HMW 7- Data 605

OMER OZEREN

Table of Contents

Problem 1

Let $X_1, X_2, \ldots, X_n$ be n mutually independent random variables, each of which is uniformly distributed on the integers from 1 to k. Let Y denote the minimum of the X_i's. Find the distribution of Y.

SOLUTION

Number of possible combinations of X_i's is k^n (choosing n values out of k options with replacement).

Consider number of combinations with at least one 1. It is equal to all combinations (k^n) minus all combinations with values between 2 and k ($(k-1)^n$). So $P(Y=1) = \frac{k^n-(k-1)^n}{k^n}$.

Consider number of combinations with at least one 2 and no 1. It is euqal to all combinations (k^n) minus all combinations with at least one 1 (see above: $k^n-(k-1)^n$) and minus all combinations with values between 3 and k ($(k-2)^n$). So $P(Y=2) = \frac{k^n-(k^n-(k-1)^n)-(k-2)^n}{k^n} = \frac{k^n-k^n+(k-1)^n-(k-2)^n}{k^n} = \frac{(k-1)^n-(k-2)^n}{k^n}$.

Similarly considering combinations without 1 or 2 and with at least one 3,

$$\begin{aligned} P(Y=3) &= \frac{k^n - (k^n - (k-1)^n) - ((k-1)^n - (k-2)^n) - (k-3)^n}{k^n} \\ &= \frac{k^n - k^n + (k-1)^n - (k-1)^n + (k-2)^n - (k-3)^n}{k^n} \\ &= \frac{(k-2)^n - (k-3)^n}{k^n} \end{aligned}$$

.

More generally, we can see that $P(Y=a) = \frac{(k-a+1)^n - (k-a)^n}{k^n}$.

Problem 2.

Your organization owns a copier (future lawyers, etc.) or MRI (future doctors). This machine has a manufacturer's expected lifetime of 10 years. This means that we expect one failure every ten years. (Include the probability statements and R Code for each part.)

a.

What is the probability that the machine will fail after 8 years?. Provide also the expected value and standard deviation. Model as a geometric. (Hint: the probability is equivalent to not failing during the first 8 years.)

Solution

Let p be the probability that the machine fails, and $q = 1 - p$ the probability that it doesn't.

We're looking for the first failure (success) after 8 years, so this is a geometric distribution. Since we're only expecting one failure in 10 years, $p = 0.1,\ q = 0.9$.

$P(X > 8) = 1 - P(X \leq 8)$

$1 - (1 - q^{i+1}) = q^{i+1} = 0.9^9 = 0.3874.$

$E[X] = \frac{1}{p} = \frac{1}{0.1} = 10.$

$Var(X) = \frac{1-p}{p^2} = \frac{0.9}{(0.1)^2} = 90.$

Standard deviation = $\sqrt{Var(X)} = \sqrt{90} \approx 9.486833.$

```
pdf <- pgeom(8, 0.1, lower.tail = F)
p <- 0.1
q <- 1 - p
ex <- p^-1
var <- q/p^2
sd <- sqrt(var)
cat(sprintf("\n %s = %f \n",
            c("Probability", "Expected Value", "Variance", "Standard
Deviation"),
```

```
            c(pdf, ex, var, sd))
    )
##
##   Probability = 0.387420
##
##   Expected Value = 10.000000
##
##   Variance = 90.000000
##
##   Standard Deviation = 9.486833
```

b.

What is the probability that the machine will fail after 8 years?. Provide also the expected value and standard deviation. Model as an exponential.

Solution

$P(X > 8) = e^{-\lambda i}$, with $\lambda = 0.1$.

$P(X > 8) = e^{-0.8} = 0.449329$.

$E[X] = \frac{1}{\lambda} = 10$.

$Var(X) = \frac{1}{\lambda^2} = \frac{1}{0.1^2} = 100$.

Standard Deviation $= \sqrt{Var(X)} = \sqrt{100} = 10$.

```
pdf <- pexp(8, 0.1, lower.tail = F)
l <- 0.1
ex <- 1/l
var <- 1/l^2
sd <- sqrt(var)
cat(sprintf("\n %s = %f \n",
            c("Probability", "Expected Value", "Variance", "Standard
Deviation"),
            c(pdf, ex, var, sd))
    )
##
##   Probability = 0.449329
##
##   Expected Value = 10.000000
##
##   Variance = 100.000000
##
##   Standard Deviation = 10.000000
```

c.

What is the probability that the machine will fail after 8 years?. Provide also the expected value and standard deviation. Model as a binomial. (Hint: 0 success in 8 years)

Solution

We're looking for 0 successes in 8 years. So, $P(0)$ with $n = 8$.

$P(0) = \binom{8}{0}(0.1)^0 \times 0.98 - 0 = (0.9)^8 = 0.4304672.$

$E[X] = np = 8 \cdot 0.1 = 0.8.$

$Var(X) = npq = 0.8 \cdot 0.9 = 0.72.$

Standard Deviation $= \sqrt{Var(X)} = \sqrt{0.72} = 0.8485281.$

```
pdf <- pbinom(0, 8, 0.1)
n <- 8
i <- 0
p <- 0.1
q <- 0.9
ex <- n*p
var <- n*p*q
sd <- sqrt(var)
cat(sprintf("\n %s = %f \n",
            c("Probability", "Expected Value", "Variance", "Standard
Deviation"),
            c(pdf, ex, var, sd))
    )
##
##  Probability = 0.430467
##
##  Expected Value = 0.800000
##
##  Variance = 0.720000
##
##  Standard Deviation = 0.848528
```

d.

What is the probability that the machine will fail after 8 years?. Provide also the expected value and standard deviation. Model as a Poisson.

Solution

Since Poisson uses averages, and we expect one failure every 10 years, we can say the average yearly failure rate is 0.1. We're looking for 0 failures in the first 8 years.

$$P_i(t) = \frac{(\lambda t)^i e^{-\lambda t}}{i!}$$

$P_0(8) = \frac{(0.1 \cdot 8)^0 e^{-0.1 \cdot 8}}{0!} = e^{-0.8} = 0.449329.$

$E[X] = Var(X) = \lambda \cdot t = 0.1 \cdot 8 = 0.8.$

Standard Deviation $= \sqrt{Var(X)} = \sqrt{0.8} = 0.8944272.$

```
lambda <- 0.1
t <- 8
i <- 0
ex <- lambda*t
var <- lambda*t
sd <- sqrt(var)
pdf <- ppois(i, t*lambda)
cat(sprintf("\n %s = %f \n",
            c("Probability", "Expected Value", "Variance", "Standard
Deviation"),
            c(pdf, ex, var, sd))
    )
##
##  Probability = 0.449329
##
##  Expected Value = 0.800000
##
##  Variance = 0.800000
##
##  Standard Deviation = 0.894427
```