# Series 10

## 1. Feynman Kac links

The price of a multi-asset option on $d$ assets is given as the conditional expectation

$$V(t,x) = \mathbb{E}\left[e^{-\int_t^T r(X_s)\,ds} g(X_T) \mid X_t = x\right],$$

where $X_t = (X_t^1, \dots, X_t^d)^\top$ is an $\mathbb{R}^d$-valued stochastic process modeling the dynamics of the $d$ assets, $r \in C^0(\mathbb{R}^d; \mathbb{R}_{\geq 0})$ is the deterministic interest rate and $g\colon \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ denotes the payoff of the option. We assume that the $i$th component of the process $X$ evolves according to

$$dX_t^i = b_i(X_t)\,dt + \sum_{j=1}^n \Sigma_{ij}(X_t) dW_t^j\,, \quad X_0^i = Z^i\,, \quad i = 1, \dots, d\,,$$

and we further assume the coefficients $b\colon \mathbb{R}^d \to \mathbb{R}^d, \Sigma\colon \mathbb{R}^d \to \mathbb{R}^{d \times n}$ satisfy the usual Lipschitz continuity and linear growth condition, i.e. there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^d$

$$\begin{aligned} |b(x) - b(y)| + |\Sigma(x) - \Sigma(y)| \leq |x - y|\,, \\ |b(x)| + |\Sigma(x)| \leq C(1 + |x|)\,. \end{aligned}$$

**a)** Let $f \in C^{1,2}(\mathbb{R} \times \mathbb{R}^d)$ with bounded derviatives in $x$. Let $\mathcal{A}$ be the infinitesimal generator of $X$

$$(\mathcal{A}f)(x) = \frac{1}{2}\operatorname{tr}[\mathcal{Q}(x) D^2 f(x)] + b(x)^\top \nabla f(x) \tag{1}$$

and assume that $r \in C^0(\mathbb{R}^d)$ is bounded. Show that the process

$$M_t := e^{-\int_0^t r(X_s)\,ds} f(t, X_t) - \int_0^t e^{-\int_0^s r(X_\tau)\,d\tau} (\partial_t f + \mathcal{A}f - rf)(s, X_s)\,ds$$

is a martingale with respect to the filtration of $W$.

*Hint:* Use Theorem 1.2.6 and Proposition 1.2.7 from the textbook.

**b)** Prove Theorem 8.1.3 in the textbook: Let $V \in C^{1,2}(J \times \mathbb{R}^d) \cap C^0(\overline{J} \times \mathbb{R}^d)$ with bounded derivatives in $x$ be a solution of

$$\partial_t V + \mathcal{A}V - rV = 0 \quad \text{in } J \times \mathbb{R}^d\,, \qquad V(T, x) = g(x) \quad \text{in } \mathbb{R}^d\,,$$

with $\mathcal{A}$ as in (1). Then, $V(t,x)$ can also be represented as

$$V(t,x) = \mathbb{E}\left[e^{-\int_t^T r(X_s)\,ds} g(X_T) \mid X_t = x\right].$$

*Hint:* Use the result from **a)**.

## 2. Basic properties of the Kronecker product

Given $\mathbf{A} \in \mathbb{R}^{n \times m}$, $\mathbf{B} \in \mathbb{R}^{s \times t}$, and $\mathbf{C} \in \mathbb{R}^{p \times q}$ for $n, m, s, t, p, q \in \mathbb{N}$, show that

**a)** If $\mathbf{A} \otimes \mathbf{B} = \mathbf{0} \in \mathbb{R}^{sn \times tm}$, then at least one of the matrices $\mathbf{A}, \mathbf{B}$ is a zero matrix.

**b)** $\mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} + \mathbf{C})$, if $s = p$ and $t = q$,

**c)** $(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$,

**d)** $\otimes$ is *not* commutative, i.e. there exists $\mathbf{A} \in \mathbb{R}^{n \times m}$, $\mathbf{B} \in \mathbb{R}^{s \times t}$ such that $\mathbf{A} \otimes \mathbf{B} \neq \mathbf{B} \otimes \mathbf{A}$.

**e)** If $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{s \times s}$ are symmetric, then $\mathbf{A} \otimes \mathbf{B}$ is also symmetric.

### 3. The pricing equation for multi-asset options

We consider the two-dimensional Black-Scholes model. Let $(W_t)_{t \geq 0}$ be a two-dimensional Brownian motion. The $i$-th component of the process $S = (S_t)_{t \geq 0}$, $S_t := (S_t^1, S_t^2)^\top$ evolves according to

$$dS_t^i = b_i(S_t)dt + \sum_{j=1}^{d} \Sigma_{ij}(S_t) dW_t^j, \quad i = 1, 2\ , \tag{2}$$

starting from $S_0 = (S_0^1, S_0^2)$, where $b_i(s) = rs_i$, $\Sigma_{ij}(s) = \mathbf{\Sigma}_{ij} s_i$, $S_0^j > 0$, $1 \leq i, j \leq 2$. We assume $r \in [-1, 1]$, $\mathbf{\Sigma}$ is a constant matrix such that $\det(\mathbf{\Sigma}) \neq 0$, and set $\boldsymbol{\mathcal{Q}} = \mathbf{\Sigma}\mathbf{\Sigma}^\top$.
The price $V(t, S_t^1, S_t^2)$ of a European basket option maturing at $T$ with a sufficiently smooth payoff $g$ is given by the conditional expectation

$$V(t, s_1, s_2) = \mathbb{E}\left[e^{-r(T-t)} g(S_T^1, S_T^2) \mid S_t^i = s_i,\ i = 1, 2\right].$$

Let $G \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let its boundary $\partial G$ be divided into two parts: $\partial G = \partial_1 G \cup \partial_2 G$. Assume that $\partial_1 G \cap \partial_2 G = \emptyset$ and $\partial_1 G \neq \emptyset$. We transform the pricing equation to time-to-maturity, log-price and localize to $G$ to obtain the following transformed PDE for $u(t, \mathbf{x}) = u(t, x_1, x_2) := V(T - t, \exp(x_1), \exp(x_2))$:

$$\begin{aligned} \partial_t u(t, \mathbf{x}) - \frac{1}{2} \nabla \cdot (\boldsymbol{\mathcal{Q}} \nabla u(t, \mathbf{x})) + \mu^\top \nabla u(t, \mathbf{x}) + ru(t, \mathbf{x}) &= 0 \quad \text{in } J \times G\ , \\ u(t, \mathbf{x}) &= 0 \quad \text{in } J \times \partial_1 G\ , \\ (\boldsymbol{\mathcal{Q}} \nabla u) \cdot \mathbf{n} &= 0 \quad \text{in } J \times \partial_2 G\ , \\ u(0, \mathbf{x}) &= g(e^{\mathbf{x}}) \quad \text{in } G\ , \end{aligned} \tag{3}$$

where $\mu := [Q_{11}/2 - r, Q_{22}/2 - r]^\top$, $g(e^{\mathbf{x}}) := (g(e^{x_1}), g(e^{x_2}))$. Also, at any point $\mathbf{x} \in \partial G$, $\mathbf{n}(\mathbf{x})$ denotes the outer unit normal vector, which means that this vector satisfies (1) it is perpendicular to the tangent line to $\partial G$ at $\mathbf{x}$ (2) its length is 1 and (3) it points outward with respect to $G$. This truncation aims at approximating the knock-out barrier option

$$V_G(t, s_1, s_2) = \mathbb{E}\left[e^{-r(T-t)} g(S_T^1, S_T^2) \mathbf{1}_{\{T < \tau_G\}} \mid S_t^i = s_i,\ i = 1, 2\right].$$

Here $\tau_G = \inf\{t > 0 \mid \log(S_t) \in \mathbb{R}^2 \setminus G\}$.

**a)** Define $V = \{v \in H^1(G) \mid v|_{\partial_1 G} = 0\}$ equipped with $H^1(G)$ norm and let $V^*$ be the dual space of $V$, show that the weak formulation of Equation (3) is as follows:

$$\begin{aligned} &\text{Find } u \in H^1(J; V^*) \cap L^2(J; V) \text{ s.t. for all } v \in V \text{ and for a.e. } t \in J : \\ &\quad \langle \partial_t u(t, \mathbf{x}), v \rangle_{H^{-1}(G) \times V} + a(u(t, \mathbf{x}), v) = 0 \\ &\qquad\qquad\qquad u(0, \mathbf{x}) = g(\exp(\mathbf{x}))\ . \end{aligned} \tag{4}$$

Here,

$$a(w,v) := \frac{1}{2}\int_G \nabla w(\mathbf{x})^\top \, \boldsymbol{\mathcal{Q}} \nabla v(\mathbf{x})\, d\mathbf{x} + \int_G \mu^\top \nabla w(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x} + r\int_G w(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x}\,.$$

*Hint:* Green's formula implies

$$\int_G v\nabla \cdot (\boldsymbol{\mathcal{Q}}\nabla u) + \nabla v \cdot (\boldsymbol{\mathcal{Q}}\nabla u)\, d\mathbf{x} = \int_{\partial G} v(\boldsymbol{\mathcal{Q}}\nabla u)\cdot \mathbf{n}\, ds.$$

This holds if $u \in H^1(G)$ and $v \in C^\infty(G)$.

**b)** Prove that $a(\cdot,\cdot)$ obtained in **a)** satisfies the Gårding inequality, i. e. under which there exist $C_1 > 0, C_2 \geq 0$ such that

$$a(v,v) \geq C_1\|v\|^2_{H^1(G)} - C_2\|v\|^2_{L^2(G)}\,.$$

*Hint:* You may use the following arithmetic-geometric mean inequality $ab \leq \epsilon/2a^2 + 1/(2\epsilon)b^2$, which is valid for any $a, b \in \mathbb{R}$ and $\epsilon > 0$ to estimate the integral $\int_G \mu^\top \nabla w(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x}$.

**c)** Assume that $g(\cdot,\cdot)$ satisfies the polynomial growth condition (see (8.10) in the textbook). Prove that there exists $C, \gamma_1, \gamma_2 > 0$ depending only on $\boldsymbol{\mathcal{Q}}, \mu, T, g$ such that if $G \supset B_R(0) := \{(x_1,x_2) \in \mathbb{R}^2, \sqrt{x_1^2 + x_2^2} < R\}$ for some $R > 0$, then for any $s_1, s_2 \in \mathbb{R}^+$,

$$|V_G(t,s_1,s_2) - V(t,s_1,s_2)| \leq C\exp(-\gamma_1 \cdot R + \gamma_2 \max(s_1,s_2)).$$

*Hint:* Use Theorem 8.3.1 in the textbook, which is stated using log-price.

**Due: Wednesday, May 15th, at 2pm.**