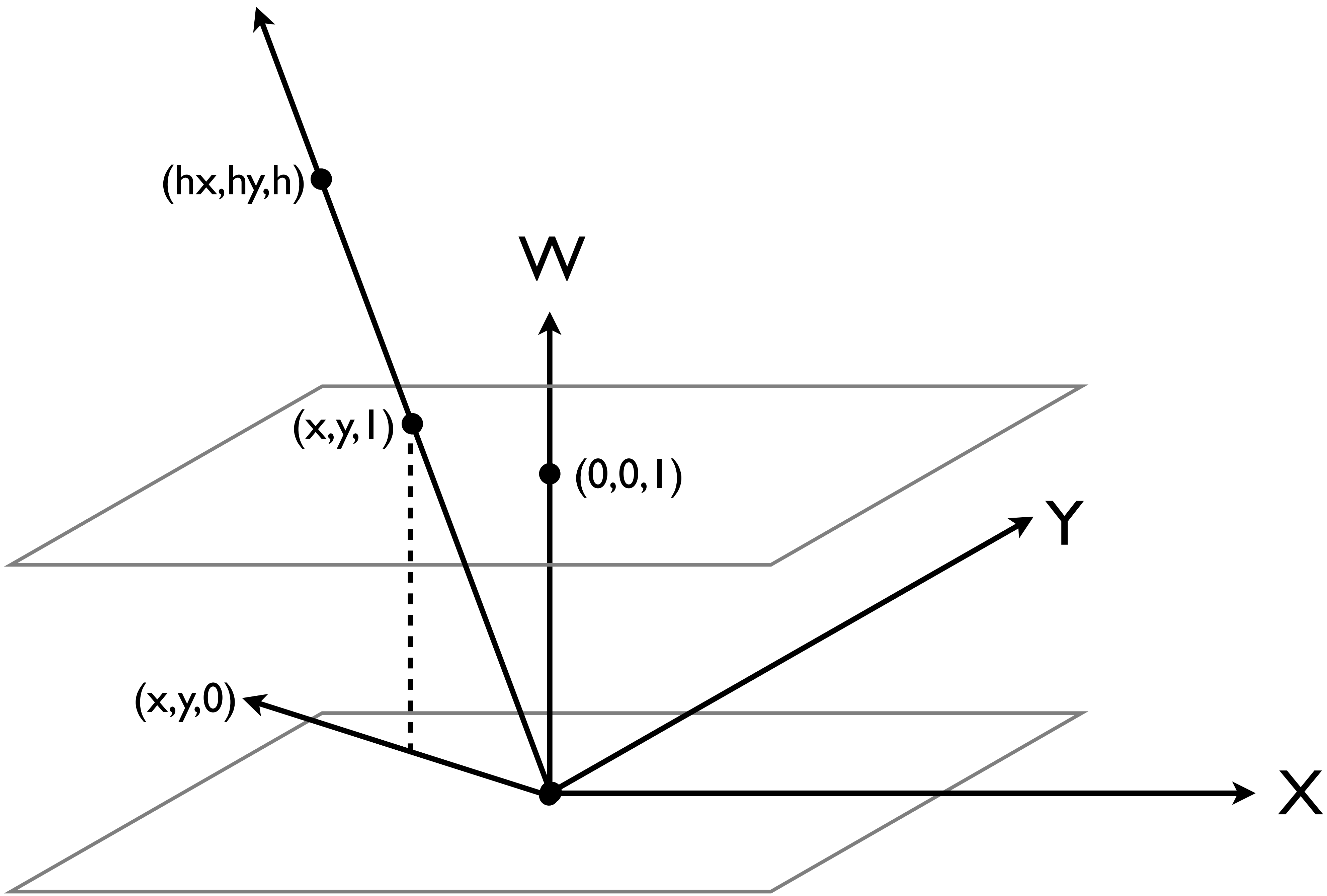
Translation

- Translate(a,b): $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x+a \\ y+b \end{bmatrix}$
- Problem: cannot represent translation using 2 by 2 matrices!
- Solution: *homogeneous coordinates!* Embed the affine space of dimension n in a space of dimension n+1
  - (x,y) -> (x,y,1)
- Use a 3 by 3 linear transformation:

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

The Projective Plane



The Projective Plane

- A point in the projective plane P^2 is represented by 3 coordinates, at least one of which is non-zero.
- Two 3-vectors $\mathbf{a},\mathbf{b}$ represent the same point in P^2 iff $\mathbf{a} = h\mathbf{b}$, where h is a non-zero scalar.
- A 2D point (x, y) in the Euclidean plane corresponds to all of the 3-vectors (hx,hy,h) in P^2, such as $(x,y,1)$.
  Note: this is a one-to-many correspondence!
- Geometric interpretation: each point (x,y) corresponds to a ray in 3D, from the origin $(0,0,0)$ through the point $(x,y,1)$

The Projective Plane

- A 2D vector (x, y) in the Euclidean plane corresponds to all of the 3-vectors $(hx, hy, 0)$ in P^2, such as $(x, y, 0)$.
- A 3-vector whose 3rd coordinate is 0 has two geometric interpretations (which are really the same):
  - A direction
  - A point at infinity

Homogeneous Coordinates

- Translate(a,b):

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

- Shear(a,b):

$$\begin{bmatrix} 1 & a & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+ay \\ bx+y \\ 1 \end{bmatrix}$$

- Scale(a,b):

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ by \\ 1 \end{bmatrix}$$

- Rotate(θ):

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ 1 \end{bmatrix}$$

Homogeneous Matrices

- All of the 2D transformations we have seen so far can now be written as follows:

$$\begin{bmatrix} a & b & m \\ c & d & n \\ 0 & 0 & 1 \end{bmatrix}$$

- What happens when the last row is not $[0, 0, 1]$?

Combining Transformations

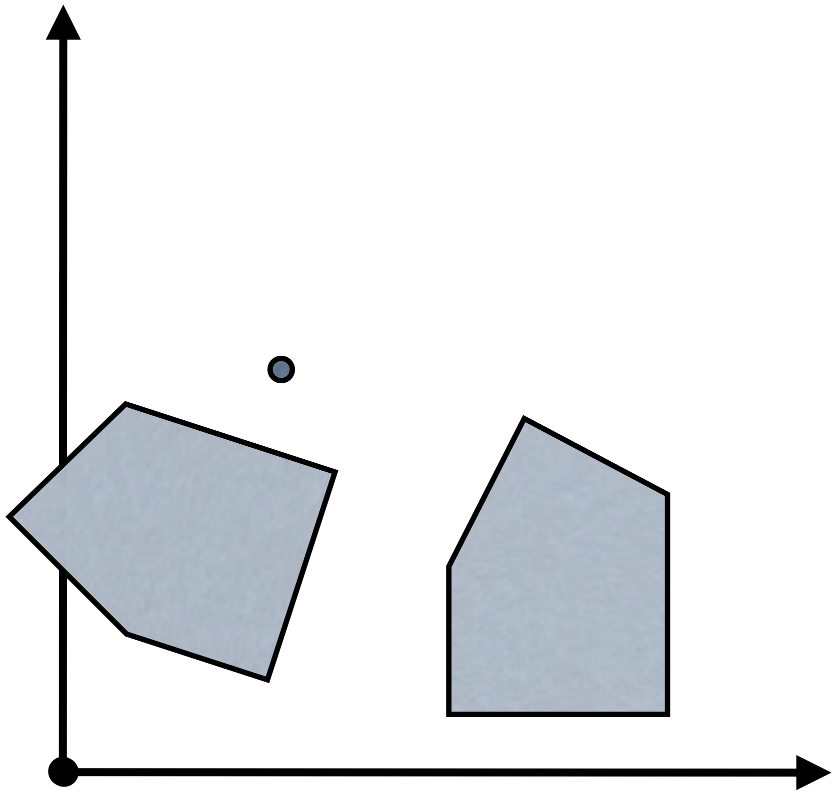
- A sequence of transformations can be collapsed into a single matrix using matrix multiplication:

$$T_1 T_2 T_3 \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = T_{1,2,3} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Is the order of transformations important?

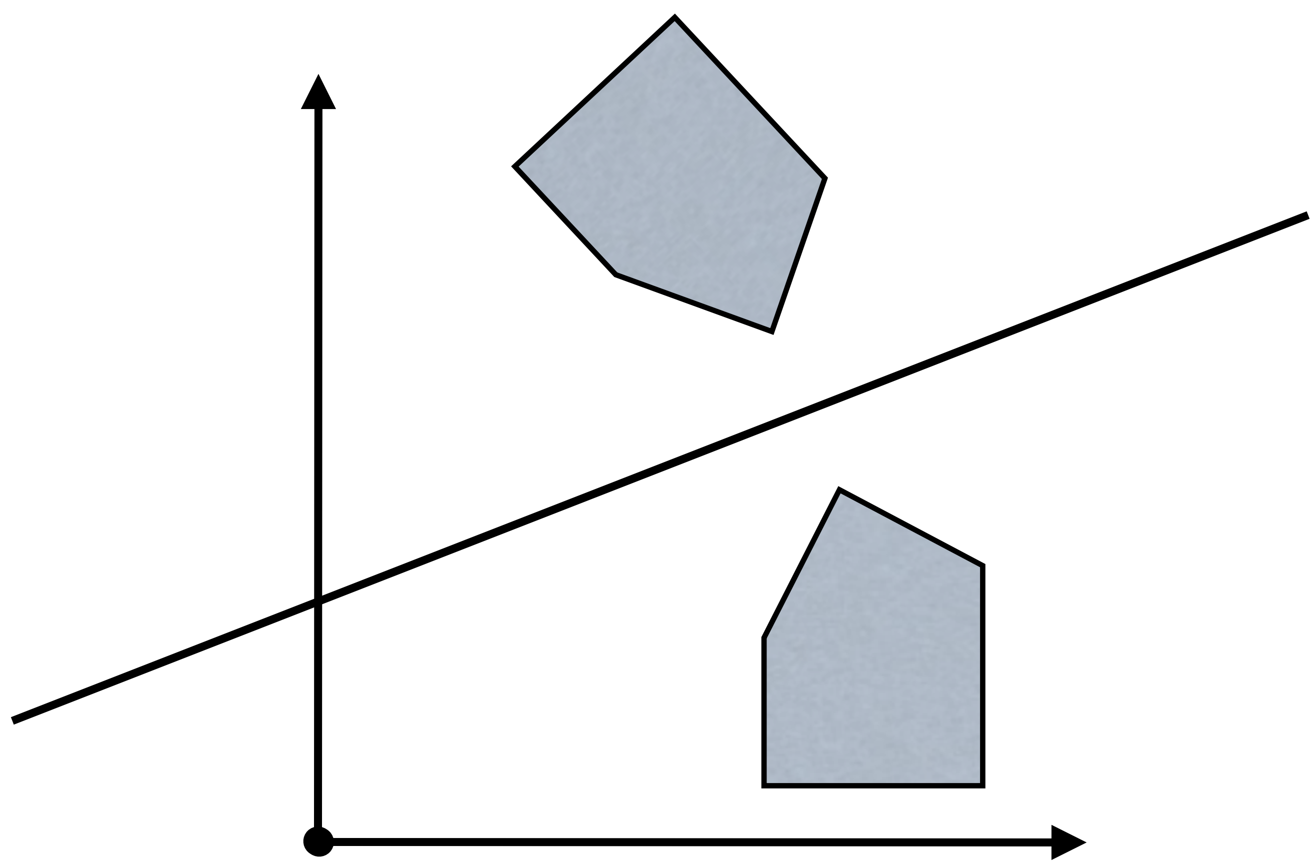
Example 1

- How to rotate about an arbitrary point ?



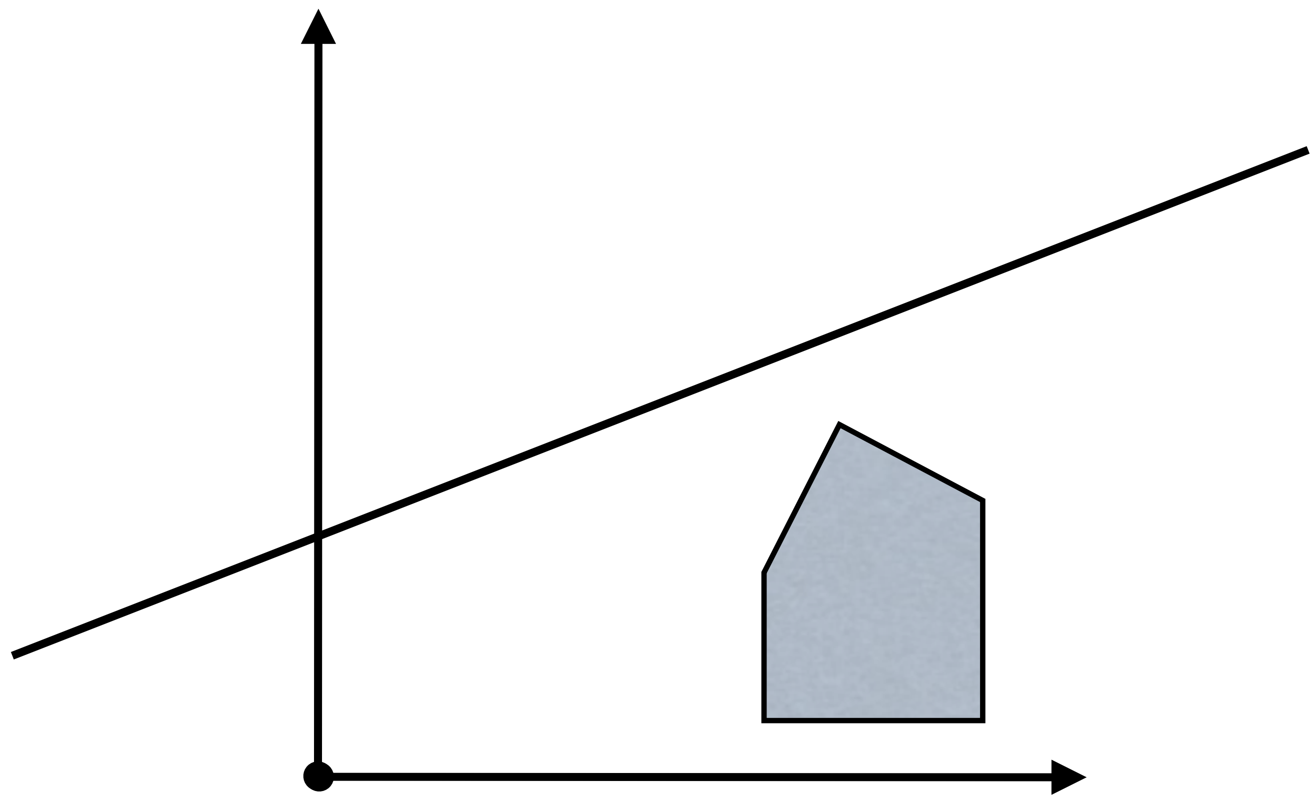
Example 2

- How to reflect through an arbitrary line ?



Example 2

- How to reflect through an arbitrary line ?



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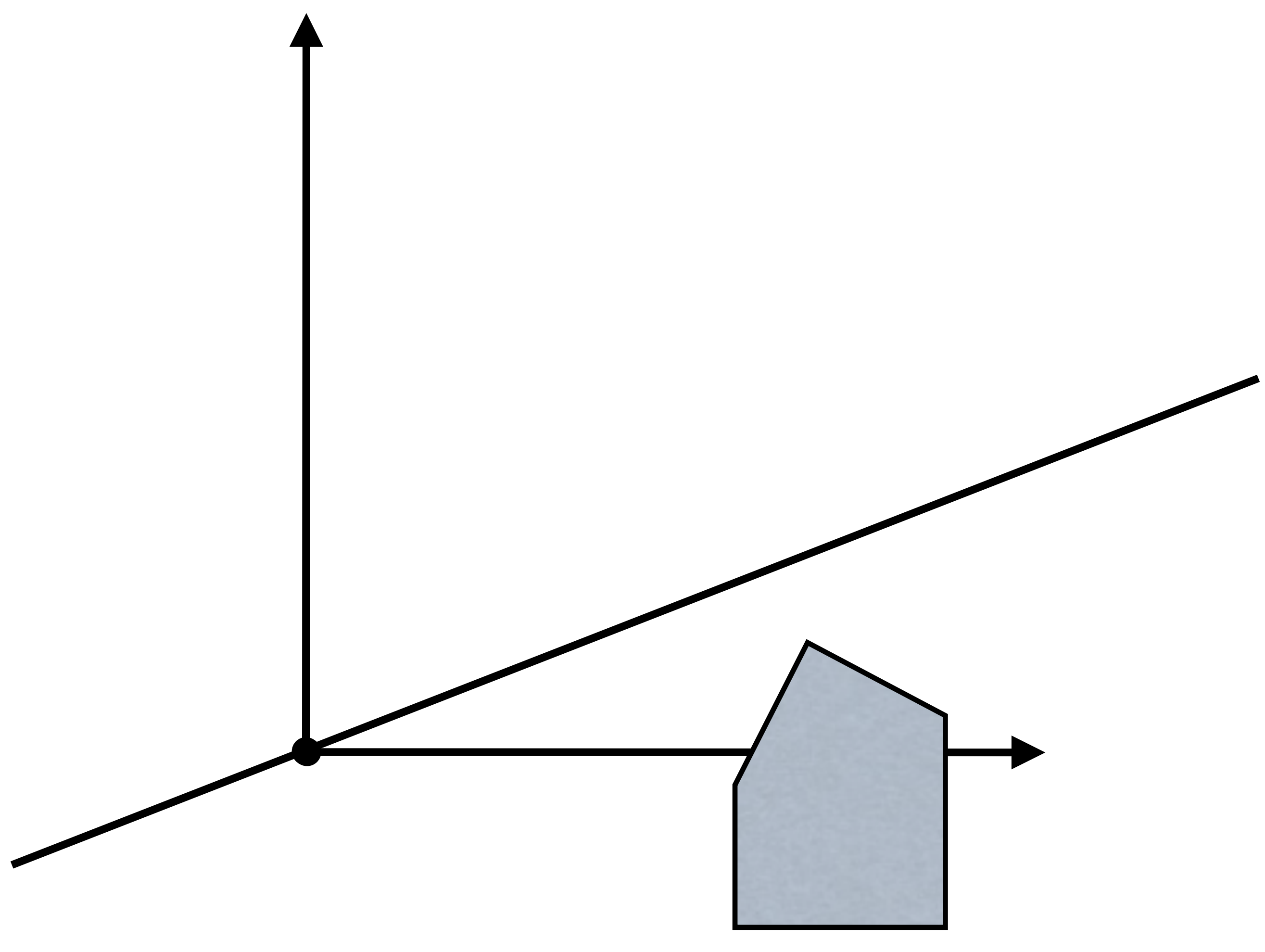

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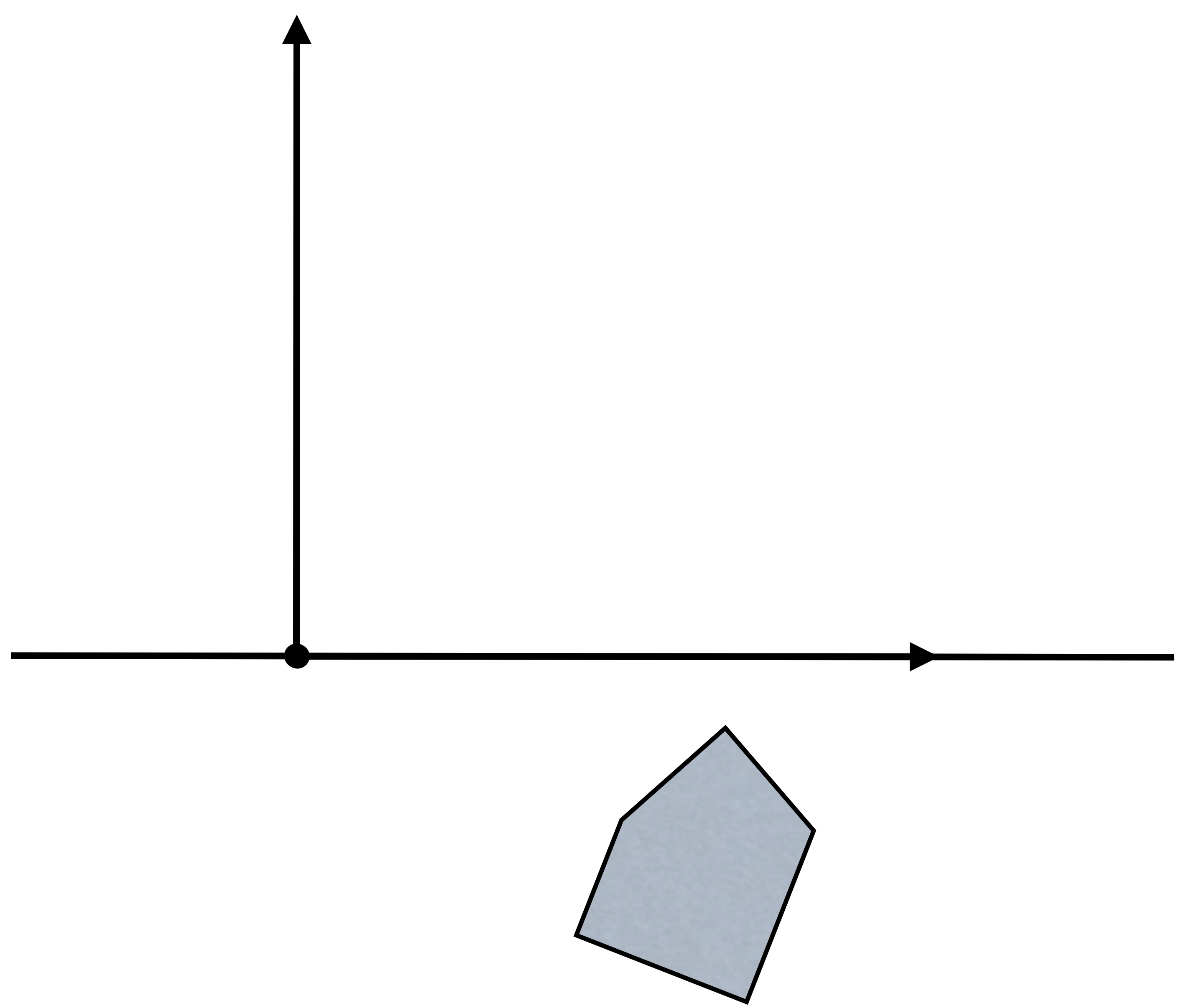

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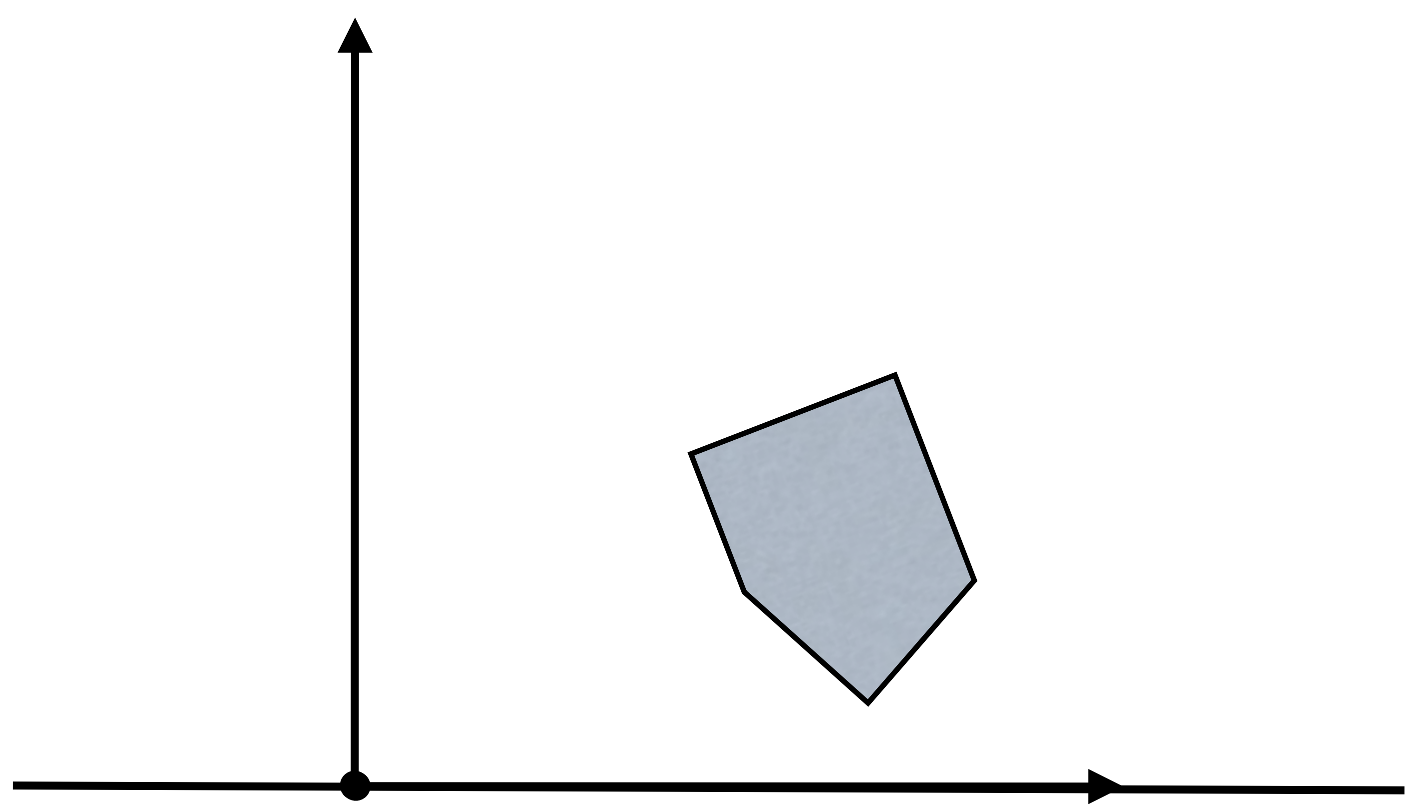

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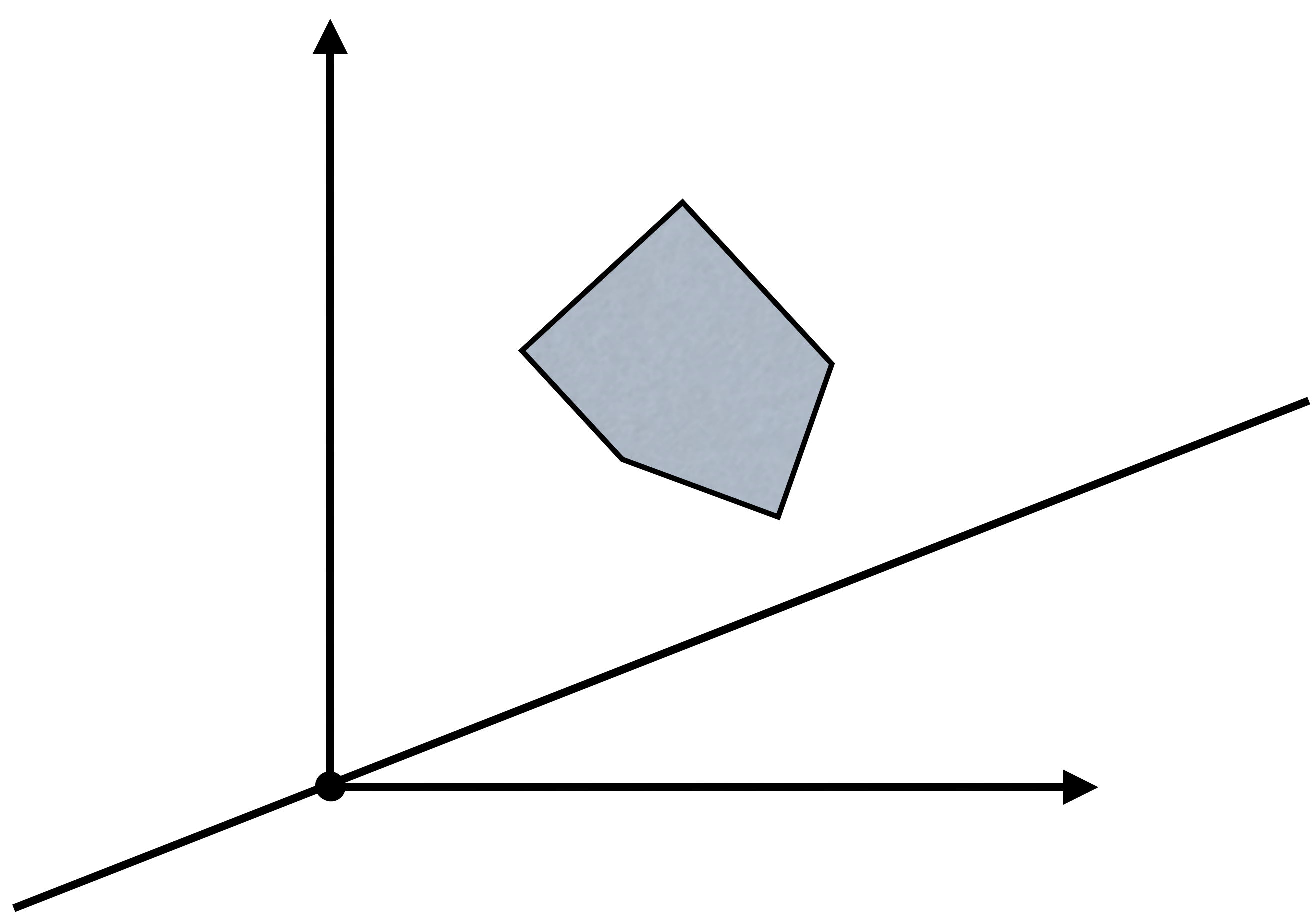

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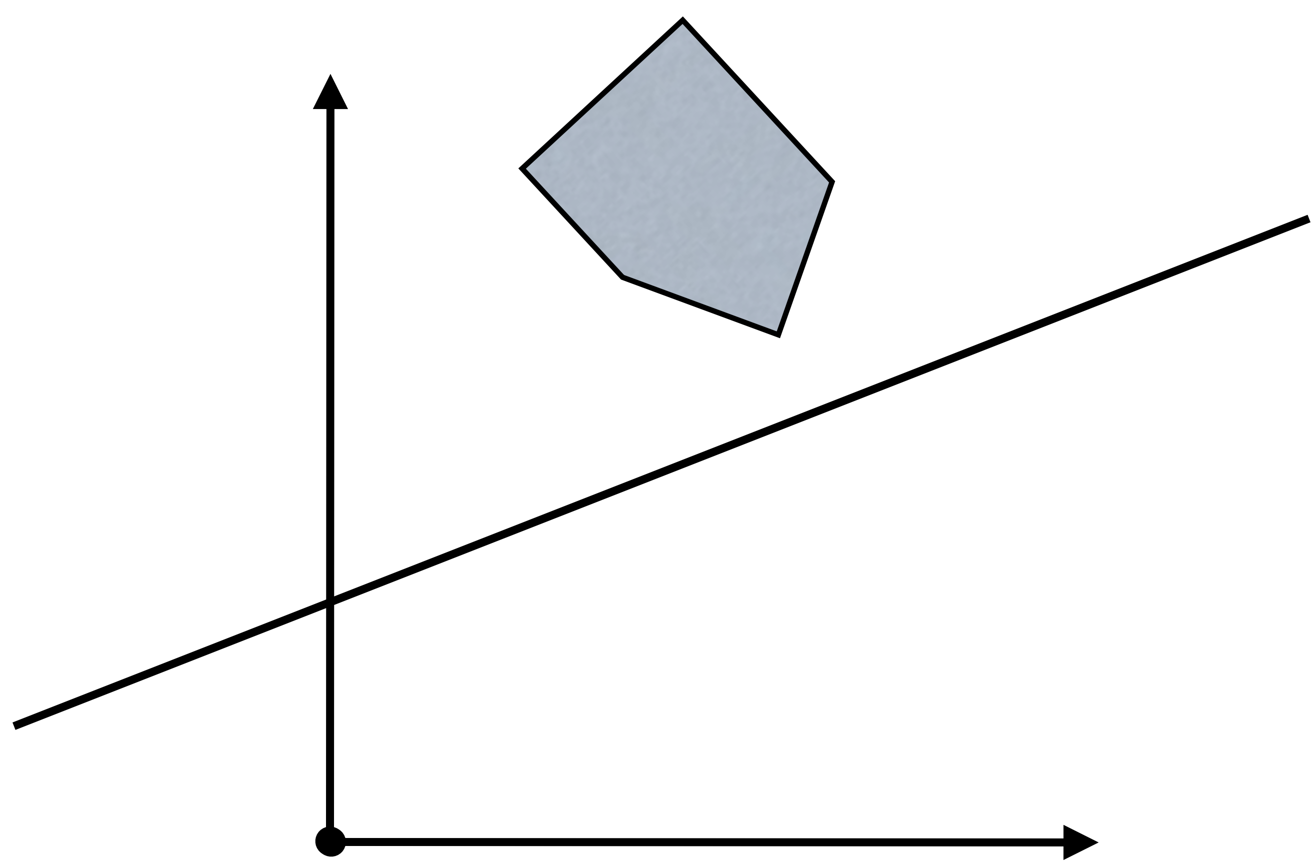


Affine Transformations: Definition

- Let $T : A_1 \rightarrow A_2$, where A_1 and A_2 are affine spaces.
- Then T is said to be an affine transformation if:
  - T maps vectors to vectors and points to points
  - T is a linear transformation on vectors
  - If $\mathbf{p}$ is a point and $\mathbf{u}$ is a vector:
    $T(\mathbf{p} + \mathbf{u}) = T(\mathbf{p}) + T(\mathbf{u})$

Affine Transformations: Properties

- Affine transformations preserve affine combinations of points. In other words, given an affine transformation T and a point $\mathbf{p}$:

$$\mathbf{p} = \alpha_1 \mathbf{p}_1 + \cdots + \alpha_k \mathbf{p}_k$$

- it holds that: $T(\mathbf{p}) = \alpha_1 T(\mathbf{p}_1) + \cdots + \alpha_k T(\mathbf{p}_k)$
- Intersections between lines are preserved.
- Parallel lines are preserved.

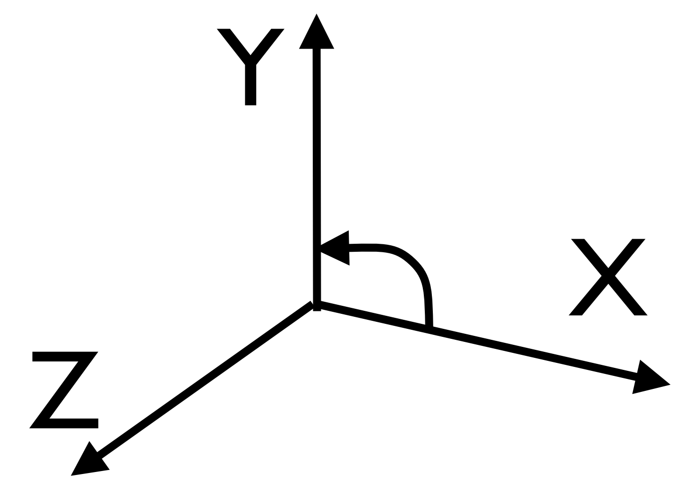
Special Cases

- **Note**: affine transformation, in general, do not preserve lengths or angles.
- **Rigid-body transformations**:
  - preserve angles and lengths
  - an arbitrary sequence of translations and rotations
- **Similarity transformations**:
  - preserve angles, but not lengths
  - rigid transformations + uniform scaling

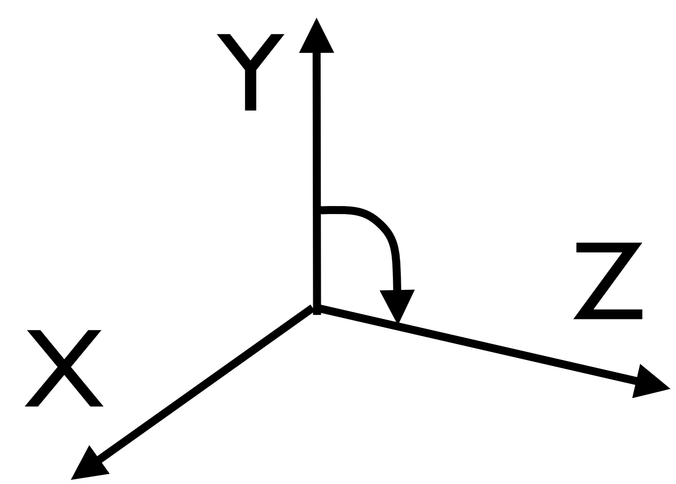
Transformations in 3D

3D Coordinate Systems

- Right-handed coordinate system:



- Left-handed coordinate system:



3D Transformations

- A point is represented by a 3D column vector: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$
- Homogeneous coordinates: $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$
- Transformations are 4 by 4 matrices: $\begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3D Transformations

- Translation: $$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ z+c \\ 1 \end{bmatrix}$$

- Scaling: $$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \\ 1 \end{bmatrix}$$

3D Shearing

- Shearing: $$\begin{bmatrix} 1 & a & b & 0 \\ c & 1 & d & 0 \\ e & f & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + ay + bz \\ cx + y + dz \\ ex + fy + z \\ 1 \end{bmatrix}$$

- The change in each coordinate is a linear combination of all three

- Transforms a cube into a general parallelepiped

3D Rotation

- Rotation about the x-axis:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y\cos\theta - z\sin\theta \\ y\sin\theta + z\cos\theta \\ 1 \end{bmatrix}$$

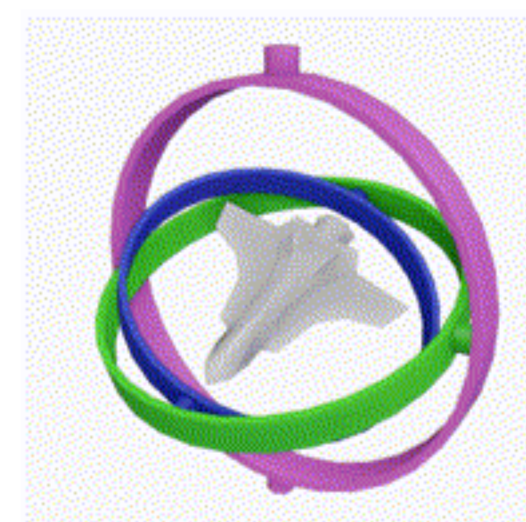
- Rotations about each of the other two axes are defined similarly.
- Rotations are orthogonal matrices, preserving distances and angles.

Rotation About an Arbitrary Axis

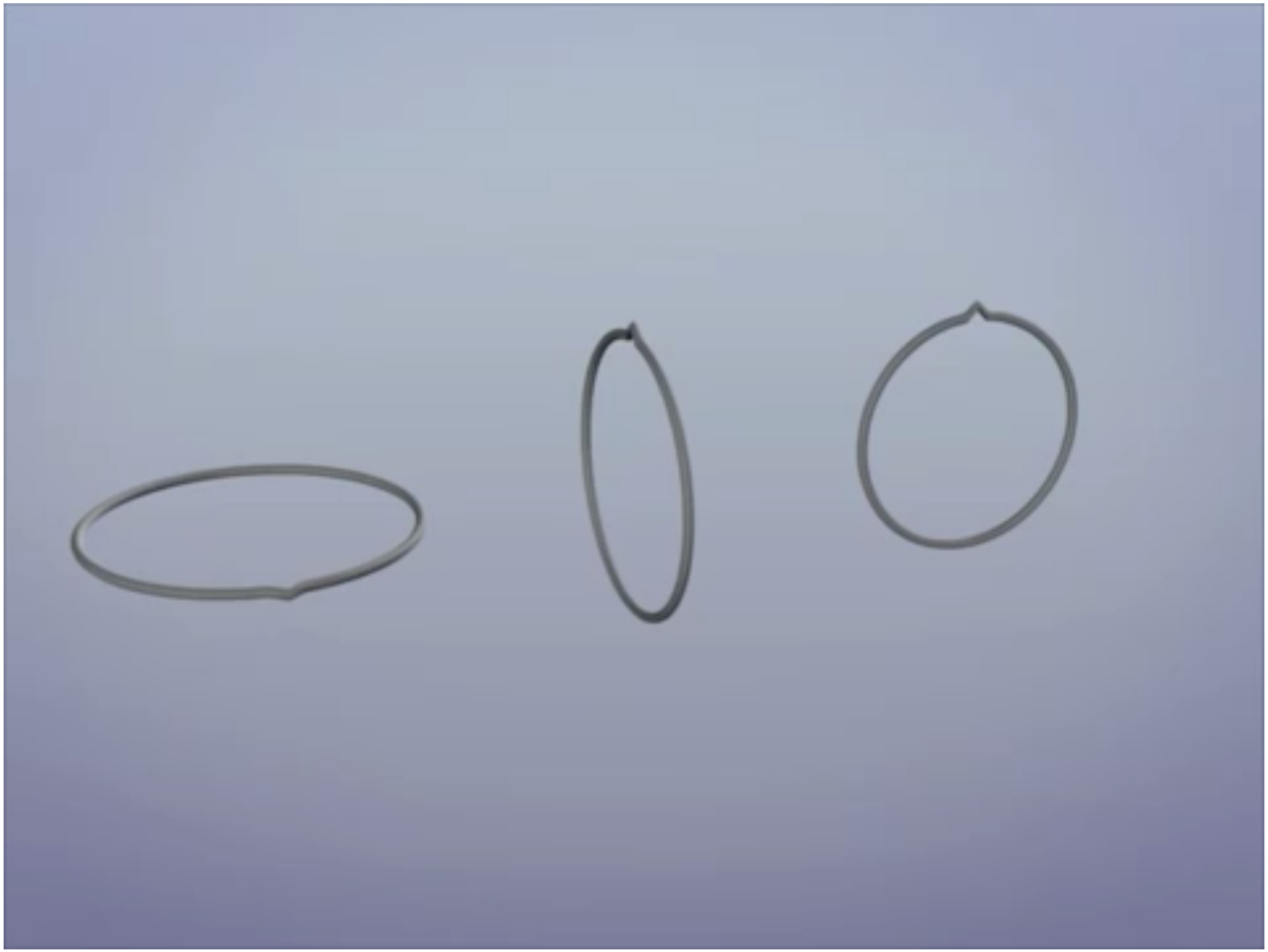
- The idea: make the arbitrary axis coincident with one of the coordinate axes, rotate, and then transform back:
  - Translate rotation axis to pass through the origin;
  - Rotate about the X axis into the XZ plane;
  - Rotate about the Y axis into the YZ plane - rotation axis is now aligned with the Z axis;
  - Rotate about the Z axis by the desired angle;
  - Apply inverse rotations about the Y and X axes;
  - Apply inverse translation.

Euler Angles

- **Euler's rotation theorem**: any rotation or sequence of rotations of a rigid body about a fixed point is equivalent to a single rotation by a given angle θ about a fixed axis (called the *Euler axis*) that runs through the fixed point.
- **Euler angles**: the orientation of an object is specified by rotation angles about the X, Y, Z axes (performed in this order)
- Euler angle interpolation is not intuitive
- Gimbal lock



Euler Angles



Quaternions

- Rotations may be represented by unit quaternions: $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$
- a, b, c, d are real numbers, $a^2 + b^2 + c^2 + d^2 = 1$
- $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$

  $\mathbf{ij} = \mathbf{k} = -\mathbf{ji} \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj} \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}$

- Rotation by angle ϕ about the unit vector [b, c, d] corresponds to the quaternion:

$$q = \cos\frac{\phi}{2} + b\sin\frac{\phi}{2}\mathbf{i} + c\sin\frac{\phi}{2}\mathbf{j} + d\sin\frac{\phi}{2}\mathbf{k}$$

Quaternions (cont'd)

- The rotation matrix corresponding to a quaternion:

$$R = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{pmatrix}$$

- Performing successive rotations corresponds to multiplying unit quaternions
- Unit quaternions can be thought of as points on the unit sphere in 4D
- Interpolating between orientations can be done by spherical linear interpolation (slerp) between unit quaternions!

3D Reflection

- Through the xy plane: $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
- Reflections through the xz and the yz planes are defined similarly
- How can we reflect through some arbitrary plane?

Transforming Planes

- One way to transform a plane is by transforming any three non-collinear points on the plane.
- Another way is to transform the plane equation coefficients [A,B,C,D] directly:

$$Ax + By + Cz + D = \begin{bmatrix} A & B & C & D \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

- Given a transformation T that transforms [x,y,z] to [x',y',z'] find A', B', C', and D', such that:

$$\begin{bmatrix} A' & B' & C' & D' \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = 0$$

(continued)

- Note that $$\begin{bmatrix} A & B & C & D \end{bmatrix} T^{-1} T \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$
- Thus, the transformation that we should apply to the plane equation is: $$\left(T^{-1}\right)^T \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix}$$
- This is how we transform normal vectors!