Series 6

1. Log-moneyness

For European put or call options with strike $K > 0$ and spot price $s > 0$, it is convenient to introduce the so–called *log–moneyness* variable $y \coloneqq \log(s/K)$. Hereafter, $V(t, s)$ denotes the value of an European put option in a Black–Scholes market with parameters $r \in \mathbb{R}$ and $\sigma > 0$, i.e. the solution of the terminal value problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 V}{\partial s^2} + rs\dfrac{\partial V}{\partial s} - rV = 0 & \text{on } [0,T) \times \mathbb{R}_+\,, \\ V(T,s) = (K-s)_+ & \forall s \in \mathbb{R}_+\,, \end{cases}$$

We assume that for fixed $t > 0$, $s \mapsto V(t, s)$ is in $C^2(\mathbb{R})$.

a) Show that there holds

$$V(t,s) = Kw(T-t, \log(s/K))\,,$$

where $w(t, y)$ is the solution to the parabolic PDE

$$\begin{cases} \partial_t w(t,y) - \mathcal{A}_y^{BS} w(t,y) + rw(t,y) = 0 & \text{in } (0,T] \times \mathbb{R}\,, \\ w(0,y) = (1-e^y)_+ & \text{on } \{0\} \times \mathbb{R}\,, \end{cases} \tag{1}$$

with differential operator given by

$$\mathcal{A}_y^{BS} f \coloneqq \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial y^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial f}{\partial y}\,, \quad f \in C^2(\mathbb{R})\,.$$

Remark: Note that the initial condition in Equation (1) does not depend on K.

2. Fractional Sobolev spaces

We have seen up to this point of the course the function spaces like $L^2(G)$, $H^1(G)$, $H_0^1(G)$ and $H^2(G)$, for a given bounded, open interval $G \subset \mathbb{R}$. These spaces arise as $\mathcal{V}$ and $\mathcal{H}$ in variational formulations of Kolmogorov PDEs for diffusion models in finance, for instance in the Black-Scholes model. However, to characterize payoff functions with kinks (e.g. as for call or put options), or with discontinuities (digital option), we further need fractional order spaces of functions v that satisfy $v \in L^2(G)$, but $v \notin H^1(G)$ or $v \notin H^2(G)$. Later in the semester, we will also extend the diffusion models to allow jump processes. Here, in the corresponding parabolic PDE, $\mathcal{V}$ has to be chosen as a fractional order space.

This exercise addresses the definition of the fractional order norms and further properties. Given $G = (a, b)$ for $a, b \in \mathbb{R}$ with $a < b$, we define for $s \in (0, 1)$ the fractional Sobolev spaces

$$H^s(G) \coloneqq \{v \in L^2(G) : \int_a^b \int_a^b \frac{|v(x)-v(y)|^2}{|x-y|^{2s+1}}\,dx\,dy < \infty\}\,.$$

For any noninteger $s > 1$, we further define

$$H^s(G) \coloneqq \{v \in H^{\lfloor s \rfloor}(G) : v^{(\lfloor s \rfloor)} \in H^{s-\lfloor s \rfloor}(G)\}\,.$$

There are several equivalent definitions of fractional Sobolev spaces, which we do not further specify here. The fractional Sobolev spaces are also used to analyze the regularity of (nonsmooth) initial data for parabolic PDEs (see Slides 4b).

a) Prove that for any $0 < s < t < 1$, there holds $H^s(G) \supset H^t(G)$.

b) Let $\mathbf{1}_A : G \to \{0,1\}$ denote the indicator function of any set $A \subset G$. Prove that for any $c \in G$, $\mathbf{1}_{(c,b)} \notin H^{\frac{1}{2}}(G)$ but $\mathbf{1}_{(c,b)} \in H^{\frac{1}{2}-\epsilon}(G)$ for any $\epsilon \in \left(0, \frac{1}{2}\right)$.

Hint: You can assume without loss of generality that $G = (0,1)$ and $c = 0.5$.

c) Let $K > 1$, $R > \log(K) > 0$ be fixed and consider the truncated domain $G := (-R, R)$. Prove that the payoff of a call option contract in log-price on G, given by $u_0(x) := (\exp(x) - K)_+$ for $x \in G$, satisfies $u_0 \in H^{\frac{3}{2}-\epsilon}(G)$ for $\epsilon \in \left(0, \frac{1}{2}\right)$ and $u_0 \notin H^{\frac{3}{2}}(G)$.

Hint: You may use the fact that $H^1(-R,R) \subset H^s(-R,R)$ and that $H^s(-R,R)$ is a linear space for $s \in [0,1]$ without proof.

3. Theoretical study of the CEV model

Let $J = (0,T]$ and $G = (0,R)$ with $T, R > 0$, and let $r, \sigma > 0$ and $\rho \geq 0$ be constant parameters. Consider the localized CEV model

$$\begin{cases} \partial_t v - \mathcal{A}^{\mathrm{CEV}} v + rv = 0 & \text{on } J \times G\,, \\ v(t,R) = 0 & \text{on } J\,, \\ v(0,s) = g(s) & \text{on } G\,, \end{cases} \tag{2}$$

where $g : G \to \mathbb{R}$ is the payoff function, and $\mathcal{A}^{\mathrm{CEV}}$ the infinitesimal generator given by

$$\mathcal{A}^{\mathrm{CEV}} := \frac{1}{2}\sigma^2 s^{2\rho} \partial_{ss} + rs\partial_s\,.$$

Let $\mathcal{H}_\mu := L^2(G; s^{2\mu}\, ds)$ be the Hilbert space with inner-product

$$(w,v)_\mu := \int_G w(s)v(s)s^{2\mu}\, ds\,,$$

and let $W_{\rho,\mu} := \overline{C_0^\infty}(G)^{\|\cdot\|_{\rho,\mu}}$ which is the completion of $C_0^\infty(G)$ with respect to the norm

$$\|w\|^2_{\rho,\mu} := \int_G s^{2\rho+2\mu} |\partial_s w|^2 + s^{2\mu} |w|^2\, ds\,, \quad w \in W_{\rho,\mu}\,.$$

The variational formulation of (2) reads as follows

$$\begin{aligned} &\text{Find } v \in L^2(J; W_{\rho,\mu}) \cap H^1(J; (W_{\rho,\mu})^*) \text{ such that for a.e. } t \in J\,, \\ &(\partial_t v, w)_\mu + a^{\mathrm{CEV}}_{\rho,\mu}(v,w) = 0 \qquad \forall\, w \in W_{\rho,\mu}\,, \\ &v(0,\cdot) = g(\cdot) \qquad \text{in } \mathcal{H}_\mu\,, \end{aligned} \tag{3}$$

where the bilinear form $a^{\mathrm{CEV}} : W_{\rho,\mu} \times W_{\rho,\mu} \to \mathbb{R}$, is given by

$$a^{\mathrm{CEV}}_{\rho,\mu}(u,v) = \frac{\sigma^2}{2}\left(s^{2\rho}u', v'\right)_\mu + \sigma^2(\rho+\mu)\left(s^{2\rho-1}u', v\right)_\mu - r\left(su', v\right)_\mu + r\left(u,v\right)_\mu\,.$$

a) Assume that the parameter ρ satisfies

$$\rho \in [0,1]\,,\ \rho \neq \frac{1}{2}\,, \tag{4}$$

and set $\mu = 0$. Show that the bilinear form $a^{\mathrm{CEV}}_{\rho,\mu}(\cdot,\cdot)$ is continuous.

Hint: You may use without proof the following *weighted Hardy's inequality*: For any $\phi \in C_0^\infty(G)$ and any $\epsilon > 0$ with $\epsilon \neq 1$, it holds that

$$\left(\int_0^R s^{\epsilon-2} \left|\phi(s)\right|^2 \, ds \right)^{\frac{1}{2}} \leq \frac{2}{|\epsilon - 1|} \left(\int_0^R s^{\epsilon} \left|\partial_s \phi(s)\right|^2 \, ds \right)^{\frac{1}{2}} .$$

b) Let again $\mu = 0$ and assume now $0 \leq \rho \leq \frac{1}{2}$. Show that $a_{\rho,\mu}^{\mathrm{CEV}}(\cdot,\cdot)$ is **strongly coercive** on $W_{\rho,\mu}$, i.e. there exists $c > 0$ such that for all $v \in W_{\rho,\mu}$,

$$a_{\rho,\mu}^{\mathrm{CEV}}(v,v) \geq c \left\|v\right\|_{\rho,\mu}^2 .$$

The previous two subquestions now yield the well-posedness of (3) for the case $0 \leq \rho < \frac{1}{2}$ by an application of [1, Theorem 3.2.2] for the triplet

$$W_{\rho,0} \subset L^2(G) \equiv L^2(G)^* \subset \left(W_{\rho,0}\right)^* .$$

Importantly, the case $\rho = \frac{1}{2}$, which corresponds to the Heston model and CIR process, is not covered by the above due to the fact that the Hardy inequality fails for $\epsilon = 1$.

c) Assume now that $\rho = \frac{1}{2}$ and let $-\frac{1}{2} < \mu < 0$. Show that there exist $C_1, C_2 > 0$ such that $\forall \varphi, \phi \in W_{\rho,\mu}$ the following holds:

$$\begin{aligned} \left|a_{\rho,\mu}^{\mathrm{CEV}}(\varphi,\phi)\right| &\leq C_1 \left\|\varphi\right\|_{\rho,\mu} \left\|\phi\right\|_{\rho,\mu} , \\ a_{\rho,\mu}^{\mathrm{CEV}}(\varphi,\varphi) &\geq C_2 \left\|\varphi\right\|_{\rho,\mu}^2 . \end{aligned}$$

Conclude by showing the well-posedness (existence and uniqueness of the solution) of (3) also for the case $\rho = \frac{1}{2}$.

Due: Wednesday, April 17th, at 2pm.

References

[1] Hilber, Norbert and Reichmann, Oleg and Schwab, Christoph and Winter, Christoph: *Computational Methods for Quantitative Finance: Finite Element Methods for Derivative Pricing"*, Springer Science & Business Media, 2013.