Solutions: Series 5

1. Inverse inequality for $S^1(\mathcal{T}_h)$

a) Using the substitution $x = \psi(\hat{x})$, we obtain that

$$\|u\|^2_{L^2(K)} = \int_a^b u(x)^2\,dx = \int_{-1}^1 u(\psi(\hat{x}))^2 \cdot \psi'(\hat{x})\,d\hat{x} = \frac{b-a}{2}\int_{-1}^1 \hat{u}(\hat{x})^2\,d\hat{x} = \frac{b-a}{2}\|\hat{u}\|^2_{L^2(\hat{K})}\,.$$

For $\widehat{x} \in \widehat{K}$, we have

$$\frac{d}{d\widehat{x}}\left(u(\psi(\widehat{x}))\right) = u'(\psi(\hat{x})) \cdot \frac{d}{d\widehat{x}}\psi(\widehat{x})\,,$$

hence we obtain that

$$u'(\psi(\hat{x})) = \frac{2}{b-a}\frac{d}{d\widehat{x}}\left(u(\psi(\widehat{x}))\right)\,.$$

It therefore follows that

$$\|u'\|^2_{L^2(K)} = \int_a^b u'(x)^2\,dx = \int_{-1}^1 u'(\psi(\hat{x}))^2 \cdot \psi'(\hat{x})\,d\hat{x} = \frac{2}{b-a}\int_{-1}^1 \hat{u}'(\hat{x})^2\,d\hat{x} = \frac{2}{b-a}\|\hat{u}'\|^2_{L^2(\hat{K})}\,.$$

b) Consider a sub-inteval $K_l = (x_{l-1}, x_l)$ for some $1 \le l \le N+1$ and let $\hat{K} = (-1,1)$ as in the previous question. Consider the space $S^1(\hat{K}) = \{u(x) \in C^0(\overline{\hat{K}}) : u \text{ is linear on } \hat{K}\}$. It is a finite-dimensional space, therefore the norms $\|\cdot\|_{H^1(\hat{K})}$ and $\|\cdot\|_{L^2(\hat{K})}$ on it are equivalent. This indicates that there exists a constant $C_1 > 0$ independent of a, b, h such that for any $\hat{u} \in S^1(\hat{K})$,

$$\|\hat{u}'\|_{L^2(\hat{K})} \le \|\hat{u}\|_{H^1(\hat{K})} \le C_1\|\hat{u}\|_{L^2(\hat{K})}\,. \tag{1}$$

One may alternatively show the existence of C_1 by noting that for any $\hat{u}(\hat{x}) = \alpha + \beta\hat{x}$ with $\alpha, \beta \in \mathbb{R}$,

$$\|\hat{u}'\|_{L^2(\hat{K})} = \sqrt{2}\beta\,, \quad \|\hat{u}\|_{L^2(\hat{K})} \ge \|\beta\hat{x}\|_{L^2(\hat{K})} = \frac{\sqrt{6}}{3}\beta\,.$$

Therefore one can set $C_1 = \sqrt{3}$. Now **a)** helps us to spread estimate (1) to all subintervals of $\mathcal{T}_h$: For any $u_h \in S^1(\mathcal{T}_h)$ and any $K_l, l = 1, \ldots, N+1$, we have that

$$\|(u_h)'\|^2_{L^2(K_l)} \le C_1^2 h^{-2}\|u_h\|^2_{L^2(K_l)}\,.$$

Summing over all subintervals yields

$$\|(u_h)'\|^2_{L^2(G)} \le C_1^2 h^{-2}\|u_h\|^2_{L^2(G)}\,.$$

Finally,

$$\begin{aligned}\|u_h\|^2_{H^1(G)} &= \|(u_h)'\|^2_{L^2(G)} + \|u_h\|^2_{L^2(G)} \le (1 + C_1^2 h^{-2})\|u_h\|^2_{L^2(G)} \le \left((b-a)^2 h^{-2} + C_1^2 h^{-2}\right)\|u_h\|^2_{L^2(G)} \\ &= \left((b-a)^2 + C_1^2\right)h^{-2}\|u_h\|^2_{L^2(G)}\,.\end{aligned}$$

Setting $C = \sqrt{(b-a)^2 + C_1^2}$ finishes the proof.

2. A priori estimates

a) We have, for any $x \in G = (a,b)$ and any $\epsilon \in (0,1)$,

$$|u_{K+\epsilon}(0,x) - u_K(0,x)| = \begin{cases} \epsilon & \text{if } \exp(x) \ge K + \epsilon\,, \\ \exp(x) - K & \text{if } \exp(x) \in (K, K+\epsilon)\,, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$|u_{K+\epsilon}(0,x) - u_K(0,x)| \le \epsilon\,,$$

and we could set $C_1 = \sqrt{b-a}$.

b) Let $w := u_{K+\epsilon} - u_K$. Since the PDE we treat is linear, we have that

$$\begin{aligned} \partial_t w - \partial_x(\alpha(x)\partial_x w) + \beta(x)\partial_x w + \gamma(x) w = 0 \qquad & \text{in } J \times G\,, \\ w = 0 \qquad & \text{on } J \times \partial G\,, \\ w|_{t=0} = g_{K+\epsilon}(\exp(x)) - g_K(\exp(x)) \qquad & \text{in } G\,. \end{aligned} \tag{2}$$

The continuity and the Gårding inequality for $a(\cdot,\cdot)$ justified in Exercise sheet 3 enable us to apply the well-posedness Theorem 3.2.2 in the textbook (also on Slide 2) and we obtain that there exists a constant $\widehat{C} > 0$ with

$$\begin{aligned} \sup_{t\in\bar{J}} \|u_{K+\epsilon}(t,\cdot) - u_K(t,\cdot)\|_{L^2(G)} &= \|w\|_{C^0(\bar{J};L^2(G))} \\ &\le \widehat{C}\|w|_{t=0}\|_{L^2(G)}\,. \end{aligned}$$

Now, with the result of the previous question we obtain

$$\|w|_{t=0}\|_{L^2(G)} = \|u_{K+\epsilon}(t=0) - u_K(t=0)\|_{L^2(G)} \le C_1\epsilon\,.$$

Hence,

$$\sup_{t\in\bar{J}} \|u_{K+\epsilon}(t,\cdot) - u_K(t,\cdot)\|_{L^2(G)} \le C_1\widehat{C}_2\epsilon\,.$$

Clearly, we could set $C_2 = C_1\widehat{C}$ to finish the proof.

3. Black-Scholes formula for European call options

a) Define $f(t,x) = S_0 \exp\big(\big(r - \sigma^2/2\big)\,t + \sigma x\big)$. Then,

$$df(t,W_t) = (\partial_t f + \frac{1}{2}\partial_x^2 f)\,dt + \partial_x f\,dW_t = (r - \frac{\sigma^2}{2} + \frac{\sigma^2}{2}) f\,dt + \sigma f\,dW_t = rf(t,W_t)\,dt + \sigma f(t,W_t)\,dW_t\,.$$

This implies that $S_t = f(t,W_t)$ is a solution to the SDE.

b) We have,

$$S_T = S_0 \exp\big(\big(r-\sigma^2/2\big)\,T + \sigma W_T\big) = \exp\big(\ln(S_0) + \big(r-\sigma^2/2\big)\,T + \sigma W_T\big)\,.$$

Furthermore, $W_T \sim \sqrt{T}Z$ with $Z \sim \mathcal{N}(0,1)$, and consequently

$$S_T = \exp\Big(\ln(S_0) + \big(r-\sigma^2/2\big)\,T + \sigma\sqrt{T}Z\Big)\,.$$

We define the quantities

$$\alpha_T := \ln(S_0) + \big(r-\sigma^2/2\big)\,T, \quad \beta_T := \sigma\sqrt{T}\,.$$

Then, for every $K \in (0,+\infty)$ we have that

$$\mathbb{E}\left[f\left(S_T\right)\right] = \mathbb{E}\left[\max\left\{e^{\alpha_T + \beta_T Z} - K, 0\right\}\right] = \int_{\mathbb{R}} \max\left\{e^{\alpha_T+\beta_T y} - K, 0\right\} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,\mathrm{d}y\,.$$

We observe that

$$e^{\alpha_T+\beta_T y} - K \ge 0 \Longleftrightarrow \alpha_T + \beta_T y \ge \ln(K) \Longleftrightarrow y \ge \frac{\ln(K) - \alpha_T}{\beta_T}\,,$$

and consequently

$$\begin{aligned}
\mathbb{E}_P\left[f\left(S_T\right)\right] &= \int_{\frac{\ln(K)-\alpha_T}{\beta_T}}^{\infty} \left(e^{\alpha_T+\beta_T y}-K\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\,\mathrm{d}y \\
&= e^{\alpha_T}\int_{\frac{\ln(K)-\alpha_T}{\beta_T}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\left(\beta_T y-\frac{1}{2}y^2\right)}\mathrm{d}y - K\int_{\frac{\ln(K)-\alpha_T}{\beta_T}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\,\mathrm{d}y \\
&= e^{\alpha_T}\int_{\frac{\ln(K)-\alpha_T}{\beta_T}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(y^2-2\beta_T y+\beta_T^2-\beta_T^2\right)}\mathrm{d}y - K\Phi\left(\frac{\alpha_T-\ln(K)}{\beta_T}\right) \\
&= e^{\alpha_T+\frac{1}{2}\beta_T^2}\int_{\frac{\ln(K)-\alpha_T}{\beta_T}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-\beta_T)^2}\,\mathrm{d}y - K\Phi\left(\frac{\alpha_T-\ln(K)}{\beta_T}\right).
\end{aligned}$$

By the change of variable $z=y-\beta_T$ we obtain

$$\begin{aligned}
\mathbb{E}_P\left[f\left(S_T\right)\right] &= e^{\alpha_T+\frac{1}{2}\beta_T^2}\int_{\frac{\ln(K)-\alpha_T}{\beta_T}-\beta_T}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,\mathrm{d}z - K\Phi\left(\frac{\alpha_T-\ln(K)}{\beta_T}\right) \\
&= e^{\alpha_T+\frac{1}{2}\beta_T^2}\Phi\left(\frac{\alpha_T-\ln(K)}{\beta_T}+\beta_T\right) - K\Phi\left(\frac{\alpha_T-\ln(K)}{\beta_T}\right).
\end{aligned}$$

Since

$$e^{-rT}e^{\alpha_T+\frac{1}{2}\beta_T^2} = e^{\ln(S_0)} = S_0\,,$$

we arrive at the desired formula

$$\begin{aligned}
e^{-rT}\mathbb{E}_P\left[f\left(S_T\right)\right] &= S_0\Phi\left(\frac{\alpha_T-\ln(K)}{\beta_T}+\beta_T\right) - e^{-rT}K\Phi\left(\frac{\alpha_T-\ln(K)}{\beta_T}\right) \\
&= S_0\Phi\left(\frac{\left(r-\frac{\sigma^2}{2}\right)T+\ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}}+\sigma\sqrt{T}\right) - e^{-rT}K\Phi\left(\frac{\left(r-\frac{\sigma^2}{2}\right)T+\ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}}\right) \\
&= S_0\Phi\left(\frac{\left(r+\frac{\sigma^2}{2}\right)T+\ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}}\right) - e^{-rT}K\Phi\left(\frac{\left(r-\frac{\sigma^2}{2}\right)T+\ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}}\right).
\end{aligned}$$

c) By part **a)** there holds

$$\begin{aligned}
e^{-rT}\mathbb{E}\left[f\left(S_T\right)\right] &= \int_{\mathbb{R}}\max\left\{e^{\alpha_T+\beta_T y-rT}-e^{-rT}K,0\right\}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\,\mathrm{d}y \\
&= \int_{\mathbb{R}}\max\left\{e^{\ln(S_0)-\sigma^2/2T+\sigma\sqrt{T}y}-e^{-rT}K,0\right\}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\,\mathrm{d}y.
\end{aligned}$$

It is immediate that the mapping $(S_0,r,K)\mapsto\max\left\{e^{\ln(S_0)-\sigma^2/2T+\sigma\sqrt{T}y}-e^{-rT}K,0\right\}$ is a non-decreasing function of S_0,r and a non-increasing function of K for all $y\in\mathbb{R}$. These monotonicity properties would then also hold for $e^{-rT}\mathbb{E}\left[f\left(S_T\right)\right]$, as a function of S_0,r and K since $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}$ is positive for all $y\in\mathbb{R}$.

4. Closed form solution for the Black-Scholes equation

a) We have

$$\frac{\partial u}{\partial t} = \frac{-1}{2t}u + \frac{1}{\sqrt{4\pi t}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-y)^2}{4t}\right)u_0(y)\,dy\,.$$

Fix $x\in\mathbb{R}$ and let $g(y,t):=\exp\left(-\frac{(x-y)^2}{4t}\right)u_0(y)$. Then

$$\frac{\partial g}{\partial t}(y,t) = \frac{\partial}{\partial t}\exp\left(-\frac{(x-y)^2}{4t}\right)u_0(y) = \frac{(x-y)^2}{4t^2}\exp\left(-\frac{(x-y)^2}{4t}\right)u_0(y)\,.$$

Fix any $t_0 > 0$. Then for $t \in [t_0/2, 2t_0]$, $\frac{\partial g}{\partial t}(\cdot, t)$ is bounded uniformly by the function

$$y \mapsto \frac{(x-y)^2}{t_0^2} \exp\left(-\frac{(x-y)^2}{8t_0}\right) u_0(y)$$

which (in view of the exponential bound for u_0) is integrable on $\mathbb{R}$. Hence, it holds that

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} g(y,t)\, dy = \int_{-\infty}^{\infty} \frac{\partial g}{\partial t}(y,t)\, dy\,, \quad \forall t \in [t_0/2, 2t_0]\,.$$

Since x and t_0 were arbitrary, this is true for all $x \in \mathbb{R}$ and $t > 0$. We conclude

$$\frac{\partial u}{\partial t}(x,t) = \int_{-\infty}^{\infty} \left(\frac{-1}{2t} + \frac{(x-y)^2}{4t^2}\right) \exp\left(-\frac{(x-y)^2}{4t}\right) u_0(y)\, dy\,, \quad \forall x \in \mathbb{R}\,,\ \forall t > 0\,.$$

With a similar argument, we obtain

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \frac{-(x-y)}{2t} \exp\left(-\frac{(x-y)^2}{4t}\right) u_0(y)\, dy\,, \quad \forall x \in \mathbb{R}\,,\ \forall t > 0\,,$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \left(\frac{(x-y)^2}{4t} - \frac{1}{2t}\right) \exp\left(-\frac{(x-y)^2}{4t}\right) u_0(y)\, dy\,, \quad \forall x \in \mathbb{R}\,,\ \forall t > 0\,,$$

hence it holds for all $x \in \mathbb{R}$ and $t > 0$ that

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0\,. \tag{3}$$

b) The formula given in the problem sheet is immediate using the suggested change of variables. One has

$$\left|u_0(x - t\sqrt{z})\right| \le C \exp(\kappa(x - 2\sqrt{t}z)) = Ce^{\kappa x} e^{-2\kappa\sqrt{t}z}\,.$$

Thus, $h(z,t) := e^{-z^2} u_0(x - 2\sqrt{t}z)$ is bounded by the function

$$\psi : z \mapsto Ce^{\kappa x} e^{-z^2 + 2\kappa|z|}\,,$$

uniformly for $0 < t \le 1$. Since ψ is integrable on $\mathbb{R}$, it follows by the dominated convergence theorem that

$$\lim_{t\to 0} u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} \lim_{t\to 0} u_0(x - 2\sqrt{t}z)\, dz\,.$$

The limit appearing on the right hand side is

$$\lim_{t\to 0} u(x,t) = \frac{u_0(x)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2}\, dz = u_0(x)\,,$$

since

$$\int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi}\,.$$

c) Let $g(x,\tau) := e^{-\alpha x - \beta\tau}$. We have

$$\begin{aligned}
\partial_\tau u(x,\tau) &= g(x,\tau)\cdot(-\beta V(t,s) - \frac{2}{\sigma^2}\partial_t V(t,s)) \Rightarrow \partial_t V(t,s) = \frac{\sigma^2}{2g(x,\tau)}\big(-\partial_\tau u(x,\tau) - \beta u(x,\tau)\big),\\
\partial_x u(x,\tau) &= g(x,\tau)\cdot\big(-\alpha V(t,s) + s\partial_s V(t,s)\big) \Rightarrow s\partial_s V(t,s) = \frac{1}{g(x,\tau)}\big(\partial_x u(x,\tau) + \alpha u(x,\tau)\big),\\
\partial_{xx} u(x,\tau) &= g(x,\tau)\cdot\big(\alpha^2 V(t,s) + (1-2\alpha)s\partial_s V(t,s) + s^2\partial_{ss} V(t,s)\big)\\
&\qquad \Rightarrow s^2\partial_{ss} V(t,s) = \frac{1}{g(x,\tau)}\big(\partial_{xx} u(x,\tau) + (2\alpha-1)\partial_x u(x,\tau) + \alpha(\alpha-1)u(x,\tau)\big).
\end{aligned}$$

Inserting the expressions for $\partial_t V, s\partial_s V, s^2\partial_{ss}V$ and $V = \frac{1}{g}u$ into the Black-Scholes equation, multiplying with $-\frac{2g}{\sigma^2}$ and ordering w.r.t. the derivatives yields

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} + \left(1 - \frac{2r}{\sigma^2} - 2\alpha\right)\frac{\partial u}{\partial x} + \left(\beta + (1-\alpha)(\alpha + \frac{2r}{\sigma^2})\right)u = 0.$$

This PDE reduces to the heat equation if we set the parameters α, β to

$$\begin{aligned} \alpha &= \frac{1}{2} - \frac{r}{\sigma^2}, \\ \beta &= (\alpha - 1)\left(\alpha + \frac{2r}{\sigma^2}\right) = -\left(\frac{1}{2} + \frac{r}{\sigma^2}\right)^2. \end{aligned}$$

d) We deduce from the previous questions that V satisfies the Black-Scholes PDE if and only if u satisfies the heat equation. Furthermore, the condition $V(T,s) = (s-K)_+$ is equivalent to $u(x,0) = e^{-\alpha x}(e^x - K)_+$, which satisfies the exponential bound:

$$u(x,0) \le e^{(1-\alpha)x} \quad \forall x \in \mathbb{R}\,.$$

From the previous questions, we know that

$$u(x,\tau) := \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4\tau}\right) u_0(y)\, dy\,,$$

satisfies the heat equation with initial condition u_0. Thus choosing

$$u_0(x) = e^{-\alpha x}(e^x - K)_+\,,$$

we have constructed a closed-form solution of the PDE, given by

$$\begin{aligned} u(x,\tau) &= \frac{1}{\sqrt{4\pi\tau}} \int_{\ln K}^{\infty} e^{-\frac{(x-y)^2}{4\tau}} e^{(1-\alpha)y} \mathrm{d}y - \frac{K}{\sqrt{4\pi\tau}} \int_{\ln K}^{\infty} e^{-\frac{(x-y)^2}{4\tau}} e^{-\alpha y} \mathrm{d}y \\ &=: I_1 + I_2. \end{aligned}$$

Consider the integral I_1. We substitute $s := \frac{x-y}{\sqrt{2\tau}}$ to obtain

$$\begin{aligned} I_1 &= -\frac{1}{\sqrt{2\pi}} \int_{\frac{x-\ln K}{\sqrt{2\tau}}}^{-\infty} e^{-\frac{1}{2}s^2} e^{(1-\alpha)(x-\sqrt{2\tau}s)} \mathrm{d}s \\ &= e^{(1-\alpha)x} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\ln K}{\sqrt{2\tau}}} e^{-\frac{1}{2}(s+(1-\alpha)\sqrt{2\tau})^2 + (1-\alpha)^2\tau} \mathrm{d}s \\ &\overset{z:=s+(1-\alpha)\sqrt{2\tau}}{=} e^{(1-\alpha)x+(1-\alpha)^2\tau} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\ln K}{\sqrt{2\tau}}+(1-\alpha)\sqrt{2\tau}} e^{-\frac{1}{2}z^2} \mathrm{d}z \\ &= e^{(1-\alpha)x+(1-\alpha)^2\tau} N\left(\frac{x - \ln K + 2(1-\alpha)\tau}{\sqrt{2\tau}}\right), \end{aligned}$$

where for all $x \in \mathbb{R}$, we denoted

$$N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, dy\,.$$

Similarly, we have for I_2:

$$I_2 = -Ke^{-\alpha x + \alpha^2\tau} N\left(\frac{x - \ln K - 2\alpha\tau}{\sqrt{2\tau}}\right).$$

Finally, we have by definition $V(t,s) = e^{\alpha x+\beta\tau}u(x,\tau) = e^{\alpha x+\beta\tau}(I_1 + I_2)$. By noting that $(1-\alpha)^2 + \beta = 0$, $\alpha^2 + \beta = -\frac{2r}{\sigma^2}$, and using $s = e^x$, $2\tau = \sigma^2(T-t)$ and $\alpha = \frac{1}{2} - \frac{r}{\sigma^2}$ we obtain the Black-Scholes formula for a European call

$$V(t,s) = sN(d_+) - Ke^{-r(T-t)}N(d_-),$$

with

$$d_\pm = \frac{\ln\frac{s}{K} + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}.$$

5. Graded time mesh

a) We have,

$$\frac{|e(h) - e(h/2)|}{|e(h/2) - e(h/4)|} = \frac{Ch^p - C(h/2)^p + O(h^q)}{C(h/2)^p - C(h/4)^p + O(h^q)} = \frac{1 - 2^{-p} + \frac{O(h^q)}{h^p}}{2^{-p} - 4^{-p} + \frac{O(h^q)}{h^p}} \to 2^p \quad \text{as } h \to 0^+.$$

b) See the solution code for the programming part.

c) For $\beta = 1$. The estimated convergence rate is 0.4999 and thus the expected convergence rate $O(h^2 + k^2)$ is not achieved.

d) See the solution code for the programming part. The estimated convergence rate is 2.0448. We observe that we successfully recover the full convergence rate $O(h^2 + k^2)$.

We observe a lower convergence rate for $\beta = 1$ and this is because $u_0 \notin H^2(G)$. The usage of the *graded time mesh* with $\beta = 15$ successfully resolves the irregularity of the initial value and this helps us to observe the convergence of order $O(h^2 + k^2)$.