Discussion Session 6: Endogenous Growth with Human Capital Externalities

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September 1, 2023

An economy is populated by measure one of households whose preferences are:

$$\sum_{t=0}^{\infty} \beta^t \log(c_t)$$

where β is the discount factor and c_t is consumption at time t. Each household posesses human capital h_t which they supply to the market at price w_t. There is a final goods producer who uses the following technology:

$$c_t = \phi_t h_t E_t$$

where ϕ_t is the (endogenous) fraction of the household's human capital supplied to the market and E_t is an externality described below. Consumption is produced by competitive markets that hire human capital, sell output to the households, and take E_t as given each period.

The externality is given by $E_t = H_t^{\eta}$, where H_t is the average human capital in the economy at t and $\eta > 0$ is a constant. Intuitively, the externality captures the idea that production is more efficient when the average worker is more knowledgable. Households can also supply their labor to the education sector, which is also competitive, and which has production function

$$x_t = A(1-\phi_t)h_t$$

where x_t is the output of new human capital and $A > 0$ is a constant. The law of motion for human capital is given by $h_{t+1} = h_t + x_t$.

A **Formulate the household's problem as a dynamic programming problem. Define a recursive competitive equilibrium.**

The household's recursive problem is

$$\begin{aligned} V(H,h) = &\max_{h',c,x} \log(c) + \beta V(H',h') \\ s.t.\ \ & c + p(H)x = w(H)h \\ & h' = h + x \\ & H' = G(H) \end{aligned}$$

And the firms problem is

$$\max_{h_C^d} \quad h_C^d E - w(H) h_C^d$$

$$\max_{h_X^d} \quad p(H) A h_X^d - w(H) h_X^d$$

A recursive competitive equilibrium is

(a) A value function $V(H,h)$ and policy functions $c(H,h)$, $x(H,h)$, and $h'(H,h)$ for the household.

(b) Decision rules $h_C^d(H)$ for the consumption good firm

(c) Decision rules $h_X^d(H)$ for the education firm

(d) Price functions $w(H)$ and $p(H)$.

(e) A perceived law of motion $\hat{G}(H)$.

such that

(a) Given (4) and (5), (1) solves the household's problem.

(b) Given (4), (2) solves the consumption good firm's problem

(c) Given (4), (3) solves the education firm's problem

(d) Markets clear:
   - $h_C^d(H) + h_x^d(H) = H$
   - $x(H,H) = A h_X^d$
   - $c(H,H) = h_C^d E$

(e) Perceptions are correct: $\hat{G}(H) = h'(H,H)$.

B **Derive the optimality conditions of households and firms and characterize the equilibrium.**

The Lagrangian of the household isÇ

$$\mathcal{L} = \log(c) + \beta V(H',h') + \lambda \left[w(H)h - c - p(H)[h' - h] \right]$$

with FOCs satisfying:

$$
\begin{aligned}
c &: \quad \frac{1}{c} - \lambda = 0 \\
h' &: \quad \beta V_{h'}(H', h') - \lambda p(H) = 0
\end{aligned}
$$

In order to derive $V_{h'}(H', h')$, rewrite the value function substituting in the constraints and, using the envelope condition, take derivative of the value function ignoring the (indirect) reoptimization effect:

$$V(H,h) = \log\left(w(H)h - p(H)\left[h'(H,h) - h\right]\right) + \beta V(H', h')$$

which implies:

$$V_h(H,h) = \frac{w(H) + p(H)}{c(H,h)}$$

Combining the two FOCs and using the result derived above yields the euler equation of this problem:

$$\frac{p(H)}{c(H,h)} = \beta\left[\frac{w(H')}{c(H',h')} + \frac{p(H')}{c(H',h')}\right]$$

The left hand side in this equation denotes the marginal utility of consumption equivalent of buying one extra unit of human capital. The right hand side is the discounted marginal utility of consuming tomorrow additional wage from investing in one extra unit of human capital as well as the consumption equivalent of one extra unit of human capital tomorrow. At the optimal, these two values must be the same.

For the firm's optimality conditions, this problem is standard. In equilibrium, it must be that:

$$E = w(H) = p(H)A$$

If the wages are not the same across sectors, then there is no production in one of the sectors, which cannot be an equilibrium and will not satisfy the euler equation (which requires positive consumption and investment in all periods). If the wages do not equal marginal products of labor, then it cannot be an equilibrium either, because, in this case, demand for labor will be either zero or infinity.

C **Characterize the balanced growth path of the economy. Characterize the growth rates of consumption and human capital on the balanced growth path.**

We will characterize everything in terms of gross growth rates. That is, for any variable x_t, $x_{t+1} = g_x x_t$ for every $t > t_0$, where t_0 is the first period when the economy is in a balanced growth path.

Note that in the balanced growth path (BGP) we are looking for a **constant** growth rate. Therefore, in the BGP, we can always solve for endogenous variables recursively in terms of growth rates and starting values:

$$x_t = g_x x_{t-1} = g_x^2 x_{t-2} = \cdots = g_x^{(t-t_0)} x_{t_0}$$

First, we note that in BGP ϕ must constant. Suppose not. If $g_\phi > 1$, then $\lim_{t\to\infty} \phi_t = g_\phi^t \phi_{t_0} = \infty$. However, since $\phi \in [0,1]$, this is a contradiction, so $g_\phi \leq 1$. Now, suppose $g_\phi > 1$. Then $\lim_{t\to\infty} \phi_t = g_\phi^t \phi_{t_0} = 0$. However, this means that the share of labor allocated to production of the consumption good is zero, which implies that there is no production of such a good. But this cannot happen in equilibrium, since $\lim_{c\to 0} u'(c) = \infty$. Therefore, $g_\phi = 1$.

Our main variables of interest are c and h. From the production function from consumption goods, we have that $c = \phi HE$. At the balanced growth path, we can write this production function as:

$$\frac{c_t}{H_t E_t} = \phi_t \iff \frac{g_c^{(t-t_0)} c_{t_0}}{g_E^{(t-t_0)} E_{t_0} \cdot g_h^{(t-t_0)} H_{t_0}} = g_\phi^{(t-t_0)} \phi_{t_0}$$

dividing both sides by ϕ_{t_0} results in

$$\frac{g_c^{(t-t_0)}}{g_E^{(t-t_0)} \cdot g_h^{(t-t_0)}} = g_\phi^{(t-t_0)} \iff \frac{g_c}{g_E \cdot g_h} = g_\phi$$

Now, note that $E = H^\eta$, which, by the same logic, implies that $g_E = g_h^\eta$. Additionally, we have shown above that $g_\phi = 1$. Therefore:

$$g_c = g_h^{1+\eta}$$

The first order condition of the final goods firm implies that $E = w$. Substituting for the definition of E yields $H^\eta = w$. Therefore, using similar steps, we can show $g_w = g_h^\eta$.

Using condition for education firm: $pA = w$, therefore $g_p = g_w = g_h^\eta$. Similarly, using production function for education: $x = A(1-\phi)H$, and therefore: $g_x = g_h$

Now we notice that all growth rates are in terms of g_h, so to fully characterize the BGP, we need to characterize g_h.

In order to find g_h, we use the optimality condition for the household:

$$\beta\frac{(w'+p')}{c'}=\frac{p}{c} \implies \frac{c'}{c}=\beta\left(\frac{w'}{p}+\frac{p'}{p}\right)$$

Since from education firm $p=\frac{w}{A}$:

$$g_c=\beta\left(Ag_w+g_p\right)$$

Plugging for these growth rates form before:

$$g_h^{\eta+1}=\beta\left(Ag_h^{\eta}+g_h^{\eta}\right)$$

Rearranging:

$$g_h=\beta\left(A+1\right)$$

Then we have the following Balanced Growth Path:

$$\begin{aligned} g_c &= g_h^{1+\eta} \\ g_p &= g_w = g_h^{\eta} \\ g_x &= g_h \\ g_h &= \beta\left(A+1\right) \end{aligned}$$

We can also characterize ϕ. Let's look at accumulation equation for an arbitrary $t>t_0$:

$$h_{t+1}=h_t+x_t \iff g_h^{t-t_0+1}h_{t_0}=g_h^{t-t_0}h_{t_0}+g_x^{t-t_0}x_{t_0}$$

Plugging in the production function for $x_{t_0}=A(1-\phi)h_{t_0}$ and the fact that $g_x=g_h$:

$$g_h^{t-t_0+1}h_{t_0}=g_h^{t-t_0}h_{t_0}+g_h^{t-t_0}A(1-\phi)h_{t_0}$$

dividing both sides through by h_{t_0}:

$$g_h^{t-t_0+1}=g_h^{t-t_0}(1+A(1-\phi)) \implies g_h=1+A(1-\phi)$$

Since we know $g_h=\beta(A+1)$ we can solve for the value of ϕ in BGP:

$$\phi^*=\frac{(1-\beta)(A+1)}{A}$$

Initial conditions follow from this ϕ, and the initial allocation of human capital h_{t_0} in the economy.

D **Now imagine resources are allocated by a benevolent social planner who internalizes the externality. Characterize the balanced growth path under the planner's solution, including the growth rates of c_t and h_t. How do the social planner's allocation and market allocation differ?**

A social planner solves:

$$
\begin{aligned}
V(H) = \max_{C,H',\Phi} & \left\{ \log(C) + \beta V(H') \right\} \\
\text{s.t.} \quad & C = \Phi H \cdot H^{\eta} \\
& X = A(1-\Phi)H \\
& H' = H + X
\end{aligned}
$$

$$
\mathcal{L} = \left\{ \log(\Phi H^{1+\eta}) + \beta V(H') + \lambda(H - H' + A(1-\Phi)H) \right\}
$$

with FOCs

$$
\begin{aligned}
\Phi \quad &: \quad \frac{1}{\Phi H^{1+\eta}} H^{1+\eta} - AH\lambda = 0 \\
H' \quad &: \quad \beta V'(H') - \lambda = 0
\end{aligned}
$$

which implies:

$$
\frac{1}{A\Phi H} = \beta V'(H')
$$

Using the envelope condition (this time it will be convenient to use the trick of taking the derivative of the Lagrangia wrt to H, because the algebra gets nasty when replacing all the constraints in):

$$
V'(H') = \frac{1+\eta}{H'} + \lambda'(1 + A(1-\Phi')) = \frac{1+\eta}{H'} + \frac{1}{A\Phi' H'}(1 + A(1-\Phi'))
$$

Therefore:

$$
\begin{aligned}
\frac{1}{A\Phi H} &= \beta\left(\frac{1+\eta}{H'} + \frac{1}{A\Phi' H'}(1 + A(1-\Phi'))\right) \\
\iff \frac{1}{\Phi H} &= \beta\left(\frac{A(1+\eta)\Phi' + 1 + A(1-\Phi')}{H'\Phi'}\right) \\
\frac{1}{\Phi H} &= \beta\left(\frac{A\Phi' + A\eta\Phi' + 1 + A - A\Phi'}{H'\Phi'}\right) \\
\frac{1}{\Phi H} &= \beta\left(\frac{A\eta\Phi' + 1 + A}{H'\Phi'}\right) \\
\frac{1}{\Phi H} &= \beta\left(\frac{A\eta}{H'} + \frac{(1+A)}{H'\Phi'}\right) \\
\iff \frac{H'}{H} &= \beta\left(A\eta\Phi + (1+A)\frac{\Phi}{\Phi'}\right)
\end{aligned}
$$

E Characterize BGP under planner's problem

Following the same reasoning as before, we note that Φ is constant in the BGP, meaning its growth rate is $g_\Phi = 1$.

From the production function from consumption goods, we have that $\frac{C}{H^{1+\eta}} = \Phi$ but since Φ is constant, this gives us that $g_C = g_H^{1+\eta}$.

Now using production function for education: $X = A(1-\Phi)H$, and therefore: $g_X = g_H$.

We need to find g_H to have BGP fully characterized. From human capital accumulation equation: $g_H = 1 + \frac{X}{H}$ Plugging in the production function of $X = A(1-\Phi)H$ we get: $g_H = 1 + A(1-\Phi)$.

The last step is to derive Φ. Start from:

$$
\begin{aligned}
\frac{H'}{H} &= \beta\left(A\eta\Phi + \beta\frac{(1+A)}{H'}\frac{\Phi}{\Phi'}\right) \\
g_H &= \beta\left(A\eta\Phi + (1+A)g_\Phi^{-1}\right) = \beta\left(A\eta\Phi + (1+A)\right)
\end{aligned}
$$

Since we know $g_H = 1 + A(1-\Phi)$:

$$
1 + A(1-\Phi) = \beta\left(A\eta\Phi + (1+A)\right) \iff \Phi = \frac{(1+A)(1-\beta)}{A(1+\beta\eta)}
$$

From here we can retrieve g_h:

$$g_H = 1 + A(1-\Phi) = \beta(1+A)\frac{(1+\eta)}{1+\beta\eta}$$

Notice from here that since $\frac{(1+\eta)}{1+\beta\eta} > 1$ we have that g_H will be bigger under social planner (in RCE, $g_H = \beta(1+A)$). This is because when deciding H, social planner takes into account the externality that a higher H will increase the productivity for the final firm, so g_H will be higher. Notice also that here when $\eta = 0$ (no externality), we get the same growth rate as for the RCE.