Solutions: Series 9

1. Localization for barrier options

a) Let $M_T = \sup_{\tau\in[t,T]} X_\tau$. Then, by the polynomial growth condition,

$$\begin{aligned}|v_{do}(t,x) - v_R(t,x)| &\le \mathbb{E}\left[g(e^{X_T})1_{\{\tau_B > T \ge \tau_G\}} \mid X_t = x\right] \\ &\le C\,\mathbb{E}\left[e^{qM_T}1_{\{T\ge\tau_G\}} \mid X_t = x\right] .\end{aligned}$$

The statement now follows as in the proof of Theorem 4.3.1 in the textbook: It suffices to show that there exists a constant $C(T,\sigma) > 0$ and $\gamma_1,\gamma_2 > 0$ such that

$$\mathbb{E}\left[e^{qX_T}1_{\{X_T>R\}} \mid X_t = x\right] \le C(T,\sigma,r)e^{-\gamma_1 R+\gamma_2 x} .$$

We have for $\mu = r - \sigma^2/2$, with the transition probability p_{T-t},

$$\begin{aligned}\mathbb{E}\left[e^{qX_T}1_{\{X_T>R\}} \mid X_t = x\right] &= \int_{\mathbb{R}} e^{q(z+x)}1_{\{z+x>R\}}p_{T-t}(z)\,dz \\ &\le e^{qx}\int_{\mathbb{R}} e^{qz}1_{\{z+x>R\}}\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}e^{-(z-\mu(T-t))^2/2\sigma^2(T-t)}\,dz \\ &\le C_1(T,\sigma,r)e^{qx}\int_{\mathbb{R}} e^{(q+\mu/\sigma^2)z}1_{\{z+x>R\}}e^{-z^2/2\sigma^2(T-t)}\,dz \\ &\le C_1(T,\sigma,r)e^{qx}\int_{\mathbb{R}} e^{-(\eta-q-\mu/\sigma^2)z}e^{\eta z}1_{\{z+x>R\}}e^{-z^2/2\sigma^2(T-t)}\,dz \\ &\le C_1(T,\sigma,r)e^{qx-(\eta-q-\mu/\sigma^2)(R-x)}\int_{\mathbb{R}} e^{\eta z}e^{-z^2/2\sigma^2(T-t)}\,dz \\ &\le C_1(T,\sigma,r)e^{-\gamma_1 R+\gamma_2 x}\int_{\mathbb{R}} e^{\eta z}e^{-z^2/2\sigma^2(T-t)}\,dz\,,\end{aligned}$$

for some $\eta > 0$ arbitrary, $\gamma_1 = \eta - q - \mu/\sigma^2$ and $\gamma_2 = \gamma_1 + q$. As $\int_{\mathbb{R}} e^{\eta z}e^{-z^2/2\sigma^2(T-t)}\,dz < \infty$ for any $\eta > 0$, we obtain the result by choosing $\eta > q + \mu^2/2$.

2. Barrier options in the Black–Scholes market

a) By the linearity of the conditional expectation,

$$\begin{aligned}V^{\text{Eur}}_{\text{ui}}(t,s) + V^{\text{Eur}}_{\text{uo}}(t,s) &= \mathbb{E}\big[e^{-r(T-t)}g(S_T)1_{\{T>\tau_B\}} \mid S_t = s\big] + \mathbb{E}\big[e^{-r(T-t)}g(S_T)1_{\{T\le\tau_B\}} \mid S_t = s\big] \\ &= \mathbb{E}\big[e^{-r(T-t)}g(S_T)(1_{\{T>\tau_B\}} + 1_{\{T\le\tau_B\}}) \mid S_t = s\big].\end{aligned}$$

We observe that $1_{\{T\le\tau_B\}}$ is the complement of $\{T \le \tau_B\}$, which implies that $1_{\{T>\tau_B\}}+1_{\{T\le\tau_B\}} \equiv 1$. Hence,

$$\mathbb{E}\big[e^{-r(T-t)}g(S_T)(1_{\{T>\tau_B\}} + 1_{\{T\le\tau_B\}}) \mid S_t = s\big] = \mathbb{E}\big[e^{-r(T-t)}g(S_T) \mid S_t = s\big] = V^{\text{Eur}}(t,s),$$

which implies the claim.

b) Let $H = L^2(G)$ and $V = H^1_0(G)$. The variational formulation corresponding to the localized PDE in **b)** reads

$$\begin{aligned}&\text{Find } v_R \in L^2(J;V)\cap H^1(J;V^*) \text{ such that for a.e. } t\in J \\ &\langle\partial_t v_R, u\rangle_{V^*,V} + a^{\text{BS}}(v_R,u) = 0 \qquad \forall\, u\in V \\ &v_R(0,x) = (e^x-1)_+ \text{ for every } x\in G.\end{aligned}$$

The bilinear form a^{BS} is for every $w, v \in V$ given by

$$a^{\mathrm{BS}}(w,u) = \frac{\sigma^2}{2}\int_G w'v'\mathrm{d}x + \left(\frac{\sigma^2}{2} - r\right)\int_G w'v\mathrm{d}x + r\int_G wv\mathrm{d}x.$$

c) We have the following representation for u and v: $u(t,x) = \sum_{i=1}^N u_i(t)b_i(x)$ and $v(x) = \sum_{i=1}^N v_i b_i(x)$. Then, the discrete variational formulation is given for $\underline{\mathbf{u}}(t) = (u_1(t), \ldots, u_N(t))^T$ and every $\underline{\mathbf{v}} = (v_1, \ldots, v_N)^T \in \mathbb{R}^N$ by

$$\begin{aligned}
\sum_{i,j}(\partial_t u_i(t)b_i, v_j\, b_j)_{L^2(G)} + \frac{1}{2}\sigma^2\sum_{i,j}&(u_i(t)b_i', v_j\, b_j')_{L^2(G)} \\
&+ (\frac{\sigma^2}{2} - r)\sum_{i,j}(u_i(t)b_i', v_j\, b_j)_{L^2(G)} \\
&+ r\sum_{i,j}(u_i(t)b_i, v_j\, b_j)_{L^2(G)} = 0 \\
\sum_{i,j}\partial_t u_i(t)(b_i, b_j)_{L^2(G)}v_j + \frac{1}{2}\sigma^2\sum_{i,j}&u_i(t)(b_i', b_j')_{L^2(G)}v_j \\
&+ (\frac{\sigma^2}{2} - r)\sum_{i,j}u_i(t)(b_i', b_j)_{L^2(G)}v_j \\
&+ r\sum_{i,j}u_i(t)(b_i, b_j)_{L^2(G)}v_j = 0
\end{aligned}$$

Therefore, setting $\mathbf{M}_{i,j} = (b_j, b_i)_{L^2(G)}$ and $\mathbf{S}_{i,j} = (b_j', b_i')_{L^2(G)}$ and $\mathbf{B}_{i,j} = (b_j', b_i)_{L^2(G)}$ and $\mathbf{A} = \frac{1}{2}\sigma^2\mathbf{S} + (\frac{\sigma^2}{2} - r)\mathbf{B} + r\mathbf{M}$, we get

$$\mathbf{M}\partial_t\underline{\mathbf{u}}(t) + \mathbf{A}\underline{\mathbf{u}}(t) = 0.$$

We discretize in time with the ϑ-scheme and get ($\vartheta \in [0,1]$)

$$\mathbf{M}\frac{\underline{\mathbf{u}}^m - \underline{\mathbf{u}}^{m-1}}{k} + (\vartheta\underline{\mathbf{u}}^m + (1-\vartheta)\mathbf{A}\underline{\mathbf{u}}^{m-1}) = 0$$

which leads to

$$(\mathbf{M} + k\vartheta\mathbf{A})\underline{\mathbf{u}}^m = (\mathbf{M} - k(1-\vartheta)\mathbf{A})\underline{\mathbf{u}}^{m-1}.$$

Here, $\underline{\mathbf{u}}^m = \underline{\mathbf{u}}(t_m)$. So we have for the variational form in matrix formulation:

Find $\underline{\mathbf{u}}^m \in \mathbb{R}^N$ such that for all $m = 1, \ldots, M$

$$(\mathbf{M} + k\vartheta\mathbf{A})\underline{\mathbf{u}}^m = (\mathbf{M} - k(1-\vartheta)\mathbf{A})\underline{\mathbf{u}}^{m-1}$$

and $\underline{\mathbf{u}}_i^0 = (e^{x_i} - 1)_+$.

d) See the solution code.

3. Caplet in the CIR model

a) First, we rewrite the PDE in time-to-maturity, i.e. we write down the PDE satisfied by $u_0(t,r) := V_0(T_1 - t, r)$.

$$\left\{\begin{array}{rcll}
\partial_t u_0 - \frac{1}{2}\sigma^2 r\partial_{rr}u_0 + (\beta r - \alpha)\partial_r u_0 + ru_0 & = & 0 & \text{in } (0,T_1)\times G\,, \\
u_0 & = & 0 & \text{in } (0,T_1)\times\{R\}\,, \\
u_0(0,\cdot) & = & g_0 & \text{in } G\,.
\end{array}\right. \tag{1}$$

At a fixed time $t \in (0, T_1]$, we multiply the first line of (1) by an $r^{2\mu}w$, where μ is a parameter to be selected and $w \in C_0^\infty(G)$ is a test function, and integrate from $r = 0$ to $r = R$. We find

$$(\partial_t u_0, r^{2\mu} w)_{L^2(G)} - \frac{1}{2}\sigma^2 (r\partial_{rr} u_0, r^{2\mu} w)_{L^2(G)} + ((\beta r - \alpha)\partial_r u_0, r^{2\mu} w)_{L^2(G)} + (r u_0, r^{2\mu} v)_{L^2(G)} = 0\,.$$

Finally, applying integration by parts, one has

$$(r\partial_{rr} u_0, r^{2\mu} w)_{L^2(G)} = -(\partial_r u_0, r^{2\mu+1}\partial_r w)_{L^2(G)} - (2\mu + 1)(\partial_r u_0, r^{2\mu} w)_{L^2(G)}\,,$$

without boundary term, since $w \in C_0^\infty(G)$. For $\varphi, \phi \in C_0^\infty(G)$, we define the bilinear form

$$\begin{aligned} a^{\mathrm{CIR}}_{1/2,\mu}(\varphi, \phi) :=& \frac{1}{2}\sigma^2 (r^{1+2\mu}\partial_r \varphi, \partial_r \phi)_{L^2(G)} + \sigma^2 \left(\frac{1}{2} + \mu\right)(r^{2\mu}\partial_r\varphi, \phi)_{L^2(G)} \\ &- ((\alpha - \beta r) r^{2\mu}\partial_r \varphi, \phi)_{L^2(G)} + (r^{2\mu+1}\varphi, \phi)_{L^2(G)}\,. \end{aligned} \tag{2}$$

The continuity of $a^{\mathrm{CIR}}_{1/2,\mu}$ in $W_{1/2,\mu} \times W_{1/2,\mu}$ follows by the Cauchy-Schwarz inequality and the weighted Hardy's inequality with $\epsilon = 1 + 2\mu \neq 1$: Let $\varphi, \phi \in C_0^\infty$. Then

$$\begin{aligned} |(r^{1+2\mu}\partial_r\varphi, \partial_r\phi)_{L^2(G)}| &\le \|\varphi\|_{1/2,\mu}\|\phi\|_{1/2,\mu}\,, \\ |(r^{2\mu}\partial_r\varphi, \phi)_{L^2(G)}| \le \|\varphi\|_\mu \|r^{-1/2}\phi\|_\mu &\le \frac{1}{|\mu|}\|\varphi\|_{1/2,\mu}\|\partial_r\phi\|_\mu \le \frac{1}{|\mu|}\|\varphi\|_{1/2,\mu}\|\phi\|_{1/2,\mu}\,, \\ |(r^{2\mu+1}\partial_r\varphi, \phi)_{L^2(G)}| &\le \sqrt{R}\|r\varphi\|_\mu\|\varphi\|_\mu \le \sqrt{R}\|\varphi\|_{1/2,\mu}\|\phi\|_{1/2,\mu}\,, \\ |(r^{2\mu+1}\varphi, \phi)_{L^2(G)}| &\le R\|\varphi\|_\mu\|\varphi\|_\mu \le R\|\varphi\|_{1/2,\mu}\|\phi\|_{1/2,\mu}\,. \end{aligned}$$

It follows that

$$|a^{\mathrm{CIR}}_{1/2,\mu}(\varphi, \phi)| \le C(R, \alpha, \beta, \sigma, \mu)\|\varphi\|_{1/2,\mu}\|\phi\|_{1/2,\mu}\,.$$

Hence, we may extend the bilinear form $a^{\mathrm{CIR}}_{1/2,\mu}(\cdot,\cdot)$ from $C_0^\infty(G)$ to $W_{1/2,\mu}$ by continuity. It follows that $u_0 \in W_{1/2,\mu}$ is a solution of the variational problem for $a^{\mathrm{CIR}}_{1/2,\mu}$ as in (2). Finally, if we define the functions $a, b, c\colon G \to \mathbb{R}$ by

$$\begin{aligned} a(r) &:= \frac{1}{2}\sigma^2 r^{1+2\mu}\,, \\ b(r) &:= \sigma^2\left(\frac{1}{2} + \mu\right) r^{2\mu} + (\beta r - \alpha) r^{2\mu}\,, \\ c(r) &:= r^{2\mu+1}\,, \end{aligned}$$

we obtain the desired form of $a^{\mathrm{CIR}}_{1/2,\mu}(\cdot,\cdot)$.

b) By definition, one has

$$L(T, T_1, r_T) = \frac{1 - B(T, T_1, r_T)}{(T_1 - T)B(T, T_1, r_T)} \quad \text{and} \quad B(T, T_1, r_T) = V_0(T, r_T)\,.$$

Hence

$$V_1(t, r) = \mathbb{E}\left[e^{-\int_t^{T_1} r_s\, ds}\tilde{g}(V_0(T, r_T)) \mid r_t = r\right]\,, \quad \text{with} \quad \tilde{g}(x) = (T_1 - T)\left(\frac{1 - x}{(T_1 - T)x} - K\right)_+\,.$$

c) We write $e^{-\int_t^{T_1} r_s\, ds} = e^{-\int_t^{T} r_s\, ds}e^{-\int_T^{T_1} r_s\, ds}$ and remark that the term $e^{-\int_t^{T} r_s\, ds}\tilde{g}(V_0(T, r_T))$ is measurable with respect to $\mathcal{G}_T$. Hence

$$V_1(t, r) = \mathbb{E}\left[e^{-\int_t^T r_s\, ds}\tilde{g}_1(V_0(T, r_T))\mathbb{E}\left[e^{-\int_T^{T_1} r_s\, ds} \mid \mathcal{G}_T\right] \mid r_t = r\right]\,.$$

Notice that, by the Markov property of r_t,

$$\mathbb{E}\left[e^{-\int_T^{T_1} r_s\,ds} \mid \mathcal{G}_T\right] = \mathbb{E}\left[e^{-\int_T^{T_1} r_s\,ds} \mid r_T\right] = V_0(T, r_T)\,,$$

by definition of V_0. Hence

$$V_1(t,r) = \mathbb{E}\left[e^{-\int_t^{T} r_s\,ds} g_1(V_0(T, r_T)) \mid r_t = r\right]\,,$$

with $g_1(x) = x\tilde{g}_1(x)$.

d) The function u_1 is the solution of the variational problem

$$\begin{aligned}
&\text{Find } u_1 \in L^2(J_1; W_{1/2,\mu}) \cap H^1(J_1; \mathcal{H}_\mu) \text{ such that, for all } v \in W_{1/2,\mu}\,,\\
&\begin{cases} (\partial_t u_1, v)_\mu + a^{\mathrm{CIR}}_{1/2,\mu}(u_1, v) = 0 & \text{a.e. in } J\,,\\ u_1(0,\cdot) = g_1(V_0(T,\cdot)) & \text{a.e. in } G\,, \end{cases}
\end{aligned}$$

where $J_1 = (0, T)$. We note that, by definition of u_0,

$$u_1(0, r) = g_1(u_0(T_1 - T, r))\,.$$