Exam [maximal number of points: 89 pts.]

1. General questions [in total 15 pts.]

a) [1 pt.] Let $G = (-1, 1)$. Give an example of a function $f \in L^2(G)$ which does not belong to $H^1(G)$. State the definition precisely; no proofs are required.

b) [1 pt.] Consider a Swing option in a Black–Scholes market with fixed refracting time $\delta > 0$. Justify the necessity of $\delta > 0$.

c) [2 pts.] What kind of convergence behavior do you expect for the price of a digital option in the Black-Scholes model using the θ-scheme in time and a piecewise linear continuous Finite Element approximation in space in the $L^2(J, L^2(D))$-norm, where D is an appropriately chosen domain of interest and $J = (0, 1)$. Is there a method to improve the rate? If so, name it.

d) [2 pts.] Consider the Heston and the CEV model. State one major difference from a numerical perspective and one major difference from a financial perspective.

e) [2 pts.] For which values of θ is the θ-scheme unconditionally stable? For which values of θ is the θ-scheme conditionally stable? What is this condition?

f) [1 pt.] For $d \in \mathbb{N}$ consider a d-dimensional Black-Scholes market model and let $a^{\mathrm{BS}}: H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d) \to \mathbb{R}$ be the corresponding bilinear form. Give a sufficient condition so that a^{BS} is continuous and satisfies a Gårding inequality.

g) [6 pts.] Let X be a Lévy process with characteristic triplet (σ^2, γ, ν) being the $\log$-price process. Assume $\sigma > 0$. Suppose that $\nu(\mathrm{d}z) = k(z)\mathrm{d}z$ and that the density $k(z)$ satisfies, for $\alpha \in (0, 2)$, $\beta_+ > 1$, $\beta_- > 0$, $C > 0$,

$$k(z) \leq Ce^{-\beta_-|z|}, \qquad z < -1, \tag{1}$$

$$k(z) \leq Ce^{-\beta_+ z}, \qquad z > 1, \tag{2}$$

$$k(z) \leq C\frac{1}{|z|^{1+\alpha}}, \qquad 0 < |z| < 1. \tag{3}$$

i) Explain why in the context of European plain vanilla option pricing the condition $\beta_+ > 1$ is required.

ii) What is the difference in the algebraic structure of the discretization of the problem between the two cases $k(z) = 0$ for every $z \in \mathbb{R}$ and $k(z) > 0$ for every $z \in \mathbb{R}$. What effect does this have on the computational cost of the solution?

iii) Why is the value $\alpha = 2$ not admissible?

iv) State a necessary and sufficient condition on the triplet (σ^2, γ, ν) for which e^X is a martingale.

2. Pricing of multidimensional options [in total 37 pts.]

We consider the d-dimensional Black-Scholes model under a risk-neutral measure $\mathbb{Q}$. Let $d \in \mathbb{N}$, $d \geq 1$ and let $(W_t)_{t\geq 0}$ be a d-dimensional Brownian motion. The i-th component of the process $S = (S_t)_{t\geq 0}$, $S_t := (S_t^1, \dots, S_t^d)^\top$ evolves according to

$$\mathrm{d}S_t^i = b_i(S_t)\mathrm{d}t + \sum_{j=1}^{d} \Sigma_{ij}(S_t)\mathrm{d}W_t^j, \quad i = 1, \dots, d\ , \tag{4}$$

starting from $S_0 = (S_0^1, \dots, S_0^d)$, where $b_i(s) = rs_i$, $\Sigma_{ij}(s) = \mathbf{\Sigma}_{ij}s_i$, $S_0^j > 0$, $1 \leq i, j \leq d$. We assume $r \in [-1, 1]$, $\mathbf{\Sigma}_{ij} \geq 0$ constant, and set $\boldsymbol{\mathcal{Q}} = \mathbf{\Sigma}\mathbf{\Sigma}^\top$.
The price $V(t, S_t^1, \dots, S_t^d)$ of a European basket option maturing at T with a sufficiently smooth payoff g is given by the conditional expectation

$$V(t, s_1, \dots, s_d) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}g(S_T^1, \dots, S_T^d) \mid S_t^i = s_i,\ i = 1, \dots, d\right].$$

a) [1 pts.] Name at least one limitation of the Black-Scholes model (4).

b) [4 pts.] State the pricing PDE for $V(t, s_1, \dots, s_d)$ (no proofs are required). Define the infinitesimal generator $\mathcal{A}$ of the process S.

c) [6 pts.] We transform the pricing equation to time-to-maturity, log-price and localize to a bounded domain $G := (-R, R)^d$ to obtain the following transformed PDE for $u(t, \mathbf{x}) := u(t, x_1, \dots, x_d) := V(T - t, \exp(x_1), \dots, \exp(x_d))$:

$$\begin{aligned} \partial_t u(t, \mathbf{x}) - \frac{1}{2}\nabla \cdot (\boldsymbol{\mathcal{Q}}\nabla u(t, \mathbf{x})) + \boldsymbol{\mu}^\top \nabla u(t, \mathbf{x}) + ru(t, \mathbf{x}) = 0 &\quad \text{in } J \times G\ , \\ u(t, \mathbf{x}) = 0 &\quad \text{in } J \times \partial G\ , \\ u(0, x_1, \dots, x_d) = g(e^{x_1}, \dots, e^{x_d}) &\quad \text{in } G\ , \end{aligned} \tag{5}$$

where $\mu := [Q_{11}/2 - r,\ \dots,\ Q_{dd}/2 - r]^\top$.
Find the weak formulation of Equation (5).
Show that the bilinear form $a(\cdot, \cdot)$ associated with the operator in log-price is given by

$$a(w, v) := \frac{1}{2}\int_G \nabla w(\mathbf{x})^\top\ \boldsymbol{\mathcal{Q}}\nabla v(\mathbf{x})\, d\mathbf{x} + \int_G \mu^\top \nabla w(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x} + r\int_G w(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x}\ .$$

d) [3 pts.] Give non-trivial sufficient conditions on $\boldsymbol{\mathcal{Q}}$ and r under which conditions the bilinear form $a(\cdot, \cdot)$ obtained in subproblem **c)** satisfies the Gårding inequality, i. e. under which conditions there exist $C_1, C_2 > 0$ such that

$$a(v, v) \geq C_1\|v\|_{H^1(G)}^2 - C_2\|v\|_{L^2(G)}^2\ . \tag{6}$$

Prove that under these conditions, $a(\cdot, \cdot)$ satisfies the Gårding inequality.

e) [2 pts.] Assume we discretize the weak form using piecewise linear Finite Elements. What is the main difficulty that comes up in the numerical treatment for $d \geq 3$ and which does not come up for $d \leq 2$?

From now on, let $d = 2$ and

$$\boldsymbol{Q} = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{pmatrix},$$

with $\sigma_1, \sigma_2 > 0$, $\rho \in [-1, 1]$, for the initial data $s_1, s_2 > 0$.

f) [7 pts.] What is the matrix representation $\mathbf{A}$ of the discretization of the bilinear form $a(\cdot, \cdot)$ on the Finite Element space V_N given by

$$V_N = \text{span}\left\{\varphi_i := b_{i_1}(x_1)b_{i_2}(x_2) \mid 1 \le i_k \le N_k,\ k = 1, 2\right\},$$

where $b_{i_k}(x_k)$, $k = 1, 2$, are the univariate hat-functions?
Prove that the stiffness matrix $\mathbf{A}$ is given by the formula

$$\mathbf{A} = \left(\frac{Q_{11}}{2}\mathbf{S}^1 + \mu_1\mathbf{B}^1 + r\mathbf{M}^1\right) \otimes \mathbf{M}^2 + \left(-Q_{12}\mathbf{B}^1 + \mu_2\mathbf{M}^1\right) \otimes \mathbf{B}^2 + \frac{Q_{22}}{2}\mathbf{M}^1 \otimes \mathbf{S}^2\,, \quad (7)$$

Define the matrices $\mathbf{M}^j, \mathbf{B}^j, \mathbf{S}^j$ (where $j = 1, 2$).

g) [6 pts.] Suppose now that $N_1 = N_2 = N$, and that you can use a function `C = matrices(f1, f2, f3, N)` that takes as input three function handles `f1 = @(x)(...)`, `f2 = @(x)(...)`, `f3 = @(x)(...)`, and an integer `N` and returns a matrix $C \in \mathbb{R}^{N\times N}$ such that

$$C_{ij} = \int_{-R}^{R} f_1(x)b_i'(x)b_j'(x)dx + \int_{-R}^{R} f_2(x)b_i'(x)b_j(x)dx + \int_{-R}^{R} f_3(x)b_i(x)b_j(x)dx.$$

Complete the following snippet to compute the matrix $\mathbf{A}$ defined in (7).

```
L = 5;
N = 2.^(L+1)-1;

sigma = [.1; .3];
rho = -.6;
Q=[sigma(1)^2, sigma(1)*sigma(2)*rho; sigma(1)*sigma(2)*
    rho, sigma(2)^2];

%% complete the lines below
S = ...
M = ...
B = ...

A = ...
```

h) [5 pts.] Define g_N to be the $L^2(G)$ projection of g in V_N, i.e., $g_N \in V_N$ such that

$$\int_G g_N(x_1, x_2)v_N(x_1, x_2)dx_1dx_2 = \int_G g(x_1, x_2)v_N(x_1, x_2)dx_1dx_2, \qquad \forall v_N \in V_N.$$

Let $\widehat{\mathbf{g}} \in \mathbb{R}^{N^2}$ be the vector such that $g_N = \sum_{i=1}^{N^2} \widehat{\mathbf{g}}_i\varphi_i(x_1, x_2)$.

Suppose that you are given a function `w = int2dbasis(f, N)` which takes as inputs a function handle `f = @(x)(...)` and an integer `N` and returns a vector $\mathbf{v} \in \mathbb{R}^{N^2}$ such that $\mathbf{v}_i = \int_G f(x_1, x_2)\varphi_i(x_1, x_2)dx_1dx_2$, for $i = 1, \ldots, N^2$. You can also use the function `C = matrices(f1, f2, f3, N)` introduced above.

Complete the following snippet that takes as input a handle `g = @(x)(...)` and an integer `N` and returns $\widehat{\mathbf{g}}$.

```
function gvec = L2proj(g, N)
  %% compute gvec=ĝ here
  ....

  gvec = ...
end
```

i) [3 pts.] Suppose that we are pricing a digital option with payoff $g(s_1, s_2) = \mathbb{1}_{[-1,1]^2}(s_1, s_2)$. and that we use, for time-stepping, the Crank-Nicolson scheme on a uniform time mesh $\{0 = t_0, \ldots, t_N = T\}$, with $t_k = k\Delta t$.

What convergence rate with respect to the *total* number of degrees of freedom N_{2D} would you expect? Can this be improved? If yes, how?

3. Pricing of American options [in total 27 pts.] We consider an American put option in a Black-Scholes market and in log price. For $R, T > 0$, let $G = (-R, R)$, $J = (0, T]$, and consider the truncated problem

$$\begin{array}{rcll} \partial_t v_R - \mathcal{A}^{\mathrm{BS}} v_R + r v_R & \geq & 0 & \text{in } J \times G \\ v_R(t,x) & \geq & g(e^x)|_G & \text{in } J \times G \\ (\partial_t v_R - \mathcal{A}^{\mathrm{BS}} v_R + r v_R)((g \circ \exp)|_G - v_R) & = & 0 & \text{in } J \times G \\ v_R(0,x) & = & g(e^x)|_G & \text{in } G \\ v_R(t, \pm R) & = & g(e^{\pm R}) & \text{in } J\,. \end{array} \tag{8}$$

The variational formulation of the truncated problem for the excess to payoff $u_R = v_R - g \circ \exp|_G$ reads then

$$\begin{array}{l} \text{Find } u_R \in L^2(J; H_0^1(G)) \cap H^1(J; H^{-1}(G)) \text{ such that } u_R(t,\cdot) \in \mathcal{K}_{0,R} \text{ and} \\ \langle \partial_t u_R, v - u_R \rangle + a^{\mathrm{BS}}(u_R, v - u_R) \geq -a^{\mathrm{BS}}(g \circ \exp, v - u_R),\ \forall v \in \mathcal{K}_{0,R}, \\ u_R(0) = 0, \end{array} \tag{9}$$

with

$$\mathcal{K}_{0,R} := \{ v \in H_0^1(G) :\ v \geq 0 \ \text{ a.e. } \ x \in G \}.$$

For $n \in \mathbb{N}$, introduce the uniform grid $\{-R = x_0, \ldots, x_j = -R + j(2R)/(n+1), \ldots, x_{n+1} = R\}$ and let the finite element space be

$$V_n = \operatorname{span}\{ b_i : i = 1, \ldots, n \} \subset H_0^1(G),$$

where b_i is the classic hat function such that $b_i(x_j) = \delta_{ij}$.

a) [4 pts.] Give the algebraic formulation of the approximation of problem (9) with finite elements and with the θ-scheme in time. You don't need to explicitly compute the entries of the matrices, but you need to define them.

The following listing gives the implementation of a function that takes as input the number of intervals n, R, T, the strike price K, the rate r, and the volatility σ, and returns the price of an American put option in a Black-Scholes market and the exercise boundary.

Listing 1: Solution of an American put problem in a Black-Scholes market

```
function [u, fb] = amput_BS(n, R, T, K, r, sigma)
  h = 2*R/(n+1);
  x = linspace(-R,R,n+2)';
  S = exp(x);
  M = ceil(T/h);
  k = T/M;

  e = ones(n,1);
  Am = h/6*spdiags([e, 4*e, e], -1:1, n, n);
  As = 1/h*spdiags([-e, 2*e, -e], -1:1, n, n);
  Ac = spdiags([-0.5*e, zeros(n, 1), 0.5*e], -1:1, n, n);
```

```matlab
    A = 0.5*sigma^2*As + (0.5*sigma^2-r)*Ac + r*Am;

    payoff = max(K-S, 0);
    u = zeros(n+2, 1);
    fb = zeros(M+1, 1);
    fb(1) = K;

    dof = 2:n+1;

    f = zeros(n+2, 1);
    j = find(x<=log(K), 1, 'last');
    f(1) = -r*K*h/2;
    f(2:j-1) = - r*K*h;
    f(j) = sigma^2/2*K*(x(j+1)-log(K))/h - r*K/2*(2*h-1/h*(x(j+1)-log(K))*(x(j+1)-log(K)));
    f(j+1) = sigma^2/2*K*(x(j)<log(K))*(-x(j)+log(K))/h - r*K/2*(1/h*(log(K)-x(j))*(log(K)-x(j)));

    B = Am+k*A;
    C = Am;

    for i=1:M
        % The function x = psor(A, b, x0) solves
        % with initial guess x0
        % a linear complementary problem given by
        % x'(Ax - b) = 0, x >= 0, Ax - b >= 0
        u(dof) = psor(B, k*f(dof) + C*u(dof), zeros(n,1));

        J = find(u(dof) > 1.e-6);
        fb(i+1) = S(J(1));
    end
    u = u + payoff;
end
```

b) [1 pt.] Indicate what time-stepping scheme is being used in Listing 1.

c) [6 pts.] Suppose that the payoff $g(s) = \max(K - s, 0)$ is replaced by a long butterfly payoff:

i) Write the long butterfly payoff for strikes K_0, K_1, K_2.

ii) Explain, in financial terms, the goal of a long butterfly strategy.

iii) Indicate how you would change Listing 1 to compute the price of an American option with butterfly payoff. Indicate the lines you would change and what would replace them.

Consider now, for the American put option, the case where the constant volatility is replaced by a model with local volatility. The value of the put option $V(t,s)$ is given by

$$V(t,s) := \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\big[e^{-r(\tau-t)} g(S_\tau) \mid S_t = s\big],$$

where S_t is given as the solution of

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma(S_t)S_t\,\mathrm{d}W_t, \quad S_0 = s \geq 0 \tag{10}$$

and $\mathcal{T}_{t,T}$ denotes the set of all stopping times for S_t, where $\sigma(s) = \widehat{\sigma}\min(1+s, 2)$, $s \in (0,\infty)$, for a fixed $\widehat{\sigma} > 0$.

d) [2 pts.] Is the solution to the SDE (10) unique? Explain why.

e) [2 pts.] State the system of inequalities satisfied by $v(t,s) = V(T-t,s)$.

f) [6 pts.] State the localized variational formulation in excess-to-payoff coordinates and log-price on the domain $G = (-R,R)$, $R > 0$. In particular, define the corresponding bilinear form $a^{\mathrm{LV}}(\cdot,\cdot)$.

g) [6 pts.] Indicate how you would change Listing 1 to compute the price of the American put option in a local volatility model. Fix $\widehat{\sigma} = 1$. You can use the same time-stepping method as in Listing 1; furthermore, you can assume you can use a function `C = matrices(f1, f2, f3, N)` that takes as input three function handles `f1 = @(x)(...)`, `f2 = @(x)(...)`, `f3 = @(x)(...)`, and an integer `N` and returns a matrix $C \in \mathbb{R}^{N\times N}$ such that

$$C_{ij} = \int_{-R}^{R} f_1(x) b_i'(x) b_j'(x) dx + \int_{-R}^{R} f_2(x) b_i'(x) b_j(x) dx + \int_{-R}^{R} f_3(x) b_i(x) b_j(x) dx.$$

Indicate what lines you would change and how you would change them.