Series 5

1. Inverse inequality for $S^1(\mathcal{T}_h)$

a) Let $K = (a, b)$ be a bounded open interval and $\hat{K} = (-1, 1)$. Consider the affine linear transformation $\psi\colon \hat{K} \to K;\ \hat{x} \mapsto \frac{b-a}{2}\hat{x} + \frac{a+b}{2}$. Let $u \in C^1(K)$ and define the function $\hat{u}\colon \hat{K} \to \mathbb{R}; \hat{x} \mapsto u(\psi(\hat{x}))$. Show that it holds

i) $\|u\|_{L^2(K)} = \left(\frac{b-a}{2}\right)^{1/2} \|\hat{u}\|_{L^2(\hat{K})}$,

ii) $\|u'\|_{L^2(K)} = \left(\frac{2}{b-a}\right)^{1/2} \|\hat{u}'\|_{L^2(\hat{K})}$.

b) Let $\mathcal{T}_h = \{a = x_0 < x_1 < x_2 < \cdots < x_{N+1} = b\}$ be a mesh on G with uniform mesh width $h := \frac{1}{N+1}$. Setting $K_l := (x_{l-1}, x_l)$, we have that $|K_l| = h$, for all $l = 1, \ldots, N+1$. Define

$$S^1(\mathcal{T}_h) := \{u(x) \in C^0(G) : u|_{K_l} \text{ is linear on } K_l \in \mathcal{T}_h \text{ for } l = 1, \ldots, N+1\}\,.$$

Show that there exists a constant $C > 0$ independent of h, but possibly depending on $|b - a|$, such that

$$\forall u_h \in S^1(\mathcal{T}_h): \quad \|u_h\|_{H^1(G)} \le Ch^{-1} \|u_h\|_{L^2(G)}\,.$$

Hint: Any two norms $\|\cdot\|_1, \|\cdot\|_2$ on a finite-dimensional real vector space V are equivalent, i.e. there exist $c, C > 0$ such that

$$\forall x \in V : c\,\|x\|_2 \le \|x\|_1 \le C\,\|x\|_2\,.$$

2. A priori estimates

We restudy the parabolic equation introduced in Exercise sheet 3 with a particular initial value. Let $G = (a, b)$, for some $a, b, \in \mathbb{R}, a < b$ and $J = (0, 1)$. The equation reads:

$$\begin{aligned}
\partial_t u(t,x) - \partial_x(\alpha(x)\partial_x u(t,x)) + \beta(x)\partial_x u(t,x) + \gamma(x)u(t,x) &= f(t,x) && \text{in } J \times G\\
u &= 0 && \text{on } J \times \partial G\\
u|_{t=0} &= g_K(\exp(x)) && \text{in } G,
\end{aligned}$$

where $K \in G$ and

$$g_K(x) := \begin{cases} 0, & x < K\,,\\ x - K, & x \ge K\,. \end{cases}$$

Moreover, $f \in L^2(J, H^{-1}(G))$, $\alpha, \gamma \in C(\overline{G})$ and $\beta \in C^1(\overline{G})$ such that for a constant $\underline{\alpha} > 0$ the bound $\alpha(x) > \underline{\alpha}$ holds for all $x \in G$. The weak formulation is as follows: Find $u_K \in L^2(J; H_0^1(G)) \cap H^1(J; H^{-1}(G))$ such that $u_K(0, x) = g_K(\exp(x))$ for $x \in G$ and

$$\frac{d}{dt}(u_K, v)_{L^2(G)} + a(u_K, v) = (f, v)_{L^2(G)}\,, \quad \forall v \in H_0^1(G)\,. \tag{1}$$

Here $a(u, v) : H_0^1(G) \times H_0^1(G) \to \mathbb{R}, (u, v) \mapsto \int_G \alpha(x)\partial_x u \partial_x v + \beta(x)(\partial_x u)v + \gamma(x)uv\,dx$. Recall that we justified the continuity and the Gårding inequality for $a(\cdot, \cdot)$ in problem 4a) of Exercise sheet 3.

a) Prove that there exists a constant $C_1 > 0$ such that for any $\epsilon \in (0,1)$,

$$\|u_{K+\epsilon}(0,\cdot) - u_K(0,\cdot)\|_{L^2(G)} \leq C_1 \epsilon\,.$$

Hint: For $x \in G$, consider the point-wise difference $|u_{K+\epsilon}(0,x) - u_K(0,x)|$.

b) Prove that for any $\epsilon \in (0,1)$, there exists a constant $C_2 > 0$ independent of ϵ with

$$\sup_{t \in \bar{J}} \|u_{K+\epsilon}(t,\cdot) - u_K(t,\cdot)\|_{L^2(G)} \leq C_2 \epsilon\,.$$

Hint: Derive a PDE for $u_{K+\epsilon} - u_K$ and use the a priori estimate from the lecture.

3. Black-Scholes formula for European call options

Let $\mathbb{F} := \mathbb{F}^W$ be the filtration on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ generated by the Wiener process $W = (W_t, t \in [0,T])$ for a $T \in (0,\infty)$, cf. Definitions 1.2.1/1.2.2 in [1]. Let $r \in \mathbb{R}$ and $S_0, \sigma > 0$ be fixed parameters. The *geometric Brownian motion* is the stochastic process $S = (S_t, t \in [0,T])$ given by

$$S_t = S_0 \exp\left(\left(r - \sigma^2/2\right) t + \sigma W_t\right), \quad t \in [0,T]\,. \tag{2}$$

a) Use Itô's formula to show that the geometric Brownian motion (2) is a solution to the stochastic differential equation (SDE)

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t\,, \quad S_0 = s_0 > 0\,, \quad t \in [0,T]\,.$$

In fact, S_t is the $\mathbb{P}$-a.s. unique t-continuous solution. In the Black-Scholes model, S models the price of an underlying with spot price $S_0 > 0$, where $r \in \mathbb{R}$ is the fixed interest rate and $\sigma > 0$ ist the volatility parameter.

b) Let

$$f(S_T) = \max\{S_T - K, 0\}$$

be the payoff function of a European call option with strike price $K > 0$. Derive the Black-Scholes formula for this European call option

$$e^{-rT}\mathbb{E}\left[f(S_T)\right] = S_0 \Phi\left(\frac{\left(r + \frac{\sigma^2}{2}\right)T + \ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\left(r - \frac{\sigma^2}{2}\right)T + \ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}}\right),$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ denotes the $\mathcal{N}_{0,1}$-distribution function, i.e.

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,dy\,, \quad x \in \mathbb{R}\,.$$

Hint: You may use without proof that for any $t \in [0,T]$ and $\beta \in \mathbb{R}$ there holds

$$\mathbb{E}\left[e^{\beta W_t}\right] = e^{\frac{1}{2}\beta^2 t}\,.$$

c) Show that the value of the European call option is a non-decreasing function of S_0, r and a non-increasing function of K.

4. Closed form solution for the Black-Scholes equation

a) Fix a given function $u_0 \in C(\mathbb{R})$, define $u : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ as

$$u(x,t) := \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp\left(\frac{-(x-y)^2}{4t}\right) u_0(y)\, dy\,, \quad \forall t > 0,\ x \in \mathbb{R}\,.$$

Show that $\frac{\partial u}{\partial t}(t,x) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0$ for all $t > 0$ and $x \in \mathbb{R}$.

Here and in the following question, we assume that for some $C > 0$ and $\kappa > 0$, there holds $|u_0(x)| \le Ce^{\kappa x}$ for all $x \in \mathbb{R}$.

b) Use the change of variables $z = \frac{x-y}{2\sqrt{t}}$ to prove that

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} u_0\left(x - 2\sqrt{t}z\right)\, dz\,,$$

and deduce the value of the limit $\lim_{t\to 0} u(x,t)$ for each $x \in \mathbb{R}$.
Hint: The function $\psi : z \mapsto Ce^{\kappa x} e^{-z^2+2\kappa|z|}$ is integrable on $\mathbb{R}$ and it holds that $\psi \ge 0$.

c) Consider the Black-Scholes equation for the price $V(t,s)$ of a European call option at time t and stock price s given by

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 V}{\partial s^2} + rs\dfrac{\partial V}{\partial s} - rV = 0 & \text{on } [0,T) \times \mathbb{R}_+, \\ V(T,s) = (s-K)_+ & \forall s \in \mathbb{R}_+. \end{cases} \tag{3}$$

where r is the risk-free interest rate, σ the volatility, and K the strike price. Consider the change of variables

$$s = e^x\,, \quad t = T - \frac{2\tau}{\sigma^2} \quad \text{and} \quad V(t,s) = e^{\alpha x + \beta\tau} u(x,\tau)\,.$$

Show that for some appropriate choice of the parameters α and β, it holds that

$$\frac{\partial u}{\partial \tau}(x,\tau) - \frac{\partial^2 u}{\partial x^2}(x,\tau) = 0\,, \quad \forall x \in \mathbb{R}\,,\ \forall \tau \in [0, \frac{\sigma^2}{2}T)\,.$$

d) Determine $u(x,0)$. Using the previous questions, deduce a closed-form solution for problem (3) (the Black-Scholes formula).

5. Graded time mesh

We restudy the heat equation from Exercise sheet 2, now with nonsmooth initial data. As was shown in the lecture (see Slides 4b), this non-smoothness deteriorates convergence rates of the numerical methods. Graded time meshes, which apply a high temporal refinements near the singular initial condition, are a powerful tool to remedy this issue.

In the following, set $G = J = (0,1)$ and consider

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = xt & \text{on } G \times J\,, \\ u(x,0) = u_0(x) & \forall x \in G\,, \\ u(0,t) = u(1,t) = 0 & \forall t \in J\,. \end{cases} \tag{4}$$

Its variational formulation reads: Find $u \in L^2(J;\mathcal{V}) \cap H^1(J;\mathcal{V}^*)$ such that for almost every $t \in J$,

$$\begin{aligned} \left\langle \frac{d}{dt}u(t), v \right\rangle_{\mathcal{V}^*,\mathcal{V}} + \left(\frac{d}{dx}u(t), \frac{d}{dx}v \right)_{L^2(G)} &= \int_G xt\, v(x)dx\,, \quad \forall v \in \mathcal{V}\,, \\ u(0) &= u_0\,, \quad \text{in } \mathcal{H}\,. \end{aligned} \tag{5}$$

Here we set $\mathcal{V} := H_0^1(G)$ and $\mathcal{H} := L^2(G)$ and set $u_0 \in \mathcal{H}$. Recall that the existence and uniqueness of the weak solution u to (5) are guaranteed.

Please note that for general u_0 we do not have a closed-form solution to the equation we want to solve now. In previous exercise sheets, we usually examined the convergence rate of a numerical method by evaluating the difference between the numerical solution and the exact solution to a given PDE. In most situations, however, we do not know the exact solution. The following subquestion provides a way to study the convergence rate without any knowledge on the exact solution.

We recall the Landau notation for the asymptotic behavior of real-valued mappings. Let $p > 0$. For any function $f : \mathbb{R} \to \mathbb{R}$, we say that $f = O(h^p)$ as $h \to 0^+$ if there exist $\delta, M > 0$ such that $|\frac{f(h)}{h^p}| < M$ for any $h \in (0, \delta)$.

a) Let $e : (0,1) \to \mathbb{R}$ be such that $e(h) = Ch^p + O(h^q)$ for some $0 < C, p, q < +\infty$ with $p < q$. Prove that $\frac{|e(h)-e(h/2)|}{|e(h/2)-e(h/4)|} \to 2^p$ as $h \to 0$.

This result tells us that if we obtain the numerical solution $\underline{u}_h$ of a PDE with $h = \frac{1}{N+1} \in (0,1)$ as the discretization parameter and if the convergence rate is of the form $O(h^p)$, we could estimate, without knowing the exact solution u, the convergence rate p by computing $\log\left(\frac{\|\underline{u}_h - \underline{u}_{h/2}\|_h}{\|\underline{u}_{h/2} - \underline{u}_{h/4}\|_{h/2}}\right) / \log(2)$, where we denote by $\|\cdot\|_h$ the discrete ℓ_2-error at the common meshpoints $\frac{j}{N+1}$, for $j = 1, \ldots, N$:

$$\left\| \underline{u}_h - \underline{u}_{h/2} \right\|_h^2 := h \sum_{j=1}^{N} \left| u_{h,j} - u_{\frac{h}{2},2j} \right|^2 ,$$

and where $\underline{u}_h \in \mathbb{R}^N$ is the numerical solution computed by FEM at time $t = 1$ for the spatial mesh-width h.

We do not intend to solve equation (4) analytically. Instead, we will apply the method of the previous subquestion to study the convergence of the θ-method. We use the same discretization as in Question 3 of Exercise sheet 2. For a discretization parameter $h := \frac{1}{N+1} = \frac{1}{M}$, we denote the space-time approximation to the exact solution u to (4) at time $t = 1$ by $u_h(x)$. By part **a)**, we may estimate the convergence rate p of the error $e_N^M(h) := \|u(\cdot,1) - u_h(\cdot)\|_{L^2(G)}$ by computing the quantity

$$\log\left(\frac{\|\underline{u}_{h_1} - \underline{u}_{h_2}\|_{h_1}}{\|\underline{u}_{h_2} - \underline{u}_{h_3}\|_{h_2}}\right) / \log(2) \approx p\,,$$

for $h_l = \frac{1}{N_l+1}$, $N_l = 2^{7+l} - 1$ with $l = 1, 2, 3$,

b) Implement the θ-scheme for the heat equation (4) in the template `FEM_oscillation.py` with $\theta = 0.5, N = 2^{7+l} - 1$ and $M = 2^{7+l}$ with $l = 1, 2, 3$, and where $u_0 \in L^2(0,1)$ is the function

$$u_0(x) = \begin{cases} 1 & \text{if } x \in [1/4, 3/4]\,, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, implement the estimation strategy of the convergence rate proposed in **a)** and complete the function `FEM_theta`. Report the estimated convergence rate at time $t = 1$ for the discrete l^2-error. (Note that, as preparation for questions **c)** and **d)**, the function `FEM_theta` in the template takes an additional parameter `beta` as input - with a uniform temporal mesh corresponding to `beta`=1.)

c) The convergence may be improved by using a *graded time mesh.* Specifically, let $\beta \in [1, \infty)$ be a grading parameter and let the temporal mesh be given by $t_j = (\frac{j}{M})^\beta$, for $j = 0, 1, \ldots, M$. Modify your code to implement the FEM combined with a graded temporal mesh for an arbitrary grading parameter $\beta \in [1, \infty)$.

d) Repeat the experiment in **b)** for $\theta = 0.5$ and grading parameter $\beta = 15$. Compare both convergence rates and explain the difference. Why can we not expect a rate of $O(h^2 + k^2)$ for the example in **b)**.
Hint: Does u_0 belong to $H^2(0, 1)$, if so, why (not)?

Due: Wednesday, April 10th, at 2pm.

References

[1] Hilber, Norbert and Reichmann, Oleg and Schwab, Christoph and Winter, Christoph: *Computational Methods for Quantitative Finance: Finite Element Methods for Derivative Pricing"*, Springer Science & Business Media, 2013.