Series 4

The following *Lax-Milgram lemma*[1] holds:

Let H be a Hilbert space with its norm $\|\cdot\|_H$ and let $V \subset H$ be a closed subspace. Furthermore, let $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ be a bilinear form such that the following conditions hold true:

1. (Continuity) There exists $C_1 > 0$ such that $|a(u,v)| \leq C_1 \|u\|_H \|v\|_H$ for any $u, v \in H$.
2. (Coercivity) There exists $C_2 > 0$ such that $a(u,u) \geq C_2 \|u\|_H^2$ for any $u \in H$.

Then for any bounded linear functional $l \in H^*$ there exists a unique solution $u \in V$ to the following variational problem: Find $u \in V$ such that

$$a(u,v) = l(v), \quad \forall v \in V.$$

1. Spaces $H_0^1(0,1)$ and $H^{-1}(0,1)$

The space $H^{-1}(0,1)$ is defined as the dual space of $H_0^1(0,1)$ (that is, $H^{-1}(0,1)$ contains all bounded linear functionals on $H_0^1(0,1)$).

a) For any function $f \in H_0^1(0,1)$, define $\delta_{0.5}(f) = \tilde{f}(0.5)$, where $\tilde{f}$ denotes the unique continuous modification of f. Show that $\delta_{0.5} \in H^{-1}(0,1)$.

b) Given a $F \in H^{-1}(0,1)$, consider the following PDE: Find $u \in H_0^1(0,1)$ such that

$$-\Delta u = F \quad \text{on } (0,1)\,.$$

Its variational problem is of the following form: Find $u \in H_0^1(0,1)$ such that for any $v \in H_0^1(0,1)$,

$$\int_{(0,1)} u'v'\,dx = F(v)\,. \tag{1}$$

Use the Lax-Milgram lemma and the Poincaré inequality (slides 2) to show that the variational problem (1) has a unique solution $u \in H_0^1(0,1)$.

Let $F \in H^{-1}(0,1)$ be arbitrary. With u as in Equation (1), we may write $u = (-\Delta)^{-1}F$, and hence obtain a mapping $(-\Delta)^{-1} : H^{-1}(0,1) \to H_0^1(0,1)$.

c) Verify that

$$(-\Delta)^{-1}\delta_{0.5} = \begin{cases} x/2, & x \in (0,0.5)\,, \\ 1/2 - x/2, & x \in [0.5,1)\,. \end{cases}$$

d) Show that we could establish a one-to-one correspondence between $H^{-1}(0,1)$ and $H_0^1(0,1)$ with $(-\Delta)^{-1}$ by verifying that it is injective and surjective. In this sense we may identify $H_0^1(0,1)$ with its dual $H^{-1}(0,1)$.

Important remark: Do not simply regard this result as $H_0^1(0,1) = H^{-1}(0,1)$! This would lead to wrong conclusions, e.g. the Gelfand triplet $H_0^1(0,1) \subset L^2(0,1) \subset H^{-1}(0,1)$ becomes $H_0^1(0,1) = L^2(0,1) = H^{-1}(0,1)$. We will see in the following question that $H^{-1}(0,1)$, as a Hilbert space, could be equipped with an inner product different from the one used in $H_0^1(0,1)$.

[1] See e.g. Corollary 5.8 in Brezis' book.

e) Define the map $((\cdot,\cdot))\colon H^{-1}(0,1)\times H^{-1}(0,1)\to\mathbb{R};\ (f,g)\mapsto f((-\Delta)^{-1}g)$. Show that $((\cdot,\cdot))$ defines an inner product on $H^{-1}(0,1)$.

2. FEM for parabolic PDEs: Implementation

We continue studying the numerical solution of the following parabolic equation which was introduced in the previous exercise sheet for $G = J = (0,1)$. Consider:

$$\begin{aligned} \partial_t u - \partial_x(\alpha(x)\partial_x u) + \beta(x)\partial_x u + \gamma(x)u &= f(t,x) && \text{in } J\times G \\ u &= 0 && \text{on } J\times\partial G \\ u|_{t=0} &= u_0 && \text{in } G\,, \end{aligned} \tag{2}$$

where $u_0 \in L^2(G)$, $f(t,x)\in L^2(J,H^{-1}(G))$, $\alpha,\gamma\in C(\overline{G})$ and $\beta\in C^1(\overline{G})$ such that with some $\underline{\alpha}>0$ the bound $\alpha(x)>\underline{\alpha}$ holds for all $x\in G$.

Recall that for fixed $N,M\in\mathbb{N}$, we set $h=\frac{1}{N+1}, k=\frac{1}{M}$, the spatial mesh points $x_i := hi, i = 1,2,\dots,N$ and discrete time points $t_j = kj, j=0,1,\cdots,M$. We also define the spatial intervals $K_i = (h(i-1),hi)$, for $i=1,1,\dots,N+1$.

Following the same steps as for the heat equation in Problem 2 of the Exercise Sheet 2, we obtain the following full discretization using the finite element basis $V_N = \{\phi_{N,j}\}_{1\le j\le N}$ for (2):

$$\begin{aligned} &\text{Find } \underline{u}_N^m \in \mathbb{R}^N \text{ such that for } m=1,\dots,M \\ &(\mathbf{M}+k\vartheta\mathbf{A})\underline{u}_N^m = (\mathbf{M}-k(1-\vartheta)\mathbf{A})\underline{u}_N^{m-1} + k\underline{F}^m\,, \\ &\underline{F}^m = (\vartheta\underline{f}^m + (1-\vartheta)\underline{f}^{m-1})\,, \\ &\underline{u}_N^0 = \underline{u}_0\,. \end{aligned} \tag{3}$$

a) Verify that the matrices $\mathbf{M},\mathbf{A}$ are given by

$$\mathbf{M}_{i,j} = (\phi_{N,j},\phi_{N,i})_{L^2(G)} = \begin{cases} \frac{2h}{3}, & i=j \\ \frac{h}{6}, & |i-j|=1 \\ 0, & \text{otherwise} \end{cases},$$

and

$$\mathbf{A}_{i,j} = \int_G \alpha(x)\partial_x\phi_{N,i}\partial_x\phi_{N,j} + \beta(x)(\partial_x\phi_{N,j})\phi_{N,i} + \gamma(x)\phi_{N,i}\phi_{N,j}\,dx\,, \quad 1\le i,j\le N\,. \tag{4}$$

As for the heat equation, $\underline{f}^m$ is the column vector given by

$$f_i^m(t) = (f(t_m),\phi_{N,i})_{L^2(G)}\,, \quad 1\le i\le N\,.$$

Unlike for heat equation from the previous sheet, the coefficients α,β,γ are non-constant, which has to be taken into account when assembling the stiffness matrix $\mathbf{A}$.

b) Set $u = e^{-t}\sin(\pi x)$, $\alpha(x) = 1+x^2$, $\beta(x)=2x$ and $\gamma(x)=\pi^2x^2$. Assuming that u is the classical solution to (2), determine $f(t,x)$.

c) Modify the template solution file "FEM_parabolic.py" so that it is suitable for solving (2) numerically. To do this, you should implement the function "build_rigidityMatrix" which additionally takes the coefficient functions "alpha,beta,gamma" as inputs.

Use Simpson quadrature, to approximate the integral in (4) for the entries of $\mathbf{A}$ and construct the approximated matrix $\hat{\mathbf{A}}\in\mathbb{R}^{N\times N}$. The output of this Python functions shall be $\hat{\mathbf{A}}$, i.e. for $K_i = (x_{i-1},x_i)$ and $g\in C(\overline{K_i})$

$$\int_{K_i} g(x)\,dx \approx \frac{h}{6}\left[g(x_{i-1}) + 4\,g\left(\frac{x_{i-1}+x_i}{2}\right) + g(x_i)\right].$$

We now repeat the numerical tests in the third question of Exercise Sheet 2.

d) Repeat 3f) in Exercise Sheet 2 and report the result.

e) Repeat 3e) in Exercise Sheet 2 with $l = \{5, 6, 7, 8, 9\}$(N, M will then be different from those in Exercise Sheet 2!) and report the result.

Due: Wednesday, March 27th, at 2pm.

1. Spaces $H_0^1(0,1)$ and $H^{-1}(0,1)$

The space $H^{-1}(0,1)$ is defined as the dual space of $H_0^1(0,1)$ (that is, $H^{-1}(0,1)$ contains all bounded linear functionals on $H_0^1(0,1)$).

a) For any function $f \in H_0^1(0,1)$, define $\delta_{0.5}(f) = \tilde{f}(0.5)$, where $\tilde{f}$ denotes the unique continuous modification of f. Show that $\delta_{0.5} \in H^{-1}(0,1)$.

a) We need to show $\delta_{0.5}$ is linear and bounded. And $\delta_{0.5} : H^1 \to \mathbb{R}$

The last statement is clear as $f \in H_0^1(0,1)$.

Linearity: $f, g \in H_0^1(0,1)$ and $\forall \lambda \in \mathbb{R}$, we have

$$\delta_{0.5}(\lambda f + g) = (\lambda \tilde{f} + \tilde{g})(0.5) = \lambda \tilde{f}(0.5) + \tilde{g}(0.5)$$
$$= \lambda \delta_{0.5}(f) + \delta_{0.5}(g)$$

Boundedness:

$$|\delta_{0.5}(f)| = |\tilde{f}(0.5)| \leq \|\tilde{f}\|_{L^\infty(0,1)} \underset{\text{definition of } \tilde{f}.}{=} \|f\|_{L^\infty(0,1)}$$

To conclude, we use the exercise 3.1.6 which states that $\exists C = C(|G|)$ st

$$\|f\|_{L^\infty(0,1)} < C\|f\|_{H_0^1} \Rightarrow |\delta_{0.5}(f)| \leq C\|f\|_{H_0^1(0,1)}$$

$$\Rightarrow \underbrace{\|\delta_{0.5}\|}_{\text{operator norm.}} \leq C$$

b) Given a $F \in H^{-1}(0,1)$, consider the following PDE: Find $u \in H_0^1(0,1)$ such that

$$-\Delta u = F \quad \text{on } (0,1).$$

Its variational problem is of the following form: Find $u \in H_0^1(0,1)$ such that for any $v \in H_0^1(0,1)$,

$$\int_{(0,1)} u'v'\,dx = F(v). \tag{1}$$

Use the Lax-Milgram lemma and the Poincaré inequality (slides 2) to show that the variational problem (1) has a unique solution $u \in H_0^1(0,1)$.

The following *Lax-Milgram lemma*[1] holds:

Let H be a Hilbert space with its norm $\|\cdot\|_H$ and let $V \subset H$ be a closed subspace. Furthermore, let $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ be a bilinear form such that the following conditions hold true:

1. (Continuity) There exists $C_1 > 0$ such that $|a(u,v)| \leq C_1\|u\|_H\|v\|_H$ for any $u, v \in H$.
2. (Coercivity) There exists $C_2 > 0$ such that $a(u,u) \geq C_2\|u\|_H^2$ for any $u \in H$.

Then for any bounded linear functional $l \in H^*$ there exists a unique solution $u \in V$ to the following variational problem: Find $u \in V$ such that

$$a(u,v) = l(v), \quad \forall v \in V.$$

To use the Lax Milgram lemma we need a bilinear form. We define

$$a(\cdot,\cdot) : H_0^1(0,1) \times H_0^1(0,1) \to \mathbb{R}$$
$$(u,v) \longmapsto \int_0^1 u'v'\,dx$$

Now we show that a is cts and coercive.

cts: $|a(u,v)| = \left|\int_0^1 u'v'\,dx\right| \leq \int_0^1 |u'v'|\,dx = \|u'v'\|_{L^1(0,1)} \overset{\text{hölder}}{\leq} \|u'\|_{L^2} \cdot \|v'\|_{L^2}$

$\leq \|u\|_{H^1_0} \cdot \|v\|_{H^1_0}$

coercive: $a(u,u) = \int_0^1 (u')^2\,dx = \|u'\|_{L^2}^2 =$

$= \frac{1}{2}\|u'\|_{L^2}^2 + \frac{1}{2}\|u'\|_{L^2}^2 \geq$

$\geq \frac{1}{2}\|u'\|_{L^2}^2 + \frac{1}{2C}\|u\|_{L^2}^2$

$\geq \frac{1}{2}\min\{C^{-1},1\}\|u\|_{H^1_0(0,1)}^2$

Theorem (Poincaré inequality)

Assume that $G \subset \mathbb{R}$ is bounded and let $1 \leq p < \infty$. Then, there exists $C(|G|,p)$ such that

$\not{W}_0^{1,p} \subsetneq W^{1,p}(G)$

wrong on $W^{1,p}(G)$ = const: $\|u\|_{L^p} > 0$, $\|u'\|_{L^p} = 0$.

$$\forall u \in W_0^{1,p}(G): \ \|u\|_{L^p(G)} \leq C\,\|u'\|_{L^p(G)}\,.$$

$$\begin{aligned} a(u,u) &= \int_G |u'(x)|^2 \mathrm{d}x = \frac{1}{2}\|u'\|_{L^2(G)}^2 + \frac{1}{2}\|u'\|_{L^2(G)}^2 \\ &\geq \frac{1}{2C}\|u\|_{L^2(G)}^2 + \frac{1}{2}\|u'\|_{L^2(G)}^2 \\ &\geq \frac{1}{2}\min\{C^{-1},1\}\left(\|u\|_{L^2(G)}^2 + \|u'\|_{L^2(G)}^2\right) = C_1\|u\|_{H^1(G)}^2\,. \end{aligned}$$

Given $F \in H^{-1}(0,1)$, Lax-Milgram lemma implies that

$\exists!\ u \in H^1_0$ st $F(v) = a(u,v)$

Let $F \in H^{-1}(0,1)$ be arbitrary. With u as in Equation (1), we may write $u = (-\Delta)^{-1}F$, and hence obtain a mapping $(-\Delta)^{-1} : H^{-1}(0,1) \to H_0^1(0,1)$.

$F \mapsto u$

c) Verify that

$$(-\Delta)^{-1}\delta_{0.5} = \begin{cases} x/2, & x \in (0, 0.5)\,, \\ 1/2 - x/2, & x \in [0.5, 1)\,. \end{cases}$$

i.e. we need to show $u = \begin{cases} x/2 & 0 < x < 1/2 \\ 1/2 - x/2 & 1/2 \leq x < 1 \end{cases}$ is the unique

solution to the variational problem $\int_0^1 u'v'\,dx = \delta_{0.5}(v) = \tilde{v}(0.5)$

$u' = \begin{cases} 1/2 & 0 < x < 1/2 \\ -1/2 & 0 \leq x < 1 \end{cases} \Rightarrow \int_0^1 u'v'\,dx = \frac{1}{2}\int_0^{1/2} v'\,dx - \frac{1}{2}\int_{1/2}^1 v'\,dx$

$= \frac{1}{2}[\tilde{v}(1/2) - \underbrace{\tilde{v}(0)}_{=0}] - \frac{1}{2}[\underbrace{\tilde{v}(1)}_{=0} - \tilde{v}(1/2)]$

bc $v \in H^1_0$

$= \tilde{v}(1/2)$

$\Rightarrow a(u,v) = \delta_{0.5}(v)$ as intended.

d) Show that we could establish a one-to-one correspondence between $H^{-1}(0,1)$ and $H_0^1(0,1)$ with $(-\Delta)^{-1}$ by verifying that it is injective and surjective. In this sense we may identify $H_0^1(0,1)$ with its dual $H^{-1}(0,1)$.

We will show that $(-\Delta)^{-1}$ is both injective and surjective.

Notice at point b we showed that there exists a solution which implies surjectivity and also that the solution is unique which implies injectivity.

Formally: suppose $v_1, v_2 \in H^{-1}(0,1)$ st $(-\Delta)^{-1}(v_1) = u = (-\Delta)^{-1}(v_2)$

then using the PDE formulation we get:

$$v_1 = -\Delta(-\Delta)^{-1}(v_1) = -\Delta u = -\Delta(-\Delta)^{-1}(v_2) = v_2$$

which shows injectivity.

Now take $u \in H_0^1(0,1)$ we want to show $\exists F \in H_0^{-1}(0,1)$ st $(-\Delta)^{-1} F = u$.

So we take $F : H_0^1(0,1) \to \mathbb{R}$, $v \mapsto \int_0^1 u'v' \, dx$. Defined like this

F is a bounded linear operator. (Bounded by b) linearity is clear) $\Rightarrow F \in H^{-1}(0,1)$

Furthermore by the previous points, clearly $(-\Delta^{-1})F = u \Rightarrow (-\Delta)^{-1}$ surjective

Thus we found the desired F, and we can conclude that (Δ^{-1}) is indeed bijective.

e) Define the map $((\cdot,\cdot)) : H^{-1}(0,1) \times H^{-1}(0,1) \to \mathbb{R}$; $(f,g) \mapsto f((-\Delta)^{-1}g)$. Show that $((\cdot,\cdot))$ defines an inner product on $H^{-1}(0,1)$.

We verify that $(\cdot,\cdot)$ satisfies the properties of an inner product. Take $g \in H^{-1}(0,1)$

- $$(g,g) = g((-\Delta)^{-1}g) = \int_0^1 \Big(\underbrace{\big((-\Delta^{-1})g\big)'}_{\geq 0}\Big)^2 dx \geq 0$$

  and $(g,g) = 0 \Rightarrow (-\Delta)^{-1} g = 0$ (by the previous line)

  $\Rightarrow g = 0$ since $(-\Delta)^{-1}$ is bijective and $(\Delta^{-1})0 = 0$

- symmetric is clear so we just show bilinearity: $f, g_1, g_2 \in H^{-1}(0,1)$, $d \in \mathbb{R}$

$$(f, g_1 + dg_2) = \int (-\Delta^{-1}) f \cdot (-\Delta^{-1} g_1 + (-\Delta)^{-1} d g_2) = \int (-\Delta)^{-1} f (-\Delta)^{-1} g_1 + \int (-\Delta)^{-1} f (-\Delta)^{-1} d g_2$$

$$= (f, g_1) + d(f, g_2)$$

Note: $(-\Delta)^{-1} dg_2 = d(-\Delta)^{-1} g_2$ because $\Delta u = g_2 \Leftrightarrow d\Delta u = dg_2$

a) Verify that the matrices $\mathbf{M}, \mathbf{A}$ are given by

$$\mathbf{M}_{i,j} = (\phi_{N,j}, \phi_{N,i})_{L^2(G)} = \begin{cases} \frac{2h}{3}, & i = j \\ \frac{h}{6}, & |i-j| = 1 \\ 0, & \text{otherwise} \end{cases},$$

and

$$\mathbf{A}_{i,j} = \int_G \alpha(x)\partial_x\phi_{N,i}\partial_x\phi_{N,j} + \beta(x)(\partial_x\phi_{N,j})\phi_{N,i} + \gamma(x)\phi_{N,i}\phi_{N,j}\,dx\,, \quad 1 \le i,j \le N\,. \tag{4}$$

As for the heat equation, $\underline{f}^m$ is the column vector given by

$$\underline{f}_i^m(t) = (f(t_m), \phi_{N,i})_{L^2(G)}\,, \quad 1 \le i \le N\,.$$

Unlike for heat equation from the previous sheet, the coefficients α, β, γ are non-constant, which has to be taken into account when assembling the stiffness matrix $\mathbf{A}$.

b) Set $u = e^{-t}\sin(\pi x)$, $\alpha(x) = 1 + x^2$, $\beta(x) = 2x$ and $\gamma(x) = \pi^2 x^2$. Assuming that u is the classical solution to (2), determine $f(t,x)$.

a) $$\phi_{N,i} = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \text{if } x \in [x_i, x_{i+1}] \\ 0 & \text{o.w.} \end{cases}$$

Then if $|i-j| > 1 \Rightarrow (\phi_{N,i}, \phi_{N,j})_{L^2} = 0$.

For $i=j$: $$(\phi_{N,i}, \phi_{N,i})_{L^2} = \int_0^1 \left(\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\mathbb{1}_{[x_{i-1}, x_i]} + \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)\mathbb{1}_{[x_i, x_{i+1}]} \right)^2 dx$$

$$= \frac{1}{h^2}\left[\int_{x_{i-1}}^{x_i} (x - x_{i-1})^2 dx + \int_{x_i}^{x_{i+1}} (x_{i+1} - x)^2 dx \right]$$

$$= \frac{2}{3}h.$$

for $j = i+1$:

$$(\phi_{N,i}, \phi_{N,i+1})_{L^2} = \int_0^1 \left(\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\mathbb{1}_{[x_{i-1}, x_i]} + \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)\mathbb{1}_{[x_i, x_{i+1}]} \right) \cdot$$

$$\cdot \left(\left(\frac{x - x_i}{x_{i+1} - x_i}\right)\mathbb{1}_{[x_i, x_{i+1}]} + \left(\frac{x_{i+2} - x}{x_{i+2} - x_{i+1}}\right)\mathbb{1}_{[x_{i+1}, x_{i+2}]} \right) dx$$

$$= \frac{1}{h^2}\int_{x_i}^{x_{i+1}} (x_{i+1} - x)(x - x_i)\,dx = 1/_6\, h.$$

For A_{ij} we calculate the above expression w/ $$\phi_{N,i}' = \begin{cases} 1 & x \in [x_{i-1}, x_i] \\ -1 & x \in [x_i, x_{i+1}) \\ 0 & \text{o.w.} \end{cases}$$

b) we compute the PDE by plugging in the solution.

$$f(t,x) = \partial_t u - \partial_x(\alpha(x)\partial_x u) + \beta(x)\partial_x u + \gamma(x) u$$

$$= -e^{-t}\sin(\pi x) - \partial_x\left((1+x^2)\pi e^{-t}\cos(\pi x)\right) + 2x\,(\pi e^{-t}\cos(\pi x))$$
$$+ \pi^2 x^2 e^{-t}\sin(\pi x)$$

$$= (\pi^2 + 2\pi^2 x^2 - 1)\, e^{-t}\sin(\pi x)$$