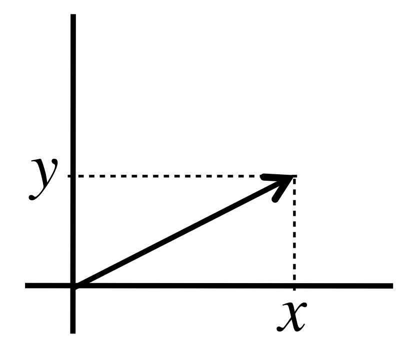
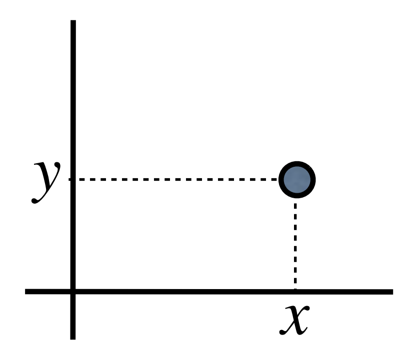
Basic Geometry Review

Basic Geometric Entities

- Scalars - real numbers
  - sizes/lengths, angles
- Vectors - typically 2D, 3D, 4D
  - directions



- Points - typically 2D, 3D, 4D
  - positions



Spaces

- ***Scalar field***: formed by scalars and the operations between them (+,*).

- ***Vector space:*** formed by vectors, scalars, and the operations between them.

- ***Note:*** a purely abstract vector space has no notion of: distance, size, angle, or point!

Euclidean vector spaces

- A *Euclidean space* is a linear space with a distance metric based on inner product:

$$a \cdot b = \langle a, b \rangle = \sum_{i=1}^{n} a_i b_i$$

$$|a - b| = \sqrt{\langle a - b, a - b \rangle} = \sqrt{(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2}$$

- *Cartesian coordinates*: a standard orthonormal basis:
  - Basis vectors have length 1
  - Basis vectors are pairwise perpendicular (orthogonal)

Operations on Vectors

- Vector size/length/norm: $$\|\mathbf{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

- Unit vector = pure direction $$\|\mathbf{v}\| = 1$$

- General vector = size * direction

- Vector normalization: $$\mathbf{v} \leftarrow \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Operations on Vectors

- Addition $$\mathbf{u}+\mathbf{v}=[u_1+v_1,\ldots,u_n+v_n]^{\mathrm{T}}$$

- Multiplication by a scalar $$s\mathbf{u}=[su_1,\ldots,su_n]^{\mathrm{T}}$$

- Dot product (scalar product)

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^{\mathrm{T}}\mathbf{v}=u_1v_1+\cdots+u_nv_n$$

$$\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\,\|\mathbf{v}\|\cos(\mathbf{u},\mathbf{v})$$

Operations on Vectors

- Cross product (vector product)

$$\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ i & j & k \end{bmatrix} = i \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - j \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + k \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

- What is the geometric meaning of this operation?
  - A perpendicular vector of size $\|\mathbf{u}\| \|\mathbf{v}\| \sin(\mathbf{u}, \mathbf{v})$

Affine spaces

- An *affine space* adds the notion of points to the vector space: A = (P,V), where V is a vector space and P is a set of points:
  - for every point p and vector v: $p+v \in P$
  - for every two points p,q: $p-q \in V$
- By choosing a basis $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ and designating an origin point $\mathbf{a}$, we define an *affine frame*:

$$\mathbf{p} = \mathbf{a} + \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

Operations on Points

- Can't add points to each other…
  - but can add a vector to a point $\mathbf{a} + \mathbf{v} = \mathbf{b}$
  - also can subtract points $\mathbf{v} = \mathbf{b} - \mathbf{a}$

- Can't multiply a point by a scalar…
  - but can take the affine combination of two points:
    where $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$ $\quad s + t = 1$
  - Why?

$$\begin{aligned}\mathbf{c} &= (1-t)\mathbf{a} + t\mathbf{b} \\ &= \mathbf{a} + t(\mathbf{b} - \mathbf{a})\end{aligned}$$

Operations on Points

- What about an affine combination of n>2 points?

$$
\begin{aligned}
\mathbf{c} &= \alpha_1 \mathbf{p}_1 + \cdots + \alpha_n \mathbf{p}_n \\
&= (1 - \alpha_2 - \cdots - \alpha_n)\mathbf{p}_1 + \cdots + \alpha_n \mathbf{p}_n \\
&= \mathbf{p}_1 + \alpha_2(\mathbf{p}_2 - \mathbf{p}_1) + \cdots + \alpha_n(\mathbf{p}_n - \mathbf{p}_1)
\end{aligned}
$$

- Another legal combination of points is their vector combination:

$$
\mathbf{v} = \alpha_1 \mathbf{p}_1 + \cdots + \alpha_n \mathbf{p}_n, \quad \text{where } \sum_{i=1}^{n} \alpha_i = 0
$$

Lines and Planes

- A line in 2D can be defined by:
  - 1D affine space: all affine combinations of two points
  - implicit equation: $f(x,y) = ax + by + c = 0$
  - parametric equation: $\mathrm{L}(t) = \mathrm{O} + t\mathrm{D}$

- A plane can be defined by:
  - 2D affine space: all affine combinations of three points
  - implicit equation: $f(x,y,z) = ax + by + cz + d = 0$
  - parametric equation: $\mathrm{P}(s, t) = \mathrm{O} + s\mathrm{D}_1 + t\mathrm{D}_2$