Chapter 6

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

6.1 INTRODUCTION

A physical situation concerned with the rate of change of one quantity with respect to another gives rise to a differential equation.

Consider the first order ordinary differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

with the initial condition

$$y(x_0) = y_0 \tag{2}$$

Many analytical techniques exist for solving such equations, but these methods can be applied to solve only a selected class of differential equations.

However, a majority of differential equations appearing in physical problems cannot be solved analytically. Thus it becomes imperative to discuss their solution by numerical methods.

In numerical methods, we do not proceed in the hope of finding a relation between variables but we find the numerical values of the dependent variable for certain values of independent variable.

It must be noted that even the differential equations which are solvable by analytical methods can be solved numerically as well.

6.2 INITIAL-VALUE AND BOUNDARY-VALUE PROBLEMS

Problems in which all the conditions are specified at the initial point only are called **initial-value problems**. For example, the problem given by eqns. (1) and (2) is an initial value problem.

Problems involving second and higher order differential equations, in which the conditions at two or more points are specified, are called **boundary-value problems**.

To obtain a unique solution of n^{th} order ordinary differential equation, it is necessary to specify n values of the dependent variable and/or its derivative at specific values of independent variable.

6.3 SINGLE STEP AND MULTI-STEP METHODS

The numerical solutions are obtained step-by-step through a series of equal intervals in the independent variable so that as soon as the solution y has been obtained at $x = x_i$, the next step consists of evaluating y_{i+1} at $x = x_{i+1}$. The methods which require only the numerical value y_i in order to compute the next value y_{i+1} for solving eqn. (1) given above are termed as **single step methods.**

The methods which require not only the numerical value y_i but also at least one of the past values $y_{i-1}, y_{i-2}, \ldots\ldots$ are termed as **multi-step methods.**

6.4 COMPARISON OF SINGLE-STEP AND MULTI-STEP METHODS

The single step method has obvious advantages over the multi-step methods that use several past values $(y_n, y_{n-1}, \ldots\ldots, y_{n-p})$ and that require initial values $(y_1, y_2, \ldots\ldots, y_n)$ that have to be calculated by another method.

The major disadvantage of single-step methods is that they use many more evaluations of the derivative to attain the same degree of accuracy compared with the multi-step methods.

6.5 NUMERICAL METHODS OF SOLUTION OF O.D.E.

In this chapter we will discuss various numerical methods of solving ordinary differential equations.

We know that these methods will yield the solution in one of the two forms:

(*a*) A series for y in terms of powers of x from which the value of y can be obtained by direct substitution.

(b) A set of tabulated values of x and y.

Picard's method and Taylor's method belong to class (a) while those of Euler's, Runge-Kutta, Adams-Bashforth, Milne's, etc. belong to class (b). Methods which belong to class (b) are called **step-by-step methods** or **marching methods** because the values of y are computed by short steps ahead for equal intervals of the independent variable.

In Euler's and Runge-Kutta methods, the interval range h should be kept small, hence they can be applied for tabulating y only over a limited range. To get functional values over a wider range, the Adams-Bashforth, Milne, Adams-Moulton, etc. methods may be used since they use finite differences and require starting values, usually obtained by Taylor's series or Runge-Kutta methods.

6.6 PICARD'S METHOD OF SUCCESSIVE APPROXIMATIONS

Picard was a distinguished Professor of Mathematics at the university of Paris, France. He was famous for his research on the Theory of Functions.

Consider the differential equation

$$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0 \tag{3}$$

Integrating eqn. (3) between the limits x_0 and x and the corresponding limits y_0 and y, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx$$

$$\Rightarrow \qquad y - y_0 = \int_{x_0}^{x} f(x, y)\, dx$$

or,
$$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx \tag{4}$$

In equation (4), the unknown function y appears under the integral sign. This type of equation is called integral equation.

This equation can be solved by the method of successive approximations or iterations.

To obtain the first approximation, we replace y by y_0 in the R.H.S. of eqn. (4).

Now, the first approximation is

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

The integrand is a function of x alone and can be integrated.

8. Approximate y and z by using Picard's method for the solution of simultaneous differential equations

$$\frac{dy}{dx} = 2x + z, \quad \frac{dz}{dx} = 3xy + x^2z$$

with $y = 2$, $z = 0$ at $x = 0$ up to third approximation.

9. Using Picard's method, obtain the solution of $\frac{dy}{dx} = x(1 + x^3y)$, $y\ (0) = 3$

Tabulate the values of $y(0.1)$, $y(0.2)$.

6.8 EULER'S METHOD

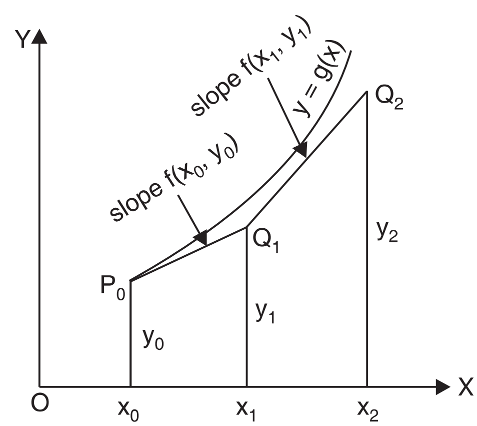
Euler's method is the simplest one-step method and has a limited application because of its low accuracy. This method yields solution of an ordinary differential equation in the form of a set of tabulated values.

In this method, we determine the change Δy is y corresponding to small increase in the argument x. Consider the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{7}$$

Let $y = g(x)$ be the solution of (7). Let $x_0, x_1, x_2, ...$ be equidistant values of x.

In this method, we use the property that in a small interval, a curve is nearly a straight line. Thus at the point (x_0, y_0), we approximate the curve by the tangent at the point (x_0, y_0).



The eqn. of the tangent at $P_0(x_0, y_0)$ is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{P_0} (x - x_0) = f(x_0, y_0)\,(x - x_0)$$

$$\Rightarrow \qquad y = y_0 + (x - x_0)\, f(x_0, y_0) \tag{8}$$

This gives the y-coordinate of any point on the tangent. Since the curve is approximated by the tangent in the interval (x_0, x_1), the value of y on the curve corresponding to $x = x_1$ is given by the above value of y in eqn. (8) approximately.

Putting $x = x_1 (= x_0 + h)$ in eqn. (8), we get

$$y_1 = y_0 + hf(x_0, y_0)$$

Thus Q_1 is (x_1, y_1)

Similarly, approximating the curve in the next interval (x_1, x_2) by a line through $Q_1(x_1, y_1)$ with slope $f(x_1, y_1)$, we get

$$y_2 = y_1 + hf(x_1, y_1)$$

In general, it can be shown that,

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$$

This is called Euler's Formula.

A great disadvantage of this method lies in the fact that if $\frac{dy}{dx}$ changes rapidly over an interval, its value at the beginning of the interval may give a poor approximation as compared to its average value over the interval and thus the value of y calculated from Euler's method may be in much error from its true value. These errors accumulate in the succeeding intervals and the value of y becomes erroneous.

NOTE *In Euler's method, the curve of the actual solution $y = g(x)$ is approximated by a sequence of short lines. The process is very slow. If h is not properly chosen, the curve $P_0Q_1Q_2$ ...... of short lines representing numerical solution deviates significantly from the curve of actual solution.*

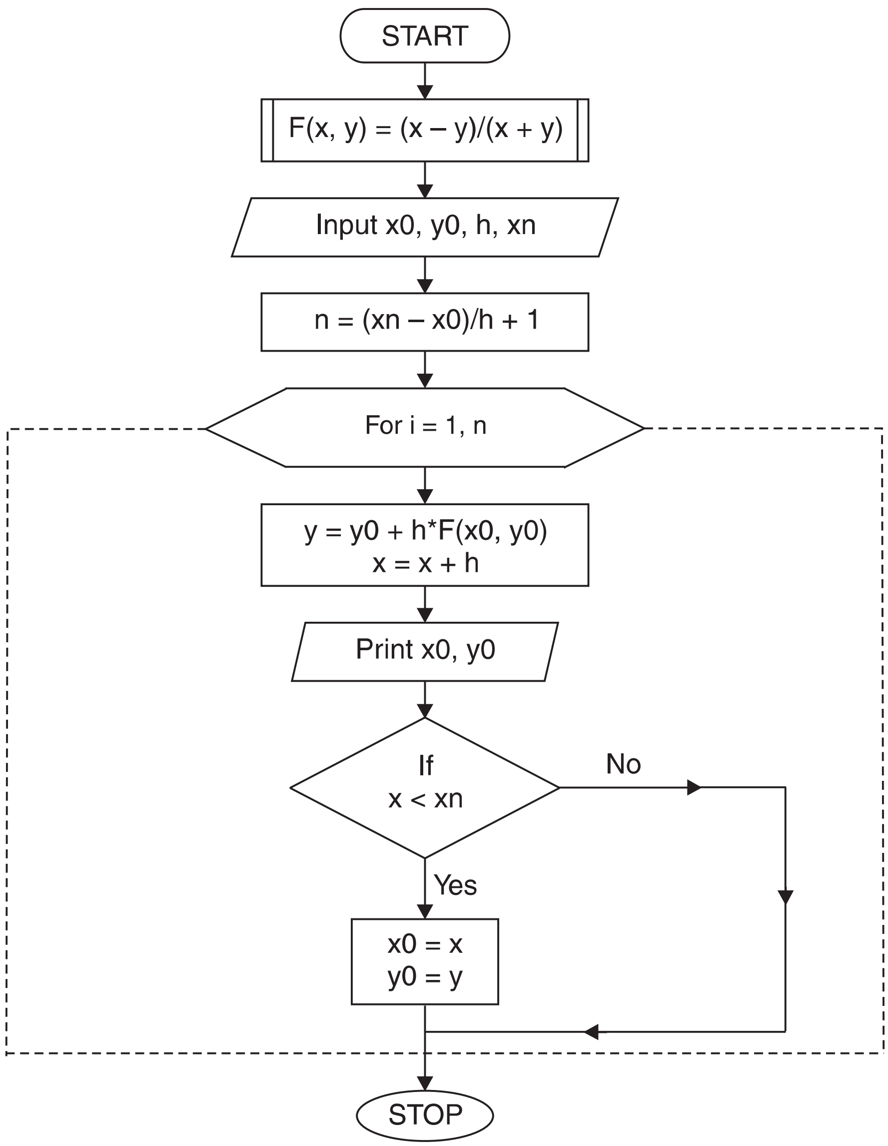
To avoid this error, **Euler's modified method** is preferred because in this, we consider the curvature of the actual curve inplace of approximating the curve by sequence of short lines.

6.9 ALGORITHM OF EULER'S METHOD

```
1. Function F(x,y)=(x-y)/(x+y)
2. Input x0,y0,h,xn
3. n=((xn-x0)/h)+1
4. For i=1,n
5. y=y0+h*F(x0,y0)
6. x=x+h
```

```
 7. Print x0,y0
 8. If x<xn then
          x0=x
          y0=y
    ELSE
 9. Next i
10. Stop
```

6.10 FLOW-CHART OF EULER'S METHOD



6.11 PROGRAM OF EULER'S METHOD

```
#include<stdio.h>
#define F(x,y) (x-y)/(x+y)
main ( )
{
int i,n;
float x0,y0,h,xn,x,y;
printf("\n Enter the values: x0,y0,h,xn: \n");
scanf ("%f%f%f%f",&x0,&y0,&h,&xn);
n=(xn-x0)/h+1;
for (i=1;i<=n;i++)
 {
 y=y0+h*F(x0,y0);
 x=x0+h;
 printf("\n X=%f  Y=%f",x0,y0);
   if(x<xn)
   {
   x0=x;
   y0=y;
   }
 }
return;
}
```

6.11.1 Output

```
Enter the values: x0,y0,h,xn:
0 1 0.02 0.1
X=0.000000 Y=1.000000
X=0.020000 Y=0.980000
X=0.040000 Y=0.960800
X=0.060000 Y=0.942399
X=0.080000 Y=0.924793
X=0.100000 Y=0.907978
```

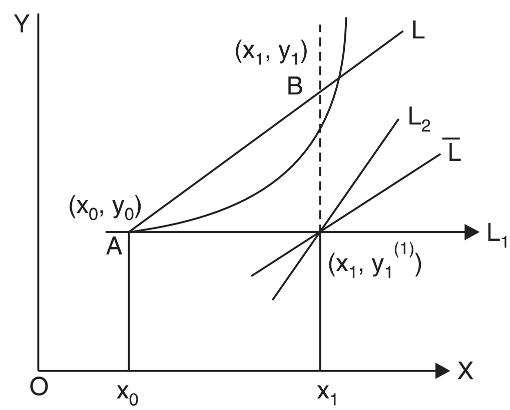
6.11.2 Notations used in the Program

(*i*) $\mathbf{x_0}$ is the initial value of x.
(*ii*) $\mathbf{y_0}$ is the initial value of y.
(*iii*) **h** is the spacing value of x.
(*iv*) $\mathbf{x_n}$ is the last value of x at which value of y is required.

6.12 MODIFIED EULER'S METHOD

The modified Euler's method gives greater improvement in accuracy over the original Euler's method. Here the core idea is that we use a line through (x_0, y_0) whose slope is the average of the slopes at (x_0, y_0) and $(x_1, y_1^{(1)})$ where $y_1^{(1)} = y_0 + hf(x_0, y_0)$. This line approximates the curve in the interval (x_0, x_1).

Geometrically, if L_1 is the tangent at (x_0, y_0), L_2 is a line through $(x_1, y_1^{(1)})$ of slope $f(x_1, y_1^{(1)})$ and $\overline{L}$ is the line through $(x_1, y_1^{(1)})$ but with a slope equal to the average of $f(x_0, y_0)$ and $f(x_1, y_1^{(1)})$ then the line L through (x_0, y_0) and parallel to $\overline{L}$ is used to approximate the curve in the interval (x_0, x_1). Thus the ordinate of the point B will give the value of y_1. Now, the eqn. of the line AL is given by



$$y_1 = y_0 + (x_1 - x_0)\left[\frac{f(x_0, y_0) + f(x_1, y_1^{(1)})}{2}\right]$$

$$= y_0 + h\left[\frac{f(x_0, y_0) + f(x_1, y_1^{(1)})}{2}\right]$$

A generalised form of Euler's modified formula is

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f\{x_1, y_1^{(n)}\}]\ ;\ n = 0, 1, 2, \ldots\ldots$$

where $y_1^{(n)}$ is the n^{th} approximation to y_1.

The above iteration formula can be started by choosing $y_1^{(1)}$ from Euler's formula

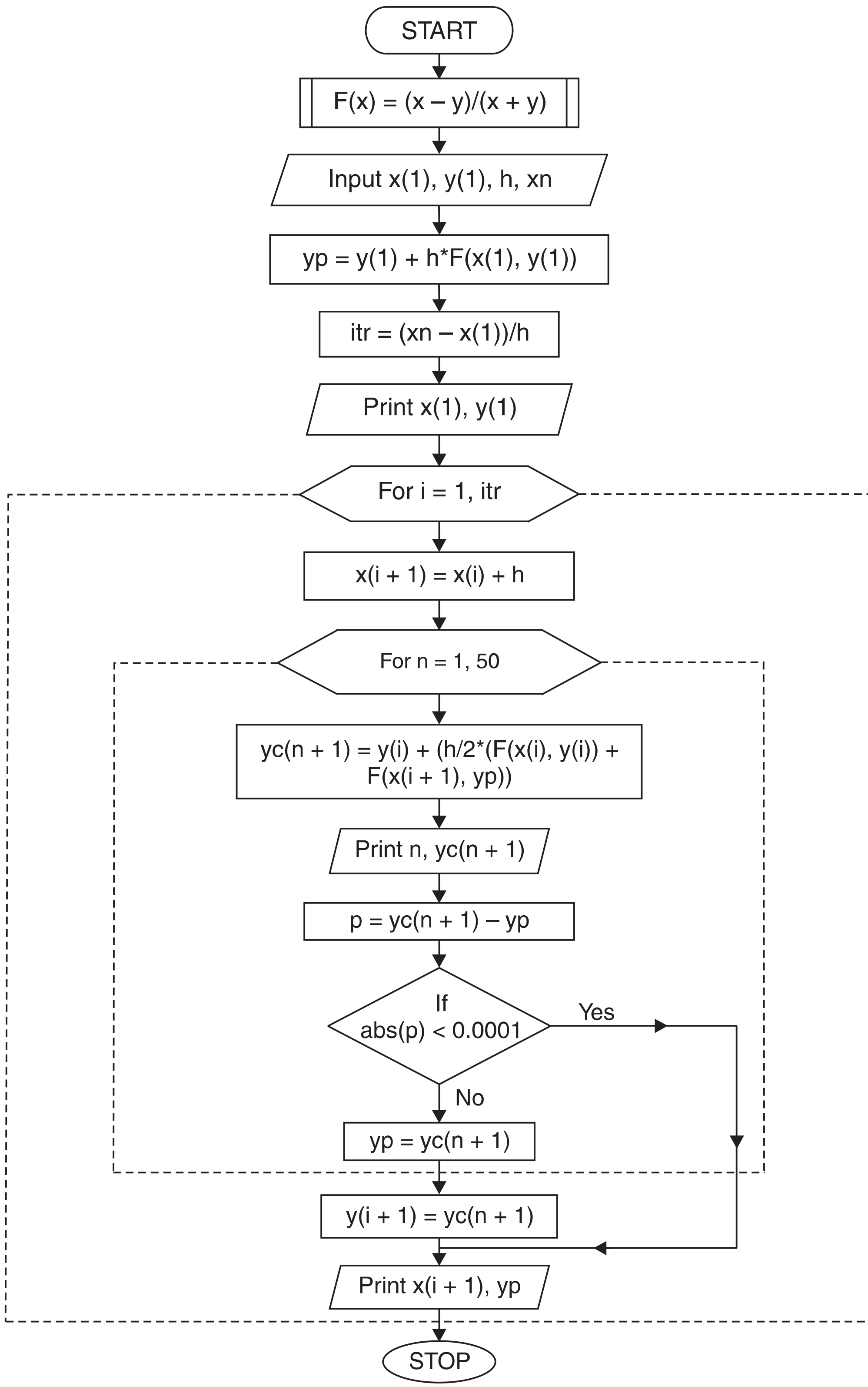
$$y_1^{(1)} = y_0 + hf(x_0, y_0)$$

Since this formula attempts to correct the values of y_{n+1} using the predicted value of y_{n+1} (by Euler's method), it is classified as a one-step predictor-corrector method.

6.13 ALGORITHM OF MODIFIED EULER'S METHOD

```
 1. Function F(x)=(x-y)/(x+y)
 2. Input x(1),y(1),h,xn
 3. yp=y(1)+h*F(x(1),y(1))
 4. itr=(xn-x(1))/h
 5. Print x(1),y(1)
 6. For i=1,itr
 7. x(i+1)=x(i)+h
 8. For n=1,50
 9. yc(n+1)=y(i)+(h/2*(F(x(i),y(i))+F(x(i+1),yp))
10. Print n,yc(n+1)
11. p=yc (n+1)-yp
12. If abs(p)<.0001 then
             goto Step 14
        ELSE
             yp=yc(n+1)
13. Next n
14. y(i+1)=yc(n+1)
15. print x(i+1),yp
16. Next i
17. Stop
```

6.14 FLOW-CHART OF MODIFIED EULER'S METHOD



6.15 PROGRAM OF MODIFIED EULER'S METHOD

```
#include<stdio.h>
#include<math.h>
#define F(x,y) (x-y)/(x+y)
main ()
{
int i,n,itr ;
float x[5],y[50],yc[50],h,yp,p,xn;
printf("\n Enter the values: x[1],y[1],h,xn:\n");
scanf("%f%f%f%f",&x[1],&y[1],&h,&xn);
yp=y[1]+h*F(x[1],y[1]);
itr=(xn-x[1])/h;
printf("\n\n X=%f Y=%f\n",x[1],y[1];
for (i=1;i<=itr;i++)
 {
  x[i+1]=x[i]+h;
  for (n=1;n<=50;n++)
      {
yc[n+1]=y[i]+(h/2.0)*(F(x[i],y[i])+F(x[i+1],yp));
printf("\nN=%d Y=%f",n,yc[n+1]);
p=yc[n+1]-yp;
if(fabs (p)<0.0001)
      goto next;
 else
 yp=yc[n+1];
    }
   next:
y[i+1]=yc[n+1];
printf("\n\n X=%f Y=%f\n",x[i+1], yp);
  }
return;
}
```

6.15.1 Output

```
Enter the values: x[1],y[1],h,xn:
0 1 0.02 0.06
     X=0.000000    Y=1.000000
N=1  Y=0.980400
N=2  Y=0.980400
     X=0.020000    Y=0.980400
N=1  Y=0.961584
N=2  Y=0.961598
     X=0.040000    Y=0.961584
N=1  Y=0.943572
N=2  Y=0.943593
     X=0.060000    Y=0.943572
```

6.15.2 Notations used in the Program

(*i*) **x(1)** is an array of the initial value of x.

(*ii*) **y(1)** is an array of the initial value of y.

(*iii*) **h** is the spacing value of x.

(*iv*) $\mathbf{x_n}$ is the last value of x at which value of y is required.

EXAMPLES

Example 1. *Given* $\frac{dy}{dx} = \frac{y-x}{y+x}$ *with* $y = 1$ *for* $x = 0$*. Find* y *approximately for* $x = 0.1$ *by Euler's method.*

Sol. We have

$$\frac{dy}{dx} = f(x, y) = \frac{y-x}{y+x} \; ; x_0 = 0, y_0 = 1, h = 0.1$$

Hence the approximate value of y at $x = 0.1$ is given by

$$y_1 = y_0 + hf(x_0, y_0) \qquad | \text{ using } y_{n+1} = y_n + hf(x_n, y_n)$$

$$= 1 + (.1) + \left(\frac{1-0}{1+0}\right) = 1.1$$

Much better accuracy is obtained by breaking up the interval 0 to 0.1 into five steps. The approximate value of y at $x_A = .02$ is given by,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + (.02)\left(\frac{1-0}{1+0}\right) = 1.02$$

At $x_B = 0.04$, $\quad y_2 = y_1 + hf(x_1, y_1)$

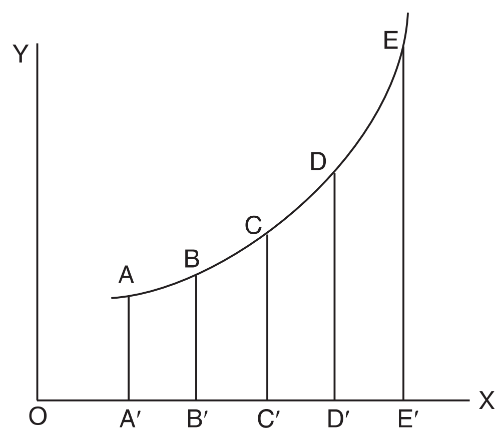
$$= 1.02 + (.02)\left(\frac{1.02 - .02}{1.02 + .02}\right) = 1.0392$$

At $x_C = .06$, $\quad y_3 = 1.0392 + (.02)\left(\dfrac{1.0392 - .04}{1.0392 + .04}\right) = 1.0577$

At $x_D = .08$, $\quad y_4 = 1.0577 + (.02)\left(\dfrac{1.0577 - .06}{1.0577 + .06}\right) = 1.0756$

At $x_E = .1$, $\quad y_5 = 1.0756 + (.02)\left(\dfrac{1.0756 - .08}{1.0756 + .08}\right) = 1.0928$

Hence $y = 1.0928$ when $x = 0.1$



Example 2. *Solve the equation* $\dfrac{dy}{dx} = 1 - y$ *with the initial condition* $x = 0, y = 0$ *using Euler's algorithm and tabulate the solutions at* $x = 0.1, 0.2, 0.3$.

Sol. Here, $\quad f(x, y) = 1 - y$

Taking $h = 0.1, x_0 = 0, y_0 = 0$, we obtain

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 0 + (.1)(1 - 0) = .1$$

$$\therefore \quad y(0.1) = 0.1$$

Again, $y_2 = y_1 + hf(x_1, y_1)$

$= 0.1 + (0.1)(1 - .1)$

$= 0.1 + .09 = .19$

$\therefore$ $y(0.2) = 0.19$

Again, $y_3 = y_2 + hf(x_2, y_2)$

$= .19 + (.1)(1 - .19)$

$= .19 + (.1)(.81) = .271$

$\therefore$ $y(0.3) = .271$

Tabulated values are

x	$y(x)$
0	0
0.1	0.1
0.2	0.19
0.3	0.271

Example 3. *Using Euler's modified method, obtain a solution of the equation*

$$\frac{dy}{dx} = x + |\sqrt{y}| = f(x, y)$$

with initial condition $y = 1$ at $x = 0$ for the range $0 \le x \le 0.6$ in steps of 0.2.

Sol. Here $f(x, y) = x + |\sqrt{y}|$; $x_0 = 0$, $y_0 = 1$, $h = .2$

$\therefore$ $f(x_0, y_0) = x_0 + |\sqrt{y_0}| = 0 + 1 = 1$

We have $y_1^{(1)} = y_0 + hf(x_0, y_0)$

$= 1 + (.2) . 1 = 1.2$

$\therefore$ $f(x_1, y_1^{(1)}) = x_1 + |\sqrt{y_1^{(1)}}|$

$= 0.2 + |\sqrt{1.2}| = 1.2954$

The second approximation to y_1 is

$$y_1^{(2)} = y_0 + h\left[\frac{f(x_0, y_0) + f\{x_1, y_1^{(1)}\}}{2}\right]$$

$$= 1 + (0.2)\left(\frac{1 + 1.2954}{2}\right) = 1.2295$$

Again, $f\{x_1, y_1^{(2)}\} = x_1 + |\sqrt{y_1}^{(2)}| = 0.2 + \sqrt{1.2295} = 1.3088$

So, $y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\{x_1, y_1^{(2)}\}]$

$= 1 + \frac{0.2}{2}[1 + 1.3088] = 1.2309$

We have $f\{x_1, y_1^{(3)}\} = 0.2 + \sqrt{1.2309} = 1.309$

Then $y_1^{(4)} = 1 + \frac{.2}{2}[1 + 1.309] = 1.2309$

Since, $y_1^{(4)} = y_1^{(3)}$ hence $y_1 = 1.2309$

Now, $y_2^{(1)} = y_1 + hf(x_1, y_1)$

$= 1.2309 + (0.2)[0.2 + \sqrt{1.2309}]$

$= 1.4927$ $\qquad | \because \quad x_1 = 0.2$

$f\{x_2, y_2^{(1)}\} = x_2 + \sqrt{y_2^{(1)}} = 0.4 + \sqrt{1.4927}$

$= 1.622$ $\qquad | \because \quad x_2 = 0.4$

Then, $y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f\{x_2, y_2^{(1)}\}]$

$= 1.2309 + \frac{0.2}{2}[(.2 + \sqrt{1.2309}) + 1.622] = 1.524$

Now, $y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f\{x_2, y_2^{(2)}\}]$

$= 1.2309 + \frac{0.2}{2}[(.2 + \sqrt{1.2309}) + (.4 + \sqrt{1.524})]$

$= 1.5253$

$y_2^{(4)} = 1.2309 + \frac{0.2}{2}[(.2 + \sqrt{1.2309}) + (.4 + \sqrt{1.5253})]$

Since, $y_2^{(4)} = y_2^{(3)}$ hence $y_2 = 1.5253$

Now, $y_3^{(1)} = y_2 + hf(x_2, y_2)$

$= 1.5253 + (0.2)[.4 + \sqrt{1.5253}] = 1.8523$

$y_3^{(2)} = y_2 + \frac{h}{2}[f(x_2, y_2) + f\{x_3, y_3^{(1)}\}]$

$= 1.5253 + \frac{0.2}{2}[(.4 + \sqrt{1.5253}) + (.6 + \sqrt{1.8523})]$

$= 1.8849$

Similarly, $y_3^{(3)} = 1.8861 = y_3^{(4)}$

Since, $y_3^{(3)} = y_3^{(4)}$

Hence, we take $y_3 = 1.8861$.

Example 4. *Given that* $\frac{dy}{dx} = \log_{10}(x + y)$ *with the initial condition that* $y = 1$ *when* $x = 0$. *Find* y *for* $x = 0.2$ *and* $x = 0.5$ *using Euler's modified formula.*

Sol. Let $x = 0$, $x_1 = 0.2$, $x_2 = .5$ then $y_0 = 1$

y_1 and y_2 are yet to be computed.

Here, $f(x, y) = \log (x + y)$

$\therefore$ $f(x_0, y_0) = \log 1 = 0$

$\therefore$ $y_1^{(1)} = y_0 + hf(x_0, y_0) = 1$

$$f\{x_1, y_1^{(1)}\} = \log \{x_1 + y_1^{(1)}\} = \log (.2 + 1) = \log (1.2)$$

$\therefore$
$$y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f\{x_1, y_1^{(1)}\}]$$

$$= 1 + \frac{.2}{2} [0 + \log (1.2)] = 1.0079$$

Also,
$$y_1^{(3)} = 1 + \frac{.2}{2} [0 + \log (.2 + 1.0079)] = 1.0082$$

$$y_1^{(4)} = 1 + \frac{.2}{2} [0 + \log (.2 + 1.0082)] = 1.0082$$

Since, $y_1^{(4)} = y_1^{(3)}$ hence $y_1 = 1.0082$

To obtain y_2, the value of y at $x = 0.5$, we take,

$$y_2^{(1)} = y_1 + hf(x_1, y_1)$$

$$= 1.0082 + 0.3 \log (.2 + 1.0082)$$

$$= 1.0328 \qquad (\because \quad h = .5 - .2 = .3 \text{ here})$$

Now,
$$y_2^{(2)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f\{x_2, y_2^{(1)}\}]$$

$$= 1.0082 + \frac{.3}{2} [\log (.2 + 1.0082) + \log (.5 + 1.0328)]$$

$$= 1.0082 + 0.0401 = 1.0483$$

Also,
$$y_2^{(3)} = 1.0082 + \frac{.3}{2} [\log (.2 + 1.0082) + \log (.5 + 1.0483)]$$

$$= 1.0082 + .0408 = 1.0490$$

Similarly, $y_2^{(4)} = 1.0490$

Since, $y_2^{(3)} = y_2^{(4)}$ hence, $y_2 = 1.0490$.

Example 5. *Given :* $\frac{dy}{dx} = x - y^2$ *; y(.2) = 0.2, find y(.4) by modified Euler's method correct to 3 decimal places, taking h = 0.2.*

Sol. Here, $f(x, y) = x - y^2$; $x_0 = 0.2$, $y_0 = .02$ and $h = 0.2$

Let $x_1 = 0.4$ then we are to find $y_1 = y(0.4)$

We have $f(x_0, y_0) = x_0 - y_0^2 = 0.2 - (.02)^2 = 0.2 - .0004 = 0.1996$

$$\therefore \quad y_1^{(1)} = y_0 + hf(x_0, y_0) = .02 + (.2)(.1996) = .060$$

$$f\{x_1, y_1^{(1)}\} = x_1 - \{y_1^{(1)}\}^2 = .4 - (.06)^2 = .3964$$

$$\therefore \quad y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\{x_1, y_1^{(1)}\}]$$

$$= .02 + \frac{.2}{2}[.1996 + .3964] = .0796 \simeq .080$$

Now, $f\{x_1, y_1^{(2)}\} = x_1 - [y_1^{(2)}]^2 = .4 - (.08)^2 = .3936$

$$\therefore \quad y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\{x_1, y_1^{(2)}\}]$$

$$= .02 + \frac{.2}{2}[.1996 + .3936] = .07932 \simeq .079$$

$$f\{x_1, y_1^{(3)}\} = x_1 - [y_1^{(3)}]^2 = .4 - (.079)^2 = .3938$$

$$\therefore \quad y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\{x_1, y_1^{(3)}\}]$$

$$= .02 + \frac{.2}{2}[.1996 + .3938] = .0793 \simeq .079$$

Since $y_1^{(3)} = y_1^{(4)}$ hence $y_1 = .079$.

ASSIGNMENT 6.2

1. Find y for $x = 0.2$ and $x = 0.5$ using modified Euler's method, given that

$$\frac{dy}{dx} = \log_e(x + y); \quad y(0) = 1$$

2. Taking $h = .05$, determine the value of y at $x = 0.1$ by Euler's modified method, given that,

$$\frac{dy}{dx} = x^2 + y; \quad y(0) = 1$$

3. Given $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$, find $y(.02)$, $y(.04)$ and $y(.06)$ using Euler's modified method.

4. Apply Euler's method to the initial value problem $\frac{dy}{dx} = x + y$, $y(0) = 0$ at $x = 0$ to $x = 1.0$ taking $h = 0.2$.

5. Use Euler's method with $h = 0.1$ to solve the differential equation $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$ in the range $x = 0$ to $x = 0.3$.

6. Solve for y at $x = 1.05$ by Euler's method, the differential equation $\frac{dy}{dx} = 2 - \left(\frac{y}{x}\right)$ where $y = 2$ when $x = 1$. (Take $h = 0.05$).

7. Use Euler's modified method to compute y for $x = .05$ and $.10$. Given that $\frac{dy}{dx} = x + y$ with the initial condition $x_0 = 0$, $y_0 = 1$. Give the correct result up to 4 decimal places.

8. Using Euler's method, compute $y(0.04)$ for the differential eqn. $\frac{dy}{dx} = -y$; $y(0) = 1$. Take $h = 0.01$.

9. Compute $y(0.5)$ for the differential eqn. $\frac{dy}{dx} = y^2 - x^2$ with $y(0) = 1$ using Euler's method.

10. Find $y(2.2)$ using modified Euler's method for $\frac{dy}{dx} = -xy^2$; $y(2) = 1$. Take $h = .1$.

11. Given $\frac{dy}{dx} = x^3 + y, y(0) = 1$. Compute y (0.02) by Euler's method taking $h = 0.01$.

12. Find $y(1)$ by Euler's method from the differential equation $\frac{dy}{dx} = \frac{-y}{1+x}$ when $y(0.3) = 2$. Convert up to four decimal places taking step length $h = 0.1$.

6.16 TAYLOR'S METHOD

Consider the differential equation

$$\left.\begin{aligned} \frac{dy}{dx} &= f(x, y) \\ \text{with the initial condition} \quad y(x_0) &= y_0. \end{aligned}\right\} \tag{9}$$

If $y(x)$ is the exact solution of (9) then $y(x)$ can be expanded into a Taylor's series about the point $x = x_0$ as

$$y(x) = y_0 + (x - x_0)\, y_0' + \frac{(x - x_0)^2}{2!}\, y_0'' + \frac{(x - x_0)^3}{3!}\, y_0''' + \ldots\ldots \tag{10}$$

where dashes denote differentiation with respect to x.

Differentiating (9) successively with respect to x, we get

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y} = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)f \tag{11}$$

$$\therefore \qquad y''' = \frac{d}{dx}(y'') = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\frac{\partial^2 f}{\partial x \partial y} + f\frac{\partial^2 f}{\partial y \partial x} + f\left(\frac{\partial f}{\partial y}\right)^2 + f^2\frac{\partial^2 f}{\partial y^2} \tag{12}$$

and so on.

Putting $x = x_0$ and $y = y_0$ in the expressions for y', y'', y''', ... and substituting them in eqn. (10), we get a power series for $y(x)$ in powers of $x - x_0$.

i.e.,

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \frac{(x - x_0)^3}{3!}y_0''' + \ldots\ldots \tag{13}$$

Putting $x = x_1 (= x_0 + h)$ in (13), we get

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots\ldots \tag{14}$$

Here y_0', y_0'', y_0''', ... can be found by using (9) and its successive differentiations (11) and (12) at $x = x_0$. The series (14) can be truncated at any stage if h is small.

After obtaining y_1, we can calculate y_1', y_1'', y_1''', ... from (9) at $x_1 = x_0 + h$.

Now, expanding $y(x)$ by Taylor's series about $x = x_1$, we get

$$y_2 = y_1 + hy'_1 + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots\ldots$$

Proceeding, we get

$$y_n = y_{n-1} + hy_{n-1}' + \frac{h^2}{2!}y_{n-1}'' + \frac{h^3}{3!}y_{n-1}''' + \ldots\ldots$$

Practically, this method is not of much importance because of its need of partial derivatives.

Moreover if we are interested in a better approximation with a small truncation error, the evaluation of higher order derivatives is needed which are complicated in evaluation. Besides its impracticability, it is useful in judging the degree of accuracy of the approximations given by other methods.

We can determine the extent to which any other formula agrees with the Taylor's series expansion. Taylor's method is one of those methods which yield the solution of a differential equation in the form of a power series. This method suffers from a serious disadvantage that h should be small enough so that successive terms in the series diminish quite rapidly.

6.17 TAYLOR'S METHOD FOR SIMULTANEOUS I ORDER DIFFERENTIAL EQUATIONS

Simultaneous differential equations of the type

$$\frac{dy}{dx} = f(x, y, z) \tag{15}$$

and

$$\frac{dz}{dx} = \phi(x, y, z) \tag{16}$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$

can be solved by Taylor's method.

If h is the step-size then

$$y_1 = y(x_0 + h) \quad \text{and} \quad z_1 = z(x_0 + h)$$

Taylor's algorithm for (15) and (16) gives

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots\ldots \tag{17}$$

and

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \ldots\ldots \tag{18}$$

Differentiating (15) and (16) successively, we get y'', y''', ..., z'', z''', ... etc. So the values y_0'', y_0''', ... and z_0'', z_0''', ... can be obtained.

Substituting them in (17) and (18), we get y_1, z_1 for the next step.

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots\ldots$$

and

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!} z_1'' + \frac{h^3}{3!} z_1''' + \ldots\ldots.$$

Since y_1 and z_1 are known, y_1', y_1'', y_1'''......., z_1', z_1'', z_1''', ... can be calculated. Hence y_2 and z_2 can be obtained. Proceeding in this manner, we get other values of y, step-by-step.

EXAMPLES

Example 1. *Use Taylor's series method to solve*

$$\frac{dy}{dx} = x + y; \; y(1) = 0$$

numerically up to x = 1.2 with h = 0.1. Compare the final result with the value of explicit solution.

Sol. Here, $x_0 = 1, y_0 = 0$

$$y' = x + y \qquad i.e., \qquad y_0' = x_0 + y_0 = 1$$

$$\Rightarrow \qquad y'' = 1 + y' \qquad i.e., \qquad y_0'' = 1 + y_0' = 2$$

$$\Rightarrow \qquad y''' = y'' \qquad i.e., \qquad y_0''' = y_0'' = 2$$

$$\Rightarrow \qquad y^{(iv)} = y''' \qquad i.e., \qquad y_0^{(iv)} = 2$$

$$\Rightarrow \qquad y^{(v)} = y^{(iv)} \qquad i.e., \qquad y_0^{(v)} = 2$$

By Taylor's series, we have

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{(iv)} + \ldots\ldots$$

$$y(1 + h) = 0 + (0.1)\, 1 + \frac{(0.1)^2}{2!}\, 2 + \frac{(0.1)^3}{3!}\, 2 + \frac{(0.1)^4}{4!}\, 2 + \ldots\ldots$$

$$\Rightarrow \qquad y(1.1) = 0.1103081 = 0.110 \text{ (app.)}$$

Also, $x_1 = x_0 + h = 1.1$

Again, $y_1' = x_1 + y_1 = 1.1 + 0.11 = 1.21$

$$y_1'' = 1 + y_1' = 1 + 1.21 = 2.21$$

$$y_1''' = y_1'' = 2.21$$

$$y_1^{(iv)} = 2.21$$

$$y_1^{(v)} = 2.21$$

Now, $$y(1.1 + h) = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots\ldots$$

$$= 0.11 + (0.1)\,(1.21) + \frac{(0.1)^2}{2}\,(2.21) + \ldots\ldots$$

$$\Rightarrow \qquad y(1.2) = 0.232 \text{ (app.)}$$

The analytical solution of the given differential equation is

$$y = -x - 1 + 2e^{x-1}$$

when $x = 1.2$, we get

$$y = -1.2 - 1 + 2e^{0.2} = 0.242.$$

Example 2. *For the differential eqn.,* $\dfrac{dy}{dx} = -xy^2$, *y (0) = 2. Calculate y(0. 2) by Taylor's series method retaining four non-zero terms only.*

Sol. Here $\quad x_0 = 0, y_0 = 2$ Also $y' = -xy^2$

Taylor's series for $y(x)$ is given by

$$y(x) = y_0 + xy_0' + \frac{x^2}{2}y_0'' + \frac{x^3}{6}y_0''' + \frac{x^4}{24}y_0^{(iv)} + \frac{x^5}{120}y_0^{(v)} + \ldots\ldots. \quad (19)$$

The values of the derivatives y_0', y_0'', ..., etc. are obtained as follows:

$$y' = -xy^2 \qquad y_0' = -x_0y_0^2 = 0$$

$$y'' = -y^2 - 2xyy' \qquad y_0'' = -2^2 - 0 = -4$$

$$y''' = -4yy' - 2xy'^2 - 2xyy'' \qquad y_0''' = 0$$

$$y^{(iv)} = -6y'^2 - 6y'y'' - 6xy'y'' - 2xyy''' \qquad y_0^{(iv)} = 48$$

$$y^{(v)} = -24y'y'' - 8yy''' - 6xy''^2 - 8xy'y''' - 2xyy^{(iv)} \qquad y_0^{(v)} = 0$$

$$y^{(vi)} = -40y'y''' - 30y''^2 - 10\,yy^{(iv)} - 20xy''y''' - 10xy'\,y^{(iv)} - 2xyy^{(v)}. \qquad y_0^{(vi)} = -1440$$

We stop here as we shall get four non-zero terms in the Taylor's series (19).

$$\therefore \quad y(x) = 2 + \frac{x^2}{2}(-4) + \frac{x^4}{24}(48) + \frac{x^6}{720}(-1440) + \ldots\ldots$$

$$= 2 - 2x^2 + 2x^4 - 2x^6 + \ldots\ldots.$$

$$\therefore \quad y(0.2) = 2 - 2(0.2)^2 + 2(0.2)^4 - 2\,(0.2)^6 + \ldots\ldots$$

$$= 2 - 0.08 + 0.0032 - 0.000128 = 1.923072$$

$$\simeq 1.9231 \quad \text{correct up to four decimal places.}$$

Example 3. *From the Taylor's series, for y(x), find y(0.1) correct to four decimal places if y(x) satisfies* $\dfrac{dy}{dx} = x - y^2$ *and y(0) = 1. Also find y(0.2).*

Sol. Here $x_0 = 0, \quad y_0 = 1$

$$\begin{aligned}
y' &= x - y^2 & y_0' &= 0 - 1 = -1\\
y'' &= 1 - 2yy' & y_0'' &= 3\\
y''' &= -2yy'' - 2y'^2 & y_0''' &= -8\\
y^{(iv)} &= -2yy''' - 6y'y'' & y_0^{(iv)} &= 34\\
y^{(v)} &= -2yy^{(iv)} - 8y'y''' - 6y''^2 & y_0^{(v)} &= -186\\
y^{(vi)} &= -2yy^{(v)} - 10y'y^{(iv)} - 20\, y''y''' & y_0^{(vi)} &= 1192\\
y^{(vii)} &= -2yy^{(vi)} - 12y'y^{(v)} - 50\, y''y^{(iv)} & y_0^{(vii)} &= -10996 \quad \text{only for } y(0.2)\\
&\quad - 20\, y'''^2
\end{aligned}$$

Using these values, Taylor's series becomes

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 - \frac{31}{20}x^5 + \ldots\ldots \tag{20}$$

Put $\quad x = 0.1$ in (20), we get

$$y(0.1) = 0.91379 \simeq 0.9138 \qquad \text{(upto four decimal places)}$$

To determine $y(0.2)$, we have

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 - \frac{31}{20}x^5 + \frac{1192}{720}x^6 - \frac{10996}{5040}x^7 + \ldots\ldots..$$

$$= 0.8512 \quad \text{(correct to four decimal places).}$$

Example 4. *Using Taylor's series, find the solution of the differential equation $xy' = x - y$, $y(2) = 2$ at $x = 2.1$ correct to five decimal places.*

Sol. Here $x_0 = 2, y_0 = 2$

Also,
$$\begin{aligned}
y' &= 1 - \frac{y}{x} & y_0' &= 0\\
y'' &= -\frac{y'}{x} + \frac{y}{x^2} & y_0'' &= -0 + \frac{2}{4} = \frac{1}{2}\\
y''' &= -\frac{y''}{x} + \frac{2y'}{x^2} - \frac{2y}{x^3} & y_0''' &= \frac{-3}{4}\\
y^{(iv)} &= -\frac{y'''}{x} + \frac{3y''}{x^2} - \frac{6y'}{x^3} + \frac{6y}{x^4} & y_0^{(iv)} &= \frac{3}{2} \text{ and so on.}
\end{aligned}$$

Putting these values in Taylor's series, we get

$$y(2 + h) = 2 + \frac{h^2}{4} - \frac{h^3}{8} + \frac{h^4}{16} + \ldots\ldots.$$

Put $\quad h = 0.1$, we get

$$y\,(2.1) = 2.00238 \text{ (correct to 5 decimal places).}$$

Example 5. *Find y(1) for* $\dfrac{dy}{dx} = 2y + 3e^x$ *, y(0) = 0. Also check the value.*

Sol. Here $\quad x_0 = 0, y_0 = 0$

$$\begin{aligned} y'(x) &= 2y + 3e^x & y_0' &= 3, & y_0'' &= 9, \\ y''(x) &= 2y' + 3e^x & y_0''' &= 21, & y_0^{(iv)} &= 45 \\ &\vdots \quad \vdots & & & & \\ &\vdots \quad \vdots & y_0^{(v)} &= 93, & y_0^{(vi)} &= 189 \\ y^{(viii)}(x) &= 2y^{(vii)} + 3e^x & y_0^{(vii)} &= 381, & y_0^{(viii)} &= 765 \end{aligned}$$

Now, $$y(h) = 3h + \frac{9}{2}h^2 + \frac{7}{2}h^3 + \frac{15}{8}h^4 + \frac{31}{40}h^5 + \frac{21}{80}h^6 + \frac{127}{1680}h^7 + \frac{17}{896}h^8 + \ldots\ldots.$$

Put $\quad h = 1, \; y(1) = 14.01$

Exact solution.

$$\frac{dy}{dx} - 2y = 3e^x$$

Solution is $\quad ye^{-2x} = -\,3e^{-x} + c$

$$x = 0, y = 0 \quad \therefore \quad c = 3$$

$\therefore \quad ye^{-2x} = -\,3e^{-x} + 3$

$\Rightarrow \quad y = 3(e^{2x} - e^x)$

when $x = 1$,

$$y = 3(e^2 - e) = 14.01 \text{ correct to two decimal places.}$$

Example 6. *Solve the simultaneous equations*

$$y' = 1 + xyz, \quad y\,(0) = 0$$

$$z' = x + y + z, \quad z(0) = 1.$$

Sol. Differentiating the given equations

$$y'' = yz + xy'z + xyz', \qquad y''' = 2y'z + 2yz' + 2xy'z' + xy''z + xyz''$$

$$z'' = 1 + y' + z', \qquad z''' = y'' + z''$$

with $x = 0, y = 0, z = 1$; we get $y' = 1, y'' = 0, y''' = 2$

Also $\qquad z' = 1, z'' = 3, z''' = 3$

Hence, $\qquad y(x) = x + \dfrac{x^3}{3}$ and $z(x) = 1 + x + \dfrac{3}{2}x^2 + \dfrac{1}{2}x^3.$

ASSIGNMENT 6.3

1. Compute y for $x = 0.1$ and 0.2 correct to four decimal places given: $y' = y - x, y(0) = 2$.

2. Solve by Taylor's method, $y' = x^2 + y^2, y(0) = 1$ compute $y(0.1)$.

3. Solve by Taylor's method: $y' = y - \dfrac{2x}{y}$; $y(0) = 1$. Also compute $y(0.1)$.

4. Using Taylor series method, solve $\dfrac{dy}{dx} = x^2 - y, y(0) = 1$ at $x = 0.1, 0.2, 0.3$ and 0.4. Compare the values with exact solution.

5. Solve $\dfrac{dy}{dx} = x + z, \quad \dfrac{dz}{dx} = x - y^2$ with $y(0) = 2, z(0) = 1$ to get $y(0.1), y(0.2), z(0.1)$ and $z(0.2)$ approximately by Taylor's algorithm.

6. Given the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x^2 + y}$ with $y(4) = 4$

Obtain $y(4.1)$ and $y(4.2)$ by Taylor's series method.

6.18 RUNGE-KUTTA METHODS

More efficient methods in terms of accuracy were developed by two German Mathematicians **Carl Runge** (1856-1927) and **Wilhelm Kutta** (1867-1944). These methods are well-known as Runge-Kutta methods. They are distinguished by their orders in the sense that they agree with Taylor's series solution up to terms of h^r where r is the order of the method.

These methods do not demand prior computation of higher derivatives of $y(x)$ as in Taylor's method. In place of these derivatives, extra values of the given function $f(x, y)$ are used.

The fourth order Runge-Kutta method is used widely for finding the numerical solutions of linear or non-linear ordinary differential equations.

Runge-Kutta methods are referred to as single step methods. The major disadvantage of Runge-Kutta methods is that they use many more evaluations of the derivative $f(x, y)$ to obtain the same accuracy compared with multi-step methods. A class of methods known as Runge-Kutta methods combines the advantage of high order accuracy with the property of being one step.

6.18.1 First Order Runge-Kutta Method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y);\ \ y\,(x_0) = y_0 \tag{21}$$

Euler's method gives

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0' \tag{22}$$

Expanding by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!}\,y_0'' + \ldots\ldots. \tag{23}$$

Comparing (22) and (23), it follows that Euler's method agrees with Taylor's series solution up to the term in h. Hence *Euler's method is the first order Runge-Kutta method*.

6.18.2 Second Order Runge-Kutta Method

Consider the differential equation

$$y' = f(x, y) \text{ with the initial condition } y(x_0) = y_0$$

Let h be the interval between equidistant values of x then in II order Runge-Kutta method, the first increment in y is computed from the formulae

$$k_1 = hf\,(x_0, y_0)$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

$$\Delta y = \tfrac{1}{2}(k_1 + k_2)$$

taken in the given order.

Then, $\quad x_1 = x_0 + h$

$$y_1 = y_0 + \Delta y = y_0 + \tfrac{1}{2}\,(k_1 + k_2)$$

In a similar manner, the increment in y for the second interval is computed by means of the formulae,

$$k_1 = hf\,(x_1, y_1)$$

$$k_2 = hf\,(x_1 + h, y_1 + k_1)$$

$$\Delta y = \tfrac{1}{2}(k_1 + k_2)$$

and similarly for the next intervals.

The inherent error in the second order Runge-Kutta method is of order h^3.

6.18.3 Third Order Runge-Kutta Method

This method gives the approximate solution of the initial value problem

$$\frac{dy}{dx} = f(x, y);\ y(x_0) = y_0 \text{ as}$$

$$\left.\begin{aligned} y_1 &= y_0 + \delta y \\ \text{where} \quad \delta y &= \frac{h}{6}(k_1 + 4k_2 + k_3) \end{aligned}\right\} \tag{24}$$

Here,
$$k_1 = f(x_0, y_0)$$

$$k_2 = f\left\{x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right\}$$

$$k_3 = f(x_0 + h, y_0 + k'); \qquad k' = hf(x_0 + h, y_0 + k_1)$$

Formula (24) can be generalized for successive approximations. Expression in (24) agrees with Taylor's series expansion for y_1 up to and including terms in h^3. This method is also known as Runge's method.

6.19 FOURTH ORDER RUNGE-KUTTA METHOD

The fourth order Runge-Kutta Method is one of the most widely used methods and is particularly suitable in cases when the computation of higher derivatives is complicated.

Consider the differential equation $y' = f(x, y)$ with the initial condition $y(x_0) = y_0$. Let h be the interval between equidistant values of x, then the first increment in y is computed from the formulae

$$\left.\begin{aligned} k_1 &= hf(x_0, y_0) \\ k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\ k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\ k_4 &= hf(x_0 + h, y_0 + k_3) \\ \Delta y &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}\right\} \tag{25}$$

taken in the given order.

Then,
$$x_1 = x_0 + h \quad \text{and} \quad y_1 = y_0 + \Delta y$$

In a similar manner, the increment in y for the II interval is computed by means of the formulae

$$k_1 = hf\,(x_1, y_1)$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = hf\,(x_1 + h, y_1 + k_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and similarly for the next intervals.

This method is also simply termed as *Runge-Kutta's* method.

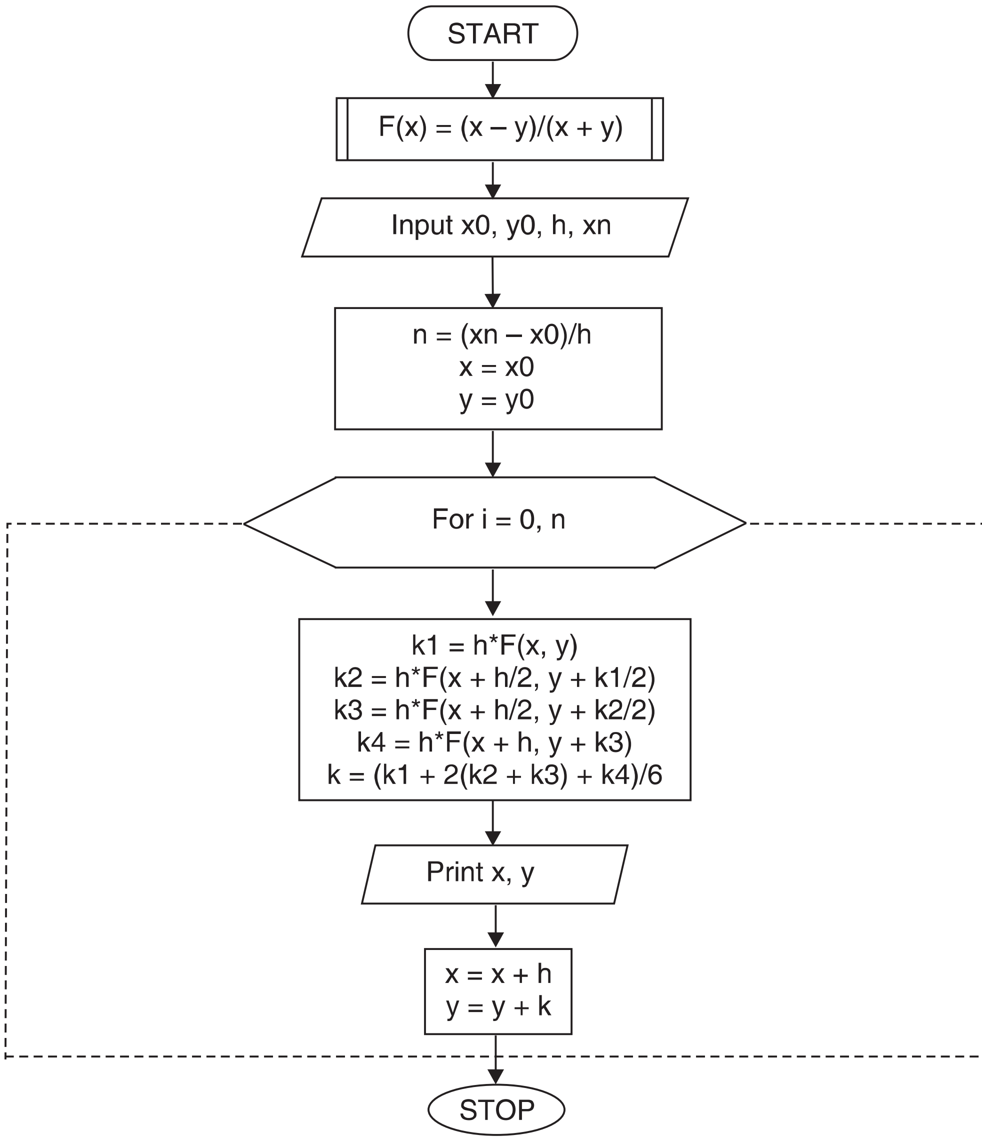
It is to be noted that the calculations for the first increment are exactly the same as for any other increment. The change in the formula for the different intervals is only in the values of x and y to be substituted. Hence, to obtain Δy for the n^{th} interval, we substitute $x_{n-1}, y_{n-1,}$ in the expressions for k_1, k_2, etc.

The inherent error in the fourth order Runge-Kutta method is of the order h^5.

6.19.1 Algorithm of Runge-Kutta Method

```
 1. Function  F(x)=(x-y)/(x+y)
 2. Input  x0,y0,h,xn
 3. n=(xn-x0)/h
 4. x=x0
 5. y=y0
 6. For i=0, n
 7. k1=h*F(x,y)
 8. k2=h*F(x+h/2,y+k1/2)
 9. k3=h*F(x+h/2,y+k2/2)
10. k4=h*F(x+h,y+k3)
11. k=(k1+(k2+k3)2+k4)/6
12. Print x,y
13. x=x+h
14. y=y+k
15. Next i
16. Stop
```

6.19.2 Flow-Chart of Runge-Kutta Method



6.19.3 Program of Runge-Kutta Method

```
#include<stdio.h>
#define F(x,y)  (x-y)/(x+y)
main()
{
 int i,n;
 float  x0,y0,h,xn,k1,k2,k3,k4,x,y,k;
 printf("\n Enter the values: x0,y0,h,xn:\n");
```

```
scanf("%f%f%f%f", &x0,&y0,&h,&xn);
n=(xn-x0)/h;
x=x0;
y=y0;
for(i=0;i<=n;i++)
     {
     k1=h*F(x,y);
     k2=h*F(x+h/2.0,y+k1/2.0);
     k3=h*F(x+h/2.0,y+k2/2.0);
     k4=h*F(x+h,y+k3);
     k=(k1+(k2+k3)*2.0+k4)/6.0;
     printf("\n X=%f Y=%f", x, y);
     x=x+h;
     y=y+k;
}
return;
}
```

6.19.4 Output

```
Enter the values: x0,y0,h,xn:
0 1 0.02 0.1
X=0.000000    Y=1.000000
X=0.020000    Y=0.980000
X=0.040000    Y=0.960816
X=0.060000    Y=0.942446
X=0.080000    Y=0.924885
X=0.100000    Y=0.908128
```

Notations used in the Program

(*i*) $\mathbf{x_0}$ is the initial value of x.
(*ii*) $\mathbf{y_0}$ is the initial value of y.
(*iii*) **h** is the spacing value of x.
(*iv*) $\mathbf{x_n}$ is the last value of x at which value of y is required.

6.20 RUNGE-KUTTA METHOD FOR SIMULTANEOUS FIRST ORDER EQUATIONS

Consider the simultaneous equations

$$\frac{dy}{dx} = f_1(x, y, z)$$

$$\frac{dz}{dx} = f_2(x, y, z)$$

With the initial condition $y(x_0) = y_0$ and $z(x_0) = z_0$. Now, starting from (x_0, y_0, z_0), the increments k and l in y and z are given by the following formulae:

$$k_1 = hf_1(x_0, y_0, z_0); \qquad l_1 = hf_2(x_0, y_0, z_0)$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right); \qquad l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right); \qquad l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3); \qquad l_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4); \qquad l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

Hence $\quad y_1 = y_0 + k, \qquad z_1 = z_0 + l$

To compute y_2, z_2, we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in the above formulae.

EXAMPLES

Example 1. *Solve the equation* $\frac{dy}{dx} = x + y$ *with initial condition* $y(0) = 1$ *by Runge-Kutta rule, from* $x = 0$ *to* $x = 0.4$ *with* $h = 0.1$.

Sol. Here $f(x, y) = x + y, h = 0.1, x_0 = 0, y_0 = 1$

We have,

$$k_1 = hf(x_0, y_0) = 0.1\,(0 + 1) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = 0.1\,(0.05 + 1.05) = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = 0.1105$$

$$k_4 = hf(x_0 + h,\ \ y_0 + k_3) = 0.12105$$

$$\therefore \qquad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.11034$$

Thus, $x_1 = x_0 + h = 0.1$ and $y_1 = y_0 + \Delta y = 1.11034$

Now for the second interval, we have

$$k_1 = hf(x_1, y_1) = 0.1\,(0.1 + 1.11034) = 0.121034$$

$$k_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right) = 0.13208$$

$$k_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right) = 0.13263$$

$$k_4 = hf(x_1 + h,\ \ y_1 + k_3) = 0.14429$$

$$\therefore \qquad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.132460$$

Hence $x_2 = 0.2$ and $y_2 = y_1 + \Delta y = 1.11034 + 0.13246 = 1.24280$

Similarly, for finding y_3, we have

$$k_1 = hf(x_2, y_2) = 0.14428$$

$$k_2 = 0.15649$$

$$k_3 = 0.15710$$

$$k_4 = 0.16999$$

Repeating the above process

$\therefore \qquad y_3 = 0.13997$

and for $\qquad y_4 = y(0.4)$, we calculate

$$k_1 = 0.16997$$
$$k_2 = 0.18347$$
$$k_3 = 0.18414$$
$$k_4 = 0.19838$$

$\therefore \qquad y_4 = 1.5836$

Example 2. *Given* $\dfrac{dy}{dx} = y - x$, *$y(0) = 2$. Find $y(0.1)$ and $y(0.2)$ correct to four decimal places (use both II and IV order methods).*

Sol. By II order Method

To find y(0.1)

Here $\qquad y' = f(x, y) = y - x$, $x_0 = 0$, $y_0 = 2$ and $h = 0.1$

Now, $\qquad k_1 = hf(x_0, y_0) = 0.1(2 - 0) = 0.2$

$$k_2 = hf(x_0 + h, y_0 + k_1) = 0.21$$

$\therefore \qquad \Delta y = \dfrac{1}{2}(k_1 + k_2) = 0.205$

Thus, $\qquad x_1 = x_0 + h = 0.1 \quad$ and $\quad y_1 = y_0 + \Delta y = 2.205$

To find y(0.2) we note that,

$$x_1 = 0.1, y_1 = 2.205, h = 0.1$$

For interval II, we have

$$k_1 = hf(x_1, y_1) = 0.2105$$
$$k_2 = hf(x_1 + h, y_1 + k_1) = 0.22155$$

$\therefore \qquad \Delta y = \dfrac{1}{2}(k_1 + k_2) = 0.216025$

Thus, $\qquad x_2 = x_1 + h = 0.2 \quad$ and $\quad y_2 = y_1 + \Delta y = 2.4210$

Hence $\qquad y(0.1) = 2.205, \quad y(0.2) = 2.421.$

By IV order method- As before

$$k_1 = 0.2, k_2 = 0.205,$$
$$k_3 = hf(x_0 + h/2, y_0 + k_{2/2}) = 0.20525$$

and $\qquad k_4 = hf(x_0 + h, y_0 + k_3) = 0.210525$

$\therefore \qquad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2052$

Thus, $\qquad x_1 = x_0 + h = 0 + 0.1 = 0.1$

$$y_1 = y_0 + \Delta y = 2 + 0.2052 = 2.2052$$

Now to determine $y_2 = y(0.2)$, we note that

$$x_1 = x_0 + h = 0.1, y_1 = 2.2052, h = 0.1$$

For interval II, $\quad k_1 = hf(x_1, y_1) = 0.21052$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.21605$$

$$k_3 = hf\left(x_1 + h/2, y_1 + \frac{k_2}{2}\right) = 0.216323$$

and $\qquad k_4 = hf(x_1 + h, y_1 + k_3) = 0.221523$

$\therefore \qquad \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.21613$

Thus, $\qquad x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

and $\qquad y_2 = y_1 + \Delta y = 2.2052 + 0.21613 = 2.4213$

Hence $\qquad y(0.1) = 2.2052, \quad y(0.2) = 2.4213.$

Example 3. *Solve* $\frac{dy}{dx} = yz + x, \; \frac{dz}{dx} = xz + y;$

given that $y(0) = 1, z(0) = -1$ *for* $y(0.1), z(0.1)$.

Sol. Here, $\quad f_1(x, y, z) = yz + x$

$$f_2(x, y, z) = xz + y$$

$$h = 0.1, x_0 = 0, y_0 = 1, z_0 = -1$$

$$k_1 = hf_1(x_0, y_0, z_0) = h(y_0 z_0 + x_0) = -0.1$$

$$l_1 = hf_2(x_0, y_0, z_0) = h(x_0 z_0 + y_0) = 0.1$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= hf_1(0.05, 0.95, -0.95) = -0.08525$$

$$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= hf_2(0.05, 0.95, -0.95) = 0.09025$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= hf_1(0.05, 0.957375, -0.954875) = -0.0864173$$

$$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= hf_2(0.05, 0.957375, -0.954875) = -0.0864173$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3) = -0.073048.$$

$$l_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.0822679$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0860637$$

$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 0.0907823$$

$\therefore$ $$y_1 = y(0.1) = y_0 + k = 1 - 0.0860637 = 0.9139363$$

$$z_1 = z(0.1) = z_0 + k = -1 + 0.0907823 = -0.9092176$$

ASSIGNMENT 6.4

1. Use the Runge-Kutta Method to approximate y when $x = 0.1$ given that $x = 0$ when $y = 1$ and $\frac{dy}{dx} = x + y$.

2. Apply the Runge-Kutta Fourth Order Method to solve $10\frac{dy}{dx} = x^2 + y^2$; $y(0) = 1$ for $0 < x \le 0.4$ and $h = 0.1$.

3. Use Runge-Kutta Fourth Order Formula to find $y(1.4)$ if $y(1) = 2$ and $\frac{dy}{dx} = xy$. Take $h = 0.2$.

4. Prove that the solution of $y' = y$, $y(0) = 1$ by Second Order Runge-Kutta Method yields

$$y_m = \left(1 + h + \frac{h^2}{2}\right)^m.$$

5. Solve $y' = \frac{1}{x + y}$, $y(0) = 1$ for $x = 0.5$ to $x = 1$ by Runge-Kutta Method ($h = 0.5$).

6. Solve $y' = -xy^2$ and By Runge-Kutta Fourth Order Method, find $y(0.6)$ given that $y = 1.7231$ at $x = 0.4$. Take $h = 0.2$.