Chapter 4 INTERPOLATION

4.1 INTRODUCTION

According to Theile, *'Interpolation is the art of reading between the lines of the table'*.

It also means insertion or filling up intermediate terms of the series.

Suppose we are given the following values of $y = f(x)$ for a set of values of x:

x:	x_0	x_1	x_2	......	x_n
y:	y_0	y_1	y_2	......	y_n

Thus the process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called **interpolation.**

Hence interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable, while the process of computing the value of the function outside the given range is called **extrapolation.**

4.2 ASSUMPTIONS FOR INTERPOLATION

1. There are no sudden jumps or falls in the values during the period under consideration.
2. The rise and fall in the values should be uniform.

   For example, if we are given data regarding deaths in various years in a particular town and some of the observations are for the years in which epidemic or war overtook the town, then interpolation methods are not applicable.
3. When we apply calculus of finite differences, we assume that the given set of observations is capable of being expressed in a polynomial form.

   If the function $f(x)$ is known explicitly, the value of y corresponding to any value of x can be found easily.

   If the function $f(x)$ is not known, it is necessary to find a simpler function, say $\phi(x)$, such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. This process is called interpolation. If $\phi(x)$ is a polynomial, then the process is called polynomial interpolation and $\phi(x)$ is called the interpolating polynomial.

4.3 ERRORS IN POLYNOMIAL INTERPOLATION

Let the function $y(x)$ defined by $(n + 1)$ points (x_i, y_i) $i = 0, 1, 2, \ldots\ldots, n$ be continuous and differentiable $(n + 1)$ times and let $y(x)$ be approximated by a polynomial $\phi_n(x)$ of degree not exceeding n such that

$$\phi_n(x_i) = y_i; \; i = 0, 1, 2, \ldots.., n \tag{1}$$

The problem lies in finding the accuracy of this approximation if we use $\phi_n(x)$ to obtain approximate values of $y(x)$ at some points other than those defined above.

Since the expression $y(x) - \phi_n(x)$ vanishes for $x = x_0, x_1, \ldots\ldots, x_n$, we put

$$y(x) - \phi_n(x) = \text{L}\ \Pi_{n+1}(x) \tag{2}$$

where
$$\Pi_{n+1}(x) = (x - x_0)(x - x_1) \ldots\ldots (x - x_n) \tag{3}$$

and L is to be determined such that equation (2) holds for any intermediate value of x say x' where $x_0 < x' < x_n$.

Clearly,
$$\text{L} = \frac{y(x') - \phi_n(x')}{\Pi_{n+1}(x')} \tag{4}$$

Construct a function,
$$\text{F}(x) = y(x) - \phi_n(x) - \text{L}\ \Pi_{n+1}(x) \tag{5}$$

where L is given by (4).

4.14 NEWTON'S FORMULAE FOR INTERPOLATION

Newton's formula is used for constructing the interpolation polynomial. It makes use of divided differences. This result was first discovered by the Scottish mathematician James Gregory (1638–1675) a contemporary of Newton.

Gregory and Newton did extensive work on methods of interpolation but now the formula is referred to as Newton's interpolation formula. Newton has derived general forward and backward difference interpolation formulae.

4.15 NEWTON'S GREGORY FORWARD INTERPOLATION FORMULA

Let $y = f(x)$ be a function of x which assumes the values $f(a)$, $f(a + h)$, $f(a + 2h)$, ..., $f(a + nh)$ for $(n + 1)$ equidistant values $a, a + h, a + 2h, ..., a + nh$ of the independent variable x. Let $f(x)$ be a polynomial of n^{th} degree.

$$\text{Let } f(x) = A_0 + A_1 (x - a) + A_2 (x - a)(x - a - h)$$

$$+ A_3 (x - a)(x - a - h)(x - a - 2h) + ...$$

$$+ A_n (x - a) ...... (x - a - \overline{n-1}h) \tag{20}$$

where $A_0, A_1, A_2, ..., A_n$ are to be determined.

Put $x = a, a + h, a + 2h, ..., a + nh$ in (20) successively.

For $x = a$, $\quad f(a) = A_0 \quad (21)$

For $x = a + h$, $\quad f(a + h) = A_0 + A_1 h$

$\Rightarrow \quad f(a + h) = f(a) + A_1 h \quad$ | By (21)

$$\Rightarrow \quad A_1 = \frac{\Delta f(a)}{h} \tag{22}$$

For $x = a + 2h$,

$$f(a + 2h) = A_0 + A_1 (2h) + A_2 (2h) h$$

$$= f(a) + 2h \left\{ \frac{\Delta f(a)}{h} \right\} + 2h^2 A_2$$

$$\Rightarrow \quad 2h^2 A_2 = f(a + 2h) - 2f(a + h) + f(a) = \Delta^2 f(a)$$

$$\Rightarrow \quad A_2 = \frac{\Delta^2 f(a)}{2!\, h^2}$$

Similarly, $\quad A_3 = \dfrac{\Delta^3 f(a)}{3!\, h^3}$ and so on.

Thus, $\quad A_n = \dfrac{\Delta^n f(a)}{n!\, h^n}$.

From (20),
$$f(x) = f(a) + (x-a)\frac{\Delta f(a)}{h} + (x-a)(x-a-h)\frac{\Delta^2 f(a)}{2!\,h^2} + \ldots\ldots\ldots$$
$$+ (x-a)\ldots\ldots(x-a-\overline{n-1}\,h)\frac{\Delta^n f(a)}{n!\,h^n}$$

Put $x = a + hu \Rightarrow u = \dfrac{x-a}{h}$, we have

$$f(a+hu) = f(a) + hu\frac{\Delta f(a)}{h} + \frac{(hu)(hu-h)}{2!\,h^2}\Delta^2 f(a) + \ldots\ldots$$
$$+ \frac{(hu)(hu-h)(hu-2h)\ldots\ldots\ldots(hu-\overline{n-1}\,h)}{n!\,h^n}\Delta^n f(a)$$

$$\Rightarrow \boxed{f(a+hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \ldots + \frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\Delta^n f(a)}$$

which is the required formula.

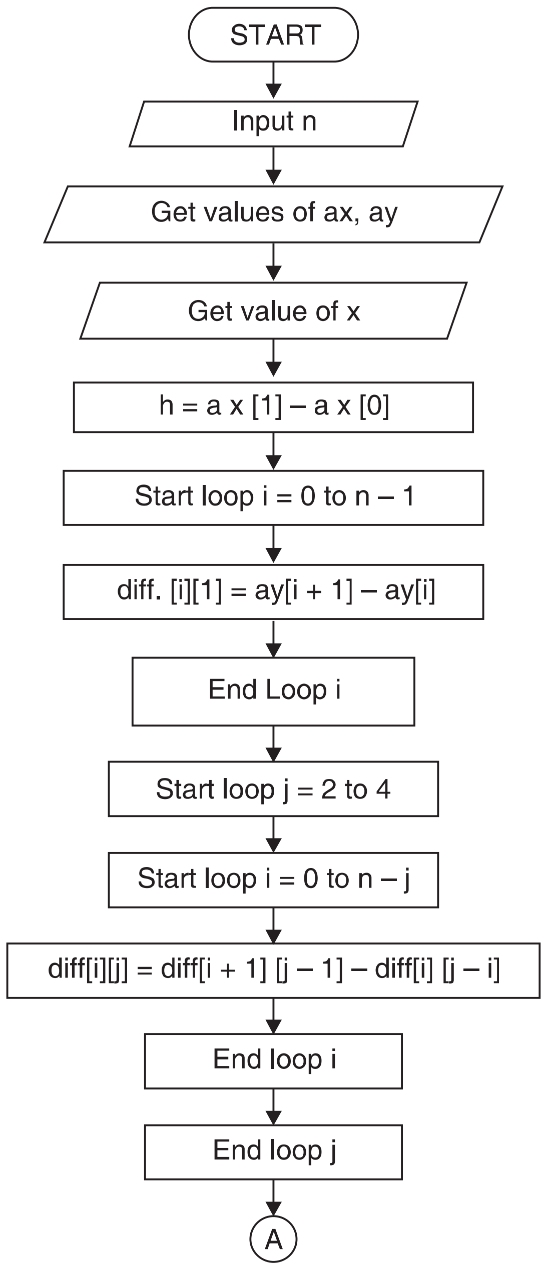
This formula is particularly useful for interpolating the values of $f(x)$ near the beginning of the set of values given. h is called the interval of difference, while Δ is forward difference operator.

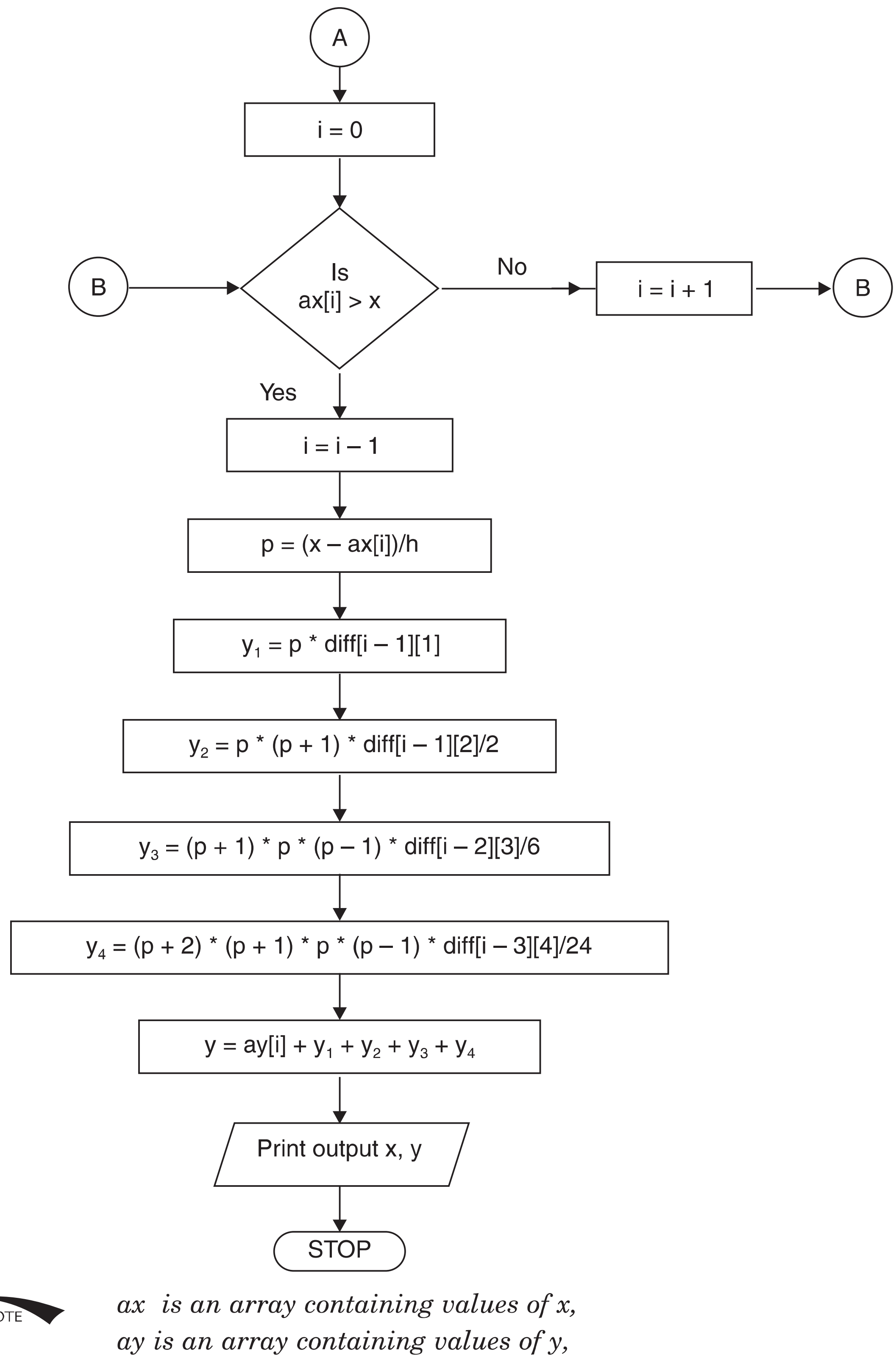
4.15.1 Algorithm for Newton's Forward Difference Formula

Step 01. Start of the program
Step 02. Input number of terms n
Step 03. Input the array ax
Step 04. Input the array ay
Step 05. h=ax[1] – ax[0]
Step 06. for i=0; i<n-1; i++
Step 07. diff[i] [1]=ay[i + 1] – ay[i]
Step 08. End Loop i
Step 09. for j=2; j<=4; j++
Step 10. for i = 0; i <n – j; i++
Step 11. diff[i][j]=diff [i + 1] [j – 1]-diff [i][j – 1]
Step 12. End Loop i
Step 13. End Loop j
Step 14. i=0
Step 15. Repeat Step 16 until ax[i]<x
Step 16. i=i + 1
Step 17. i=i – 1;

Step 18. p=(x – ax [i])/h
Step 19. y1=p*diff[i – 1][1]
Step 20. y2=p*(p+1)*diff [i – 1][2]/2
Step 21. y3=(p+1)*p*(p-1)*diff[i –2][3]/6
Step 22. y4=(p+2)*(p+1)*p*(p – 1)*diff[i – 3][4]/24
Step 23. y=ay[i]+y1+y2+y3+y4
Step 24. Print output x, y
Step 25. End of program.

4.15.2 Flow-chart





NOTE

ax is an array containing values of x,
ay is an array containing values of y,
Diff. is a two dimensional array containing difference table,
h is spacing between values of x

* **

4.15.3 Program to Implement Newton's Forward Method of Interpolation

** */

```
//... HEADER FILES DECLARATION
# include <stdio.h>
# include <conio.h>
# include <math.h>
# include <process.h>
# include <string.h>

//... MAIN EXECUTION THREAD
void main()
{
//... Variable declaration Field
//... Integer Type
int n;                          //... Number of terms
int i,j;                        //... Loop Variables

//...Floating Type
float ax[10];                   //... array limit 9
float ay[10];                   //... array limit 9
float x;                        //... User Querry
float y = 0;                    //... Initial value 0
float h;                        //... Calc. section
float p;                        //... Calc. section
float diff[20][20];             //... array limit 19,19
float y1,y2,y3,y4;              //... Formulae variables

//... Invoke Function Clear Screen
clrscr();
//... Input Section
printf("\n Enter the number of terms - ");
scanf("%d",&n);
//... Input Sequel for array X
```

```
Printf ("\n\n Enter the value in the form of x - ");
//... Input Loop for X
for  (i=0;i<n;i++)
        {
        printf("\n\n Enter the value of x%d - ",i+1);
        scanf("%f",&ax[i]);
        }

//... Input Sequel for array Y
printf("\n\n Enter the value in the form of y - ");
//... Input Loop for Y
for  (i=0;i<n;i++)
    {
    printf ("\n\n Enter the value of y%d - ", i+1);
    scanf  ("%f",&ay [i]);
    }

//... Inputting the required value quarry
printf("\nEnter the value of x for");
printf("\nwhich you want the value of y - ");
scanf("%f",&x);
//... Calculation and Processing Section
h=ax[1]-ax[0];
for(i=0;i<n-1;i++)
    {
    diff[i][1]=ay[i+1]-ay[i];
    }
for(j=2;j<=4;j++)
    {
    for(i=0;i<n-j;i++)
            {
            diff[i][j]=diff[i+1][j-1]-diff[i][j-1];
            }
    }
```

```
    i=0;
    do   {
         i++;
         }while(ax[i]<x);
    i--;
    p=(x-ax[i])/h;
    y1=p*diff[i-1][1];
    y2=p*(p+1)*diff[i-1][2]/2;
    y3=(p+1)*p*(p-1)*diff[i-2][3]/6;
    y4=(p+2)*(p+1)*p*(p-1)*diff[i-3][4]/24;
    //... Taking Sum
    y=ay[i]+y1+y2+y3+y4;

    //... Output Section
    printf("\nwhen x=%6.4f, y=%6.8f ",x,y);
    //... Invoke User Watch Halt Function
    Printf("\n\n\n Press Enter to Exit");
    getch();
    }
//... Termination of Main Execution Thread
```

4.15.4 Output

```
Enter the number of terms - 7
Enter the value in the form of x -
Enter the value of x1 - 100
Enter the value of x2 - 150
Enter the value of x3 - 200
Enter the value of x4 - 250
Enter the value of x5 - 300
Enter the value of x6 - 350
Enter the value of x7 - 400
Enter the value in the form of y -
Enter the value of y1 - 10.63
Enter the value of y2 - 13.03
Enter the value of y3 - 15.04
```

```
Enter the value of y4 - 16.81
Enter the value of y5 - 18.42
Enter the value of y6 - 19.9
Enter the value of y7 - 21.27
Enter the value of x for which you want the value of y-218
When X=218.0000, Y=15.69701481
Press Enter to Exit
```

EXAMPLES

Example 1. *Find the value of sin 52° from the given table:*

θ°	45°	50°	55°	60°
sin θ	0.7071	0.7660	0.8192	0.8660

Sol. $a = 45°, h = 5, x = 52$

$$\therefore \quad u = \frac{x-a}{h} = \frac{7}{5} = 1.4$$

Difference table is:

$x°$	$10^4 y$	$10^4 \Delta y$	$10^4 \Delta^2 y$	$10^4 \Delta^3 y$
45°	$\boxed{7071}$			
		$\boxed{589}$		
50°	7660		$\boxed{-57}$	
		532		$\boxed{-7}$
55°	8192		– 64	
		468		
60°	8660			

By forward difference formula,

$$f(a + hu) = f(a) + u\,\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$\Rightarrow \quad 10^4 f(x) = 10^4 f(a) + 10^4\, u\,\Delta f(a) + 10^4 \frac{u(u-1)}{2!}\Delta^2 f(a) + 10^4 \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$\Rightarrow \quad 10^4 f(52) = 10^4 f(45) + (1.4)\, 10^4 \Delta f(45) + \frac{(1.4)(1.4-1)}{2!} 10^4 \Delta^2 f(45)$$

$$+ \frac{(1.4)(1.4-1)(1.4-2)}{3!} 10^4 \Delta^3 f(45)$$

$$= 7071 + (1.4)(589) + \frac{(1.4)(.4)}{2}(-57) + \frac{(1.4)(.4)(-.6)}{6}(-7)$$

$$= 7880$$

$\therefore \quad f(52) = .7880$. Hence, $\sin 52° = 0.7880$.

Example 2. *The population of a town in the decimal census was as given below. Estimate the population for the year 1895.*

Year x:	*1891*	*1901*	*1911*	*1921*	*1931*
Population y: (in thousands)	*46*	*66*	*81*	*93*	*101*

Sol. Here $a = 1891$, $h = 10$, $a + hu = 1895$

$\Rightarrow \quad 1891 + 10u = 1895 \quad \Rightarrow \quad u = 0.4$

The difference table is as under:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	$\boxed{46}$				
		$\boxed{20}$			
1901	66		$\boxed{-5}$		
		15		$\boxed{2}$	
1911	81		-3		$\boxed{-3}$
		12		-1	
1921	93		-4		
		8			
1931	101				

Applying Newton's forward difference formula,

$$y(1895) = y(1891) + u\,\Delta y(1891) + \frac{u(u-1)}{2!}\Delta^2 y(1891)$$

$$+ \frac{u(u-1)(u-2)}{3!}\Delta^3 y(1891)$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y(1891)$$

$$\Rightarrow \quad y(1895) = 46 + (.4)(20) + \frac{(.4)(.4-1)}{2}(-5)$$

$$+ \frac{(.4)(.4-1)(.4-2)}{6}(2) + \frac{(.4)(.4-1)(.4-2)(.4-3)}{24}(-3)$$

$$\Rightarrow \quad y(1895) = 54.8528 \text{ thousands}$$

Hence the population for the year 1895 is **54.8528 thousands** approximately.

Example 3. *The values of f(x) for x = 0, 1, 2, ..., 6 are given by*

x:	*0*	*1*	*2*	*3*	*4*	*5*	*6*
f(x):	*2*	*4*	*10*	*16*	*20*	*24*	*38*

Estimate the value of f(3.2) using only four of the given values. Choose the four values that you think will give the best approximation.

Sol. Last four values of $f(x)$ for $x = 3, 4, 5, 6$ are taken into consideration so that 3.2 occurs in the beginning of the table.

Here $a = 3$, $h = 1, x = 3.2$ $\therefore$ $a + h\,u = 3.2$

i.e., $3 + 1 \times u = 3.2$ or $u = 0.2$

The difference table is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
3	$\boxed{16}$			
		$\boxed{4}$		
4	20		$\boxed{0}$	
		4		$\boxed{10}$
5	24		10	
		14		
6	38			

Applying Newton's forward difference formula,

$$f(3.2) = f(3) + u\,\Delta f(3) + \frac{u(u-1)}{2!}\Delta^2 f(3) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(3)$$

$$= 16 + (.2)(4) + \frac{(.2)(.2-1)}{2}(0) + \frac{(.2)(.2-1)(.2-2)}{6}(10) = 17.28.$$

Example 4. *From the following table, find the value of* $e^{0.24}$

x: *0.1*

0.2

0.3

Sol. The difference table is:

x	$10^5 y$	$10^5 \Delta y$	$10^5 \Delta^2 y$	$10^5\ \Delta^3 y$	$10^4 \Delta^4 y$
0.1	$\boxed{110517}$				
		$\boxed{11623}$			
0.2	122140		$\boxed{1223}$		
		12846		$\boxed{127}$	
0.3	134986		1350		$\boxed{17}$
		14196		144	
0.4	149182		1494		
		15690			
0.5	164872				

Here $\quad h = 0.1. \quad \therefore \quad 0.24 = 0.1 + 0.1 \times u \quad$ or $\quad u = 1.4$

Newton-Gregory forward formula is

$$y(.24) = y(.1) + u\,\Delta\, y(.1) + \frac{u(u-1)}{2!}\Delta^2\, y(.1) + \frac{u(u-1)(u-2)}{3!}\Delta^3 y(.1) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4\, y(.1)$$

$$\Rightarrow\ 10^5\, y(.24) = 10^5\, y(.1) + u\ 10^5\ \Delta y(.1) + \frac{u(u-1)}{2!}\ 10^5\ \Delta^2 y(.1) + \frac{u(u-1)(u-2)}{3!}\ 10^5\ \Delta^3 y(.1) + \frac{u(u-1)(u-2)(u-3)}{4!}\ 10^5\ \Delta^4 y(.1)$$

$$\Rightarrow\ 10^5\, y(.24) = 110517 + (1.4)(11623) + \frac{(1.4)(1.4-1)}{2}\,(1223) + \frac{(1.4)(1.4-1)(1.4-2)}{3!}\,(127) + \frac{(1.4)(1.4-1)(1.4-2)(1.4-3)}{4!}\,(17)$$

$$= 127124.9088$$

$$\therefore \quad y(.24) = 1.271249088$$

Hence, $\quad e^{.24} = 1.271249088.$

Example 5. *From the following table of half-yearly premiums for policies maturing at different ages, estimate the premium for policies maturing at age of 46.*

Age	*45*	*50*	*55*	*60*	*65*
Premium (in dollars)	*114.84*	*96.16*	*83.32*	*74.48*	*68.48*

Sol. The difference table is:

Age *(x)*	*Premium* *(in dollars)* *(y)*	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
45	$\boxed{114.84}$				
		$\boxed{-18.68}$			
50	96.16		$\boxed{5.84}$		
		– 12.84		$\boxed{-1.84}$	
55	83.32		4		$\boxed{.68}$
		– 8.84		– 1.16	
60	74.48		2.84		
		– 6			
65	68.48				

Here $h = 5, a = 45, a + hu = 46$

$\therefore \quad 45 + 5u = 46 \quad \Rightarrow \quad u = .2$

By Newton's forward difference formula,

$$y_{46} = y_{45} + u\,\Delta y_{45} + \frac{u(u-1)}{2!}\Delta^2 y_{45} + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{45} + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_{45}$$

$$= 114.84 + (.2)(-18.68) + \frac{(.2)(.2-1)}{2!}(5.84) + \frac{(.2)(.2-1)(.2-2)}{3!}(-1.84) + \frac{(.2)(.2-1)(.2-2)(.2-3)}{4!}(.68)$$

$$= 110.525632$$

Hence the premium for policies maturing at the age of 46 is **\$ 110.52.**

Example 6. *From the table, estimate the number of students who obtained scores between 40 and 45.*

Scores:	*30—40*	*40—50*	*50—60*	*60—70*	*70—80*
Number of students:	*31*	*42*	*51*	*35*	*31.*

Sol. The difference table is:

Scores less than (x)	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	$\boxed{31}$				
		$\boxed{42}$			
50	73		$\boxed{9}$		
		51		$\boxed{-25}$	
60	124		– 16		$\boxed{37}$
		35		12	
70	159		– 4		
		31			
80	190				

We shall find y_{45}, number of students with scores less than 45.

$$a = 40,\ h = 10,\ a + hu = 45.$$

$\therefore \quad 40 + 10u = 45 \quad \Rightarrow \quad u = .5$

By Newton's forward difference formula,

$$y(45) = y(40) + u\,\Delta\, y(40) + \frac{u(u-1)}{2!}\,\Delta^2\, y(40)$$

$$+ \frac{u(u-1)(u-2)}{3!}\,\Delta^3 y(40) + \frac{u(u-1)(u-2)(u-3)}{4!}\,\Delta^4\, y(40)$$

$$= 31 + (.5)(42) + \frac{(.5)(.5-1)}{2}\,(9) + \frac{(.5)(.5-1)(.5-2)}{6}\,(-25)$$

$$+ \frac{(.5)(.5-1)(.5-2)(.5-3)}{24}\,(37)$$

$$= 47.8672 \approx 48$$

Hence, the number of students getting scores less than 45 = 48

By the number of students getting scores less than 40 = 31

Hence, the number of students getting scores between 40 and 45 = 48 – 31 = 17.

Example 7. *Find the cubic polynomial which takes the following values:*

x:	*0*	*1*	*2*	*3*
f(x):	*1*	*2*	*1*	*10.*

Sol. Let us form the difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	$\boxed{1}$			
		$\boxed{1}$		
1	2		$\boxed{-2}$	
		-1		$\boxed{12}$
2	1		10	
		9		
3	10			

Here, $h = 1$. Hence, using the formula,

$$x = a + hu$$

and choosing $a = 0$, we get $x = u$

$\therefore$ By Newton's forward difference formula,

$$y = y_0 + x\,\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0$$

$$= 1 + x(1) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12)$$

$$= 2x^3 - 7x^2 + 6x + 1$$

Hence, the required cubic polynomial is

$$y = f(x) = 2x^3 - 7x^2 + 6x + 1.$$

Example 8. *The following table gives the scores secured by100 students in the Numerical Analysis subject:*

Range of scores:	*30—40*	*40—50*	*50—60*	*60—70*	*70—80*
Number of students:	*25*	*35*	*22*	*11*	*7*

Use Newton's forward difference interpolation formula to find.

(*i*) *the number of students who got scores more than 55.*

(*ii*) *the number of students who secured scores in the range between 36 and 45.*

Sol. The given table is re-arranged as follows:

Scores obtained	*Number of students*
Less than 40	25
Less than 50	60
Less than 60	82
Less than 70	93
Less than 80	100

(i) Here, $a = 40$, $h = 10$, $a + hu = 55$

$\therefore$ $40 + 10u = 55$ $\Rightarrow$ $u = 1.5$

First, we find the number of students who got scores less than 55.

The difference table follows:

Scores obtained less than	*Number of students = y*	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	$\boxed{25}$				
		$\boxed{35}$			
50	60		$\boxed{-13}$		
		22		$\boxed{2}$	
60	82		-11		$\boxed{5}$
		11		7	
70	93		-4		
		7			
80	100				

Applying Newton's forward difference formula,

$$y_{55} = y_{40} + u\,\Delta y_{40} + \frac{u(u-1)}{2!}\Delta^2 y_{40} + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{40} + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_{40}$$

$$= 25 + (1.5)(35) + \frac{(1.5)(.5)}{2!}(-13) + \frac{(1.5)(.5)(-.5)}{3!}(2) + \frac{(1.5)(.5)(-.5)(-1.5)}{4!}(5)$$

$$= 71.6171875 \approx 72$$

There are 72 students who got scores less than 55.

$\therefore$ Number of students who got scores more than 55 = 100 – 72 = 28

(ii) To calculate the number of students securing scores between 36 and 45, take the difference of y_{45} and y_{36}.

$$u = \frac{x-a}{h} = \frac{36-40}{10} = -.4$$

Also, $$u = \frac{45-40}{10} = .5$$

Newton's forward difference formula:

$$y_{36} = y_{40} + u\,\Delta y_{40} + \frac{u(u-1)}{2!}\Delta^2 y_{40} + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{40} + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_{40}$$

$$= 25 + (-.4)(35) + \frac{(-.4)(-1.4)}{2!}(-13) + \frac{(-.4)(-1.4)(-2.4)}{3!}(2) + \frac{(-.4)(-1.4)(-2.4)(-3.4)}{4!}(5) = 7.864 \approx 8$$

Also, $$y_{45} = y_{40} + u\,\Delta y_{40} + \frac{u(u-1)}{2!}\Delta^2 y_{40} + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{40} + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_{40}$$

$$= 25 + (.5)(35) + \frac{(.5)(-.5)}{2}(-13) + \frac{(.5)(-.5)(-1.5)}{6}(2) + \frac{(.5)(-.5)(-1.5)(-2.5)}{24}(5)$$

$$= 44.0546 \approx 44.$$

Hence, the number of students who secured scores between 36 and 45 is $y_{45} - y_{36} = 44 - 8 = 36$.

Example 9. *The following are the numbers of deaths in four successive ten year age groups. Find the number of deaths at 45—50 and 50—55.*

Age group:	*25—35*	*35—45*	*45—55*	*55—65*
Deaths:	*13229*	*18139*	*24225*	*31496.*

Sol. Difference table of cumulative frequencies:

Age upto x	*Number of deaths* $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
35	$\boxed{13229}$			
		$\boxed{18139}$		
45	31368		$\boxed{6086}$	
		24225		$\boxed{1185}$
55	55593		7271	
		31496		
65	87089			

Here, $h = 10, a = 35, a + hu = 50$

$\therefore \quad 35 + 10u = 50 \quad \Rightarrow \quad u = 1.5$

By Newton's forward difference formula,

$$y_{50} = y_{35} + u\,\Delta y_{35} + \frac{u(u-1)}{2!}\Delta^2 y_{35} + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{35}$$

$$= 13229 + (1.5)(18139) + \frac{(1.5)(.5)}{2}(6086) + \frac{(1.5)(.5)(-.5)}{6}(1185)$$

$$= 42645.6875 \approx 42646$$

$\therefore$ Deaths at ages beween 45 – 50 are 42646 – 31368 = 11278

and Deaths at ages between 50 – 55 are 55593 – 42646 = 12947.

Example 10. *If p, q, r, s are the successive entries corresponding to equidistant arguments in a table, show that when the third differences are taken into account, the entry corresponding to the argument half way between the arguments at q and r is* $A + \left(\frac{B}{24}\right)$*, where A is the arithmetic mean of q and r and B is arithmetic mean of 3q – 2p – s and 3r – 2s – p.*

Sol. $A = \frac{q+r}{2} \quad \Rightarrow \quad q + r = 2A$

$$B = \frac{(3q-2p-s)+(3r-2s-p)}{2} = \frac{3q+3r-3p-3s}{2}$$

$$= \frac{3(q+r)}{2} - \frac{3(p+s)}{2}$$

Let the entries $p, q, r,$ and s correspond to $x = a, a + h, a + 2h,$ and $a + 3h,$ respectively. Then the value of the argument lying half way between $a + h$ and $a + 2h$ will be $a + h + \left(\frac{h}{2}\right)$ *i.e.,* $a + \frac{3h}{2}$.

Hence $a + mh = a + \frac{3}{2}h \quad \Rightarrow \quad m = \frac{3}{2}$

Let us now construct the difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
a	p			
		$q - p$		
$a + h$	q		$r - 2q + p$	
		$r - q$		$s - 3r + 3q - p$
$a + 2h$	r		$s - 2r + q$	
		$s - r$		
$a + 3h$	s			

Using Newton's Gregory Interpolation formula up to third difference only and taking $m = 3/2$, we get

$$f\left(a+\frac{3}{2}h\right) = f(a) + \frac{3}{2}\Delta f(a) + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2}\Delta^2 f(a) + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{6}\Delta^3 f(a)$$

$$= p + \frac{3}{2}(q-p) + \frac{3}{8}(r-2q+p) - \frac{1}{16}(s-3r+3q-p)$$

$$= \frac{(16p - 24q - 24p + 6r - 12q + 6p - s + 3r - 3q + p)}{16}$$

$$= \frac{1}{16}(-p + 9q + 9r - s) = \frac{9}{16}(q+r) - \left(\frac{p+s}{16}\right)$$

$$= \frac{9}{16}(2A) - \frac{2}{3}\left(\frac{3A - B}{16}\right)$$

$$= \frac{9}{8}A - \frac{1}{8}A + \frac{B}{24} = A + \frac{B}{24}.$$

ASSIGNMENT 4.4

1. The following table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

x:	100	150	200	250	300	350	400
y:	10.63	13.03	15.04	16.81	18.42	19.9	21.27

Use Newton's forward formula to find y when $x = 218$ ft.

2. If l_x represents the number of persons living at age x in a life table, find, as accurately as the data will permit, l_x for values of $x = 35$, 42 and 47. Given

$$l_{20} = 512,\ l_{30} = 390,\ l_{40} = 360,\ l_{50} = 243.$$

3. The values of $f(x)$ for $x = 0, 1, 2, \ldots\ldots, 6$ are given by

x:	0	1	2	3	4	5	6
$f(x)$:	1	3	11	31	69	131	223

Estimate the value of $f(3.4)$, using only four of the given values.

4. Given that:

x:	1	2	3	4	5	6
$y(x)$:	0	1	8	27	64	125

Find the value of $f(2.5)$.

5. Ordinates $f(x)$ of a normal curve in terms of standard deviation x are given as

x:	1.00	1.02	1.04	1.06	1.08
$f(x)$:	0.2420	0.2371	0.2323	0.2275	0.2227

Find the ordinate for standard deviation $x = 1.025$.

6. Using Newton's formula for interpolation, estimate the population for the year 1905 from the table:

Year	*Population*
1891	98,752
1901	132,285
1911	168,076
1921	195,690
1931	246,050

7. Find the number of students from the following data who secured scores not more than 45

Scores range:	30—40	40—50	50—60	60—70	70—80
Number of students:	35	48	70	40	22

8. Find the number of men getting wages between \$ 10 and \$ 15 from the following table:

Wages (in \$):	0—10	10—20	20—30	30—40
Frequency:	9	30	35	42

9. Following are the scores obtained by 492 candidates in a certain examination

Scores	*Number of candidates*
0—40	210
40—45	43
45—50	54
50—55	74
55—60	32
60—65	79

Find out the number of candidates

(*a*) who secured scores more than 48 but not more than 50;

(*b*) who secured scores less than 48 but not less than 45.

10. Use Newton's forward difference formula to obtain the interpolating polynomial $f(x)$, satisfying the following data:

x:	1	2	3	4
$f(x)$:	26	18	4	1

If another point $x = 5$, $f(x) = 26$ is added to the above data, will the interpolating polynomial be the same as before or different. Explain why.

11. The table below gives value of tan x for $.10 \le x \le .30$.

x:	.10	.15	.20	.25	.30
tan x:	.1003	.1511	.2027	.2553	.3093

Evaluate tan 0.12 using Newton's forward difference formula

12. (*i*) Estimate the value of $f(22)$ from the following available data:

x:	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

(*ii*) Find the cubic polynomial which takes the following values:

$$y(0) = 1, \quad y(1) = 0, \quad y(2) = 1 \text{ and } y(3) = 10$$

Hence or otherwise obtain $y(4)$.

(*iii*) Use Newton's method to find a polynomial $p(x)$ of lowest possible degree such that $p(n) = 2^n$ for $n = 0, 1, 2, 3, 4$.

4.16 NEWTON'S GREGORY BACKWARD INTERPOLATION FORMULA

Let $y = f(x)$ be a function of x which assumes the values $f(a)$, $f(a + h)$, $f(a + 2h)$, ..., $f(a + nh)$ for $(n + 1)$ equidistant values $a, a + h, a + 2h, ..., a + nh$ of the independent variable x.

Let $f(x)$ be a polynomial of the n^{th} degree.

$$\text{Let,} \quad f(x) = A_0 + A_1(x - a - nh) + A_2 (x - a - nh)(x - a - \overline{n-1}\, h) + ...$$

$$+ A_n (x - a - nh)(x - a - \overline{n-1}\, h) ...... (x - a - h)$$

where $A_0, A_1, A_2, A_3, ..., A_n$ are to be determined. (23)

Put $x = a + nh, a + \overline{n-1}\, h, ..., a$ in (23) respectively.

Put $x = a + nh$, then $f(a + nh) = A_0$ (24)

Put $x = a + (n - 1) h$, then

$$f(a + \overline{n-1}\, h) = A_0 - h A_1 = f(a + nh) - h A_1 \qquad | \text{ By (24)}$$

$$\Rightarrow \qquad A_1 = \frac{\nabla f(a + nh)}{h} \tag{25}$$

Put $x = a + (n - 2)h$, then

$$f(a + \overline{n-2}\, h) = A_0 - 2hA_1 + (-2h)(-h) A_2$$

$$\Rightarrow \qquad 2\,!\,h^2 A_2 = f(a + \overline{n-2}\; h) - f(a + nh) + 2\nabla f(a + nh)$$

$$= \nabla^2 f(a + nh)$$

$$A_2 = \frac{\nabla^2 f(a+nh)}{2\,!\,h^2} \tag{26}$$

Proceeding, we get

$$A_n = \frac{\nabla^n\; f(a+nh)}{n\,!\,h^n} \tag{27}$$

Substituting the values in (24), we get

$$f(x) = f(a + nh) + (x - a - nh)\,\frac{\nabla f(a+nh)}{h} + \ldots\ldots$$

$$+ (x - a - nh)\,(x - a - \overline{n-1}\; h)$$

$$\ldots\ldots (x - a - h)\,\frac{\nabla^n f(a+nh)}{n\,!\,h^n} \tag{28}$$

Put $x = a + nh + uh,$ then

$$x - a - nh = uh$$

and $\quad x - a - (n - 1)h = (u + 1)h$

$$\vdots$$

$$x - a - h = (u + \overline{n-1})\; h$$

$\therefore$ (28) becomes,

$$f(x) = f(a + nh) + u\;\nabla f(a + nh) + \frac{u(u+1)}{2\,!}\;\nabla^2 f(a + nh)$$

$$+ \ldots\ldots + uh\,.\,(u + 1)h \ldots\ldots (u + \overline{n-1})(h)\,\frac{\nabla^n f(a+nh)}{n\,!\,h^n}$$

or

$$\boxed{f(a + nh + uh) = f(a + nh) + u\;\nabla f(a + nh) + \frac{u(u+1)}{2\,!}\;\nabla^2 f(a + nh) + \ldots\ldots + \frac{u(u+1)\ldots\ldots(u+\overline{n-1})}{n\,!}\;\nabla^n f(a + nh)}$$

which is the required formula.

This formula is useful when the value of $f(x)$ is required near the end of the table.

4.16.1 Algorithm for Newton's Backward Difference formula

Step 01. Start of the program.

Step 02. Input number of terms n

Step 03. Input the array ax

Step 04. Input the array ay

Step 05. h=ax[1]-ax[0]

Step 06. for i=0; i<n–1; i++

Step 07. diff[i][1]=ay[i+1]–ay[i]

Step 08. End Loop i

Step 09. for j = 2; j < = 4; j + +

Step 10. for i=0; i<n–j; i++

Step 11. diff[i][j]=diff[i+1][j–1]–diff [i][j–1]

Step 12. End Loop i

Step 13. End Loop j

Step 14. i=0

Step 15. Repeat Step 16 until (!ax[i]<x)

Step 16. i=i+1

Step 17. x0=mx[i]

Step 18. sum=0

Step 19. y0=my[i]

Step 20. fun=1

Step 21. p=(x–x0)/h

Step 22. sum=y0

Step 23. for k=1; k<=4; k++

Step 24. fun=(fun*(p–(k–1)))/k

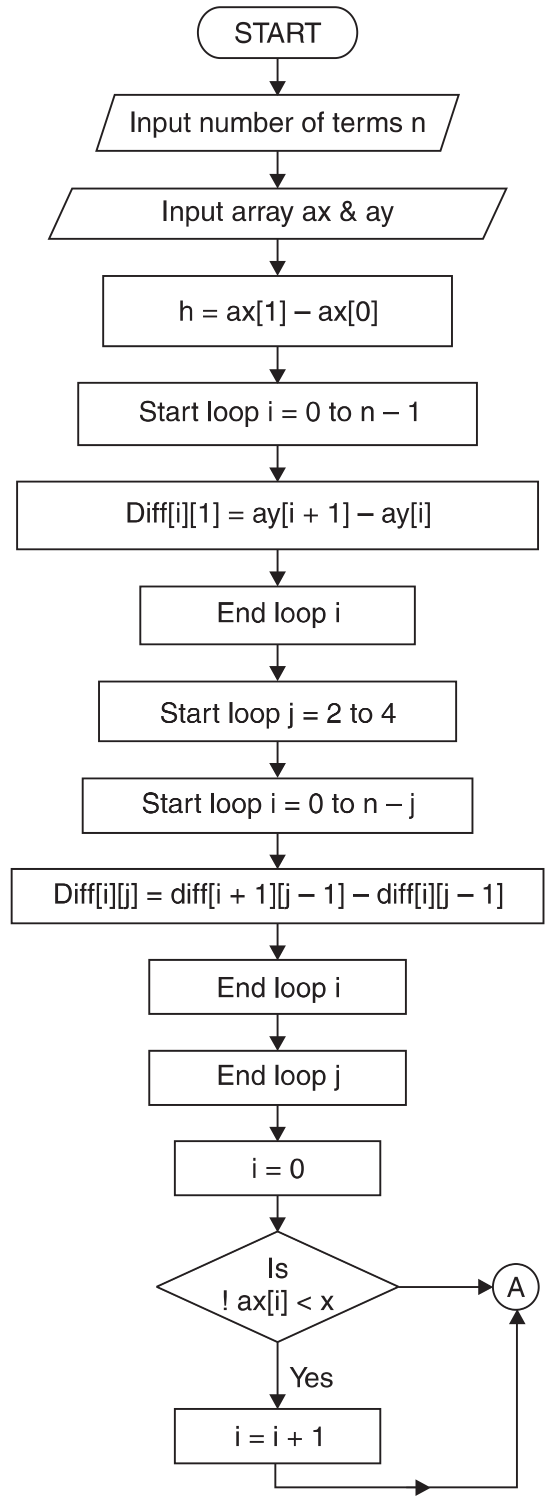
Step 25. sum=sum+fun*diff[i][k]

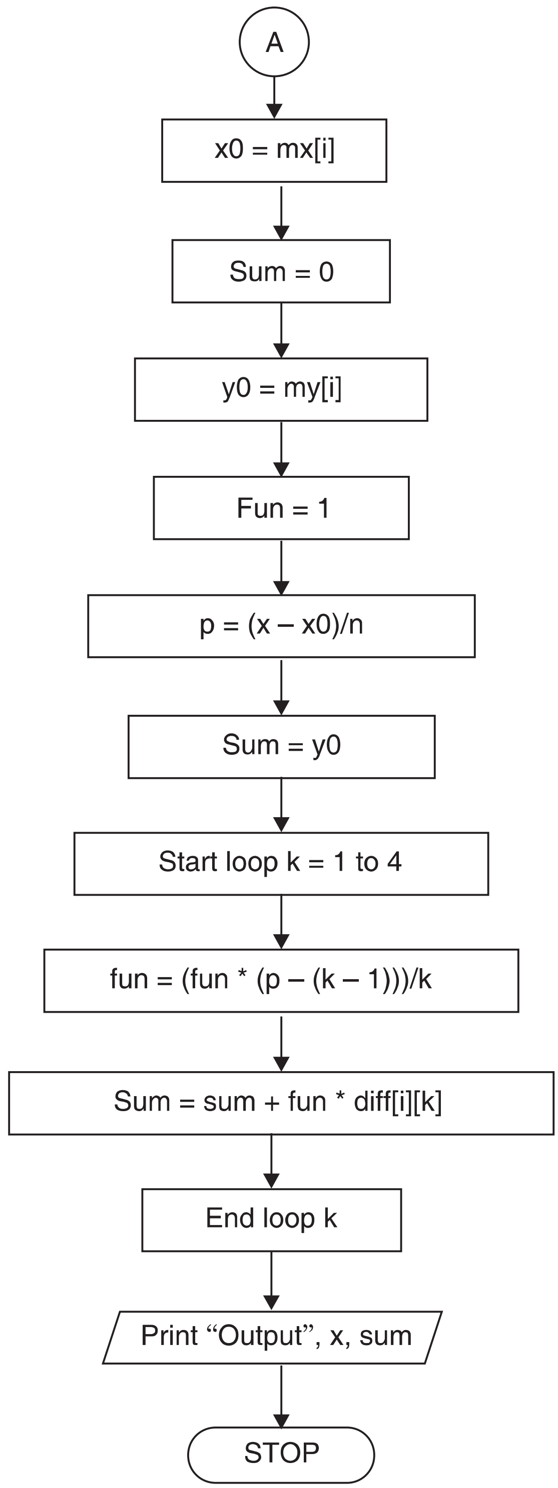
Step 26. End loop k

Step 27. Print Output x,sum

Step 28. End of Program

4.16.2 Flow-chart




A
x0 = mx[i]
Sum = 0
y0 = my[i]
Fun = 1
p = (x – x0)/n
Sum = y0
Start loop k = 1 to 4
fun = (fun * (p – (k – 1)))/k
Sum = sum + fun * diff[i][k]
End loop k
Print "Output", x, sum
STOP

4.16.3 Program to Implement Newton's Backward Method of Interpolation

```
***************************************************************************/
//...HEADER FILES DECLARATION
# include <stdio.h>
# include <conio.h>
# include <math.h>
# include <process.h>
# include <string.h>

//... MAIN EXECUTION THREAD
void main()
 {
 //...Variable declaration Field
 //...Integer Type
 int n;                            //...Number of terms
 int i,j,k;                        //...Loop Variables

 //...Floating Type
 float my[10];                     //... array limit 9
 float my[10];                     //... array limit 9
 float x;                          //... User Querry
 float x0 = 0;                     //... Initial value 0
 float y0;                         //... Calc. Section
 float sum;                        //... Calc. Section
 float h;                          //... Calc. Section
 float fun;                        //... Calc. Section
 float p;                          //... Calc. Section
 float diff[20][20];               //... array limit 19,19
 float y1, y2, y3, y4;             //... Formulae variables

 //...Invoke Function Clear Screen
 clrscr();

 //...Input Section
```

```
printf("\n Enter the number of terms - ");
scanf("%d",&n);
//...Input Sequel for array X
printf("\n\n Enter the value in the form of x - ");
//...Input Loop for X
for (i=0;i<n;i++)
     {
     printf("\n\n Enter the value of x%d - ",i+1);
     scanf (“%f”,&mx[i]);
     }
//...Input Sequel for array Y
printf ("\n\n Enter the value in the form of y -");
//...Input Loop for Y
for (i=0;i<n;i++)
     {
     printf ("\n\n Enter the value of y%d - ",i+1);
     scanf ("%f",&my[i]);
     }
//...Inputting the required value query
printf ("\nEnter the value of x for");
printf("\nwhich you want the value of y - ");
scanf("%f",&x);
//...Calculation and Processing Section
h=mx[1]-mx[0];
for(i=0;i<n-1;i++)
     {
     diff[i][1]=my[i+1]-my[i];
     }
for (j=2;j<=4;j++)
     {
     for (i=0;i<n-j;i++)
          {
          diff[i][j]=diff[i+1][j-1]-diff[i][j-1];
          }
     }
```

```
i=0;
while(!mx[i]>x)
     {
     i++;
     }
x0=mx[i];
sum=0;
y0=my[i];
fun=1;
p=(x-x0)/h;
sum=y0;
for (k=1;k<=4;k++)
     {
     fun=(fun*(p-(k-1)))/k;
     sum=sum+fun*diff[i][k];
     }

//...Output Section
printf ("\nwhen x=%6.4f,y=%6.8f",x,sum);
//...Invoke User Watch Halt Function
printf("\n\n\n Press Enter to Exit");
getch( );
}
//...Termination of Main Execution Thread
```

4.16.4 Output

```
Enter the number of terms-7
Enter the value in the form of x-
Enter the value of x1 - 100
Enter the value of x2 - 150
Enter the value of x3 - 200
Enter the value of x4 - 250
Enter the value of x5 - 300
Enter the value of x6 - 350
Enter the value of x7 - 400
```

```
Enter the value in the form of y -
Enter the value of y1 - 10.63
Enter the value of y2 - 13.03
Enter the value of y3 - 15.04
Enter the value of y4 - 16.81
Enter the value of y5 - 18.42
Enter the value of y6 - 19.90
Enter the value of y7 - 21.27
Enter the value of x for which you want the value of y - 410
When x = 410.0000, y = 21.34462738
Press Enter to Exit
```

EXAMPLES

Example 1. *The population of a town was as given. Estimate the population for the year 1925.*

Year (x):	*1891*	*1901*	*1911*	*1921*	*1931*
Population (y): *(in thousands)*	*46*	*66*	*81*	*93*	*101*

Sol. Here, $a + nh = 1931, h = 10, \quad a + nh + uh = 1925$

$$\therefore \qquad u = \frac{1925 - 1931}{10} = -0.6$$

The difference table is:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
		20			
1901	66		– 5		
		15		2	
1911	81		– 3		$\boxed{-3}$
		12		$\boxed{-1}$	
1921	93		$\boxed{-4}$		
		$\boxed{8}$			
1931	$\boxed{101}$				

Applying Newton's Backward difference formula, we get

$$y_{1925} = y_{1931} + u\nabla y_{1931} + \frac{u(u+1)}{2!}\nabla^2 y_{1931} + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_{1931} + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_{1931}$$

$$= 101 + (-.6)(8) + \frac{(-.6)(.4)}{2!}(-4) + \frac{(-.6)(.4)(1.4)}{3!}(-1) + \frac{(-.6)(.4)(1.4)(2.4)}{4!}(-3)$$

$$= 96.8368 \text{ thousands.}$$

Hence the population for the year 1925 = 96836.8 ≈ 96837.

Example 2. *The population of a town is as follows:*

Year:	*1921*	*1931*	*1941*	*1951*	*1961*	*1971*
Population: (in Lakhs)	*20*	*24*	*29*	*36*	*46*	*51*

Estimate the increase in population during the period 1955 to 1961.

Sol. Here, $a + nh = 1971$, $h = 10$, $a + nh + uh = 1955$

$\therefore$ $1971 + 10u = 1955 \Rightarrow u = -1.6$

The difference table is:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1921	20					
		4				
1931	24		1			
		5		1		
1941	29		2		0	
		7		1		$\boxed{-9}$
1951	36		3		$\boxed{-9}$	
		10		$\boxed{-8}$		
1961	46		$\boxed{-5}$			
		$\boxed{5}$				
1971	$\boxed{51}$					

Applying Newton's backward difference formula, we get

$$y_{1955} = y_{1971} + u\nabla y_{1971} + \frac{u(u+1)}{2!}\nabla^2 y_{1971} + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_{1971}$$

$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_{1971} + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 y_{1971}$$

$$= 51 + (-1.6)(5) + \frac{(-1.6)(-0.6)}{2!}(-5) + \frac{(-1.6)(-0.6)(0.4)}{6}(-8)$$

$$+ \frac{(-1.6)(-0.6)(.4)(1.4)}{24}(-9) + \frac{(-1.6)(-0.6)(0.4)(1.4)(2.4)}{120}(-9)$$

$$= 39.789632$$

$\therefore$ Increase in population during period 1955 to 1961 is

$= 46 - 39.789632 = 6.210368$ Lakhs

$= 621036.8$ Lakhs.

Example 3. *In the following table, values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series.*

x:	*3*	*4*	*5*	*6*	*7*	*8*	*9*
y:	*4.8*	*8.4*	*14.5*	*23.6*	*36.2*	*52.8*	*73.9.*

Sol. The difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4		0
		16.6		$\boxed{0.5}$	
8	52.8		$\boxed{4.5}$		
		$\boxed{21.1}$			
9	$\boxed{73.9}$				

To find the first term, we use Newton's forward interpolation formula.

Here, $a = 3, \quad h = 1, \quad x = 1 \quad \therefore \quad u = \dfrac{x-a}{h} = -2$

We have $$y_1 = y_3 + u\Delta y_3 + \frac{u(u-1)}{2!}\Delta^2 y_3 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_3$$

$$= 4.8 + (-2) \times 3.6 + \frac{(-2)(-3)}{2}(2.5) + \frac{(-2)(-3)(-4)}{6}(0.5)$$

$$= 3.1$$

To obtain the tenth term, we use Newton's Backward interpolation formula

$$a + nh = 9,\ h = 1,\ a + nh + uh = 10$$

$$\therefore \quad 10 = 9 + u \quad \Rightarrow \quad u = 1$$

$$\therefore \quad y_{10} = y_9 + u\nabla y_9 + \frac{u(u+1)}{2!}\nabla^2 y_9 + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_9$$

$$= 73.9 + 21.1 + 4.5 + .5 = 100.$$

Example 4. *Given log x for x = 40, 45, 50, 55, 60 and 65 according to the following table:*

x:	*40*	*45*	*50*	*55*	*60*	*65*
log x:	*1.60206*	*1.65321*	*1.69897*	*1.74036*	*1.77815*	*1.81291*

Find the value of log 5875.

Sol. The difference table is:

x	$10^5 \log x = 10^5 y_x$	$10^5\ \nabla y_x$	$10^5\ \nabla^2 y_x$	$10^5\ \nabla^3 y_x$	$10^5 \nabla^4 y_x$	$10^5 \nabla^5 y_x$
40	160206					
		5115				
45	165321		– 539			
		4576		102		
50	169897		– 437		– 25	
		4139		77		$\boxed{5}$
55	174036		– 360		$\boxed{-20}$	
		3779		$\boxed{57}$		
60	177815		$\boxed{-303}$			
		$\boxed{3476}$				
65	$\boxed{181291}$					

Newton's Backward difference formula is

$$f(a + nh + uh) = f(a + nh) + u\nabla f(a + nh) + \frac{u(u+1)}{2!} \nabla^2 f(a + nh)$$
$$+ \frac{u(u+1)(u+2)}{3!} \nabla^3 f(a + nh) + \frac{u(u+1)(u+2)(u+3)}{4!} \nabla^4 f(a + nh)$$
$$+ \frac{u(u+1)(u+2)(u+3)(u+4)}{5!} \nabla^5 f(a + nh) \qquad (29)$$

First we shall find the value of log(58.75).

Here, $\quad a + nh = 65,\ h = 5,\ a + nh + uh = 58.75$

$\therefore \quad 65 + 5u = 58.75 \quad \Rightarrow \quad u = -1.25$

From (29),

$$10^5 f(58.75) = 181291 + (-1.25)(3476) + \frac{(-1.25)(-.25)}{2!}(-303)$$
$$+ \frac{(-1.25)(-.25)(.75)}{3!}(57) + \frac{(-1.25)(-.25)(.75)(1.75)}{4!}(-20)$$
$$+ \frac{(-1.25)(-.25)(.75)(1.75)(2.75)}{5!}(5)$$

$\Rightarrow \quad 10^5 f(58.75) = 176900.588$

$\therefore \quad f(58.75) = \log 58.75 = 176900.588 \times 10^{-5} = 1.76900588$

Hence,

$\log 5875 = 3.76900588 \qquad | \because$ Mantissa remain the same

Example 5. *Calculate the value of tan 48° 15′ from the following table:*

x°:	*45*	*46*	*47*	*48*	*49*	*50*
tan x°:	*1.00000*	*1.03053*	*1.07237*	*1.11061*	*1.15037*	*1.19175*

Sol. Here $a + nh = 50, \quad h = 1, \quad a + nh + uh = 48° \, 15' = 48.25°$

$\therefore \quad 50 + u(1) = 48.25 \quad \Rightarrow \quad u = -1.75$

The difference table is:

$x°$	10^5y	$10^5\nabla y$	$10^5\nabla^2 y$	$10^5\nabla^3 y$	$10^5\nabla^4 y$	$10^5\nabla^5 y$
45	100000					
		3553				
46	103553		131			
		3648		9		
47	107237		140		3	
		3824		12		$\boxed{-5}$
48	111061		152		$\boxed{-2}$	
		3976		$\boxed{10}$		
49	115037		$\boxed{162}$			
		$\boxed{4138}$				
50	$\boxed{119175}$					

$$y_{a+nh+uh} = y_{a+nh} + u\nabla y_{a+nh} + \frac{u(u+1)}{2}\nabla^2 y_{a+nh} + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_{a+nh}$$

$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_{a+nh} + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 y_{a+nh}$$

$$\therefore\ 10^5 y_{48.25} = 119175 + (-1.75)\times 4138 + \frac{(-1.75)\times(-0.75)}{2}\times 162$$

$$+ \frac{(-1.75)(-0.75)(0.25)}{3!}\times 10 + \frac{(-1.75)(-.75)(.25)(1.25)}{4!}(-2)$$

$$+ \frac{(-1.75)(-.75)(.25)(1.25)(2.25)}{5!}(-5)$$

$$\Rightarrow \quad 10^5\, y_{48.25} = 112040.2867$$

$$\therefore \quad y_{48.25} = \tan 48°15' = 1.120402867.$$

Example 6. *From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for a policy maturing at the age of 63:*

Age:	*45*	*50*	*55*	*60*	*65*
Premium: (in dollars)	*114.84*	*96.16*	*83.32*	*74.48*	*68.48*

Sol. The difference table is:

Age *(x)*	*Premium* *(in dollars)* *(y)*	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
45	114.84				
		– 18.68			
50	96.16		5.84		
		– 12.84		– 1.84	
55	83.32		4		$\boxed{.68}$
		– 8.84		$\boxed{-1.16}$	
60	74.48		$\boxed{2.84}$		
		$\boxed{-6}$			
65	$\boxed{68.48}$				

Here $\quad a + nh = 65, \quad h = 5, a + nh + uh = 63$

$\therefore \quad 65 + 5u = 63 \quad \Rightarrow \quad u = -.4$

By Newton's backward difference formula,

$$y(63) = y(65) + u\nabla y(65) + \frac{u(u+1)}{2!}\nabla^2 y(65) + \frac{u(u+1)(u+2)}{3!}\nabla^3 y(65) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y(65)$$

$$= 68.48 + (-.4)(-6) + \frac{(-.4)(.6)}{2}(2.84) + \frac{(-.4)(.6)(1.6)}{6}(-1.16) + \frac{(-.4)(.6)(1.6)(2.6)}{24}(.68)$$

$$= 70.585152$$

ASSIGNMENT 4.5

1. From the following table find the value of tan 17°

θ°:	0	4	8	12	16	20	24
tan θ°:	0	0.0699	0.1405	0.2126	0.2867	0.3640	0.4402

2. Find the value of an annuity at $5\frac{3}{8}\%$, given the following table:

Rate:	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6
Annuity value:	172.2903	162.8889	153.7245	145.3375	137.6483

3. The values of annuities are given for the following ages. Find the value of annuity at the age of $27\frac{1}{2}$.

Age:	25	26	27	28	29
Annuity:	16.195	15.919	15.630	15.326	15.006

4. The table below gives the value of tan x for $0.10 \leq x \leq 0.30$.

x:	0.10	0.15	0.20	0.25	0.30
$y = tan\ x$:	0.1003	0.1511	0.2027	0.2553	0.3093

Find: (*i*) tan 0.50 (*ii*) tan 0.26 (*iii*) tan 0.40.

5. Given:

x:	1	2	3	4	5	6	7	8
$f(x)$:	1	8	27	64	125	216	343	512

Find $f(7.5)$ using Newton's Backward difference formula.

6. From the following table of values of x and $f(x)$, determine

(*i*) $f(0.23)$ (*ii*) $f(0.29)$

x:	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$:	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

7. The probability integral

$$P = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{1}{2}t^2}\, dt \text{ has following values:}$$

x:	1.00	1.05	1.10	1.15	1.20	1.25
P:	0.682689	0.706282	0.728668	0.749856	0.769861	0.788700

Calculate P for $x = 1.235$.

8. In an examination, the number of candidates who obtained scores between certain limits are as follows:

Scores	*Number of candidates*
0—19	41
20—39	62
40—59	65
60—79	50
80—99	17

Estimate the number of candidates who obtained fewer than 70 scores.

9. Estimate the value of $f(42)$ from the following available data:

x:	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

10. The area A of a circle of diameter d is given for the following values:

d:	80	85	90	95	100
A:	5026	5674	6362	7088	7854

Calculate the area of a cricle of diameter 105.

11. From the following table, find y, when $x = 1.84$ and 2.4 by Newton's interpolation formula:

x:	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$y = e^x$:	5.474	6.050	6.686	7.389	8.166	9.025	9.974

12. Using Newton's backward difference formula, find the value of $e^{-1.9}$ from the following table of values of e^{-x}:

x:	1	1.25	1.50	1.75	2.00
e^{-x}:	0.3679	0.2865	0.2231	0.1738	0.1353

4.17 CENTRAL DIFFERENCE INTERPOLATION FORMULAE

We shall study now the central difference formulae most suited for interpolation near the middle of a tabulated set.

4.18 GAUSS' FORWARD DIFFERENCE FORMULA

Newton's Gregory forward difference formula is

$$f(a + hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + \ldots\ldots \quad (30)$$

Given $a = 0$, $h = 1$, we get

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(0)$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(0) + \ldots\ldots \quad (31)$$

Now, $\Delta^3 f(-1) = \Delta^2 f(0) - \Delta^2 f(-1) \Rightarrow \Delta^2 f(0) = \Delta^3 f(-1) + \Delta^2 f(-1)$

Also, $\Delta^4 f(-1) = \Delta^3 f(0) - \Delta^3 f(-1) \Rightarrow \Delta^3 f(0) = \Delta^4 f(-1) + \Delta^3 f(-1)$

and $\Delta^5 f(-1) = \Delta^4 f(0) - \Delta^4 f(-1) \Rightarrow \Delta^4 f(0) = \Delta^5 f(-1) + \Delta^4 f(-1)$ and so on.

$\therefore$ From (31),

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\{\Delta^2 f(-1) + \Delta^3 f(-1)\}$$

$$+ \frac{u(u-1)(u-2)}{3!}\{\Delta^3 f(-1) + \Delta^4 f(-1)\}$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\{\Delta^4 f(-1) + \Delta^5 f(-1)\} + \ldots\ldots$$

$$= f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{u(u-1)}{2}\left\{1 + \frac{u-2}{3}\right\}\Delta^3 f(-1)$$

$$+ \frac{u(u-1)(u-2)}{6}\left\{1 + \frac{u-3}{4}\right\}\Delta^4 f(-1) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^5 f(-1) + \ldots\ldots$$

$$= f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)$$

$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-1) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^5 f(-1) + \ldots\ldots \quad (32)$$

But, $\Delta^5 f(-2) = \Delta^4 f(-1) - \Delta^4 f(-2)$

$\therefore$ $\Delta^4 f(-1) = \Delta^4 f(-2) + \Delta^5 f(-2)$

then (32) becomes,

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)$$

$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\{\Delta^4 f(-2) + \Delta^5 f(-2)\}$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^5 f(-1) + \ldots\ldots$$

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2) + \ldots\ldots$$

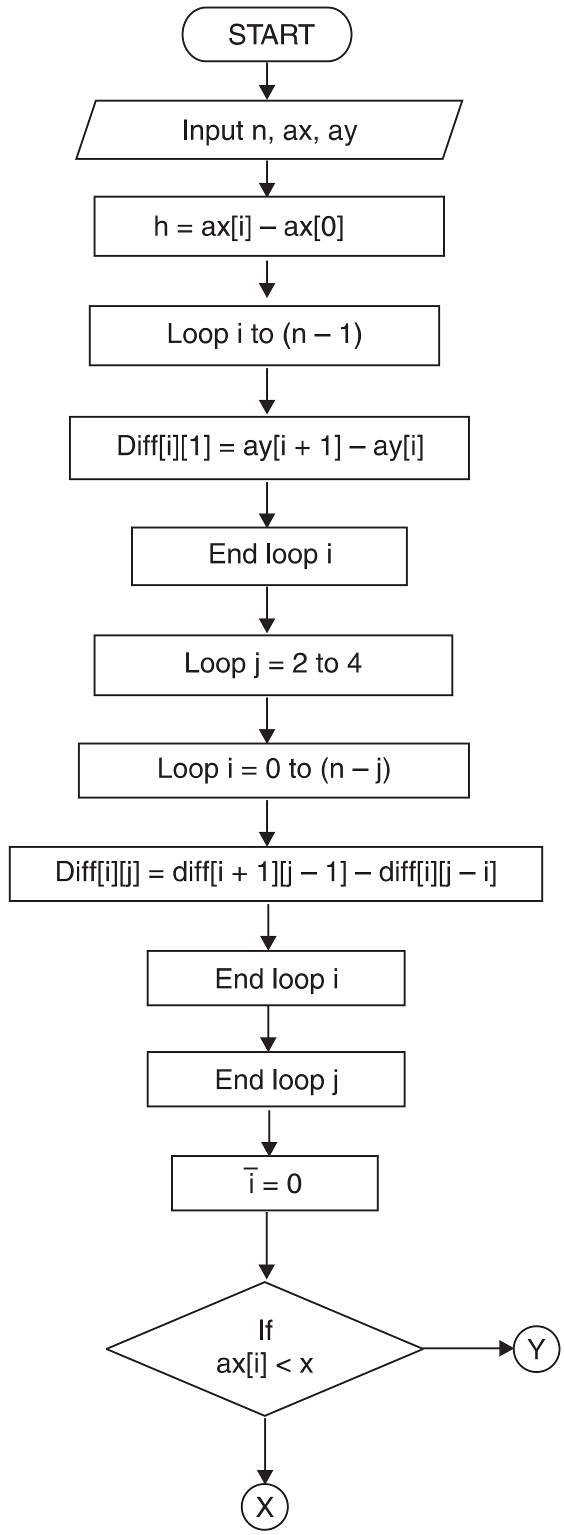
This is called **Gauss' forward difference formula.**

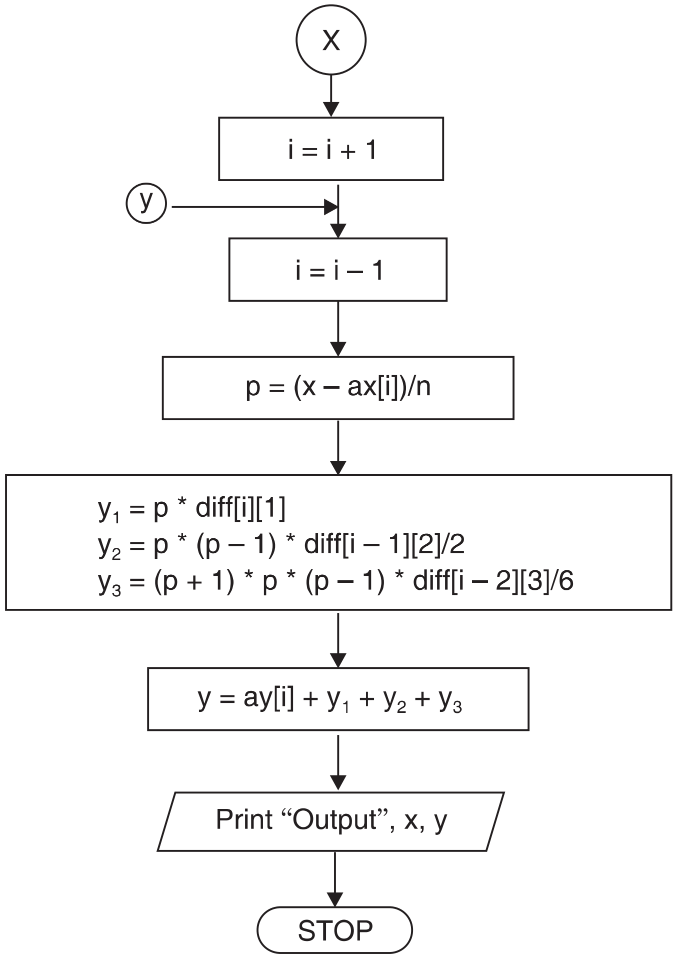
NOTE *This formula is applicable when u lies between 0 and* $\frac{1}{2}$.

4.18.1 Algorithm

Step 01. Start of the program.
Step 02. Input number of terms n
Step 03. Input the array ax
Step 04. Input the array ay
Step 05. h=ax[1]-ax[0]
Step 06. for i=0;i<n–1;i++
Step 07. diff[i][1]=ay[i+1]-ay[i]
Step 08. End Loop i
Step 09. for j=2;j<=4;j++
Step 10. for i=0;i<n–j;i++
Step 11. diff[i][j]=diff[i+1][j–1]–diff[i][j–1]
Step 12. End Loop i
Step 13. End Loop j
Step 14. i=0
Step 15. Repeat Step 16 until ax[i]<x
Step 16. i=i+1
Step 17. i=i–1;
Step 18. p=(x–ax[i])/h
Step 19. y1=p*diff[i][1]
Step 20. y2=p*(p–1)*diff[i–1][2]/2
Step 21. y3=(p+1)*p*(p-1)*diff[i–2][3]/6
Step 22. y4=(p+1)*p*(p–1)*(p–2)*diff[i–3][4]/24
Step 23. y=ay[i]+y1+y2+y3+y4
Step 24. Print Output x,y
Step 25. End of Program

4.18.2 Flow-chart






```
/* ***********************************************************************
```

4.18.3 Program to Implement Gauss's Forward Method of Interpolation

```
*********************************************************************** */

//...HEADER FILES DECLARATION
# include <stdio.h>
# include <conio.h>
# include <math.h>
# include <process.h>
# include <string.h>
//...MAIN EXECUTION THREAD
void main()
 {
 //...Variable declaration Field
```

```
//...Integer Type
int n;
int i,j;
//...Floating Type
float ax[10];                      //...array limit 9
float ax[10];                      //...array limit 9
float x;
float nr,dr;
float y=0;                         //...Initial value 0
float h;
float p;
float diff[20][20];                //...array limit 19,19
float y1,y2,y3,y4;

//...Invoke Function Clear Screen
clrscr();

//...Input Section
printf("\n Enter the number of terms - ");
scanf("%d",&n);
//...Input Sequel for array X
printf("\n\n Enter the value in the form of x - ");
//...Input loop for Array X
for (i=0;i<n;i++)
     {
     printf("\n\n Enter the value of x%d - ",i+i);
     scanf("%f”,&ax[i]);
     }
printf("\n\n Enter the value in the form of y - ");
//...Input Loop for Array Y
for(i=0;i<n;i++)
     {
     printf("\n\n Enter the value of y%d-",i+1);
     scanf("%f",&ay[i]);
     }
```

```
//...Inputting the required value query
printf("\nEnter the value of x for");
printf("\nwhich you want the value of y-");
scanf ("%f",&x);
//... Calculation and Processing Section
h=ax[1]-ax[0];
for(i=0;i<n-1;i++)
     {
     diff[i][1]=ay[i+1]-ay[i];
     }
for(j=2;j<=4;j++)
     {
     for(i=0;i<n-j;i++)
          {
          diff[i][j]=diff[i+1][j-1]-diff[i][j-1];
          }
     }
i=0;
do   {
     i++;
     }while(ax[i]<x);
i--;
p=(x-ax[i])/h;
y1=p*diff[i][1];
y2=p*(p-1)*diff[i-1][2]/2;
y3=(p+1)*p*(p-1)*diff[i-2][3]/6;
y4=(p+1)*p*(p-1)*(p-2)*diff[i-3][4]/24;
//...Taking Sum
y=ay[i]+y1+y2+y3+y4;
//...Output Section
printf("\nwhen x=%6.4f,y=%6.8f ",x,y);
//... Invoke User Watch Halt Function
printf("\n\n\n Press Enter to Exit");
getch();
}
//...Termination of Main Execution Thread
```

4.18.4 Output

```
Enter the number of terms - 7
Enter the value in the form of x -
Enter the value of x1 - 1.00
Enter the value of x2 - 1.05
Enter the value of x3 - 1.10
Enter the value of x4 - 1.15
Enter the value of x5 - 1.20
Enter the value of x6 - 1.25
Enter the value of x7 - 1.30
Enter the value in the form of y -
Enter the value of y1 - 2.7183
Enter the value of y2 - 2.8577
Enter the value of y3 - 3.0042
Enter the value of y4 - 3.1582
Enter the value of y5 - 3.3201
Enter the value of y6 - 3.4903
Enter the value of y7 - 3.6693
Enter the value of x for
which you want the value of y - 1.17
When x = 1.17, y = 3.2221
Press Enter to Exit
```

EXAMPLES

Example 1. *Apply a central difference formula to obtain f(32) given that:*

$$f(25) = 0.2707 \qquad f(35) = 0.3386$$

$$f(30) = 0.3027 \qquad f(40) = 0.3794.$$

Sol. Here $a + hu = 32$ and $h = 5$

Take origin at 30 $\therefore$ $a = 30$ then $u = 0.4$

The forward difference table is:

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
– 1	25	.2707			
			.032		
0	30	$\boxed{.3027}$		$\boxed{.0039}$	
			$\boxed{.0359}$		$\boxed{.0010}$
1	35	.3386		.0049	
			.0408		
2	40	.3794			

Applying Gauss' forward difference formula, we have

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)$$

$$\therefore \quad f(.4) = .3027 + (.4)(.0359) + \frac{(.4)(.4-1)}{2!}(.0039) + \frac{(1.4)(.4)(.4-1)}{3!}(.0010)$$

$$= 0.316536.$$

Example 2. *Use Gauss' forward formula to find a polynomial of degree four which takes the following values of the function f(x):*

x:	*1*	*2*	*3*	*4*	*5*
f(x):	*1*	*– 1*	*1*	*– 1*	*1*

Sol. Taking origin at 3 and $h = 1$

$$a + hu = x$$

$$\Rightarrow \quad 3 + u = x \quad \Rightarrow \quad u = x - 3$$

The difference table is:

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
– 2	1	1				
			– 2			
– 1	2	– 1		4		
			2		– 8	
0	3	$\boxed{1}$		$\boxed{-4}$		$\boxed{16}$
			$\boxed{-2}$		$\boxed{8}$	
1	4	– 1		4		
			2			
2	5	1				

Gauss' forward difference formula is

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)\,u(u-1)}{3!}\Delta^3 f(-1)$$
$$+ \frac{(u+1)\,u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$
$$= 1 + (x-3)(-2) + \frac{(x-3)(x-4)}{2}(-4) + \frac{(x-2)(x-3)(x-4)}{6}(8)$$
$$+ \frac{(x-2)(x-3)(x-4)(x-5)}{24}(16)$$
$$= 1 - 2x + 6 - 2x^2 + 14x - 24 + \frac{4}{3}(x^3 - 9x^2 + 26x - 24)$$
$$+ \frac{2}{3}(x^4 - 14x^3 + 71x^2 - 154x + 120)$$

$$\therefore \quad \boxed{F(x) = \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31}$$

Example 3. *The values of e^{-x} at $x = 1.72$ to $x = 1.76$ are given in the following table:*

x:	*1.72*	*1.73*	*1.74*	*1.75*	*1.76*
e^{-x}:	*0.17907*	*0.17728*	*0.17552*	*0.17377*	*0.17204*

Find the value of $e^{-1.7425}$ using Gauss' forward difference formula.

Sol. Here taking the origin at 1.74 and $h = 0.01$.

$$\therefore \qquad x = a + uh$$

$$\Rightarrow \qquad u = \frac{x-a}{h} = \frac{1.7425 - 1.7400}{0.01} = 0.25$$

The difference table is as follows:

u	*x*	$10^5 f(x)$	$10^5 \Delta f(x)$	$10^5 \Delta^2 f(x)$	$10^5 \Delta^3 f(x)$	$10^5 \Delta^4 f(x)$
– 2	1.72	17907				
			– 179			
– 1	1.73	17728		3		
			– 176		– 2	
0	1.74	$\boxed{17552}$		$\boxed{1}$		$\boxed{3}$
			$\boxed{-175}$			
1	1.75	17377		2	$\boxed{1}$	
			– 173			
2	1.76	17204				

Gauss's forward formula is

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)$$

$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$

$$\therefore \quad 10^5 f(.25) = 17552 + (.25)(-175) + \frac{(.25)(-.75)}{2}(1) + \frac{(1.25)(.25)(-.75)}{6}(1)$$

$$+ \frac{(1.25)(.25)(-.75)(-1.75)}{24}(3)$$

$$= 17508.16846$$

$$\therefore \quad f(0.25) = e^{-1.7425} = 0.1750816846.$$

Example 4. *Apply Gauss's forward formula to find the value of* u_9, *if* $u_0 = 14$, $u_4 = 24$, $u_8 = 32$, $u_{12} = 35$, $u_{16} = 40$.

Sol. The difference table is (taking origin at 8):

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(u)$	$\Delta^4 f(x)$
– 2	0	14				
			10			
– 1	4	24		– 2		
			8		– 3	
0	8	$\boxed{32}$		$\boxed{-5}$		$\boxed{10}$
			$\boxed{3}$		$\boxed{7}$	
1	12	35		2		
			5			
2	16	40				

Here $a = 8,\ h = 4,\quad a + hu = 9$

$\therefore \quad 8 + 4u = 9 \;\Rightarrow\; u = .25$

Gauss' forward difference formula is

$$f(.25) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)$$

$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 f(-2)$$

$$= 32 + (.25)(3) + \frac{(.25)(-.75)}{2}(-5) + \frac{(1.25)(.25)(-.75)}{6}(7)$$

$$+ \frac{(1.25)(.25)(-.75)(-1.75)}{24}(10)$$

$= 33.11621094$

Hence $u_9 = 33.11621094.$

ASSIGNMENT 4.6

1. Apply Gauss's forward formula to find the value of $f(x)$ at $x = 3.75$ from the table:

x:	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$:	24.145	22.043	20.225	18.644	17.262	16.047.

2. Given that

x:	25	30	35	40	45
log x:	1.39794	1.47712	1.54407	1.60206	1.65321

Find the value of log 3.7, using Gauss's forward formula.

3. Find the value of $f(41)$ by applying Gauss's forward formula from the following data:

x:	30	35	40	45	50
$f(x)$:	3678.2	2995.1	2400.1	1876.2	1416.3

4. From the following table, find the value of $e^{1.17}$ using Gauss forward formula:

x:	1	1.05	1.10	1.15	1.20	1.25	1.30
e^x:	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693

5. From the following table find y when $x = 1.45$

x:	1.0	1.2	1.4	1.6	1.8	2.0
y:	0.0	– .112	– .016	.336	.992	2.0

4.19 GAUSS'S BACKWARD DIFFERENCE FORMULA

Newton's Gregory forward difference formula is

$$f(a + hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \ldots\ldots \tag{33}$$

Put $a = 0$, $h = 1$, we get

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(0) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(0) + \ldots\ldots \tag{34}$$

Now, $\Delta f(0) = \Delta f(-1) + \Delta^2 f(-1)$

$\Delta^2 f(0) = \Delta^2 f(-1) + \Delta^3 f(-1)$

$\Delta^3 f(0) = \Delta^3 f(-1) + \Delta^4 f(-1)$

$\Delta^4 f(0) = \Delta^4 f(-1) + \Delta^5 f(-1)$ and so on.

$\therefore$ From (34),

$$f(u) = f(0) + u\,[\Delta f(-1) + \Delta^2 f(-1)] + \frac{u(u-1)}{2!}\,[\Delta^2 f(-1) + \Delta^3 f(-1)]$$
$$+ \frac{u(u-1)(u-2)}{3!}\,[\Delta^3 f(-1) + \Delta^4 f(-1)]$$
$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\,[\Delta^4 f(-1) + \Delta^5 f(-1)] + \ldots\ldots \qquad (35)$$
$$= f(0) + u\Delta f(-1) + u\left(1 + \frac{u-1}{2}\right)\Delta^2 f(-1)$$
$$+ \frac{u(u-1)}{2}\left(1 + \frac{u-2}{3}\right)\Delta^3 f(-1)$$
$$+ \frac{u(u-1)(u-2)}{6}\left\{1 + \frac{u-3}{4}\right\}\Delta^4 f(-1) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^5 f(-1) + \ldots\ldots$$
$$= f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)\,u(u-1)}{3!}\Delta^3 f(-1)$$
$$+ \frac{(u+1)\,u(u-1)(u-2)}{4!}\Delta^4 f(-1) + \ldots\ldots \qquad (36)$$

Again, $\Delta^3 f(-1) = \Delta^3 f(-2) + \Delta^4 f(-2)$

and $\Delta^4 f(-1) = \Delta^4 f(-2) + \Delta^5 f(-2)$ and so on

$\therefore$ (36) gives

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\{\Delta^3 f(-2)$$
$$+ \Delta^4 f(-2)\}$$
$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\{\Delta^4 f(-2) + \Delta^5 f(-2)\} + \ldots\ldots$$

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)\,u(u-1)}{3!}\Delta^3 f(-2) + \frac{(u+2)\,(u+1)u\,(u-1)}{4!}\Delta^4 f(-2) + \ldots\ldots \quad (37)$$

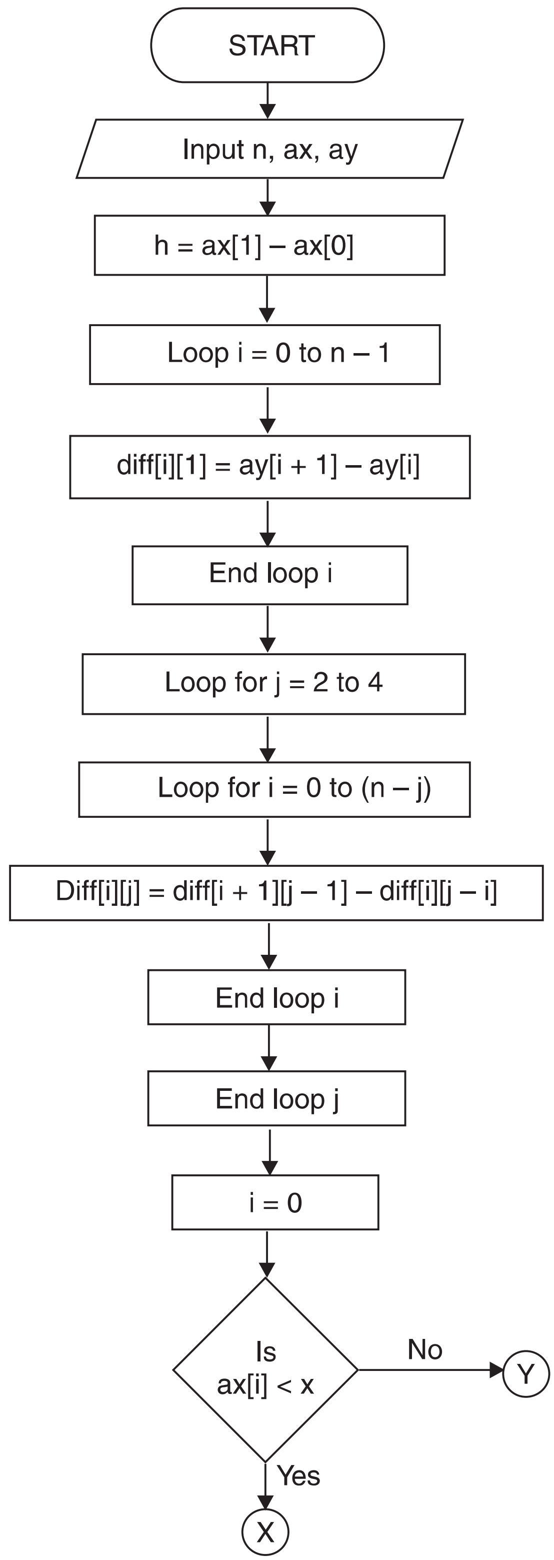
This is known as **Gauss' backward difference formula.**

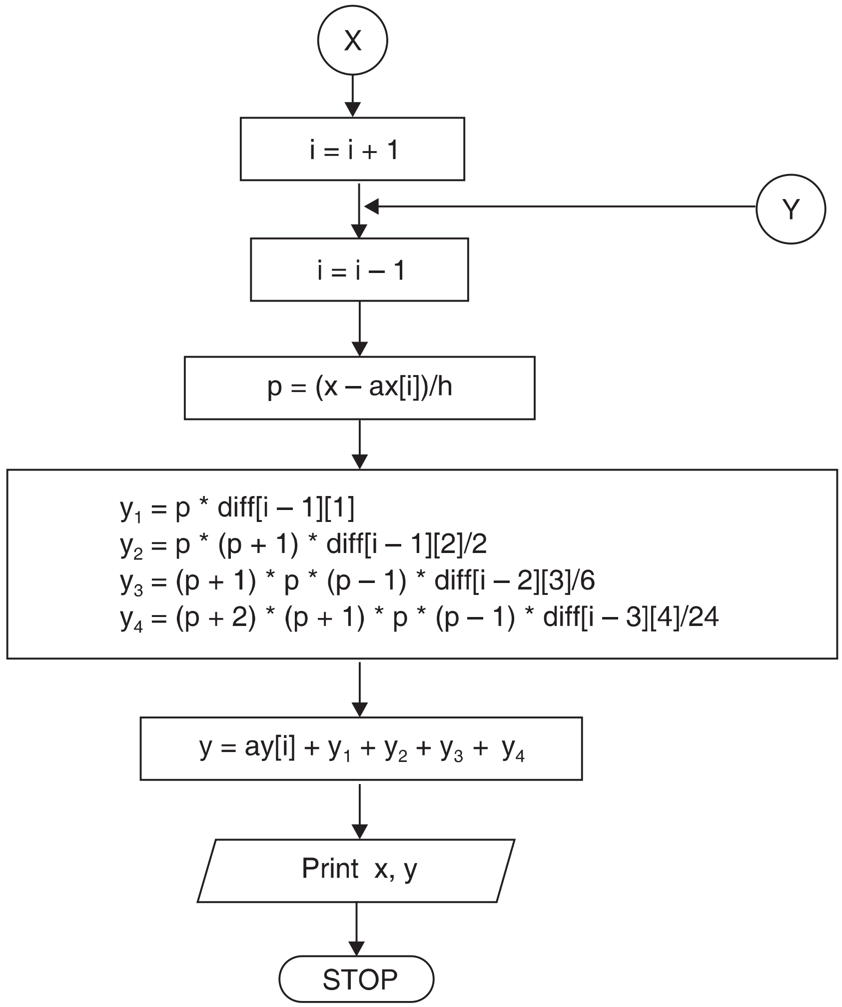
This formula is useful when u lies between $-\frac{1}{2}$ and 0.

4.19.1 Algorithm of Gauss's Backward Formula

Step 01. Start of the program.
Step 02. Input number of terms n
Step 03. Input the array ax
Step 04. Input the array ay
Step 05. h=ax[1]-ax[0]
Step 06. for i=0;i<n-l;i++
Step 07. diff[i][1]=ay[i+1]-ay[i]
Step 08. End Loop i
Step 09. for j=2;j<=4;j++
Step 10. for i=0;i<n–j;i++
Step 11. diff[i][j]=diff[i+1][j–1]–diff[i][j–1]
Step 12. End Loop i
Step 13. End Loop j
Step 14. i=0
Step 15. Repeat Step 16 until ax[i]<x
Step 16. i=i+1
Step 17. i=i–1;
Step 18. p=(x–ax[i])/h
Step 19. y1=p*diff[i-1][1]
Step 20. y2=p*(p+1)*diff[i–1][2]/2
Step 21. y3=(p+1)*p*(p-1)*diff[i–2][3]/6
Step 22. y4=(p+2)*(p+1)*p*(p–1)*diff[i–3][4]/24
Step 23. y=ay[i]+y1+y2+y3+y4
Step 24. Print Output x,y
Step 25. End of Program

4.19.2 Flow-chart






```
/* ******************************************************************
```

4.19.3 Program to Implement Gauss's Backward Method of Interpolation

```
******************************************************************/*
    //...HEADER FILES DECLARATION
    # include <stdio.h>
    # include <conio.h>
    # include <math.h>
    # include <process.h>
    # include <string.h>

    //...MAIN EXECUTION THREAD

    void main()
```

```
{
//...Variable declaration Field
//...Integer Type
int n;                          //... No. of terms
int i,j;                        //... Loop Variables

//...Floating Type
float ax[10];                   //... array limit 9
float ay[10];                   //... array limit 9
float x;                        //... User Querry
float y=0;                      //... Initial value 0
float h;                        //... Calc. section
float p;                        //... Calc. section
float diff[20][20];             //... array limit 19, 19
float y1,y2,y3,y4;              //... Formulae variables

//... Invoke Function Clear Screen
clrscr();

//... Input Section
printf("\n Enter the number of terms - ");
scanf("%d",&n);
//... Input Sequel for array X
printf("\n\n Enter the value in the form of x - ");
//... Input loop for X
for (i=0;i<n;i++)
    {
    printf("\n\n Enter the value of x%d-",i+1);
    scanf("%f",&ax[i]);
    }
//...Input Sequel for array Y
printf("\n\n Enter the value in the form of y-");
//...Input Loop for Y
for(i=0;i<n;i++)
    {
    printf("\n\n Enter the value of y%d-",i+1);
```

```
    scanf("%f",&ay[i]);
    }

//... Inputting the required value query
printf("\nEnter the value of x for");
printf("\nwhich you want the value of y - ");
scanf("%f",&x);
//... Calculation and Processing Section
h=ax[1]-ax[0];
for(i=0;i<n-1;i++)
    {
    diff[i][1]=ay[i+1]-ay[i];
    }
for(j=2;j<=4;j++)
    {
    for(i=0;i<n-j;i++)
        {
        diff[i][j]=diff[i+1][j-1]-diff[i][j-1];
        }
    }
i=0;
do      {
        i++;
        }while (ax[i]<x);
i--;
p=(x-ax[i])/h;
y1=p*diff[i-1][1];
y2=p*(p+1)*diff[i-1][2]/2;
y3=(p+1)*p*(p-1)*diff[i-2][3]/6;
y4=(p+2)*(p+1)*p*(p-1)*diff[i-3][4]/24;
//... Taking Sum
y=ay[i]+y1+y2+y3+y4;
//... Output Section
printf("\nwhen x=%6.1f,y=%6.4f ",x,y);
//... Invoke User Watch Halt Function
printf("\n\n\n Press Enter to Exit");
```

```
getch();
}
//... Termination of Main Execution Thread
```

4.19.4 Output

```
Enter the number of terms - 7
Enter the value in the form of x -
Enter the value of x1 - 1.00
Enter the value of x2 - 1.05
Enter the value of x3 - 1.10
Enter the value of x4 - 1.15
Enter the value of x5 - 1.20
Enter the value of x6 - 1.25
Enter the value of x7 - 1.30
Enter the value in the form of y -
Enter the value of y1 - 2.1783
Enter the value of y2 - 2.8577
Enter the value of y3 - 3.0042
Enter the value of y4 - 3.1582
Enter the value of y5 - 3.3201
Enter the value of y6 - 3.4903
Enter the value of y7 - 3.6693
Enter the value of x for
which you want the value of y - 1.35
When x = 1.35, y=3.8483
Press Enter to Exit
```

EXAMPLES

Example 1. *Given that*

$$\sqrt{12500} = 111.803399, \sqrt{12510} = 111.848111$$

$$\sqrt{12520} = 111.892806, \sqrt{12530} = 111.937483$$

Show by Gauss's backward formula that $\sqrt{12516} = 111.8749301.$

Sol. Taking the origin at 12520

$$\therefore \qquad u = \frac{x-a}{h} = \frac{12516 - 12520}{10} = -\frac{4}{10} = -0.4$$

Gauss's backward formula is

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)\,u}{2!}\,\Delta^2 f(-1)$$

$$+ \frac{(u+1)\,u(u-1)}{3!}\,\Delta^3 f(-2) + \ldots\ldots \qquad (38)$$

The difference table is:

u	x	$10^6\,f(x)$	$10^6\Delta\,f(x)$	$10^6\,\Delta^2\,f(x)$	$10^6\,\Delta^3\,f(x)$
– 2	12500	111803399			
			44712		
– 1	12510	111848111		– 17	
			$\boxed{44695}$		$\boxed{-1}$
0	12520	$\boxed{111892806}$		$\boxed{-18}$	
			44677		
1	12530	111937483			

From (38),

$$10^6 f(-.4) = 111892806 + (-.4)(44695)$$

$$+ \frac{(.6)(-.4)}{2!}(-18) + \frac{(.6)(-.4)(-1.4)}{3!}(-1)$$

$$= 111874930.1$$

$$\therefore \qquad f(-.4) = 111.8749301$$

Hence, $\sqrt{12516} = 111.8749301.$

Example 2. *Find the value of cos 51° 42′ by Gauss's backward formula. Given that*

x:	*50°*	*51°*	*52°*	*53°*	*54°*
cos x:	*0.6428*	*0.6293*	*0.6157*	*0.6018*	*0.5878.*

Sol. Taking the origin at 52° and $h = 1$

$$\therefore \qquad u = (x - a) = 51°\,42' - 52° = -\,18' = -\,0.3°$$

Gauss's backward formula is

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\,\Delta^2 f(-1) + \frac{(u+1)\,u\,(u-1)}{3!}\,\Delta^3 f(-2)$$

$$+ \frac{(u+2)(u+1)\,u\,(u-1)}{4!}\,\Delta^4 f(-2) \qquad (39)$$

The difference table is as below:

u	x	$10^4\ f(x)$	$10^4\ \Delta\ f(x)$	$10^4\ \Delta^2\ f(x)$	$10^4\ \Delta^3\ f(x)$	$10^4\ \Delta^4\ f(x)$
– 2	50°	6428				
			– 135			
– 1	51°	6293		– 1		
			$\boxed{-136}$		$\boxed{-2}$	
0	52°	$\boxed{6157}$		$\boxed{-3}$		$\boxed{4}$
			– 139		2	
1	53°	6018		– 1		
			– 140			
2	54°	5878				

From (39),

$$10^4 f(-.3) = 6157 + (-.3)(-136) + \frac{(.7)(-.3)}{2!}(-3) + \frac{(.7)(-.3)(-1.3)}{3!}(-2)$$

$$+ \frac{(1.7)(.7)(-.3)(-1.3)}{4!}(4)$$

$$= 6198.10135$$

$\therefore$ $\qquad f(-.3) = .619810135$

Hence $\qquad \cos 51°42' = 0.619810135.$

Example 3. *Using Gauss's backward interpolation formula, find the population for the year 1936 given that*

Year:	*1901*	*1911*	*1921*	*1931*	*1941*	*1951*
Population: (*in thousands*)	*12*	*15*	*20*	*27*	*39*	*52*

Sol. Taking the origin at 1941 and $h = 10$,

$$x = a + uh \quad \therefore \quad u = \frac{x-a}{h} = \frac{1936-1941}{10} = -0.5$$

Gauss's backward formula is

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2)$$

$$+ \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2) + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 f(-3) \quad (40)$$

The difference table is:

u	$f(u)$	$\Delta f(u)$	$\Delta^2 f(u)$	$\Delta^3 f(u)$	$\Delta^4 f(u)$	$\Delta^5 f(u)$
– 4	12					
		3				
– 3	15		2			
		5		0		
– 2	20		2		3	
		7		3		– 10
– 1	27		5		– 7	
		$\boxed{12}$		$\boxed{-4}$		
0	$\boxed{39}$		$\boxed{1}$			
		13				
1	52					

From (40),

$$f(-.5) = 39 + (-.5)(12) + \frac{(.5)(-.5)}{2}(1) + \frac{(.5)(-.5)(-1.5)}{6}(-4)$$

$$= 32.625 \text{ thousands}$$

Hence, the population for the year 1936 = 32625

Example 4. *$f(x)$ is a polynomial of degree four and given that*

$$f(4) = 270,\ f(5) = 648,\ \Delta f(5) = 682,\ \Delta^3 f(4) = 132.$$

Find the value of $f(5.8)$ using Gauss's backward formula.

Sol.

$$\Delta f(5) = f(6) - f(5)$$

$$\therefore \quad f(6) = f(5) + \Delta f(5) = 648 + 682 = 1330$$

$$\Delta^3 f(4) = (E - 1)^3 f(4) = f(7) - 3f(6) + 3f(5) - f(4) = 132$$

$$\therefore \quad f(7) = 3f(6) - 3f(5) + f(4) + 132$$

$$= 3 \times 1330 - 3 \times 648 + 270 + 132 = 2448.$$

The difference table is (Taking origin at 6):

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
– 2	4	270			
			378		
– 1	5	648		304	
			$\boxed{682}$		$\boxed{132}$
0	6	$\boxed{1330}$		$\boxed{436}$	
			1118		
1	7	2448			

Here, $a = 6, \quad h = 1, \quad a + hu = 5.8$

$\therefore \qquad 6 + u = 5.8 \quad \Rightarrow \quad u = -.2$

Gauss's backward formula is

$$f(-.2) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2)$$

$$= 1330 + (-.2)(682) + \frac{(.8)(-.2)}{2}(436) + \frac{(.8)(-.2)(-1.2)}{6}(132)$$

$$= 1162.944$$

$\therefore \qquad f(5.8) = 1162.944.$

ASSIGNMENT 4.7

1. The population of a town in the years 1931, ..., 1971 are as follows:

Year:	1931	1941	1951	1961	1971
Population: *(in thousands)*	15	20	27	39	52

Find the population of the town in 1946 by applying Gauss's backward formula.

2. Apply Gauss's backward formula to find the value of $(1.06)^{19}$ if $(1.06)^{10} = 1.79085$, $(1.06)^{15} = 2.39656$, $(1.06)^{20} = 3.20714$, $(1.06)^{25} = 4.29187$ and $(1.06)^{30} = 5.74349$.

3. Given that

x:	50	51	52	53	54
tan x:	1.1918	1.2349	1.2799	1.3270	1.3764

Using Gauss's backward formula, find the value of tan 51° 42′.

4. Interpolate by means of Gauss's backward formula, the population of a town for the year 1974 given that:

Year:	1939	1949	1959	1969	1979	1989
Population: *(in thousands)*	12	15	20	27	39	52

5. Apply Gauss's backward formula to find sin 45° from the following table:

$\theta°$:	20	30	40	50	60	70	80
sin θ:	0.34202	0.502	0.64279	0.76604	0.86603	0.93969	0.98481