Solutions: Series 2

1. Sobolev spaces and Banach-space valued functions

a) "$\Longrightarrow$": Sobolev embedding says that $u \in H^1(0,1) \subset C^0[0,1]$(see section "Sobolev spaces" of lecture slide 2). Therefore $u_1(a) = u_2(a)$.
"$\Longleftarrow$": Assume that $u_1(a) = u_2(a)$, we claim that $u \in L^2(0,1)$ and that u is weakly differentiable with its weak derivative $u' \in L^2(0,1)$. We will show that u' could defined in the following way:

$$u'(t) = \begin{cases} u_1'(t), & t \in (0,a), \\ 1, & t = a, \\ u_2'(t), & t \in (a,1). \end{cases}$$

Note that it is not a must to let $u'(a) = 1$: Since weak derivative is defined using Lebesgue integral, it is allowed to modify u' on a zero-measure set. Clearly $u, u' \in L^2(0,1)$. For any fixed $\phi \in C_0^1(0,1)$, we have

$$\begin{aligned} \int_0^1 u'(t)\phi(t)\,dt &= \int_0^a u_1'(t)\phi(t)\,dt + \int_a^1 u_2'(t)\phi(t)\,dt \\ &= [u_1\phi]_{t=0}^{t=a} - \int_0^a u_1(t)\phi'(t)\,dt + [u_2\phi]_{t=a}^{t=1} - \int_a^1 u_2(t)\phi'(t)\,dt \\ &= (u_1(a) - u_2(a))\phi(a) - \int_0^1 u(t)\phi'(t)\,dt. \\ &= -\int_0^1 u(t)\phi(t)\,dt. \end{aligned}$$

Therefore u' is the weak derivative of u.

b) To check this, we note that for any $g \in Y^*$ and any $t \neq T$,

$$|g(f(t)) - g(f(T))| = |g(f(t) - f(T))| \leq \|g\|_{Y^*} \cdot \|f(t) - f(T)\|_Y$$

So, by the continuity of $f(t)$,

$$|g(f(t)) - g(f(T))| \to \|g\|_{Y^*} \cdot 0 = 0 \text{ as } t \to T.$$

Therefore $g(f(t)) \in C[0,1]$ and is Lebesgue-measurable. By Theorem of Pettis we obtain the measurability of $f(t)$.

c) Note that for any $t \in [0,1]$,

$$V_A(t) = \begin{cases} x - t, & x \in (0, t - A), \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

Here we set $(0, t-A) = \emptyset$ if $t \leq A$.

i) $A \in (-1,1)$.

To let $V_A \in C([0,1]; L^2(0,1))$, we have to make sure that (1): $V_A(t) \in L^2(0,1)$ for any $t \in [0,1]$ and (2): $V_A(t)$ is continuous.

We study these two conditions. Since for any $A \in (-1,1)$ and any $T \in [0,1]$, $|v_A| \leq 1$, we have

$$\|V_A(T)\|^2_{L^2(0,1)} = \int_0^1 v_A(T,x)^2\,dx \leq 1.$$

So the first condition is satisfied. Moreover, for any fixed $T \in [0,1]$, we have

$$\begin{aligned}
0 &\leq \lim_{t\to T} \|V_A(t) - V_A(T)\|_{L^2(0,1)} \\
&= \lim_{t\to T} \|v_A(t,x) - v_A(T,x)\|_{L^2(0,1)} = \lim_{t\to T} \|(x-t)1_{\{t-x>A\}} - (x-T)1_{\{T-x>A\}}\|_{L^2(0,1)} \\
&\leq \lim_{t\to T} \|(x-t)1_{\{T-x>A\}} - (x-T)1_{\{T-x>A\}}\|_{L^2(0,1)} + \lim_{t\to T} \|(x-t)1_{(\min(t-A,T-A),\max(T-A,t-A))}\|_{L^2(0,1)} \\
&\leq \lim_{t\to T} \|(x-t) - (x-T)\|_{L^2(0,1)} + \lim_{t\to T} |T-t|^{0.5} = 0.
\end{aligned} \tag{2}$$

Therefore the second condition is also satisfied and $V_A \in C([0,1]; L^2(0,1))$ for any $A \in (-1,1)$.

ii) $A = 0$.

To let $V_A \in C([0,1]; H^1(0,1))$, we have to make sure that (1): $V_A(t) \in H^1(0,1)$ for any $t \in [0,1]$ and (2): $V_A(t)$ is continuous.

We consider fulfillment of the first condition. By Sobolev embedding, for any fixed $t \in (0,1)$, $V_A(t) \in H^1(0,1) \subset C^0[0,1]$. Therefore, A must be selected such that $V_A(t) = v_A(t,x)$ is continuous with respect to x for any $t \in [0,1]$. In view of (1), this is possible only if $A = 0$. Moreover, if $A = 0$, then for any $t \in [0,1]$, $V_0(t) = v_0(t,x)$ is a continuous piecewise linear function, which, by the result of exercise 1a), belongs to $H^1(0,1)$. Therefore the first condition is fulfilled only when $A = 0$.

Now we check if $A = 0$ satisfies the second condition. We have that, by following the solution to 1a), $\partial_x v_0(T,x) = 1_{\{x<T\}}$. Here ∂_x denotes the weak derivative. Therefore,

$$\begin{aligned}
\|V_0(t) - V_0(T)\|^2_{H^1(0,1)} &= \|v_0(t,x) - v_0(T,x)\|^2_{H^1(0,1)} \\
&= \|v_0(t,x) - v_0(T,x)\|^2_{L^2(0,1)} + \|\partial_x(v_0(t,x) - v_0(T,x))\|^2_{L^2(0,1)} \\
&= \|v_0(t,x) - v_0(T,x)\|^2_{L^2(0,1)} + |T-t|.
\end{aligned}$$

Here, as $t \to T$, both terms converge to 0 (The convergence of the second term is obvious while the argument for the first term is the same as in (2)). Therefore $A = 0$ also satisfies the second condition.

Note that V_A with any $A \in (-1,1)$ is measurable, due to the result in 1b).

2. Finite element discretization for the heat equation

a) Let u be a smooth solution of the PDE. Then, for all $v \in H_0^1(G)$, we may write

$$\int_G \partial_t u(t,x)v(x) - \partial_{xx}u(t,x)v(x)\,dx = \int_G f(t,x)v(x)\,dx\,.$$

By employing integration-by-parts on the second term and using the homogeneous boundary conditions satisfied by v we obtain the result.

b) Fix some column vector $V \in \mathbb{R}^N$ and let

$$v_N(x) = \sum_{i=1}^N V_i \phi_{N,i}(x)\,.$$

Injecting this expression and

$$u_N(t,x) = \sum_{i=1}^N u_{N,i}(t)\phi_{N,i}(x)$$

in the variational formulation, we get, for all $t \in J$,

$$\frac{d}{dt}\left(\sum_{j=1}^{N} u_{N,j}(t)\phi_{N,j}, \sum_{i=1}^{N} V_i\phi_{N,i}\right)_{L^2(G)} + a\left(\sum_{j=1}^{N} u_{N,j}(t)\phi_{N,j}, \sum_{i=1}^{N} V_i\phi_{N,i}\right) = \left(f(t), \sum_{i=1}^{N} V_i\phi_{N,i}\right)_{L^2(G)} .$$

Developing the sums and using linearity, we find, for each $t \in J$,

$$\sum_{i=1}^{N}\sum_{j=1}^{N}(V_i\ (\phi_{N,j}, \phi_{N,i})_{L^2(G)}\frac{d\, u_{N,j}}{dt}(t) + V_i\, a\,(\phi_{N,j}, \phi_{N,i})\, u_{N,j}(t)) = \sum_{i=1}^{N} V_i\ (f(t), \phi_{N,i})_{L^2(G)}\ , \quad (3)$$

i.e.

$$V^T\left(\mathbf{M}\frac{d\,\underline{u}_N}{dt}(t) + \mathbf{A}\underline{u}_N(t)\right) = V^T\underline{F}(t)\,,$$

where $\mathbf{M}$ and $\mathbf{A}$ are the matrices given by

$$\mathbf{M}_{i,j} = (\phi_{N,i}, \phi_{N,j})_{L^2(G)}\ , \quad \mathbf{A}_{i,j} = a\,(\phi_{N,i}, \phi_{N,j})\ , \quad 1 \le i, j \le N\,,$$

and $\underline{\mathrm{F}}(t)$ is the column vector given by

$$\underline{\mathrm{F}}_i(t) = (f(t), \phi_{N,i})_{L^2(G)}\ , \quad 1 \le i \le N\,.$$

Since Eq. (3) holds with any choice of column vector $V \in \mathbb{R}^N$, we deduce that

$$\mathbf{M}\frac{d\,\underline{u}_N}{dt}(t) + \mathbf{A}\underline{u}_N(t) = \underline{\mathrm{F}}(t)\,, \quad \forall t \in J\,.$$

c) We write the classical ϑ-scheme:

$$\mathbf{M}\frac{\underline{u}_N^{m+1} - \underline{u}_N^m}{k} + \vartheta\mathbf{A}\underline{u}_N^{m+1} + (1-\vartheta)\mathbf{A}\underline{u}_N^m = \vartheta\underline{\mathrm{F}}(t_{m+1}) + (1-\vartheta)\underline{\mathrm{F}}(t_m)\,.$$

This leads to

$$\mathbf{B}_\vartheta = \mathbf{M} + k\vartheta\mathbf{A}\,,$$
$$\mathbf{C}_\vartheta = \mathbf{M} - k(1-\vartheta)\mathbf{A}\,,$$
$$\underline{\mathrm{F}}_\vartheta^m = k\vartheta\underline{\mathrm{F}}(t_{m+1}) + k(1-\vartheta)\underline{\mathrm{F}}(t_m)\,.$$

3. Implementation on Python

a) Note that for each $i \in \{1, \dots, N\}$, we have

$$\phi_{N,i} = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x \in [x_{i-1}, x_i]\,, \\[2ex] \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x \in [x_i, x_{i+1}]\,, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

In exercise 2.b), we obtained

$$\mathbf{M}_{i,j} = \int_0^1 \phi_{N,i}(x)\phi_{N,j}(x)\,dx\,.$$

This is obviously 0 if $|i-j|>1$. If $i=j$,

$$\mathbf{M}_{i,j} = \frac{1}{h^2}\int_{x_{i-1}}^{x_i} (x-x_{i-1})^2\,dx + \frac{1}{h^2}\int_{x_i}^{x_{i+1}} (x_{i+1}-x)^2\,dx\,.$$

Clearly, both integrals have the same value. Using the change of variables $\tau = x - x_{i-1}$ in the first one, we get

$$\mathbf{M}_{i,i} = \frac{2}{h^2}\int_0^h \tau^2 d\tau = \frac{2h}{3}\,.$$

For $i \in \{1,\dots,N-1\}$ and $j=i+1$, we have

$$\mathbf{M}_{i,i+1} = \frac{1}{h^2}\int_{x_i}^{x_{i+1}} (x_{i+1}-x)(x-x_i)\,dx\,,$$

which, using $\tau = x - x_i$, leads to

$$\mathbf{M}_{i,i+1} = \frac{1}{h^2}\int_0^h \tau(h-\tau)\,d\tau = \frac{1}{h^2}\left[h\frac{\tau^2}{2} - \frac{\tau^2}{3}\right]_0^h = \frac{h}{6}\,.$$

Since $\mathbf{M}$ is symmetric, we conclude that

$$\mathbf{M} = h\begin{pmatrix} 2/3 & 1/6 & 0 & \dots & \dots & \dots & 0 \\ 1/6 & 2/3 & 1/6 & \dots & \dots & \dots & 0 \\ 0 & 1/6 & 2/3 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 2/3 & 1/6 & 0 \\ 0 & 0 & \dots & \dots & 1/6 & 2/3 & 1/6 \\ 0 & 0 & \dots & \dots & 0 & 1/6 & 2/3 \end{pmatrix}.$$

To determine $\mathbf{A}$, we note that $\mathbf{A}_{i,j} = \int_0^1 \phi'_{N,i}(x)\phi'_{N,j}(x)\,dx$. For any $i = 1,2,\dots,N$,

$$\phi'_{N,i}(x) = \frac{1}{h}(1_{(x_{i-1},x_i)} - 1_{(x_i,x_{i+1})}).$$

Therefore,

$$\mathbf{A}_{i,j} = \begin{cases} \frac{2}{h} & i=j\,, \\ -\frac{1}{h} & |i-j|=1\,, \\ 0 & \text{otherwise,} \end{cases}$$

and we have

$$\mathbf{A} = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & \dots & \dots & \dots & 0 \\ -1 & 2 & -1 & \dots & \dots & \dots & 0 \\ 0 & -1 & 2 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 2 & -1 & 0 \\ 0 & 0 & \dots & \dots & -1 & 2 & -1 \\ 0 & 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix}.$$

To see that the matrices $\mathbf{A}, \mathbf{M}$ are nonsingular, note that $\mathbf{M}$ is strictly diagonally dominant $(1/6+1/6<2/3)$ and hence invertible and for $\mathbf{A}$ it holds that $\det(\mathbf{A}) = h^{-N}(N+1) > 0$. See the solution code for the programming part.

b) The boundary and initial conditions could be easily checked by forcing $x = 0, 1$ and $t = 0$. Moreover, we have for $u(t, x) = e^{-t} x \sin(\pi x)$,

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial t^2} \\
&= -e^{-t} x \sin(\pi x) - \partial_x (e^{-t} \sin(\pi x) + \pi e^{-t} x \cos(\pi x)) \\
&= -e^{-t} x \sin(\pi x) - \pi e^{-t} \cos(\pi x) - \pi e^{-t} \cos(\pi x) + \pi^2 e^{-t} x \sin(\pi x) \\
&= e^{-t} \big((\pi^2 - 1) x \sin(\pi x) - 2\pi \cos(\pi x) \big) = f(t, x).
\end{aligned}$$

Therefore u is the solution. For the programming part, see solution code.

c) For each i, we have

$$\int_0^1 f(t,x)\phi_{N,i}(x)\,dx = \int_{x_{i-1}}^{x_i} \frac{f(t,x)(x - x_i)}{h}\,dx + \int_{x_i}^{x_{i+1}} \frac{f(t,x)(x_{i+1} - x)}{h}\,dx\,.$$

Since the integrand in each term is smooth we can use the Simpson rule on each interval. This gives

$$\begin{aligned}
\int_0^1 f(t,x)\phi_{N,i}(x)\,dx = &\frac{h}{6}\left(f(x_{i-1}) \times 0 + 4 f(x_i - h/2) \times \frac{1}{2} + f(x_i) \times 1 \right) \\
&+ \frac{h}{6}\left(f(x_i) \times 1 + 4 f(x_i + h/2) \times \frac{1}{2} + f(x_{i+1}) \times 0 \right) + O(h^5) \\
=& h\frac{f(t, x_i - h/2) + f(t, x_i) + f(t, x_i + h/2)}{3} + O(h^5)\,.
\end{aligned}$$

For the programming part, see solution code.

d) See the solution code.

e) The experimental order of convergence (eoc) observed for $\theta = 1$ is approximately 0.839 with respect to k while for $\theta = 0.5$ is 1.996. With $\theta = 0.3$ the numerical scheme does not converge.

Since $\frac{h}{k}$ is a constant, we verify the convergence order $O(h^2 + k^2)$ for $\theta = 0.5$. In the case that $\theta = 1$, the error from time discretization will be dominating(we have theoretically the convergence order $O(h^2 + k)$, as $\frac{h}{k}$ is a constant, the error from time discretization converges in a slower speed than that of spatial discretization). $\theta = 0.3$ will lead to instability. In fact, for $\theta \in [0, \frac{1}{2})$ we require that $\frac{h^2}{k}$ shall not be too small to guarantee the stability(see Chapter 3.5 of the textbook).

As is stated, the numerical scheme with $\theta = 0.5$("Crank-Nicolson scheme") achieves second-order accuracy with respect to time step k(Convergence order is $O(h^2 + k^2)$), which is better than the schemes of other θ(Convergence order is only $O(h^2 + k)$). This second-order accuracy could be easily verified by Taylor's expansion around time level $t_m + \theta k$.

f) We observe convergence for $\theta = 0.5, 1$(eocs are 1.007 and 1.022 correspondingly with respect to k). Since $\frac{h^2}{k}$ is a constant, we verify the convergence order $O(h^2 + k)$ for $\theta = 1$. For $\theta = 0.5$, the error from spatial discretization is now dominating.

Moreover, the method is not stable with $\theta = 0.3$ and the reason is the same as before.