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Exercise 27 (Frequently asked questions (FAQ) about multistep-methods)

To solve the IVP $x' = f(t,x)$, $x(t_0) = x_0$ on an equidistant time grid with stepsize τ we consider linear multistep-methods of the form

$$\varrho(E)x_\tau = \tau\sigma(E)f_\tau \qquad \text{with} \qquad \varrho(\xi) = \alpha_k\xi^k + \alpha_{k-1}\xi^{k-1} + \cdots + \alpha_0$$

with $\alpha_k \neq 0$ and $|\alpha_0| + |\beta_0| > 0$.
Prove the following statements or find counter examples:

(a) If $\alpha_j = \alpha_{k-j}$ for all $j = 1, \ldots, k$ and the method is (zero-)stable then all roots of ϱ are simple and they are on the boundary of the unit disk.

(b) If (ϱ_1, σ_1) and (ϱ_2, σ_2) are two k-step methods of order p then for all $\lambda \in [0; 1]$ also (ϱ, σ) with $\varrho := \lambda\varrho_1 + (1-\lambda)\varrho_2$ is a k-step method of order p.

(c) Let (ϱ_1, σ_1) and (ϱ_2, σ_2) be two (zero-)stable and consistent multistep-methods. For all $\lambda \in [0; 1]$ the methods (ϱ, σ) with $\sigma := \lambda\sigma_1 + (1-\lambda)\sigma_2$ and $\varrho := \lambda\varrho_1 + (1-\lambda)\varrho_2$ are also multistep methods. Then (ϱ, σ) is stable.

Exercise 28

Find all $y \in C^1[a,b]$ with $y(a) = y_a$, $y(b) = y_b$ and $\displaystyle\int_a^b x^2(y')^2\,dx = \min!$ for the following cases:

(a) $y(1) = 1$ and $y(2) = 1/2$.

(b) $y(-2) = -1/2$ and $y(1) = 1$.

Exercise 29

Consider the IVP

$$x'' = 100x, \qquad x(0) = 1, \quad x'(0) = s$$

with the solution $x(t; s)$. Let $\bar{s} = s(1+\epsilon)$ (with $0 < \epsilon < 1$).

(a) Compute for $s = -10$ the relative error of $x(3; \bar{s})$.

(b) Give reasons why it's not a good idea to solve the boundary value problem

$$x'' = 100x, \qquad x(0) = 1, \quad x(3) = e^{-30}$$

by numerically searching roots for

$$F(s) = x(3; s) - e^{-30} \stackrel{!}{=} 0.$$