Series 1

Notation:

- $\mathbb{R}_+ := (0, +\infty)$.
- For any square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ for $n \in \mathbb{N}$, we denote by $|\boldsymbol{A}|$ its determinant.
- For $n, m \in \mathbb{N} \cup \{0\}$ let
  $C^n_m([0,1]\times[0,1]) := \left\{u : [0,1] \times [0,1] \to \mathbb{R} : u, \partial_x u, \partial_x^2 u, \ldots, \partial_x^n u, \partial_t u, \ldots, \partial_t^m u \in C\left([0,1] \times [0,1]\right)\right\}.$

1. Classification of PDEs

a) Determine the type (elliptic, parabolic, hyperbolic) of the following PDEs.

1.
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rs\frac{\partial V}{\partial s} - rV = 0, \quad (t, s) \in [0, +\infty) \times \mathbb{R}_+, \quad \sigma > 0, r \in \mathbb{R}. \tag{1}$$

2.
$$\frac{\partial^2 u}{\partial x_1^2} + x_1 \frac{\partial^2 u}{\partial x_2^2} + \frac{1}{2}\frac{\partial u}{\partial x_2} = 0, \quad (x_1, x_2) \in \mathbb{R}^2. \tag{2}$$

b) Consider the following second-order linear PDE:

$$-\left(a_{11}(x_1,x_2)\frac{\partial^2 u}{\partial x_1^2} + 2a_{12}(x_1,x_2)\frac{\partial^2 u}{\partial x_1 x_2} + a_{22}(x_1,x_2)\frac{\partial^2 u}{\partial x_2^2}\right) = f(x_1,x_2), \quad (x_1,x_2) \in \mathbb{R}^2. \tag{3}$$

Here $a_{11}, a_{12}, a_{22} \in C^0(\mathbb{R}^2)$. Justify that at any point $(x_1, x_2) \in \mathbb{R}^2$, the PDE (3) is

1. elliptic, if and only if $\begin{vmatrix} a_{11}(x_1,x_2) & a_{12}(x_1,x_2) \\ a_{12}(x_1,x_2) & a_{22}(x_1,x_2) \end{vmatrix} > 0$,
2. hyperbolic, if and only if $\begin{vmatrix} a_{11}(x_1,x_2) & a_{12}(x_1,x_2) \\ a_{12}(x_1,x_2) & a_{22}(x_1,x_2) \end{vmatrix} < 0$.

c) Does there exist a second-order linear PDE of the form (3) such that it is elliptic on a non-empty set $D \subsetneq \mathbb{R}^2$ and is hyperbolic on $\mathbb{R}^2 \setminus D$?

Hint: Use the fact that the only two subsets of $\mathbb{R}^2$ which are both open and closed in the standard topology of $\mathbb{R}^2$ are $\emptyset$ and $\mathbb{R}^2$.

2. Finite-Difference method for the heat equation

Consider the following heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{on } (0,1) \times (0,1)\,, \\ u(x,0) = u_0(x) & \forall x \in (0,1)\,, \\ u(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0 & \forall t \in (0,1)\,. \end{cases} \tag{4}$$

We assume that $u_0 \in C^2([0,1])$.

a) Show that for any solution $u \in C_1^2([0,1]\times[0,1])$ to (4), the function $f(t) := \int_0^1 u(x,t)^2 dx,\ t \in [0,1]$ is non-increasing on $[0,1]$. Show furthermore that for any $u_0 \in C^2([0,1])$, there exists at most one smooth solution $u \in C_1^2([0,1]\times[0,1])$ to (4).

Let $N, M \in \mathbb{N}$. For all $i = 0,\dots,N$ and $m = 0,\dots,M$ with $N, M \in \mathbb{N}$, we seek approximations of the values

$$u(x_i, t_m) := u(ih, mk)$$

on the spatio-temporal mesh $\{(ih, mk)\}_{i=0,\dots,N, m=0,\dots,M}$ where $h = \frac{1}{N}$ is the mesh width and $k = \frac{1}{M}$ is the time step. The approximations are denoted by u_i^m, are are given for each $m \in \{0,\dots,M-1\}$ as the solutions to the linear system of equations

$$\begin{cases} \dfrac{u_i^{m+1} - u_i^m}{k} - \dfrac{u_{i-1}^m - 2u_i^m + u_{i+1}^m}{h^2} & = & 0 \qquad \text{for } 0 < i \le N\,, \\[2ex] \dfrac{-u_{N-1}^{m+1} + u_{N+1}^{m+1}}{2h} & = & 0\,, \\[2ex] u_0^{m+1} & = & 0\,, \end{cases} \tag{5}$$

with initial values $u_i^0 = u_0(ih)$ ("explicit method"). Take note that u_{N+1}^m for $m \in \{0,\dots,M-1\}$ are fictitious values introduced to suitably treat the Neumann boundary condition.

b) Assume that $u \in C_2^4([0,1]\times[0,1])$. Show that there holds

$$\begin{aligned} \frac{\partial u}{\partial t}\Big|_{(x,t)=(x_i,t_m)} &= \frac{u_i^{m+1} - u_i^m}{k} + O(k)\,, \\ \frac{\partial u}{\partial x}\Big|_{(x,t)=(x_N,t_m)} &= \frac{u_{N+1}^m - u_{N-1}^m}{2h} + O(h^2), \\ \frac{\partial^2 u}{\partial x^2}\Big|_{(x,t)=(x_i,t_m)} &= \frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{h^2} + O(h^2)\,. \end{aligned}$$

c) After eliminating the variables u_0^m and u_{N+1}^m, rewrite the system (5) in the form

$$\underline{u}^{m+1} = \boldsymbol{C}\underline{u}^m\,, \quad \text{where } \underline{u}^m := \begin{pmatrix} u_1^m \\ u_2^m \\ \cdots \\ u_N^m \end{pmatrix} \in \mathbb{R}^N\,.$$

Show that $\boldsymbol{C}$ is given by $\boldsymbol{C} = \boldsymbol{I} + \nu \boldsymbol{G}$ for some coefficient $\nu > 0$ where $\boldsymbol{I}$ is the $N \times N$ identity matrix and

$$\boldsymbol{G} = \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & \ddots & (0) & \vdots \\ 0 & 1 & -2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & (0) & \ddots & 1 & -2 & 1 \\ 0 & \dots & \dots & 0 & 2 & -2 \end{pmatrix}.$$

d) Consider an alternative way to discretize (4) which is described below ("implicit method").

$$\begin{cases} \dfrac{u_i^{m+1} - u_i^m}{k} - \dfrac{u_{i-1}^{m+1} - 2u_i^{m+1} + u_{i+1}^{m+1}}{h^2} &= 0 \qquad \text{for } 0 < i \leq N\,, \\[2ex] \dfrac{-u_{N-1}^{m+1} + u_{N+1}^{m+1}}{2h} &= 0\,, \\[2ex] u_0^{m+1} &= 0\,, \end{cases} \tag{6}$$

After eliminating the variables u_0^{m+1} and u_{N+1}^{m+1}, rewrite the system (5) in the form

$$\boldsymbol{C}\underline{u}^{m+1} = \underline{u}^m\,, \quad \text{where } \underline{u}^m := \begin{pmatrix} u_1^m \\ u_1^m \\ \dots \\ u_N^m \end{pmatrix}.$$

Show that $\boldsymbol{C}$ is given by $\boldsymbol{C} = \boldsymbol{I} - \nu\boldsymbol{G}$ for some coefficient $\nu > 0$.

3. Implementation in Python

We implement Finite Difference Method to (4) using Python. From now on we set $u_0(x) = \sin\left(\frac{\pi}{2}x\right)$. All the coding tasks in this exercise shall be finished in the template `1_exercise3_template.py`.

a) Verify that $u(x,t) = e^{-\frac{\pi^2}{4}t}\sin\left(\frac{\pi}{2}x\right)$ is the solution to (4). Use this result to complete the Python function "`exact_solution_at_1(x)`", which receives a vector of grid points and computes a vector containing the value of $u(x,1)$ at these points.

b) Complete the Python functions "`eulerexplicit(N,M)`" and "`eulerimplicit(N,M)`" which compute the numerical approximation of $u(x,1)$ using the numerical scheme (5) and (6) in matrix form. The parameters of each function (N,M) describe the grid on $(0,1)\times(0,1)$ as in Exercise 2 and the return value u_M shall be the numerical solution of u at $t=1$, which is an array with N elements (not including the value at $x=0$).

c) Test both functions with the template using the parameters $N = 2^l$ and $M = 2\times 4^l$ with $l = \{2,3,4,5,6\}$. Use the template to obtain the convergence rate with respect to h^2+k and generate the plot describing the convergence rate. You first need to modify the block "error analysis". Follow the comments in the codes.

d) Perform the same test using the parameters $N = 2^l$ and $M = 4^l$ with $l = \{2,3,4,5,6\}$ and report on the performances of the two methods. You first need to modify the block "error analysis". Follow the comments in the codes. What happens to the discrete L^2 error of the numerical solution with scheme (5)? Check the error stored in the array "`l2errorexplicit`".

Due: Wednesday, March 7th, at 2pm.