Linz 6th, Chapter 4
Properties of Regular Languages

1. Additional Closure Properties
   A. Complement
   B. Intersection
2. Elementary Questions about Regular Lang
   1. Membership, empty, finite, equality
3. Pumping Lemma
   A. Proof
   B. Applications. Just one, the famous one: $a^n b^n$

We have proven

Regular languages are closed under:

Union

Concatenation

Star operation

Reverse

Namely, for regular languages L_1 and L_2 :

$$\left.\begin{array}{ll}\text{Union} & L_1 \cup L_2 \\ \text{Concatenation} & L_1 L_2 \\ \text{Star operation} & L_1{}^* \\ \text{Reverse} & L_1{}^R\end{array}\right\}\ \text{Regular Languages}$$

We will prove

Regular languages are closed under:

Complement

Intersection

Namely, for regular languages L_1 and L_2:

$$\left.\begin{array}{ll} \text{Complement} & \overline{L_1} \\ \\ \text{Intersection} & L_1 \cap L_2 \end{array}\right\} \text{Regular Languages}$$

Complement

Theorem: For regular language L the complement $\overline{L}$ is regular

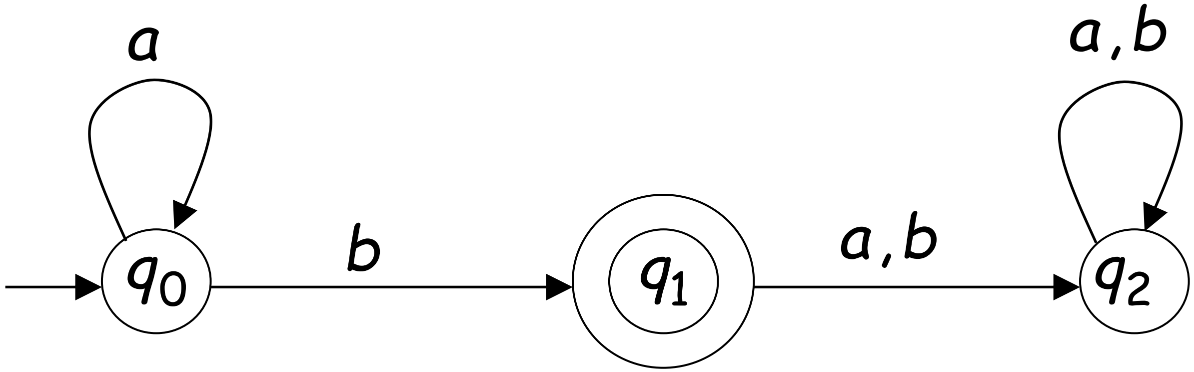
Proof: Take DFA that accepts L and make

- nonfinal states final
- final states nonfinal

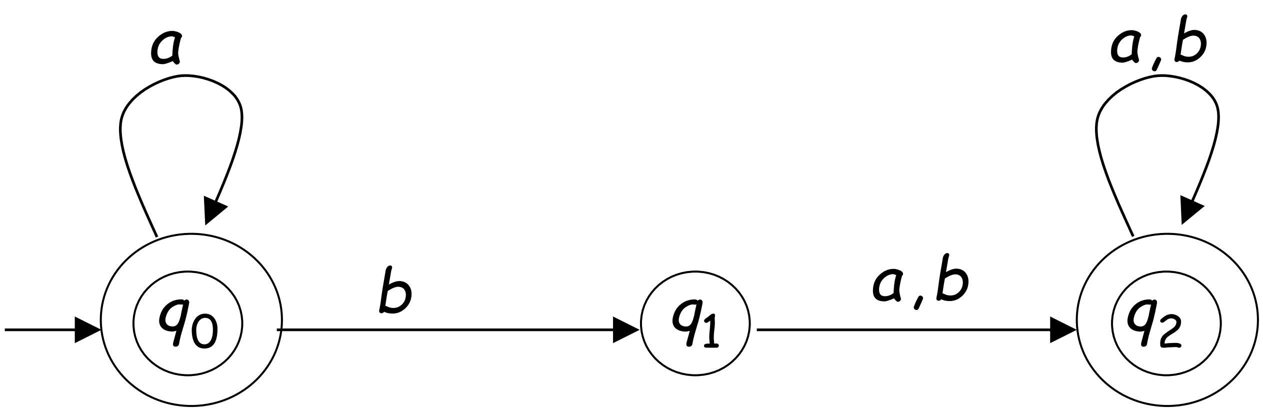
Resulting DFA accepts $\overline{L}$

Example:

$$L = L(a^*b)$$



$$\overline{L} = L(a^* + a^*b(a+b)(a+b)^*)$$



Intersection

Theorem: For regular languages L_1 and L_2 the intersection $L_1 \cap L_2$ is regular

Proof: Apply DeMorgan's Law:

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

$L_1 \,,\; L_2$ regular

$\Rightarrow$ $\overline{L_1} \,,\; \overline{L_2}$ regular

$\Rightarrow$ $\overline{L_1} \cup \overline{L_2}$ regular

$\Rightarrow$ $\overline{\overline{L_1} \cup \overline{L_2}}$ regular

$\Rightarrow$ $L_1 \cap L_2$ regular

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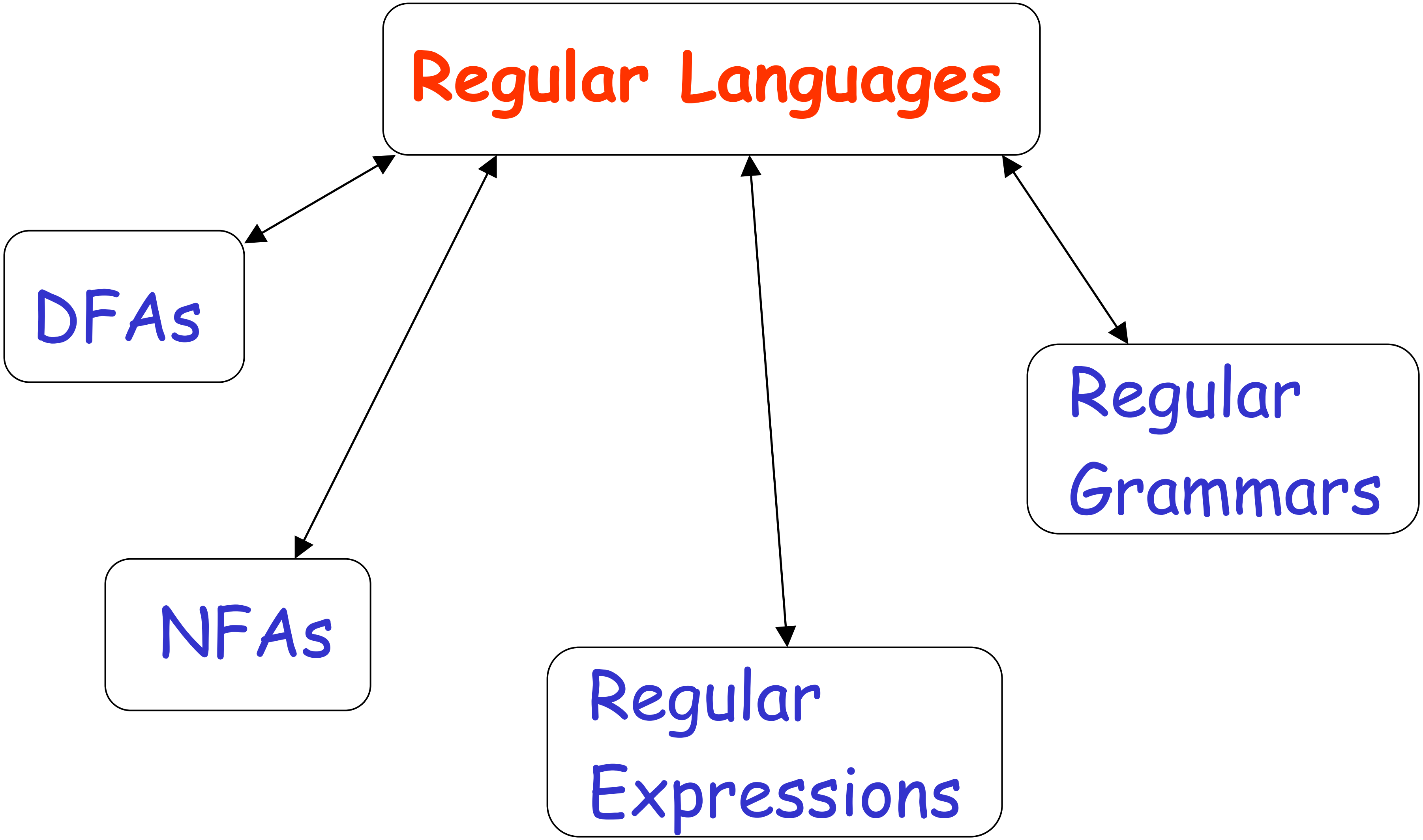
Standard Representations of Regular Languages

Standard Representations of Regular Languages



When we say: We are given a Regular Language L

We mean: Language L is in a standard representation

Elementary Questions

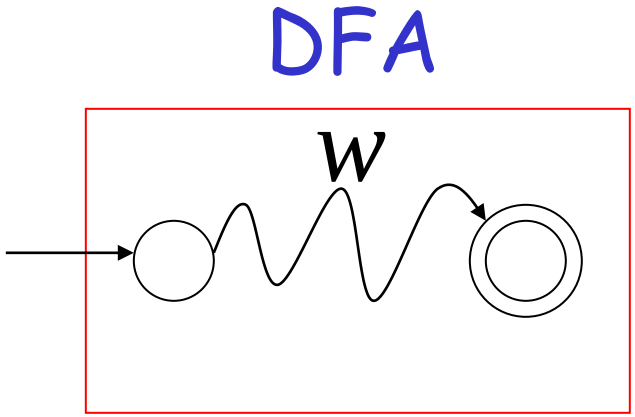
about

Regular Languages

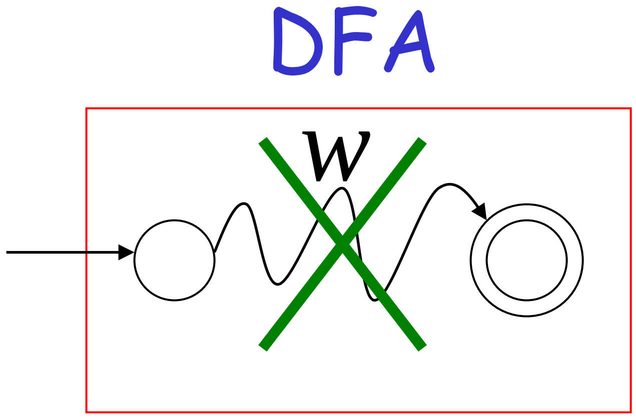
Membership Question

Question: Given regular language L
and string w
how can we check if $w \in L$?

Answer: Take the DFA that accepts L
and check if w is accepted



$w \in L$

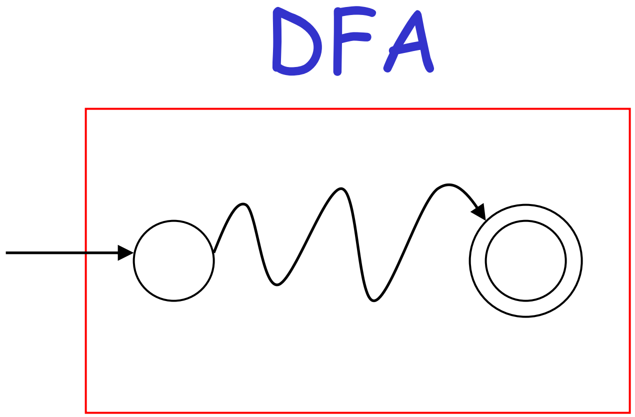


$w \notin L$

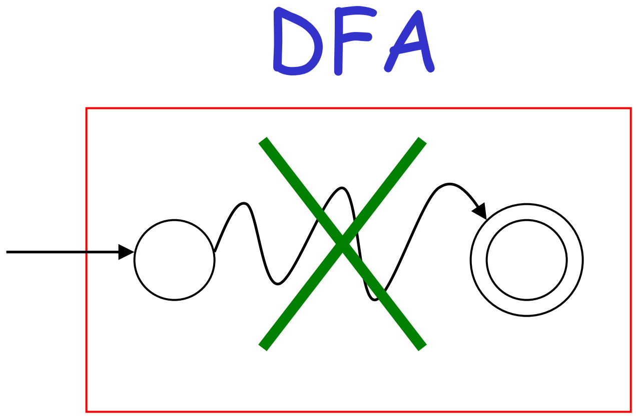
Question: Given regular language L
how can we check
if L is empty: $(L = \varnothing)$?

Answer: Take the DFA that accepts L

Check if there is a path from
the initial state to a final state



$$L \neq \emptyset$$



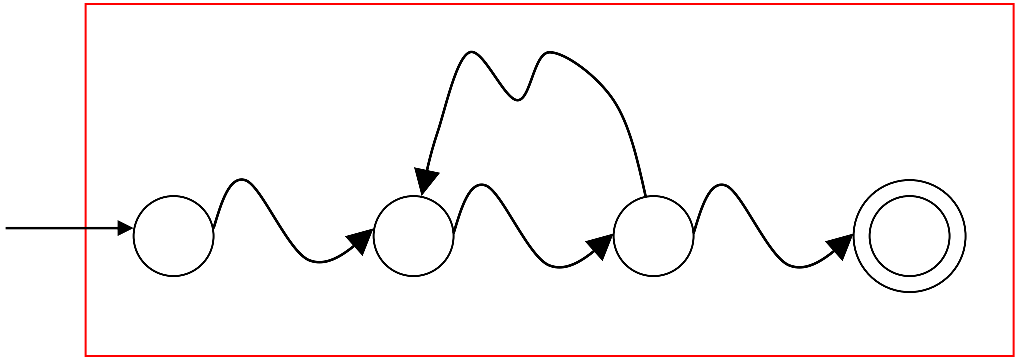
$$L = \emptyset$$

Question: Given regular language L how can we check if L is finite?

Answer: Take the DFA that accepts L

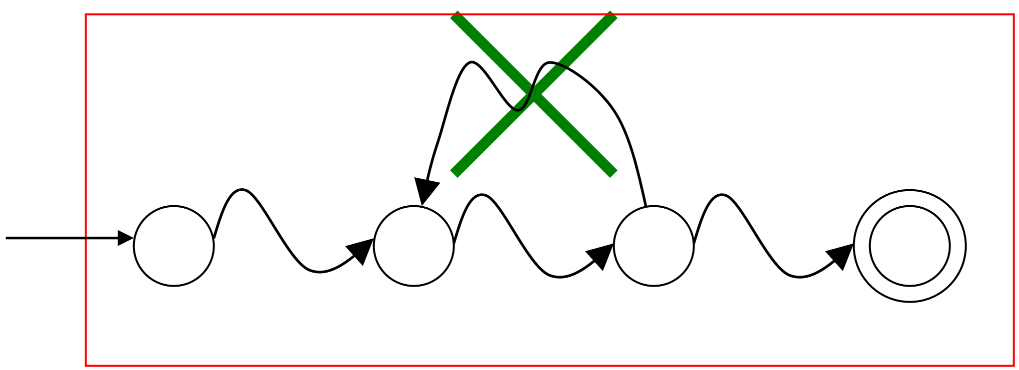
Check if there is a walk with cycle from the initial state to a final state

DFA



L is infinite

DFA



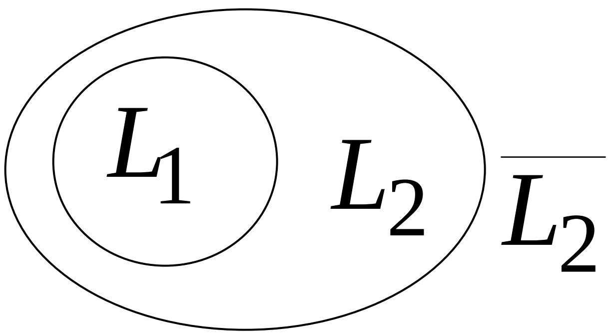
L is finite

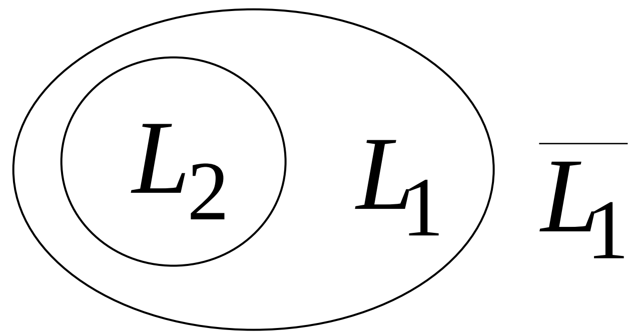
Question: Given regular languages L_1 and L_2 how can we check if $L_1 = L_2$?

Answer: Find if $(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \emptyset$

$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \emptyset$$

$L_1 \cap \overline{L_2} = \emptyset$ **and** $\overline{L_1} \cap L_2 = \emptyset$





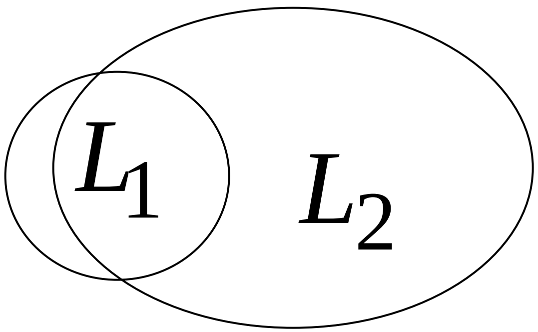
$L_1 \subseteq L_2$ $\qquad$ $L_2 \subseteq L_1$

$$L_1 = L_2$$

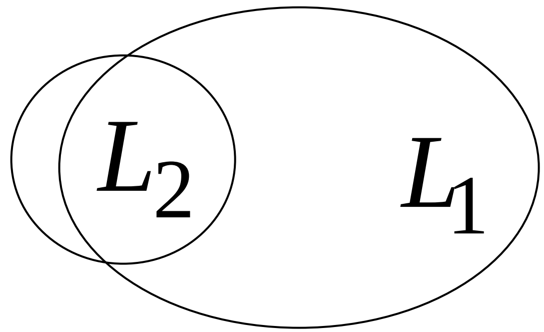
$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) \neq \emptyset$$

$$\Downarrow$$

$L_1 \cap \overline{L_2} \neq \emptyset$ or $\overline{L_1} \cap L_2 \neq \emptyset$



$$L_1 \not\subset L_2$$



$$L_2 \not\subset L_1$$

$$\Downarrow$$

$$L_1 \neq L_2$$

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Non-regular languages

Non-regular languages

$$\{a^n b^n : \quad n \geq 0\}$$

$$\{ww^R : \quad w \in \{a,b\}^*\}$$

Regular languages

$$a^*b$$

$$b^*c + a$$

$$b + c(a+b)^*$$

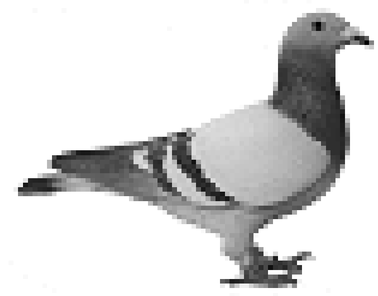
$$etc...$$

How can we prove that a language L is not regular?

Prove that there is no DFA that accepts L

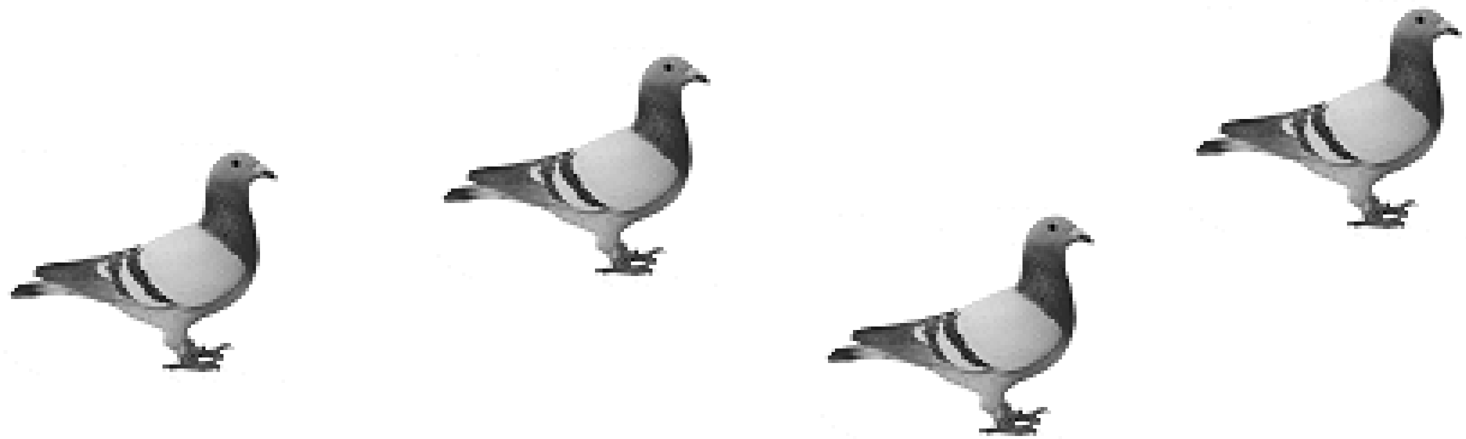
Problem: this is not easy to prove

Solution: the Pumping Lemma !!!

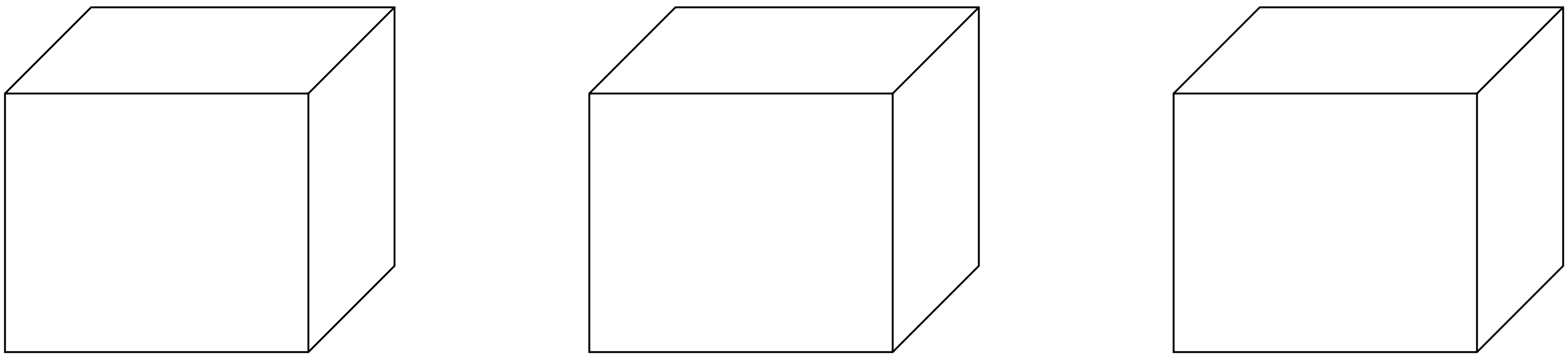


The Pigeonhole Principle

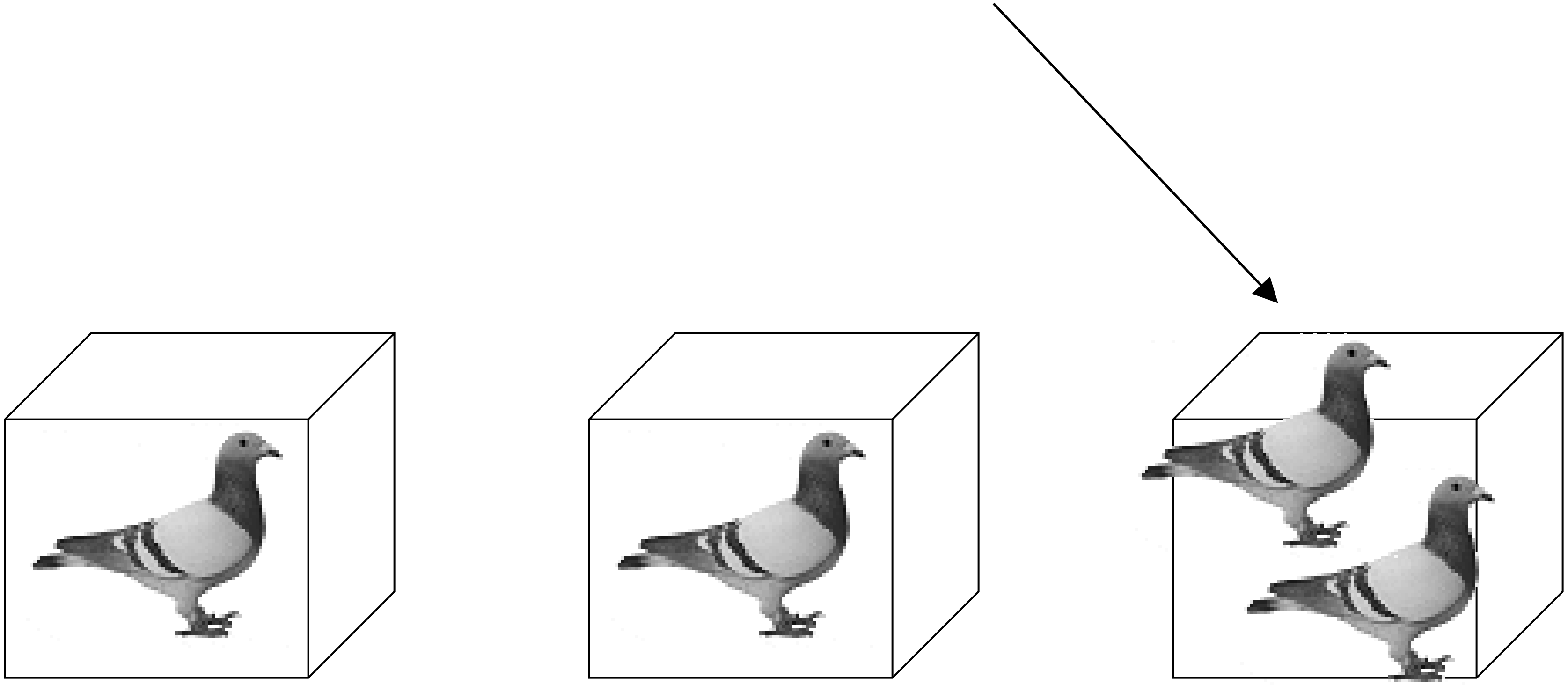
4 pigeons



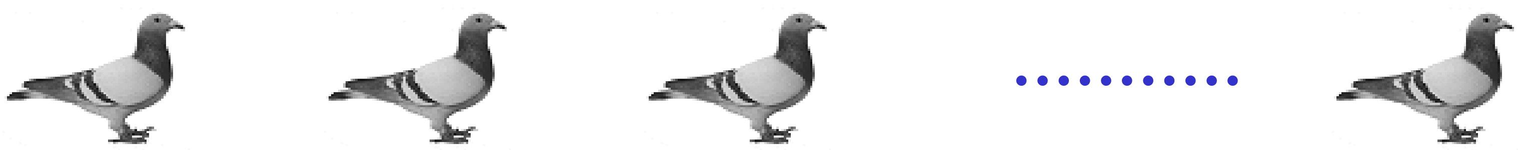
3 pigeonholes



A pigeonhole must
contain at least two pigeons

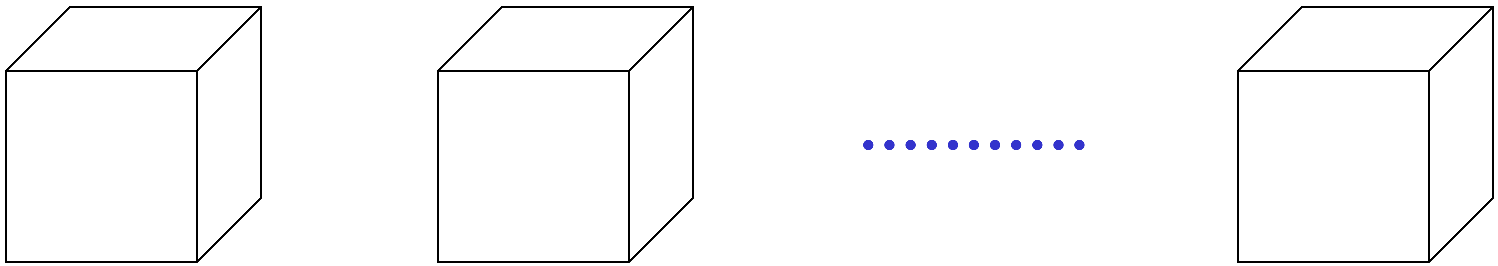


n pigeons



m pigeonholes

$$n > m$$



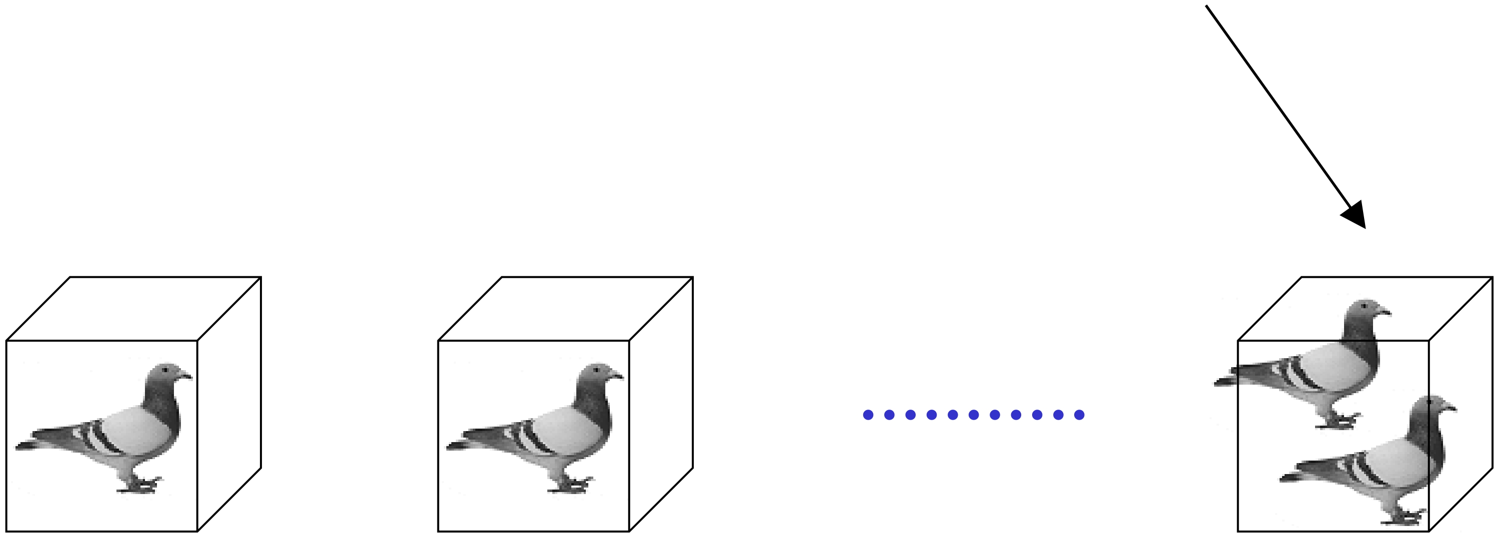
The Pigeonhole Principle

n pigeons

m pigeonholes

$$n > m$$

There is a pigeonhole with at least 2 pigeons

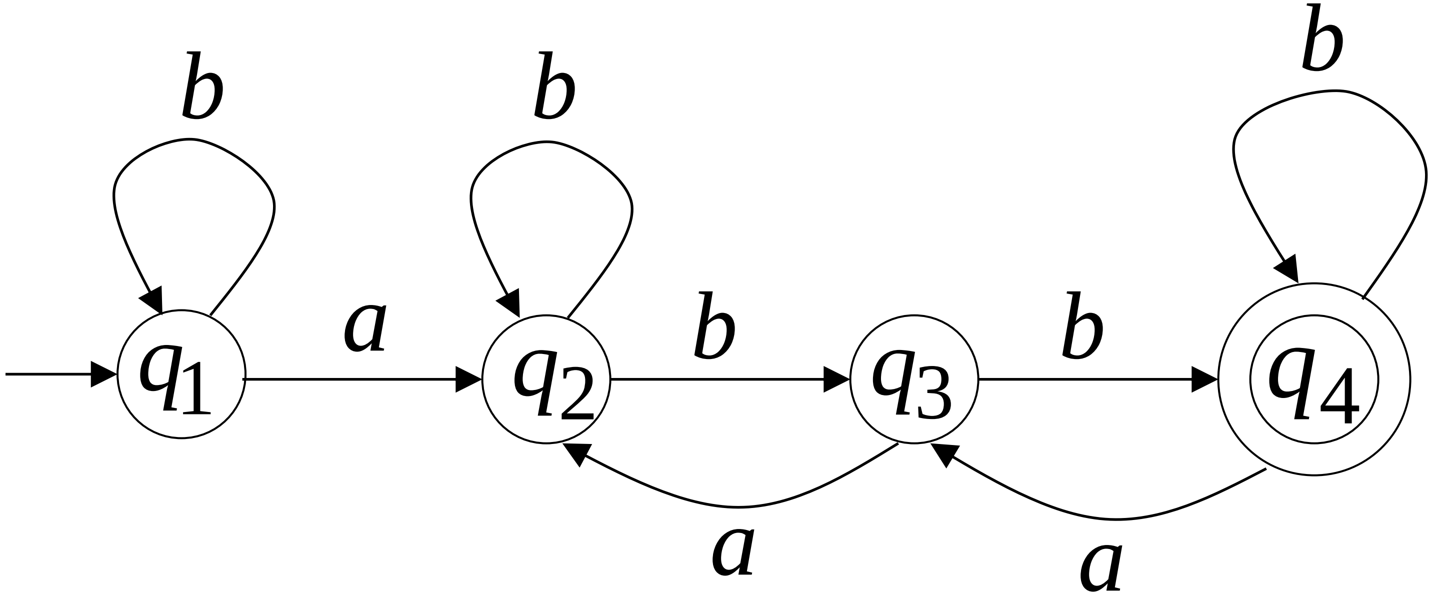


The Pigeonhole Principle

and

DFAs

DFA with 4 states

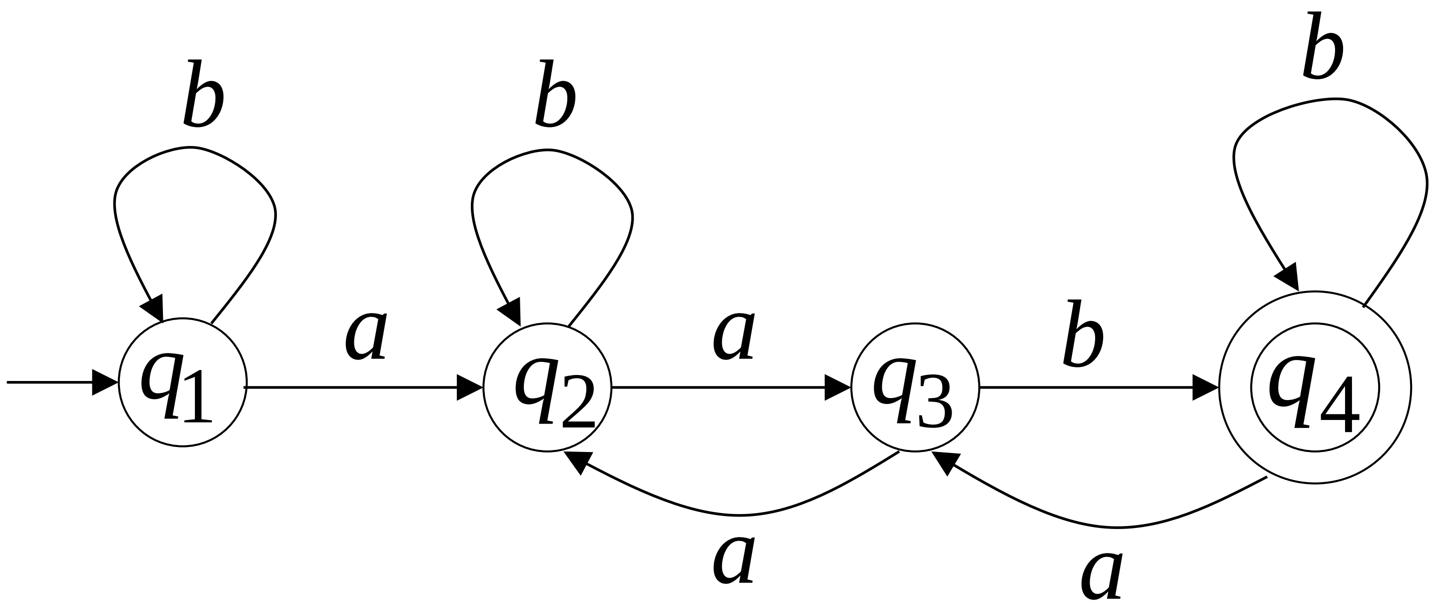


In walks of strings: a

aa

aab

no state is repeated



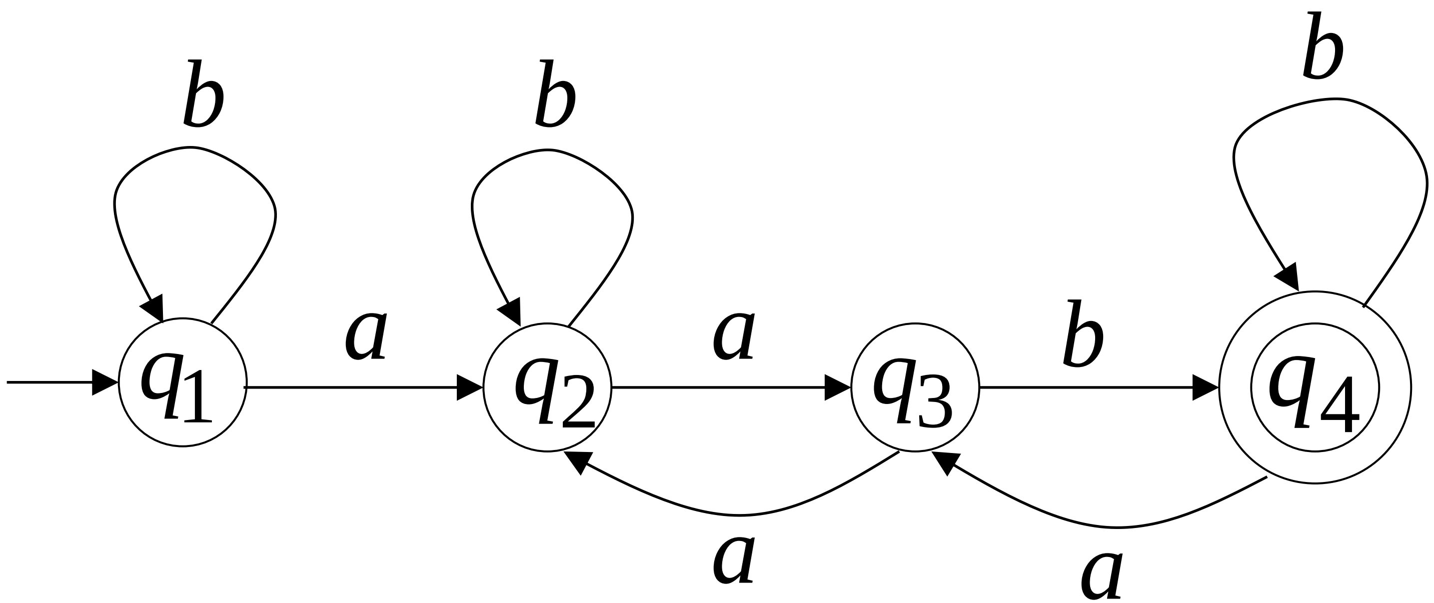
In walks of strings: $aabb$

$bbaa$

$abbabb$

$abbbabbabb...$

a state is repeated



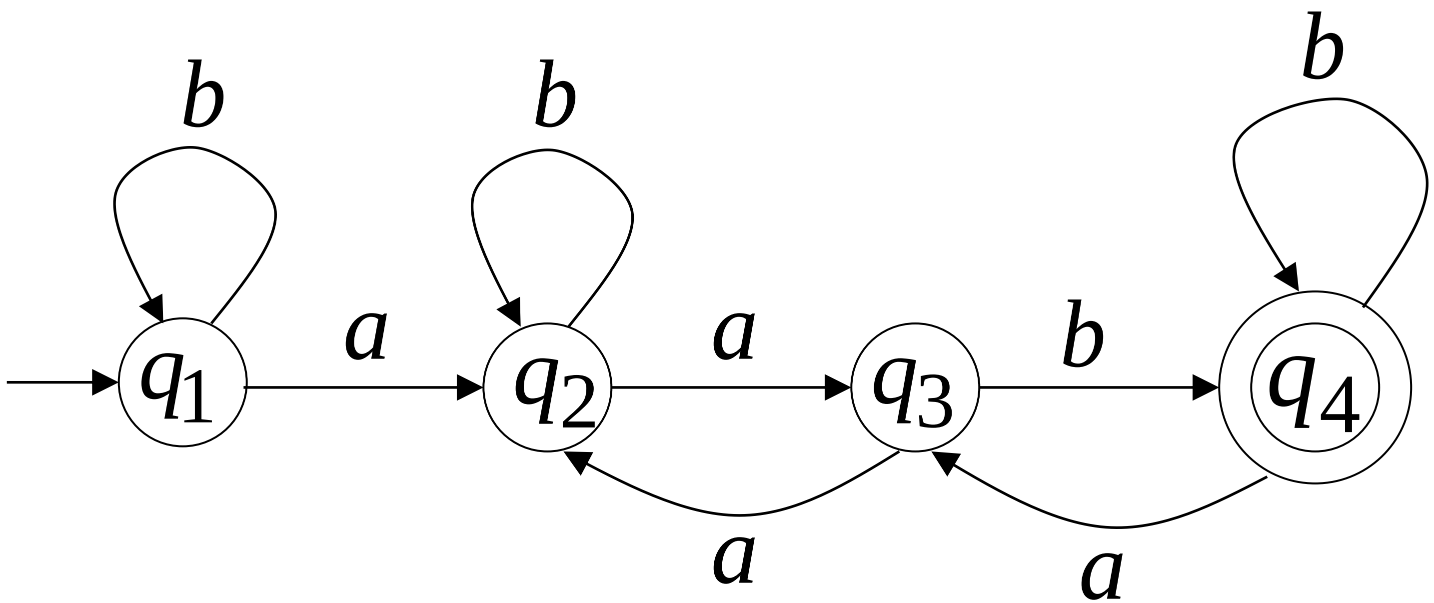
In walks of strings: $aabb$

$bbaa$

$abbabb$

$abbbabbabb...$

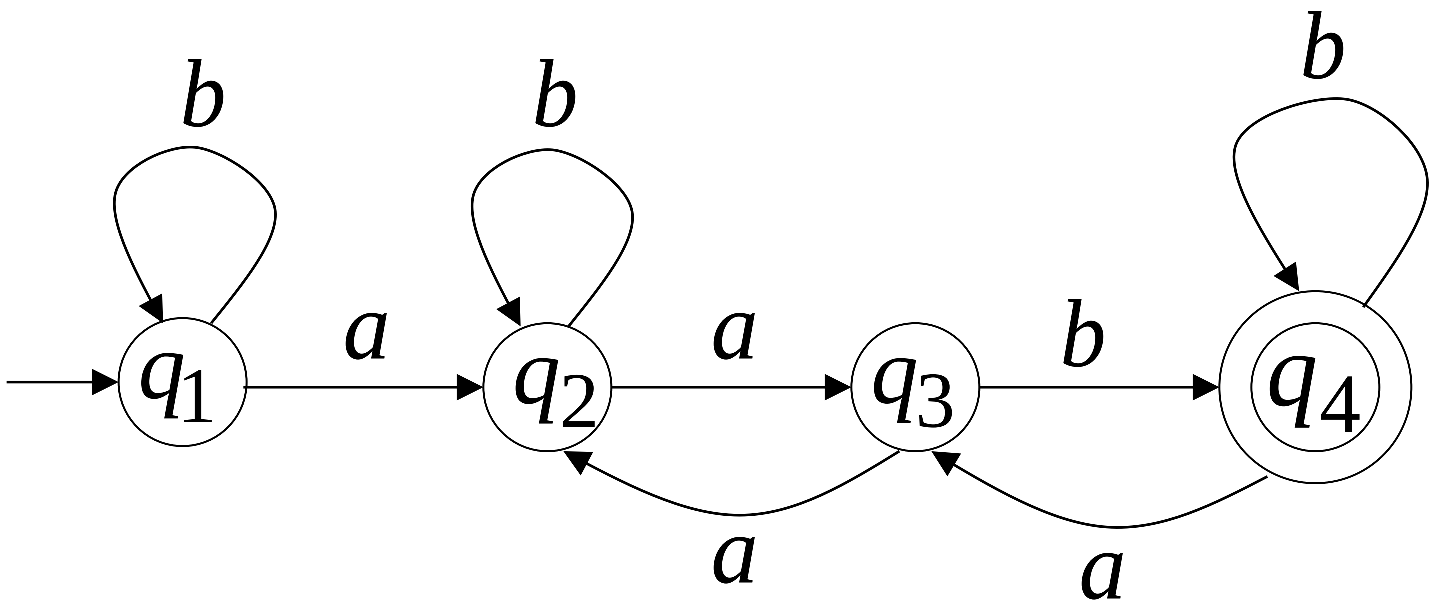
a state is repeated



If string w has length $|w| \geq 4$:

Then the transitions of string w are more than the states of the DFA

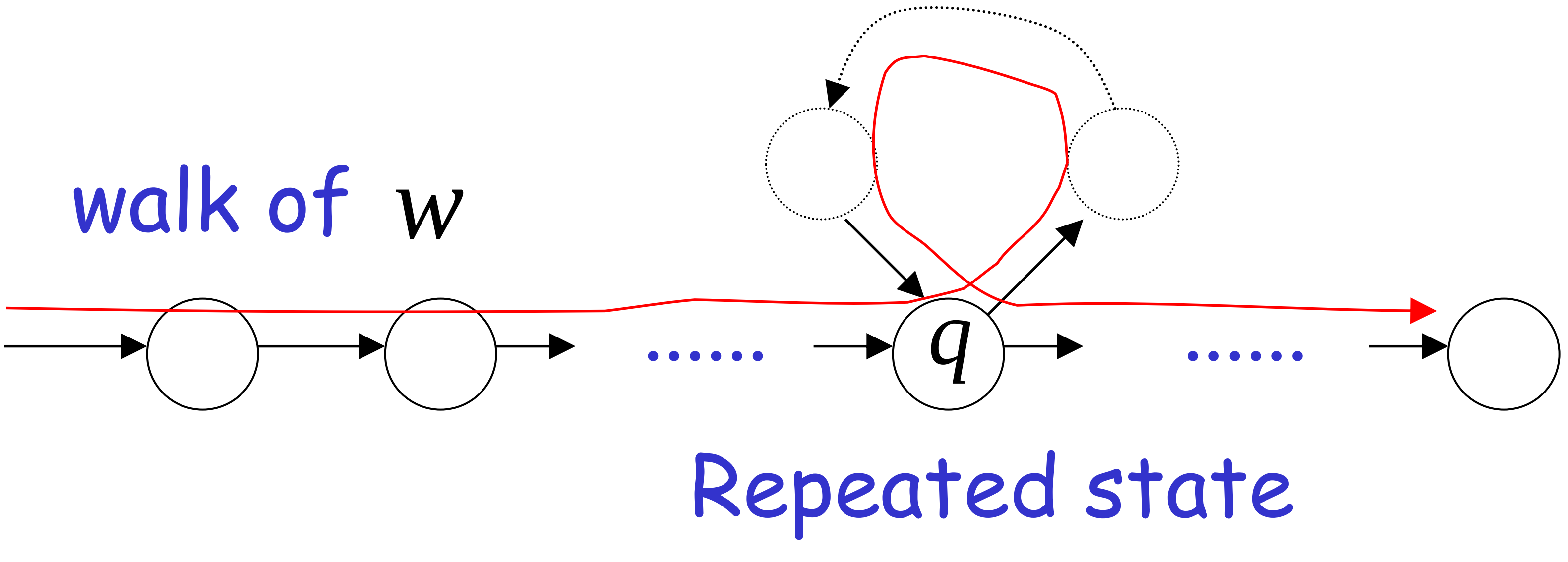
Thus, a state must be repeated



In general, for any DFA:

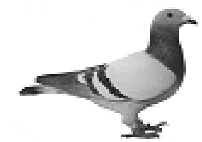
String w has length $\geq$ number of states

A state q must be repeated in the walk of w

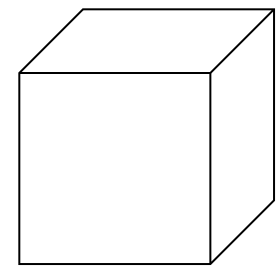


In other words for a string w:

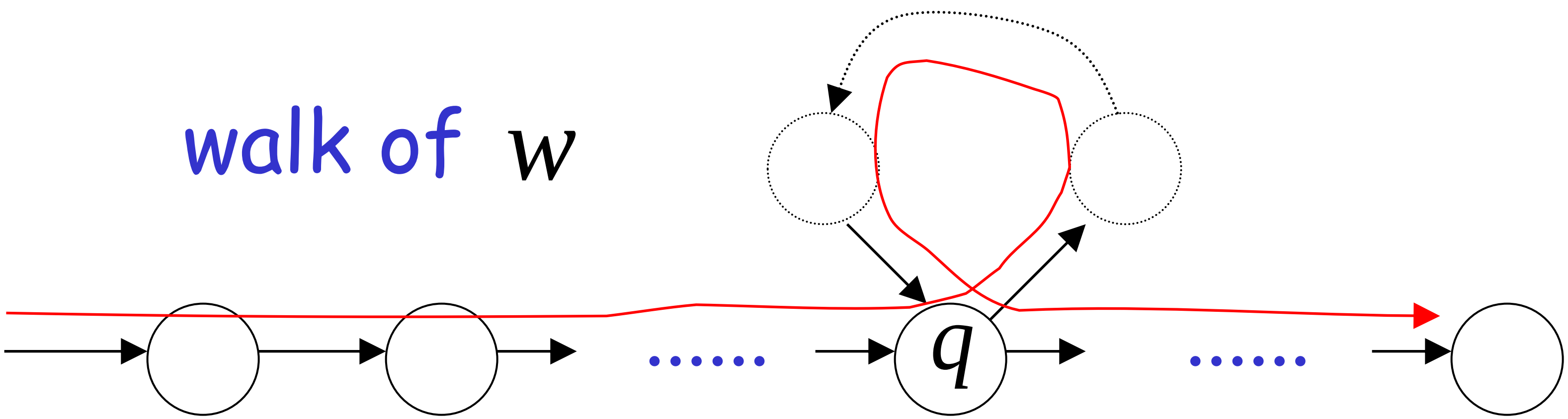
$\xrightarrow{a}$ transitions are pigeons



q states are pigeonholes



walk of w

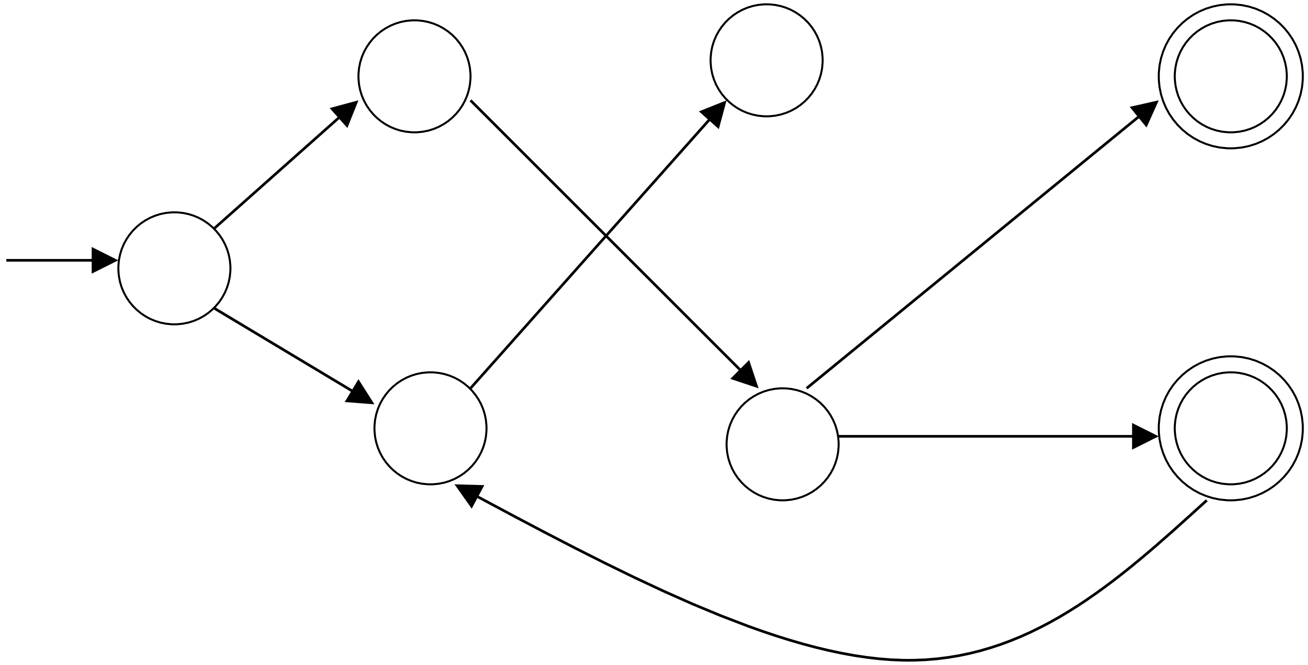


Repeated state

The Pumping Lemma

Take an infinite regular language L

DFA that accepts L



m
states

Take string w with $w \in L$

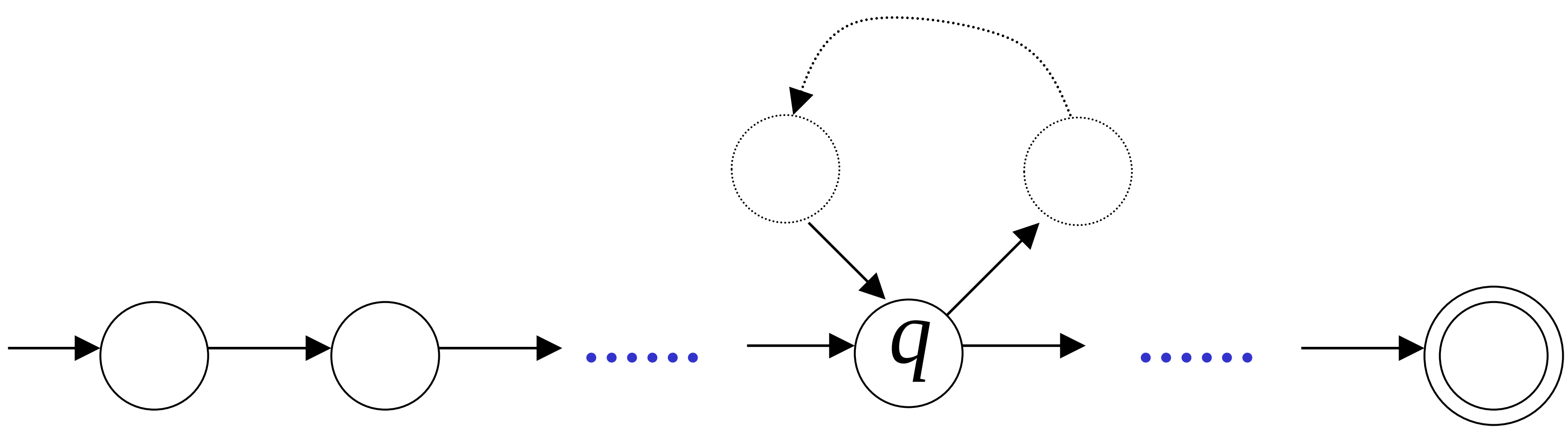
There is a walk with label w:

walk w

If string w has length $|w| \geq m$ number of states of DFA

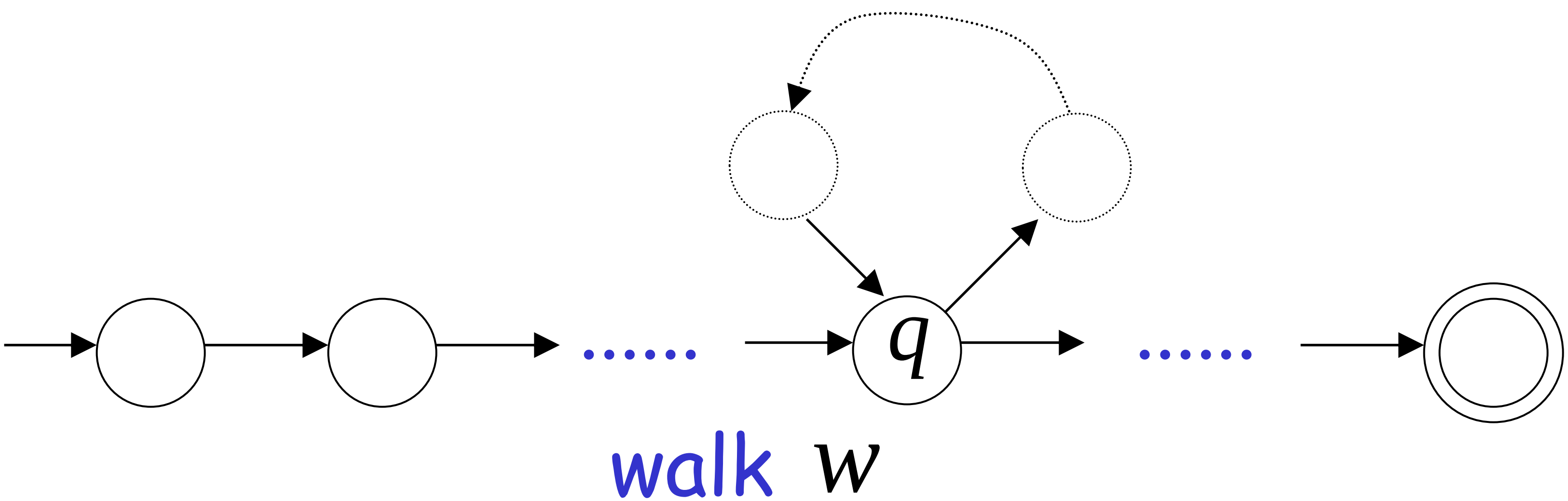
then, from the pigeonhole principle:

a state q is repeated in the walk w

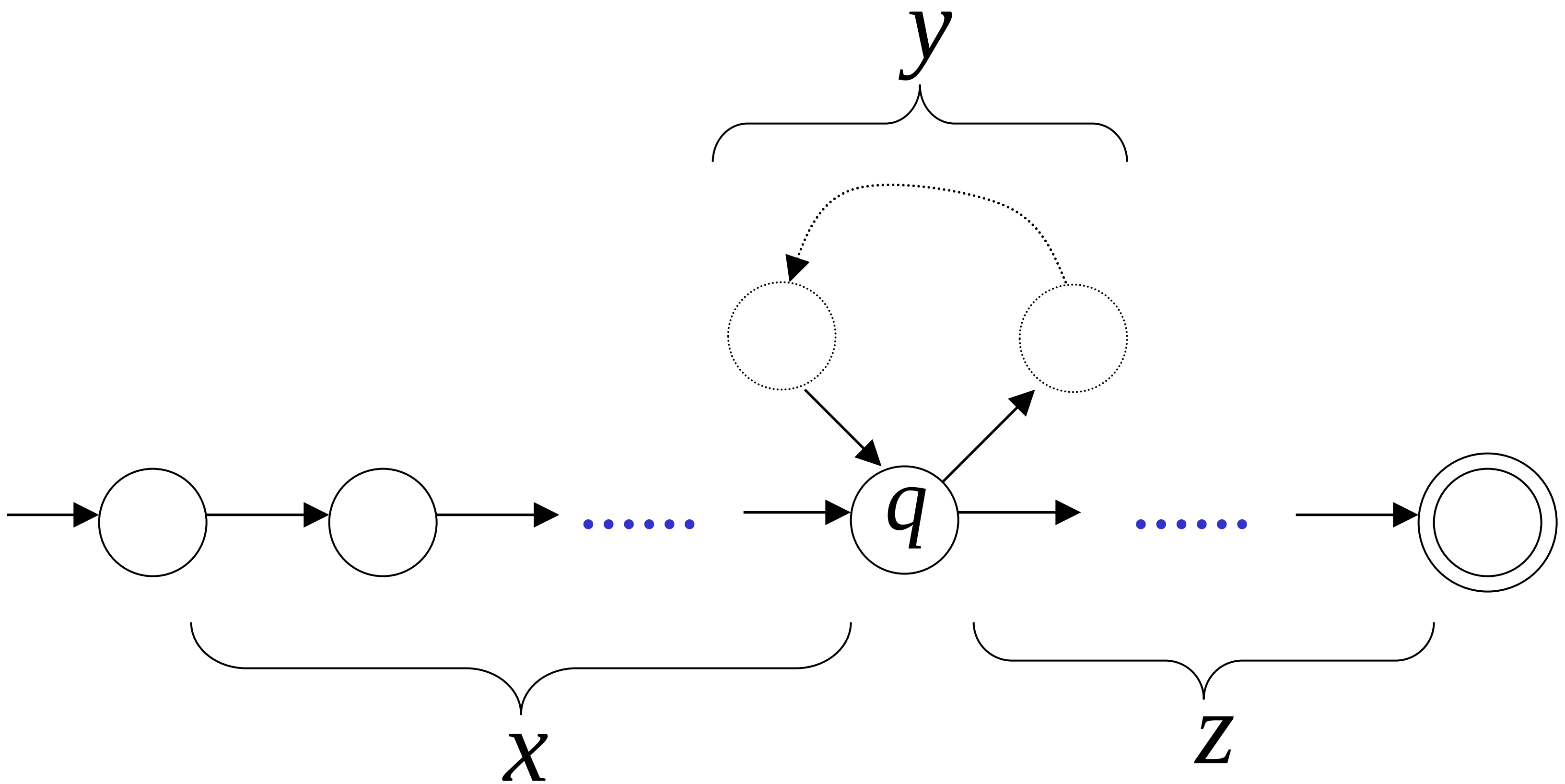


walk w

Let q be the first state repeated

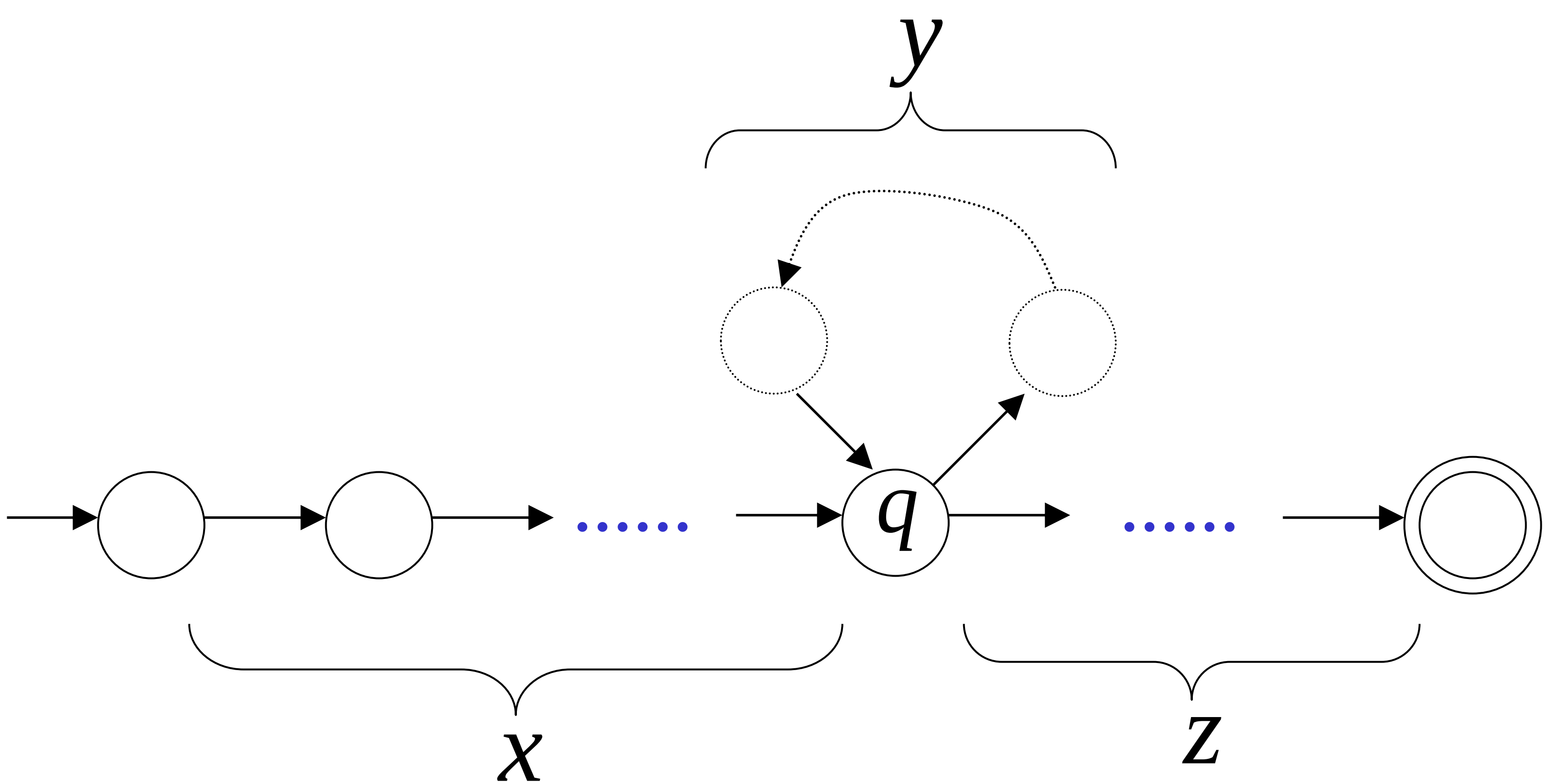


Write $w = x\ y\ z$

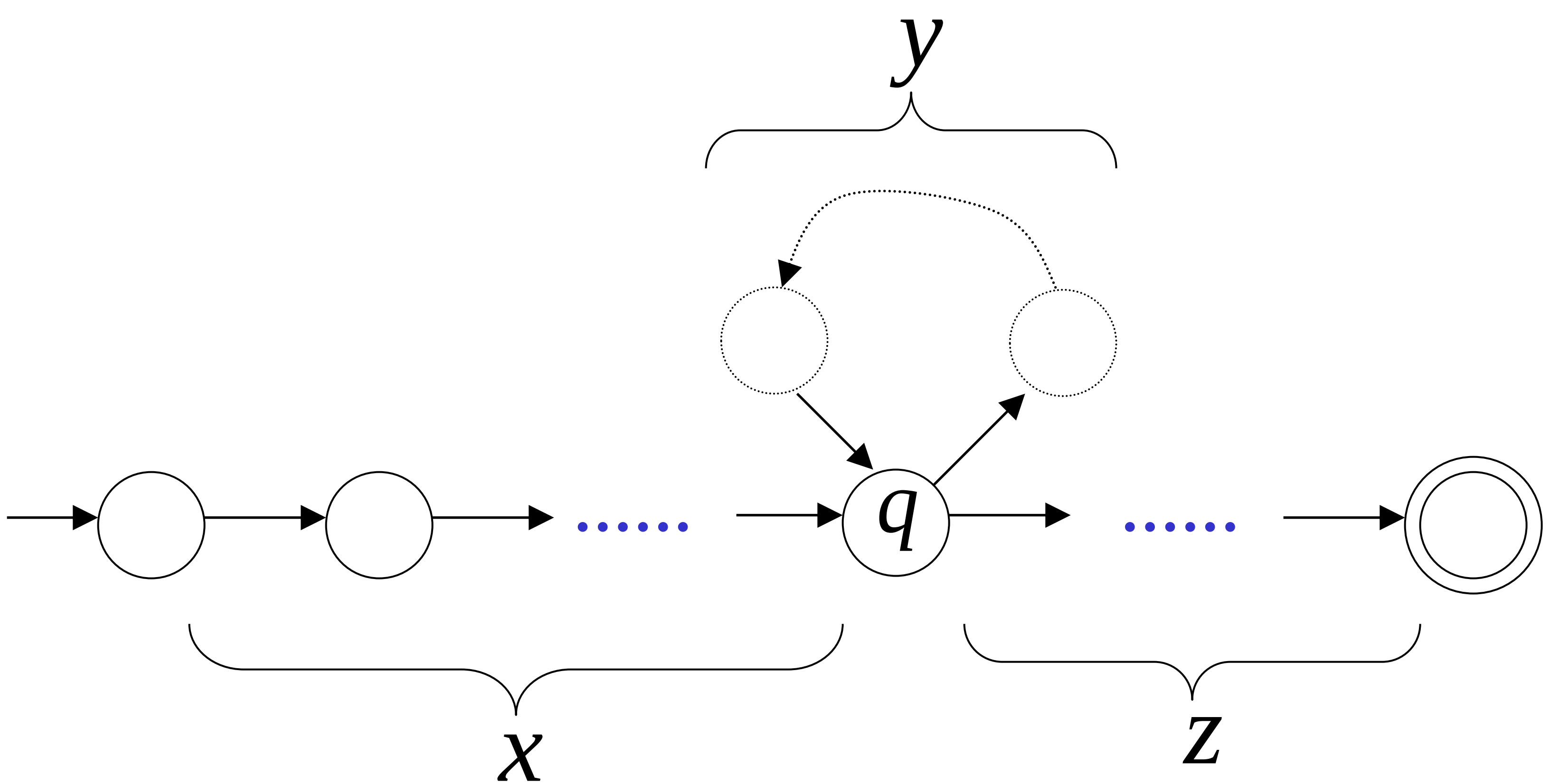


Observations: length $|x\,y| \leq m$ number of states of DFA

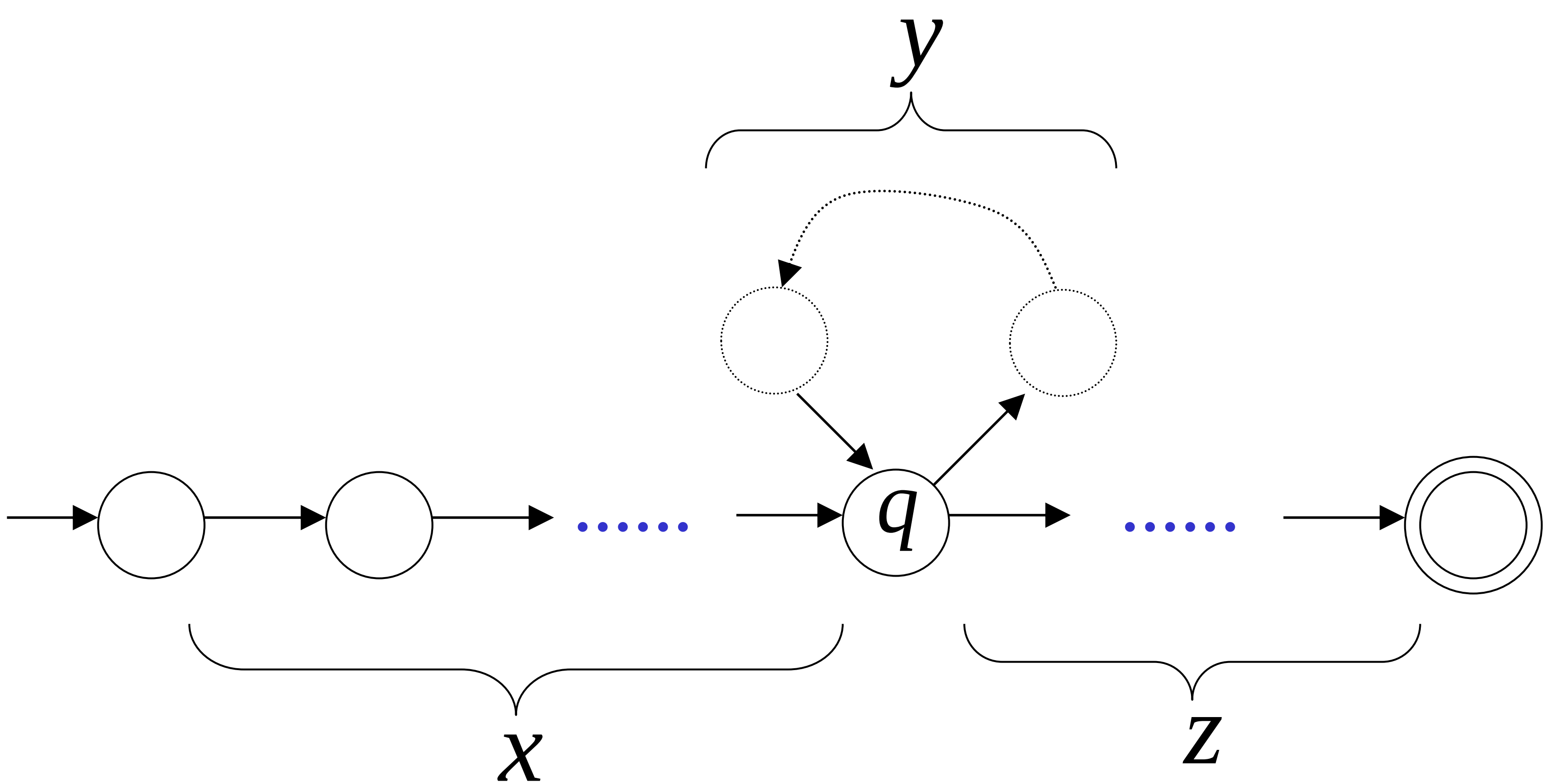
length $|y| \geq 1$



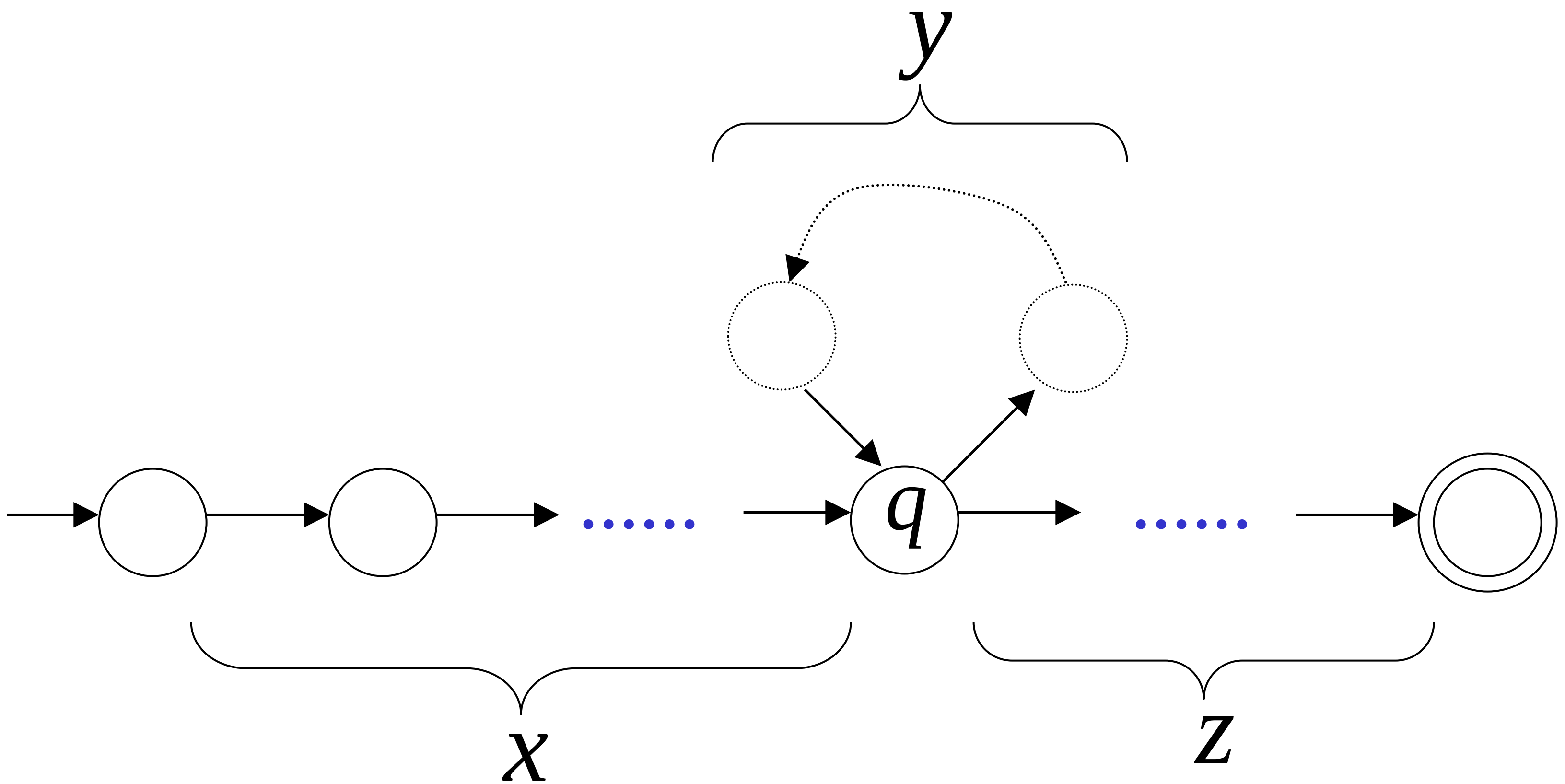
Observation: The string $x\ z$ is accepted



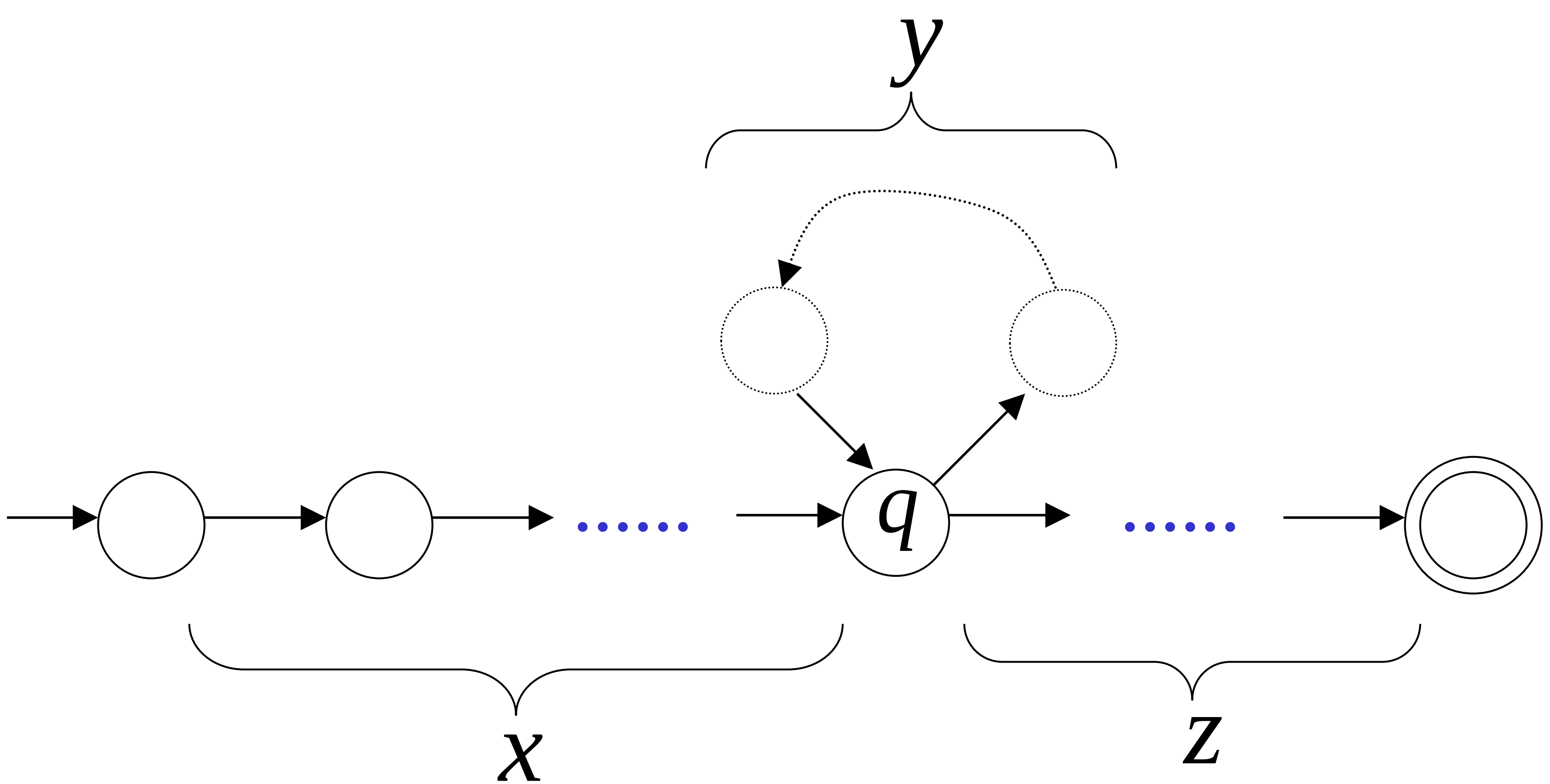
Observation: The string $x\ y\ y\ z$ is accepted



Observation: The string $x\,y\,y\,y\,z$ is accepted

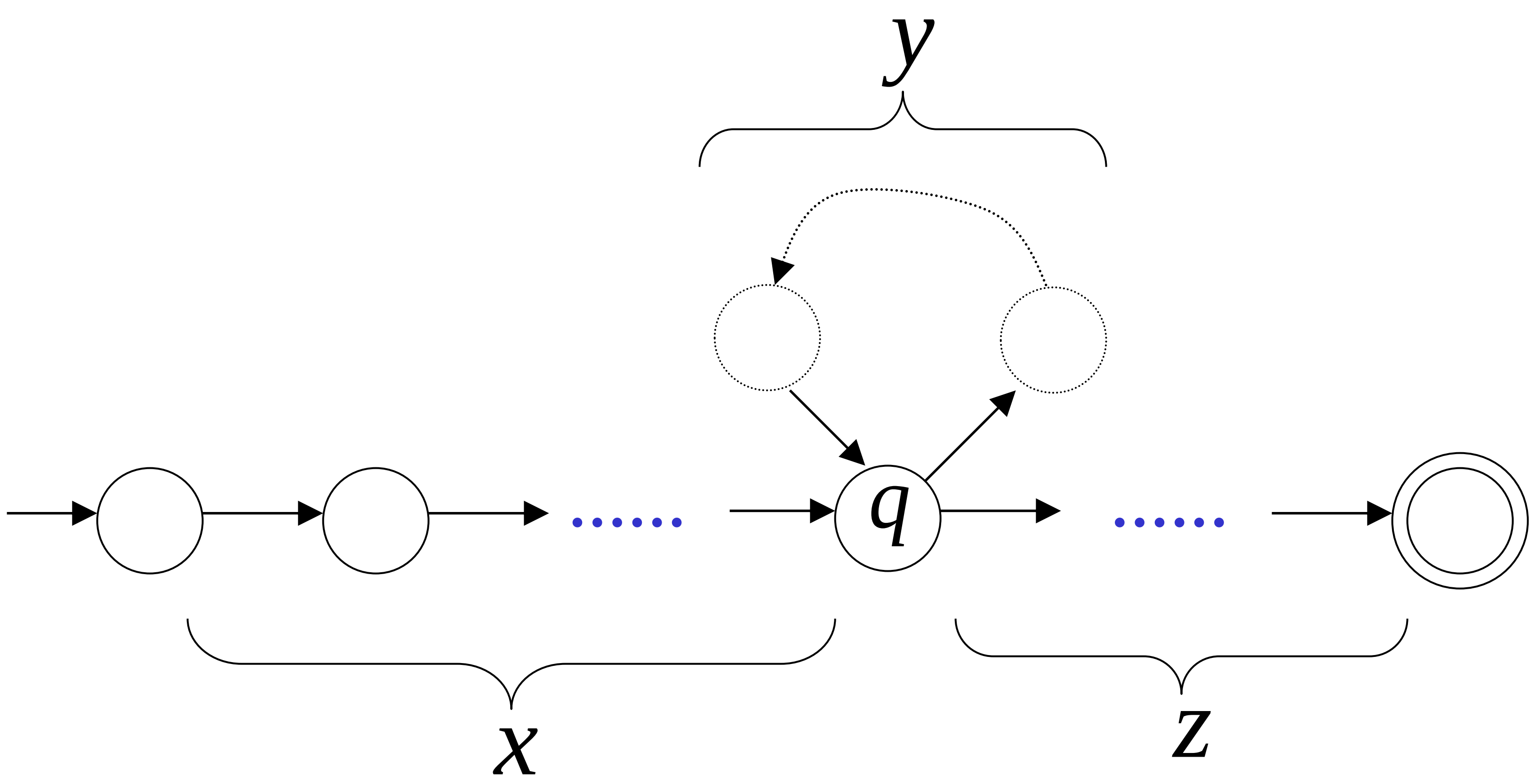


In General: The string $x\,y^i\,z$ is accepted $i = 0, 1, 2, \ldots$



In General: $x\,y^i\,z \in L \qquad i = 0, 1, 2, \ldots$

The original language



In other words, we described:

The Pumping Lemma !!!

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\, y\, z$
- with $|x\, y| \leq m$ and $|y| \geq 1$
- such that: $x\, y^i\, z \in L \qquad i = 0, 1, 2, \ldots$

Applications

of

the Pumping Lemma

Theorem: The language $L = \{a^n b^n : n \geq 0\}$

is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^n : n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^n b^n : n \geq 0\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

We pick $w = a^m b^m$

Write: $a^m b^m = x\ y\ z$

From the Pumping Lemma
it must be that length $|\, x\ y\,| \leq m, \quad |\, y\,| \geq 1$

$$xyz = a^m b^m = \underbrace{\overbrace{a...a}^{}_{x}\,\underbrace{a...a}_{y}\,\overbrace{a...a}^{m}}\ \overbrace{b...b}^{m}$$

$$xyz = a^m b^m = \underbrace{a...a}_{x}\,\underbrace{a...a}_{y}\,\underbrace{a...ab...b}_{z}$$

Thus: $y = a^k, \quad k \geq 1$

$$x\,y\,z = a^m b^m \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $\quad x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $\quad x\,y^2\,z \in L$

$$x\,y\,z = a^m b^m \qquad\qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\,y^2\,z \;\in\; L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}\underbrace{a...a}_{z}}^{m+k}\underbrace{\overbrace{b...b}^{m}}_{} \in L$$

Thus: $a^{m+k} b^m \in L$

$$a^{m+k}b^m \in L \qquad k \geq 1$$

BUT: $L = \{a^n b^n : n \geq 0\}$

$$\Downarrow$$

$$a^{m+k}b^m \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular language $\{a^n b^n : \; n \geq 0\}$