Series 8

1. Theoretical study of the PSOR algorithm

a) Consider a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and two vectors $b, c \in \mathbb{R}^n$ and consider the following linear complementarity problem (LCP): Find $\underline{x} \in \mathbb{R}^N$ such that

$$\begin{aligned} &\mathbf{A}\underline{x} \geq \underline{b}\,, \\ &\underline{x} \geq \underline{c}\,, \\ &(\underline{x} - \underline{c})^\top(\mathbf{A}\underline{x} - \underline{b}) = \underline{0}\,. \end{aligned} \tag{1}$$

Show that the LCP (1) is equivalent to the following problem: Find $\underline{x} \in \mathbb{R}^n$ such that

$$\begin{aligned} &(\underline{y} - \underline{x})^\top(\mathbf{A}\underline{x} - \underline{b}) \geq \underline{0}\,, \quad \forall y \in \mathcal{K}\,, \\ &\underline{x} \in \mathcal{K}\,, \end{aligned} \tag{2}$$

where

$$\mathcal{K} = \{\underline{y} \in \mathbb{R}^n : \underline{y} \geq \underline{c}\}\,.$$

b) Assume now the following:

i) There exist constants $C_1, C_2 > 0$ such that $C_1 \underline{v}^\top \underline{v} \leq \underline{v}^\top \mathbf{A} \underline{v} \leq C_2 \underline{v}^\top \underline{v}$.

ii) The matrix $\mathbf{A}$ is diagonally dominant, i.e. $|A_{ii}| > \sum_{j\neq i} |A_{ij}|\,, \forall i$.

Let $\psi = \psi(\omega)$ be the mapping

$$\begin{aligned} &\psi \colon \underline{x} \mapsto \underline{z}: \\ &\forall i = 1, \ldots, n, \quad z_i = \max(y_i, c_i), \quad \text{and } y_i \text{ is given by} \\ &\frac{1}{\omega}\mathbf{A}_{ii} y_i + \sum_{j<i} \mathbf{A}_{ij} z_j = b_i + \left(\frac{1}{\omega} - 1\right)\mathbf{A}_{ii} x_i - \sum_{j>i} \mathbf{A}_{ij} x_j\,. \end{aligned}$$

In the lecture it was shown that for each $0 < \omega \leq 1$, ψ is a self-mapping contraction and hence admits a unique fixed-point $\underline{x}^* = \underline{x}^*(\omega)$. Show that if $\underline{x}$ is the unique solution to the LCP (1), then $\underline{x}$ is the unique fixed point of ψ, i.e. $\psi(\underline{x}) = \underline{x}$. In particular, this shows that the fixed-point $\underline{x}$ of $\psi(\omega)$ is independent of the choice of ω.

Hint: Let $\underline{u}$ be the solution to (1) and denote $\underline{z} = \psi(\underline{u})$. Show by induction on the components, that $\underline{z} = \underline{u}$.

2. Lookback option

In this exercise, we consider a first example of an *exotic option* in the Black-Scholes model: Let the price process of a given underlying be modeled by the SDE

$$dS_t = rS_t dt + \sigma S_t dW_t$$

for a Brownian motion W and with initial price $S_0 = s$, interest rate $r \geq 0$ and volatility $\sigma > 0$. A *Lookback call option* written at time 0 with strike K and maturity T yields the payoff

$$H(S) = \max\left(\max_{t\in[0,T]} S_t - K, 0\right)$$

at maturity.

The goal of this exercise is to verify that for $K \geq S_0 = s$, the following closed-form expression for the fair value of a lookback option at time $t = 0$ holds true:

$$\mathbb{E}[e^{-rT}H(S)|S_0 = s] = s\Phi(d) - e^{-rT}K\Phi(d-\sigma\sqrt{T}) + e^{-rT}\frac{\sigma^2}{2r}s\left[-\left(\frac{s}{K}\right)^{\frac{-2r}{\sigma^2}}\Phi\left(d - \frac{2r}{\sigma}\sqrt{T}\right) + e^{rT}\Phi(d)\right],$$

where

$$d = \frac{\log(\frac{s}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

and Φ cumulative density function of the standard normal distribution.

a) Calculate the price of a digital lookback option

$$\mathbb{E}\left[\mathbf{1}_{\{\max_{t\in[0,T]} S_t \geq K\}}\right].$$

Hint: For a given constant $\alpha \in \mathbb{R}$, denote the supremum of a Brownian motion with drift α as the random variable $Y = \sup_{t\in[0,T]} \alpha t + W_t$. Use without proof that Y has the distribution function

$$P(Y \leq y) = \Phi\left(\frac{y - \alpha T}{\sqrt{T}}\right) - \exp(2\alpha y)\Phi\left(\frac{-y - \alpha T}{\sqrt{T}}\right)$$

for $y > 0$ and $P(Y \leq y) = 0$ for $y \leq 0$.

b) Show that

$$\mathbb{E}\left[(\max_{t\in[0,T]} S_t)\mathbf{1}_{\{\max_{t\in[0,T]} S_t \geq K\}}\right] = \left(1 - \frac{\sigma^2}{2r}\right)s\left(\frac{K}{s}\right)^{\frac{2r}{\sigma^2}}\Phi\left(d - \frac{2r}{\sigma}\sqrt{T}\right) + \left(1 + \frac{\sigma^2}{2r}\right)e^{rT}s\Phi(d).$$

Hint: Derive the probability density function of Y from the hint to part **2a)**.

c) Combine the results from **2a)** and **2b)** to deduce the desired expression for the price of the lookback option.

3. Dimension reduction for the Asian option pricing equation

Consider an Asian option on a stock under the Black–Scholes model with volatility $\sigma > 0$, risk-free interest rate $r \geq 0$ and spot price $s_0 > 0$. The value function $V : J \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is given for the interval $J = (0, T)$ by

$$V(t, s, y) = \mathbb{E}\left[e^{-r(T-t)}g(S_T, Y_T) \mid S_t = s, Y_t = y\right], \quad \text{for} \quad t \in J,\ s, y \in \mathbb{R}_+,$$

where $g : \mathbb{R}_+ \to \mathbb{R}_{\geq 0}$ is the payoff function, and $Y_t := \int_0^t S_\tau \mathrm{d}\tau$ for all $t \in J$. Then, the vector (S, Y) is the solution to two-dimensional SDE

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t, & t &\in J, & S_0 &= s_0, \\ dY_t &= S_t dt, & t &\in J, & Y_0 &= 0. \end{aligned}$$

Under appropriate smoothness assumptions, the corresponding backward Kolmogorov equation for the option value $V : J \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ reads

$$\begin{cases} \partial_t V + \dfrac{1}{2}\sigma^2 s^2 \partial_{ss} V + rs\partial_s V + s\partial_y V - rV = 0 & \text{on} \quad J \times \mathbb{R}_+ \times \mathbb{R}_+, \\ V(T, s, y) = g(s, y) & \text{on} \quad \mathbb{R}_+ \times \mathbb{R}_+. \end{cases} \tag{3}$$

By switching to the variables $(t, s, y) \to (t, z)$ with a suitable change of variable $z = \xi(t, s, y) = \zeta(s, y) + q(t)$ with $\zeta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and $q : J \to \mathbb{R}$, one may transform (3) to

$$\begin{cases} \partial_t H + \dfrac{1}{2}\sigma^2 (q(t) - z)^2 \partial_{zz} H + r(q(t) - z)\partial_z H = 0 & \text{on} \quad J \times \mathbb{R}\,, \\ \hfill H(T, z) = \max\{z, 0\} & \text{on} \quad \mathbb{R}\,, \end{cases} \tag{4}$$

where $sH(t, \xi(t, s, y)) = V(t, s, y)$.

a) Derive (4) from (3) in the case of a *fixed strike call* for $g(s, y) = (\frac{y}{T} - K)_+$ using $z = \xi(t, s, y) = \frac{1}{s}(\frac{y}{T} - K) + q(t)$ with $q(t) = 1 - \frac{t}{T}$.

b) Derive (4) from (3) in the case of a *floating strike call* for $g(s, y) = (s - \frac{y}{T})_+$ using $z = \xi(t, s, y) = \frac{1}{s}(-\frac{y}{T}) + q(t)$ with $q(t) = \frac{t}{T}$.

Due: Wednesday, May 1st, at 2pm.