5.2 NUMERICAL DIFFERENTIATION

In the case of numerical data, the functional form of $f(x)$ is not known in general. First we have to find an appropriate form of $f(x)$ and then obtain its derivatives. So **"Numerical Differentiation"** is concerned with the method of finding the successive derivatives of a function at a given argument, using the given table of entries corresponding to a set of arguments, equally or unequally spaced. Using the theory of interpolation, a suitable interpolating polynomial can be chosen to represent the function to a good degree of approximation in the given interval of the argument.

For the proper choice of interpolation formula, the criterion is the same as in the case of interpolation problems. In the case of equidistant values of x, if the derivative is to be found at a point near the beginning or the end of the given set of values, Newton's forward or backward difference formula should be used accordingly. Also if the derivative is to be found at a point near the middle of the given set of values, then any one of the central difference formulae should be used. However, if the values of the function are not known at equidistant values of x, Newton's divided difference or Lagrange's formula should be used.

5.3 FORMULAE FOR DERIVATIVES

(1) **Newton's forward difference interpolation formula is**

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots \qquad (1)$$

where $$u = \frac{x-a}{h} \qquad (2)$$

Differentiating eqn. (1) with respect to u, we get

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \ldots \qquad (3)$$

Differentiating eqn. (2) with respect to x, we get

$$\frac{du}{dx} = \frac{1}{h} \qquad (4)$$

We know that

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \ldots\right] \qquad (5)$$

Sol. Since the values are correct to four decimals, it follows that

$$\varepsilon = 0.5 \times 10^{-4}$$

$$\text{Truncation error} = \frac{1}{6h}\left|\frac{\Delta^3 y_{-1} + \Delta^3 y_0}{2}\right| = \frac{1}{1.2}\left(\frac{0.0361 + 0.0441}{2}\right)$$

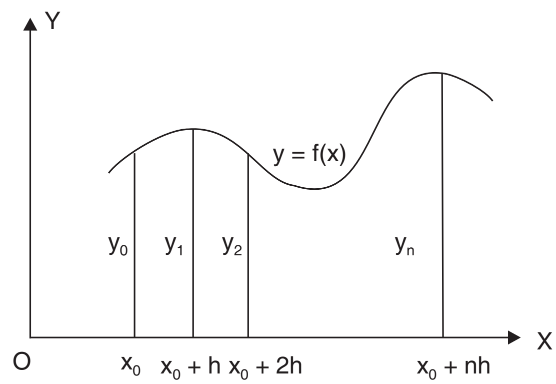
| See difference table in Example 6

$$= 0.03342$$

$$\text{Rounding error} = \frac{3\varepsilon}{2h} = \frac{3 \times 0.5 \times 10^{-4}}{2 \times 0.2} = 0.00038.$$

5.6 NUMERICAL INTEGRATION

Given a set of tabulated values of the integrand $f(x)$, determining the value of $\int_{x_0}^{x_n} f(x)\, dx$ is called numerical integration. The given interval of integration is subdivided into a large number of subintervals of equal width h and the function tabulated at the points of subdivision is replaced by any one of the interpolating polynomials like Newton-Gregory's, Stirling's, Bessel's over each of the subintervals and the integral is evaluated. There are several formulae for numerical integration which we shall derive in the sequel.



5.7 NEWTON-COTE'S QUADRATURE FORMULA

Let $I = \int_a^b y\, dx$, where y takes the values $y_0, y_1, y_2, \ldots\ldots, y_n$ for $x = x_0, x_1, x_2, \ldots\ldots, x_n$.

Let the interval of integration (a, b) be divided into n equal sub-intervals, each of width $h = \frac{b-a}{n}$ so that

$$x_0 = a,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ ....,\ x_n = x_0 + nh = b.$$

$$\therefore \qquad I = \int_{x_0}^{x_0+nh} f(x)\, dx$$

Since any x is given by $x = x_0 + rh$ and $dx = hdr$

$$\therefore \qquad I = h \int_0^n f(x_0 + rh)\, dr$$

$$= h \int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + ...\right] dr$$

[by Newton's forward interpolation formula]

$$= h \left[ry_0 + \frac{r^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{r^4}{4} - r^3 + r^2\right)\Delta^3 y_0 + ...\right]_0^n$$

$$= nh \left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + ...\right] \qquad (49)$$

This is a **general quadrature formula** and is known as **Newton-Cote's quadrature formula.** A number of important deductions *viz.* Trapezoidal rule, Simpson's one-third and three-eighth rules, Weddle's rule can be immediately deduced by putting $n = 1, 2, 3$, and 6, respectively, in formula (49).

5.8 TRAPEZOIDAL RULE (n = 1)

Putting $n = 1$ in formula (49) and taking the curve through (x_0, y_0) and (x_1, y_1) as a polynomial of degree one so that differences of an order higher than one vanish, we get

$$\int_{x_0}^{x_0+h} f(x)\, dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}\,[2y_0 + (y_1 - y_0)] = \frac{h}{2}\,(y_0 + y_1)$$

Similarly, for the next sub-interval $(x_0 + h, x_0 + 2h)$, we get

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}(y_1 + y_2), \ldots\ldots, \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots + y_{n-1})]}$$

which is known as **Trapezoidal rule.** By increasing the number of subintervals, thereby making h very small, we can improve the accuracy of the value of the given integral.

5.9 SIMPSON'S ONE-THIRD RULE (n = 2)

Putting $n = 2$ in formula (49) and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a polynomial of degree two so that differences of order higher than two vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= \frac{2h}{6}[6y_0 + 6(y_1 - y_0) + (y_2 - 2y_1 + y_0)]$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4), \ldots\ldots,$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\boxed{\begin{aligned}\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\,[(y_0 + y_n) &+ 4(y_1 + y_3 + \ldots + y_{n-1}) \\ &+ 2(y_2 + y_4 + \ldots + y_{n-2})]\end{aligned}}$$

which is known as **Simpson's one-third rule.**

While using this formula, the given interval of integration must be divided into an even number of sub-intervals, since we find the area over two sub-intervals at a time.

5.10 SIMPSON'S THREE-EIGHTH RULE (n = 3)

Putting $n = 3$ in formula (49) and taking the curve through (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3) as a polynomial of degree three so that differences of order higher than three vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}\,[8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)]$$

$$= \frac{3h}{8}\,[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly, $$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\,[y_3 + 3y_4 + 3y_5 + y_6], \ldots$$

$$\int_{x_0+(n-3)h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\,[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Adding the above integrals, we get

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\,[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \ldots\ldots + y_{n-3})]}$$

which is known as **Simpson's three-eighth rule.**

While using this formula, the given interval of integration must be divided into sub-intervals whose number n is a multiple of 3.

5.11 BOOLE'S RULE

Putting $n = 4$ in formula (49) and neglecting all differences of order higher than four, we get

$$\int_{x_0}^{x_0+4h} f(x)\,dx = h\int_0^4 \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 \right] dr$$

| By Newton's forward interpolation formula

$$= 4h\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} \right]_0^4$$

$$= 4h\left[y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{3}{2}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0 \right]$$

$$= \frac{2h}{45}\,(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)$$

Similarly, $\displaystyle\int_{x_0+4h}^{x_0+8h} f(x)\,dx = \frac{2h}{45}\,(7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8)$ and so on.

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 4, we get

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}\,[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32\,y_5 + 12y_6 + 32y_7 + 14y_8 + \ldots\ldots]}$$

This is known as **Boole's rule.**

While applying Boole's rule, the **number of sub-intervals should be taken as a multiple of 4.**

5.12 WEDDLE'S RULE (n = 6)

Putting $n = 6$ in formula (49) and neglecting all differences of order higher than six, we get

$$\int_{x_0}^{x_0+6h} f(x)\,dx = h\int_0^6 \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 \right.$$

$$+ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 y_0$$

$$\left. + \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 y_0 \right] dr$$

$$= h\left[ry_0 + \frac{r^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{r^4}{4} - r^3 + r^2\right)\Delta^3 y_0 \right.$$

$$+ \frac{1}{24}\left(\frac{r^5}{5} - \frac{3r^4}{2} + \frac{11r^3}{3} - 3r^2\right)\Delta^4 y_0$$

$$+ \frac{1}{120}\left(\frac{r^6}{6} - 2r^5 + \frac{35r^4}{4} - \frac{50r^3}{3} + 12r^2\right)\Delta^5 y_0$$

$$\left. + \frac{1}{720}\left(\frac{r^7}{7} - \frac{5r^6}{2} + 17r^5 - \frac{225r^4}{4} + \frac{274r^3}{3} - 60r^2\right)\Delta^6 y_0 \right]_0^6$$

$$= 6h\left[y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{41}{20}\Delta^4 y_0 \right.$$

$$\left. + \frac{11}{20}\Delta^5 y_0 + \frac{41}{840}\Delta^6 y_0 \right]$$

$$= \frac{6h}{20}\left[20y_0 + 60\Delta y_0 + 90\Delta^2 y_0 + 80\Delta^3 y_0 + 41\Delta^4 y_0 \right.$$

$$\left. + 11\Delta^5 y_0 + \frac{41}{42}\Delta^6 y_0 \right]$$

$$= \frac{3h}{10}\,[20y_0 + 60(y_1 - y_0) + 90(y_2 - 2y_1 + y_0)$$
$$+ 80(y_3 - 3y_2 + 3y_1 - y_0)$$
$$+ 41(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)$$
$$+ 11\,(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0)$$
$$+ (y_6 - 6y_5 + 15y_4 - 20y_3$$
$$+ 15y_2 - 6y_1 + y_0)] \qquad \left[\because \frac{41}{42} \simeq 1\right]$$

$$= \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

Similarly,

$$\int_{x_0 + 6h}^{x_0 + 12h} f(x)\, dx = \frac{3h}{10} [y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$$

....

....

$$\int_{x_0 + (n-6)h}^{x_0 + nh} f(x)\, dx = \frac{3h}{10} [y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

Adding the above integrals, we get

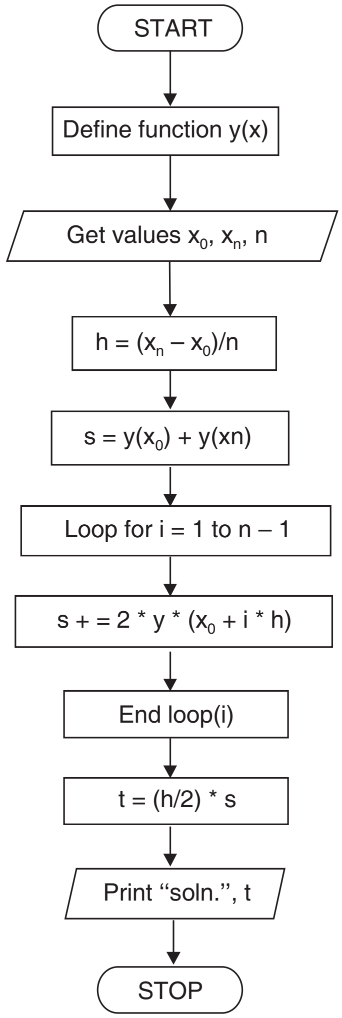
$$\boxed{\begin{aligned}\int_{x_0}^{x_0 + nh} f(x)\, dx = \frac{3h}{10} [&y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 \\ &+ 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + 2y_{12} + ...]\end{aligned}}$$

which is known as **Weddle's rule.** Here **n must be a multiple of 6.**

5.13 ALGORITHM OF TRAPEZOIDAL RULE

Step 01. Start of the program.
Step 02. Input Lower limit a
Step 03. Input Upper Limit b
Step 04. Input number of sub intervals n
Step 05. h=(b-a)/n
Step 06. sum=0
Step 07. sum=fun(a)+fun(b)
Step 08. for i=1; i<n; i++
Step 09. sum +=2*fun(a+i)
Step 10. End Loop i
Step 11. result =sum*h/2;
Step 12. Print Output result
Step 13. End of Program
Step 14. Start of Section fun
Step 15. temp = 1/(1+(x*x))
Step 16. Return temp
Step 17. End of Section fun.

5.14 FLOW-CHART FOR TRAPEZOIDAL RULE



/* **

5.15 PROGRAM TO IMPLEMENT TRAPEZOIDAL METHOD OF NUMERICAL INTEGRATION

** */

```
//... HEADER FILES DECLARATION
# include <stdio.h>
# include <conio.h>
# include <math.h>
# include <process.h>
# include <string.h>

//... Function Prototype Declaration
float fun(float);
//... Main Execution Thread
void main()
{
//... Variable Declaration Field
//... Floating Type
float result=1;
float a,b;
float h,sum;
//... Integer Type

int i,j;
int n;

//... Invoke Clear Screen Function
clrscr();

//... Input Section
//... Input Range

printf("\n\n Enter the range - ");
printf("\n\n Lower Limit a - ");
scanf("%f" ,&a);
```

```
printf("\n\n Upper Limit b - ");
scanf("%f" ,&b);
//... Input Number of subintervals
printf("\n\n Enter number of subintervals - ");
scanf("%d" ,&n);
//... Calculation and Processing Section
h=(b-a)/n;
sum=0;
sum=fun(a)+fun(b);
for(i=1;i<n;i++)
     {
     sum+=2*fun(a+i);
     }
result=sum*h/2;
//... Output Section
printf("n\n\n\n Value of the integral is %6.4f\t",result);
//...Invoke User Watch Halt Function
printf("\n\n\n Press Enter to Exit");
getch();
}
//... Termination of Main Execution Thread
//... Function Body
float fun(float x)
{
float temp;
temp = 1/(1+(x*x));
return temp;
}
//... Termination of Function Body
```

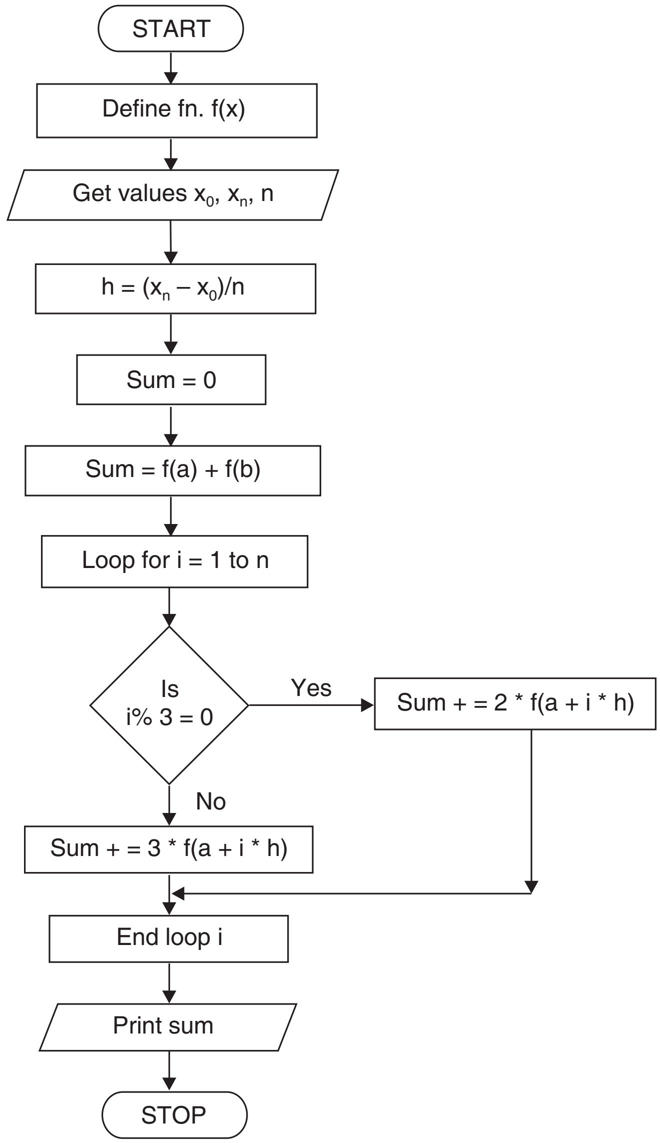
5.16 OUTPUT

```
Enter the range -
Lower Limit a - 0
Upper Limit b - 6
Enter number of subintervals - 6
Value of the integral is 1.4108
Press Enter to Exit
```

5.17 ALGORITHM OF SIMPSON'S 3/8th RULE

Step 01. Start of the program.
Step 02. Input Lower limit a
Step 03. Input Upper limit b
Step 04. Input number of sub itervals n
Step 05. h = (b – a)/n
Step 06. sum = 0
Step 07. sum = fun(a) + fun (b)
Step 08. for i = 1; i < n; i++
Step 09. if i%3=0:
Step 10. sum + = 2*fun(a + i*h)
Step 11. else:
Step 12. sum + = 3*fun(a+(i)*h)
Step 13. End of loop i
Step 14. result = sum*3*h/8
Step 15. Print Output result
Step 16. End of Program
Step 17. Start of Section fun
Step 18. temp = 1/(1+(x*x))
Step 19. Return temp
Step 20. End of section fun

5.18 FLOW-CHART OF SIMPSON'S 3/8th RULE



/**

5.19 PROGRAM TO IMPLEMENT SIMPSON'S 3/8th METHOD OF NUMERICAL INTEGRATION

**/

```
//... HEADER FILES DECLARATION
# include <stdio.h>
# include <conio.h>
# include <math.h>
# include <process.h>
# include <string.h>
//... Function Prototype Declaration

float fun(float);

//... Main Execution Thread

void main()
{
//... Variable Declaration Field

//... Floating Type

float result=1;
float a,b;
float h,sum;

//...Integer Type

int i,j;
int n;

//...Invoke Clear Screen Function

clrscr();

//...Input Section

//...Input Range
printf("\n\n Enter the range - ");
printf("\n\n Lower Limit a - ");
scanf("%f" ,&a);
```

```
printf("\n\n Upper Limit b - ");
scanf("%f" ,&b);

//...Input Number of Subintervals
printf("\n\n Enter number of subintervals - ");
scanf("%d" ,&n);

//...Calculation and Processing Section
h=(b-a)/n;

sum=0;
sum=fun(a)+fun(b);

for(i=1;i<n;i++)
    {
    if(i%3==0)
        {
        sum+=2*fun(a+i*h)
        }
    else
        {
        sum+=3*fun(a+(i)*h);
        }
    }
result=sum*3*h/8;

//... Output Section
printf("\n\n\n\n Value of the integral is %6.4f\t",result);

//... Invoke User Watch Halt Function
printf("\n\n\n Press Enter to Exit");
getch();
}

//... Termination of Main Execution Thread
//... Function Body
```

```
float fun(float x)
 {
 float temp;
 temp=1/(1+(x*x));
 return temp;
 }
//... Termination of Function Body
```

5.20 OUTPUT

```
Enter the range -
Lower Limit a - 0
Upper Limit b - 6
Enter number of subintervals - 6
Value of the integral is 1.3571
Press Enter to Exit
```

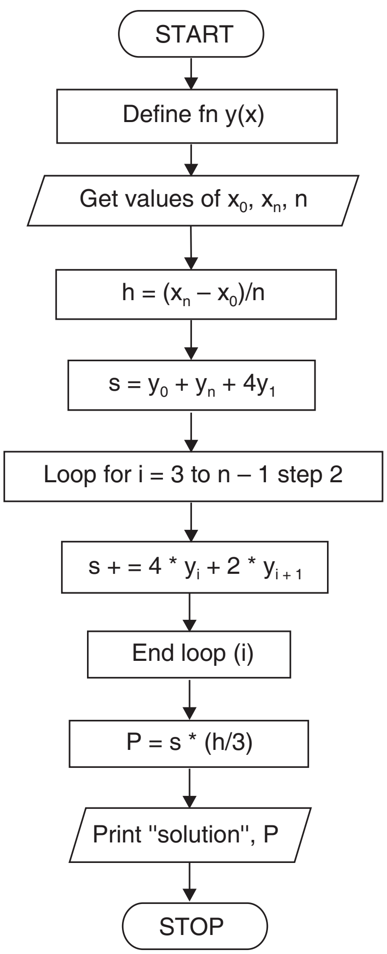
5.21 ALGORITHM OF SIMPSON'S 1/3rd RULE

Step 01. Start of the program.
Step 02. Input Lower limit a
Step 03. Input Upper limit b
Step 04. Input number of subintervals n
Step 05. h=(b–a)/n
Step 06. sum=0
Step 07. sum=fun(a)+4*fun(a+h)+fun(b)
Step 08. for i=3; i<n; i + = 2
Step 09. sum + = 2*fun(a+(i – 1)*h) + 4*fun(a+i*h)
Step 10. End of loop i
Step 11. result=sum*h/3
Step 12. Print Output result
Step 13. End of Program
Step 14. Start of Section fun
Step 15. temp = 1/(1+(x*x))

Step 16. Return temp

Step 17. End of Section fun

5.22 FLOW-CHART OF SIMPSON'S 1/3rd RULE



```
/* ***********************************************************
```

5.23 PROGRAM TO IMPLEMENT SIMPSON'S 1/3rd METHOD OF NUMERICAL INTEGRATION

```
*********************************************************** */

//... HEADER FILES DECLARATION

# include <stdio.h>
# include <conio.h>
# include <math.h>
# include <process.h>
# include <string.h>

//... Function Prototype Declaration

float fun(float);

//... Main Execution Thread

void main()
 {
 //...Variable Declaration Field
 //... Floating Type
 float result=1;
 float a,b;
 float h,sum;

 //... Integer Type

 int i,j;
 int n;

 //... Invoke Clear Screen Function

 clrscr();

 //... Input Section

 //...Input Range
 printf("\n\n Enter the range - ");
```

```
printf("\n\n Lower Limit a - ");
scanf("%f" ,&a);

printf("\n\n Upper Limit b - ");
scanf("%f" ,&b);
//... Input Number of Subintervals
printf("\n\n Enter number of subintervals - ");
scanf("%d",&n);

//... Calculation and Processing Section

h=(b-a)/n;

sum=0;
sum=fun(a)+4*fun(a+h)fun(b);
for(i=3;i<n;i+=2)
     {
     sum+=2*fun(a+(i-1)*h)+4*fun(a+i*h);
     }

result=sum*h/3;

//... Output Section
printf("\n\n\n\n Value of the integral is %6.4f\t",result);

//... Invoke User Watch Halt Function
printf("\n\n\n Press Enter to Exit");
getch();
}

//... Termination of Main Execution Thread

//... Function Body

float fun(float x)
{

float temp;
temp=1/(1+(x*x));
return temp;
}
//... Termination of Function Body
```

5.24 OUTPUT

```
Enter the range -
Lower Limit a - 0
Upper Limit b - 6
Enter number of subintervals - 6
Value of the integral is 1.3662
Press Enter to Exit
```

EXAMPLES

Example 1. *Use Trapezoidal rule to evaluate* $\int_0^1 x^3\,dx$ *considering five sub-intervals.*

Sol. Dividing the interval (0, 1) into 5 equal parts, each of width $h = \frac{1-0}{5}$ = 0.2, the values of $f(x) = x^3$ are given below:

x:	0	0.2	0.4	0.6	0.8	1.0
$f(x)$:	0	0.008	0.064	0.216	0.512	1.000
	y_0	y_1	y_2	y_3	y_4	y_5

By Trapezoidal rule, we have

$$\int_0^1 x^3\,dx = \frac{h}{2}\,[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)]$$

$$= \frac{0.2}{2}\,[(0 + 1) + 2(0.008 + 0.064 + 0.216 + 0.512)]$$

$$= 0.1 \times 2.6 = 0.26.$$

Example 2. *Evaluate* $\int_0^1 \frac{dx}{1+x^2}$ *using*

(*i*) *Simpson's* $\frac{1}{3}$ *rule taking* $h = \frac{1}{4}$

(*ii*) *Simpson's* $\frac{3}{8}$ *rule taking* $h = \frac{1}{6}$

(*iii*) *Weddle's rule taking* $h = \frac{1}{6}$

Hence compute an approximate value of π *in each case.*

Sol. (*i*) The values of $f(x) = \dfrac{1}{1+x^2}$ at $x = 0, \dfrac{1}{4}, \dfrac{2}{4}, \dfrac{3}{4}, 1$ are given below:

x:	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$f(x)$:	1	$\frac{16}{17}$	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

By Simpson's $\dfrac{1}{3}$ rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\,[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{1}{12}\left[(1+0.5) + 4\left\{\frac{16}{17} + .64\right\} + 2(0.8)\right] = 0.785392156$$

Also $$\int_0^1 \frac{dx}{1+x^2} = \Big[\tan^{-1} x\Big]_0^1 = \tan^{-1} 1 = \frac{\pi}{4}$$

$\therefore$ $$\frac{\pi}{4} \simeq 0.785392156 \quad \Rightarrow \quad \pi \simeq 3.1415686$$

(*ii*) The values of $f(x) = \dfrac{1}{1+x^2}$ at $x = 0, \dfrac{1}{6}, \dfrac{2}{6}, \dfrac{3}{6}, \dfrac{4}{6}, \dfrac{5}{6}, 1$ are given below:

x:	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$f(x)$:	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\dfrac{3}{8}$ rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}\,[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\left(\frac{1}{6}\right)}{8}\left[\left(1+\frac{1}{2}\right)+3\left\{\frac{36}{37}+\frac{9}{10}+\frac{9}{13}+\frac{36}{61}\right\}+2\left(\frac{4}{5}\right)\right]$$

$$= 0.785395862$$

Also, $\displaystyle\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$

$\therefore \qquad \dfrac{\pi}{4} = 0.785395862$

$\Rightarrow \qquad \pi = 3.141583$

(*iii*) By Weddle's rule, using the values as in (*ii*),

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{10}\,(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

$$= \frac{3\left(\frac{1}{6}\right)}{10}\left\{1+5\left(\frac{36}{37}\right)+\frac{9}{10}+6\left(\frac{4}{5}\right)+\frac{9}{13}+5\left(\frac{36}{61}\right)+\frac{1}{2}\right\}$$

$$= 0.785399611$$

Since $\displaystyle\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$

$\therefore \qquad \dfrac{\pi}{4} = 0.785399611$

$\Rightarrow \qquad \pi = 3.141598.$

Example 3. *Evaluate*

$$\int_0^6 \frac{dx}{1+x^2} \text{ by using}$$

(*i*) *Simpson's one-third rule*

(*ii*) *Simpson's three-eighth rule*

(*iii*) *Trapezoidal rule*

(*iv*) *Weddle's rule.*

Sol. Divide the interval (0, 6) into six parts each of width $h = 1$.

The values of $f(x) = \dfrac{1}{1+x^2}$ are given below:

x:	0	1	2	3	4	5	6
$f(x)$:	1	0.5	0.2	0.1	$\frac{1}{17}$	$\frac{1}{26}$	$\frac{1}{37}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(*i*) By Simpson's one-third rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\,[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}\left[\left(1+\frac{1}{37}\right)+4\left(0.5+0.1+\frac{1}{26}\right)+2\left(0.2+\frac{1}{17}\right)\right]$$

$$= 1.366173413.$$

(*ii*) By Simpson's three-eighth rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\,[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}\left[\left(1+\frac{1}{37}\right)+3\left(.5+.2+\frac{1}{17}+\frac{1}{26}\right)+2(.1)\right]$$

$$= 1.357080836.$$

(*iii*) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\,[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}\left[\left(1+\frac{1}{37}\right)+2\left(.5+.2+.1+\frac{1}{17}+\frac{1}{26}\right)\right]$$

$$= 1.410798581.$$

(*iv*) By Weddle's rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}\,[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= \frac{3}{10}\left[1 + 5(.5) + .2 + 6(.1) + \frac{1}{17} + 5\left(\frac{1}{26}\right) + \frac{1}{37}\right]$$

$$= 1.373447475.$$

Example 4. *The speed, v meters per second, of a car, t seconds after it starts, is shown in the following table:*

t	0	12	24	36	48	60	72	84	96	108	120
v	0	3.60	10.08	18.90	21.60	18.54	10.26	5.40	4.50	5.40	9.00

Using Simpson's rule, find the distance travelled by the car in 2 minutes.

Sol. If s meters is the distance covered in t seconds, then

$$\frac{ds}{dt} = v$$

$$\therefore \qquad \Big[s\Big]_{t=0}^{t=120} = \int_0^{120} v\,dt$$

since the number of sub-intervals is **10 (even).** Hence, by using Simpson's $\frac{1}{3}$rd rule,

$$\int_0^{120} v\,dt = \frac{h}{3}\,[(v_0 + v_{10}) + 4(v_1 + v_3 + v_5 + v_7 + v_9) + 2(v_2 + v_4 + v_6 + v_8)]$$

$$= \frac{12}{3}\,[(0 + 9) + 4(3.6 + 18.9 + 18.54 + 5.4 + 5.4)$$
$$+ 2(10.08 + 21.6 + 10.26 + 4.5)]$$

$$= 1236.96 \text{ meters.}$$

Hence, the distance travelled by car in 2 minutes is 1236.96 meters.

Example 5. *Evaluate* $\int_{0.6}^{2} y\,dx$, *where y is given by the following table:*

x:	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y:	1.23	1.58	2.03	4.32	6.25	8.36	10.23	12.45.

Sol. Here the number of subintervals is 7, which is neither even nor a multiple of 3. Also, this number is neither a multiple of 4 nor a multiple of 6, hence using Trapezoidal rule, we get

$$\int_{0.6}^{2} y\,dx = \frac{h}{2}\,[(y_0 + y_7) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)]$$

$$= \frac{0.2}{2}\,[(1.23 + 12.45) + 2(1.58 + 2.03 + 4.32 + 6.25 + 8.36 + 10.23)]$$

| Here $h = 0.2$

$$= 7.922.$$

Example 6. *Find* $\int_{1}^{11} f(x)\,dx$, *where f(x) is given by the following table, using a suitable integration formula.*

x:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
f(x):	*543*	*512*	*501*	*489*	*453*	*400*	*352*	*310*	*250*	*172*	*95*

Sol. Since the number of subintervals is 10 (even) hence we shall use Simpson's $\frac{1}{3}$rd rule.

$$\int_{1}^{11} f(x)\,dx = \frac{h}{3}\,[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$

$$= \frac{1}{3}\,[(543 + 95) + 4(512 + 489 + 400 + 310 + 172) + 2(501 + 453 + 352 + 250)]$$

$$= \frac{1}{3}\,[638 + 7532 + 3112] = 3760.67.$$

Example 7. *Evaluate* $\int_{0}^{1} \frac{dx}{1+x}$ *by dividing the interval of integration into 8 equal parts. Hence find* $\log_e 2$ *approximately.*

Sol. Since the interval of integration is divided into an even number of subintervals, we shall use Simpson's one-third rule.

Here, $$y = \frac{1}{1+x} = f(x)$$

$$y_0 = f(0) = \frac{1}{1+0} = 1,\quad y_1 = f\left(\frac{1}{8}\right) = \frac{1}{1+\frac{1}{8}} = \frac{8}{9},\quad y_2 = f\left(\frac{2}{8}\right) = \frac{4}{5}$$

$$y_3 = f\left(\frac{3}{8}\right) = \frac{8}{11},\quad y_4 = f\left(\frac{4}{8}\right) = \frac{2}{3},\quad y_5 = f\left(\frac{5}{8}\right) = \frac{8}{13}$$

$$y_6 = f\left(\frac{6}{8}\right) = \frac{4}{7}, \qquad y_7 = f\left(\frac{7}{8}\right) = \frac{8}{15} \qquad \text{and} \quad y_8 = f(1) = \frac{1}{2}$$

Hence the table of values is

x:	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	1
y:	1	$\frac{8}{9}$	$\frac{4}{5}$	$\frac{8}{11}$	$\frac{2}{3}$	$\frac{8}{13}$	$\frac{4}{7}$	$\frac{8}{15}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Simpson's $\frac{1}{3}$rd rule,

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{3}\ [(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{1}{24}\left[\left(1+\frac{1}{2}\right) + 4\left(\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\right) + 2\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right)\right]$$

| Here $h = 1/8$

$$= 0.69315453$$

Since, $$\int_0^1 \frac{dx}{1+x} = \Big[\log_e (1+x)\Big]_0^1 = \log_e 2$$

$\therefore$ $\log_e 2 = 0.69315453.$

Example 8. *Find, from the following table, the area bounded by the curve and the x-axis from x = 7.47 to x = 7.52.*

x:	*7.47*	*7.48*	*7.49*	*7.50*	*7.51*	*7.52*
f(x):	*1.93*	*1.95*	*1.98*	*2.01*	*2.03*	*2.06.*

Sol. We know that

$$\text{Area} = \int_{7.47}^{7.52} f(x)\, dx$$

with $h = 0.01$, the trapezoidal rule gives,

$$\text{Area} = \frac{.01}{2}\ [(1.93 + 2.06) + 2(1.95 + 1.98 + 2.01 + 2.03)]$$

$$= 0.09965.$$

Example 9. *Use Simpson's rule for evaluating*

$$\int_{-0.6}^{0.3} f(x)\, dx$$

from the table given below:

x:	– 0.6	– 0.5	– 0.4	– 0.3	– 0.2	– 0.1	0	.1	.2	.3
f(x):	4	2	5	3	– 2	1	6	4	2	8

Sol. Since the number of subintervals is 9(a multiple of 3), we will use Simpson's 3/8$^{\text{th}}$ rule here.

$$\therefore \qquad \int_{-0.6}^{0.3} f(x)\, dx = \frac{3(.1)}{8}\ [(4 + 8) + 3\{2 + 5 + (-2) + 1 + 4 + 2\} + 2(3 + 6)]$$

$$= 2.475.$$

Example 10. *Evaluate* $\int_1^2 e^{-\frac{1}{2}x}\, dx$ *using four intervals.*

Sol. The table of values is:

x:	1	1.25	1.5	1.75	2
$y = e^{-x/2}$:	.60653	.53526	.47237	.41686	.36788
	y_0	y_1	y_2	y_3	y_4

Since we have four (even) subintervals here, we will use Simpson's $\frac{1}{3}$rd rule.

$$\therefore \qquad \int_1^2 e^{-\frac{1}{2}x}\, dx = \frac{h}{3}\ [(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{.25}{3}\ [(.60653 + .36788) + 4(.53526) + .41686) + 2(.47237)]$$

$$= 0.4773025.$$

Example 11. *Find* $\int_0^6 \frac{e^x}{1+x}\, dx$ *approximately using Simpson's* $\frac{3}{8}$*th rule on integration.*

Sol. Divide the given integral of integration into 6 equal subintervals, the arguments are 0, 1, 2, 3, 4, 5, 6; $h = 1$.

$$f(x) = \frac{e^x}{1+x};\ y_0 = f(0) = 1$$

$$y_1 = f(1) = \frac{e}{2}, \qquad y_2 = f(2) = \frac{e^2}{3}, \qquad y_3 = f(3) = \frac{e^3}{4},$$

$$y_4 = f(4) = \frac{e^4}{5}, \qquad y_5 = f(5) = \frac{e^5}{6}, \qquad y_6 = f(6) = \frac{e^6}{7}$$

The table is as below:

x:	0	1	2	3	4	5	6
y:	1	$\frac{e}{2}$	$\frac{e^2}{3}$	$\frac{e^3}{4}$	$\frac{e^4}{5}$	$\frac{e^5}{6}$	$\frac{e^6}{7}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Applying Simpson's three-eighth rule, we have

$$\int_0^6 \frac{e^x}{1+x}\,dx = \frac{3h}{8}\ [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}\left[\left(1+\frac{e^6}{7}\right) + 3\left(\frac{e}{2}+\frac{e^2}{3}+\frac{e^4}{5}+\frac{e^5}{6}\right) + 2\,\frac{e^3}{4}\right]$$

$$= \frac{3}{8}\ [(1 + 57.6327) + 3(1.3591 + 2.463 + 10.9196 + 24.7355 + 2(5.0214)]$$

$$= 70.1652.$$

NOTE *It is not possible to evaluate* $\int_0^6 \frac{e^x}{1+x}\,dx$ *by using usual calculus method. Numerical integration comes to our rescue in such situations.*

Example 12. *A train is moving at the speed of 30 m/sec. Suddenly brakes are applied. The speed of the train per second after t seconds is given by*

Time (t):	0	5	10	15	20	25	30	35	40	45
Speed (v):	30	24	19	16	13	11	10	8	7	5

Apply Simpson's three-eighth rule to determine the distance moved by the train in 45 seconds.

Sol. If s meters is the distance covered in t seconds, then

$$\frac{ds}{dt} = v \qquad \Rightarrow \qquad \Big[\, s \,\Big]_{t=0}^{t=45} = \int_0^{45} v\,dt$$

Since the number of subintervals is **9 (a multiple of 3)** hence by using Simpson's $\left(\frac{3}{8}\right)^{\text{th}}$ rule,

$$\int_0^{45} v\,dt = \frac{3h}{8}\,[(v_0 + v_9) + 3(v_1 + v_2 + v_4 + v_5 + v_7 + v_8) + 2(v_3 + v_6)]$$

$$= \frac{15}{8}\,[(30 + 5) + 3(24 + 19 + 13 + 11 + 8 + 7) + 2(16 + 10)]$$

$$= 624.375 \text{ meters.}$$

Hence the distance moved by the train in 45 seconds is **624.375** meters.

Example 13. *Evaluate* $\int_0^4 \frac{dx}{1+x^2}$ *using Boole's rule taking*

(*i*) *h = 1* (*ii*) *h = 0.5*

Compare the results with the actual value and indicate the error in both.

Sol. (*i*) Dividing the given interval into 4 equal subintervals (*i.e.,* $h = 1$), the table is as follows:

x:	0	1	2	3	4
y:	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$
	y_0	y_1	y_2	y_3	y_4

using Boole's rule,

$$\int_0^4 y\,dx = \frac{2h}{45}\,[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$

$$= \frac{2(1)}{45}\left[7(1) + 32\left(\frac{1}{2}\right) + 12\left(\frac{1}{5}\right) + 32\left(\frac{1}{10}\right) + 7\left(\frac{1}{17}\right)\right]$$

$$= 1.289412 \text{ (approx.)}$$

$$\therefore \qquad \int_0^4 \frac{dx}{1+x^2} = 1.289412.$$

(*ii*) Dividing the given interval into 8 equal subintervals (*i.e.*, $h = 0.5$), the table is as follows:

x:	0	.5	1	1.5	2	2.5	3	3.5	4
y:	1	0.8	0.5	$\frac{4}{13}$	.2	$\frac{4}{29}$	.1	$\frac{4}{53}$	$\frac{1}{17}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

using Boole's rule,

$$\int_0^4 y dx = \frac{2h}{45}[7(y_0) + 32(y_1) + 12(y_2) + 32(y_3) + 7(y_4) + 7(y_4) + 32(y_5) + 12(y_6) + 32(y_7) + 7(y_8)]$$

$$= \frac{1}{45}\left[7(1) + 32(.8) + 12(.5) + 32\left(\frac{4}{13}\right) + 7(.2) + 7(.2) + 32\left(\frac{4}{29}\right) + 12(.1) + 32\left(\frac{4}{53}\right) + 7\left(\frac{1}{17}\right)\right]$$

$$= 1.326373$$

$$\therefore \quad \int_0^4 \frac{dx}{1+x^2} = 1.326373$$

But the actual value is

$$\int_0^4 \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_0^4 = \tan^{-1}(4) = 1.325818$$

$$\text{Error in result I} = \left(\frac{1.325818 - 1.289412}{1.325818}\right) \times 100 = 2.746\%$$

$$\text{Error in result II} = \left(\frac{1.325818 - 1.326373}{1.325818}\right) \times 100 = -0.0419\%.$$

Example 14. *A river is 80 m wide. The depth 'y' of the river at a distance 'x' from one bank is given by the following table:*

x:	*0*	*10*	*20*	*30*	*40*	*50*	*60*	*70*	*80*
y:	*0*	*4*	*7*	*9*	*12*	*15*	*14*	*8*	*3*

Find the approximate area of cross-section of the river using

(*i*) *Boole's rule.*

(*ii*) *Simpson's* $\frac{1}{3}$*rd rule.*

Sol. The required area of the cross-section of the river

$$= \int_0^{80} y\,dx$$

Here the number of sub intervals is 8.

(*i*) *By Boole's rule,*

$$\int_0^{80} y\,dx = \frac{2h}{45}\,[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4 + 7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8]$$

$$= \frac{2\,(10)}{45}\,[7(0) + 32(4) + 12(7) + 32(9) + 7(12) + 7(12) + 32(15) + 12(14) + 32(8) + 7(3)]$$

$$= 708$$

Hence the required area of the cross-section of the river = 708 sq. m.

(*ii*) *By Simpson's* $\frac{1}{3}$*rd rule*

$$\int_0^{80} y\,dx = \frac{h}{3}\,[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{10}{3}\,[(0 + 3) + 4(4 + 9 + 15 + 8) + 2(7 + 12 + 14)]$$

$$= 710$$

Hence the required area of the cross-section of the river = 710 sq. m.

Example 15. *Evaluate* $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\,dx$ *approximately using Weddle's rule correct to 4 decimals.*

Sol. Let $f(x) = \sin x - \log x + e^x$. Divide the given interval of integration into 12 equal parts so that the arguments are: 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4.

The corresponding entries are

$y_0 = f(0.2) = 3.0295,\quad y_1 = f(0.3) = 2.8494,\quad y_2 = f(0.4) = 2.7975,$

$y_3 = f(0.5) = 2.8213,\quad y_4 = f(0.6) = 2.8976,\quad y_5 = f(0.7) = 3.0147$

$y_6 = f(0.8) = 3.1661,\quad y_7 = f(0.9) = 3.3483,\quad y_8 = f(1) = 3.5598,$

$y_9 = f(1.1) = 3.8001,\quad y_{10} = f(1.2) = 4.0698,\quad y_{11} = f(1.3) = 4.3705$

$y_{12} = f(1.4) = 4.7042$

Now, by Weddle's rule,

$$\int_{0.2}^{1.4} f(x)\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 + y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$$

$$= \frac{3}{10}(0.1)[3.0295 + 14.2470 + 2.7975 + 16.9278 + 2.8976 + 15.0735 + 3.1661 + 3.1661 + 16.7415 + 3.5598 + 22.8006 + 4.0698 + 21.8525 + 4.7042]$$

$$= (0.03)[135.0335] = 4.051.$$

Example 16. *A solid of revolution is formed by rotating about x-axis, the lines x = 0 and x = 1 and a curve through the points with the following coordinates.*

x:	*0*	*0.25*	*0.5*	*0.75 1*	
y:	*1*	*0.9896*	*0.9589*	*0.9089*	*0.8415*

Estimate the volume of the solid formed using Simpson's rule.

Sol. If V is the volume of the solid formed then we know that

$$V = \pi \int_0^1 y^2\,dx$$

Hence we need the values of y^2 and these are tabulated below correct to four decimal places

x	0	.25	.5	.75	1
y^2	1	.9793	.9195	.8261	.7081

with $h = 0.25$, Simpson's rule gives

$$V = \pi \frac{(0.25)}{3}[(1 + .7081) + 4(.9793 + .8261) + 2(.9195)]$$

$$= 2.8192.$$

Example 17. *A tank is discharging water through an orifice at a depth of x meter below the surface of the water whose area is A m^2. Following are the values of x for the corresponding values of A.*

A:	*1.257*	*1.39*	*1.52*	*1.65*	*1.809*	*1.962*	*2.123*	*2.295*	*2.462*	*2.650*	*2.827*
x:	*1.5*	*1.65*	*1.8*	*1.95*	*2.1*	*2.25*	*2.4*	*2.55*	*2.7*	*2.85*	*3*

Using the formula $(0.018)\ T = \int_{1.5}^{3.0} \frac{A}{\sqrt{x}}\, dx$, *calculate T, the time (in seconds) for the level of the water to drop from 3.0 m to 1.5 m above the orifice.*

Sol. Here $h = 0.15$

The table of values of x and the corresponding values of $\frac{A}{\sqrt{x}}$ is

x	1.5	1.65	1.8	1.95	2.1	2.25	2.4	2.55	2.7	2.85	3
$y = \frac{A}{\sqrt{x}}$	1.025	1.081	1.132	1.182	1.249	1.308	1.375	1.438	1.498	1.571	1.632

Using Simpson's $\frac{1}{3}$rd rule, we get

$$\int_{1.5}^{3} \frac{A}{\sqrt{x}}\, dx = \frac{.15}{3}[(1.025+1.632) + 4(1.081 + 1.182 + 1.308 + 1.438 + 1.571) + 2(1.132 + 1.249 + 1.375 + 1.498)]$$

$$= 1.9743$$

Using the formula

$$(0.018)\mathrm{T} = \int_{1.5}^{3} \frac{A}{\sqrt{x}}\, dx$$

We get $\quad 0.018\mathrm{T} = 1.9743 \quad \Rightarrow \quad \mathrm{T} = 110$ sec. (approximately).

Example 18. *Using the following table of values, approximate by Simpson's rule, the arc length of the graph* $y = \frac{1}{x}$ *between the points* $(1, 1)$ *and* $\left(5, \frac{1}{5}\right)$

x:	*1*	*2*	*3*	*4*	*5*
$\sqrt{\frac{1+x^4}{x^4}}$:	*1.414*	*1.031*	*1.007*	*1.002*	*1.001.*

Sol. The given curve is

$$y = \frac{1}{x}$$

$$\therefore \qquad \frac{dy}{dx} = -\frac{1}{x^2}$$

$$\therefore \qquad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{x^4}} = \sqrt{\frac{1+x^4}{x^4}}$$

$\therefore$ The arc length of the curve between the points (1, 1) and $\left(5, \frac{1}{5}\right)$

$$= \int_1^5 \sqrt{\frac{1+x^4}{x^4}}\, dx$$

$$= \frac{h}{3}[(1.414 + 1.001) + 4(1.031 + 1.002) + 2(1.007)]$$

$$= \frac{1}{3}(2.415 + 8.132 + 2.014) = 4.187$$

Example 19. *From the following values of $y = f(x)$ in the given range of values of x, find the position of the centroid of the area under the curve and the x-axis*

x:	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y:	1	4	8	4	1

Also find

(*i*) *the volume of solid obtained by revolving the above area about x-axis.*

(*ii*) *the moment of inertia of the area about x-axis.*

Sol. Centroid of the plane area under the curve $y = f(x)$ is given by $(\bar{x}, \bar{y})$ where

$$\left.\begin{aligned} \bar{x} &= \frac{\int_0^1 xy\, dx}{\int_0^1 y\, dx} \\ \text{and} \qquad \bar{y} &= \frac{\int_0^1 \frac{y}{2} . y\, dx}{\int_0^1 y\, dx} = \frac{\int_0^1 \frac{y^2}{2}\, dx}{\int_0^1 y\, dx} \end{aligned}\right] \qquad (50)$$

From the given data, we obtain

x:	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y:	1	4	8	4	1
xy:	0	1	4	3	1
$\frac{y^2}{2}$:	$\frac{1}{2}$	8	32	8	$\frac{1}{2}$

$\therefore$ By Simpson's rule,

$$\int_0^1 xy\,dx = \frac{(1/4)}{3}[(0+1)+4(1+3)+2(4)] = \frac{25}{12}$$

$$\int_0^1 \frac{y^2}{2}\,dx = \frac{1}{12}\left[\left(\frac{1}{2}+\frac{1}{2}\right)+4(8+8)+2(32)\right] = \frac{129}{12}$$

$$\int_0^1 y\,dx = \frac{1}{12}[(1+1)+4(4+4)+2(8)] = \frac{50}{12}$$

From (50),
$$\bar{x} = \frac{25/12}{50/12} = \frac{1}{2} = 0.5$$

$$\bar{y} = \frac{129/12}{50/12} = \frac{129}{50} = 2.58$$

$\therefore$ Centroid is the point (0.5, 2.58).

(*i*) We know that

$$\text{V} = \text{Volume} = \pi \int_0^1 y^2\,dx$$

$\therefore$ Required volume $= \pi . 2\int_0^1 \frac{y^2}{2}\,dx = 2\pi \times \frac{129}{12} = 67.5442$

(*ii*) We know that moment of inertia of the area about the x-axis is given by

$$\text{M.I.} = \frac{1}{3}\rho \int_a^b y^3\,dx$$

where ρ is the mass per unit area.

Table for y^3 is

x:	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y:	1	4	8	4	1
y^3:	1	64	512	64	1

$$\int_0^1 y^3\, dx = \frac{1}{12}[(1 + 1) + 4(64 + 64) + 2(512)] = \frac{769}{6}$$

$$\therefore \qquad \text{Reqd. M.I.} = \frac{1}{3}\rho\left(\frac{769}{6}\right) = \frac{769}{18}\rho = 42.7222\,\rho.$$

Example 20. *A reservoir discharging water through sluices at a depth h below the water surface, has a surface area A for various values of h as given below:*

h (in meters):	*10*	*11*	*12*	*13*	*14*
A (in sq. meters):	*950*	*1070*	*1200*	*1350*	*1530*

If t denotes time in minutes, the rate of fall of the surface is given by

$$\frac{dh}{dt} = -\frac{48}{A}\sqrt{h}$$

Estimate the time taken for the water level to fall from 14 to 10 m above the sluices.

Sol. From $\qquad \dfrac{dh}{dt} = -\dfrac{48}{\text{A}}\sqrt{h}$, we have

$$dt = -\frac{\text{A}}{48}\frac{dh}{\sqrt{h}}$$

Integration yields,

$$t = -\frac{1}{48}\int_{14}^{10}\frac{\text{A}}{\sqrt{h}}\,dh = \frac{1}{48}\int_{10}^{14}\frac{\text{A}}{\sqrt{h}}\,dh$$

Here, $y = \dfrac{\text{A}}{\sqrt{h}}$. The table of values is as follows:

h:	10	11	12	13	14
A:	950	1070	1200	1350	1530
$\frac{A}{\sqrt{h}}$:	300.4164	322.6171	346.4102	374.4226	408.9097

Applying Simpson's $\frac{1}{3}$rd rule, we have

$$\text{time } t = \frac{1}{48} \cdot \frac{1}{3}[(300.4164 + 408.9097) + 4(322.6171 + 374.4226) + 2(346.4102)]$$

$$= 29.0993 \text{ minutes.}$$

ASSIGNMENT 5.2

1. Evaluate $\int_1^2 \frac{1}{x}\, dx$ by Simpson's $\frac{1}{3}$rd rule with four strips and determine the error by direct integration.

2. Evaluate the integral $\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta$ by dividing the interval into 6 parts.

3. Evaluate $\int_4^{5.2} \log_e x\, dx$ by Simpson's $\frac{3}{8}$th rule. Also write its programme in 'C' language.

4. Evaluate $\int_{30°}^{90°} \log_{10} \sin x\, dx$ by Simpson's $\frac{1}{3}$rd rule by dividing the interval into 6 parts.

5. Evaluate $\int_4^{5.2} \log_e x\, dx$ using

(*i*) Trapezoidal rule (*ii*) Weddle's rule.

6. Evaluate using Trapezoidal rule

(*i*) $\int_0^{\pi} t \sin t\, dt$ (*ii*) $\int_{-2}^{2} \frac{t\, dt}{5 + 2t}$

7. Evaluate $\int_3^7 x^2 \log x\, dx$ taking 4 strips.

8. The velocities of a car running on a straight road at intervals of 2 minutes are given below:

Time (in minutes):	0	2	4	6	8	10	12
Velocity (in km/hr):	0	22	30	27	18	7	0

Apply Simpson's rule to find the distance covered by the car.

9. Evaluate $\int_0^1 \cos x\, dx$ using $h = 0.2$.

10. Evaluate $\int_0^4 e^x\, dx$ by Simpson's rule, given that $e = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.6$ and compare it with the actual value.

11. Find an approximate value of $\log_e 5$ by calculating to 4 decimal places, by Simpson's $\frac{1}{3}$rd rule, $\int_0^5 \frac{dx}{4x+5}$ dividing the range into 10 equal parts.

12. Use Simpson's rule, taking five ordinates, to find an approximate value of $\int_1^2 \sqrt{x - \frac{1}{x}}\, dx$ to 2 decimal places.

13. Evaluate $\int_0^{\pi/2} \sqrt{\sin x}\, dx$ given that

x:	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
$\sqrt{sin\ x}$:	0	0.5087	0.7071	0.8409	0.9306	0.9878	1

14. The velocity of a train which starts from rest is given by the following table, time being reckoned in minutes from the start and speed in kilometers per hour:

Minutes:	0	2	4	6	8	10	12	14	16	18	20
Speed (km/hr):	0	10	18	25	29	32	20	11	5	2	0

Estimate the total distance in 20 minutes. $\left[\textbf{Hint:}\ \text{Here step-size } h = \frac{2}{60}\right]$

15. A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the following table. Using Simpson's $\frac{1}{3}$rd rule, find the velocity of the rocket at $t = 80$ seconds.

$t(sec)$:	0	10	20	30	40	50	60	70	80
$f(cm/sec^2)$:	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67.

16. A curve is drawn to pass through the points given by the following table:

x:	1	1.5	2	2.5	3	3.5	4
y:	2	2.4	2.7	2.8	3	2.6	2.1

Find

(*i*) Center of gravity of the area.

(*ii*) Volume of the solid of revolution.

(*iii*) The area bounded by the curve, the x-axis and lines $x = 1$, $x = 4$.

17. In an experiment, a quantity G was measured as follows:

G(20) = 95.9, G(21) = 96.85, G(22) = 97.77

G(23) = 98.68, G(24) = 99.56, G(25) = 100.41, G(26) = 101.24.

Compute $\int_{20}^{26} G(x)\, dx$ by Simpson's and Weddle's rule, respectively.

18. Using the data of the following table, compute the integral $\int_{0.5}^{1.1} xy\,dx$ by Simpson's rule:

x:	0.5	0.6	0.7	0.8	0.9	1.0	1.1
y:	0.4804	0.5669	0.6490	0.7262	0.7985	0.8658	0.9281

19. Find the value of $\log_e 2$ from $\int_0^{1.} \frac{x^2}{1+x^3}\,dx$ using Simpson's $\frac{1}{3}$rd rule by dividing the range of integration into four equal parts. Also find the error.

20. Use Simpson's rule dividing the range into ten equal parts to show that

$$\int_0^1 \frac{\log(1+x^2)}{1+x^2}\,dx = 0.173$$

21. Find by Weddle's rule the value of the integral

$$I = \int_{0.4}^{1.6} \frac{x}{\sinh x}\,dx$$

by taking 12 sub-intervals.

22. Evaluate $\int_{0.5}^{0.7} x^{1/2} e^{-x}\,dx$ approximately by using a suitable formula.

23. (*i*) Compute the integral

$$I = \sqrt{\frac{2}{\pi}} \int_0^1 e^{-(x^2/2)}\,dx$$

Using Simpson's $\frac{1}{3}$rd rule, taking $h = 0.125$.

(*ii*) Compute the value of I given by

$$I = \int_{0.2}^{1.5} e^{-x^2}\,dx$$

Using Simpson's $\left(\frac{1}{3}\right)$ rule with four subdivisions.

24. Using Simpson's $\frac{1}{3}$rd rule, Evaluate the integrals:

(*i*) $\int_{1.0}^{1.8} \frac{e^x + e^{-x}}{2}\,dx$ (taking $h = 0.2$)

(*ii*) $\int_0^{\pi/2} \frac{dx}{\sin^2 x + \frac{1}{4}\cos^2 x}$

25. Evaluate $\int_0^1 \sqrt{\sin x + \cos x}\, dx$ correct to two decimal places using seven ordinates.

26. Use Simpson's three-eighths rule to obtain an approximate value of

$$\int_0^{0.3} (1 - 8x^3)^{1/2}\, dx$$

27. Evaluate $\int_0^{1/2} \frac{dx}{\sqrt{1 - x^2}}$ using Weddle's rule.

28. Evaluate $\int_0^1 \frac{x^2 + 2}{x^2 + 1}\, dx$ using Weddle's rule correct to four places of decimals.

29. Using $\frac{3}{8}$th Simpson's rule,

Evaluate: $\int_0^6 \frac{dx}{1 + x^4}$.

30. Apply Simpson's $\frac{1}{3}$rd rule to evaluate the integral

$$\mathrm{I} = \int_0^1 e^x\, dx \text{ by choosing step size } h = 0.1$$

Show that this step size is sufficient to obtain the result correct to five decimal places.

31. (*i*) Obtain the global truncation error term of trapezoidal method of integration.

(*ii*) Compute the approximate value of the integral

$$l = \int (1 + x + x^2)\, dx$$

Using Simpson's rule by taking interval size h as 1. Write a C program to implement.

32. The function $f(x)$ is known at one point x^* in the interval $[a, b]$. Using this value, $f(x)$ can be expressed as

$$f(x) = p_0(x) + f'\{\xi(x)\}\,(x - x^*) \quad \text{for} \quad x \in (a, b)$$

where $p_0(x)$ is the zeroth-order interpolating polynomial $p_0(x) = f(x^*)$ and $\xi(x) \in (a, b)$. Integrate this expression from a to b to derive a quadrature rule with error term. Simplify the error term for the case when $x^* = a$.

5.25 EULER-MACLAURIN'S FORMULA

This formula is based on the expansion of operators. Suppose $\Delta F(x) = f(x)$, then an operator Δ^{-1}, called inverse operator, is defined as

$$F(x) = \Delta^{-1} f(x) \tag{51}$$

Also, $\Delta F(x) = f(x)$ gives

$$F(x_1) - F(x_0) = f(x_0)$$

Similarly,

$$\begin{aligned} F(x_2) - F(x_1) &= f(x_1) \\ \vdots \quad\quad \vdots \quad &\quad \vdots \\ F(x_n) - F(x_{n-1}) &= f(x_{n-1}) \end{aligned}$$

On adding,

$$F(x_n) - F(x_0) = \sum_{i=0}^{n-1} f(x_i) \tag{52}$$

where $x_0, x_1, ..., x_n$ are the $(n + 1)$ equidistant values of x with difference h.

From (51),

$$\begin{aligned} F(x) &= (E - 1)^{-1} f(x) \\ &= (e^{hD} - 1)^{-1} f(x) \\ &= \left[\left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + ...\right) - 1\right]^{-1} f(x) \\ &= \left[hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + ...\right]^{-1} f(x) \\ &= (hD)^{-1}\left[1 + \left(\frac{hD}{2!} + \frac{h^2D^2}{3!} + ...\right)\right]^{-1} f(x) \\ &= \frac{1}{h} D^{-1}\left[1 - \left(\frac{hD}{2!} + \frac{h^2D^2}{3!} + ...\right)\right. \\ &\quad \left. + \frac{(-1)(-2)}{2!}\left(\frac{hD}{2!} + \frac{h^2D^2}{3!} + ...\right)^2 + ...\right] f(x) \\ &= \frac{1}{h} D^{-1}\left[1 - \frac{hD}{2} + \frac{h^2D^2}{12} - \frac{h^4D^4}{720} + ...\right] f(x) \end{aligned}$$

$$F(x) = \frac{1}{h}\int f(x)\,dx - \frac{1}{2} f(x) + \frac{h}{12} f'(x) - \frac{h^3}{720} f'''(x) + ... \tag{53}$$

Putting $x = x_n$ and $x = x_0$ in (53) and then subtracting, we get

$$F(x_n) - F(x_0) = \frac{1}{h}\int_{x_0}^{x_n} f(x)\,dx - \frac{1}{2}[f(x_n) - f(x_0)] + \frac{h}{12}[f'(x_n) - f'(x_0)]$$

$$- \frac{h^3}{720}[f'''(x_n) - f'''(x_0)] + \ldots\ldots$$

$$\Rightarrow \qquad \sum_{i=0}^{n-1} f(x_i) = \frac{1}{h}\int_{x_0}^{x_n} f(x)\,dx - \frac{1}{2}[f(x_n) - f(x_0)] + \frac{h}{12}[f'(x_n) - f'(x_0)]$$

$$- \frac{h^3}{720}[f'''(x_n) - f'''(x_0)] + \ldots\ldots \quad | \text{ using (52)}$$

$$\Rightarrow \quad \frac{1}{h}\int_{x_0}^{x_n} f(x)\,dx = \sum_{i=0}^{n-1} f(x_i) + \frac{1}{2}[f(x_n) - f(x_0)] - \frac{h}{12}[f'(x_n) - f'(x_0)]$$

$$+ \frac{h^3}{720}[f'''(x_n) - f'''(x_0)] - \ldots\ldots \qquad (54)$$

or
$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[y_0 + 2y_1 + 2y_2 + \ldots\ldots + y_n]$$

$$- \frac{h^2}{12}(y_n' - y_0') + \frac{h^4}{720}(y_n''' - y_0''') - \ldots\ldots$$

$$= \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots + y_{n-1})]$$

$$- \frac{h^2}{12}(y_n' - y_0') + \frac{h^4}{720}(y_n''' - y_0''') - \ldots\ldots \qquad (55)$$

which is called **Euler-Maclaurin's formula.** The first term on the R.H.S. of (55) represents the approximate value of the integral obtained from trapezoidal rule and the other terms denote the successive corrections to this value.

This formula is often used to find the sum of a series of the form

$$y(x_0) + y(x_0 + h) + y(x_0 + 2h) + \ldots\ldots + y(x_0 + nh).$$

5.26 GAUSSIAN QUADRATURE FORMULA

Consider the numerical evaluation of the integral

$$\int_a^b f(x)\,dx \qquad (56)$$

So far, we studied some integration formulae which require values of the function at equally spaced points of the interval. Gauss derived a formula which uses the same number of function values but with different spacing and gives better accuracy.

Gauss's formula is expressed in the form

$$\int_{-1}^{1} \mathrm{F}(u)\, du = \mathrm{W}_1\, \mathrm{F}(u_1) + \mathrm{W}_2\, \mathrm{F}(u_2) + \ldots\ldots + \mathrm{W}_n\, \mathrm{F}(u_n)$$

$$= \sum_{i=1}^{n} \mathrm{W}_i\, \mathrm{F}(u_i) \tag{57}$$

where W_i and u_i are called the weights and abscissae respectively. The formula has an advantage that the abscissae and weights are symmetrical with respect to the middle point of the interval.

In equation (57), there are altogether $2n$ arbitrary parameters and therefore the weights and abscissae can be determined so that the formula is exact when $\mathrm{F}(u)$ is a polynomial of degree not exceeding $2n - 1$. Hence, we start with

$$\mathrm{F}(u) = \mathrm{C}_0 + \mathrm{C}_1 u + \mathrm{C}_2 u^2 + \mathrm{C}_3 u^3 + \ldots\ldots + \mathrm{C}_{2n-1} u^{2n-1} \tag{58}$$

Then from (57),

$$\int_{-1}^{1} \mathrm{F}(u)\, du = \int_{-1}^{1} (\mathrm{C}_0 + \mathrm{C}_1 u + \mathrm{C}_2 u^2 + \mathrm{C}_3 u^3 + \ldots\ldots + \mathrm{C}_{2n-1} u^{2n-1})\, du$$

$$= 2\,\mathrm{C}_0 + \frac{2}{3}\mathrm{C}_2 + \frac{2}{5}\mathrm{C}_4 + \ldots\ldots. \tag{59}$$

Set $u = u_i$ in (58), we get

$$\mathrm{F}(u_i) = \mathrm{C}_0 + \mathrm{C}_1 u_i + \mathrm{C}_2 u_i^2 + \mathrm{C}_3 u_i^3 + \ldots\ldots + \mathrm{C}_{2n-1} u_i^{2n-1}$$

From (57),

$$\begin{aligned}\int_{-1}^{1} \mathrm{F}(u)\, du &= \mathrm{W}_1 (\mathrm{C}_0 + \mathrm{C}_1 u_1 + \mathrm{C}_2 u_1^2 + \ldots\ldots.. + \mathrm{C}_{2n-1} u_1^{2n-1}) \\ &\quad + \mathrm{W}_2 (\mathrm{C}_0 + \mathrm{C}_1 u_2 + \mathrm{C}_2 u_2^2 + \ldots\ldots + \mathrm{C}_{2n-1} u_2^{2n-1}) \\ &\quad + \mathrm{W}_3 (\mathrm{C}_0 + \mathrm{C}_1 u_3 + \mathrm{C}_2 u_3^2 + \ldots\ldots + \mathrm{C}_{2n-1} u_3^{2n-1}) + \ldots\ldots \\ &\quad + \mathrm{W}_n (\mathrm{C}_0 + \mathrm{C}_1 u_n + \mathrm{C}_2 u_n^2 + \ldots\ldots + \mathrm{C}_{2n-1} u_n^{2n-1})\end{aligned}$$

which can be written as

$$\begin{aligned}\int_{-1}^{1} \mathrm{F}(u)\, du &= \mathrm{C}_0 (\mathrm{W}_1 + \mathrm{W}_2 + \ldots\ldots. + \mathrm{W}_n) + \mathrm{C}_1(\mathrm{W}_1 u_1 + \mathrm{W}_2 u_2 \\ &\quad + \mathrm{W}_3 u_3 + \ldots\ldots + \mathrm{W}_n u_n) + \mathrm{C}_2(\mathrm{W}_1 u_1^2 + \mathrm{W}_2 u_2^2 \\ &\quad + \mathrm{W}_3 u_3^2 + \ldots\ldots + \mathrm{W}_n u_n^2) + \ldots\ldots \\ &\quad + \mathrm{C}_{2n-1}(\mathrm{W}_1 u_1^{2n-1} + \mathrm{W}_2 u_2^{2n-1} \\ &\quad + \mathrm{W}_3 u_3^{2n-1} + \ldots\ldots + \mathrm{W}_n u_n^{2n-1})\end{aligned} \tag{60}$$

Now equations (59) and (60) are identical for all values of C_i and hence comparing the coefficients of C_i, we obtain $2n$ equations

$$\left.\begin{aligned} W_1 + W_2 + W_3 + \ldots\ldots. + W_n &= 2 \\ W_1 u_1 + W_2 u_2 + W_3 u_3 + \ldots\ldots. + W_n u_n &= 0 \\ W_1 u_1^2 + W_2 u_2^2 + W_3 u_3^2 + \ldots\ldots.. + W_n u_n^2 &= \frac{2}{3} \\ \vdots \qquad \vdots \qquad \vdots \\ W_1 u_1^{2n-1} + W_2 u_2^{2n-1} + W_3 u_3^{2n-1} + \ldots\ldots.. + W_n u_n^{2n-1} &= 0 \end{aligned}\right\} \quad (61)$$

in $2n$ unknowns W_i and u_i ($i = 1, 2, \ldots\ldots, n$).

The abscissae u_i and the weights W_i are extensively tabulated for different values of n.

The table up to $n = 5$ is given below:

n	$\pm u_i$	W_i
2	0.57735, 02692	1.0
	0.0	0.88888 88889
3	0.77459 66692	0.55555 55556
4	0.33998 10436	0.65214 51549
	0.86113 63116	0.34785 48451
	0.0	0.56888 88889
5	0.53846 93101	0.47862 86705
	0.90617 98459	0.23692 68851

In general case, the limits of integral in (56) have to be changed to those in (57) by transformation

$$x = \frac{1}{2} u (b - a) + \frac{1}{2}(a + b).$$

5.27 NUMERICAL EVALUATION OF SINGULAR INTEGRALS

The various numerical integration formulae we have discussed so far are valid if integrand $f(x)$ can be expanded by a polynomial or, alternatively can be expanded in a Taylor's series in the interval $[a, b]$. In a case where function has a singularity, the preceding formulae cannot be applied and special methods will have to be adopted.

5.28 EVALUATION OF PRINCIPAL VALUE INTEGRALS

Consider, $$\mathrm{I}(f) = \int_a^b \frac{f(x)}{x-t}\,dx \tag{62}$$

which is singular at $t = x$.

Its Principal value,

$$\mathrm{P(I)} = \lim_{\varepsilon \to 0}\left[\int_a^{t-\varepsilon} \frac{f(x)}{x-t}\,dx + \int_{t+\varepsilon}^b \frac{f(x)}{x-t}\,dx\right];\ a < t < b \tag{63}$$

$$= \mathrm{I}(f) \text{ (for } t < a \quad \text{or} \quad t > b)$$

Set $x = a + uh$ and $t = a + kh$ in (1), we get

$$\mathrm{P(I)} = \mathrm{P}\int_0^p \frac{f(a+hu)}{u-k}\,du \tag{64}$$

Replacing $f(a + hu)$ by Newton's forward difference formula at $x = a$ and simplifying, we get

$$\mathrm{I}(f) = \sum_{j=0}^{\infty} \frac{\Delta^j f(a)}{j\,!}\,\mathrm{C}_j \tag{65}$$

where the constants C_j are given by

$$\mathrm{C}_j = \mathrm{P}\int_0^p \frac{(u)_j}{u-k}\,du \tag{66}$$

In (66), $(u)_0 = 1, \quad (u)_1 = u, \quad (u)_2 = u\,(u-1)$ etc.

Various approximate formulae can be obtained by truncating the series on R.H.S. of (65).

Eqn. (65) may be written as

$$\mathrm{I}_n(f) = \sum_{j=0}^{n} \frac{\Delta^j f(a)}{j\,!}\,\mathrm{C}_j \tag{67}$$

We obtain rules of orders 1, 2, 3, ... etc. by setting n = 1, 2, 3, ... respectively.

(*i*) **Two point rule** (n = 1): $\mathrm{I}_1(f) = \sum_{j=0}^{1} \frac{\Delta^j f(a)}{j\,!}\,\mathrm{C}_j$

$$= \mathrm{C}_0 f(a) + \mathrm{C}_1 \Delta f(a)$$

$$= (\mathrm{C}_0 - \mathrm{C}_1) f(a) + \mathrm{C}_1 f(a+h) \tag{68}$$

(*ii*) **Three-point rule** ($n = 2$):

$$\mathrm{I}_2(f) = \sum_{j=0}^{2} \frac{\Delta^j f(a)}{j\,!}\, \mathrm{C}_j = \mathrm{C}_0 f(a) + \mathrm{C}_1\, \Delta f(a) + \mathrm{C}_2\, \Delta^2 f(a)$$

$$= \left(\mathrm{C}_0 - \mathrm{C}_1 + \frac{1}{2}\mathrm{C}_2\right) f(a) + (\mathrm{C}_1 - \mathrm{C}_2)\, f(a+h)$$

$$+ \frac{1}{2}\, \mathrm{C}_2\, f(a+2h) \qquad (69)$$

In above relations (68) and (69), values of C_j are given by,

$$\mathrm{C}_0 = \log_e \left|\frac{p-k}{k}\right|$$

$$\mathrm{C}_1 = p + \mathrm{C}_0\, k$$

$$\mathrm{C}_2 = \frac{1}{2}\, p^2 + p\,(k-1) + \mathrm{C}_0\, k\,(k-1)\,.$$

EXAMPLES

Example 1. *Apply Euler-Maclaurin formula to evaluate*

$$\frac{1}{51^2} + \frac{1}{53^2} + \frac{1}{55^2} + \ldots\ldots + \frac{1}{99^2}.$$

Sol. Take $\quad y = \dfrac{1}{x^2}$, $x_0 = 51$, $h = 2$, $n = 24$, we have

$$y' = -\frac{2}{x^3}, \qquad y''' = -\frac{24}{x^5}$$

Then from Euler-Maclaurin's formula,

$$\int_{51}^{99} \frac{dx}{x^2} = \frac{2}{2}\left[\frac{1}{51^2} + \frac{2}{53^2} + \frac{2}{55^2} + \ldots\ldots + \frac{2}{97^2} + \frac{1}{99^2}\right]$$

$$- \frac{(2)^2}{12}\left[\frac{(-2)}{(99)^3} - \frac{(-2)}{(51)^3}\right] + \frac{(2)^4}{720}\left[\frac{(-24)}{(99)^5} - \frac{(-24)}{(51)^5}\right]$$

$$\therefore \qquad \frac{1}{51^2} + \frac{2}{53^2} + \frac{2}{55^2} + \ldots\ldots + \frac{2}{97^2} + \frac{1}{99^2}$$

$$= \int_{51}^{99} \frac{dx}{x^2} + \frac{2}{3}\left[\frac{1}{(51)^3} - \frac{1}{(99)^3}\right] - \frac{8}{15}\left[\frac{1}{(51)^5} - \frac{1}{(99)^5}\right] + \ldots\ldots$$

$$\Rightarrow \quad 2\left[\frac{1}{51^2}+\frac{1}{53^2}+\frac{1}{55^2}+\ldots\ldots+\frac{1}{99^2}\right]$$

$$= \int_{51}^{99}\frac{dx}{x^2}+\left(\frac{1}{51^2}+\frac{1}{99^2}\right)+\frac{2}{3}\left[\frac{1}{(51)^3}-\frac{1}{(99)^3}\right]$$

$$-\frac{8}{15}\left[\frac{1}{(51)^5}-\frac{1}{(99)^5}\right]+\ldots\ldots$$

$$\Rightarrow \quad \frac{1}{(51)^2}+\frac{1}{(53)^2}+\frac{1}{(55)^2}+\ldots\ldots+\frac{1}{(99)^2}$$

$$= \frac{1}{2}\int_{51}^{99}\frac{dx}{x^2}+\frac{1}{2}\left[\frac{1}{(51)^2}+\frac{1}{(99)^2}\right]+\frac{1}{3}\left[\frac{1}{(51)^3}-\frac{1}{(99)^3}\right]$$

$$-\frac{4}{15}\left[\frac{1}{(51)^5}-\frac{1}{(99)^5}\right]+\ldots\ldots$$

$$= \frac{1}{2}\left(-\frac{1}{x}\right)_{51}^{99}+\frac{1}{2}\left[\frac{1}{(51)^2}+\frac{1}{(99)^2}\right]+\frac{1}{3}\left[\frac{1}{(51)^3}-\frac{1}{(99)^3}\right]$$

$$-\frac{4}{15}\left[\frac{1}{(51)^5}-\frac{1}{(99)^5}\right]+\ldots\ldots.$$

$$= 0.00475 + 0.000243 + 0.0000022 + \ldots\ldots$$

$$= 0.00499 \text{ approximately.}$$

Example 2. *Using Euler-Maclaurin's formula, find the value of* $\log_e 2$ *from* $\int_0^1 \frac{dx}{1+x}$.

Sol. Take $\qquad y = \dfrac{1}{1+x},\ x_0 = 0,\ n = 10,\ h = 0.1,$

we have $\qquad y' = -\dfrac{1}{(1+x)^2} \quad \text{and} \quad y''' = \dfrac{-6}{(1+x)^4}$

Then from Euler-Maclaurin's formula, we have

$$\int_0^1 \frac{dx}{1+x} = \frac{0.1}{2}\left[\frac{1}{1+0}+\frac{2}{1+0.1}+\frac{2}{1+0.2}+\frac{2}{1+0.3}+\frac{2}{1+0.4}\right.$$

$$\left.+\frac{2}{1+0.5}+\frac{2}{1+0.6}+\frac{2}{1+0.7}+\frac{2}{1+0.8}+\frac{2}{1+0.9}+\frac{1}{1+1}\right]$$

$$-\frac{(0.1)^2}{12}\left[\frac{(-1)}{(1+1)^2}-\frac{(-1)}{(1+0)^2}\right]+\frac{(0.1)^4}{720}\left[\frac{(-6)}{(1+1)^4}-\frac{(-6)}{(1+0)^4}\right]$$

$$= 0.693773 - 0.000625 + 0.000001 = 0.693149$$

Also, $\int_0^1 \frac{dx}{1+x} = \left| \log(1+x) \right|_0^1 = \log 2$

Hence $\log_e 2 = 0.693149.$

Example 3. *Evaluate* $\int_0^{\pi/2} \sin x \, dx$ *using the Euler-Maclaurin formula.*

Sol. $$\int_0^{\pi/2} \sin x \, dx = \frac{h}{2}[y_0 + 2y_1 + 2y_2 + ... + 2y_{n-1} + y_n]$$

$$+\frac{h^2}{12}+\frac{h^4}{720}+\frac{h^6}{30240}+ ......$$

To evaluate the integral, let us take $h = \frac{\pi}{4}$.

Then we obtain,

$$\int_0^{\pi/2} \sin x \, dx = \frac{\pi}{8}(0+2+0)+\frac{\pi^2}{192}+\frac{\pi^4}{184320}+ ......$$

$$= \frac{\pi}{4}+\frac{\pi^2}{192}+\frac{\pi^4}{184320} \text{ (approximately)}$$

$$= 0.785398 + 0.051404 + 0.000528 = 0.837330$$

If we take $h = \frac{\pi}{8}$, we get

$$\int_0^{\pi/2} \sin x \, dx = \frac{\pi}{16}[0 + 2(0.382683 + 0.707117 + 0.923879) + 1]$$

$$= 0.987119 + 0.012851 + 0.000033 = 1.000003.$$

Example 4. *Use Euler-Maclaurin's formula to prove that*

$$\sum_{1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}.$$

Sol. By Euler–Maclaurin's formula,

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[y_0 + 2y_1 + 2y_2 + \ldots\ldots\ldots + 2y_{n-1} + y_n] - \frac{h^2}{12}(y_n' - y_0')$$

$$+ \frac{h^4}{720}(y_n''' - y_0''') - \frac{h^6}{30240}(y_n^{(v)} - y_0^{(v)}) + \ldots..$$

$$\Rightarrow \quad \frac{1}{2}y_0 + y_1 + y_2 + \ldots\ldots + y_{n-1} + \frac{1}{2}y_n$$

$$= \frac{1}{h}\int_{x_0}^{x_n} y\,dx + \frac{h}{12}(y_n' - y_0') - \frac{h^3}{720}(y_n''' - y_0''')$$

$$+ \frac{h^5}{30240}(y_n^{(v)} - y_0^{(v)}) - \ldots\ldots. \qquad (70)$$

Here $\quad y(x) = x^2,\ y'(x) = 2x$ and $h = 1$

$\therefore$ From (70),

$$\text{Sum} = \int_1^n x^2\,dx + \frac{1}{2}(n^2 + 1) + \frac{1}{12}(2n - 2)$$

$$\Bigg|\ \because\ \frac{1}{2}y_0 = \frac{1}{2},\ \frac{1}{2}y_n = \frac{n^2}{2}$$

$$= \frac{1}{3}(n^3 - 1) + \frac{1}{2}(n^2 + 1) + \frac{1}{6}(n - 1) = \frac{n(n+1)(2n+1)}{6}.$$

Example 5. *Find* $\int_0^1 x\,dx$ *by Gaussian formula.*

Sol. Let us change the limits as

$$x = \frac{1}{2}u(1 - 0) + \frac{1}{2}(1 + 0) = \frac{1}{2}(u + 1)$$

This gives,

$$\text{I} = \frac{1}{4}\int_{-1}^{1}(u + 1)\,du = \frac{1}{4}\sum_{i=1}^{n} \text{W}_i\, \text{F}(u_i)$$

where $\qquad \text{F}(u_i) = u_i + 1$

For simplicity, let $n = 4$ and using the abscissae and weights corresponding to $n = 4$ in the table, we get

$$I = \frac{1}{4}[(-0.86114 + 1)(0.34785) + (-0.33998 + 1)(0.65214) + (0.33998 + 1)(0.65214) + (0.86114 + 1)(0.34785)]$$

$$= 0.49999 \ldots\ldots$$

where the abscissae and weights have been rounded to 5 decimal places.

Example 6. *Show that the integration formula* $\int_0^h f(x)\,dx = hf\left(\frac{h}{2}\right)$ *is exact for all polynomials of degree less than or equal to 1. Obtain an estimate for the truncation error.*

If $|f''(x)| < 1$ *for all x, then find the step size h so that the truncation error is less than* 10^{-3}.

Sol. If $f(x) = k$ (a constant or zero degree polynomial) then the result is obvious since

$$\int_0^h f(x)\,dx = kh \qquad (71)$$

and

$$hf\left(\frac{h}{2}\right) = hk \qquad (72)$$

$\therefore$ From (71) and (72),

$$\int_0^h f(x)\,dx = hf\left(\frac{h}{2}\right)$$

If $f(x)$ is a polynomial of degree one then

$$f(x) = ax + b$$

$$\int_0^h f(x)\,dx = \int_0^h (ax + b)\,dx = \frac{ah^2}{2} + bh \qquad (73)$$

$$hf\left(\frac{h}{2}\right) = h\left(\frac{ah}{2} + b\right) = \frac{ah^2}{2} + bh \qquad (74)$$

From (73) and (74), we have the result.

Now,
$$\int_0^h y\,dx = \int_0^h \left[y_0 + (x - x_0)\,y_0' + \frac{(x - x_0)^2}{2}\,y_0'' + \ldots\ldots\right] dx$$

$$= hy_0 + \frac{h^2}{2!} y_0' + \frac{h^3}{3!} y_0'' + \ldots.. \tag{75}$$

(where $x - x_0 = h$)

Also, $$hf\left(\frac{h}{2}\right) = h\left[y_0 + \frac{h}{2} y_0' + \frac{\left(\frac{h}{2}\right)^2}{2!} y_0'' + \ldots..\right] \tag{76}$$

(75) – (76) gives the truncation error

$$= h^3\left(\frac{1}{6} - \frac{1}{8}\right) y_0'' \text{ (nearly)}$$

Now, $$\left|\frac{h^3}{24} y_0''\right| < \frac{1}{24} h^3$$

$$\Rightarrow \quad \frac{1}{24} h^3 < 10^{-3} \quad \text{or} \quad |\, h^3 \,| < 24 \times 10^{-3} = 0.024$$

$$\Rightarrow \quad -\sqrt[3]{0.024} < h < \sqrt[3]{0.024}.$$

Example 7. *Find λ such that the quadrature formula* $\int_0^1 \frac{f(x)}{\sqrt{x}} dx \approx Af(0) + Bf(\lambda)$ *+ Cf(1) may be exact for polynomials of degree 3.*

Sol. $$\int_0^1 \frac{f(x)}{\sqrt{x}} dx = \mathrm{A}f(0) + \mathrm{B}f(\lambda) + \mathrm{C}f(1)$$

Set $f(x) = 1, x, x^2$ and x^3 in turn,

$$2 = \mathrm{A} + \mathrm{B} + \mathrm{C} \tag{77}$$

$$\frac{2}{3} = \mathrm{B}\lambda + \mathrm{C} \tag{78}$$

$$\frac{2}{5} = \mathrm{B}\lambda^2 + \mathrm{C} \tag{79}$$

$$\frac{2}{7} = \mathrm{B}\lambda^3 + \mathrm{C} \tag{80}$$

Subtracting (78) from (79), we get

$$\mathrm{B}\lambda(\lambda - 1) = -\frac{4}{15}$$

Subtracting (79) from (80), we get

$$B\lambda^2(\lambda - 1) = -\frac{4}{35}$$

$$\therefore \qquad \lambda = \frac{3}{7}.$$

Example 8. *Determine W_0, W_1 and W_2 as functions of α such that the error R in*

$$\int_{-1}^{1} f(x)\, dx = W_0 f(-\alpha) + W_1 f(0) + W_2 f(\alpha) + R,\ \alpha \neq 0$$

Vanishes when f(x) is an arbitrary polynomial of degree at most 3. Show that the precision is five when $\alpha = \sqrt{\frac{3}{5}}$ and three otherwise.

Compute the error R when $\alpha = \sqrt{\frac{3}{5}}$.

Sol. $\int_{-1}^{1} f(x)\, dx = W_0 f(-\alpha) + W_1 f(0) + W_2 f(\alpha)$ is exact for $f(x) = 1, x, x^2, x^3$.

$$f(x) = 1 \quad \Rightarrow \quad W_0 + W_1 + W_2 = 2$$

$$f(x) = x \quad \Rightarrow \quad W_0 = W_2$$

$$f(x) = x^2 \quad \Rightarrow \quad 2W_0\alpha^2 = \frac{2}{3}$$

$$f(x) = x^3 \quad \Rightarrow \quad W_0 = W_2$$

Solving, we find

$$W_0 = W_2 = \frac{1}{3\alpha^2}, \qquad W_1 = 2\left(1 - \frac{1}{3\alpha^2}\right)$$

Choosing $f(x) = x^4$, we get

$$\frac{2}{5} = 2W_0\alpha^4 = \frac{2}{3}\alpha^2$$

$$\Rightarrow \qquad \alpha = \sqrt{\frac{3}{5}}$$

With this value, $f(x) = x^5$ gives exact value.

$\therefore$ The precision is 5.

If $\quad \alpha \neq \sqrt{\frac{3}{5}}$ the precision is 3.

With $\quad \alpha = \sqrt{\frac{3}{5}}$, we have

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\sqrt{\frac{3}{5}}\right)\right]+\frac{8}{9}f(0)+\text{R}$$

Hence the error term R is given by

$$\text{R} = \frac{2}{7!}f^{(vi)}(0) + \text{terms involving higher order derivatives}$$

$$= \frac{f^{(vi)}(0)}{2520}.$$

Example 9. *Determine a, b and c such that the formula*

$$\int_0^h f(x)\,dx = h\left\{af(0)+bf\left(\frac{h}{3}\right)+cf(h)\right\}$$

is exact for polynomials of as high order as possible and determine the order of truncation error.

Sol. Making the method exact for polynomials of degree up to 2, we get

For $f(x) = 1$: $\quad h = h(a+b+c) \quad \Rightarrow \quad a+b+c=1$

For $f(x) = x$: $\quad \frac{h^2}{2} = h\left(\frac{bh}{3}+ch\right) \quad \Rightarrow \quad \frac{b}{3}+c=\frac{1}{2}$

For $f(x) = x^2$: $\quad \frac{h^3}{3} = h\left(\frac{bh^2}{9}+ch^2\right) \quad \Rightarrow \quad \frac{b}{9}+c=\frac{1}{3}$

Solving above eqns., we get

$$a = 0,\ b = \frac{3}{4},\ c = \frac{1}{4}$$

Truncation error of the formula $= \frac{c}{3!}f'''(\xi); \qquad 0 < \xi < h$

and

$$c = \int_0^h x^3\,dx - h\left(\frac{bh^3}{27}+ch^3\right) = -\frac{h^4}{36}$$

Hence, we have

Truncation error $= -\dfrac{h^4}{216} f'''(\xi) = 0\,(h^4).$

ASSIGNMENT 5.3

1. Using Euler-Maclaurin's formula, evaluate

(*i*) $\dfrac{1}{400} + \dfrac{1}{402} + \dfrac{1}{404} + \ldots\ldots + \dfrac{1}{500}$ (*ii*) $\dfrac{1}{(201)^2} + \dfrac{1}{(203)^2} + \dfrac{1}{(205)^2} + \ldots\ldots + \dfrac{1}{(299)^2}$.

2. Prove that $\sum_{1}^{n} x^3 = \left\{\dfrac{n(n+1)}{2}\right\}^2$ applying Euler-Maclaurin's formula.

3. Use Euler-Maclaurin's formula to find the value of π from the formula

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}.$$

4. Find the sum of the fourth powers of first n natural numbers by means of Euler-Maclaurin's formula.

OR

Prove that, $\sum_{0}^{n} i^4 = \dfrac{n^5}{5} + \dfrac{n^4}{2} + \dfrac{n^3}{3} + \dfrac{n}{30}.$

5. Sum the series $\dfrac{1}{100} + \dfrac{1}{101} + \dfrac{1}{102} + \dfrac{1}{103} + \dfrac{1}{104}.$

6. Determine α, β, γ and δ such that the relation

$$y'\left(\frac{a+b}{2}\right) = \alpha y(a) + \beta y(b) + \gamma y''(a) + \delta y''(b)$$

is exact for polynomials of as high degree as possible.

7. Find the values of α_0, α_1, α_2 so that the given rule of differentiation

$$f'(x_0) = \alpha_0 f_0 + \alpha_1 f_1 + \alpha_2 f_2 \quad (x_k = x_0 + kh)$$

is exact for $f \in P_2$.

8. Find the values a, b, c such that the truncation error in the formula

$$\int_{-h}^{h} f(x)\,dx = h\,[af(-h) + bf(0) + af(h) + h^2 c\,\{f'(-h) - f'(h)\}]$$

is minimized.

9. Show that $\sum_{i=1}^{n} i^7 + \sum_{i=1}^{n} i^5 = 2\left(\sum_{i=1}^{n} i^3\right)^2.$

10. Evaluate: $\sum_{m=0}^{\infty} \dfrac{1}{(10+m)^2}$ by applying Euler-Maclaurin's formula.