Last Name		Mark
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Studies		
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1	2	3		Points

- Fill in this cover sheet first.
- Put your student card on the table.
- Start a new page for each question and write your full name on each one of the sheets.
- Write your name as a comment line on top of each codefile you submit.
- Solutions written in pencil, red or green will be considered invalid. Please do not use Tipp-Ex.
- All mobile phones must be switched off and must be stored in your pocket or your bag during the exam.
- You can start Matlab in Linux by clicking the “Actitivies” icon in the top left corner of the screen and then by searching for “Matlab” (top middle of the screen) and then press “enter”.
- Work in the folder “/questions”. Here, a backup of saved files will be created automatically.

Please turn over.

Exam [maximal number of points: 70 pt]

1. General questions [in total 13 pt]

a) [3 pt] Set $G := (-1, 1)$ and consider the usual Finite Element space

$$S^1_{\mathcal{T}} = \mathrm{span}\{b_i : i = 0, \ldots, N+1\} \subset H^1(G),$$

where $\mathcal{T}$ is a partition of $\overline{G}$. State a linear mapping $P\colon L^2(G) \to S^1_{\mathcal{T}}$ such that $P(v) = v$ for every $v \in S^1_{\mathcal{T}}$ by specifying the coefficients of $P(v)$ in the hat function basis $\{b_i : i = 0, \ldots, N+1\}$ for arbitrary $v \in L^2(G)$. Prove that P is well-defined.

b) [2 pt] What does it mean for the θ-scheme to be conditionally resp. unconditionally stable? For which values of θ in the θ-scheme are these forms of stability satisfied? State the condition if applicable.

c) [3 pt] Let $\{W_t : t \geq 0\}$ denote a one-dimensional Wiener process. Consider the stochastic differential equation

$$\mathrm{d}S_t = b(t, S_t)\,\mathrm{d}t + \sigma(t, S_t)\,\mathrm{d}W_t, \qquad S_0 = s_0,$$

with deterministic initial value $s_0 > 0$. Under which assumptions on the functions $b, \sigma\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are existence and uniqueness of a solution$\{S_t : t \geq 0\}$ to this equation guaranteed? Provide one example which satisfies these conditions and another example which does not fulfill them.

d) [4 pt] Let X be a Lévy process with characteristic triplet (σ^2, ν, γ) to be used as log-price process. Let $\sigma > 0$. Suppose that $\nu(\mathrm{d}z) = k(z)\mathrm{d}z$ and that for $\alpha \in (0, 2)$, $\beta_+ > 1$, $\beta_- > 0$

$$k(z) = c\left(\frac{e^{-\beta_+|z|}}{|z|^{1+\alpha}}\mathbb{1}_{\{z>0\}} + \frac{e^{-\beta_-|z|}}{|z|^{1+\alpha}}\mathbb{1}_{\{z<0\}}\right), \quad \text{for every } z \in \mathbb{R}.$$

i) Provide a financial interpretation of the case $\beta_+ > \beta_-$.
ii) Describe from a numerical perspective the cases $c = 0$ and $c > 0$.
iii) Modify the density $k(z)$ such that only upward jumps are obtained.
iv) Why is the value $\alpha = 2$ not admissible?

e) [1 pt] Consider the Black and Scholes model in one dimension. You have computed values of the price of a European put option with strike $K = 1$ already. How can you use these values to obtain the option price for $K \neq 1$?

Please turn over.

Continued overleaf.

2. Black–Scholes with dividends [in total 33 pt]

Consider the Black-Scholes model with a continuously paid dividend $\delta \geq 0$, i.e., the stock price satisfies the following SDE

$$dS_t = (r - \delta)S_t dt + \sigma S_t dW_t, \quad S_0 = s > 0,$$

where $r > \delta \geq 0$, $\sigma > 0$.

a) [2 pt] State the PDE satisfied by the price $V(t, s)$ of a European call option with maturity $T > 0$ and strike $K > 0$ in this market model.

b) [3 pt] To get a numerically tractable problem, we change to time-to-maturity $t \to T - t$ and log-price coordinates $x = \log(s)$, restrict the PDE to a bounded domain $G := (-R, R)$, $R > 0$ sufficiently large, and impose homogeneous Dirichlet boundary conditions. Let $J := (0, T]$. Derive the corresponding PDE.

Let $H = L^2(G)$ and $V = H_0^1(G)$. The variational formulation corresponding to the localized PDE in **b)** reads

$$\begin{aligned} &\text{Find } v \in L^2(J; V) \cap H^1(J; V^*) \text{ such that for a.e. } t \in J \\ &\langle \partial_t v, u \rangle_{V^*,V} + a(v, u) = 0 \qquad \forall\, u \in V \\ &v(0, x) = \max\{e^x - K, 0\} \text{ for every } x \in G. \end{aligned} \tag{1}$$

The bilinear form $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ is given by

$$a(\phi, \varphi) = \frac{1}{2}\sigma^2(\phi', \varphi') + (\frac{\sigma^2}{2} + \delta - r)(\phi', \varphi) + r(\phi, \varphi) \quad \forall \phi, \varphi \in V.$$

c) [3 pt] We discretize in space using the mesh, $-R = x_0 < x_1 < \cdots < x_N < x_{N+1} = R$, with equidistant points $x_n := nh - R$, $n = 0, \ldots, N + 1$ and mesh width $h := \frac{2R}{N+1}$. We also discretize in time with time points $t_m := mk$, $m = 0, \ldots, M$ and time step $k := \frac{T}{M}$. The θ-scheme is used to discretize the time variable and Finite Elements is used to discretize the space variable. The used Finite Element space is spanned by the continuous hat-functions. State the matrix formulation and define precisely all parts of it. You do not have to derive it.

d) [12 pt] Modify the template `P2_main.m` to solve the problem obtained in **d)** numerically and report on the convergence rate in the $L^\infty(\widetilde{G})$-norm in the domain of interest $\widetilde{G}$ at final time T using the exact solution in `bseucalldiv.m`. Choose the parameters $K = 1$, $\sigma = 0.3$, $r = 0.07$, $\delta = 0.05$, $T = 1$, $R = 4$, $\theta = 0.5$, and $k = h$ and $\widetilde{G} = \{x \in G \mid |e^x - K| < \frac{2K}{3}\}$. You **can** use the routine `stiff.m` for the computation of the stiffness and mass matrices. You can also compute the stiffness and mass matrices directly.

In the following sub-exercises, we aim at sensitivities of the price V; here with respect to the interest rate r. Let

$$w := \partial_r v$$

be the sensitivity with respect to r; this is also referred to as the Greek Rho.

Please turn over.

e) [3 pt] Derive that w is the solution of the variational formulation: Find $w \in L^2(0,T;V) \cap H^1(0,T;V^*)$ such that $w(0) = 0$ and for a.e. $t \in (0,T)$

$$\langle \partial_t w, u \rangle_{V^* \times V} + a(w,u) = -(v,u) + (v',u) \quad \forall u \in V\,, \tag{2}$$

where v is the solution of (1).
Hint: Differentiate the pricing equation (1) with respect to r. Assume that all appearing derivatives are well-defined and the order of derivatives may be interchanged.

f) [2 pt] Use the discretization as above. Show that the variational formulation (2) can be written in matrix form

$$(\mathbf{M} + k\theta\mathbf{A})\underline{w}^{m+1} - (\mathbf{M} - k(1-\theta)\mathbf{A})\underline{w}^{m} = k\widetilde{\mathbf{A}}(\theta\underline{v}^{m+1} + (1-\theta)\underline{v}^{m}) \tag{3}$$

with $m = 0, \dots, M-1$ and $\widetilde{\mathbf{A}}_{ji} = -(b_i, b_j) + (b_i', b_j)$. Note that the matrices $\mathbf{M}$ and $\mathbf{A}$ are the usual mass and stiffness matrix, which have been introduced in the lecture.

g) [8 pt] Complete the MATLAB routine `P2_main.m` to compute w at the final time T (in time to maturity) using the parameter as above $K = 1$, $\sigma = 0.3$, $r = 0.07$, $\delta = 0.05$, $T = 1$, $R = 4$, $\theta = 0.5$, and $k = h$ and $\widetilde{G} = \left\{x \in G \mid |e^x - K| < \frac{2K}{3}\right\}$. Report on convergence rate in the $L^\infty(\widetilde{G})$-norm in the domain of interest $\widetilde{G}$ at final time T using the exact solution in `bseurho.m`.

Continued overleaf.

3. Multi-scale stochastic volatility models [in total 24 pt]

Consider the uncorrelated multi-scale model for (S_t, Y_t), where the process $\{Y_t, t \geq 0\}$ follows a mean-reverting Ornstein–Uhlenbeck process, i.e.,

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \xi(Y_t)S_t\,\mathrm{d}W_t^1 \tag{4a}$$

$$\mathrm{d}Y_t = \alpha(m - Y_t)\,\mathrm{d}t + \beta\,\mathrm{d}W_t^2, \tag{4b}$$

where $\alpha, \beta, m, r \geq 0$ are constants, $\xi\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a deterministic function, W_t^1, W_t^2 are two independent standard Brownian motions.

a) [2 pt] Show that the infinitesimal generator $\mathcal{A}^{\mathrm{MS}}$ corresponding to the (log-price) process $(X_t, Y_t) = (\ln S_t, Y_t)$ for the model in (4) is given by

$$\begin{aligned}\mathcal{A}^{\mathrm{MS}} f(x,y) &= \frac{1}{2}\xi(y)^2\partial_{xx}f(x,y) + \frac{1}{2}\beta^2\partial_{yy}f(x,y)\\ &\quad + \left(r - \frac{1}{2}\xi(y)^2\right)\partial_x f(x,y) + \alpha(m-y)\partial_y f(x,y).\end{aligned} \tag{5}$$

From now on, we consider the ***Stein–Stein model***, with

$$r := 0 \qquad \text{and} \qquad \xi(y) := |y|. \tag{6}$$

We want to find the option price

$$V(t,x,y) = \mathbb{E}\left[g(e^{X_T}) \mid X_t = x,\, Y_t = y\right],$$

for some payoff function $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$.

Let G_R be a bounded domain $G_R = (-R_1, R_1) \times (0, R_2)$ and consider the *localized* pricing equation in log-price and time to maturity, i.e., for $u(t,x,y) = V(T-t,x,y)$,

$$\partial_t u - \mathcal{A}^{\mathrm{MS}} u = 0 \quad \text{in } (0,T)\times G_R, \qquad u(0,x,y) = g(e^x) \quad \text{in } G_R, \tag{7}$$

where the operator $\mathcal{A}^{\mathrm{MS}}$ is given by (5), with r and ξ as in (6).

We impose boundary conditions on $\Gamma_{\mathrm{D}} := \{-R_1, R_1\} \times (0, R_2)$ and $\Gamma_{\mathrm{N}} = \partial G_R \backslash \Gamma_{\mathrm{D}} = (-R_1, R_1) \times \{0, R_2\}$, as follows

$$u = 0 \text{ on } (0,T)\times\Gamma_{\mathrm{D}}, \qquad \partial_y u = 0 \text{ on } (0,T)\times\Gamma_{\mathrm{N}}.$$

Assuming (6) the corresponding variational formulation reads

$$\begin{gathered}\text{Find } u \in L^2(0,T;V)\cap H^1(0,T;V^*) \text{ such that } u(0,x,y) = g(e^x) \text{ and}\\ (\partial_t u, v)_{L^2(G_R)} + a^{\mathrm{St}}(u,v) = 0 \quad \forall v \in V, \quad \text{a.e. in } (0,T),\end{gathered}$$

where $V := \overline{\{v \in C^\infty(G_R) \mid v = 0 \text{ on } \Gamma_D\}}^{\|\cdot\|_V}$, with norm

$$\|v\|_V^2 := \|y\partial_x v\|_{L^2(G_R)}^2 + \|\partial_y v\|_{L^2(G_R)}^2 + \|v\|_{L^2(G_R)}^2,$$

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and the bilinear form $a^{\mathrm{St}}\colon V \times V \to \mathbb{R}$ is given by

$$
\begin{aligned}
a^{\mathrm{St}}(v,w) := \frac{1}{2}(y^2 \partial_x v, \partial_x w)_{L^2(G_R)} + \frac{1}{2}\beta^2 (\partial_y v, \partial_y w)_{L^2(G_R)} \\
+ \frac{1}{2}(y^2 \partial_x v, w)_{L^2(G_R)} - \alpha m (\partial_y v, w)_{L^2(G_R)} + \alpha (y \partial_y v, w)_{L^2(G_R)}.
\end{aligned}
\tag{8}
$$

Discretizing in time by the θ-scheme with time step $k = \frac{T}{M}$ and in space by using the finite element space

$$
V_{N_x,N_y} := \mathrm{span}\{ b_i(x) b_j(y) \mid i = 1,\ldots,N_x;\ j = 0,\ldots,N_y+1 \},
$$

of continuous, piecewise bilinears with respect to the two-dimensional grid (x_i, y_j), $x_i = -R_1 + i h_x$, $y_j = j h_y$, with mesh width $h_x = \frac{2R_1}{N_x+1}$, $h_y = \frac{R_2}{N_y+1}$, gives the matrix formulation

$$
\begin{gathered}
\text{Given } \underline{u}_N^0 \in \mathbb{R}^N, \text{ find } \underline{u}_N^m \in \mathbb{R}^N \text{ such that, for } m = 0,\ldots,M-1, \\
(\mathbf{M} + k\theta \mathbf{A})\, \underline{u}_N^{m+1} = (\mathbf{M} - k(1-\theta)\mathbf{A})\, \underline{u}_N^m.
\end{gathered}
$$

with $\underline{u}_N^0 = \underline{g} \otimes \underline{1}$ and $g_i = g(e^{x_i})$.

b) [4 pt] Show that the stiffness matrix has the tensor product structure

$$
\mathbf{A} = \left(\frac{1}{2}\mathbf{S}^1 + \frac{1}{2}\mathbf{B}^1 \right) \otimes \mathbf{M}^{y^2} + \mathbf{M}^1 \otimes \left(\frac{1}{2}\beta^2 \mathbf{S}^2 - \alpha m\, \mathbf{B}^2 + \alpha\, \mathbf{B}^y \right),
$$

where the one-dimensional matrices above are given by

$$
\begin{aligned}
\{\mathbf{S}^1\}_{i',i} &= (b_i', b_{i'}')_{L^2(-R_1,R_1)}, & \{\mathbf{S}^2\}_{j',j} &= (b_j', b_{j'}')_{L^2(0,R_2)}, \\
\{\mathbf{B}^1\}_{i',i} &= (b_i', b_{i'})_{L^2(-R_1,R_1)}, & \{\mathbf{B}^2\}_{j',j} &= (b_j', b_{j'})_{L^2(0,R_2)}, \\
\{\mathbf{M}^1\}_{i',i} &= (b_i, b_{i'})_{L^2(-R_1,R_1)}, & \{\mathbf{M}^{y^2}\}_{j',j} &= \left(y^2 b_j, b_{j'}\right)_{L^2(0,R_2)}, \\
& & \{\mathbf{B}^y\}_{j',j} &= (y b_j', b_{j'})_{L^2(0,R_2)}.
\end{aligned}
$$

c) [13 pt] Complete the MATLAB routine `P3_main.m` to compute the price of a European call option with the Stein–Stein model assuming (6). Set $T = 1$, $K = 1$, $\alpha = 2.5$, $\beta = 0.5$ and $m = 0.06$. To implement the weighted one-dimensional matrices you may use the routine `stiff.m`.

d) [5 pt] Show that, for $\beta > 0$ and $\alpha, m \geq 0$, the bilinear form $a^{\mathrm{St}}(\,\cdot\,,\cdot\,)$ in (8) is continuous and satisfies a Gårding inequality with respect to V, i.e., there exist constants $C_1, C_2, C_3 > 0$ such that

$$
\left| a^{\mathrm{St}}(v,w) \right| \leq C_1 \|v\|_V \|w\|_V, \qquad a^{\mathrm{St}}(v,v) \geq C_2 \|v\|_V^2 - C_3 \|v\|_{L^2(G_R)}^2.
$$

Hint: The constants $C_1, C_2, C_3 > 0$ may depend on $\alpha, \beta, m, R_1, R_2$.