Discussion Session 7: The Romer Model

Carlos Góes

September 1, 2023

In this section, we will present a groundbreaking model by Nobel Laureate Paul Romer [**?**]. The model's main insight is that innovation through the introduction of new ideas and products have spillovers. That is, society as a whole benefits from innovation, even if they did not innovate individually. This happens because **ideas are not rival goods** and **ideas are only partially excludable**. For instance, my use of the Pythagorean theorem does not prevent others from making use of it and I cannot exclude others from using it if they want to.

The implication for macroeconomics is that if a technology (production function) includes a nonrival input, it cannot be constant returns to scale in all its inputs. In fact, nonrivalry of ideas give rise to **increasing returns to scale** in a context of monopolistic competition.

The model has three sectors: a research and development sector, an intermediate goods sector, and a final goods sector. Households rent their human capital and buy goods from the final goods firms.

1 Characterizing the competitive equilibrium

There is measure one of households who solve the following problem:

$$\max_{a_{t+1}, c_t} \sum_{t=0}^{\infty} \beta^t \log c_t \qquad s.t. \quad c_t + a_{t+1} \leq (1 + r_{t+1})a_t + w_t n \tag{1}$$

where c_t are units of the final good. Households supply inelastically n units of labor to the market. Households can accumulate assets a_t to smooth consumption over time. The intertemporal euler equation of this simple household problem is:

$$\frac{c_{t+1}}{c_t} = \beta(1 + r_{t+1})$$

There are many perfectly competitive **final goods producers** (FGP), all endowed with the same technology:

$$Y_t = n^{1-\alpha} \int_0^{M_t} x_t(\omega)^\alpha d\omega$$

Each period, a final goods producer sources labor n_t and infinitely many intermediate inputs $x_t(\omega)$ of each variety $\omega \in [0, M_t]$. Note that there is a time-dependent measure of varieties M_t. As we will see below, this measure increases as the research and development sector produces new ideas. They maximize their profits through the following problem:

$$\max_{n_t^d, \{x_t^d(\omega)\}_{\omega \in [0,M_t]}} (n_t^d)^{1-\alpha} \int_0^{M_t} (x_t^d(\omega))^\alpha d\omega - w_t n_t^d - \int_0^{M_t} p_t(\omega) x_t^d(\omega) d\omega \qquad (2)$$

Differentiating under the integral sign in gives the demand functions:

$$(1-\alpha)(n_t^d)^{-\alpha} \int_0^{M_t} (x_t^d(\omega))^\alpha d\omega = w_t \qquad (3)$$

$$\alpha (n_t^d)^{1-\alpha} (x_t^d(\omega))^{\alpha-1} = p_t(\omega) \text{ for each } \omega \in (0, M_t) \qquad (4)$$

Rearranging, optimal demand functions for variety ω are decreasing in prices $p_t(\omega)$ and increasing in total use of labor n_t^d:

$$x_t^d(\omega) = p(\omega)^{-\frac{1}{1-\alpha}} \cdot n_t^d \cdot \alpha^{\frac{1}{1-\alpha}} \text{ for each } \omega \in [0, M_t]$$

There is an **intermediate goods producer** that has perpetual rights over the production of each variety $\omega \in [0, M_t]$. Each of them is endowed with a linear technology that transforms one unit of the final good into one unit of the intermediate good: $x(\omega) = F(y) = y$. They take marginal costs κ and demand curves $x_t^d(\omega)$ and choose optimal prices to maximize profits:

$$\begin{aligned} \max_{p_t(\omega)} \pi_t(\omega) &= p_t(\omega) x_t^d(\omega) - \kappa x_t^d(\omega) \\ &= p_t(\omega) p(\omega)^{-\frac{1}{1-\alpha}} \cdot n_t^d \cdot \alpha^{\frac{1}{1-\alpha}} - \kappa p(\omega)^{-\frac{1}{1-\alpha}} \cdot n_t^d \cdot \alpha^{\frac{1}{1-\alpha}} \\ &= \left(p(\omega)^{-\frac{\alpha}{1-\alpha}} - \kappa p(\omega)^{-\frac{1}{1-\alpha}} \right) \cdot n_t^d \cdot \alpha^{\frac{1}{1-\alpha}} \end{aligned}$$

whose optimality condition satisfy:

$$-\frac{\alpha}{1-\alpha} p_t(\omega)^{-\frac{1}{1-\alpha}} + \frac{\kappa}{1-\alpha} p_t(\omega)^{-\frac{2-\alpha}{1-\alpha}} = 0 \implies p_t(\omega) = \frac{1}{\alpha}\kappa \qquad \forall \omega \in [0, M_t]$$

Importantly, we note that the price $p_t(\omega) = \frac{1}{\alpha}\kappa$, which is a simple markup over marginal cost, does not depend on ω. They are the same for every variety. The implication is that demanded inputs $x_t(\omega)$ and profits $\pi_t(\omega)$ will also be equal for every variety. Since labor supply n is constant in every period and is inelastically supplied at the market, then $n_t^d = n \forall t$ and in equilibrium both profits and demanded inputs for each variety ω will be constant over time.

$$
\begin{aligned}
x_t^d(\omega) &= \bar{x} = \kappa^{-\frac{1}{1-\alpha}} \cdot n \cdot \alpha^{\frac{2}{1-\alpha}} \qquad \forall \omega \in [0, M_t] \\
\pi_t(\omega) &= \bar{\pi} = \frac{1}{\alpha}\kappa\bar{x} - \kappa\bar{x} = \frac{1-\alpha}{\alpha} \cdot \kappa^{-\frac{\alpha}{1-\alpha}} \cdot n \cdot \alpha^{\frac{2}{1-\alpha}} \qquad \forall \omega \in [0, M_t]
\end{aligned}
$$

We can also calculate total output:

$$
\begin{aligned}
Y_t &= n^{1-\alpha} \int_0^{M_t} x_t(\omega)^\alpha d\omega \\
&= n^{1-\alpha} \int_0^{M_t} \bar{x}^\alpha d\omega \\
&= n^{1-\alpha} \left[\kappa^{-\frac{1}{1-\alpha}} \cdot n \cdot \alpha^{\frac{2}{1-\alpha}}\right]^\alpha \cdot \int_0^{M_t} 1 d\omega \\
&= \left(\frac{\alpha^2}{\kappa}\right)^{\frac{\alpha}{1-\alpha}} \cdot n \cdot M_t
\end{aligned}
$$

and wages:

$$
w_t n = (1-\alpha)Y_t = (1-\alpha) \cdot \left(\frac{\alpha^2}{\kappa}\right)^{\frac{\alpha}{1-\alpha}} \cdot n \cdot M_t
$$

There is a **research sector in this economy**, which creates new varieties according to the following law of motion:

$$
M_{t+1} = M_t + \eta Z_t
$$

where Z_t is investment in RD in units of the final good. Once a new variety exists, its owners have a perpetual patent over its design. The economic value of a new variety net present value of producing the new varieties and selling them as intermediate inputs to final goods producers:

$$
V_t = \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \frac{1}{1+r_{t+k}} \pi_{t+j}(\omega) \right) = \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \frac{1}{1+r_{t+k}} \bar{\pi} \right)
$$

Note that entrepreneurs invest Z_t at period t, but the additional varieties $M_{t+1} - M_t$ only materialize in the following period. They fund themselves by borrowing Z_t at the assets markets at period t and repay $(1 + r_{t+1})Z_t$ at $t + 1$. Therefore, entrepreneurs will only invest at t if the present discounted value of varieties at $t + 1$ exceeds its marginal cost:

$$
\underbrace{\eta V_{t+1} Z_t}_{\text{benefit}} - \underbrace{(1 + r_{t+1}) Z_t}_{\text{cost}} \geq 0 \iff \underbrace{\frac{1}{1 + r_{t+1}} V_{t+1}}_{\text{PV of } V_{t+1} \text{ at } t} \geq \frac{1}{\eta}
$$

2 Balanced Growth Path

Along the Balanced Growth Path, we are looking for constant growth rates of consumption g_c and output g_y. From the euler equation, it is clear that the interest rate must be constant for every period $t \geq t_0$, where t_0 is the first period in which the economy is at a balanced growth path:

$$\begin{aligned} \frac{c_{t+1}}{c_t} &= \underbrace{g_c}_{\text{constant}} = \underbrace{\beta(1+r_{t+1})}_{\text{constant}} \qquad \forall t \geq t_0 \\ \implies r_t &= r_{t+1} = r_{t+2} = \bar{r} \qquad \forall t \geq t_0 \end{aligned}$$

Therefore, along the Balanced Growth Path, the value of new varieties satisfies:

$$V_t = \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j} \frac{1}{1+r_{t+k}} \bar{\pi} \right) = \sum_{j=0}^{\infty} \left(\frac{1}{1+\bar{r}} \right)^j \bar{\pi} = \frac{1+\bar{r}}{\bar{r}} \bar{\pi} \qquad \forall t \geq t_0$$

By free-entry, varieties will be created up to the point in which:

$$\frac{1}{1+\bar{r}} V_t = \frac{1}{\eta} \implies \bar{r} = \eta \cdot \bar{\pi}$$

We can now go back to the Euler Equation and characterize g_c:

$$\begin{aligned} \frac{c_{t+1}}{c_t} = g_c &= \beta(1+\eta \cdot \bar{\pi}) \qquad \forall t \geq t_0 \\ &= \beta \left(1 + \eta \cdot \frac{1-\alpha}{\alpha} \cdot \kappa^{\frac{\alpha}{1-\alpha}} \cdot n \cdot \alpha^{\frac{2}{1-\alpha}} \right) \end{aligned}$$

We can show that $g_y = g_w = g_m$. For g_y, calculate:

$$\frac{Y_{t+1}}{Y_t} = g_y = \frac{\left(\frac{\alpha^2}{\kappa}\right)^{\frac{\alpha}{1-\alpha}} \cdot n \cdot M_{t+1}}{\left(\frac{\alpha^2}{\kappa}\right)^{\frac{\alpha}{1-\alpha}} \cdot n \cdot M_t} = \frac{M_{t+1}}{M_t} = g_m$$

Likewise, for wages:

$$\frac{w_{t+1}}{w_t} = g_w = \frac{(1-\alpha) \cdot \left(\frac{\alpha^2}{\kappa}\right)^{\frac{\alpha}{1-\alpha}} \cdot M_{t+1}}{(1-\alpha) \cdot \left(\frac{\alpha^2}{\kappa}\right)^{\frac{\alpha}{1-\alpha}} \cdot M_t} = \frac{M_{t+1}}{M_t} = g_m$$

Now turn to the law of motion for varieties. We can write it as:

$$\frac{M_{t+1}}{M_t} = g_m = 1 + \eta \frac{Z_t}{M_t} \qquad \forall t \geq t_0$$

which implies that the ratio $\frac{Z_t}{M_t}$ is a constant $(\forall t \geq t_0)$. This can only be the case if Z_t and M_t are growing at the same rate. Therefore: $g_z = g_m$.

Now turn to the resource constraint:

$$\begin{aligned}
C_t &= Y_t - Z_t - \int_0^{M_t} x_t(\omega)d\omega \\
g_c^{t-t_0} C_0 &= g_y^{t-t_0} Y_0 - g_z^{t-t_0} Z_0 - \bar{x} g_m^{t-t_0} M_0 \\
g_c^{t-t_0} C_0 &= g_m^{t-t_0} Y_0 - g_m^{t-t_0} Z_0 - \bar{x} g_m^{t-t_0} M_0 \qquad (\because g_y = g_z = g_m) \\
g_c^{t-t_0} C_0 &= g_m^{t-t_0} (Y_0 - Z_0 - \bar{x} M_0) \\
g_c^{t-t_0} &= g_m^{t-t_0} \qquad (\because C_0 = Y_0 - Z_0 - \bar{x} M_0) \\
g_c &= g_m
\end{aligned}$$

Finally, since by the asset market clearance $a_t = Z_t$, it is trivial that $g_a = g_z = g_m$. Previously, we have concluded that profits per variety ($g_\pi = 1$) and demand per variety ($g_x = 1$) are constant along the balanced growth path.

Therefore, along the BGP:

$$\begin{aligned}
g_m &= g_y = g_c = g_w = g_z = g_a \\
g_x &= g_\pi = 1
\end{aligned}$$