Normal Forms
for
Context-free Grammars

Linz 6th, Section 6.2

"Two Important Normal Forms,"

pages 171--178

Chomsky Normal Form

All productions have form:

$$A \to BC \qquad \text{and} \qquad A \to a$$

variable variable terminal

Examples:

$$S \to AS$$
$$S \to a$$
$$A \to SA$$
$$A \to b$$

Chomsky
Normal Form

$$S \to AS$$
$$S \to AAS$$
$$A \to SA$$
$$A \to aa$$

Not Chomsky
Normal Form

Conversion to Chomsky Normal Form

Example:

$$S \to ABa$$

$$A \to aab$$

$$B \to Ac$$

Not Chomsky
Normal Form

Introduce variables for terminals: T_a, T_b, T_c

$$S \to ABa$$
$$A \to aab$$
$$B \to Ac$$

$$S \to ABT_a$$
$$A \to T_aT_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Introduce intermediate variable: V_1

$$S \to ABT_a$$
$$A \to T_aT_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

$$\Longrightarrow$$

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$A \to T_aT_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Introduce intermediate variable: V_2

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$A \to T_aT_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

$$\Longrightarrow$$

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$A \to T_aV_2$$
$$V_2 \to T_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Final grammar in Chomsky Normal Form:

Initial grammar

$$S \to ABa$$

$$A \to aab$$

$$B \to Ac$$

$$S \to AV_1$$

$$V_1 \to BT_a$$

$$A \to T_a V_2$$

$$V_2 \to T_a T_b$$

$$B \to AT_c$$

$$T_a \to a$$

$$T_b \to b$$

$$T_c \to c$$

In general:

From any context-free grammar
not in Chomsky Normal Form

we can obtain:

An equivalent grammar
in Chomsky Normal Form

The Procedure

First remove:

Nullable variables

Unit productions

For every symbol a:

Add production $T_a \to a$

In productions: replace a with T_a

New variable: T_a

Replace any production $A \to C_1 C_2 \cdots C_n$

with
$$A \to C_1 V_1$$
$$V_1 \to C_2 V_2$$
$$\cdots$$
$$V_{n-2} \to C_{n-1} C_n$$

New intermediate variables: $V_1, V_2, \ldots, V_{n-2}$

Theorem: For any context-free grammar G such that the empty string is not in L(G), there is an equivalent grammar in Chomsky Normal Form

Linz, 6th, Theorem 6.6, page 172.

Observations

- Chomsky normal forms are good for parsing and proving theorems

- It is very easy to find the Chomsky normal form of any context-free grammar

Greibach Normal Form

All productions have form:

$$A \to a\, V_1 V_2 \cdots V_k \qquad\qquad k \geq 0$$

symbol

variables

Examples:

$$S \to cAB$$

$$A \to aA \mid bB \mid b$$

$$B \to b$$

Greibach
Normal Form

$$S \to abSb$$

$$S \to aa$$

Not Greibach
Normal Form

Conversion to Greibach Normal Form:

$$S \to abSb$$
$$S \to aa$$

$$S \to aT_bST_b$$
$$S \to aT_a$$
$$T_a \to a$$
$$T_b \to b$$

Greibach
Normal Form

Theorem: For any context-free grammar G such that the empty string is not in $L(G)$, there is an equivalent grammar in Greibach Normal Form

Linz, 6th, Theorem 6.7, page 176.
Proof not given because it is too complicated.

Observations

- Greibach normal forms are very good for parsing

- It is hard to find the Greibach normal form of any context-free grammar

An Application
of
Chomsky Normal Forms

The CYK Membership Algorithm

J. Cocke, D. H. Younger, and T. Kasami

Input:

- Grammar G in Chomsky Normal Form

- String w

Output:

find if $w \in L(G)$

Considers every possible consecutive subsequence of letters and sets $K \in T[i,j]$ if the sequence of letters starting from i to j can be generated from the non-terminal K.

Once it has considered sequences of length 1, it goes on to sequences of length 2, and so on.

For subsequences of length 2 and greater, it considers every possible partition of the subsequence into two halves, and checks to see if there is some production *A*-> BC such that B matches the first half and C Matches the second half. If so, it records *A* as matching the whole subsequence.

Once this process is completed, the sentence is recognized by the grammar if the entire string is matched by the start symbol.

The Algorithm

Input example:

- Grammar G : $S \to AB$

$$A \to BB$$
$$A \to a$$
$$B \to AB$$
$$B \to b$$

- String w : $aabbb$

aabbb

[0:1]	[1:2]	[2:3]	[3:4]	[4:5]
[0:2]	[1:3]	[2:4]	[3:5]	
[0:3]	[1:4]	[2:5]		
[0:4]	[1:5]			
[0:5]				

aabbb

a	a	b	b	b
aa	ab	bb	bb	
aab	abb	bbb		
aabb	abbb			
aabbb				

$$S \to AB$$

$$A \to BB$$

$$A \to a$$

$$B \to AB$$

$$B \to b$$

a	a	b	b	b
A	A	B	B	B
aa	ab	bb	bb	
aab	abb	bbb		
aabb	abbb			
aabbb				

$$S \to AB$$

$$A \to BB$$

$$A \to a$$

$$B \to AB$$

$$B \to b$$

a	a	b	b	b
A	A	B	B	B

aa	ab	bb	bb
	S,B	A	A

aab	abb	bbb

aabb	abbb

aabbb

$$S \to AB$$

$$A \to BB$$

$$A \to a$$

$$B \to AB$$

$$B \to b$$

a A	a A	b B	b B	b B
aa	ab S,B	bb A	bb A	
aab S,B	abb A	bbb S,B		
aabb A	abbb S,B			
aabbb S,B				

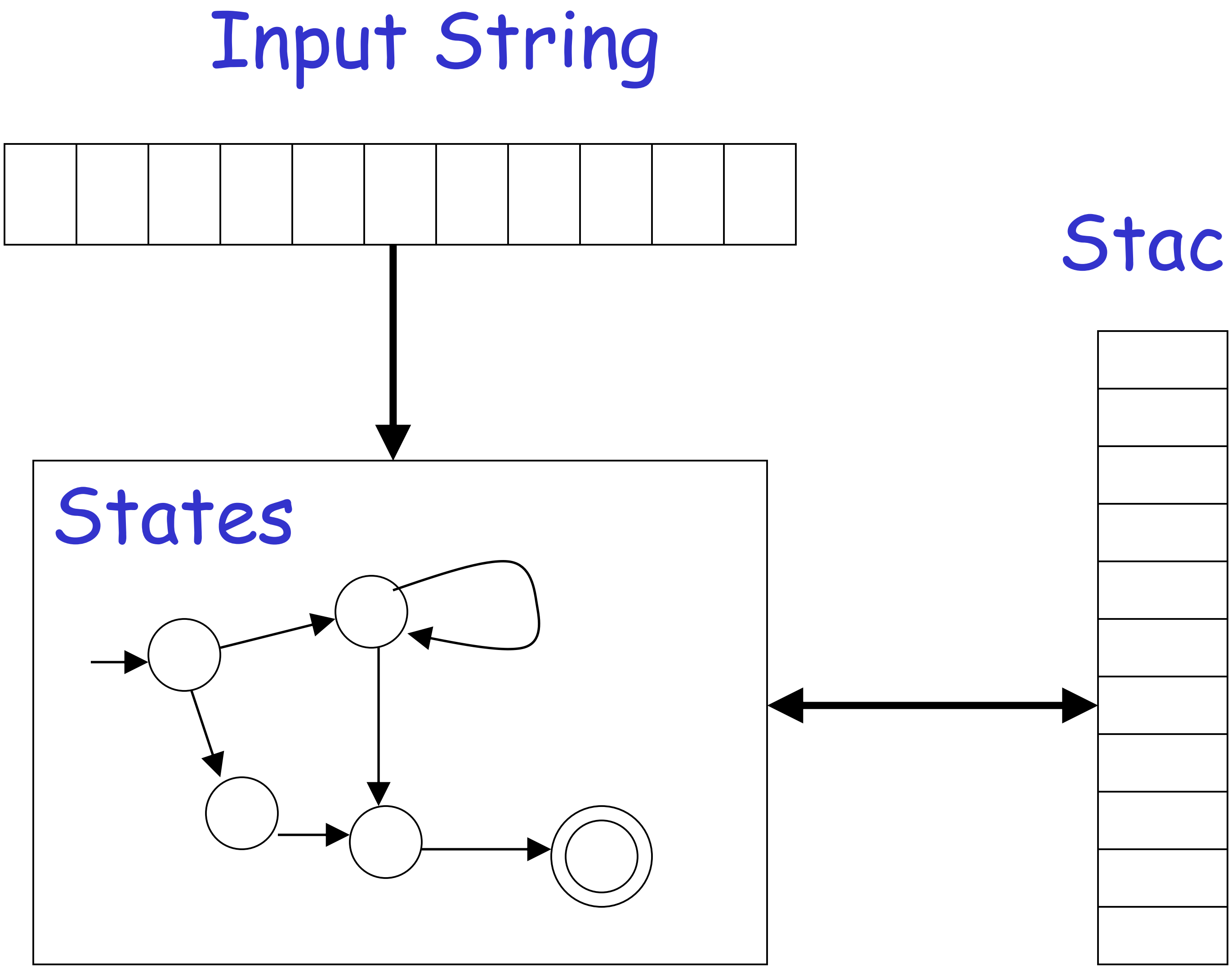
Therefore: $aabbb \in L(G)$

Time Complexity: $|w|^3$

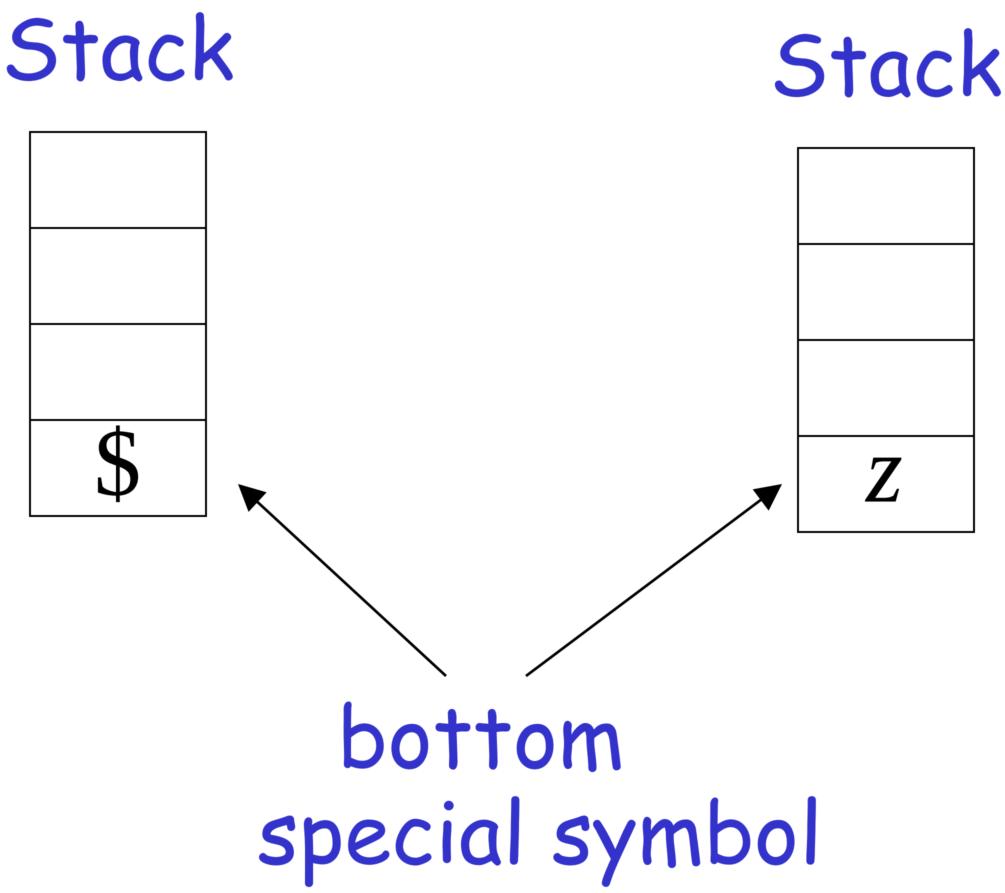
Observation: The CYK algorithm can be easily converted to a parser

Pushdown Automata
PDAs

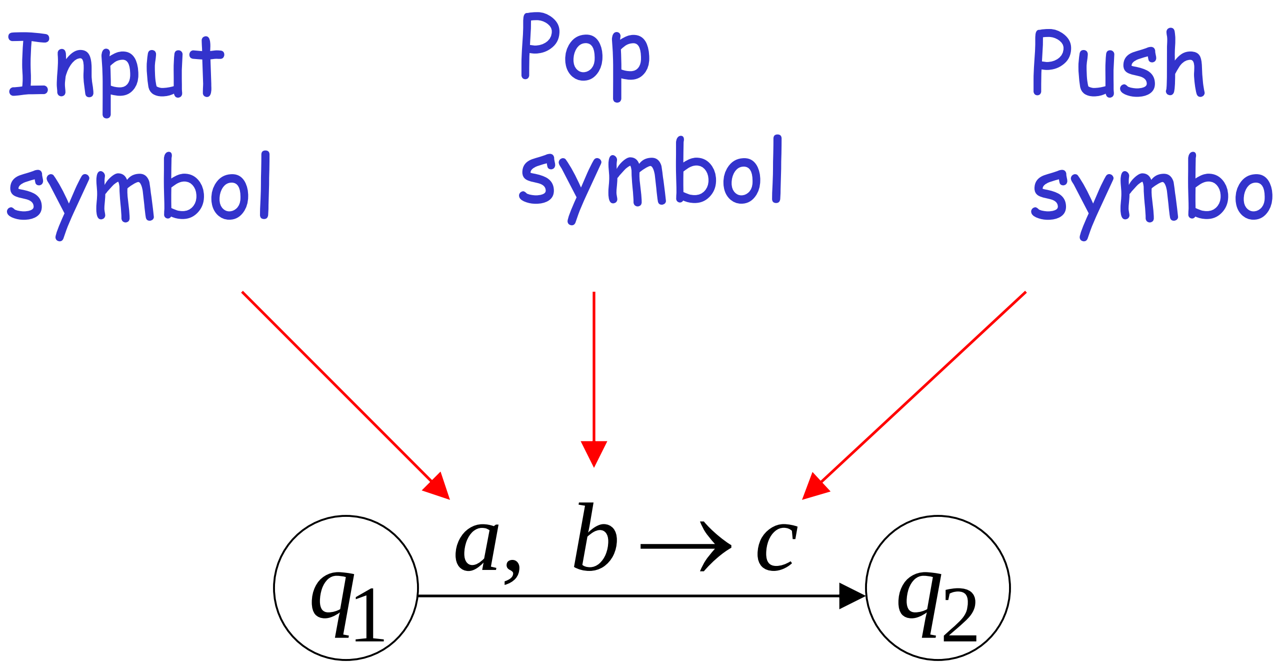
Pushdown Automaton -- PDA

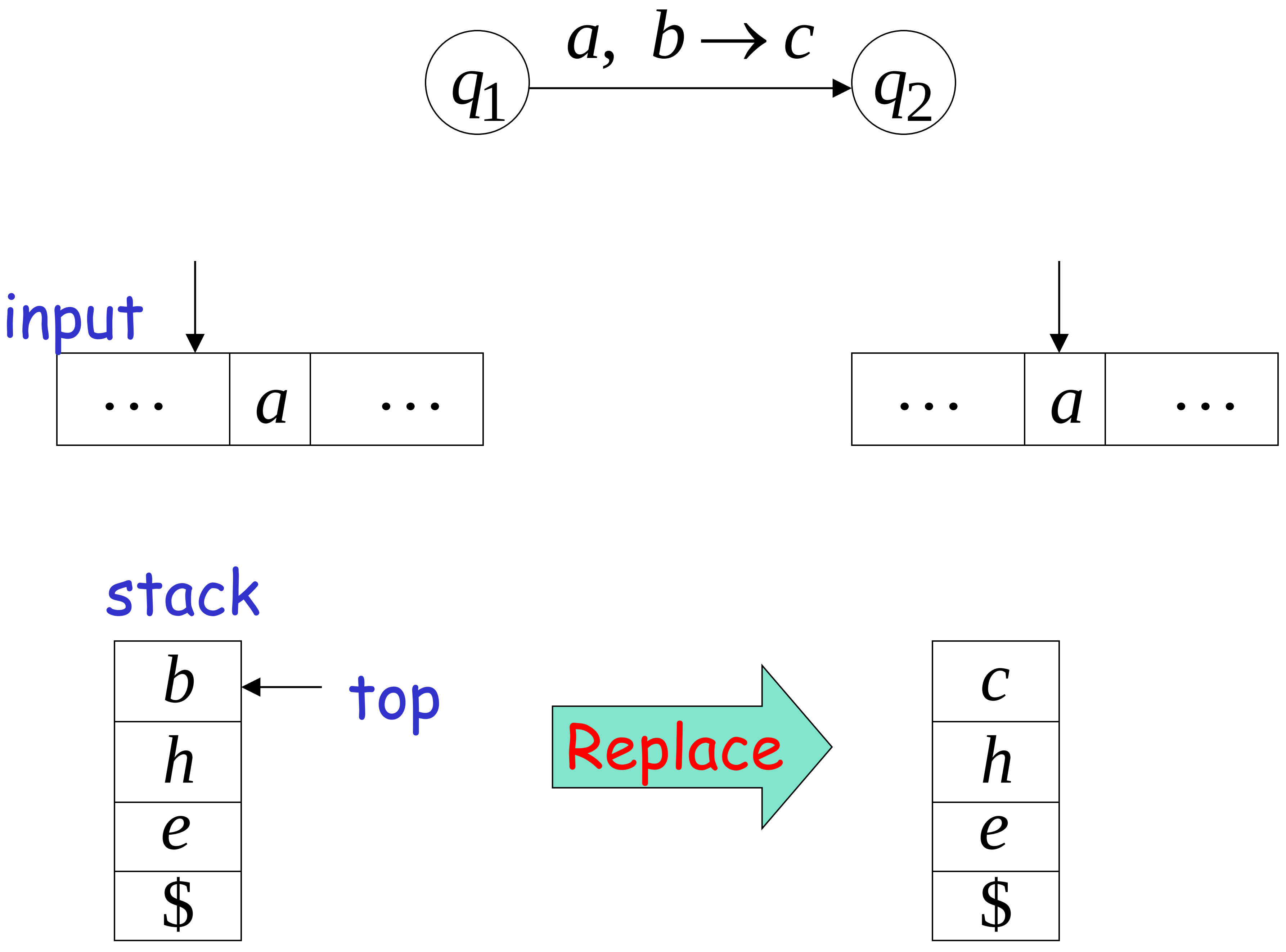


Initial Stack Symbol

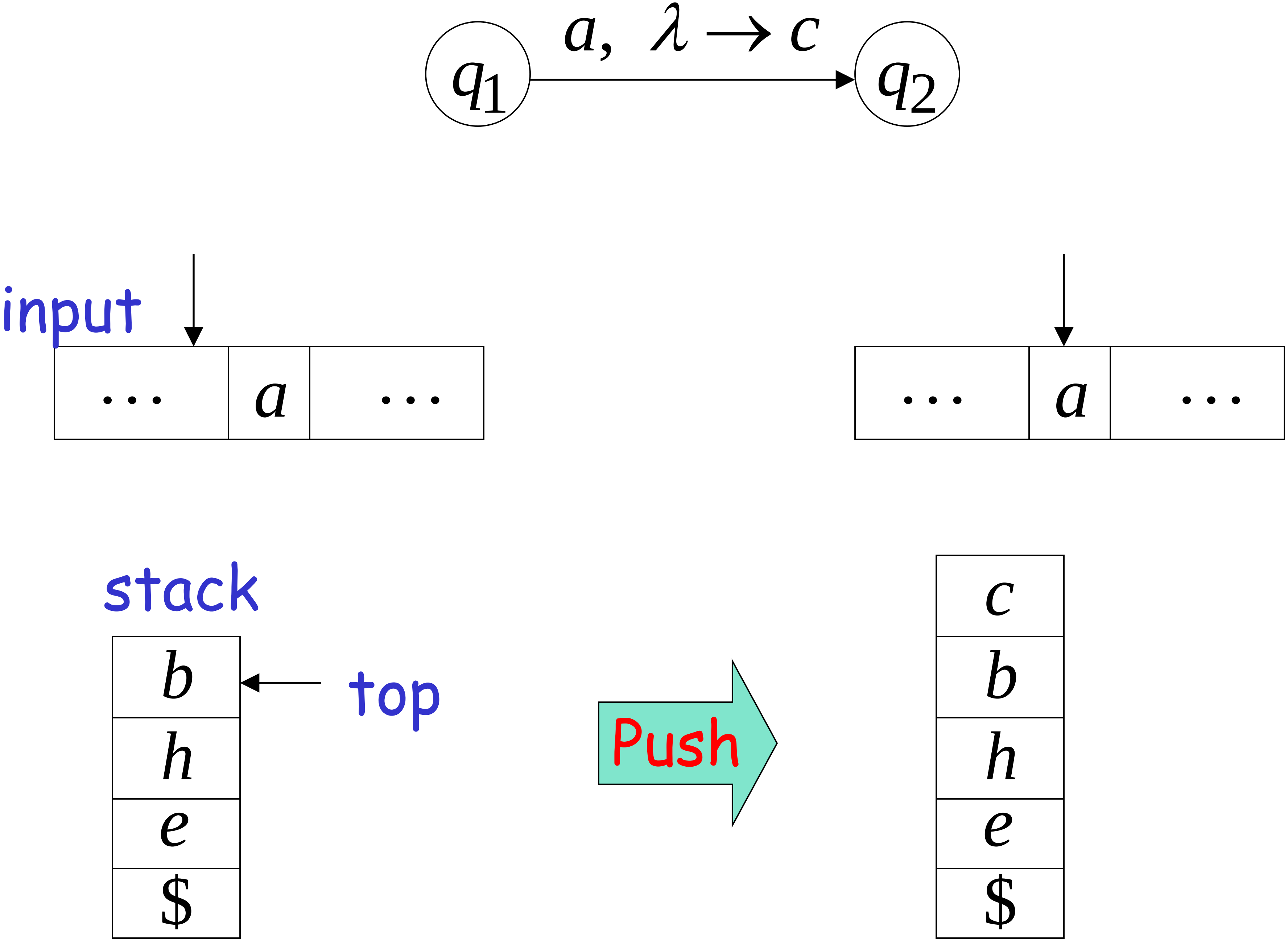


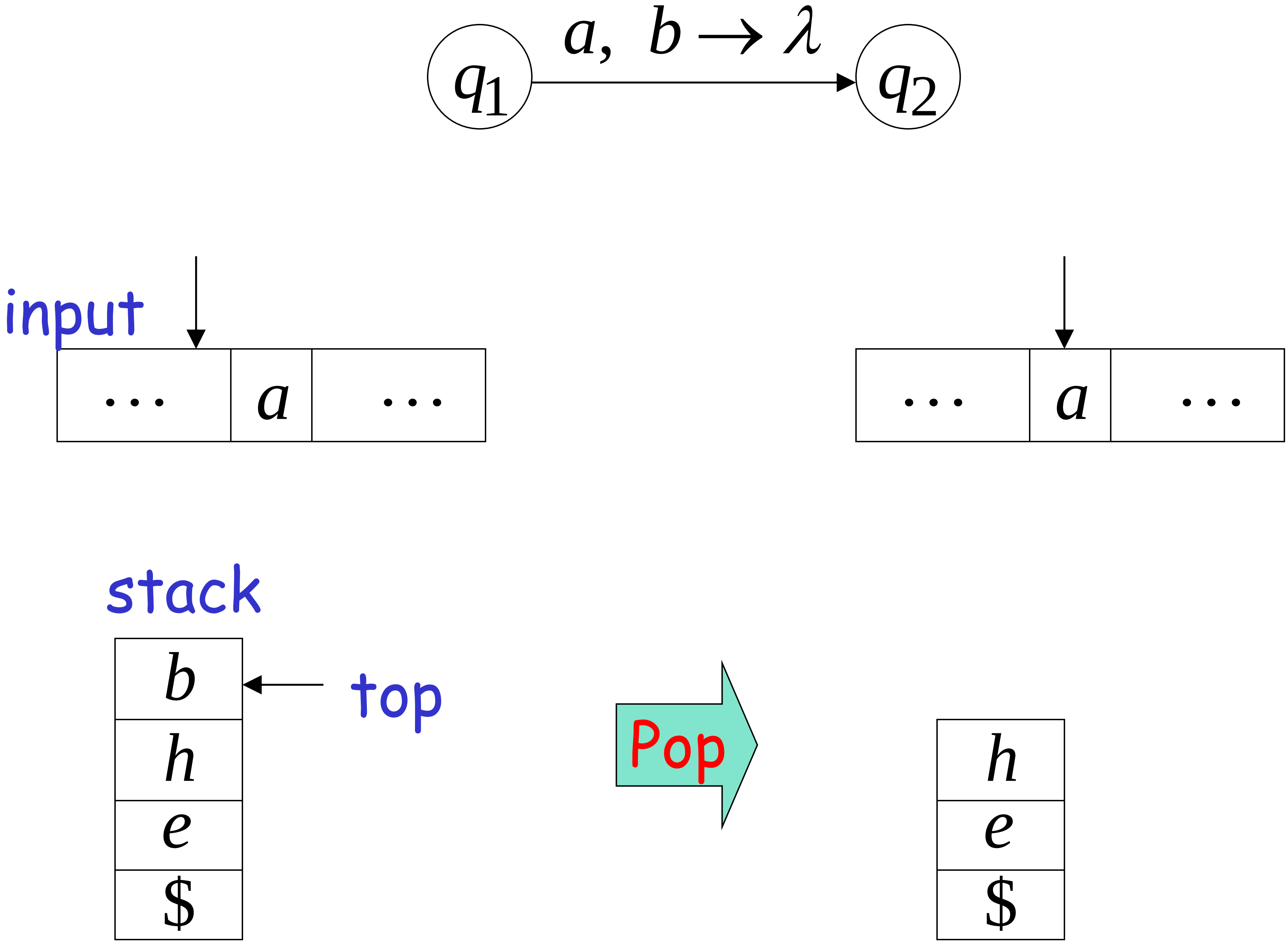
The States

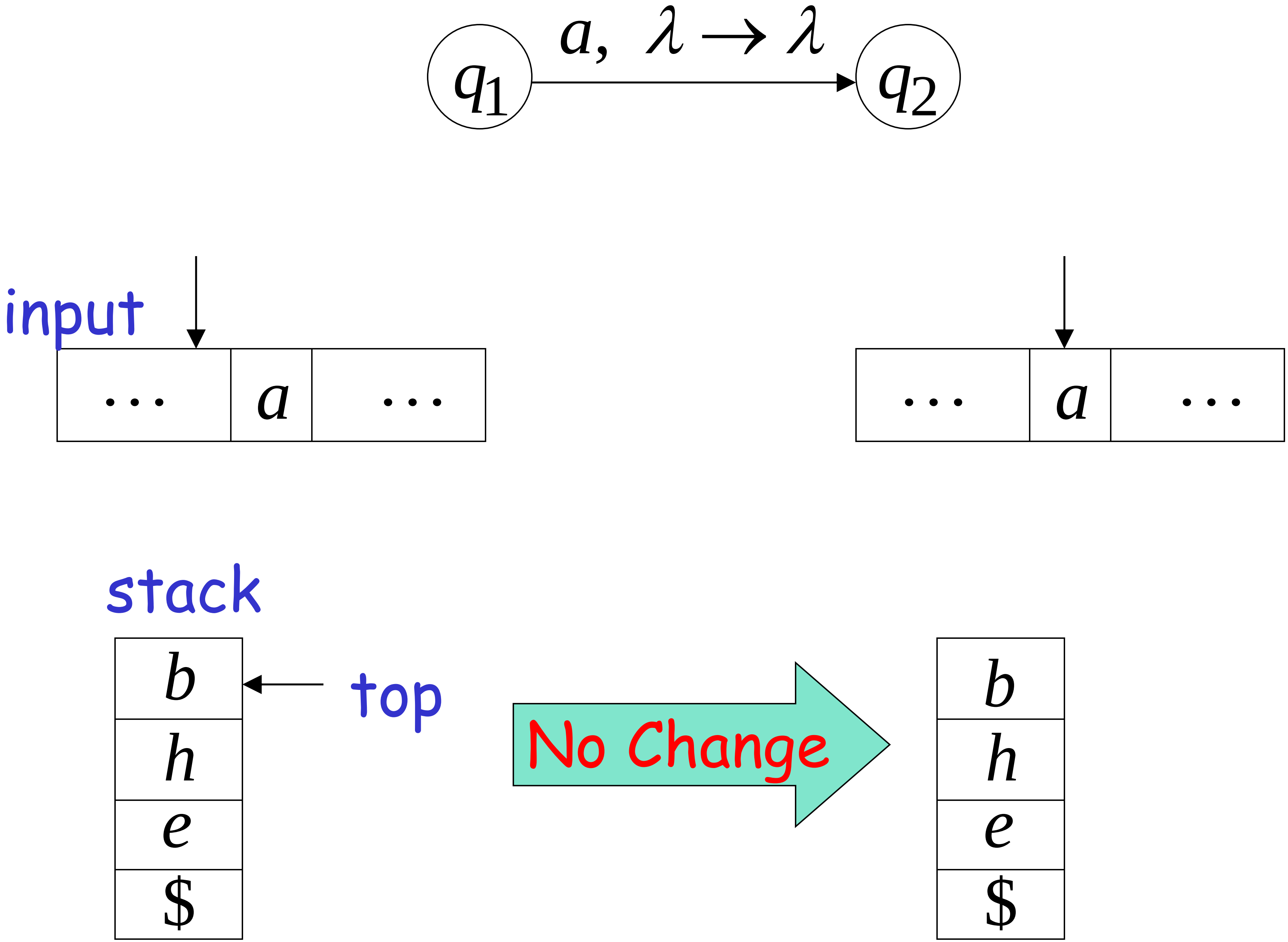




$a, \ b \rightarrow c$
q_1
q_2
input
$\cdots$ a $\cdots$
$\cdots$ a $\cdots$
stack
top
b
h
e
$\$$
Replace
c
h
e
$\$$

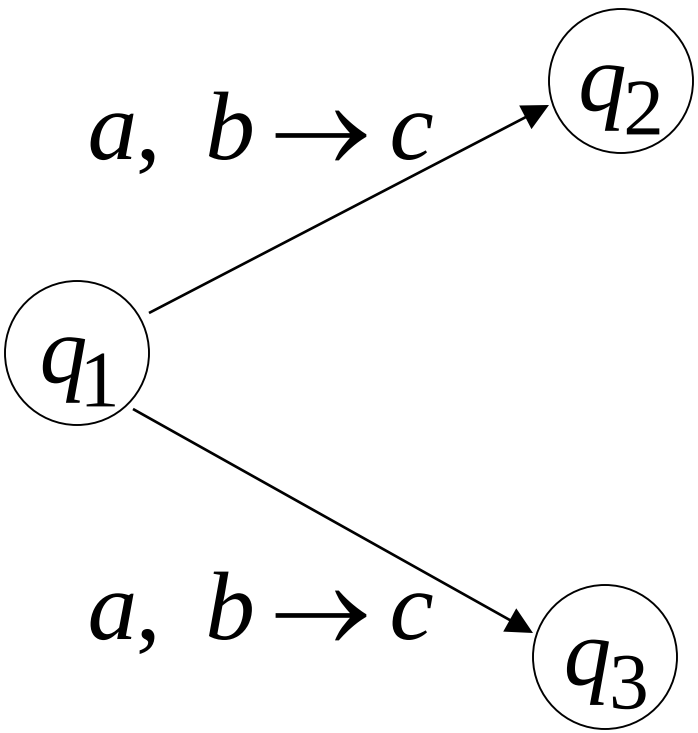

$a, \lambda \rightarrow c$
q_1
q_2
input
$\cdots$ a $\cdots$
$\cdots$ a $\cdots$
stack
b
h
e
$\$$
top
Push
c
b
h
e
$\$$

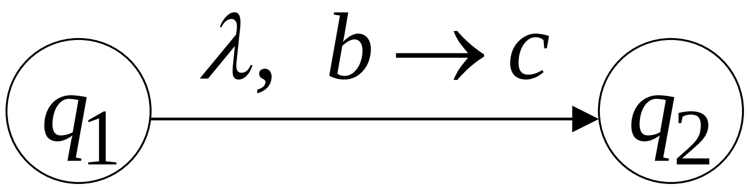

$a, \; b \rightarrow \lambda$
q_1
q_2
input
$\cdots$ a $\cdots$
$\cdots$ a $\cdots$
stack
top
b
h
e
$\$$
Pop
h
e
$\$$



$a, \lambda \rightarrow \lambda$
q_1
q_2
input
$\cdots$ a $\cdots$
$\cdots$ a $\cdots$
stack
top
b h e $\$$
No Change
b h e $\$$

Non-Determinism



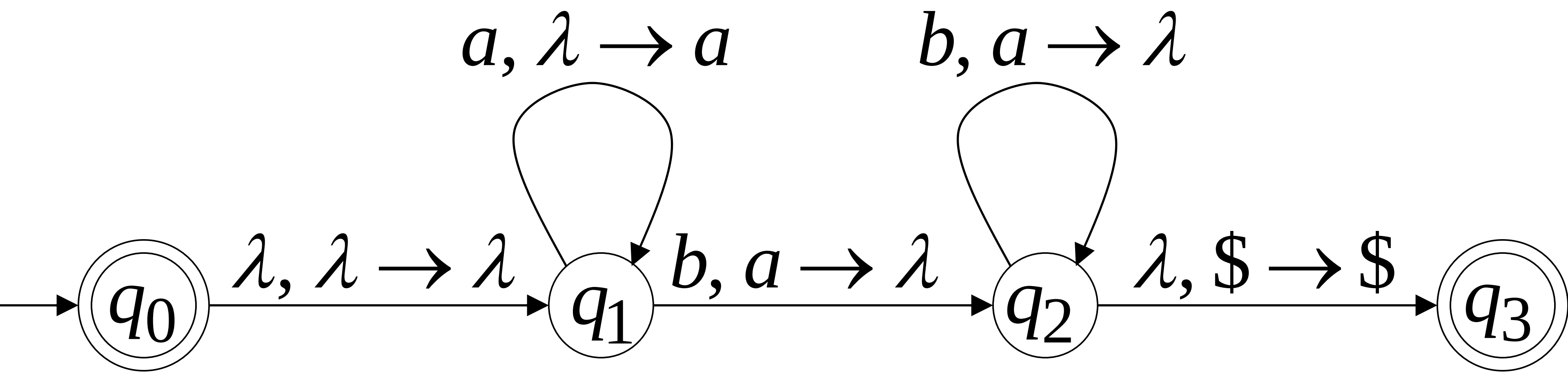


$\lambda -$ transition

These are allowed transitions in a Non-deterministic PDA (NPDA)

NPDA: Non-Deterministic PDA

Example:



Execution Example: Time 0

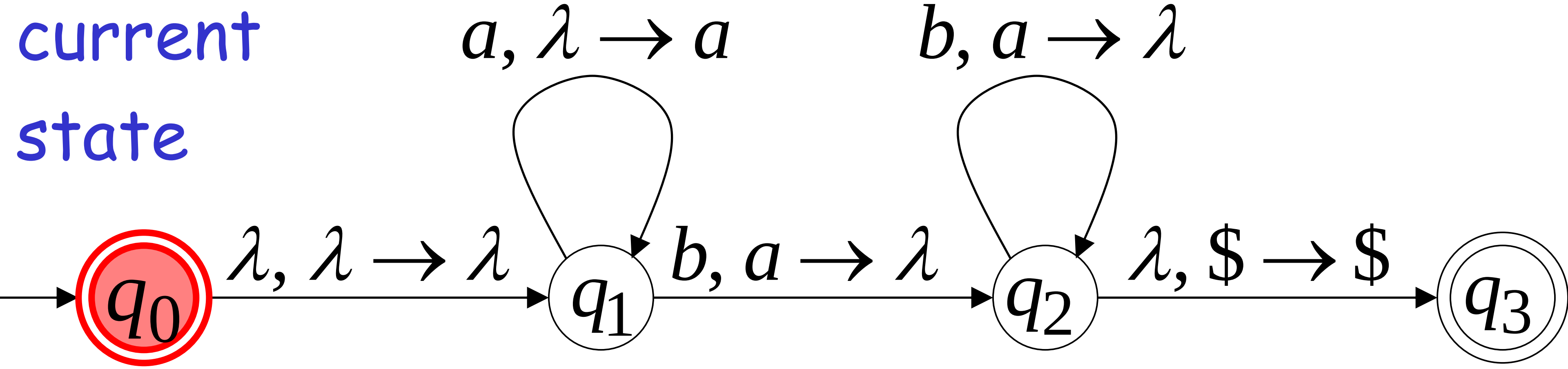
Input

a	a	a	b	b	b

$\$$

Stack

current state



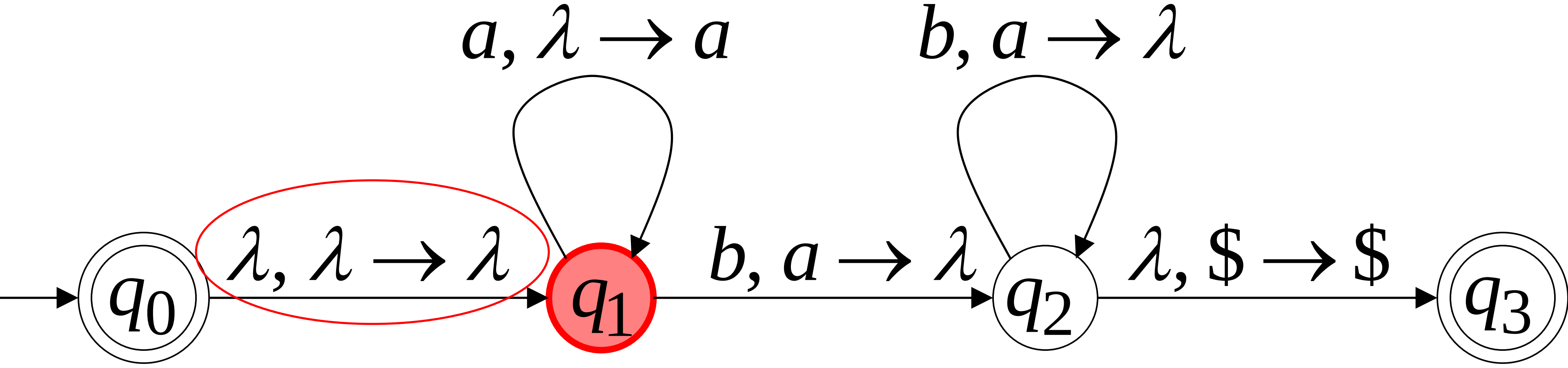
Time 1

Input

a	a	a	b	b	b

$\$$

Stack



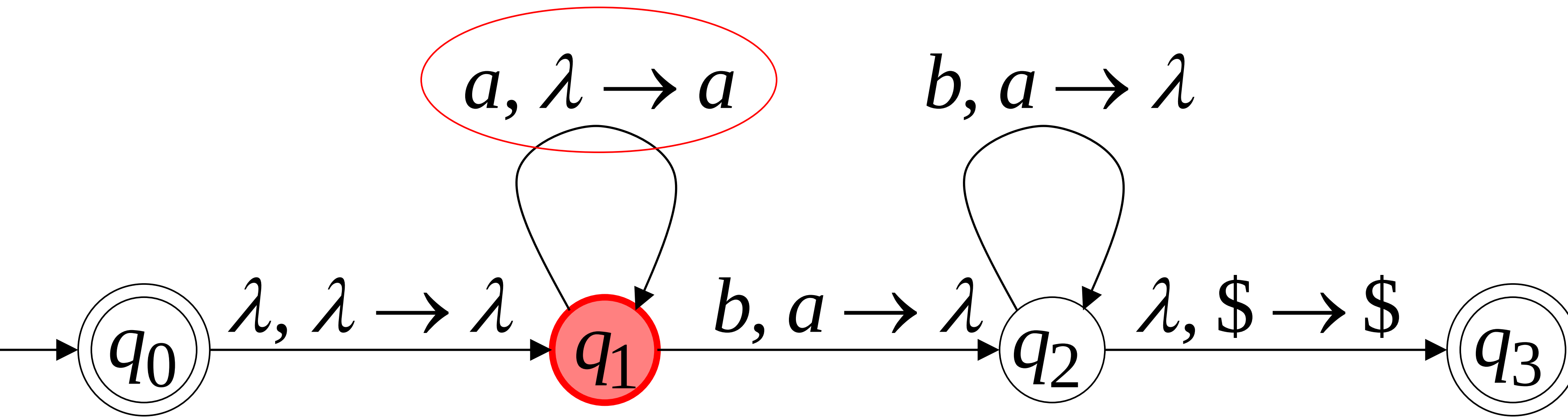
Time 2

Input

a	a	a	b	b	b

a
$\$$

Stack



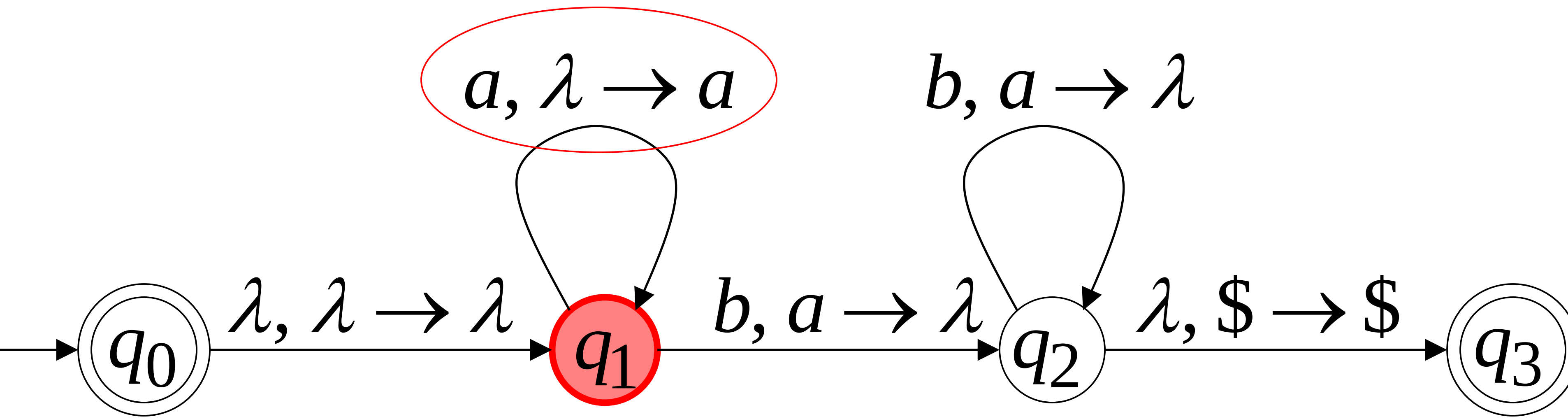
Time 3

Input

a	a	a	b	b	b

a
a
$\$$

Stack



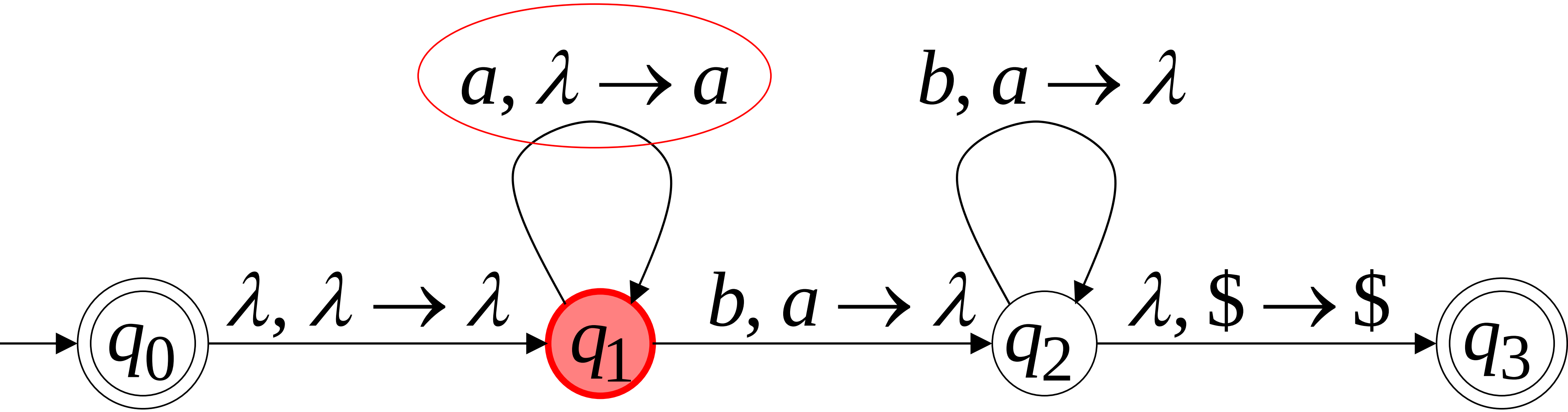
Time 4

Input

a	a	a	b	b	b

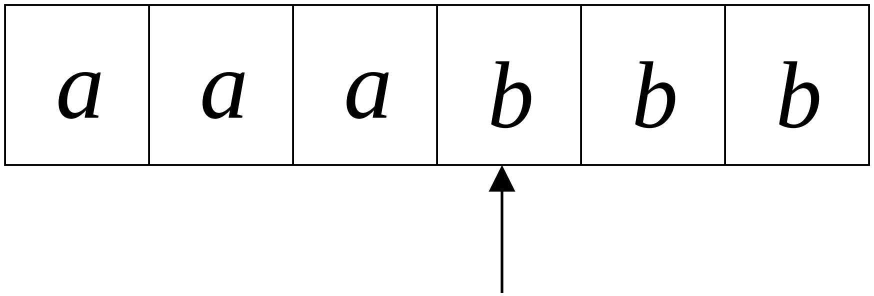
a
a
a
$\$$

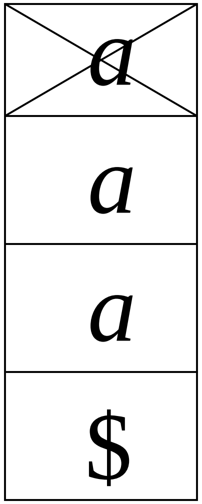
Stack



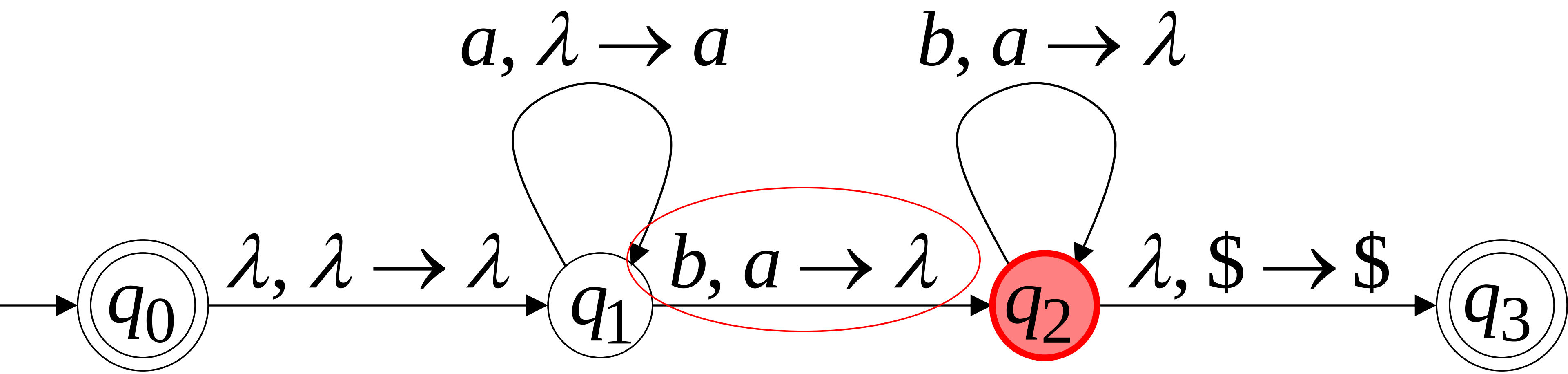
Time 5

Input





Stack



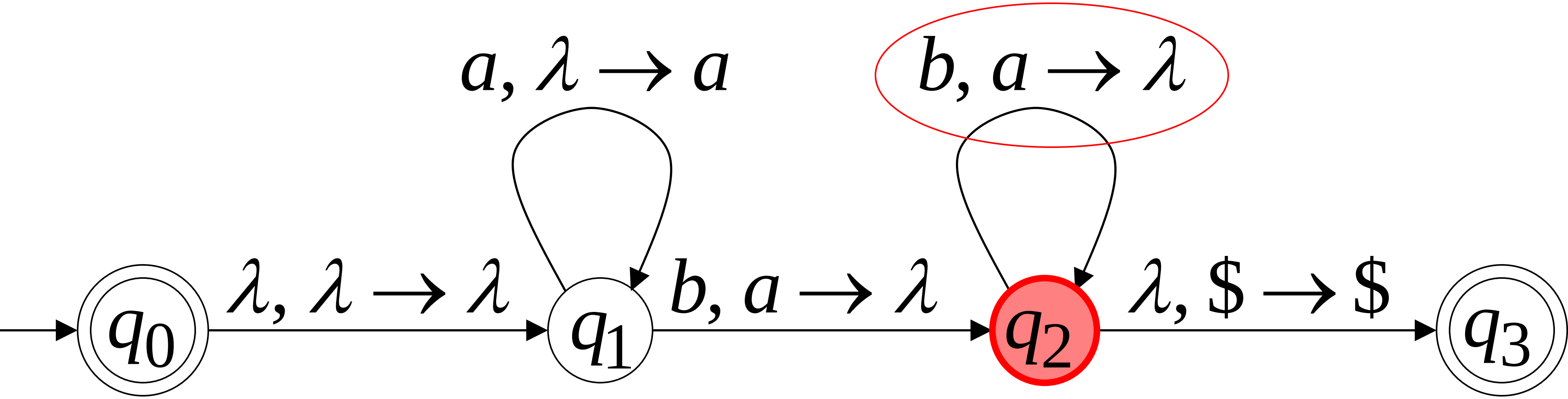
Time 6

Input

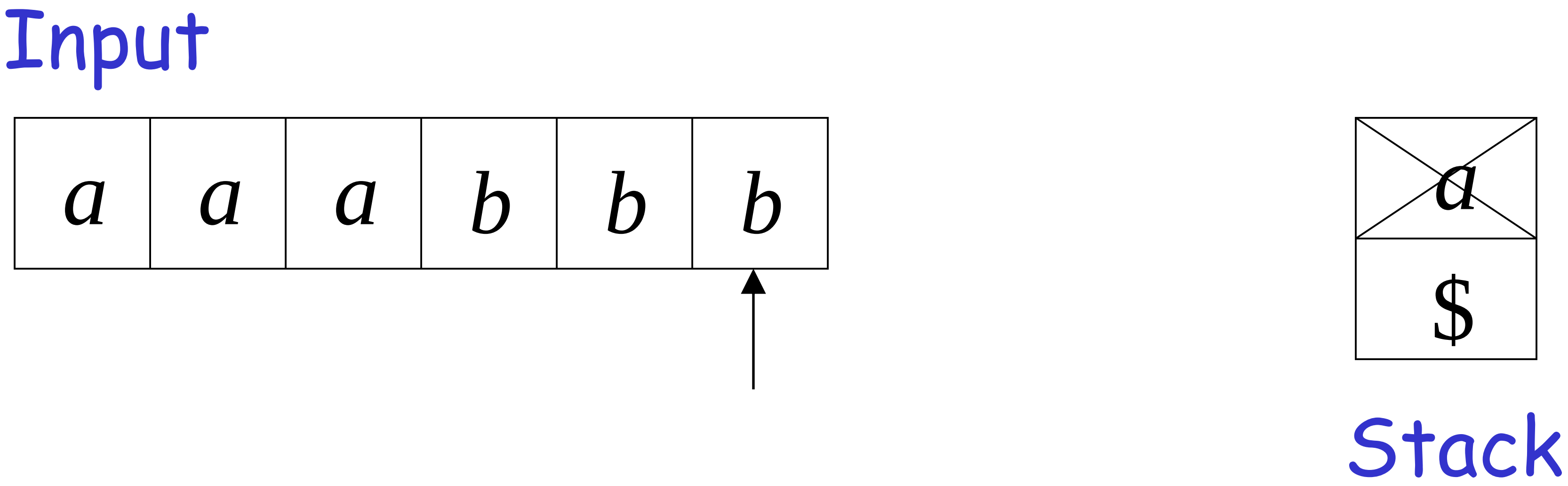
a	a	a	b	b	b

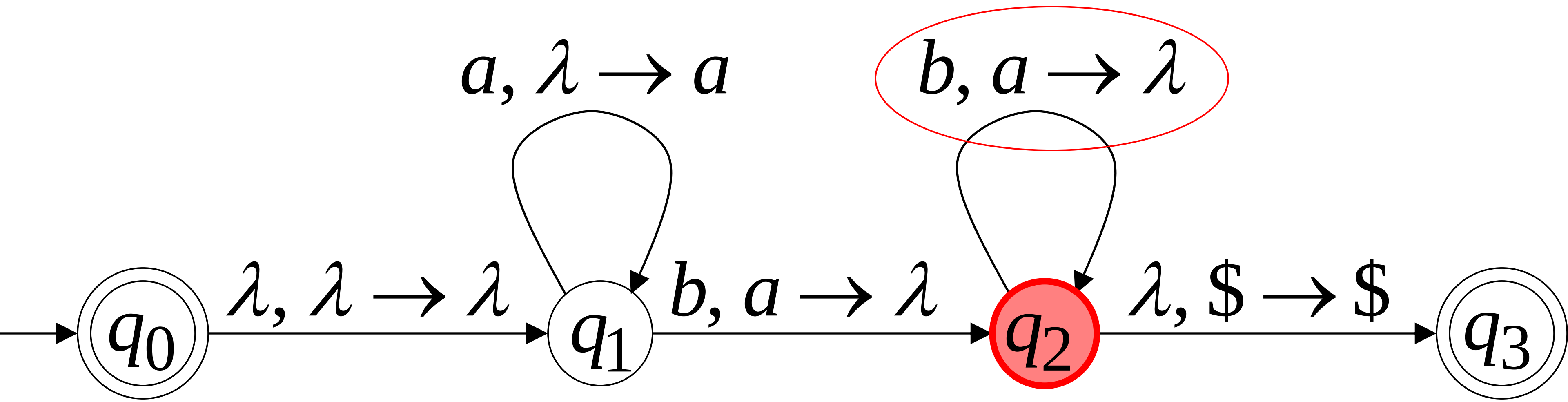
$\cancel{a}$
a
$\$$

Stack

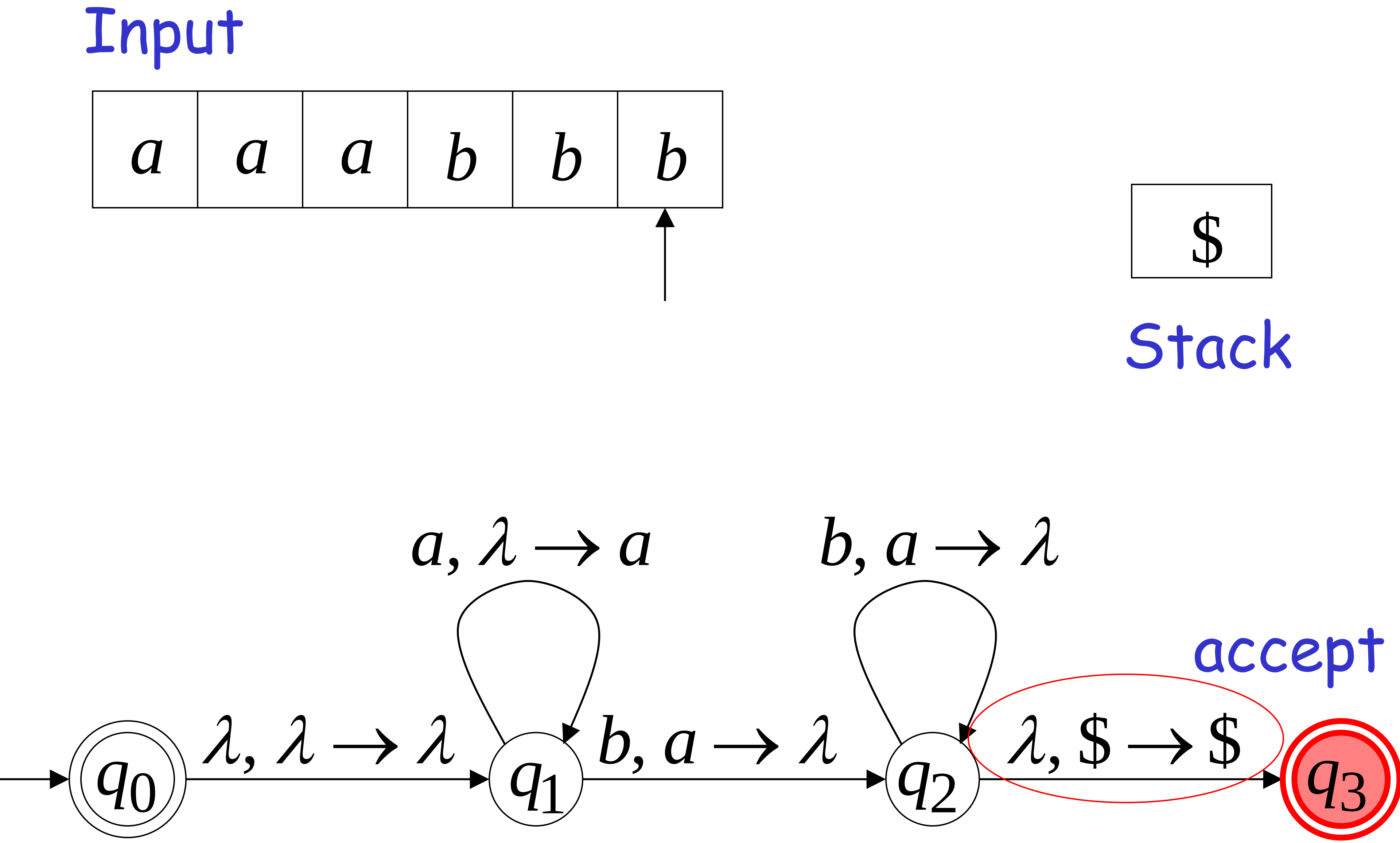


Time 7


Input
a
a
a
b
b
b
a
$
Stack


$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$b, a \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2
q_3

Time 8



A string is accepted if there is
a computation such that:

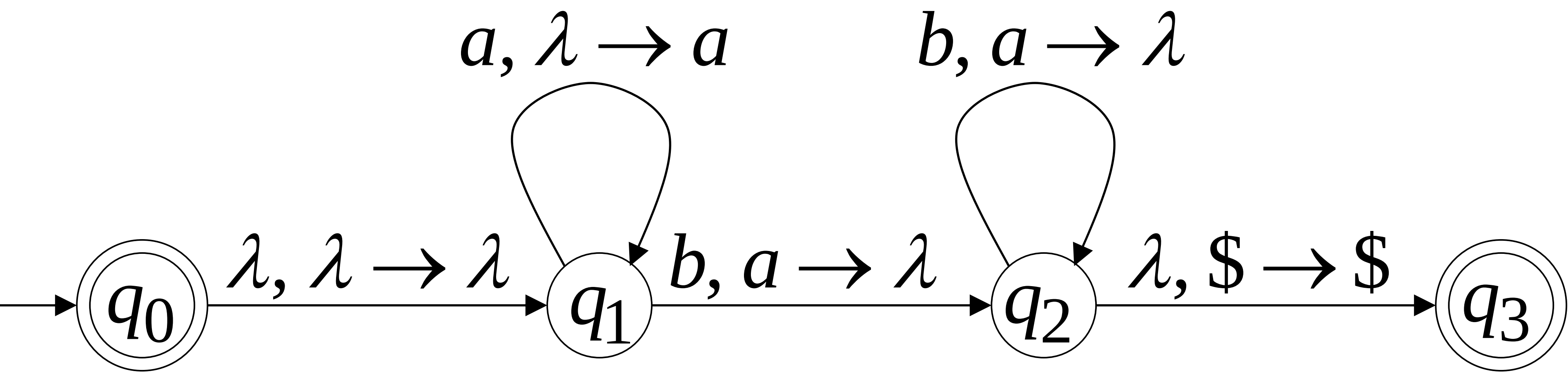
- All the input is consumed

- The last state is a final state

At the end of the computation,
we do not care about the stack contents

The input string $aaabbb$

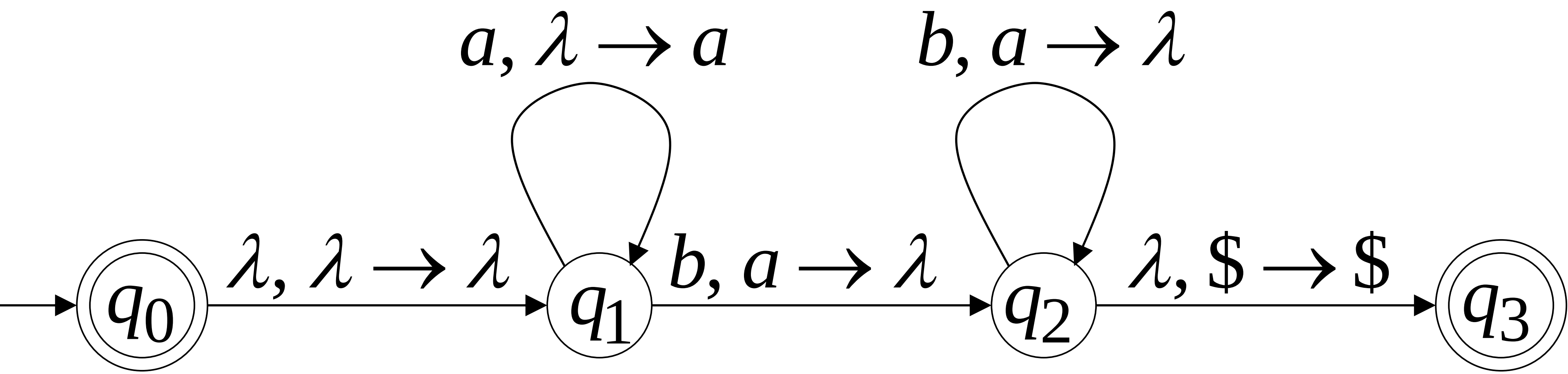
is accepted by the NPDA:



In general,

$$L = \{a^n b^n : n \geq 0\}$$

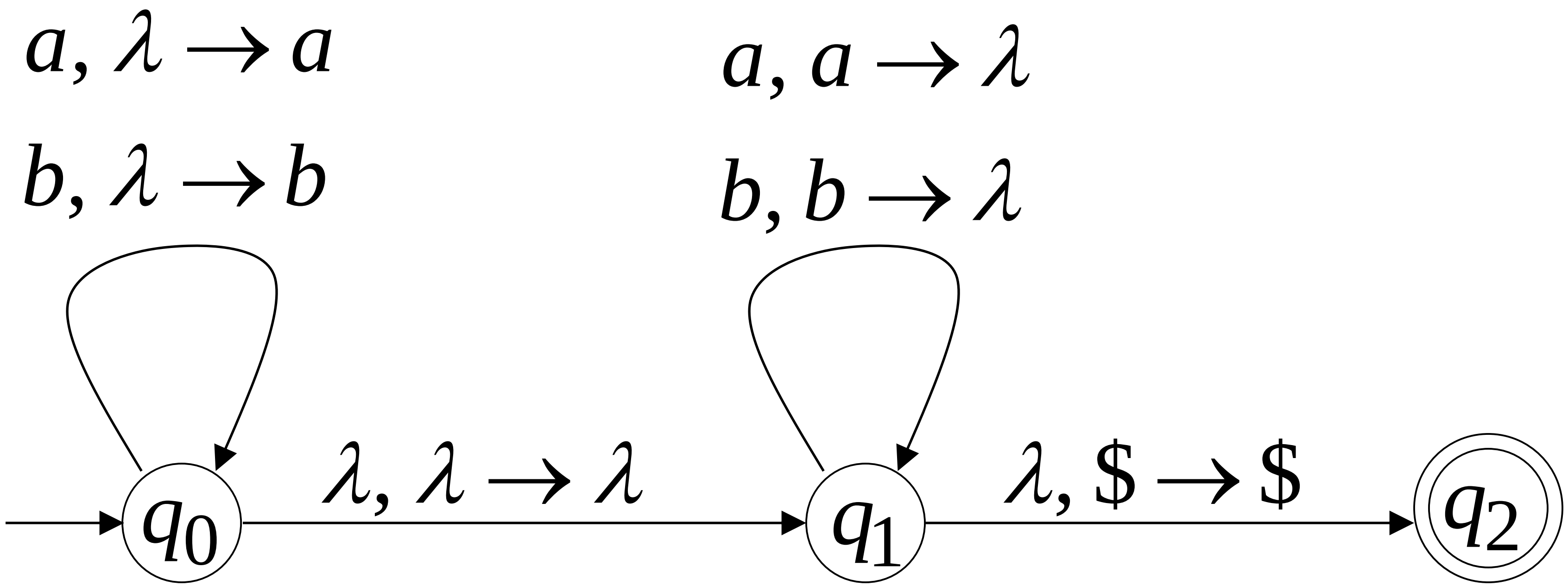
is the language accepted by the NPDA:



Another NPDA example

NPDA M

$$L(M) = \{ww^R\}$$



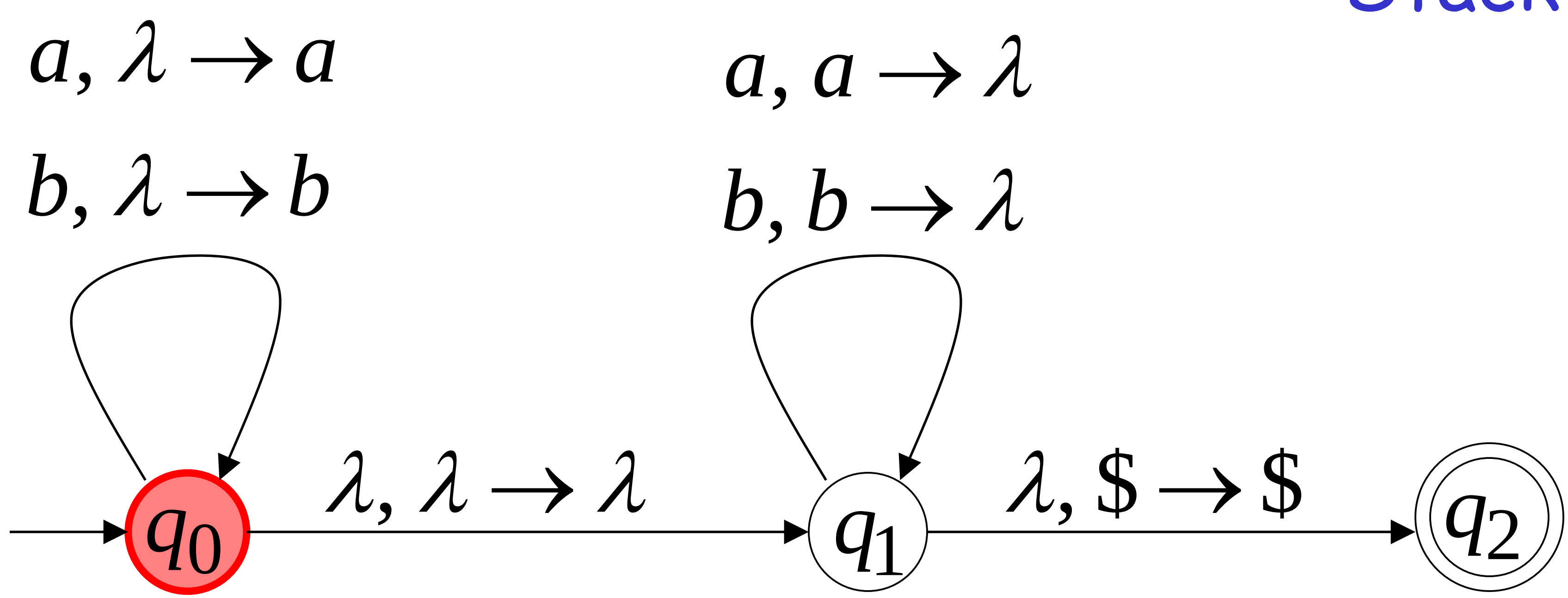
Execution Example: Time 0

Input

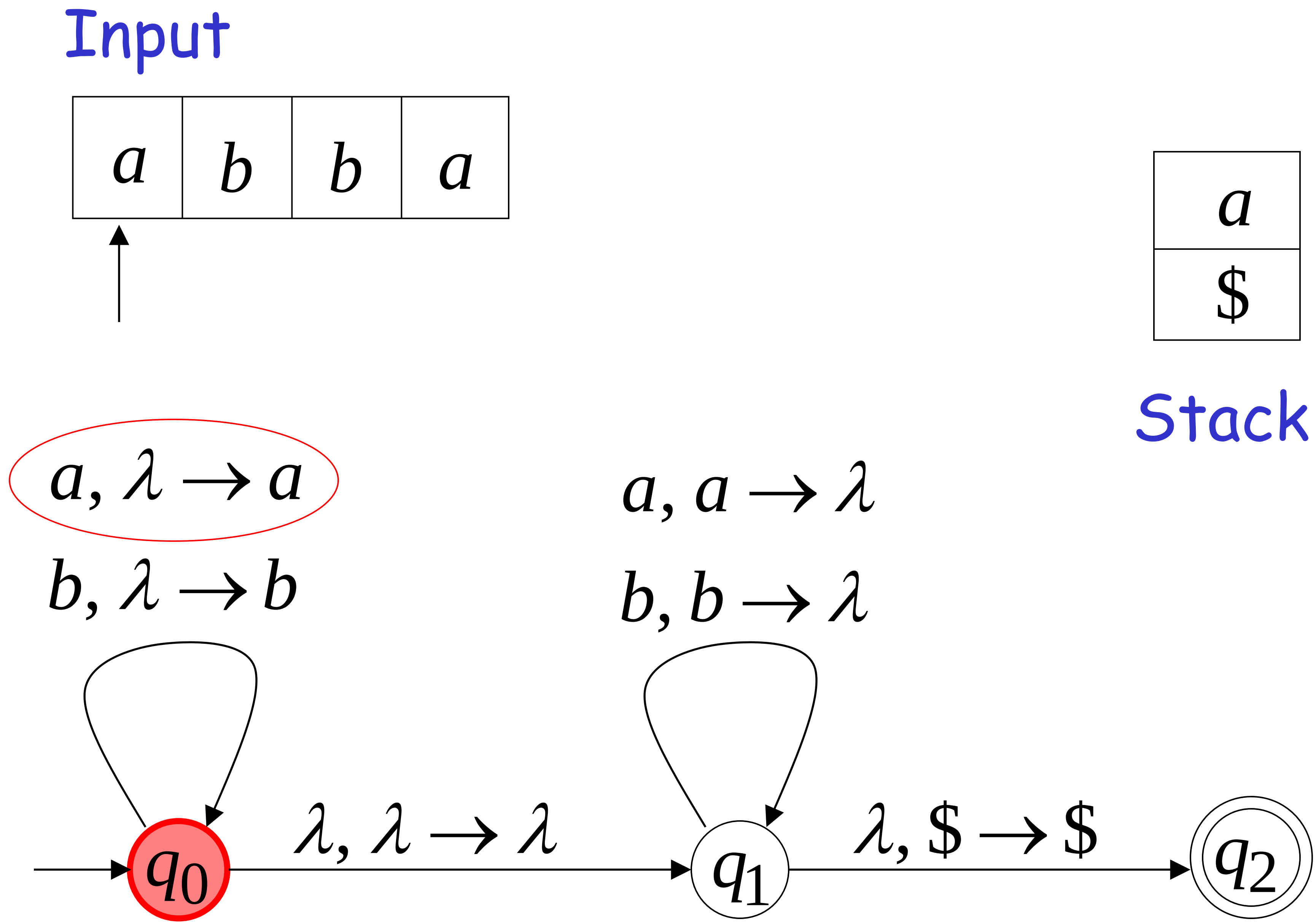
a	b	b	a

$\$$

Stack

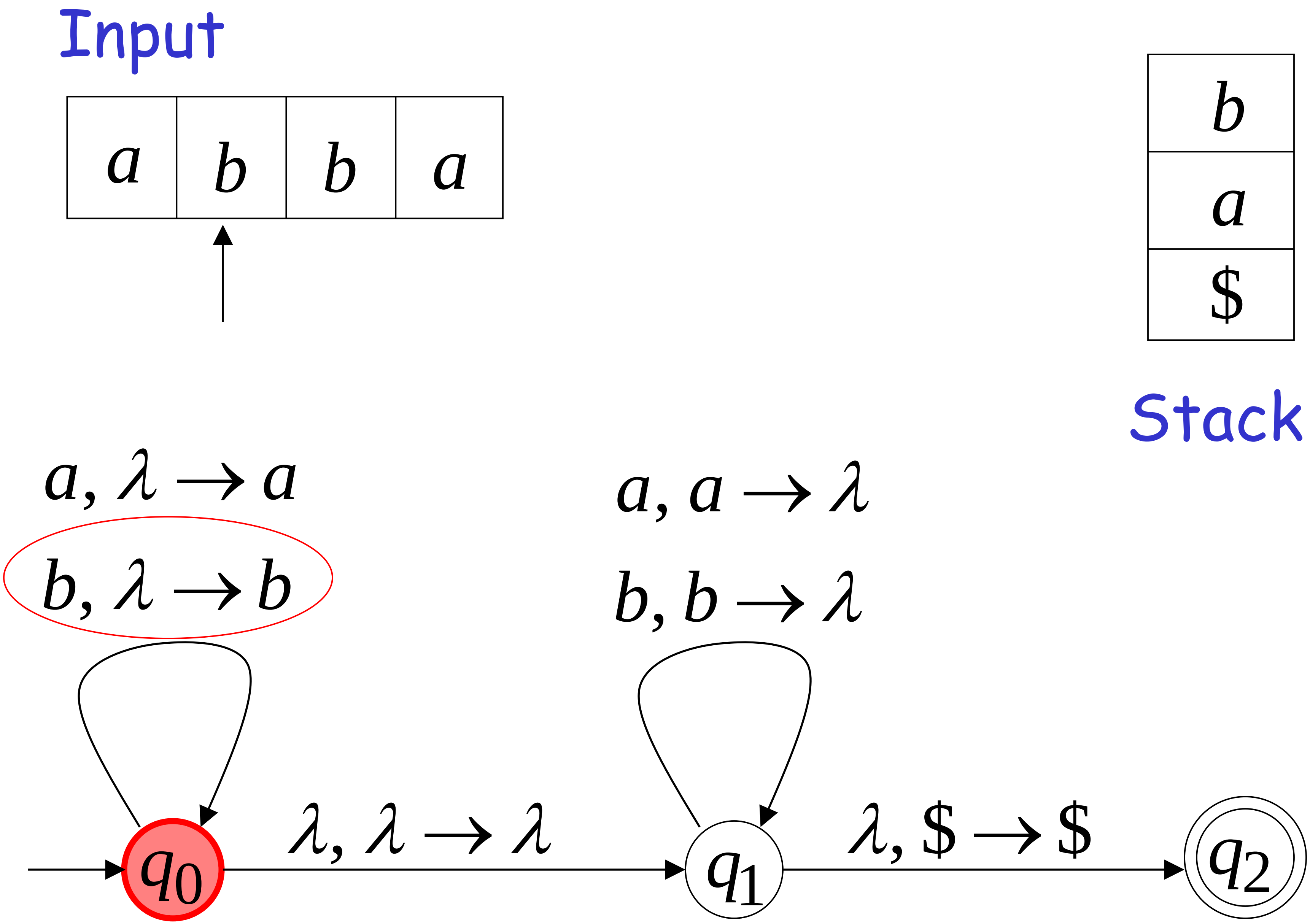


Time 1



Input
a b b a
a
$
Stack
$a, \lambda \to a$
$b, \lambda \to b$
$a, a \to \lambda$
$b, b \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \$ \to \$$
q_0
q_1
q_2

Time 2



Input
a b b a
b
a
$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 3

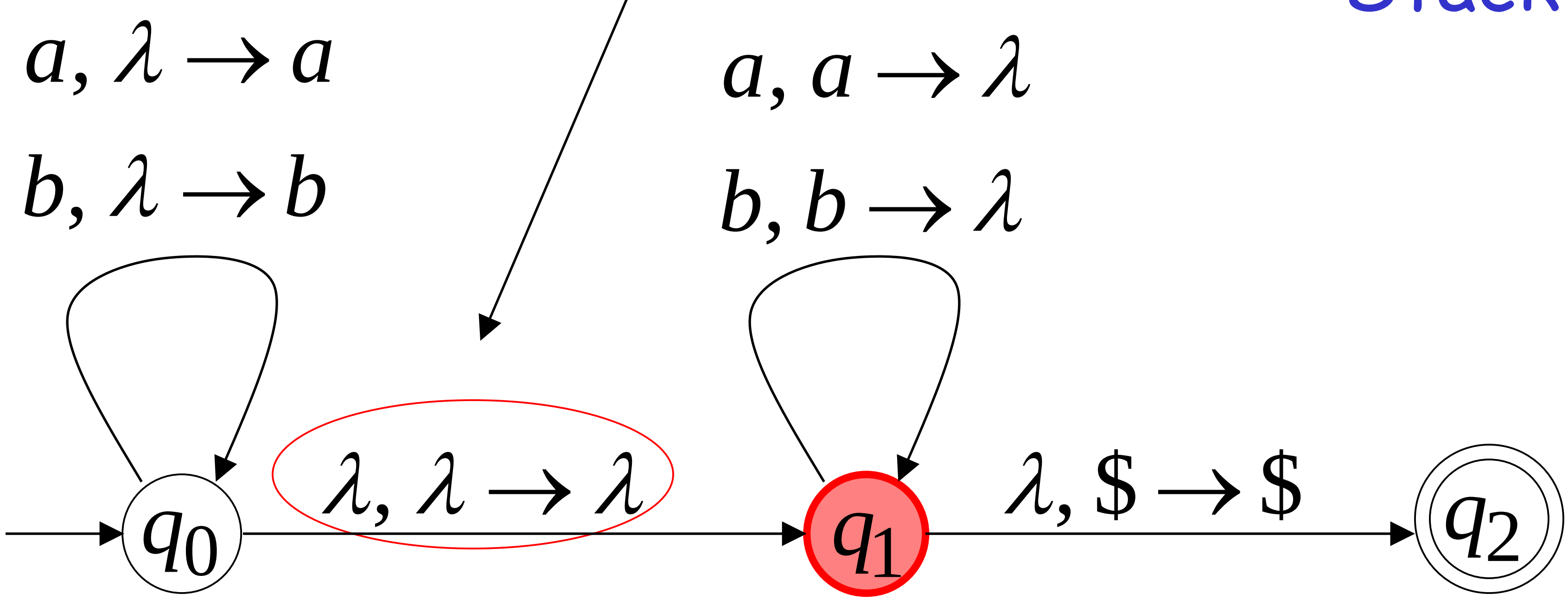
Input

a	b	b	a

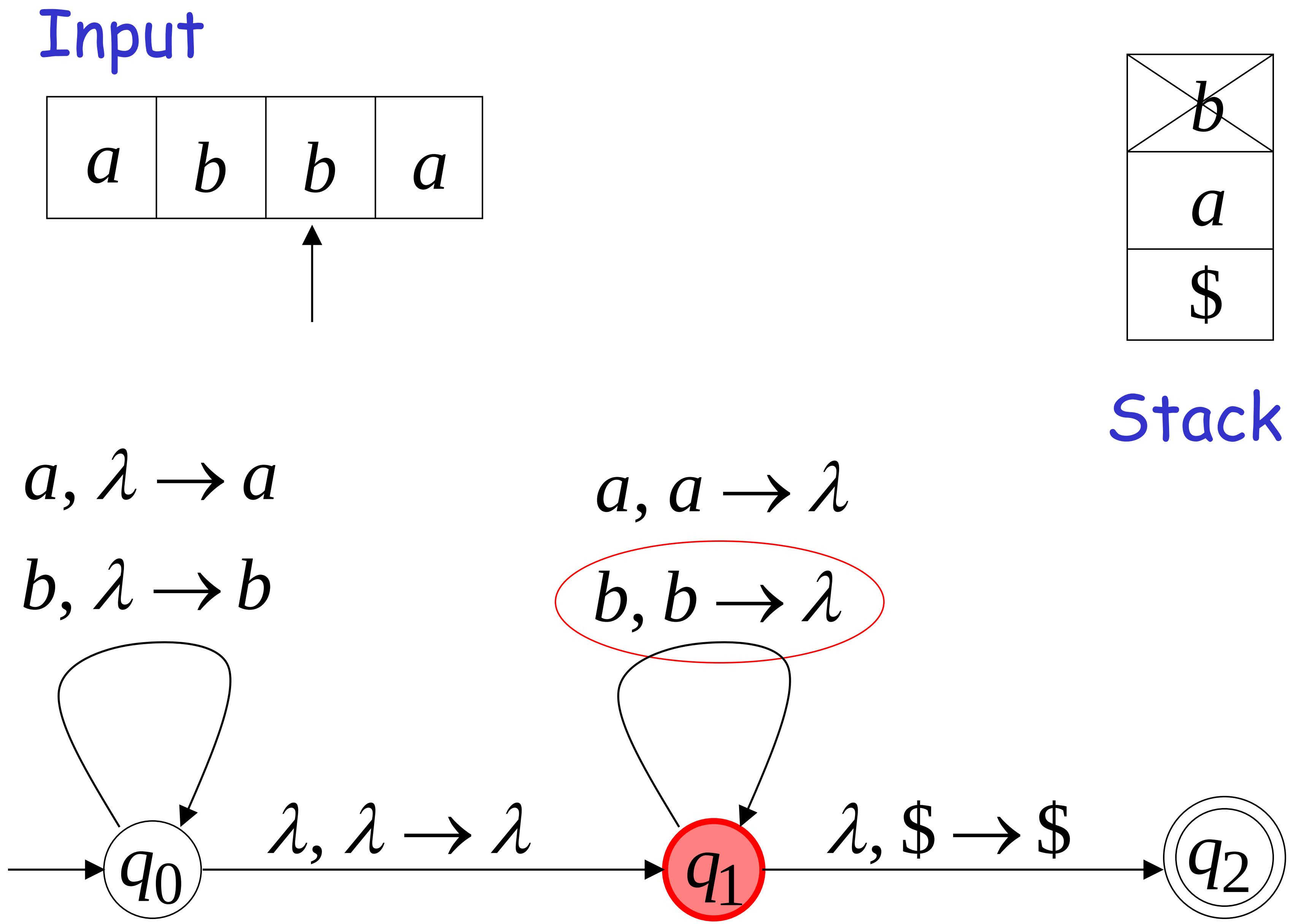
Guess the middle of string

b
a
$\$$

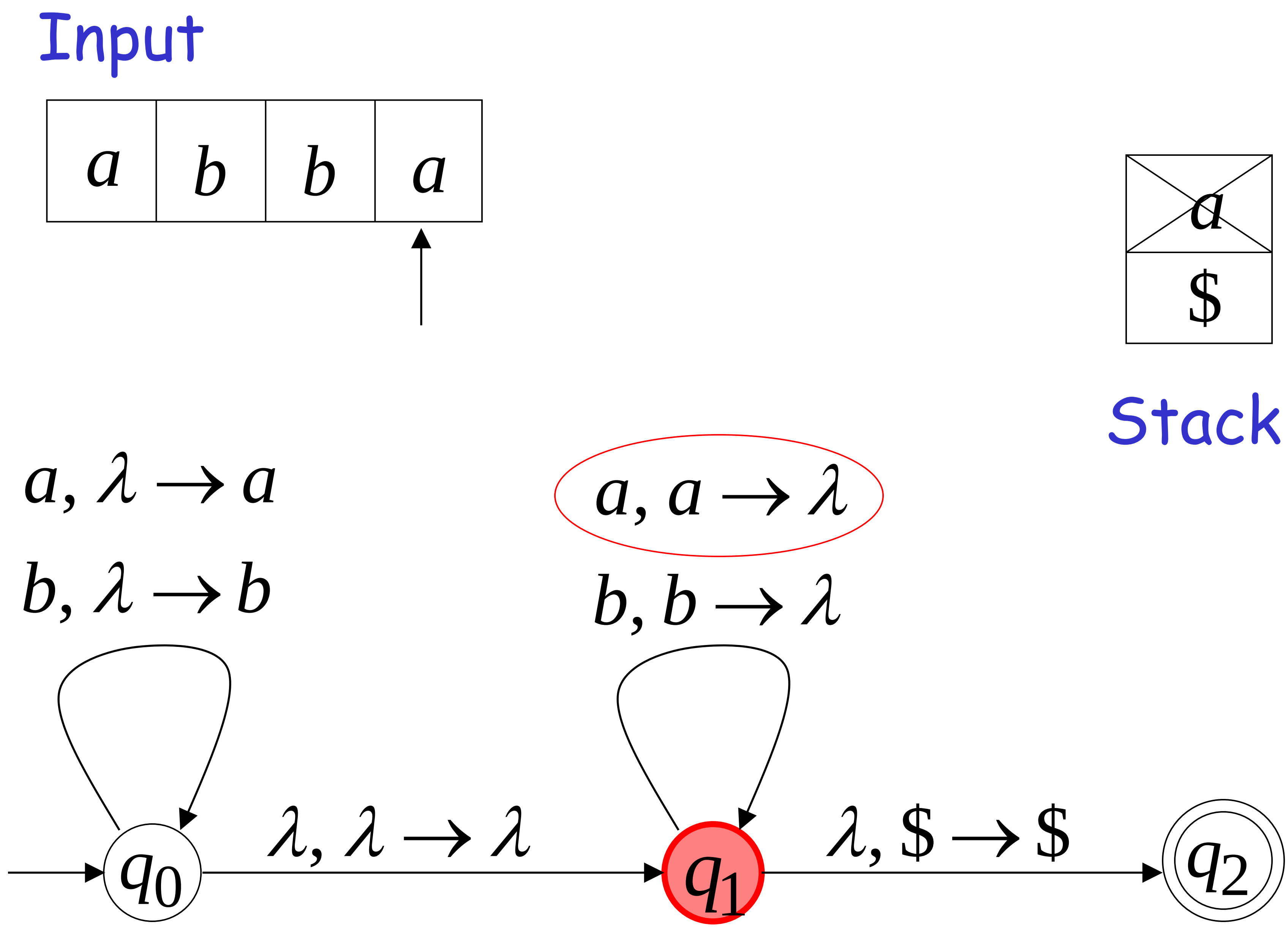
Stack



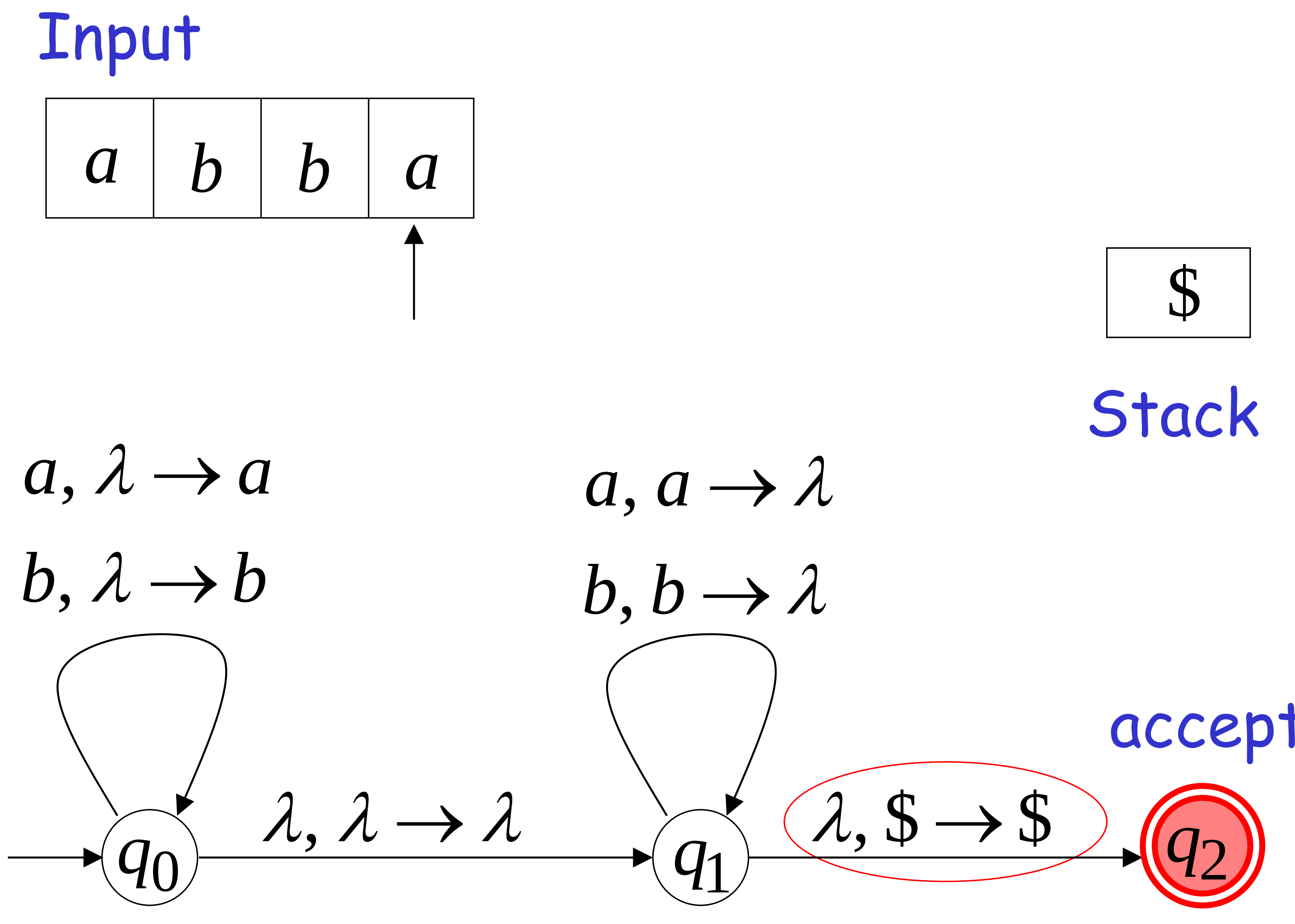
Time 4



Time 5

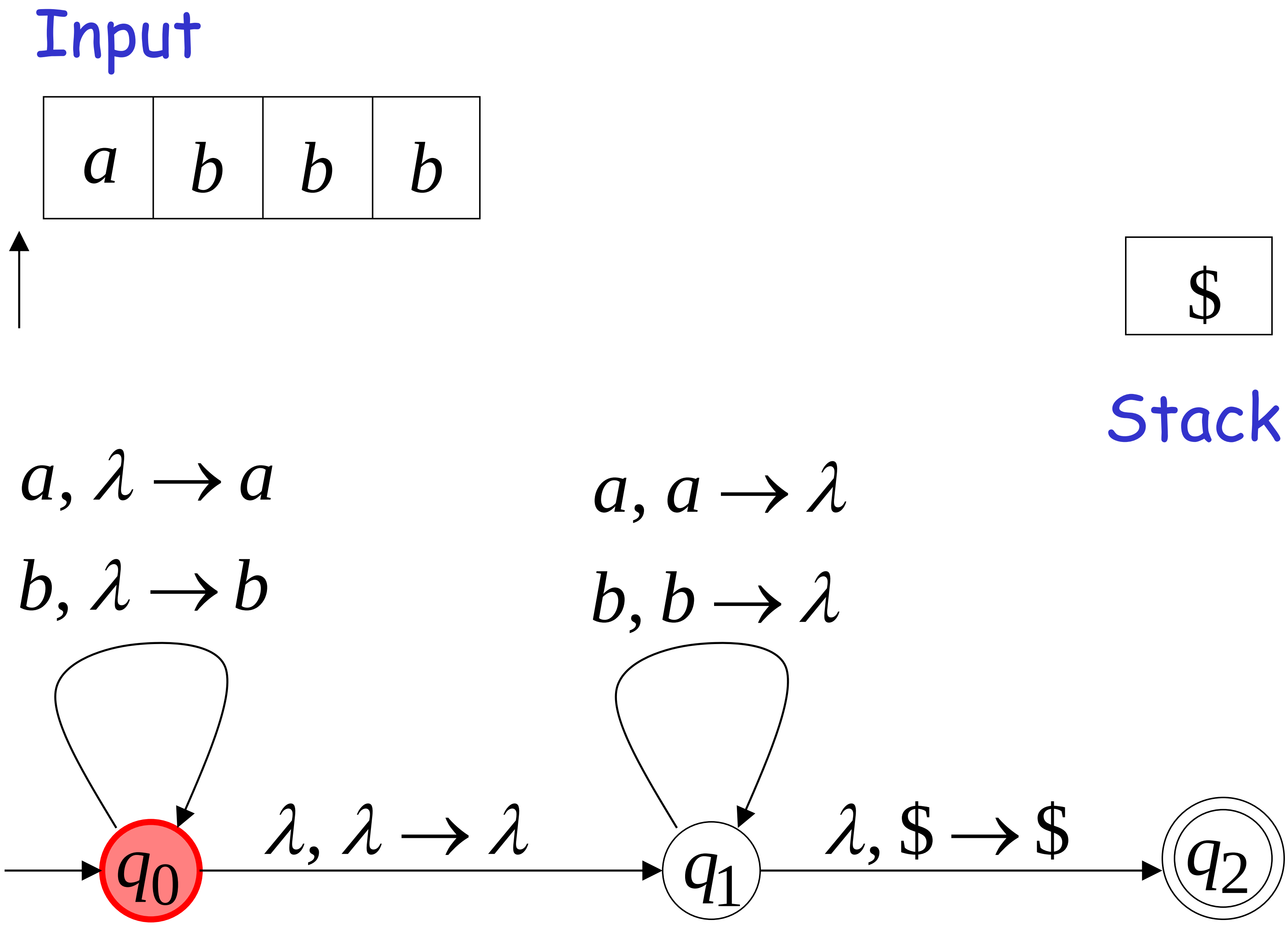

Input
a b b a
a
$
Stack
$a, \lambda \to a$
$b, \lambda \to b$
$a, a \to \lambda$
$b, b \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \$ \to \$$
q_0
q_1
q_2

Time 6



Input
a
b
b
a
$
Stack
$a, \lambda \to a$
$b, \lambda \to b$
$a, a \to \lambda$
$b, b \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \$ \to \$$
accept
q_0
q_1
q_2

Rejection Example: Time 0



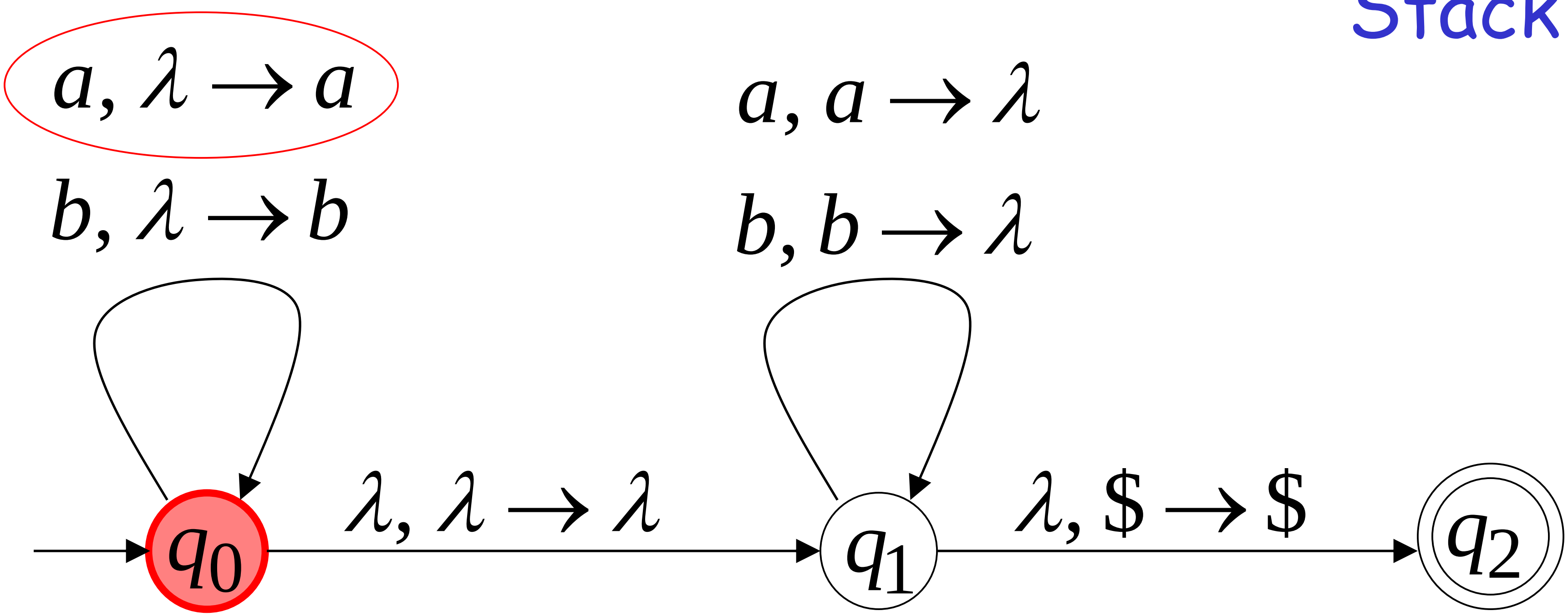
Time 1

Input

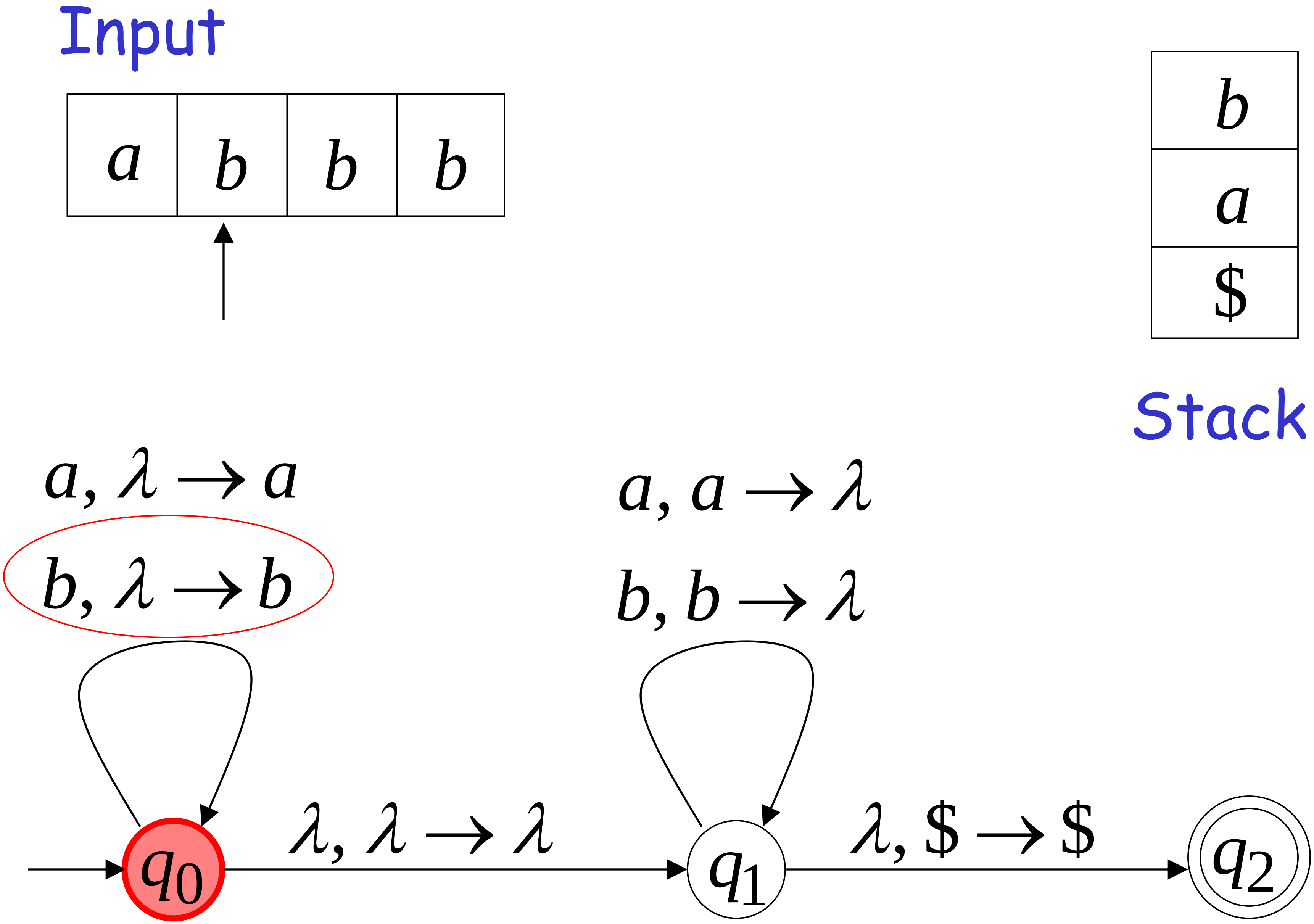
a	b	b	b

a
$\$$

Stack



Time 2



Input
a b b b
b
a
$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 3

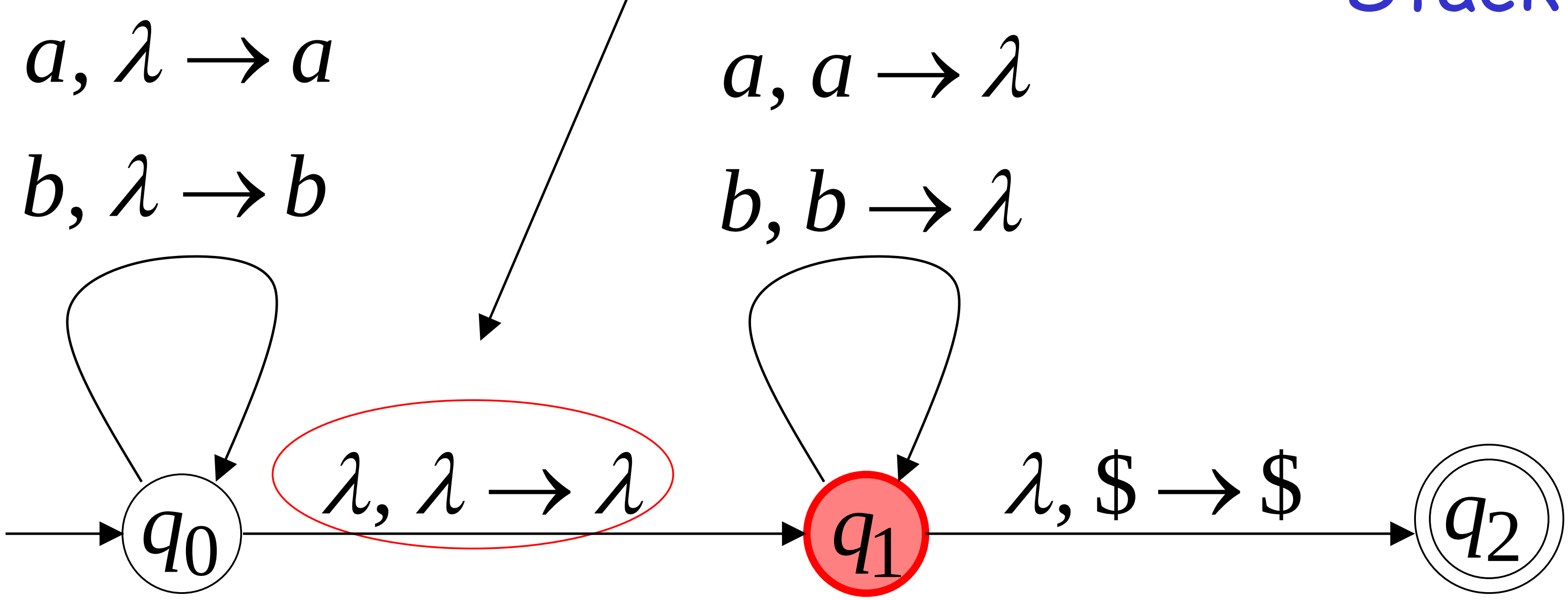
Input

a	b	b	b

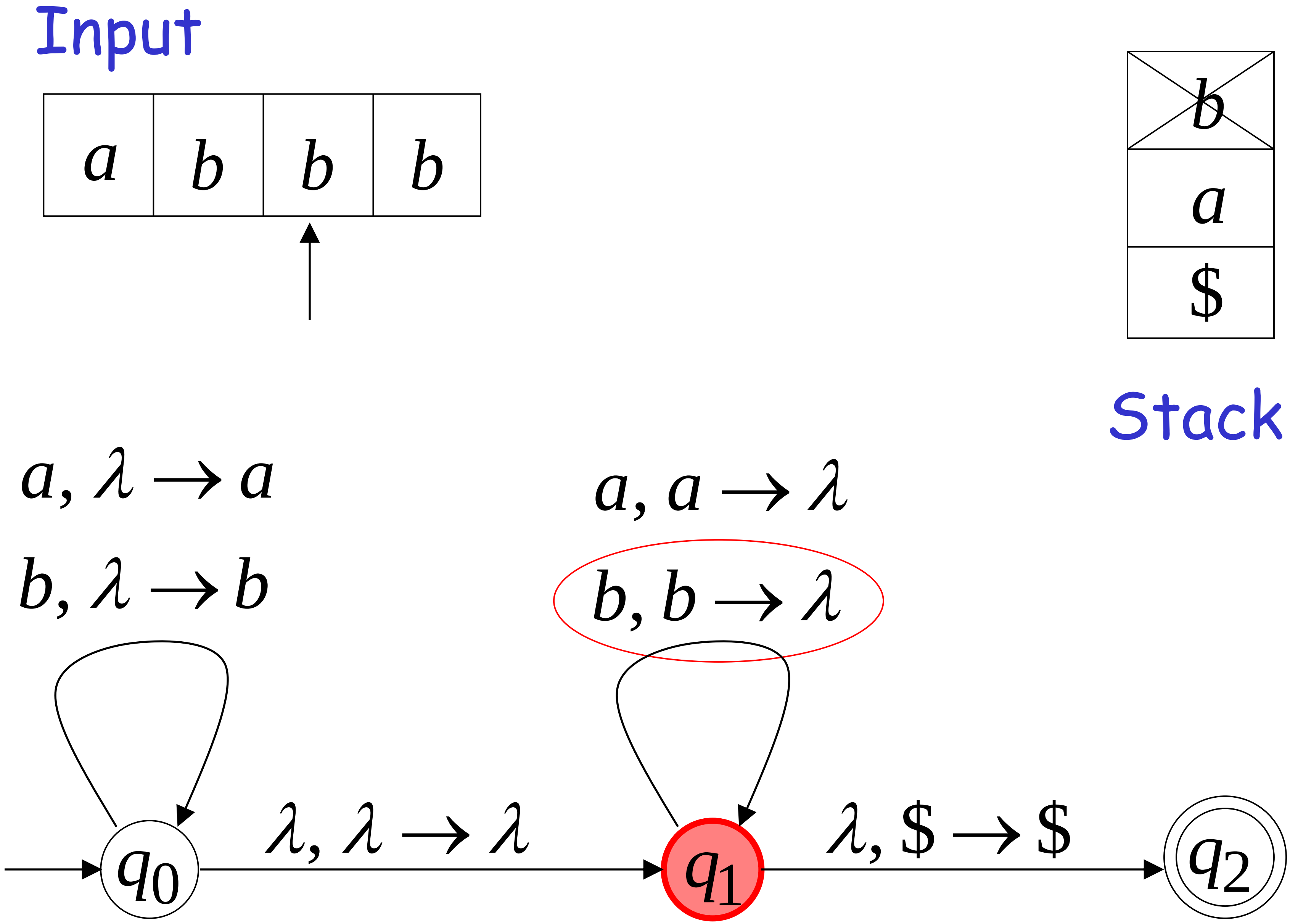
Guess the middle of string

b
a
$\$$

Stack



Time 4



Input
a b b b
$\cancel{b}$
a
$\$$
Stack
$a, \lambda \to a$
$b, \lambda \to b$
$a, a \to \lambda$
$b, b \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \$ \to \$$
q_0
q_1
q_2

Time 5

There is no possible transition.

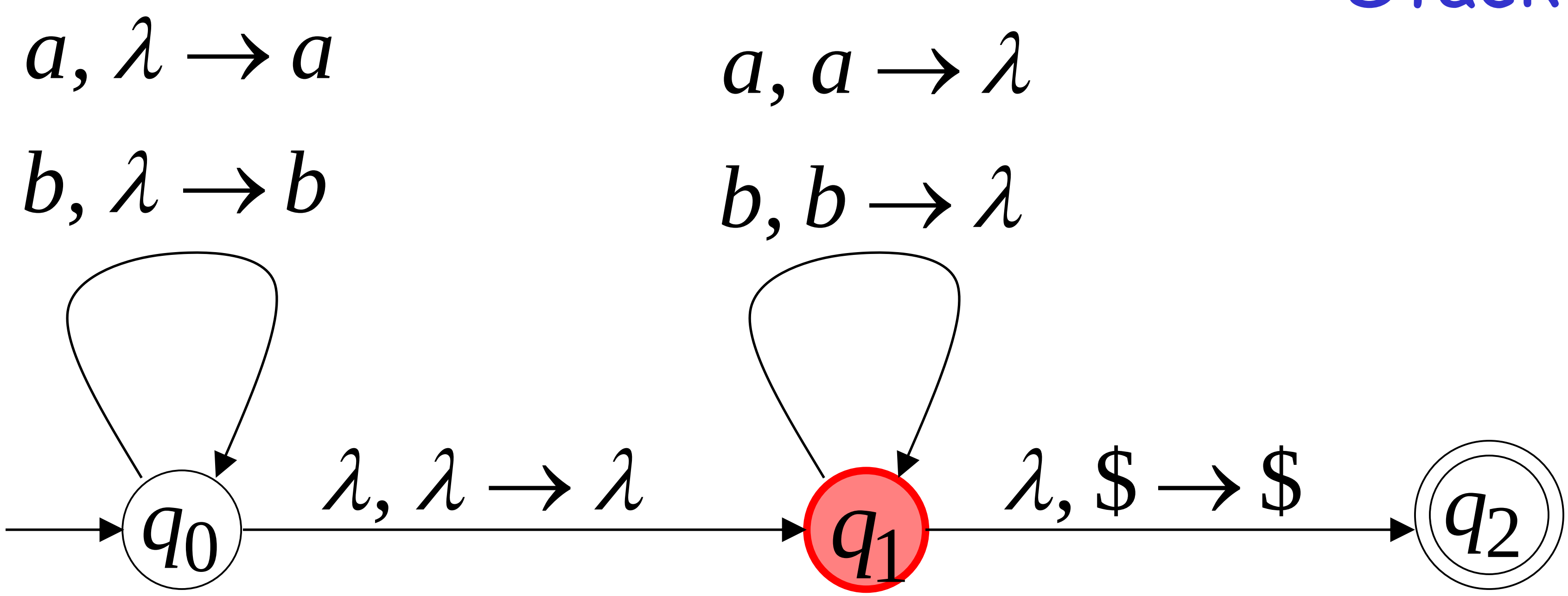
Input

a	b	b	b

Input is not consumed

a
$\$$

Stack



Another computation on same string:

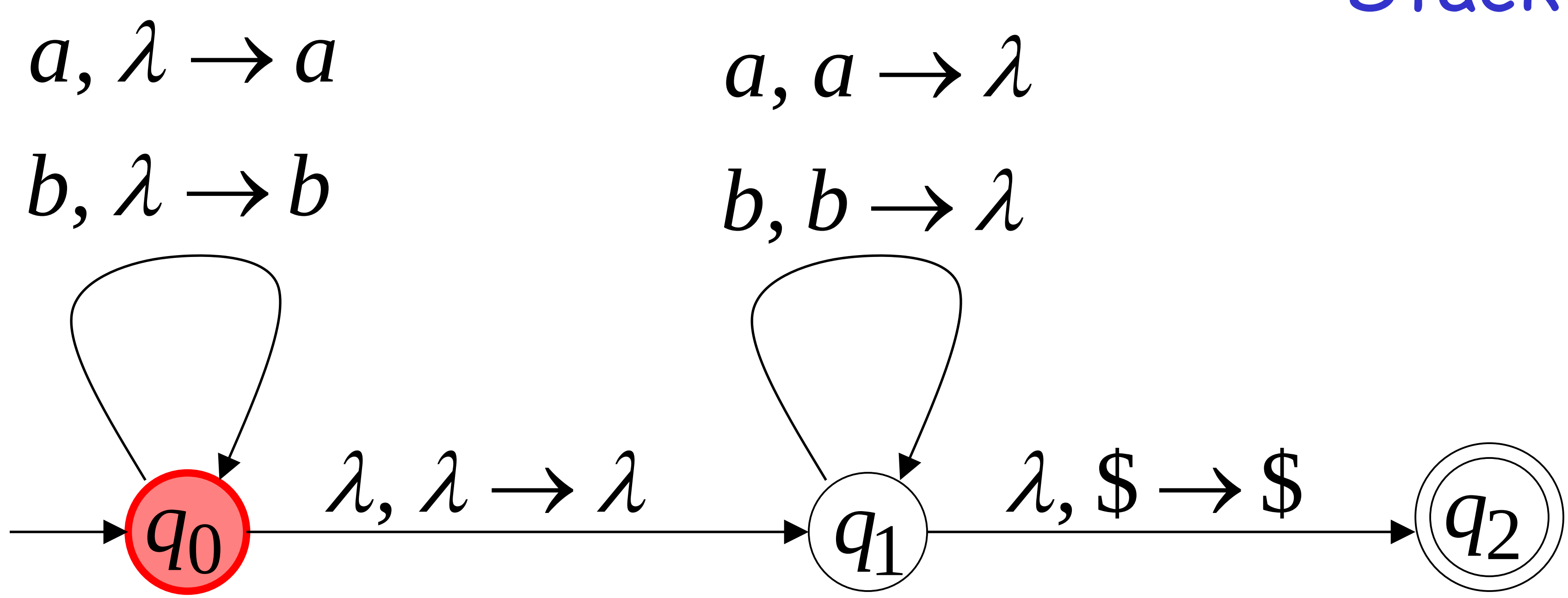
Input

Time 0

a	b	b	b

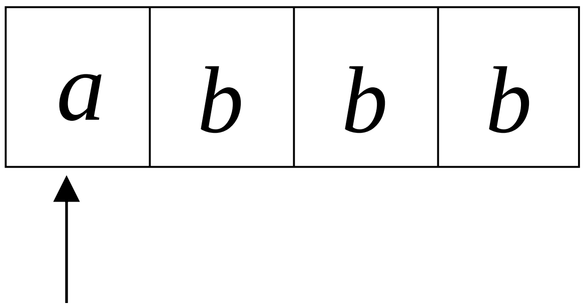
$\$$

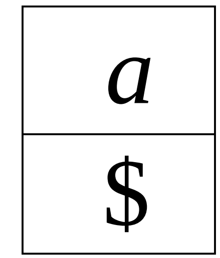
Stack



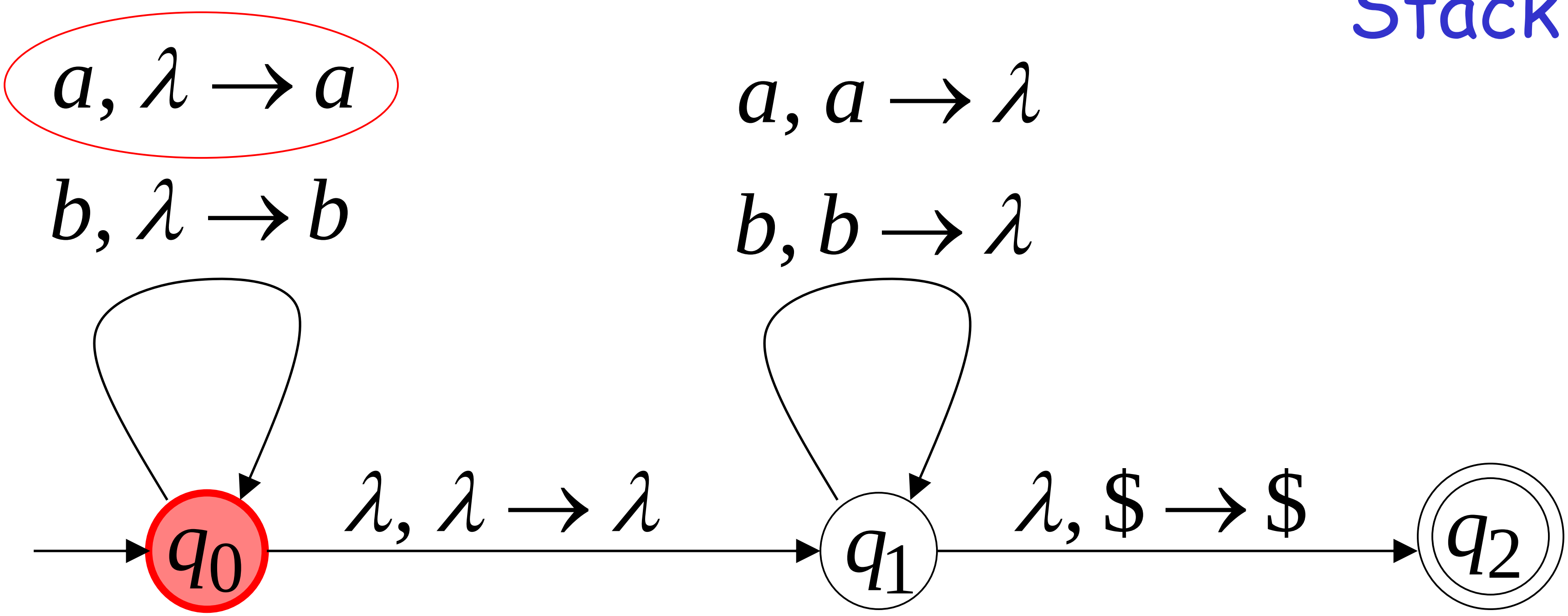
Time 1

Input



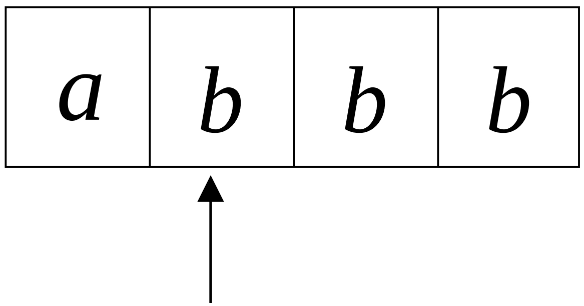


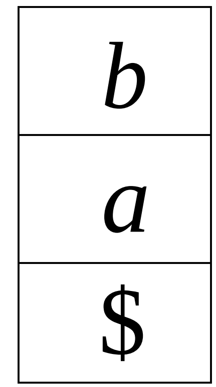
Stack



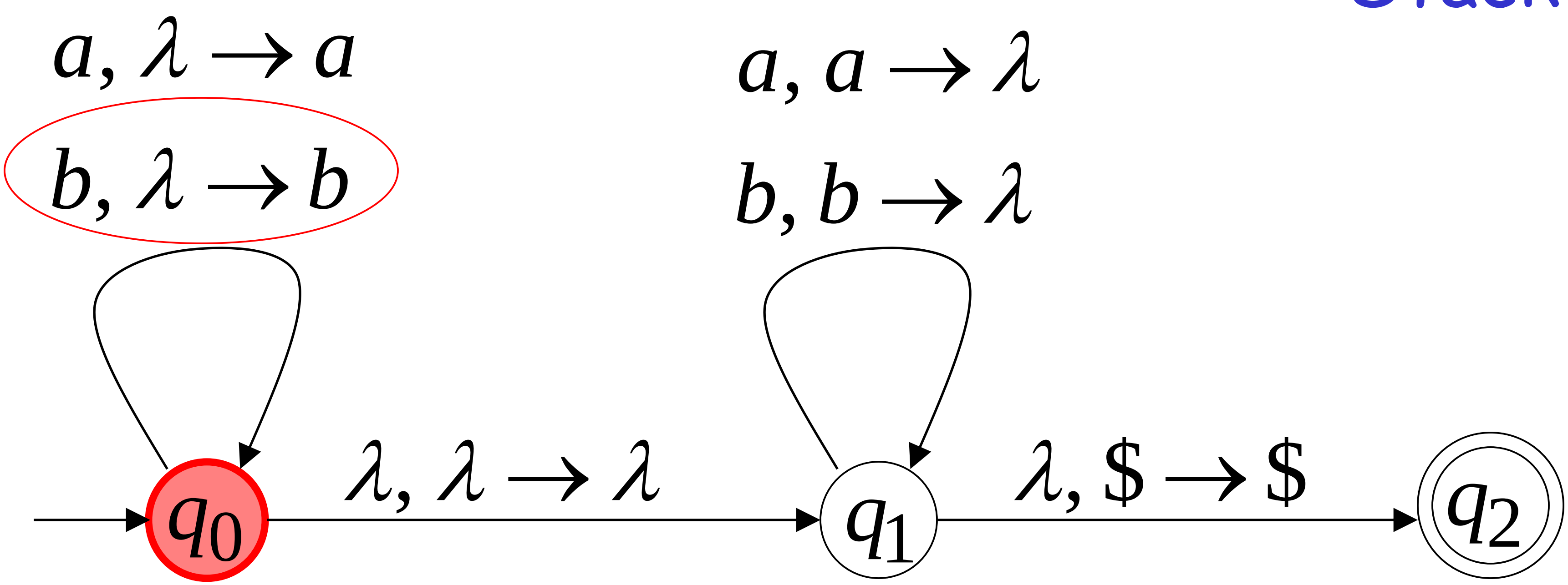
Time 2

Input





Stack



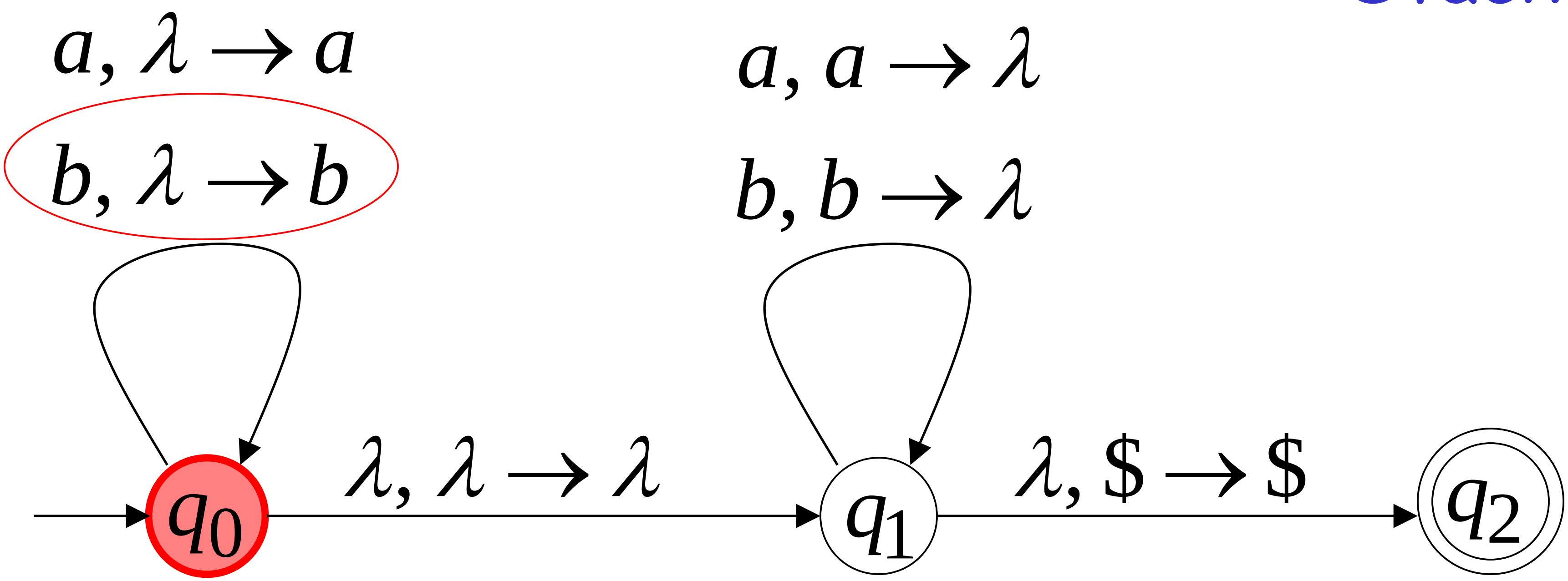
Time 3

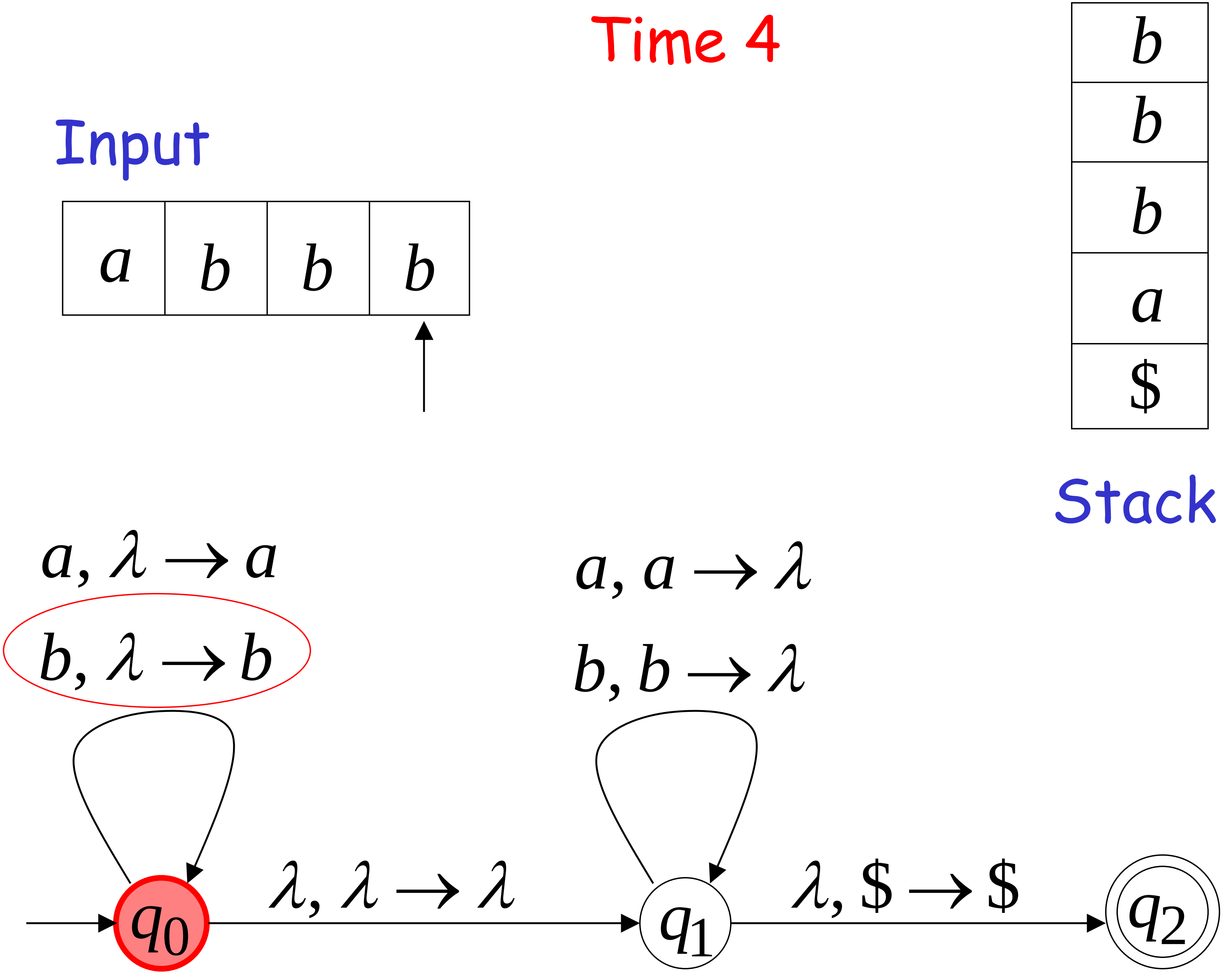
Input

a	b	b	b

b
b
a
$\$$

Stack




Time 4
Input
a b b b
b
b
b
a
$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 5

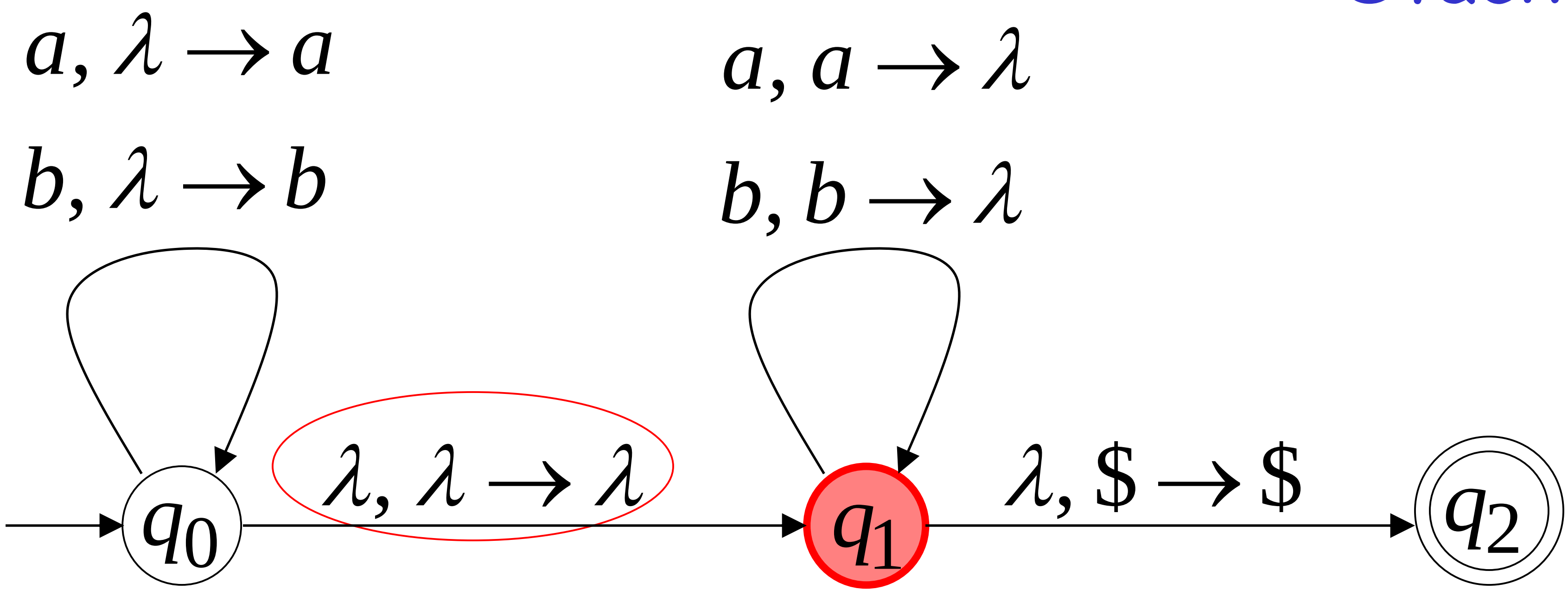
Input

a	b	b	b

No final state is reached

b
b
b
a
$\$$

Stack



There is no computation that accepts string $abbb$

$$abbb \notin L(M)$$