Solutions: Series 7

1. Implementation of the CEV model in Python

a) We define the discrete variational problem as: find $u_N \in V_{N+1}$ such that

$$\begin{cases} \left(\dfrac{\partial u_N}{\partial t}(t,\cdot), v_N\right) + a^{\text{CEV}}_{\rho,\mu}(u_N(t,\cdot), v_N) &= 0 \quad \forall v_N \in V_{N+1}\,, \\ u_N(0,\cdot) &= u_{N,0}\,, \end{cases}$$

where $u_{N,0}$ is the L^2 projection of g on V_{N+1} (or the piecewise linear interpolant when g is continuous). Introduce the mass and stiffness matrices

$$\mathbf{M}^{\text{CEV}}_{i,j} := (b_{j-1}, b_{i-1})_\mu\,,$$

$$\mathbf{A}^{\text{CEV}}_{i,j} := a^{\text{CEV}}_{\rho,\mu}(b_{j-1}, b_{i-1})_\mu\,.$$

Then, the variational problem is equivalent to the differential equation system

$$\begin{cases} \mathbf{M}^{\text{CEV}}\dfrac{d}{dt}\underline{u_N} + \mathbf{A}^{\text{CEV}}\underline{u_N} &= 0 \\ \underline{u_N}(0) &= \underline{u_N}_0 \end{cases}$$

where $\underline{u_N}(t)$ (resp. $\underline{u_N}_0$) is the column vector of coefficients of $u_N(t,\cdot)$ (resp. $u_{N,0}$) in the basis $\{b_i\}_{0\le i\le N}$. For the time discretization, we approximate the vector $\underline{u_N}(t_m)$ by the vector $\underline{u}^m$ defined by

$$\underline{u}^0 = \underline{u_N}_0\,,$$
$$\mathbf{B}_\vartheta \underline{u}^{m+1} = \mathbf{C}_\vartheta \underline{u}^m\,,$$

where

$$\mathbf{B}_\vartheta = \mathbf{M}^{\text{CEV}} + k\vartheta\mathbf{A}^{\text{CEV}}\,, \quad \mathbf{B}_\vartheta = \mathbf{M}^{\text{CEV}} - k(1-\vartheta)\mathbf{A}^{\text{CEV}}\,.$$

b) We split the integral to be evaluated into two integrals over (x_{i-1}, x_i) and (x_i, x_{i+1}) and apply Gaussian quadrature on these intervals, when the intergrand has its support on (x_{i-1}, x_{i+1}) and we apply Gaussian quadrature on (x_{i-1}, x_i), when the integrand has its support on (x_{i-1}, x_i). By following this strategy we obtain

$$\begin{aligned}(\beta b_i', b_i) = &\frac{1}{2}\left(\beta\left(x_{i-1}+p_1\right)w_1 + \beta\left(x_{i-1}+p_2\right)w_2\right) \\ &- \frac{1}{2}\left(\beta\left(x_i+p_1\right)w_2 + \beta\left(x_i+p_2\right)w_1\right)\,,\end{aligned}$$

for $i = 1, 2, \ldots, N$ and

$$(\beta b_0', b_0) = -\frac{1}{2}\left(\beta\left(p_1\right)w_2 + \beta\left(p_2\right)w_1\right).$$

Also, for $i = 0, 1, \ldots, N-1$,

$$(\beta b_i', b_{i+1}) = -\frac{1}{2}\left(\beta\left(x_i+p_1\right)w_1 + \beta\left(x_i+p_2\right)w_2\right)$$

$$(\beta b_{i+1}', b_i) = \frac{1}{2}\left(\beta\left(x_i+p_1\right)w_2 + \beta\left(x_i+p_2\right)w_1\right).$$

Finally,

$$\begin{aligned}(\gamma b_i, b_i) = &\frac{h}{2}\left(\gamma\left(x_{i-1}+p_1\right)w_1^2 + \gamma\left(x_{i-1}+p_2\right)w_2^2\right) \\ &+ \frac{h}{2}\left(\gamma\left(x_i+p_1\right)w_2^2 + \gamma\left(x_i+p_2\right)w_1^2\right)\end{aligned}$$

for $i = 1, 2, \ldots, N$ and

$$(\gamma b_0, b_0) = \frac{h}{2}\left(\gamma\left(p_1\right) w_2^2 + \gamma(p2) w_1^2\right)$$

$$(\gamma b_i, b_{i+1}) = (\gamma b_{i+1}, b_i) = \frac{h}{12}\left(\gamma\left(x_i + p_1\right) + \gamma\left(x_i + p_2\right)\right).$$

For the programming part, see the solution code.

c) See the solution code.

d) See the solution code.

2. FEM for American put in a BS market

a) First, we directly have $u_R \in \mathcal{K}_{0,R}$ since $v_R \geq \tilde{g}$. It remains to show that for all $v \in \mathcal{K}_{0,R}$, there holds, for all $t \in (0, T]$ (for $t = 0$, the next inequality is satisfied as an equality by the initial condition)

$$\langle \partial_t u_R(t,\cdot), v - u_R(t,\cdot)\rangle + a^{\mathrm{BS}}(u_R(t,\cdot), v - u_R(t,\cdot)) \geq -a^{\mathrm{BS}}(\tilde{g}, v - u_R(t,\cdot)).$$

From now on, we fix $t > 0$ and (abusively) write u_R instead of $u_R(t,\cdot)$ for conciseness. The previous inequality reads equivalently

$$\langle \partial_t v_R, v - u_R\rangle + a^{\mathrm{BS}}(v_R, v - u_R) \geq 0\,,$$

where we used that $\partial_t \tilde{g} = 0$ and $u_R + \tilde{g} = v_R$. This is, in turn, equivalent to

$$\langle \partial_t v_R, v\rangle + a^{\mathrm{BS}}(v_R, v) \geq \langle \partial_t v_R, u_R\rangle + a^{\mathrm{BS}}(v_R, u_R)\,. \tag{1}$$

For any function f of class C^2 and for all $\varphi \in H_0^1(G)$, we can write (using the usual integration by parts):

$$\langle \partial_t f, \varphi\rangle + a^{\mathrm{BS}}(f, \varphi) = \int_G \left(\partial_t f - \mathcal{A}^{\mathrm{BS}} f + r f\right)\varphi\,.$$

We can apply this argument in Eq. (1), since v_R is of class C^2 while both v and u_R are in $H_0^1(G)$. After this transformation, the inequality we have to show becomes

$$\int_G v\left(\partial_t v_R - \mathcal{A}^{\mathrm{BS}} v_R + r v_R\right) \geq \int_G u_R\left(\partial_t v_R - \mathcal{A}^{\mathrm{BS}} v_R + r v_R\right)\,.$$

This indeed holds since, on the one hand $v \geq 0$ and by the first equation of the Partial Differential Inequality,

$$\left(\partial_t v_R - \mathcal{A}^{\mathrm{BS}} v_R + r v_R\right) \geq 0$$

and on the other hand, by the third equation,

$$u_R\left(\partial_t v_R - \mathcal{A}^{\mathrm{BS}} v_R + r v_R\right) = 0\,.$$

b) Let

$$V_N = \left\{u \in C^0(G) \;\middle|\; u(\pm R) = 0 \text{ and } \forall i \in \{0, \ldots, N\},\ u_{|s \in (x_i, x_{i+1})} \text{ is a linear function}\right\},$$

and let b_i be the element of V_N defined by

$$b_i(x_j) = \delta_{i,j}\,.$$

Furthermore, let $\mathbf{M}$ and $\mathbf{A}^{\mathrm{BS}}$ be the mass and stiffness matrices defined by

$$\mathbf{M}_{i,j} = \int_G b_i(x) b_j(x)\,dx\,, \quad \mathbf{A}^{\mathrm{BS}}_{i,j} = a^{\mathrm{BS}}(b_j, b_i)\,.$$

These matrices were calculated in exercise 2 of the previous exercise sheet. The variational inequality can then be written using the space $W_N \coloneqq \mathcal{K}_{0,R} \cap V_N$: find $u_N \in W_N$ such that, for all $w_N \in W_N$,

$$\begin{cases} \langle \partial_t u_N, w_N - u_N \rangle + a^{\mathrm{BS}}(u_N, w_N - u_N) & \geq -a^{\mathrm{BS}}(\tilde{g}, w_N - u_N)\,, \\ u_N(0) & = 0\,. \end{cases}$$

In matrix form, this reads

$$(\underline{w}_N - \underline{u}_N)^T \mathbf{M}\, \underline{\dot{u}}_N + (\underline{w}_N - \underline{u}_N) \mathbf{A}^{\mathrm{BS}} \underline{u}_N \geq (\underline{w}_N - \underline{u}_N)^T \underline{G}\,,$$

where $\underline{u}_N$ and $\underline{w}_N$ are the column vectors of coefficients of u_N and w_N in the basis $\{b_i\}_{1 \leq i \leq N}$, $\underline{\dot{u}}_N$ is the time derivative of $\underline{u}_N$, and $\underline{G}$ is the column vector whose components are given by

$$\underline{G}_i = -a^{\mathrm{BS}}(\tilde{g}, b_i)\,.$$

Next, we define approximations $\underline{u}_N^m$ of the vectors $\underline{u}_N(t_m)$. Using the approximation (recall that we want to derive a backward scheme)

$$\underline{\dot{u}}_N(t_{m+1}) \approx \frac{\underline{u}_N(t_{m+1}) - \underline{u}_N(t_m)}{k}\,,$$

we obtain the following iterative scheme

$$(\underline{w}_N - \underline{u}_N^{m+1})^T \mathbf{M}\, \frac{\underline{u}_N^{m+1} - \underline{u}_N^m}{k} + (\underline{w}_N - \underline{u}_N^{m+1}) \mathbf{A}^{\mathrm{BS}} \underline{u}_N^{m+1} \geq (\underline{w}_N - \underline{u}_N^{m+1})^T \underline{G}\,.$$

Hence, we have to find, for each $m \geq 0$, the vector $\underline{u}_N^{m+1} \in \mathbb{R}^N_{\geq 0}$ such that, for all $\underline{w}_N \in \mathbb{R}^N_{\geq 0}$,

$$(\underline{w}_N - \underline{u}_N^{m+1})^T \left(\mathbf{M} + k\mathbf{A}^{\mathrm{BS}}\right) \underline{u}_N^{m+1} \geq (\underline{w}_N - \underline{u}_N^{m+1})^T \left(k\underline{G} + \mathbf{M}\underline{u}_N^m\right)\,,$$

where $\underline{u}_N^0 = 0$. By setting $\mathbf{B} = \mathbf{M} + k\mathbf{A}^{\mathrm{BS}}$, $\underline{F}^m = k\underline{G} + \mathbf{M}\underline{u}_N^m$ and applying Lemma 5.3.1 we observe that this is equivalent to finding $\underline{u}_N^{m+1} \in \mathbb{R}^N$ for $m = 0, \ldots, M-1$ such that

$$\begin{aligned} &\mathbf{B}\underline{u}_N^{m+1} \geq \underline{F}^m\,, \\ &\underline{u}_N^{m+1} \geq \underline{0}\,, \\ &(\underline{u}_N^{m+1})^\top \left(\mathbf{B}\underline{u}_N^{m+1} - \underline{F}^m\right) = 0. \end{aligned} \tag{2}$$

c) See the solution code.

d) See the solution code. You shall observe that the American option value is greater than the European option value.