DATA 605 - Homework 8

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Chapter 7.2 Exercise 11

A company buys 100 lightbulbs, each of which has an exponential lifetime of 1000 hours. What is the expected time for the first of these bulbs to burn out?

SOLUTION

ANS: let x1,x2…xn be independent exponential random variable.

$$X_i = \mu . e^{-\mu x} where x \geq 0$$

$$P(X > x) = e^{-\mu x}$$

$$so, E(X) = 1/\sum_{1}^{n} \mu\, i$$

X = first of these bulbs to burn out

n=100

$$\mu i = 1/1000$$

i =1,2,…100

So, E(X) = 10 HRS.

Chapter 7.2 Exercise 14

Assume that X_1 and X_2 are independent random variables, each having an exponential density with parameter λ. Show that $Z = X_1 - X_2$ has density $f_Z(z) = (1/2)e^{-\lambda|z|}$.

SOLUTION

$f_Z(z) = (1/2)e^{-\lambda|z|}$ can be re-written as $f_Z(z) = \begin{cases} (1/2)e^{-\lambda z}, & \text{if } z \geq 0, \\ (1/2)e^{\lambda z}, & \text{if } z < 0. \end{cases}$

Since X_1 and X_2 have exponential density, their PDF is

$f_{X_1}(x) = f_{X_2}(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$

$$\begin{aligned} f_Z(z) &= f_{X_1+(-X_2)}(z) \\ &= \int_{-\infty}^{\infty} f_{-X_2}(z - x_1) f_{X_1}(x_1) dx_1 \\ &= \int_{-\infty}^{\infty} f_{X_2}(x_1 - z) f_{X_1}(x_1) dx_1 \\ &= \int_{-\infty}^{\infty} \lambda\, e^{-\lambda(x_1 - z)} \lambda e^{-\lambda x_1} dx_1 \\ &= \int_{-\infty}^{\infty} \lambda^2\, e^{-\lambda x_1 + \lambda z} e^{-\lambda x_1} dx_1 \\ &= \int_{-\infty}^{\infty} \lambda^2\, e^{\lambda z - \lambda x_1 - \lambda x_1} dx_1 \\ &= \int_{-\infty}^{\infty} \lambda^2\, e^{\lambda(z - 2x_1)} dx_1 \end{aligned}$$

Consider $z = x_1 - x_2$, then $x_2 = x_1 - z$.

If $z \geq 0$, then $x_2 = (x_1 - z) \geq 0$, and $x_1 \geq z$, and, using WolframAlpha, $f_Z(z) = \int_z^{\infty} \lambda^2\, e^{\lambda(z-2x_1)} dx_1 = \frac{1}{2}\lambda e^{-\lambda z}$.

If $z < 0$, then $x_2 = (x_1 - z) \geq 0$, and $x_1 \geq 0$, and $f_Z(z) = \int_0^{\infty} \lambda^2\, e^{\lambda(z-2x_1)} dx_1 = \frac{1}{2}\lambda e^{\lambda z}$.

Combining two sides we get $f_Z(z) = \begin{cases} (1/2)e^{-\lambda z}, & \text{if } z \geq 0, \\ (1/2)e^{\lambda z}, & \text{if } z < 0. \end{cases}$

Chapter 8.2 Exercise 1

Let X be a continuous random variable with mean $\mu = 10$ and variance $\sigma^2 = 100/3$. Using Chebyshev's Inequality, find an upper bound for the following probabilities.

a. $P(|X - 10| \geq 2)$
b. $P(|X - 10| \geq 5)$
c. $P(|X - 10| \geq 9)$
d. $P(|X - 10| \geq 20)$

SOLUTION

Chebyshev Inequality: $P(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$.

Per problem, $\mu = 10$ and $\sigma = \sqrt{\frac{100}{3}} = \frac{10}{\sqrt{3}}$.

If $\epsilon = k\sigma$, then $k = \frac{\epsilon}{\sigma} = \frac{\epsilon\sqrt{3}}{10}$.

(a) P(|X - 10|>=2).

Let us find k first

```
s = 10/sqrt(3)
#sigma
s

## [1] 5.773503

#value of k
k = 2/s
k

## [1] 0.3464102

#value of k square -- upper bound
ksq = 1/(k*k)
ksq

## [1] 8.333333
```

S0, P(|X - 10|>=2) <= 8.33 which is not true.

(b) P(|X - 10| >= 5).

```
s = 10/sqrt(3)
#sigma
s

## [1] 5.773503

#value of k
k = 5/s
k

## [1] 0.8660254

#value of k square  -- upper bound
ksq = 1/(k*k)
ksq

## [1] 1.333333
```

So, P(|X - 10|>=5) <= 1.33 which is not true

(c) P(|X - 10| >= 9).

```
s = 10/sqrt(3)
#sigma
s

## [1] 5.773503

#value of k
k = 9/s
k

## [1] 1.558846

#value of k square  -- upper bound
ksq = 1/(k*k)
ksq

## [1] 0.4115226
```

So, P(|X - 10|>=9) <= 0.4115 which is not true

(d) P(|X - 10| >= 20).

```
s = 10/sqrt(3)
#sigma
s

## [1] 5.773503

#value of k
k = 20/s
k

## [1] 3.464102

#value of k square  -- upper bound
ksq = 1/(k*k)
ksq

## [1] 0.08333333
```

So, P(|X - 10|>=20) <= 0.083 which is not true.