Series 3

Recall the following Sobolev embedding theorem: Assume $G \subset \mathbb{R}$ is a bounded open interval and let $u \in W^{1,p}(G)$. Then, $\exists \tilde{u} \in C^0(\bar{G})$ such that

$$u = \tilde{u} \text{ a.e. on } G$$
$$\forall x_1, x_2 \in \bar{G}\colon \tilde{u}(x_2) - \tilde{u}(x_1) = \int_{x_1}^{x_2} u'(\xi)\, d\xi\,.$$

1. Sobolev embedding and Poincaré inequality

a) Let $G = (a,b) \subset \mathbb{R}$. In the lecture we have seen that there exists a constant $C = C(|G|) > 0$ such that

$$\forall u \in H_0^1(G)\colon \ \|u\|_{L^2(G)} \leq C\, \|u'\|_{L^2(G)}\ .$$

Determine the best constant C_{opt} for this inequality, which indicates that for any $\widehat{C} > C_{opt}$, there exists $v \in H_0^1(G)$ such that $\widehat{C}\|v\|_{L^2(G)} > \|v'\|_{L^2(G)}$. Show that it is sufficient to consider $a = 0, b = 1, G = (0,1)$.

Hint: Consider the eigenvalue problem for the Laplacian with zero Dirichlet boundary condition:

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

It can be shown that in this case all the eigenvalues λ are positive and the smallest eigenvalue λ_1 is given by *Rayleigh's formula*:

$$\lambda_1 = \min_{\substack{u \in H_0^1(0,1) \\ u \not\equiv 0}} \frac{\|u'\|^2_{L^2(0,1)}}{\|u\|^2_{L^2(0,1)}}\,.$$

b) Consider the Sobolev embedding $H^1(G) \hookrightarrow L^\infty(G)$. Determine a constant $C = C(|G|) > 0$ such that

$$\forall u \in H^1(G)\colon \quad \|u\|_{L^\infty(G)} \leq C\, \|u\|_{H^1(G)}\ .$$

2. Interpolation error in $L^\infty(G)$

a) Let $p \in (1, \infty]$. For $G = (0,1)$, $u \in W^{1,p}(G)$ and a mesh

$$\mathcal{T} = \{a = x_0 < x_1 < x_2 \cdots x_{N+1} = b\}\,,$$

show that there exists $\alpha = \alpha(p)$ and a constant $C > 0$ such that

$$\|u - \mathcal{I}_N u\|_{L^\infty(G)} \leq C h^\alpha\, \|u\|_{W^{1,p}(G)}\ ,$$

where $h \coloneqq \max\{h_i : i = 1, \ldots, N+1\}$ and where $\mathcal{I}_N u$ denotes the *nodal interpolant* of u, which is defined as

$$\mathcal{I}_N u(x) = \tilde{u}(x_i) + (x - x_i) \cdot \frac{\tilde{u}(x_{i+1}) - \tilde{u}(x_i)}{x_{i+1} - x_i}\ , \quad \text{if } x \in [x_i, x_{i+1}], \quad i = 0, 1, \ldots, N\,.$$

Why is the interpolant $\mathcal{I}_N : W^{1,p}(G) \to S^1_{\mathcal{T}}$ well-defined?

3. Finite element discretization for the heat equation II

Let $J = (0,T)$, $T > 0$, $\beta \geq 1$, $G = (0,1) \subset \mathbb{R}$ and $f \in C(\overline{J}; L^2(G))$. For any $N, M \in \mathbb{N}$, we set $k = \frac{1}{M}$ and consider the spatial mesh points $x_i = \left(\frac{i}{N+1}\right)^\beta$, $i = 1, 2, \dots, N$. Let V_N be the vector space of continuous functions on G, vanishing at both ends of the interval, and which are linear on each (x_i, x_{i+1}). For each $i \in \{1, \dots, N\}$, there is a unique element $\phi_{N,i}$ of V_N satisfying

$$\phi_{N,i}(x_j) = \delta_{i,j}\,, \quad \forall j \in \{1, \dots, N\}$$

and $\{\phi_{N,i}\}_{1 \leq i \leq N}$ is a basis of V_N.

We wish to solve the heat equation with zero Dirichlet boundary conditions and with initial value $u_0 \in L^2(G)$,

$$\begin{cases} \partial_t u(t,x) - \partial_{xx} u(t,x) &= f(t,x) \qquad \text{in } J \times G\,, \\ u(t,x) &= 0 \qquad\qquad \text{on } J \times \partial G\,, \\ u(0,x) &= u_0(x) \qquad \text{in } G\,. \end{cases} \tag{1}$$

As in the previous exercise sheet, we discretize using a ϑ scheme

$$\mathbf{B}_\vartheta \underline{u}_N^{m+1} = \mathbf{C}_\vartheta \underline{u}_N^m + \underline{\mathrm{F}}_\vartheta^m\,,$$

with

$$\begin{aligned} \mathbf{B}_\vartheta &= \mathbf{M} + k\vartheta \mathbf{A}\,, \\ \mathbf{C}_\vartheta &= \mathbf{M} - k(1-\vartheta)\mathbf{A}\,, \\ \underline{\mathrm{F}}_\vartheta^m &= k\vartheta \underline{\mathrm{F}}(t_{m+1}) + k(1-\vartheta)\underline{\mathrm{F}}(t_m)\,. \end{aligned}$$

Here $\mathbf{M}$ and $\mathbf{A}$ are the matrices given by

$$\mathbf{M}_{i,j} = (\phi_{N,i}, \phi_{N,j})_{L^2(G)}\,, \quad \mathbf{A}_{i,j} = a\left(\phi_{N,i}, \phi_{N,j}\right)\,, \quad 1 \leq i,j \leq N\,,$$

and $\underline{\mathrm{F}}(t)$ is the column vector given by

$$\mathrm{F}_i(t) = (f(t), \phi_{N,i})_{L^2(G)}\,, \quad 1 \leq i \leq N\,.$$

a) Give the expression of the matrices $\mathbf{M}$, $\mathbf{A}$ and the vector $\underline{\mathrm{F}}(t)$ in terms of $h_i := x_i - x_{i-1}$, $i = 1, \dots, N+1$ and f.

b) Modify your code from Series 2, Problem 3 and implement these changes. The template `FEM_heat.py` contains the solution of Series 2, Problem 3. You will need to modify the functions `build_massMatrix`, `build_rigidityMatrix`, `build_F`, and `FEM_theta`. The entries of $\underline{\mathrm{F}}(t)$ shall be approximated via the following formula: for any $i \in 1, \dots, N$,

$$\mathrm{F}_i(t) = \frac{h_i f(t, \frac{x_{i-1}+x_i}{2})}{3} + (h_i + h_{i+1})\frac{f(t,x_i)}{6} + \frac{h_{i+1} f(t, \frac{x_i + x_{i+1}}{2})}{3}\,.$$

c) Test your code with $\theta = 1$, $\beta = 1, 1.05, 1.2$, $N = 2^l - 1$ and $M = 4^l$ with $l = \{2,3,4,5,6\}$. Study if those numerical schemes converge and report the convergence rates if they converge.

d) Test your code with $\theta = 0$, $\beta = 1, 1.05, 1.2$, $N = 2^l - 1$ and $M = 4^l$ with $l = \{2,3,4,5,6\}$. Study if those numerical schemes converge and report the convergence rates if they converge. Comment on your result.

e) Test your code with $\theta = 0$, $\beta = 1, 1.05, 1.2$, $N = 2^l - 1$ and $M = 7 \times 4^l$ with $l = \{2,3,4,5,6\}$. Study if those numerical schemes converge and report the convergence rates if they converge. Comment on your result.

4. A general second-order parabolic problem

Let $a, b \in \mathbb{R}$, $a < b$, and let $G = (a, b)$, $J = (0, 1)$. Consider the Dirichlet problem with general coefficient functions $\alpha(x), \beta(x), \gamma(x)$

$$\begin{aligned} \partial_t u - \partial_x(\alpha(x)\partial_x u) + \beta(x)\partial_x u + \gamma(x)u &= f(t, x) && \text{in } J \times G \\ u &= 0 && \text{on } J \times \partial G \\ u|_{t=0} &= u_0 && \text{in } G\,, \end{aligned} \tag{2}$$

where $u_0 \in L^2(G)$, $f(t, x) \in L^2(J, H^{-1}(G))$, $\alpha, \gamma \in C(\overline{G})$ and $\beta \in C^1(\overline{G})$ such that with some $\underline{\alpha} > 0$ the bound $\alpha(x) > \underline{\alpha}$ holds for all $x \in G$. The weak formulation is as follows: Find $u \in L^2(J; H_0^1(G)) \cap H^1(J; H^{-1}(G))$ such that $u(0) = u_0$ and $\forall v \in H_0^1(G)$,

$$\frac{d}{dt}\,(u, v)_{L^2(G)} + a(u, v) = (f, v)_{L^2(G)}\ . \tag{3}$$

Here $a(u, v) = \int_G \alpha(x)\partial_x u \partial_x v + \beta(x)(\partial_x u)v + \gamma(x)uv\, dx$ is a bilinear form.

This more general formulation is necessary e.g. for local volatililty models.

a) Prove that there exists $C_1, C_2 > 0$ and $C_3 \geq 0$ such that $a(\cdot, \cdot)$ satisfies the following:

1. $|a(u, v)| \leq C_1\|u\|_{H^1(G)} \cdot \|v\|_{H^1(G)}$ for any $u, v \in H_0^1(G)$ (Continuity)
2. $a(u, u) \geq C_2\|u\|^2_{H^1(G)} - C_3\|u\|^2_{L^2(G)}$ for any $u \in H_0^1(G)$ (Gårding inequality)

b) Prove that there exists a unique weak solution $u \in L^2(J; H_0^1(G)) \cap H^1(J; H^{-1}(G))$.

c) Assume further that $\alpha(x) \in C^1(\overline{G})$, $\gamma(x) > 0$ for all $x \in \overline{G}$ and that $f(t, x) \equiv 0$. Prove that if the solution satisfies $u \in C^2(\overline{J \times G})$, then the following holds:

$$u(t, x) \leq \max(0, \max_{x\in\overline{G}} u_0(x)) \quad \text{for any } (t, x) \in \overline{J \times G}\,.$$

d) Under the same assumptions as in the previous subquestion, prove that for any $(t, x) \in \overline{J \times G}$, $|u(t, x)| \leq \max_{x\in\overline{G}} |u_0(x)|$.

Remark: In other words, the $L^\infty(\overline{J \times G})$ norm of the (unique) solution is controlled by the $L^\infty(\overline{G})$ norm of the initial value.

Due: Wednesday, March 20th, at 2pm.