DATA 605 - Homework 9

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Question 1 (9.3 #11)

The price of one share of stock in the Pilsdorff Beer Company (see Exercise 8.2.12) is given by Y_n on the nth day of the year. Finn observes that the differences $X_n = Y_{n+1} - Y_n$ appear to be independent random variables with a common distribution having mean $\mu = 0$ and variance $\sigma^2 = \frac{1}{4}$. If $Y_1 = 100$, estimate the probability that Y_{365} is:

(a) ≥ 100.

(b) ≥ 110.

(c) ≥ 120.

Solution

$Y_{365} = Y_1 + X_1 + X_2 + \ldots + X_{364}$.

$S_n = Y_{n+1} - 100$.

If $n = 364, S_{364} = Y_{365} - 100 \;\rightarrow\; Y_{365} = S_{364} + 100$.

$E[S_{364}] = n\mu = 364 \cdot 0 = 0$.

Variance of $S_{364} = 364 \cdot \frac{1}{4} = 91 \;\rightarrow\; \sigma = \sqrt{91}$.

(a)

$P(Y_{365} \geq 100) = P(S_{364} + 100 \geq 100) = P(S_{364} \geq 0)$.

```
q <- 0
mu <- 0
sd <- sqrt(91)
pnorm(q, mean = mu, sd = sd, lower.tail = F)

## [1] 0.5
```

(b)

$P(Y_{365} \geq 110) = P(S_{364} + 100 \geq 110) = P(S_{364} \geq 10)$.

```
q <- 10
pnorm(q, mean = mu, sd = sd, lower.tail = F)

## [1] 0.1472537
```

(c)

$P(Y_{365} \geq 120) = P(S_{364} + 100 \geq 120) = P(S_{364} \geq 20)$.

```
q <- 20
pnorm(q, mean = mu, sd = sd, lower.tail = F)

## [1] 0.01801584
```

Question 2

Calculate the expected value and variance of the binomial distribution using the moment generating function.

Solution

The pmf for the binomial distribution is $\binom{n}{k} p^n q^{n-k}$.

The moment generating function of a random variable is $M_k(t) = E[e^{tn}],\ \ t \in \mathbb{R}$.

Plugging in the binomial formula, the moment generating function is

$$M_k(t) = \sum_{k=0}^{n} e^{tn} \binom{n}{k} p^n q^{n-k} = \sum_{k=0}^{n} (pe^t)^n \binom{n}{k} q^{n-k}$$

Simplifying: $M_k(t) = (q + pe^t)^n$.

If we differentiate $M_k(t)$ wrt t: $M_k'(t) = n(pe^t)(q + pe^t)^{n-1}$.

When $t = 0$: $E[k] = np(q + p)^{n-1} = np$.

For the second moment, we take the second derivative (using the product rule):

$M_k''(t) = np[e^t(pe^t + q)^{n-1} + (n-1)(pe^t + q)^{n-2}(e^t p + 0)]$

Simplifying: $M_k''(t) = npe^t(pe^t + q)^{n-2}(npe^t + q)$.

When $t = 0$:

$M_k''(0) = E[k^2] = np(q + p)^{n-2}(q + np) = np(q + np)$.

Using the formula $V(x) = E[x^2] - (E[x])^2$:

$V(k) = np(q + np) - n^2p^2 = npq$.

Question 3

Calculate the expected value and variance of the exponential distribution using the moment generating function.

Solution

Proceeding in the same manner as in question 2.

The pmf for the exponential distribution of a random variable is $\lambda e^{-\lambda x}$.

Moment generating function:

$$M_x(t) = \int_0^\infty e^{tx}\,\lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x}\, dx = \frac{\lambda}{t-\lambda}, \;\; |t| < \lambda$$

$M_x'(t) = \frac{\lambda}{(\lambda-t)^2}$.

When $t = 0$:

$E[X] = M_x'(0) = \frac{1}{\lambda}$.

Second moment = second derivative:

$E[X^2] = M_x''(0) = \frac{2\lambda}{(\lambda-t)^3} = \frac{2}{\lambda^2}$.

Using the formula $V(x) = E[X^2] - (E[X])^2$:

$V(x) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$.