply. This is a illustration of Urzawa's Steady-State Growth Theorem. In the particular case $\gamma = 1$, then:

$$\begin{aligned} L &= \left[\frac{1}{\psi}(1-\theta)\left(\frac{\tilde{C}}{\tilde{Y}}\right)^{-\gamma}\right]^{\frac{1}{1+\phi}} \\ \tilde{Y} &= \left(\frac{\tilde{K}}{\tilde{Y}}\right)^{\theta} \left[\frac{1}{\psi}(1-\theta)\left(\frac{\tilde{C}}{\tilde{Y}}\right)^{-\gamma}\right]^{\frac{1-\theta}{1+\phi}} \end{aligned}$$

We now derive the growth rates. It is trivial that if $\tilde{Y}_{t+1} = \tilde{Y}_t$, then $g_y = g_z$. This also holds true for capital: $g_k = g_z$. Finally, note that, while labor is constant, wages grow over time. From the FOC of the firm:

$$w_t L_t = (1-\theta)\tilde{Y}_t z_t = (1-\theta) Y_t \iff \frac{w_{t+1}}{w_t} = \frac{Y_{t+1}}{Y_t} = g_y = g_z$$