# Solutions: Series 6

## 1. Log-moneyness

**a)** From Theorem 4.1.4 in the textbook, we know that $V(t,s)$ is given by

$$V(t,s) = \mathbb{E}[e^{-r(T-t)} g(S_T) | S_t = s]\,.$$

By definition of $g$

$$V(t,s) = \mathbb{E}[e^{-r(T-t)} (K - S_T)_+ | S_t = s]\,,$$

and therefore

$$V(t,s) = K\mathbb{E}\left[e^{-r(T-t)} \left(1 - \frac{S_T}{K}\right)_+ | S_t = s\right]\,.$$

Now introduce the variables

$$Y_T = \log\left(\frac{S_T}{K}\right), \qquad y = \log\left(\frac{s}{K}\right), \qquad \text{and} \qquad \tau = T - t\,.$$

Then,

$$V(t,s) = K\mathbb{E}[e^{-r(T-t)}(1 - e^{Y_T})_+ | Y_t = y] = K\mathbb{E}[e^{-r(T-t)} w_0(Y_T) | Y_t = y] =: Kw(\tau, y)\,,$$

where $w_0(y) = (1 - e^y)_+$. Since $s = Ke^y$ and $t = T - \tau$, we compute this change of coordinates with the usual chain rule from calculus and obtain that

$$\begin{aligned}
K\partial_\tau w(\tau,y) &= -\partial_t V(t,s)\,,\\
K\partial_y w(\tau,y) &= Ke^y \partial_s V(t,s) = s\partial_s V(t,s)\,,\\
K\partial_{yy} w(\tau,y) &= (Ke^y)^2 \partial_{ss} V(t,s) + Ke^y \partial_s V(t,s) = s^2 \partial_{ss} V(t,s) + s\partial_s V(t,s)\,.
\end{aligned}$$

Inserting these expressions for $\partial_t V, s\partial_s V$ and $s^2\partial_{ss} V$ into the Black–Scholes equation we obtain

$$\begin{cases} \partial_\tau w(\tau,y) - \mathcal{A}_y^{BS} w(\tau,y) + rw(\tau,y) = 0 & \text{in } (0,T)\times\mathbb{R}\,,\\ w(0,y) = w_0(y) & \text{on } \{0\}\times\mathbb{R}\,, \end{cases}$$

with $w_0(y) = (1 - e^y)_+$.

## 2. Fractional Sobolev spaces

**a)** To show that $H^s(G) \supset H^t(G)$ for $0 < s < t < 1$, note that for any $v \in H^t(G)$ and $0 < s < t < 1$ it holds

$$\frac{|v(x) - v(y)|}{|x-y|^{s+\frac{1}{2}}} \le (b-a)^{t-s} \frac{|v(x) - v(y)|}{|x-y|^{t+\frac{1}{2}}}\,, \quad (x,y) \in G\times G\,.$$

This implies that

$$\int_G\int_G \frac{|v(x)-v(y)|^2}{|x-y|^{2s+1}}\,dx\,dy \le (b-a)^{2(t-s)} \int_G\int_G \frac{|v(x)-v(y)|^2}{|x-y|^{2t+1}}\,dx\,dy < \infty\,.$$

**b)** We set $G = (0,1)$ and write $v(x) = \mathbf{1}_{(0.5,1)}(x)$. We calculate

$$\begin{aligned}\int_{G\times G} \frac{|v(x)-v(y)|^2}{|x-y|^{2-2\epsilon}}\,dy\,dx &= \int_{\big((0,0.5)\times(0.5,1)\big)\cup\big((0.5,1)\times(0,0.5)\big)} \frac{1}{|y-x|^{2-2\epsilon}}\,dy\,dx \\ &= 2\int_{(0,0.5)\times(0.5,1)} \frac{1}{|y-x|^{2-2\epsilon}}\,dy\,dx \\ &= 2\int_{(-0.5,0)\times(0,0.5)} \frac{1}{|y-x|^{2-2\epsilon}}\,dy\,dx\,.\end{aligned}$$

Using the standard polar coordinate system with $r^2 = x^2 + y^2$ and $\theta = \arctan(\frac{y}{x})$, for $x > 0$, and the fact that $(0,0.5)\times(-0.5,0) \subset \{r \in (0,\frac{\sqrt{2}}{2}), \theta \in (-\frac{\pi}{2},0)\}$, we have that

$$\begin{aligned}\int_{G\times G} \frac{|v(y)-v(x)|^2}{|y-x|^{2-2\epsilon}}\,dy\,dx &= 2\int_{(-0.5,0)\times(0,0.5)} \frac{1}{|y-x|^{2-2\epsilon}}\,dy\,dx \\ &\le 2\int_{r\in(0,\sqrt{2}/2),\theta\in(-\frac{\pi}{2},0)} \frac{1}{r^{2-2\epsilon}}\cdot r\,dr\,d\theta \\ &= 2\int_{-\frac{\pi}{2}}^{0} d\theta \int_0^{\sqrt{2}/2} \frac{1}{r^{1-2\epsilon}}\,dr \\ &= \pi\int_0^{\sqrt{2}/2} \frac{1}{r^{1-2\epsilon}}\,dr\,,\end{aligned}$$

where in the second step we used that $|y-x|^2 \ge x^2 + y^2$, for $x \ge 0, y \le 0$. The last integral converges for $\epsilon \in (0,0.5)$, so $v \in H^{\frac{1}{2}-\epsilon}(G)$ for $\epsilon \in (0,0.5)$. On the other hand, to show that $v \notin H^{1/2}(G)$, note that $(0,0.5)\times(0.5,1) \supset \{r \in (0,0.5), \theta \in (-\frac{\pi}{2},0)\}$ and we have for $\epsilon = 0$ that

$$\begin{aligned}\int_{G\times G} \frac{|v(y)-v(x)|^2}{|y-x|^2}\,dy\,dx &= 2\int_{(-0.5,0)\times(0,0.5)} \frac{1}{|y-x|^2}\,dy\,dx \\ &\ge 2\int_{r\in(0,0.5),\theta\in(-\frac{\pi}{2},0)} \frac{1}{r^2\,|\cos(\theta)-\sin(\theta)|^2}\cdot r\,dr\,d\theta \\ &\ge 8\int_{-\frac{\pi}{2}}^{0} d\theta \int_0^{0.5} \frac{1}{r}\,dr \\ &= \pi\int_0^{0.5} \frac{1}{r}\,dr = +\infty\,.\end{aligned}$$

So $v \notin H^{\frac{1}{2}}(G)$.

**c)** The payoff function of a call option is given by $u_0(x) = (\exp(x) - K)_+$. It is clear that $u_0(x) \in H^1(-R,R)$ and that $u_0'(x) = \mathbf{1}_{\{x>\log(K)\}}\cdot\exp(x)$. It hence suffices to show that $u_0'(x) \in H^{\frac{1}{2}-\epsilon}(-R,R)$ for any $\epsilon \in (0,\frac{1}{2})$ but $u_0'(x) \notin H^{\frac{1}{2}}(-R,R)$. Note that $u_0'(x) = (K\cdot\mathbf{1}_{x\le\log(K)} + \mathbf{1}_{x>\log(K)}\cdot\exp(x)) - K\cdot\mathbf{1}_{x\le\log(K)} =: U_1(x) + U_2(x)$. Here $U_1(x) \in H^1(-R,R)$ is a piesewise differentiable function while $U_2(x) \in H^{\frac{1}{2}-\epsilon}(-R,R)$ for $\epsilon \in (0,\frac{1}{2})$ but $U_2(x) \notin H^{\frac{1}{2}}(-R,R)$, as was shown before. Therefore $u_0'(x) = U_1(x) + U_2(x) \in H^{\frac{1}{2}-\epsilon}(-R,R)$ for $\epsilon \in (0,\frac{1}{2})$ but not for $\epsilon = \frac{1}{2}$.

## 3. Theoretical study of the CEV model

**a)** Let us fix $u$ and $v$ in $C_0^\infty(G)$. We can then extend the reasoning to $W_{\rho,\mu}$ by density (since by definition, $C_0^\infty(0,R)$ is dense in $W_{\rho,\mu}$ with continuous inclusion). Note that for $\mu = 0$, we have that $\mathcal{H}_\mu = L^2(G)$. We estimate now each term in the bilinear form

$$a^{\mathrm{CEV}}_{\rho,0}(u,v) = \frac{\sigma^2}{2}\left(s^{2\rho}u', v'\right) + \sigma^2\rho\left(s^{2\rho-1}u', v\right) - r\,(su', v) + r\,(u,v)\ .$$

*First term.* By the Cauchy-Schwarz inequality, one has

$$\mathrm{I} := \left|(s^{2\rho}u', v')\right| \leq \|s^\rho u'\|_{L^2(G)}\ \|s^\rho v'\|_{L^2(G)} \leq \|u\|_{\rho,0}\,\|v\|_{\rho,0}\ .$$

*Second term.* Again by the Cauchy-Schwarz inequality, we have

$$\mathrm{II} := \left|(s^{2\rho-1}u', v)\right| \leq \|s^\rho u'\|_{L^2(G)}\ \left\|s^{\rho-1}v\right\|_{L^2(G)}\ .$$

The assumption $\rho \neq \frac{1}{2}$ ensures that $2\rho$, so we can apply the weighted Hardy's inequality in the hint with $\epsilon = 2\rho$ to obtain

$$\left\|s^{\rho-1}v\right\|_{L^2(G)} \leq C_1\,\|s^\rho v'\|_{L^2(G)} \leq C_1\,\|v\|_{\rho,0}\ ,$$

where $C_1 := \frac{2}{|\epsilon-1|}$. Hence,

$$\left|(s^{\rho-1}u', v)\right| \leq C_1\,\|s^\rho u'\|_{L^2(G)}\ \|s^\rho v'\|_{L^2(G)} \leq C_1\,\|u\|_{\rho,0}\ \|v\|_{\rho,0}\ .$$

*Third term.* We re-write

$$(su', v) = (s^\rho u', s^{1-\rho}v)\,.$$

By the Cauchy-Schwarz inequality, and using the fact that $s^{1-\rho} \leq R^{1-\rho}$ on $G$, we find

$$\mathrm{III} := |(su', v)| \leq R^{1-\rho}\,\|u\|_{\rho,0}\,\|v\|_{L^2(G)} \leq C_2\,\|u\|_{\rho,0}\,\|v\|_{\rho,0}\ ,$$

for $C_2 := R^{1-\rho}$.
*Fourth term.* By Cauchy-Schwarz inequality,

$$\mathrm{IV} := |(u,v)| \leq \|u\|_{L^2(G)}\,\|v\|_{L^2(G)} \leq \|u\|_{\rho,0}\,\|v\|_{\rho,0}\ .$$

Finally, by combining the four inequalities, we obtain that for $u, v \in C_0^\infty(G)$,

$$\left|a^{\mathrm{CEV}}_{\rho,0}(u,v)\right| \leq \frac{\sigma^2}{2}\mathrm{I} + \rho\sigma^2\mathrm{II} + r\mathrm{III} + r\mathrm{IV} \leq C\,\|u\|_{\rho,0}\ \|v\|_{\rho,0}\,,$$

for $C(\rho,\sigma,r) := \frac{\sigma^2}{2} + C_1\rho\sigma^2 + (C_2+1)r$. We conclude that $a^{\mathrm{CEV}}_{\rho,0}(\cdot,\cdot)$ is continuous on $W_{\rho,0}\times W_{\rho,0}$.

**b)** Fix $u$ in $C_0^\infty(G)$. Using integration by parts, we obtain that for $0 \leq \rho \leq \frac{1}{2}$,

$$(s^{2\rho-1}u', u) = \frac{1}{2}\int_0^R s^{2\rho-1}(u^2)'\,ds = -\frac{1}{2}(2\rho-1)\int_0^R s^{2\rho-2}u^2\,ds \geq 0\,.$$

where the right-hand side is well-defined and finite by Hardy's inequality (if $\rho \neq \frac{1}{2}$). Similarly,

$$-(su', u) = -\frac{1}{2}\int_0^R s(u^2)'\,ds = \frac{1}{2}\int_0^R u^2\,ds \geq 0\,.$$

It follows that

$$a^{\mathrm{CEV}}_{\rho,0}(u,u) \geq \frac{1}{2}\,\|s^\rho u'\|^2_{L^2(G)} + \frac{3}{2}r\,\|u\|^2_{L^2(G)} \geq \frac{1}{2}\min\{\sigma^2, 3r\}\,\|u\|^2_{\rho,0}\,.$$

This establishes the strong coercivity of $a^{\mathrm{CEV}}_{\rho,0}(\cdot,\cdot)$.

**c)** To show the continuity of $a^{\mathrm{CEV}}_{1/2,\mu}$ in $W_{1/2,\mu} \times W_{1/2,\mu}$, we proceed as in **a)**. Let $u, v \in C_0^\infty(G)$. Then

$$a^{\mathrm{CEV}}_{1/2,\mu}(u,v) = \frac{\sigma^2}{2}\left(su',v'\right)_\mu + \sigma^2\left(\frac{1}{2}+\mu\right)\left(u',v\right)_\mu - r\left(su',v\right)_\mu + r\left(u,v\right)_\mu \,.$$

From the Cauchy-Schwarz inequality, we obtain that

$$\left|\left(su',v'\right)_\mu\right| \le \left(\int_0^R s^{2\mu+1}(u')^2\,ds\right)^{\frac{1}{2}}\left(\int_0^R s^{2\mu+1}(v')^2\,ds\right)^{\frac{1}{2}} \le \|u\|_{1/2,\mu}\,\|v\|_{1/2,\mu}\,,$$

and

$$\left|\left(u,v\right)_\mu\right| \le \|u\|_{\mathcal{H}_\mu}\,\|v\|_{\mathcal{H}_\mu} \le \|u\|_{1/2,\mu}\,\|v\|_{1/2,\mu}\,.$$

Similarly, by the Cauchy-Schwarz inequality and using the fact that $s \le R$ on $G$,

$$\left|\left(su',v\right)_\mu\right| \le \left(\int_0^R s^{2\mu+1}(u')^2\,ds\right)^{\frac{1}{2}}\left(R\int_0^R s^{2\mu}v^2\,ds\right)^{\frac{1}{2}} \le R^{\frac{1}{2}}\,\|u\|_{1/2,\mu}\;\|v\|_{1/2,\mu}\,.$$

Moreover, again by using the Cauchy-Schwarz inequality

$$\begin{aligned}|(u',v)_\mu| = \left|\int_0^R s^{2\mu}u'v\,ds\right| &= \left|\int_0^R s^{\mu+1/2}u's^{\mu-1/2}v\,ds\right| \\ &\le \left(\int_0^R s^{2\mu+1}(u')^2\,ds\right)^{\frac{1}{2}}\left(\int_0^R s^{2\mu-1}v^2\,ds\right)^{\frac{1}{2}} \\ &\le \|u\|_{1/2,\mu}\left(\int_0^R s^{2\mu-1}v^2\,ds\right)^{\frac{1}{2}}\,.\end{aligned}$$

By our choice of $\mu \in (-\frac{1}{2},0)$, it holds that $2\mu+1 \neq 1$. Hence we may use the weighted Hardy inequality for $\epsilon = 2\mu+1$, to obtain that

$$\left(\int_0^R s^{2\mu-1}v^2\,ds\right)^{\frac{1}{2}} \le \frac{1}{|\mu|}\left(\int_0^R s^{2\mu+1}(v')^2\,ds\right)^{\frac{1}{2}} \le \frac{1}{|\mu|}\,\|v\|_{1/2,\mu}\,,$$

and thus

$$|(u',v)_\mu| \le \frac{1}{|\mu|}\,\|u\|_{1/2,\mu}\,\|v\|_{1/2,\mu}\,.$$

Finally, since $\frac{1}{2}+\mu > 0$, by combining the above estimates, we obtain

$$\begin{aligned}\left|a^{\mathrm{CEV}}_{1/2,\mu}(u,v)\right| &\le \frac{\sigma^2}{2}\left|\left(su',v'\right)_\mu\right| + \sigma^2\left(\frac{1}{2}+\mu\right)|(u',v)_\mu| + r\left|\left(su',v\right)_\mu\right| + r\left|\left(u,v\right)_\mu\right| \\ &\le \left[\frac{\sigma^2}{2} + \frac{\sigma^2}{|\mu|}\left(\frac{1}{2}+\mu\right) + rR^{\frac{1}{2}} + r\right]\|u\|_{1/2,\mu}\,\|v\|_{1/2,\mu}\,.\end{aligned}$$

We now show the strong coercivity of $a^{\mathrm{CEV}}_{1/2,\mu}(\cdot,\cdot)$. Using integration by parts, it holds that

$$(u',u)_\mu = \int_G s^{2\mu}u'u = \frac{1}{2}\int_G s^{2\mu}(u^2)' = -\mu\int_G s^{2\mu-1}u^2 > 0\,,$$

and

$$(su',u)_\mu = \int_G s^{2\mu+1}u'u = \frac{1}{2}\int_G s^{2\mu+1}(u^2)'\,ds = -\frac{1}{2}(1+2\mu)\int_G s^{2\mu}u^2\,ds < 0\,.$$

With this, we finally obtain

$$
\begin{aligned}
a^{\mathrm{CEV}}_{1/2,\mu}(u,u) &= \frac{\sigma^2}{2}\left\| s^{\frac{1}{2}+\mu} u' \right\|^2_{L^2(G)} + \sigma^2 \left(\frac{1}{2}+\mu\right)(u',v)_\mu - r\,(su',u)_\mu + r\,\|s^\mu u\|^2_{L^2(G)} \\
&= \frac{\sigma^2}{2}\left\| s^{\frac{1}{2}+\mu} u' \right\|^2_{L^2(G)} - \mu\sigma^2 \left(\frac{1}{2}+\mu\right)\int_G s^{2\mu-1}u^2\,ds + r\frac{1}{2}(1+2\mu)\int_G s^{2\mu}u^2\,ds + r\,\|s^\mu u\|^2_{L^2(G)} \\
&\geq \frac{1}{2}\min\{\sigma^2, 2r\}\,\|u\|^2_{1/2,\mu}\ ,
\end{aligned}
$$

where we used that with our choice of $\mu$ it holds that $\mu < 0$, $\frac{1}{2}+\mu > 0$ and $1+2\mu > 0$. This shows the strong coercivity of $a^{\mathrm{CEV}}_{1/2,\mu}(\cdot,\cdot)$.