Buffon’s Needle Problem

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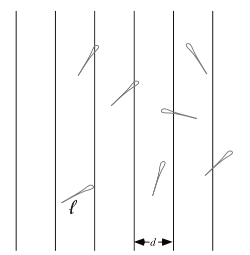
Abstract

This document explores Buffon’s Needle problem, a classic probability problem. It investigates the analytical solution as well as Monte Carlo simulations to estimate the value of π. Key observations from the simulations and their implications are discussed.

1 Problem Description

Buffon’s Needle problem is a classic probability problem that can be stated as follows:

Suppose you have a floor with equally spaced parallel lines, and you have a needle of length L that you randomly drop onto the floor. What is the probability that the needle will intersect or cross one of the lines?



This problem can be solved analytically and through Monte Carlo simulation. The key idea behind Monte Carlo simulations is to use randomness to generate a large number of potential outcomes and then analyze the statistical properties of these outcomes to approximate solutions.

To gain some intuition about the problem and the Monte Carlo simulation here is a convergence graph for the probability of needle crossing the line in this Monte Carlo experiments.

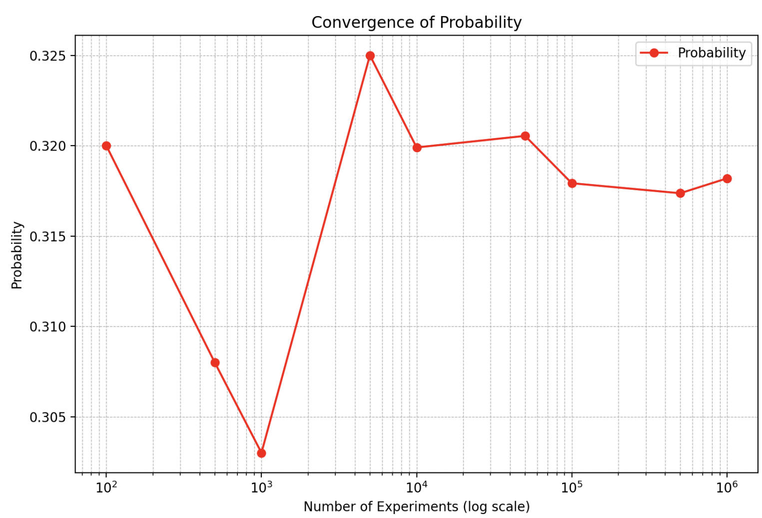


Figure 1: Simulations

2 Analytic Solution

2.1 Step 1: Define Variables

Let's define the following variables:

$$L : \text{Length of the needle}$$
$$d : \text{Distance between the parallel lines}$$
$$\theta : \text{Angle between the needle and the lines}$$
$$x : \text{The distance from the center of the needle to the closest parallel line}$$

2.2 Step 2: Geometric Setup

We consider a floor with equally spaced parallel lines and $L < d$. The objective is to find the probability that the needle crosses or intersects one of these lines.

2.3 Step 3: Expressing Probabilities

The uniform probability distribution for x between 0 and $\frac{d}{2}$ is:

$$f_X(x) = \begin{cases} \frac{2}{d} & \text{if } 0 \leq x \leq \frac{d}{2} \\ 0 & \text{elsewhere} \end{cases}$$

When x is 0, the needle is directly centered on a line. When x is $\frac{d}{2}$, it's perfectly between two lines. The uniform PDF means the needle can land anywhere in this range, but not outside of it.

The uniform probability distribution for θ between 0 and $\frac{\pi}{2}$ is:

$$f_\Theta(\theta) = \begin{cases} \frac{2}{\pi} & \text{if } 0 \leq \theta \leq \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases}$$

When θ is 0, the needle is aligned parallel to the marked lines, while $\theta = \frac{\pi}{2}$ radians means the needle is positioned perpendicular to the lines. Any angle within this range is considered an equally probable outcome.

The two random variables, x and θ, are independent. So the joint probability density function can be determined by multiplying the individual probabilities.

$$f_{X,\Theta}(x, \theta) = \begin{cases} \frac{4}{d\pi} & \text{if } 0 \leq x \leq \frac{d}{2}, 0 \leq \theta \leq \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases}$$

So the needle crosses a line if:

$$x \leq \frac{L}{2} \sin\theta$$

The probability that the needle will cross a line is given by the integral:

$$P = \int_{\theta=0}^{\frac{\pi}{2}} \int_{x=0}^{\frac{L}{2}\sin\theta} \frac{4}{d\pi} \, dx \, d\theta = \frac{2L}{d\pi}$$

2.4 Result: Probability of Intersection

The probability $P(\text{intersection})$ can be calculated as:

$$P(\text{intersection}) = \frac{2L}{\pi d}$$

This formula is derived from the geometric properties of the problem.

3 Monte Carlo Simulation

3.1 Step 1: Define Variables

Let's define the following variables for the Monte Carlo simulation:

$$\begin{aligned} L &: \text{Length of the needle} \\ d &: \text{Distance between the lines } L < d \\ n_{\text{trials}} &: \text{Number of trials} \\ \text{success\_count} &: \text{Counter for successful trials} \end{aligned}$$

3.2 Step 2: Simulation

To simulate Buffon's Needle problem using Monte Carlo, we'll follow these steps:

1. Initialize the success_count to zero.
2. Repeat the following process for each trial (n_{trials} times):
   - Generate a random position for the center of the needle, x_{center}, between 0 and $\frac{d}{2}$.
   - Generate a random angle for the needle, θ, between 0 and $\frac{\pi}{2}$ radians.
   - Check if the needle intersects a line:

     $$x_{\text{center}} \leq \frac{L}{2} * \sin(\theta)$$

     If so, increment success_count by 1.
3. Calculate the probability as:
   $$P = \frac{\text{success\_count}}{n_{\text{trials}}}$$
4. The value of π can be determined as:
   $$\pi = \frac{2L}{Pd}$$

The more trials you perform, the closer the Monte Carlo result will be to the analytical solution.

3.3 Pseudo-Code for Monte Carlo Simulation

Here is the pseudo-code for the Monte Carlo simulation:

```
Set L  // Length of the needle
Set d  // Distance between the lines
Set n_trials  // Number of trials
Set success_count to 0  // Counter for successful trials

For each trial in range from 1 to n_trials:
    Generate a random value for x_center between 0 and d/2
    Generate a random value for theta between 0 and pi/2
    If x_center is less than or equal to L/2 * sin(theta):
        Increment success_count by 1

Calculate the probability as success_count divided by n_trials
Calculate the value of pi as: pi = 2L / (probability * d)
```

4 Simulation Results

These are some simulations I performed and the estimated π values.

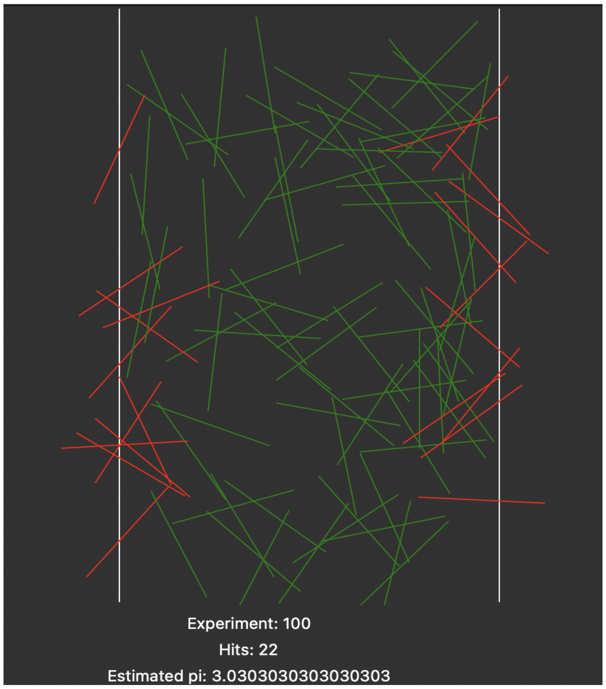


Figure 2: Simulation with 10^2 trials

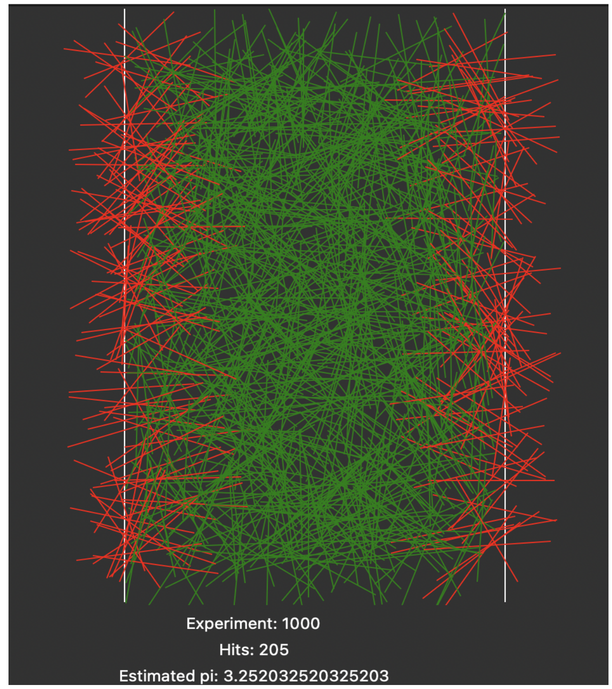


Figure 3: Simulation with 10^3 trials

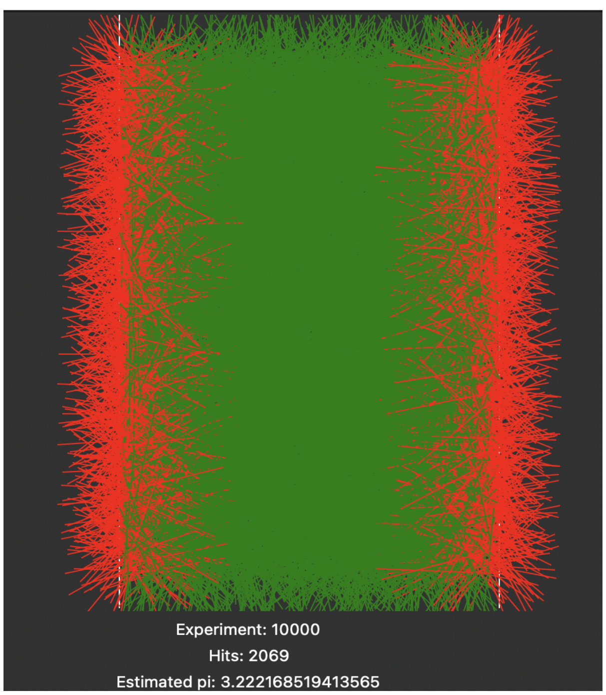


Figure 4: Simulation with 10^4 trials

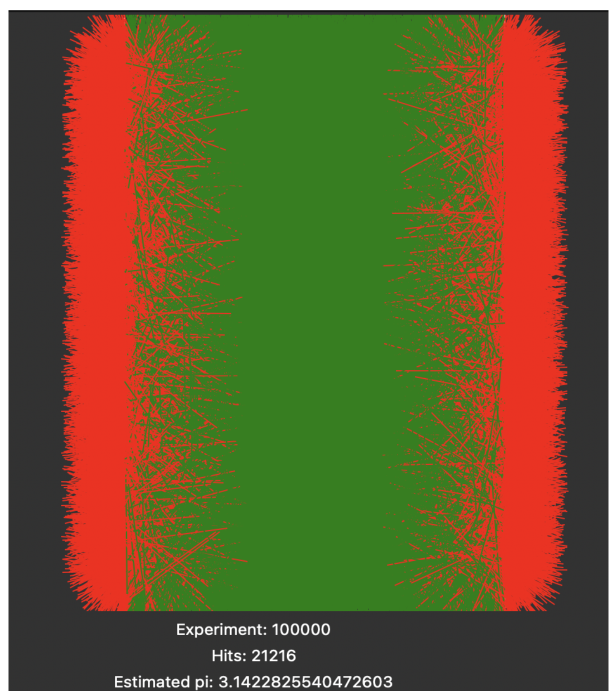


Figure 5: Simulation with 10^5 trials

5 Some More Simulation Results

For values larger than 10^6, the computational demands to draw the graphs exceeded the capabilities of my computer, so I'm summarizing the results of multiple runs without graph below:

Experiment Count	D	L	**Probability**	**Approximated** π
10^2	2	1	0.290000	3.448276
10^3	2	1	0.315000	3.174603
10^4	2	1	0.328200	3.046923
10^5	2	1	0.317100	3.153579
10^6	2	1	0.318549	3.139234
10^7	2	1	0.318456	3.140156
10^8	2	1	0.318284	3.141844
10^9	2	1	0.318308	3.141607

Key Observations:

- As the number of experiments increases (from 100 to 1 billion), the estimated probability P tends to stabilize around a specific value.
- The Approximated π converges to the true value of π (3.14159265...) as the number of experiments grows. This is consistent with the Law of Large Numbers, which suggests that the more trials we perform, the closer our estimate will be to the true value.
- The accuracy of the π approximation is influenced by the number of experiments. In the earlier experiments, with 100 and 1,000 trials, the approximated π may deviate from the true value. However, with larger experiment counts the approximation becomes increasingly accurate.
- As this is a probabilistic simulation there is a chance to underestimate or overestimate π for some of these experiments. We can never be hundred percent sure.
- As we transitioned from 10^3 trials to 10^4 trials our estimate of π deviated from the true value of π. This is a reflection of the inherent randomness in the problem.

In summary, these experiments demonstrate how Buffon's Needle simulation can be used to estimate the value of π. The results illustrate the relationship between the number of trials, the estimated probability of intersection, and the accuracy of the π approximation. As the number of trials increases, the estimate of π becomes more reliable and approaches the true mathematical constant.