Solutions: Series 1

1. Classification of PDEs

a) 1: Parabolic on the whole domain. 2: Elliptic on $\{x_1 > 0\}$, parabolic on $\{x_1 = 0\}$ and hyperbolic on $\{x_1 < 0\}$. To see this, note that this PDE could be rewritten as

$$-\sum_{i,j=0}^{2} a_{ij}(x_1, x_2)\partial_{x_i x_j} u + \sum_{i=1}^{2} b_i(x_1, x_2)\partial_{x_i} u = 0 \,,$$

with

$$\boldsymbol{A}(x_1, x_2) := \begin{pmatrix} a_{11}(x_1, x_2) & a_{12}(x_1, x_2) \\ a_{12}(x_1, x_2) & a_{22}(x_1, x_2) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -x_1 \end{pmatrix} \,, \quad \boldsymbol{b}(x_1, x_2) := \begin{pmatrix} b_1(x_1, x_2) \\ b_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \,,$$

for any $(x_1, x_2) \in \mathbb{R}^2$. Therefore,

- If $x_1 > 0$, then $\boldsymbol{A}(x_1, x_2)$ has two negative eigenvalues and thus the equation is elliptic.
- If $x_1 = 0$, then $\boldsymbol{A}(x_1, x_2)$ has a negative and a zero eigenvalues. Moreover, $\mathrm{Rank}(\boldsymbol{A}, \boldsymbol{b}) = 2$. Therefore the equation is parabolic.
- If $x_1 < 0$, then $\boldsymbol{A}(x_1, x_2)$ has a positive and a negative eigenvalues, this implies that the equation is hyperbolic.

b) We study the relation between the eigenvalues and determinant of the coefficient matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11}(x_1, x_2) & a_{12}(x_1, x_2) \\ a_{12}(x_1, x_2) & a_{22}(x_1, x_2) \end{pmatrix} \,.$$

Note that any symmetric matrix is similar to a diagonal matrix with all eigenvalues on the diagonal line and those eigenvalues are real. Let us assume that $\boldsymbol{A}$ has eigenvalues λ_1, λ_2 (they depend on x_1, x_2), then there exists a transformation matrix $\boldsymbol{P}$ (which is also orthogonal) such that $\boldsymbol{A} = \boldsymbol{P}^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \boldsymbol{P}$. Therefore $|\boldsymbol{A}| = |\boldsymbol{P}^{-1}| \cdot \lambda_1 \cdot \lambda_2 \cdot |\boldsymbol{P}| = \lambda_1\lambda_2$.

Thus:

$$\begin{aligned} |\boldsymbol{A}| > 0 &\iff \lambda_1 \cdot \lambda_2 > 0 \iff \text{the equation is elliptic,} \\ |\boldsymbol{A}| < 0 &\iff \lambda_1 \cdot \lambda_2 < 0 \iff \text{the equation is hyperbolic.} \end{aligned}$$

c) No, such a PDE does not exist.

Assume that there exists a PDE satisfying the requirement in the question. The result in b) shows that for this PDE, $|\boldsymbol{A}| > 0$ on D and $|\boldsymbol{A}| < 0$ on $\mathbb{R}^2 \setminus D$.

Clearly $|\boldsymbol{A}|$ depends continuously on the coeffcients $a_{11}, a_{12}, a_{22} \in C^0(\mathbb{R}^2)$. Therefore the domain D on which $|\boldsymbol{A}| > 0$ should be open. In a similar way we could show that $\mathbb{R}^2 \setminus D$ on which $|\boldsymbol{A}| < 0$ is open. Therefore D is both open and close and thus it shall be $\emptyset$ or $\mathbb{R}^2$, which violates the condition given in the question that D is nonempty and that $D \subsetneq \mathbb{R}^2$.

2. Finite-Difference method for the heat equation

a) Since u is smooth, so is f and

$$f'(t) = \int_0^1 2u\frac{\partial u}{\partial t}(x,t)\,dx = \int_0^1 2u\frac{\partial^2 u}{\partial x^2}(x,t)\,dx = \left[2u\frac{\partial u}{\partial x}\right]_{x=0}^{x=1} - \int_0^1 2(\frac{\partial u}{\partial x})^2\,dx = -\int_0^1 2(\frac{\partial u}{\partial x})^2\,dx \le 0\,.$$

Therefore $f(t)$ is non-increasing on $[0,1]$.

Assume that there exists two distinct solutions $u_1, u_2 \in C_1^2([0,1]\times[0,1])$ solving the equation with the same u_0. Then $u_1 - u_2 \in C_1^2([0,1]\times[0,1])$ solves the equation with $(u_1-u_2)|_{t=0} \equiv 0$.

By the first part of this exercise, $\int_0^1 (u_1(x,t)-u_2(x,t))^2\,dx$ is a non-increasing function of t on $[0,1]$. This together with the facts that $\int_0^1 (u_1(x,t)-u_2(x,t))^2\,dx \ge 0$ and $\int_0^1 (u_1(x,0)-u_2(x,0))^2\,dx = 0$ (since $(u_1-u_2)|_{t=0} \equiv 0$) gives $\int_0^1 (u_1(x,t)-u_2(x,t))^2\,dx = 0$ for any $t\in[0,1]$. Thus, $u_1(x,t) \equiv u_2(x,t)$ for $(x,t)\in[0,1]\times[0,1]$ since they both are continuous functions.

b) Taylor's expansion shows that

$$u_i^{m+1} = u_i^m + k\frac{\partial u}{\partial t}|_{(x,t)=(x_i,t_m)} + O(k^2),$$

which verifies the first approximation. Moreover,

$$\begin{aligned} u_{i+1}^m &= u_i^m + \frac{\partial u}{\partial x}|_{(x,t)=(x_i,t_m)}h + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}|_{(x,t)=(x_i,t_m)} + \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}|_{(x,t)=(x_i,t_m)} + O(h^4)\,, \\ u_{i-1}^m &= u_i^m - \frac{\partial u}{\partial x}|_{(x,t)=(x_i,t_m)}h + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}|_{(x,t)=(x_i,t_m)} - \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3}|_{(x,t)=(x_i,t_m)} + O(h^4)\,, \end{aligned}$$

which imply

$$\begin{aligned} \frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{h^2} &= \frac{\partial^2 u}{\partial x^2}|_{(x,t)=(x_i,t_m)} + O(h^2)\,, \\ \frac{\partial u}{\partial x}|_{(x,t)=(x_0,t_m)} &= \frac{u_1^m - u_{-1}^m}{2h} + O(h^2)\,. \end{aligned}$$

Therefore the second and the third approximations also hold.

c) We reformulate the numerical scheme as the following,

$$\begin{cases} u_i^{m+1} - u_i^m = u_i^m + \frac{k}{h^2}(u_{i-1}^m - 2u_i^m + u_{i+1}^m)\,, & \text{for } 0\le i<N\,, \\ u_{N-1}^{m+1} = u_{N+1}^{m+1}\,, \\ u_0^{m+1} = 0\,. \end{cases} \tag{1}$$

This scheme, after the removal of the variables u_0^m and u_{N+1}^m using the last two lines, is equivalent to $\boldsymbol{C} = \boldsymbol{I} + \frac{k}{h^2}\boldsymbol{G}$.

d) The claim is similar to the previous one and $v = \frac{k}{h^2}$, as before.

3. Implementation in Matlab

a) We first show that u satisfies the heat equation on $(0,1)\times(0,1)$. We have

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = -\frac{\pi^2}{4}e^{-\frac{\pi^2}{4}t}u + e^{-\frac{\pi^2}{4}t}\frac{\pi^2}{4}u = 0\,.$$

For the initial value we have

$$u(x,0) = \sin(\frac{\pi}{2}x) \ .$$

Finally, we verify the boundary conditions: Clearly u vanishes at $x = 0$. Moreover,

$$\frac{\partial u}{\partial x} = \frac{\pi}{2} e^{-\frac{\pi^2}{4}t} \cos(\frac{\pi}{2}x) \ .$$

Therefore,

$$\frac{\partial u}{\partial x}|_{x=1} = 0 \ .$$

b) See the solution file `1_exercise3_solution.py`.

c) For an error vector $err \in \mathbb{R}^{N+1}$, the square of the discrete L^2 norm of it is defined as:

$$\|err\|_2^2 := h \sum_{i=1}^{N+1} err_i^2 \ .$$

Here err_i denotes the i-th element of err.

The output given by the script `1_exercise3_solution.py` is the following:

i) Explicit method: Convergence rate: 1.0196

ii) Implicit method: Convergence rate: 1.0187

Two plots showing convergence rate of both methods are generated. Therefore, both schemes produce numerical solutions with convergence order $O(h^2 + k)$ in the case studied here.

d) The output is:

i) Explicit method: Error unbounded, does not converge

ii) Implicit method: Convergence rate: 1.0178

The implicit method converges with convergence rate 1.0178, this verifies the convergence order $O(h^2 + k)$ for this scheme. The explicit method does not converge and the error is increasing fast. This implies the instability of the method with the new parameters.

In fact, for the explicit method it is required that $v = \frac{k}{h^2} \leq \frac{1}{2}$, or it may lead to instability. On the other hand, the implicit method is always unconditionally stable.