Solutions: Series 8

1. Theoretical study of the PSOR algorithm

a) Assume first that $\underline{x}$ satifies the LCP: Find $\underline{x} \in \mathbb{R}^n$ such that

$$\begin{aligned} &\mathbf{A}\underline{x} \geq \underline{b}\,, \\ &\underline{x} \geq \underline{c}\,, \\ &(\underline{x}-\underline{c})^\top(\mathbf{A}\underline{x}-\underline{b}) = 0\,. \end{aligned} \tag{1}$$

The second line implies that $\underline{x} \in \mathcal{K}$. Let now $\underline{y} \in \mathcal{K}$. Then

$$(\underline{y}-\underline{x})^\top(\mathbf{A}\underline{x}-\underline{b}) \geq 0$$

is equivalent to

$$(\underline{y}-\underline{c})^\top(\mathbf{A}\underline{x}-\underline{b}) \geq (\underline{x}-\underline{c})^\top(\mathbf{A}\underline{x}-\underline{b}) = 0\,,$$

where the equality holds by the third line of (1). Moreover, by the first line and the fact that $\underline{y} \in \mathcal{K}$, it follows that

$$(\underline{y}-\underline{c})^\top(\mathbf{A}\underline{x}-\underline{b}) \geq 0\,.$$

This shows that $\underline{x}$ is a solution to the problem: Find $\underline{x} \in \mathbb{R}^n$ such that

$$\begin{aligned} &(\underline{y}-\underline{x})^\top(\mathbf{A}\underline{x}-\underline{b}) \geq 0\,, \quad \forall y \in \mathcal{K}\,, \\ &\underline{x} \in \mathcal{K}\,, \end{aligned} \tag{2}$$

where

$$\mathcal{K} = \{\underline{y} \in \mathbb{R}^n : \underline{y} \geq \underline{c}\}\,.$$

On the other hand, assume now $\underline{x}$ is a solution to (2). As $\underline{x} \in \mathcal{K}$, we have that $\underline{x} \geq \underline{c}$. By choosing $\underline{y} = \underline{c}$, it holds that

$$(\underline{x}-\underline{c})^\top(\mathbf{A}\underline{x}-\underline{b}) \leq 0.$$

Moreover, it must hold that $(\mathbf{A}\underline{x}-\underline{b}) \geq \underline{0}$. For if not, there would exist an index i such that $(\mathbf{A}\underline{x}-\underline{b}))_i < 0$ and by choosing a $\underline{y} \in \mathcal{K}$ with a large enough entry y_i, we would arrive at $(\underline{y}-\underline{x})^\top(\mathbf{A}\underline{x}-\underline{b}) < 0$, contradicting the first line of (2). Finally, as both $(\underline{x}-\underline{c}) \geq \underline{0}$ and $(\mathbf{A}\underline{x}-\underline{b}) \geq \underline{0}$, we obtain

$$0 \leq (\underline{x}-\underline{c})^\top(\mathbf{A}\underline{x}-\underline{b}) \leq 0\,,$$

i.e. $(\underline{x}-\underline{c})^\top(\mathbf{A}\underline{x}-\underline{b}) = 0$. This shows that $\underline{x}$ is a solution to the LCP (1).

b) By assumption, the diagonal entries of $\mathbf{A}$ are positive. Moreover, if $\underline{x}$ solves the LCP, it satisfies

$$\begin{aligned} &\sum_j \mathbf{A}_{ij}x_j - b_j \geq 0\,, \\ &x_i \geq c_i\,, \\ &\left(\sum_j \mathbf{A}_{ij}x_j - b_i\right)(x_i - c_i) = 0\,. \end{aligned}$$

Denote $\underline{z} = \psi(\underline{x})$. We show by induction on the indices that $\underline{z} = \underline{x}$:
If $x_1 > c_1$, then it holds

$$\sum_j \mathbf{A}_{1j}x_j - b_1 = 0\,.$$

Then since $\frac{1}{\omega}\mathbf{A}_{11} > 0$,

$$\frac{1}{\omega} A_{11} y_1 = b_1 + \left(\frac{1}{\omega} - 1\right) \mathbf{A}_{11} x_1 - \sum_{j>1} \mathbf{A}_{1j} x_j = \frac{1}{\omega} \mathbf{A}_{11} x_1 \,,$$

we have $Z_1 = Y_1 = x_1 > c_1$. On the other hand, if $x_1 = c_1$,

$$\sum_j \mathbf{A}_{1j} x_j - b_1 \geq 0 \,.$$

$$\frac{1}{\omega} \mathbf{A}_{11} y_1 = c_1 + \left(\frac{1}{\omega} - 1\right) \mathbf{A}_{11} x_1 - \sum_{j>1} \mathbf{A}_{1j} x_j \leq \frac{1}{\omega} \mathbf{A}_{11} x_1 \,,$$

which implies that $z_1 = x_1$.

Assume now that $z_j = x_j$ for $j < i$. It holds that

$$\begin{aligned} \frac{1}{\omega} \mathbf{A}_{ii} y_i &= b_i + \left(\frac{1}{\omega} - 1\right) \mathbf{A}_{ii} x_i - \sum_{j>i} \mathbf{A}_{ij} x_j - \sum_{j<i} \mathbf{A}_{ij} z_j \\ &= b_i + \frac{1}{\omega} \mathbf{A}_{ii} x_i - \sum_j \mathbf{A}_{ij} x_j \,. \end{aligned}$$

Then, if $x_i > c_i$,

$$\sum_j \mathbf{A}_{ij} x_j - b_i = 0 \,.$$

and thus

$$\frac{1}{\omega} \mathbf{A}_{ii} y_i = \frac{1}{\omega} \mathbf{A}_{ii} x_i \,.$$

It follows that $z_i = x_i$. On the other hand, if $x_i = c_i$,

$$\sum_j \mathbf{A}_{ij} x_j - b_i \geq 0 \,,$$

and therefore

$$\frac{1}{\omega} \mathbf{A}_{ii} y_i \leq \frac{1}{\omega} \mathbf{A}_{ii} x_i \,,$$

which shows $z_i = x_i$. We conclude that it holds $\underline{z} = \underline{x}$, and hence $\underline{x}$ is the unique fixed-point of ψ.

2. Lookback option

a) First note that in the Black-Scholes model, the SDE for the stock price can be solved explicitly, giving

$$S_t = s \exp\left((r - \sigma^2/2)t + \sigma W_t\right) = s \exp\left(\sigma(\alpha t + W_t)\right),$$

where we have set $\alpha = \frac{r}{\sigma} - \frac{\sigma}{2}$. We can thus write

$$\begin{aligned} \mathbb{E}[\mathbf{1}_{\{\max_{t\in[0,T]} S_t \geq K\}}] &= P(\max_{t\in[0,T]} S_t \geq K) = P\left(s \exp[\sigma \max_{t\in[0,T]} (\alpha t + W_t)] \geq K\right) \\ &= P\left(Y \geq -\frac{1}{\sigma} \log(\frac{s}{K})\right) \\ &= 1 - \left(\Phi\left(\frac{-\frac{1}{\sigma}\log(\frac{s}{K}) - \alpha T}{\sqrt{T}}\right) - \exp(-2\alpha \cdot \frac{1}{\sigma} \log(\frac{s}{K}))\Phi\left(\frac{\frac{1}{\sigma}\log(\frac{s}{K}) - \alpha T}{\sqrt{T}}\right)\right) \\ &= 1 - \Phi(-(d - \sigma\sqrt{T})) + (\frac{s}{K})^{-\frac{2r}{\sigma^2}+1} \Phi(d - \frac{2r}{\sigma}\sqrt{T}). \end{aligned}$$

b) Set $\phi(z) = \Phi'(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$, which is the density of a standard normally distributed random variable. Differentiating the distribution function of Y and simplifying $e^{2\alpha y}\phi(\frac{-y-\alpha T}{\sqrt{T}}) = \phi(\frac{y-\alpha T}{\sqrt{T}})$ gives us the density

$$f_Y(y) = \frac{d}{dy} P(Y \leq y) = \frac{2}{\sqrt{T}} \phi\left(\frac{y - \alpha T}{\sqrt{T}}\right) - e^{2\alpha y} 2\alpha \Phi\left(\frac{-y - \alpha T}{\sqrt{T}}\right).$$

Thus we calculate[1] as in a)

$$\begin{aligned}
\mathbb{E}[(\max_{t\in[0,T]} S_t)\mathbf{1}_{\{\max_{t\in[0,T]} S_t \geq K\}}] = \mathbb{E}[se^{\sigma Y}\mathbf{1}_{\{Y \geq -\frac{1}{\sigma}\log(\frac{s}{K})\}}] &= \int_{\frac{1}{\sigma}\log(\frac{K}{s})}^{\infty} se^{\sigma y} f_Y(y)\, dy \\
&= \int_{\frac{1}{\sigma}\log(\frac{K}{s})}^{\infty} se^{\sigma y} \frac{2}{\sqrt{T}} \phi\left(\frac{y - \alpha T}{\sqrt{T}}\right) dy - \int_{\frac{1}{\sigma}\log(\frac{K}{s})}^{\infty} se^{\sigma y} e^{2\alpha y} 2\alpha \Phi\left(\frac{-y - \alpha T}{\sqrt{T}}\right) dy.
\end{aligned}$$

The first integral here could be calculated by recalling $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$,

$$\begin{aligned}
\int_{\frac{1}{\sigma}\log(\frac{K}{s})}^{\infty} se^{\sigma y} \frac{2}{\sqrt{T}} \phi\left(\frac{y - \alpha T}{\sqrt{T}}\right) dy &= \int_{\frac{1}{\sigma}\log(\frac{K}{s})}^{\infty} se^{\alpha\sigma T + \frac{\sigma^2}{2}T} \frac{2}{\sqrt{T}} \phi\left(\frac{y - (\alpha + \sigma)T}{\sqrt{T}}\right) dy \\
&= 2se^{\alpha\sigma T + \frac{\sigma^2}{2}T} \Phi(d).
\end{aligned}$$

For the second integral, we integrate by parts and use that $\lim_{x\to+\infty} e^{cx}\Phi(-dx) = 0$ for any $c > 0$ and

$$d := \frac{\log\left(\frac{s}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

to obtain

$$\begin{aligned}
&-2\alpha \int_{\frac{1}{\sigma}\log(\frac{K}{s})}^{\infty} se^{(2\alpha+\sigma)y} \Phi\left(\frac{-y - \alpha T}{\sqrt{T}}\right) dy \\
&= \frac{2\alpha}{2\alpha + \sigma} s \left(\frac{K}{s}\right)^{\frac{(2\alpha+\sigma)}{\sigma}} \Phi\left(d - \frac{2r}{\sigma}\sqrt{T}\right) + \frac{-2\alpha}{2\alpha + \sigma} \int_{\frac{1}{\sigma}\log(\frac{K}{s})}^{\infty} se^{(2\alpha+\sigma)y} \frac{1}{\sqrt{T}} \phi\left(\frac{-y - \alpha T}{\sqrt{T}}\right) dy \\
&= \frac{\alpha\sigma s}{r} \left(\frac{K}{s}\right)^{\frac{2r}{\sigma^2}} \Phi\left(d - \frac{2r}{\sigma}\sqrt{T}\right) + \frac{-\alpha\sigma s}{r} e^{T\sigma(\alpha + \frac{\sigma}{2})} \Phi(d)
\end{aligned}$$

where for the last step, we have completed the square as before. Putting together the two terms, we obtain with $\alpha = \frac{r}{\sigma} - \frac{\sigma}{2}$ that

$$\mathbb{E}[(\max_{t\in[0,T]} S_t)\mathbf{1}_{\{\max_{t\in[0,T]} S_t \geq K\}}] = \left(1 - \frac{\sigma^2}{2r}\right) s \left(\frac{K}{s}\right)^{\frac{2r}{\sigma^2}} \Phi\left(d - \frac{2r}{\sigma}\sqrt{T}\right) + \left(1 + \frac{\sigma^2}{2r}\right) e^{rT} s\Phi(d).$$

c) We combine the solutions from a) and b) and use $1 - \Phi(-z) = \Phi(z)$ to obtain

$$\begin{aligned}
\mathbb{E}[H] &= \mathbb{E}[(\max_{t\in[0,T]} S_t)\mathbf{1}_{\{\max_{t\in[0,T]} S_t \geq K\}}] - K\mathbb{E}[\mathbf{1}_{\{\max_{t\in[0,T]} S_t \geq K\}}] \\
&= \left(-\frac{\sigma^2}{2r}\right) \left(\frac{s}{K}\right)^{-\frac{2r}{\sigma^2}} s\Phi\left(d - \frac{2r}{\sigma}\sqrt{T}\right) + \left(1 + \frac{\sigma^2}{2r}\right) e^{rT} s\Phi(d) - K\Phi\left(d - \sigma\sqrt{T}\right).
\end{aligned}$$

which is what we wanted to show.

[1] Note that $K \geq s$, thus the lower integration limit is non-negative and we don't have to treat the part $y < 0$, where $f_Y(y) = 0$, separately.

3. Dimension reduction for the Asian option pricing equation

a) We calculate the derivatives of $V(t,s,y) = sH(t,\xi(t,s,y))$ as follows, the arguments of V and ξ and their derivatives being (t,s,y), and those of H and its derivatives being $(t,\xi(t,s,y))$. We have that

$$\partial_t \xi = \partial_t q = -\frac{1}{T}, \quad \partial_s \xi = -\frac{1}{s^2}\left(\frac{y}{T} - K\right), \quad \partial_{ss}\xi = \frac{2}{s^3}\left(\frac{y}{T} - K\right), \quad \partial_y \xi = \frac{1}{sT}, \tag{3}$$

and

$$\begin{aligned}
\partial_t V &= s\left(\partial_t H + \partial_z H \partial_t \xi\right) &&= s\left(\partial_t H - \frac{1}{T}\partial_z H\right), \\
s\partial_s V &= s\left(H + s\partial_z H \partial_s \xi\right) &&= s\left(H + \partial_z H\left(-z + q(t)\right)\right), \\
s^2\partial_{ss} V &= s^2\left(2\partial_z H \partial_s \xi + s\partial_z H \partial_{ss}\xi + s\partial_{zz} H (\partial_s \xi)^2\right) &&= s\partial_{zz} H\left(z - q(t)\right)^2, \\
s\partial_y V &= s^2\left(\partial_z H \partial_y \xi\right) &&= \frac{s}{T}\partial_z H.
\end{aligned} \tag{4}$$

Substituting the derivatives in the oringinal PDE, we obtain the PDE

$$\begin{aligned}
0 &= s\left(\partial_t H - \frac{1}{T}\partial_z H\right) + \frac{\sigma^2}{2}s\partial_{zz}H\left(z - q(t)\right)^2 + rs\left(H + \partial_z H\left(-z + q(t)\right)\right) + \frac{s}{T}\partial_z H - rsH \\
&= \partial_t H + \frac{\sigma^2}{2}\partial_{zz}H\left(q(t) - z\right)^2 + r\partial_z H\left(-z + q(t)\right) - \frac{1}{T}\partial_z H + \frac{1}{T}\partial_z H + rH - rH \\
&= \partial_t H + \frac{\sigma^2}{2}\partial_{zz}H\left(q(t) - z\right)^2 + r\partial_z H\left(q(t) - z\right),
\end{aligned}$$

with the terminal value

$$H(T,\xi(T,s,y)) = \frac{1}{s}g(s,y) = \max\{0,z\}.$$

b) The solution of **b)** is similar. The difference is that

$$\partial_t \xi = \partial_t q = \frac{1}{T}, \quad \partial_s \xi = \frac{1}{s^2}\frac{y}{T}, \quad \partial_{ss}\xi = -\frac{2}{s^3}\frac{y}{T}, \quad \partial_y \xi = -\frac{1}{sT}.$$

Also, here

$$\begin{aligned}
\partial_t V &= s\left(\partial_t H + \partial_z H \partial_t \xi\right) &&= s\left(\partial_t H + \frac{1}{T}\partial_z H\right), \\
s\partial_s V &= s\left(H + s\partial_z H \partial_s \xi\right) &&= s\left(H + \partial_z H\left(-z + q(t)\right)\right), \\
s^2\partial_{ss} V &= s^2\left(2\partial_z H \partial_s \xi + s\partial_z H \partial_{ss}\xi + s\partial_{zz} H (\partial_s \xi)^2\right) &&= s\partial_{zz} H\left(z - q(t)\right)^2, \\
s\partial_y V &= s^2\left(\partial_z H \partial_y \xi\right) &&= -\frac{s}{T}\partial_z H.
\end{aligned} \tag{5}$$

We also insert this relation into the original PDE and obtain that

$$\begin{aligned}
0 &= s\left(\partial_t H + \frac{1}{T}\partial_z H\right) + \frac{\sigma^2}{2}s\partial_{zz}H\left(z - q(t)\right)^2 + rs\left(H + \partial_z H\left(-z + q(t)\right)\right) - \frac{s}{T}\partial_z H - rsH, \\
&= \partial_t H + \frac{\sigma^2}{2}\partial_{zz}H\left(q(t) - z\right)^2 - r\partial_z H\left(z - q(t)\right) - \frac{1}{T}\partial_z H + \frac{1}{T}\partial_z H + rH - rH \\
&= \partial_t H + \frac{\sigma^2}{2}\partial_{zz}H\left(q(t) - z\right)^2 + r\partial_z H\left(q(t) - z\right),
\end{aligned}$$

with the terminal value

$$H(T,\xi(T,s,y)) = \frac{1}{s}g(s,y) = \max\{0,z\}.$$