CSE 464
DIGITAL IMAGE PROCESSING
HOMEWORK 01

Ömer ÇEVİK

161044004

1 Question's Solution

Let's define:

$\star$ $\quad x' = x\cos(\theta) - y\sin(\theta)$

$\star$ $\quad y' = y\sin(\theta) + y\cos(\theta)$

x' is rotated of x

and

y' is rotated of y

If we represent $\boldsymbol{x}$ and $\boldsymbol{y}$ in that equations:

From 1. Equation : $x = \dfrac{x' + y\sin(\theta)}{\cos(\theta)}$ and $y = \dfrac{x\cos(\theta) - x'}{\sin(\theta)}$

From 2. Equation : $x = \dfrac{y' - y\cos(\theta)}{\sin(\theta)}$ and $y = \dfrac{y' - x\sin(\theta)}{\cos(\theta)}$

If we get together all x and y:

$$x = \frac{x' + y\sin(\theta)}{\cos(\theta)} = \frac{y' - y\cos(\theta)}{\sin(\theta)} \quad \text{and} \quad y = \frac{x\cos(\theta) - x'}{\sin(\theta)} = \frac{y' - x\sin(\theta)}{\cos(\theta)}$$

Find for y:

$$\Rightarrow \frac{x'\sin(\theta) + y\sin^2(\theta)}{\cos(\theta)\sin(\theta)} = \frac{y'\cos(\theta) - y\cos^2(\theta)}{\cos(\theta)\sin(\theta)}$$

$$\Rightarrow x'\sin(\theta) + y\sin^2(\theta) - y'\cos(\theta) + y\cos^2(\theta) = 0$$

$$\Rightarrow x'\sin(\theta) + y(\sin^2(\theta) + \cos^2(\theta)) - y'\cos(\theta) = 0$$

$$\Rightarrow x'\sin(\theta) + y - y'\cos(\theta) = 0$$

$$\Rightarrow y = -x'\sin(\theta) + y'\cos(\theta)$$

Find for x:

$$\Rightarrow \frac{x\cos^2(\theta) - x'\cos(\theta)}{\cos(\theta)\sin(\theta)} = \frac{y'\sin(\theta) - x\sin^2(\theta)}{\cos(\theta)\sin(\theta)}$$

$$\Rightarrow x\cos^2(\theta) - x'\cos(\theta) - y'\sin(\theta) + x\sin^2(\theta) = 0$$

$$\Rightarrow x(\cos^2(\theta) + \sin^2(\theta)) - x'\cos(\theta) - y'\sin(\theta) = 0$$

$$\Rightarrow x - x'\cos(\theta) - y'\sin(\theta) = 0$$

$$\Rightarrow x = x'\cos(\theta) + y'\sin(\theta)$$

$$\bigtriangledown^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} \quad \textbf{and} \quad \frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'}$$

We can take derivation of x for equation result.

$$\frac{\partial x}{\partial x'} = \frac{\partial x}{\partial x'}(x' \cos(\theta) + y' \sin(\theta)) = \cos(\theta)$$

We can take derivation of y for equation result.

$$\frac{\partial y}{\partial x'} = \frac{\partial y}{\partial x'}(-x' \sin(\theta) + y' \cos(\theta)) = -\sin(\theta)$$

So our equation becomes to : $\dfrac{\partial f}{\partial x'} = \dfrac{\partial f}{\partial x}\cos(\theta) + \dfrac{\partial f}{\partial y}(-\sin(\theta))$

Taking second derivation of ***x'*** :

- $$\frac{\partial^2 f}{\partial x'^2} = \frac{\partial}{\partial x'}(\frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}(-\sin(\theta)))$$
  $$= \frac{\partial^2 f}{\partial x^2}\cos^2(\theta) + \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})\sin(\theta)\cos(\theta) - \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})\sin(\theta)\cos(\theta) + \frac{\partial^2 f}{\partial y^2}\sin^2(\theta)$$

Using **x** equation we can find derivation of **x** to **y'**:

$$\tfrac{\partial x}{\partial y'} = \tfrac{\partial}{\partial y'}(x' \cos(\theta) + y' \sin(\theta)) = \sin(\theta)$$

Using **y** equation we can find derivation of **y** to **y'**:

$$\tfrac{\partial y}{\partial y'} = \tfrac{\partial}{\partial y'}(-x' \sin(\theta) + y' \cos(\theta)) = \cos(\theta)$$

So our equation becomes to :

$$\tfrac{\partial f}{\partial y'} = \tfrac{\partial f}{\partial x}\tfrac{\partial x}{\partial y} + \tfrac{\partial f}{\partial y}\tfrac{\partial y}{\partial y'} = \tfrac{\partial f}{\partial x}\sin(\theta) + \tfrac{\partial f}{\partial y}\cos(\theta)$$

Taking second derivation of $\dfrac{\partial f}{\partial x'} = \dfrac{\partial f}{\partial x}\cos(\theta) + \dfrac{\partial f}{\partial y}(-\sin(\theta))$:

- $$\frac{\partial^2 f}{\partial x'^2} = \frac{\partial}{\partial x'}(\frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}(-\sin(\theta)))$$
  $$= \frac{\partial^2 f}{\partial x^2}\sin^2(\theta) + \frac{\partial}{\partial x}(\frac{\partial f}{\partial y})\sin(\theta)\cos(\theta) + \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})\cos^2(\theta) + \frac{\partial^2 f}{\partial y^2}\sin^2(\theta)$$

In the end, if we sum equations which are signed as black points we will see the Laplacian operator has independency from rotation as a proof.

$$\frac{\partial^2 f}{\partial x^2}\cos^2(\theta)+\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})\sin(\theta)\cos(\theta)-\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})\sin(\theta)\cos(\theta)+\frac{\partial^2 f}{\partial y^2}\sin^2(\theta)$$

$$+$$

$$\frac{\partial^2 f}{\partial x^2}\sin^2(\theta)+\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})\sin(\theta)\cos(\theta)+\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})\cos^2(\theta)+\frac{\partial^2 f}{\partial y^2}\sin^2(\theta)$$

$$\triangledown^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial x'^2}+\frac{\partial^2 f}{\partial y'^2} \qquad \underline{\textbf{Proved} \checkmark}$$

2 Question's Solution

The Minkowski distance is very common:

$$\forall p = (p_1, p_2), q = (q_1, q_2) \ \epsilon \ \mathbb{Z}^2 \ d(p, q) = (|p_1 - q_1|^r + |p_2 - q_2|^r)^{\frac{1}{r}}$$

For $r = 1$ it is known as Manhattan, city-block or L_1 distance.

⊛ Then Minkowski distance formula becomes Manhattan/city-block/L_1 distance formula: $d(p, q) = |p_1 - q_1| + |p_2 - q_2|$

Let a, b and c are pixel dots.

$a = (a_1, a_2)$, $b = (b_1, b_2)$, $c = (c_1, c_2)$

⊛ The Triangular Inequality is : $d(a, c) \leq d(a, b) + d(b, c)$

Using Manhattan formula:

$d(a, c) = |a_1 - c_1| + |a_2 - c_2|$
$d(a, b) = |a_1 - b_1| + |a_2 - b_2|$
$d(b, c) = |b_1 - c_1| + |b_2 - c_2|$

With Triangular Inequality:

⋙ $|a_1 - c_1| + |a_2 - c_2| \leq |a_1 - b_1| + |a_2 - b_2| + |b_1 - c_1| + |b_2 - c_2|$

If we think that 3 x 3 image space and the parts of this space can be called as A, B, C, D, E, F, G, H and I.

G	H	I
D	E	F
A	B	C

Let the c is the corner of [EFIH] and a is the corner of [ABED]. We can find the positions of b.

A : $b_1 \leq a_1 \leq c_1$ and $b_2 \leq a_2 \leq c_2$
$-a_1 + c_1 - a_2 + c_2 \leq a_1 - b_1 + a_2 - b_2 + -b_1 + c_1 - b_2 + c_2$
$0 \leq a_1 - b_1 + a_2 - b_2$ **True**

B : $a_1 \leq b_1 \leq c_1$ and $b_2 \leq a_2 \leq c_2$
$-a_1 + c_1 - a_2 + c_2 \leq -a_1 + b_1 + a_2 - b_2$ - $b_1 + c_1 - b_2 + c_2$
$0 \leq a_2 - b_2$ **True**

C : $a_1 \leq c_1 \leq b_1$ and $b_2 \leq a_2 \leq c_2$
$-a_1 + c_1 - a_2 + c_2 \leq -a_1 + b_1 + a_2 - b_2 + b_1 - c_1 - b_2 + c_2$
$0 \leq b_1 - c_1 + a_2 - b_2$ **True**

D : $b_1 \leq a_1 \leq c_1$ and $a_2 \leq b_2 \leq c_2$
$-a_1 + c_1 - a_2 + c_2 \leq a_1 - b_1 - a_2 + b_2 + -b_1 + c_1 - b_2 + c_2$
$0 \leq a_1 - b_1$ **True**

E : $a_1 \leq b_1 \leq c_1$ and $a_2 \leq b_2 \leq c_2$
$-a_1 + c_1 - a_2 + c_2 \leq -a_1 + b_1 - a_2 + b_2$ - $b_1 + c_1 - b_2 + c_2$
$0 \leq 0$ **True**

F : $a_1 \leq c_1 \leq b_1$ and $a_2 \leq b_2 \leq c_2$
$-a_1 + c_1 - a_2 + c_2 \leq -a_1 + b_1 - a_2 + b_2 + b_1 - c_1 - b_2 + c_2$
$0 \leq b_1 - c_1$ **True**

G : $b_1 \leq a_1 \leq c_1$ and $a_2 \leq c_2 \leq b_2$
$-a_1 + c_1 - a_2 + c_2 \leq a_1 - b_1 - a_2 + b_2$ - $b_1 + c_1 + b_2 - c_2$
$0 \leq a_1 - b_1 + b_2 - c_2$ **True**

H : $a_1 \leq b_1 \leq c_1$ and $a_2 \leq c_2 \leq b_2$
$-a_1 + c_1 - a_2 + c_2 \leq -a_1 + b_1 - a_2 + b_2$ - $b_1 + c_1 + b_2 - c_2$
$0 \leq b_2 - c_2$ **True**

I : $a_1 \leq c_1 \leq b_1$ and $a_2 \leq c_2 \leq b_2$
$-a_1 + c_1 - a_2 + c_2 \leq -a_1 + b_1 - a_2 + b_2 + b_1 - c_1 + b_2 - c_2$
$0 \leq b_1 - c_1 + b_2 - c_2$ **True**

For all these 9 positions we can figure out that the formula is working.

Also there are 4 different rotations of **a** and **c** to find **b**. But if other rotations be showed we can see the distance between that points will never change. To prove the distance that 9 positions are enough.

Proved ✓

3 Question's Solution

Let define C_{3x3} as our solved matrix.

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

And this C matrix must be multiply by B matrix to create equations of C matrix's values.

$$\bigstar \overbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & 1 \end{bmatrix}}^{C} \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}^{B} = \overbrace{\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}^{A} \quad \clubsuit \overbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & 1 \end{bmatrix}}^{C} \overbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}^{B} = \overbrace{\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}}^{A} \quad \blacklozenge \overbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & 1 \end{bmatrix}}^{C} \overbrace{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}}^{B} = \overbrace{\begin{bmatrix} -4 \\ 4 \\ 1 \end{bmatrix}}^{A}$$

For $\bigstar$ we can create this equations:

$$1c_{11} + 2c_{12} + 1c_{13} = 2 \quad \Rightarrow \quad c_{11} + 2c_{12} + c_{13} = 2$$
$$1c_{21} + 2c_{22} + 1c_{23} = 2 \quad \Rightarrow \quad c_{21} + 2c_{22} + c_{23} = 2$$

For $\clubsuit$ we can create this equations:

$$2c_{11} + 1c_{12} + 1c_{13} = -1 \quad \Rightarrow \quad 2c_{11} + c_{12} + c_{13} = -1$$
$$2c_{21} + 1c_{22} + 1c_{23} = 4 \quad \Rightarrow \quad 2c_{21} + c_{22} + c_{23} = 4$$

For $\blacklozenge$ we can create this equations:

$$3c_{11} + 1c_{12} + 1c_{13} = -4 \quad \Rightarrow \quad 3c_{11} + c_{12} + c_{13} = -4$$
$$3c_{21} + 1c_{22} + 1c_{23} = 4 \quad \Rightarrow \quad 3c_{21} + c_{22} + c_{23} = 4$$

With that equations we can find each values of C matrix.

$$\begin{array}{ll} c_{11} + 2c_{12} + c_{13} = 2 & c_{21} + 2c_{22} + c_{23} = 2 \\ 2c_{11} + c_{12} + c_{13} = -1 & 2c_{21} + c_{22} + c_{23} = 4 \\ 3c_{11} + c_{12} + c_{13} = -4 & 3c_{21} + c_{22} + c_{23} = 4 \end{array}$$

Using Cramer Method we find these results:

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

$$= (1 * 1 * 1) + (2 * 1 * 3) + (1 * 2 * 1) - (3 * 1 * 1) - (1 * 1 * 1) - (1 * 2 * 2) = 1 \quad \checkmark$$

To find $c_{11} : \Delta_{11} = \begin{vmatrix} 2 & 2 & 1 \\ -1 & 1 & 1 \\ -4 & 1 & 1 \end{vmatrix}$

$= (2{*}1{*}1)+(2{*}1{*}(-4))+(1{*}(-1){*}1)-((-4){*}1{*}1)-(1{*}1{*}2)-(1{*}(-1){*}2) = -3$
$\Rightarrow c_{11} = -3/1 = -3$

To find $c_{12} : \Delta_{12} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 3 & -4 & 1 \end{vmatrix}$

$= (1{*}(-1){*}1)+(2{*}1{*}3)+(1{*}2{*}(-4))-(3{*}(-1){*}1)-((-4){*}1{*}1)-(1{*}2{*}2) = 0$
$\Rightarrow c_{12} = 0/1 = 0$

To find $c_{13} : \Delta_{13} = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -1 \\ 3 & 1 & -4 \end{vmatrix}$

$= (1{*}1{*}(-4))+(2{*}(-1){*}3)+(2{*}2{*}1)-(3{*}1{*}2)-(1{*}(-1){*}1)-((-4){*}2{*}2) = 5$
$\Rightarrow c_{13} = 5/1 = 5$

To find $c_{21} : \Delta_{21} = \begin{vmatrix} 2 & 2 & 1 \\ 4 & 1 & 1 \\ 4 & 1 & 1 \end{vmatrix}$

$= (2 * 1 * 1) + (2 * 1 * 4) + (1 * 4 * 1) - (4 * 1 * 1) - (1 * 1 * 2) - (1 * 4 * 2) = 0$
$\Rightarrow c_{21} = 0/1 = 0$

To find $c_{22} : \Delta_{22} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 4 & 1 \end{vmatrix}$

$= (1 * 4 * 1) + (2 * 1 * 3) + (1 * 2 * 4) - (3 * 4 * 1) - (4 * 1 * 1) - (1 * 2 * 2) = -2$
$\Rightarrow c_{22} = -2/1 = -2$

To find $c_{23} : \Delta_{23} = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 4 \\ 3 & 1 & 4 \end{vmatrix}$

$= (1 * 1 * 4) + (2 * 4 * 3) + (2 * 2 * 1) - (3 * 1 * 2) - (1 * 4 * 1) - (4 * 2 * 2) = 6$
$\Rightarrow c_{23} = 6/1 = 6$

$$c_{11} = -3 \qquad c_{12} = 0 \qquad c_{13} = 5 \qquad c_{21} = 0 \qquad c_{22} = -2 \qquad c_{23} = 6$$

So our C matrix is:

$$C = \begin{bmatrix} -3 & 0 & 5 \\ 0 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix} \quad and \quad C^{-1} = \frac{1}{|C|}.C^T = \frac{1}{|C|}\begin{vmatrix} \begin{vmatrix} -2 & 6 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 5 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 5 \\ -2 & 6 \end{vmatrix} \\ \begin{vmatrix} 0 & 6 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -3 & 5 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -3 & 5 \\ 0 & 6 \end{vmatrix} \\ \begin{vmatrix} 0 & -2 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} -3 & 0 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} -3 & 0 \\ 0 & -2 \end{vmatrix} \end{vmatrix}$$

Using Sarrus Method :

$|C| = ((-3) * (-2) * 1) + (0 * 6 * 0) + (5 * 0 * 0) - (0 * (-2) * 5) - (0 * 6 * (-3)) - (1 * 0 * 0) = 6$

Using Adjoint Matrix Method:

$$C^{-1} = \frac{1}{6}\begin{bmatrix} \begin{vmatrix} -2 & 6 \\ 0 & 1 \end{vmatrix} = -2 & \begin{vmatrix} 0 & 5 \\ 0 & 1 \end{vmatrix} = 0 & \begin{vmatrix} 0 & 5 \\ -2 & \end{vmatrix} = 10 \\ \begin{vmatrix} 0 & 6 \\ 0 & 1 \end{vmatrix} = 0 & \begin{vmatrix} -3 & 5 \\ 0 & 1 \end{vmatrix} = -3 & \begin{vmatrix} -3 & 5 \\ 0 & 6 \end{vmatrix} = 18 \\ \begin{vmatrix} 0 & -2 \\ 0 & 0 \end{vmatrix} = 0 & \begin{vmatrix} -3 & 0 \\ 0 & 0 \end{vmatrix} = 0 & \begin{vmatrix} -3 & 0 \\ 0 & -2 \end{vmatrix} = 6 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -2 & 0 & 10 \\ 0 & -3 & 18 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\Rightarrow C^{-1} = \begin{bmatrix} -\frac{1}{3} & 0 & \frac{5}{3} \\ 0 & -\frac{1}{2} & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

4 Question's Solution

In the definition of dilation operation we know that if **B** is a structuring element then it has decomposition:

$\check{B} = \mathrm{H}_1 \oplus \mathrm{H}_2 \oplus \cdots \oplus \mathrm{H}_N$

The dilation of X by $\check{B}$ can be performed:

$X \oplus \check{B} = X \oplus (\mathrm{H}_1 \oplus \mathrm{H}_2 \oplus \cdots \oplus \mathrm{H}_N)$

From associative of dilation operation:

$= ((((X \oplus \mathrm{H}_1) \oplus \mathrm{H}_2) \oplus \cdots) \oplus \mathrm{H}_N)$

In here $\check{B} = \mathrm{H}_1 \oplus \mathrm{H}_2 \oplus \cdots \oplus \mathrm{H}_N$ where each H_N satisfies the smallness requirement of the hardware pipelineable stages.

To make this decomposition problem more concrete, we can take the structuring element $\check{B}$ to be a 7 x 5 rectangle whose corner pixels are missing. One decomposition which is not most efficient is given by:

$\check{B} = \{(0,0),(0,1)\} \oplus \{(0,0),(0,1)\} \oplus \{(0,0),(1,1)\} \oplus \{(0,0),(1,0)\} \oplus \{(0,0),(1,0)\} \oplus \{(0,0),(1,-1)\} \oplus \{(0,0),(0,-1)\} \oplus \{(0,0),(0,-1)\}$.

If each H_N in the decomposition of $\check{B}$, we see that $\check{B}$ is a 2-point set, then the resulting decomposition is said to be a 2-point decomposition. A canonical 2-point decomposition involves only H_N's for which $0 \in \mathrm{H}_N$, n = 1,...,N. Canonical 2-point decompositions are important because dilation with such a 2-point set can be accomplished by a shift and an **OR**.

We begin by stating the relationships among dilation, erosion, set union and set intersection. These relationships are all easily proven in a few steps using the definitions of the operations.

$(\mathrm{C} \cup \mathrm{D}) \oplus \check{B} = (\mathrm{C} \oplus \check{B}) \cup (\mathrm{D} \oplus \check{B})$ **Proved ✓**