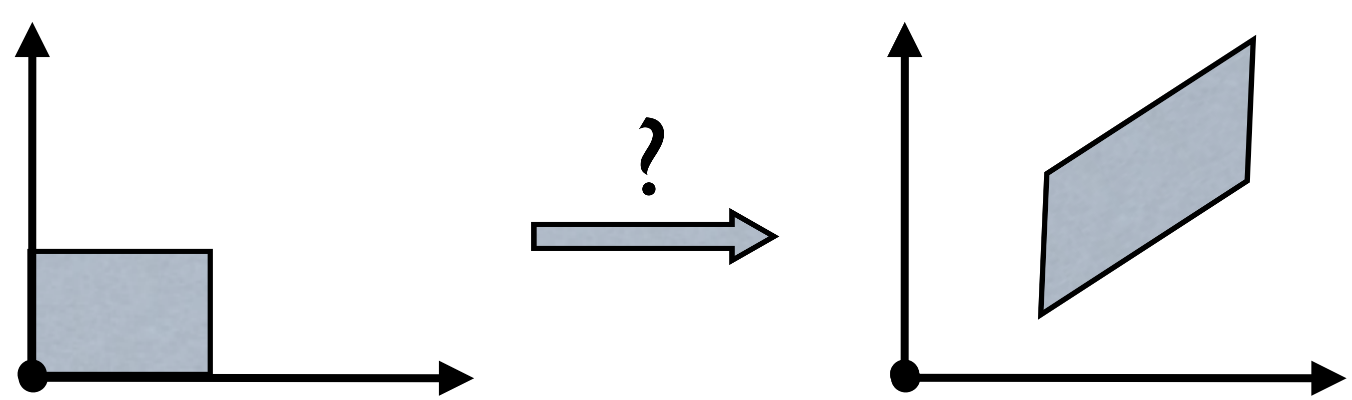
Geometric Transformations

Geometric Transformations

- Why do we need them?
  - Want to define an object in one coordinate system, then place it in another system.
  - Allow us to create multiple instances of objects.
  - Animation (time-dependent transformations).
  - Display using device independent coordinates.
  - 3D viewing (projections).

Transformations in 2D

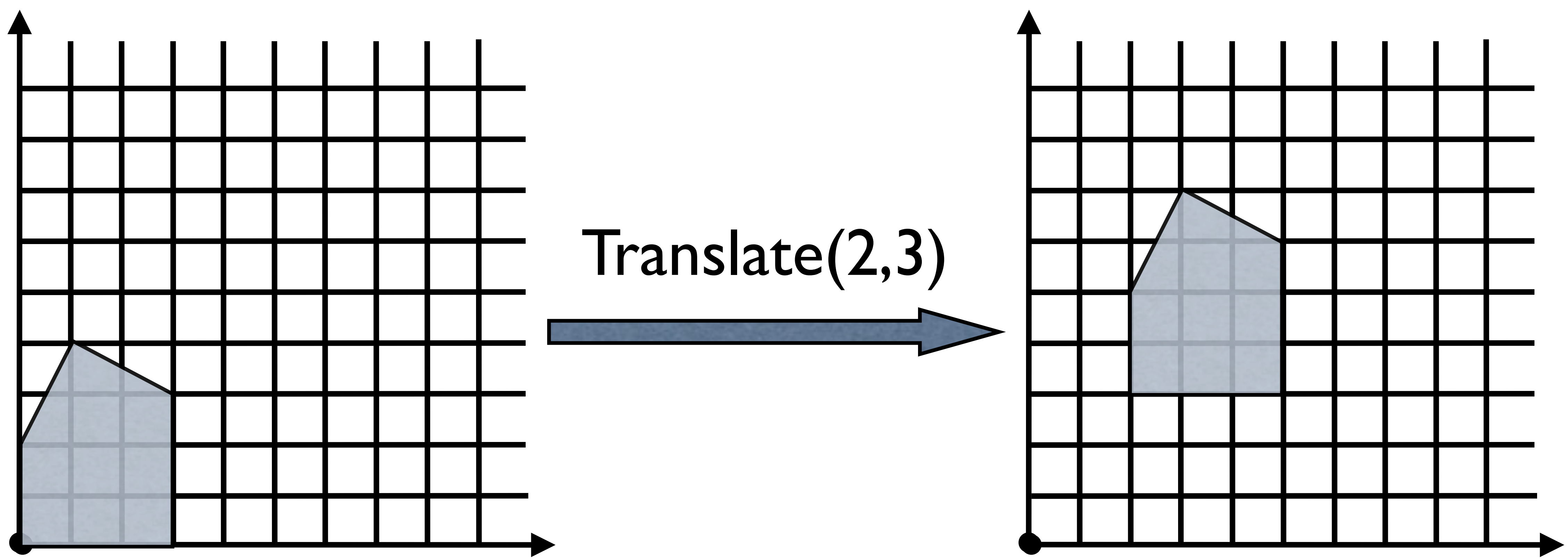
- We represent a geometric object as a set of points:
  - *Boundary representation*: the points form the boundary of the object.
  - *Solid representation*: the points form the interior of the object.
- Question: how do we transform a set of points in the plane?



Translation

- Translate(a,b):

$$(x, y) \rightarrow (x + a, y + b)$$

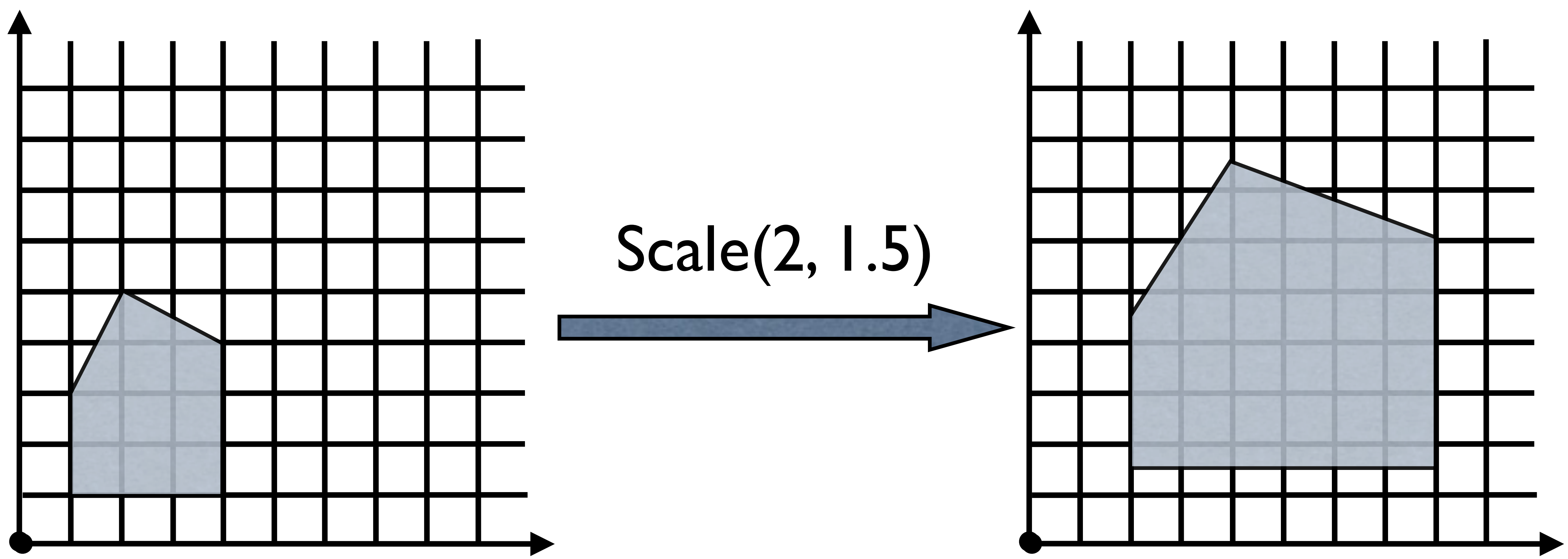


Do we transform the object, or the coordinate system?

Scaling

- Scale(a,b):

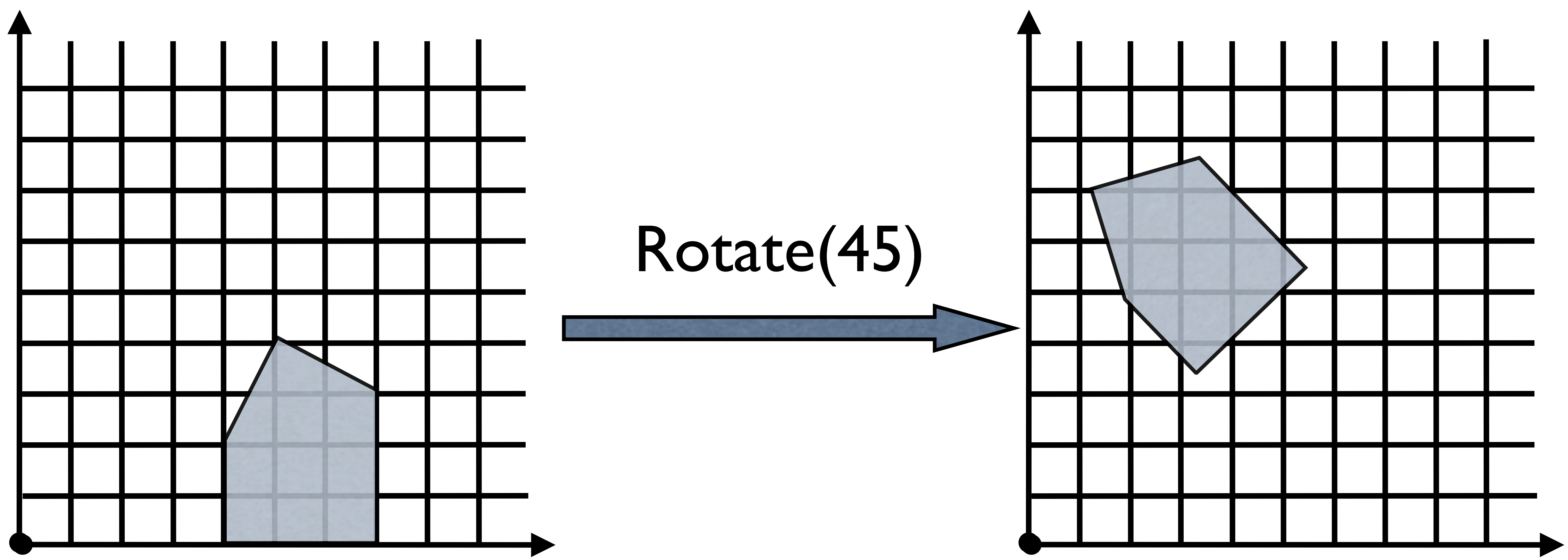
$$(x, y) \to (ax, by)$$



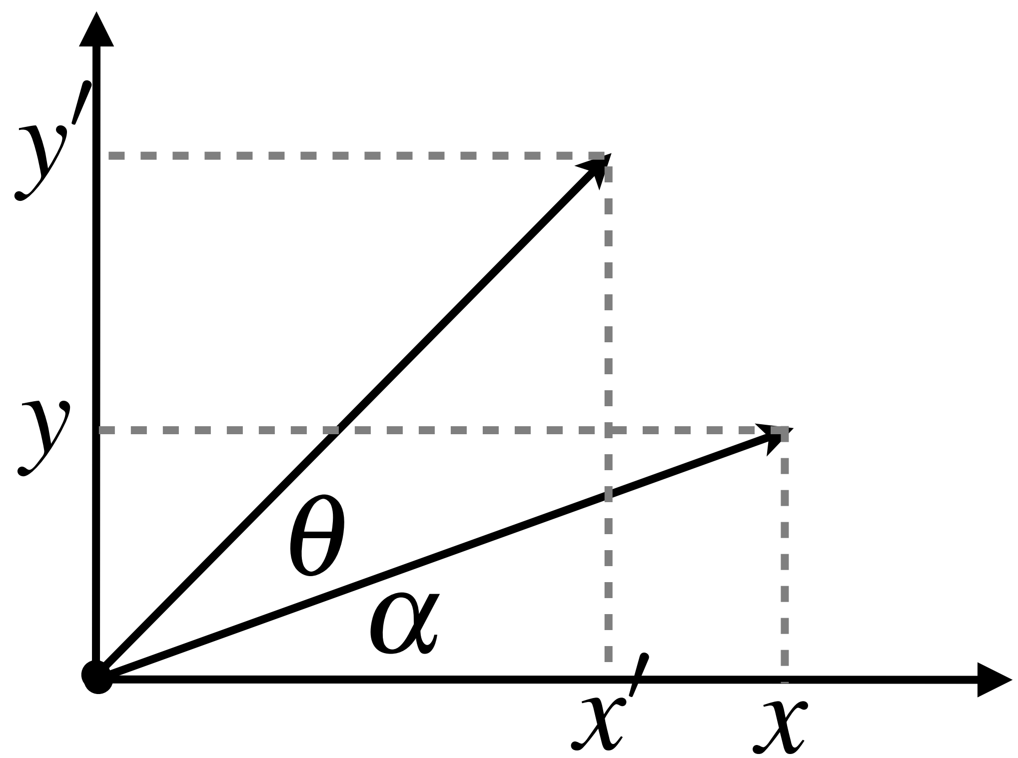
Rotation

- Rotate(θ):

$$(x, y) \rightarrow (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$$



Rotation: derivation



$$x = r\cos\alpha \qquad\qquad y = r\sin\alpha$$

$$x' = r\cos(\alpha+\theta) \qquad\qquad y' = r\sin(\alpha+\theta)$$

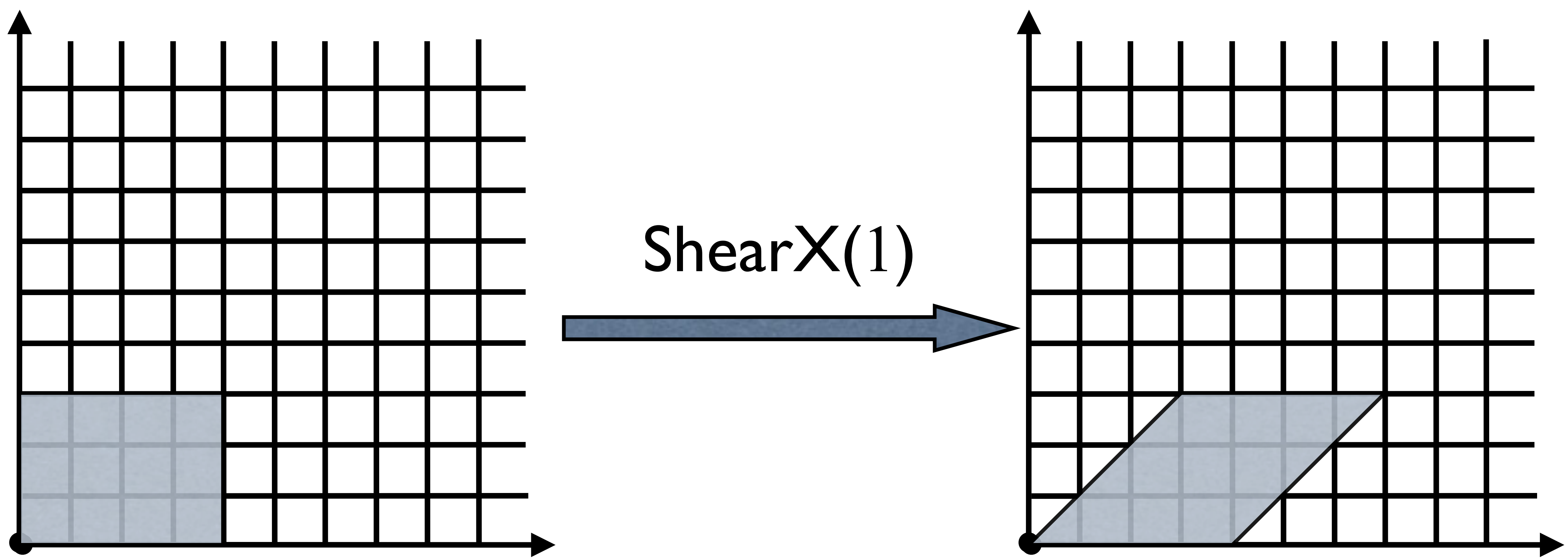
$$x' = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \qquad\qquad \cdots$$

$$x' = x\cos\theta - y\sin\theta \qquad\qquad y' = x\sin\theta + y\cos\theta$$

Shearing

- ShearX(a):

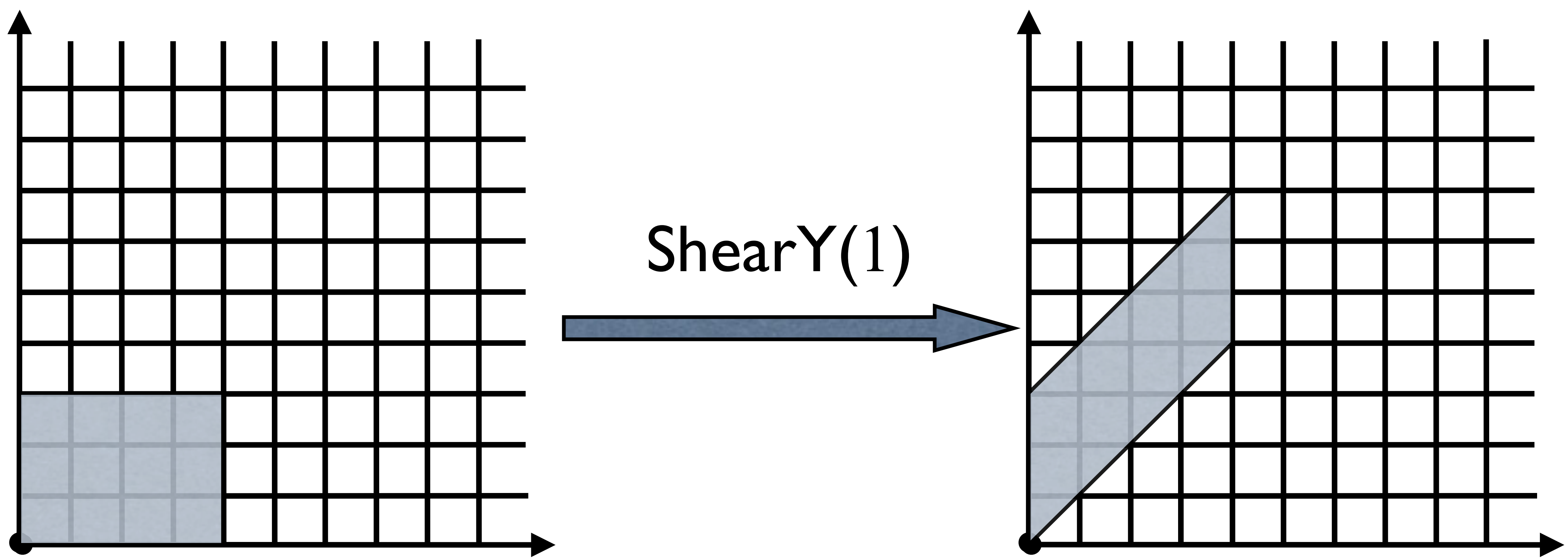
$$(x, y) \rightarrow (x + ay, y)$$



Shearing

- ShearY(b):

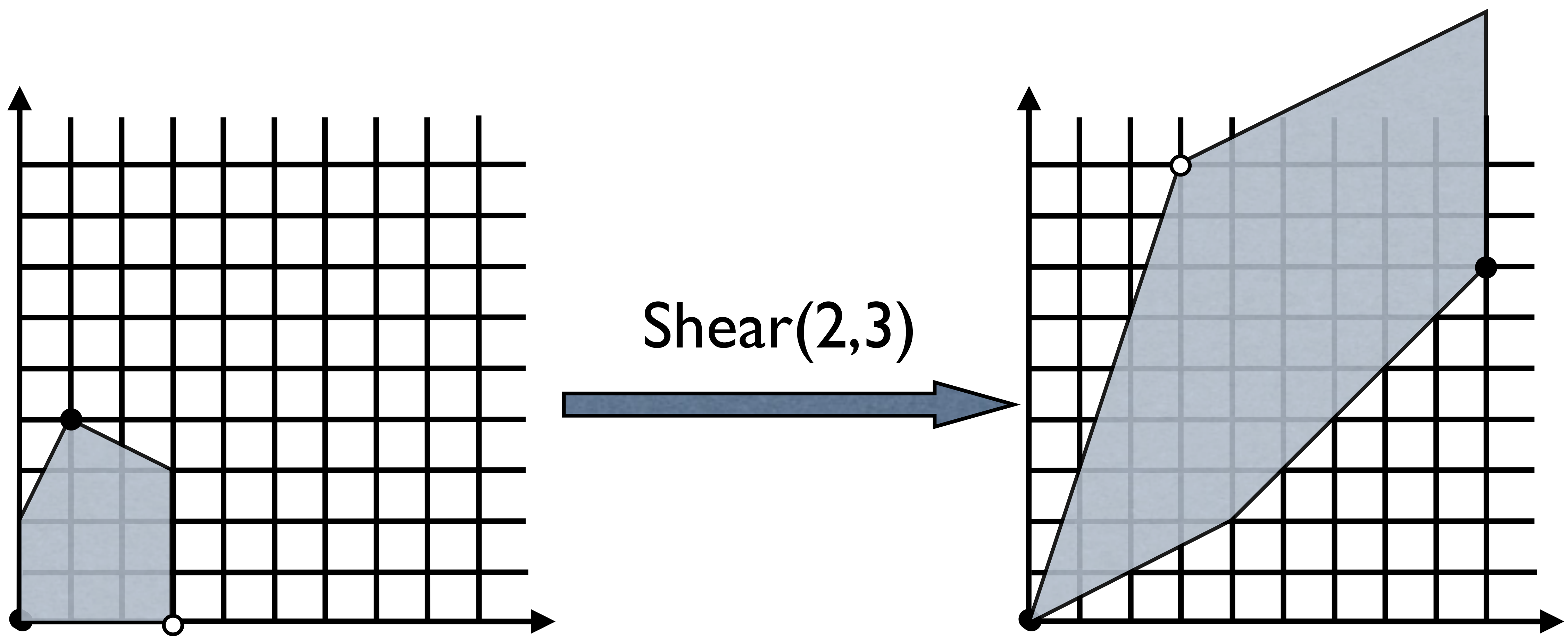
$$(x, y) \to (x, y + bx)$$



Shearing

- Shear(a,b):

$$(x, y) \to (x + ay, bx + y)$$



Matrix Notation

- Let's write a point (x,y) as a column vector of length 2:

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

- What happens when this vector is multiplied by a 2 by 2 matrix?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Scaling

- Scale(a,b):

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$$

- Uniform scaling: when a and b are equal.
- What happens when a or b are negative?

Reflection

- reflection through the y axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
- reflection through the x axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- reflection through y = x: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- reflection through y = -x: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Rotation, Shearing

- Rotate(θ):

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

- Shear(a,b):

$$\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ bx + y \end{bmatrix}$$

Translation

- Translate(a,b): $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+a \\ y+b \end{bmatrix}$
- Problem: cannot represent translation using 2 by 2 matrices!
- Solution: *homogeneous coordinates!* Embed the affine space of dimension n in a space of dimension n+1
    - (x,y) -> (x,y,1)
- Use a 3 by 3 linear transformation:

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \to \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$