More Applications

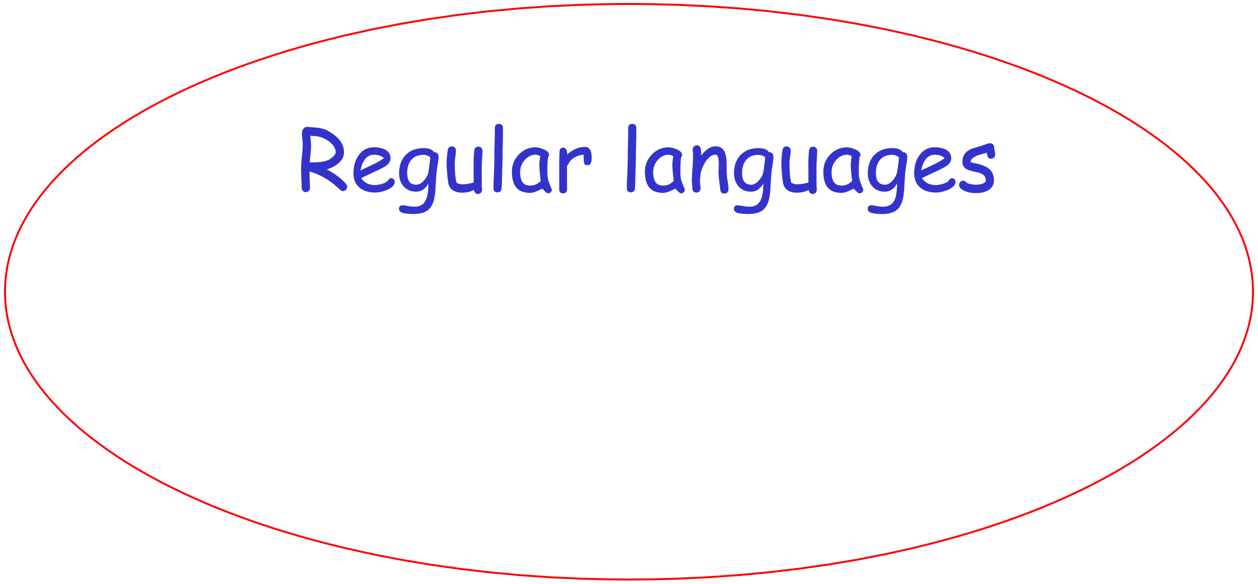
of

the Pumping Lemma

The Pumping Lemma:

- Given a ~~infinite~~ regular language L
- there exists an integer m
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\,y\,z$
- with $|x\,y| \leq m$ and $|y| \geq 1$
- such that: $x\,y^i\,z \in L$ $\quad i = 0, 1, 2, \ldots$

Non-regular languages $L = \{ww^R : w \in \Sigma^*\}$



Theorem: The language

$$L = \{ww^R : w \in \Sigma^*\} \qquad \Sigma = \{a, b\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{ww^R : w \in \Sigma^*\}$$

Assume for contradiction that L is a regular language

~~Since L is infinite~~

we can apply the Pumping Lemma

$$L = \{ww^R : w \in \Sigma^*\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$ and

length $|w| \geq m$

We pick $w = a^m b^m b^m a^m$

Write $a^m b^m b^m a^m = x\,y\,z$

From the Pumping Lemma
it must be that length $|x\,y| \leq m, \quad |y| \geq 1$

$$xyz = \overbrace{a...aa...a...a}^{m}\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m}$$

$$xyz = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{...ab...bb...ba...a}_{z}$$

Thus: $\quad y = a^k, \quad k \geq 1$

$$x\ y\ z = a^m b^m b^m a^m \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\ y^i\ z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\ y^2\ z \in L$

$$x\ y\ z = a^m b^m b^m a^m \qquad\qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\ y^2\ z \in L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}a...a}^{m+k}\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m} \in L$$

with $z = a...ab...bb...ba...a$

Thus: $a^{m+k} b^m b^m a^m \in L$

$$a^{m+k} b^m b^m a^m \quad \in L \qquad\qquad k \geq 1$$

BUT: $\quad L = \{ww^R : w \in \Sigma^*\}$

$$\Downarrow$$

$$a^{m+k} b^m b^m a^m \quad \notin L$$

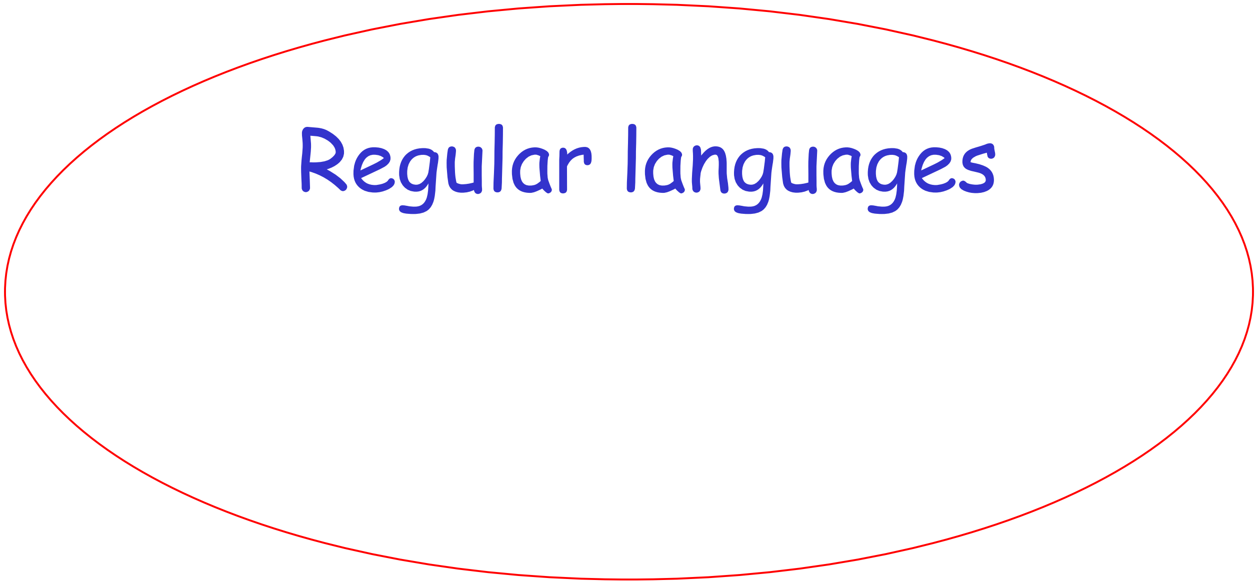
CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular languages

$$L = \{a^n b^l c^{n+l} :\ n, l \geq 0\}$$



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We pick $w = a^m b^m c^{2m}$

Write $a^m b^m c^{2m} = x\ y\ z$

From the Pumping Lemma

it must be that length $|\,x\ y\,| \le m, \quad |\,y\,| \ge 1$

$$xyz = \overbrace{a...aa...aa...a}^{m}\overbrace{b...b}^{m}\overbrace{c...cc...c}^{2m}$$

$$xyz = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...ab...bc...cc...c}_{z}$$

Thus: $y = a^k, \quad k \ge 1$

$$x\ y\ z = a^m b^m c^{2m} \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\ y^i\ z \ \in\ L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\ y^0\ z\ = xz\ \in\ L$

$$x\ y\ z = a^m b^m c^{2m} \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $\quad xz \in L$

$$xz = \underbrace{\overbrace{a...a}^{}a...a}_{x \,\ldots}\ldots$$

$$xz = \underbrace{a...a}_{x}\underbrace{\overbrace{a...a}^{m-k}\overbrace{b...b}^{m}\overbrace{c...cc...c}^{2m}}_{z} \in L$$

Thus: $\quad a^{m-k} b^m c^{2m} \quad \in L$

$$a^{m-k} b^m c^{2m} \in L \qquad k \geq 1$$

BUT: $L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$

$$\Downarrow$$

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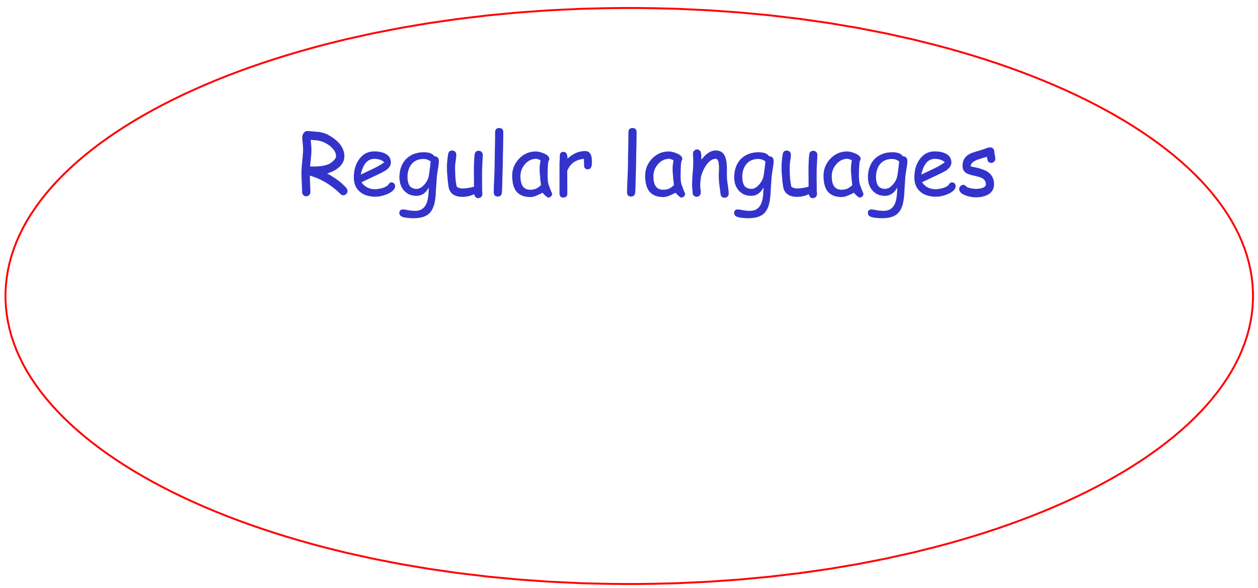
Non-regular languages $L = \{a^{n!} : n \geq 0\}$



Theorem: The language $L = \{a^{n!} : \ n \geq 0\}$

is not regular

$$n! \ = \ 1 \cdot 2 \cdots (n-1) \cdot n$$

Proof: Use the Pumping Lemma

$$L = \{a^{n!} : \ n \geq 0\}$$

Assume for contradiction

that L is a regular language

~~Since L is infinite~~

we can apply the Pumping Lemma

$$L = \{a^{n!} : \; n \geq 0\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

We pick $w = a^{m!}$

Write $a^{m!} = x\ y\ z$

From the Pumping Lemma

it must be that length $|x\ y| \leq m, \quad |y| \geq 1$

$$xyz = a^{m!} = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}a...a}^{m}\,\overbrace{a...aa...a}^{m!-m}$$

with z = the part after y, from $a...a$ to the end.

Thus: $y = a^k, \quad 1 \leq k \leq m$

$$x\,y\,z = a^{m!} \qquad\qquad y = a^{k}, \;\; 1 \le k \le m$$

From the Pumping Lemma: $x\,y^{i}\,z \;\in\; L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^{2}\,z \;\in\; L$

$$x\,y\,z = a^{m!} \qquad\qquad y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $x\,y^2\,z \in L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}a...a}^{m+k}\overbrace{a...aa...a}^{m!-m} \in L$$

(where $z = a...aa...aa...a$, the part after the second y)

Thus: $a^{m!+k} \in L$

$$a^{m!+k} \in L \qquad\qquad 1 \leq k \leq m$$

Since: $L = \{a^{n!} : \ n \geq 0\}$

There must exist p such that:

$$m!+k = p!$$

However: $m!+k \leq m!+m$ for $m>1$

$$\leq m!+m!$$

$$< m!m + m!$$

$$= m!(m+1)$$

$$= (m+1)!$$

$$\Downarrow$$

$$m!+k < (m+1)!$$

$$\Downarrow$$

$m!+k \neq p!$ for any p

$$a^{m!+k} \in L \qquad 1 \leq k \leq m$$

BUT: $L = \{a^{n!}: \ n \geq 0\}$

$$\Downarrow$$

$$a^{m!+k} \notin L$$

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Lex

Lex: a lexical analyzer

- A Lex program recognizes strings

- For each kind of string found
  the lex program takes an action

Input

```
Var = 12 + 9;
if (test > 20)
   temp = 0;
else
   while (a < 20)
      temp++;
```

→ **Lex program** →

Output

Identifier: Var
Operand: =
Integer: 12
Operand: +
Integer: 9
Semicolumn: ;
Keyword: if
Parenthesis: (
Identifier: test
....

In Lex strings are described with regular expressions

Lex program

Regular expressions

"+"

"-"

"="

/* operators */

"if"

"then"

/* keywords */

Lex program

Regular expressions

(0|1|2|3|4|5|6|7|8|9)+   /* integers */

(a|b|..|z|A|B|...|Z)+   /* identifiers */

integers

(0|1|2|3|4|5|6|7|8|9)+

[0-9]+

identifiers

(a|b|..|z|A|B|...|Z)+

[a-zA-Z]+

Each regular expression
has an associated action (in C code)

Examples:

Regular expression	Action
\n	linenum++;
[0-9]+	prinf("integer");
[a-zA-Z]+	printf("identifier");

Default action: ECHO;

Prints the string identified to the output

A small program

```
%%
[ \t\n]          ;    /*skip spaces*/

[0-9]+           printf("Integer\n");

[a-zA-Z]+        printf("Identifier\n");
```

Input

1234	test	
var	566	78
9800		

Output

Integer
Identifier
Identifier
Integer
Integer
Integer

Another program

```
%{
int linenum = 1;
%}

%%

[ \t]           ;    /*skip spaces*/
\n              linenum++;
[0-9]+          prinf("Integer\n");
[a-zA-Z]+       printf("Identifier\n");
.               printf("Error in line: %d\n",
                        linenum);
```

Input

```
1234    test
var  566     78
9800   +
temp
```

Output

```
Integer
Identifier
Identifier
Integer
Integer
Integer
Error in line: 3
Identifier
```

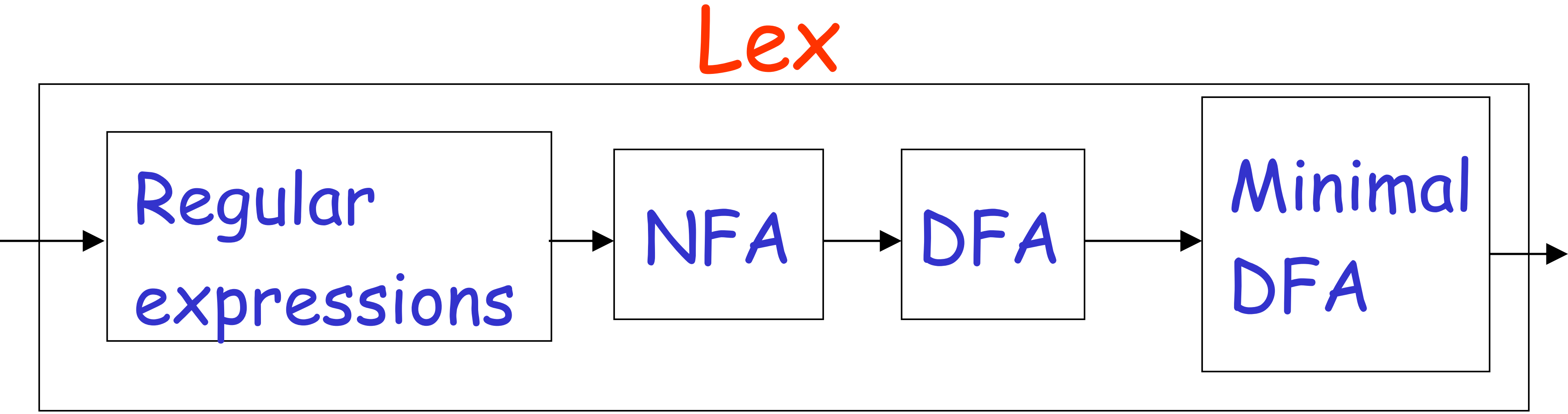
Lex matches the longest input string

Example: Regular Expressions "if"

"ifend"

Input:	ifend	if	ifn
Matches:	"ifend"	"if"	nomatch

Internal Structure of Lex



The final states of the DFA are associated with actions