Solutions: Series 3

1. Sobolev embedding and Poincaré inequality

a) Let $C > 0$ such that

$$\int_0^1 (u(x))^2 dx \leq C^2 \int_0^1 (u'(x))^2 dx\,.$$

Then, we have

$$\frac{1}{C^2} \leq \frac{\int_G |u'(x)|^2\, dx}{\int_G |u(x)|^2\, dx}\,,$$

and it follows that the optimal constant $C_{opt} > 0$ satisfies

$$\frac{1}{C_{opt}^2} = \inf_{\substack{u\in H_0^1(G)\\ u\not\equiv 0}} \frac{\|u'\|^2_{L^2(G)}}{\|u\|^2_{L^2(G)}}\,.$$

We show that the problem can be reduced to the unit interval (a, b) via a scaling argument by using the affine linear transformation $x \mapsto (b-a)x + a$. For any $u \in H_0^1(G)$ we have

$$\int_a^b (u'(x))^2 dx = (b-a)\int_0^1 \left(u'\left(\frac{y-a}{b-a}\right)\right)^2 dy = (b-a)^3 \int_0^1 \left(u\left(\frac{y-a}{b-a}\right)'\right)^2 dy\,,$$
$$\int_a^b (u(x))^2 dx = (b-a)\int_0^1 \left(u\left(\frac{y-a}{b-a}\right)\right)^2 dy\,.$$

It follows that

$$\inf_{\substack{u\in H_0^1(G)\\ u\not\equiv 0}} \frac{\|u'\|^2_{L^2(G)}}{\|u\|^2_{L^2(G)}} = (b-a)^2 \inf_{\substack{u\in H_0^1(0,1)\\ u\not\equiv 0}} \frac{\|u'\|^2_{L^2(0,1)}}{\|u\|^2_{L^2(0,1)}}\,. \tag{1}$$

In the above equality we recognize the Rayleigh quotient. By the hint, it holds that if the Rayleigh quotient is minimized by a function u, then $\frac{\|u\|^2_{L^2(0,1)}}{\|u'\|^2_{L^2(0,1)}}$ is the first eigenvalue of the eigenvalue problem for the Laplacian with Dirichlet boundary conditions,

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

The ODE with its boundary conditions being ignored has the general solution with $C_1, C_2 \in \mathbb{R}$

$$u(x) = C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x)\,.$$

Using the boundary conditions for u, we obtain that the eigenvalues are given by $\lambda_n = n^2\pi^2$ for $n \in \mathbb{N}$ and in particular $\lambda_1 = \pi^2$. Finally, using (1), we obtain that $C_{opt}(G) = \frac{b-a}{\pi}$.

b) We provide here two solutions to this subquestion.

Solution 1: Let $G = (a, b)$ and $u \in H^1(G)$. Then for any $x \in (a, b)$, using the Cauchy-Schwarz inequality we obtain

$$\tilde{u}(x)^2 = \left(\tilde{u}(y) + \int_y^x u'(t)dt\right)^2 \leq 2\tilde{u}(y)^2 + 2\left(\int_y^x u'(t)dt\right)^2$$
$$\leq 2\tilde{u}(y)^2 + 2\,|x-y|\,\|u'\|^2_{L^2(G)}\,.$$

Integrating over y gives

$$\begin{aligned}(b-a)\tilde{u}(x)^2 &\leq 2\,\|u\|_{L^2(G)}^2 + 2\,\|u'\|_{L^2(G)}^2 \int_a^b |x-y|\,dy \\ &\leq 2\max\left\{1, \frac{(b-a)^2}{2}\right\} \|u\|_{H^1(G)}^2 \,.\end{aligned}$$

We therefore obtain that

$$\|u\|_{L^\infty(G)} \leq \left[\frac{2}{b-a}\max\left\{1, \frac{(b-a)^2}{2}\right\}\right]^{1/2} \|u\|_{H^1(G)} \,.$$

As $u \in H^1(G)$ was arbitrarily selected, we may choose $C := \left[\frac{2}{b-a}\max\left\{1, \frac{(b-a)^2}{2}\right\}\right]^{1/2}$.

Solution 2: Fix $u \in H^1(G)$. By the Sobolev embedding theorem, it holds that $H^1(G) \subset C(\bar{G})$. Therefore, the function $|\tilde{u}|$ reaches its minimum value at some point $c \in [a,b]$. We claim that $|\tilde{u}(c)| \leq \frac{\|u\|_{L^2(G)}}{\sqrt{b-a}}$. For otherwise, it would hold that

$$\|u\|_{L^2(G)} > \left(\int_G \left(\frac{\|u\|_{L^2(G)}}{\sqrt{b-a}}\right)^2 dx\right)^{\frac{1}{2}} = \|u\|_{L^2(G)} \,,$$

which is a contradiction. Therefore, for any $x \in G$,

$$\begin{aligned}|\tilde{u}(x)| \leq |\tilde{u}(c)| + \left|\int_c^x u'(y)\,dy\right| &\leq \frac{\|u\|_{L^2(G)}}{\sqrt{b-a}} + \int_G |u'(y)|\,dy \\ &\leq \frac{\|u\|_{L^2(G)}}{\sqrt{b-a}} + \left(\int_G dy\right)^{\frac{1}{2}} \cdot \left(\int_G |u'(y)|^2\,dy\right)^{\frac{1}{2}} \\ &\leq \frac{\|u\|_{L^2(G)}}{\sqrt{b-a}} + \sqrt{b-a}\cdot \|u'\|_{L^2(G)} \\ &\leq \left(\frac{1}{\sqrt{b-a}} + \sqrt{b-a}\right) \|u\|_{H^1(G)} \,.\end{aligned}$$

Note: For the optimal choice of the constant, see `https://www.jstor.org/stable/2157159`

2. Interpolation error in $L^\infty(G)$

a) By the Sobolev embedding theorem, it holds that for any $1 \leq p \leq \infty$, $W^{1,p}(G) \subset C(\bar{G})$ and hence any $u \in W^{1,p}(G)$ admits a continuous representative $\tilde{u} \in C(\bar{G})$. In particular, this implies that the interpolation operator $\mathcal{I}_N$ is well-defined.

To show the inequality, note that for each $i = 1, \ldots, N+1$ and $\forall x \in [x_{i-1}, x_i]$, we have

$$\begin{aligned}|\tilde{u}(x) - \mathcal{I}_N u(x)| &= \left|\tilde{u}(x_{i-1}) + \int_{x_{i-1}}^x u'(u)dy - \left(\tilde{u}(x_{i-1}) + (x - x_{i-1})\frac{\tilde{u}(x_i) - \tilde{u}(x_{i-1})}{x_i - x_{i-1}}\right)\right| \\ &\leq \int_{x_{i-1}}^x |u'(y)|\,dy + |x - x_{i-1}|\,\frac{1}{h_i}\int_{x_{i-1}}^{x_i} |u'(s)|\,ds \\ &\leq 2\left(\int_{x_{i-1}}^{x_i} |u'(y)|^p\,dy\right)^{\frac{1}{p}} \left(\int_{x_{i-1}}^{x_i} 1^q dy\right)^{\frac{1}{q}} \\ &\leq 2(h_i)^{\frac{1}{q}}\,\|u'\|_{L^p(x_{i-1},x_i)} \\ &\leq 2(h_i)^{\frac{1}{q}}\,\|u\|_{W^{1,p}(G)} \,,\end{aligned}$$

where q is the Hölder conjugate given by $\frac{1}{q}+\frac{1}{p}=1$. It follows that

$$\begin{aligned}\|u-\mathcal{I}_N u\|_{L^\infty(G)} &= \max_{i=1,\dots,N+1}\|u-\mathcal{I}_N u\|_{\infty,[x_{i-1},x_i]}\\ &\le 2\left(\max_{i=1,\dots,N+1}(h_i)^{\frac{1}{q}}\right)\|u\|_{W^{1,p}(G)}\,.\end{aligned}$$

3. Finite element discretization for the heat equation II

a) As in Series 2, Problem 2, we obtain that

$$\mathbf{M}_{i,j}=\int_0^1 \phi_{N,i}(x)\phi_{N,j}(x)\,dx\,,$$

where for each $i\in\{1,\dots,N\}$,

$$\phi_{N,i}=\begin{cases}\dfrac{x-x_{i-1}}{x_i-x_{i-1}} & x\in[x_{i-1},x_i]\,,\\[2ex] \dfrac{x_{i+1}-x}{x_{i+1}-x_i} & x\in[x_i,x_{i+1}]\,,\\[2ex] 0 & \text{otherwise}\,.\end{cases}$$

Computing the integrals yields 0 if $|i-j|>1$. If $i=j$, we have that

$$\begin{aligned}\mathbf{M}_{i,j}&=\frac{1}{h_i^2}\int_{x_{i-1}}^{x_i}(x-x_{i-1})^2\,dx+\frac{1}{h_{i+1}^2}\int_{x_i}^{x_{i+1}}(x_{i+1}-x)^2\,dx\\ &=\frac{h_i}{3}+\frac{h_{i+1}}{3}\,.\end{aligned}$$

For $i\in\{1,\dots,N-1\}$, we have

$$\mathbf{M}_{i+1,i}=\mathbf{M}_{i,i+1}=\frac{1}{h_{i+1}^2}\int_{x_i}^{x_{i+1}}(x_{i+1}-x)(x-x_i)\,dx=\frac{h_{i+1}}{6}\,,$$

and thus

$$\mathbf{M}_{i,j}=\begin{cases}\frac{h_i}{3}+\frac{h_{i+1}}{3} & |i-j|=0\,,\\[2ex] \frac{h_{\max(i,j)}}{6} & |i-j|=1\,,\\[2ex] 0 & \text{otherwise}\,.\end{cases}$$

Similarly, the entries of $\mathbf{A}$ are given by:

$$\mathbf{A}_{i,j}=\int_0^1 \phi_{N,i}'(x)\phi_{N,j}'(x)\,dx\,,$$

which gives us

$$\mathbf{A}_{i,j}=\begin{cases}\frac{1}{h_i}+\frac{1}{h_{i+1}} & |i-j|=0\,,\\[2ex] -\frac{1}{h_{\max(i,j)}} & |i-j|=1\,,\\[2ex] 0 & \text{otherwise}\,.\end{cases}$$

The entries $F_i(t)$ for $i=1,\dots,N$ of the vector $\underline{\mathrm{F}}(t)$ are given by

$$F_i(t)=\frac{1}{h_i}\int_{x_{i-1}}^{x_i}f(t,x)(x-x_{i-1})\,dx+\frac{1}{h_{i+1}}\int_{x_i}^{x_{i+1}}f(t,x)(x_{i+1}-x)\,dx\,.$$

b) See the solution code.

c) All of the three cases lead to convergence: The corresponding convergence rates are: For $\beta = 1$, 1.022; For $\beta = 1.05$, 1.015; For $\beta = 1.2$, 1.001.

These coincides with the facts given in the lecture that the implicit scheme ensures unconditional stability and that the convergence rate is $O(h^2 + k)$. Also, the case $\beta = 1$ is the uniform-mesh case which we studied in the previous exercise sheet and the same rate is obtained here.

d) Now none of these cases converges. This is due to the instability as the temporal steps are too large compared to the spatial steps: Recall that in the lecture, we have seen that a sufficient condition for stability of the θ-scheme is the time-step restriction

$$k \le C \frac{h_{min}^2}{1 - 2\theta}, \quad 0 \le \theta < 1/2. \tag{2}$$

The graded mesh enforces h_{min} to be sufficiently small such that the above condition is easily violated.

e) The setting $\beta = 1$ or 1.05 leads to convergence: The corresponding convergence rates are: For $\beta = 1.05$, 1.001; For $\beta = 1$, 1.005. Divergence occurs if $\beta = 1.2$.

Here, since $\beta = 1.05$ is closer to 1 compared to $\beta = 1.2$, with larger M like the setting in this subquestion it is still possible that (2) is ensured.

4. A general second-order parabolic problem

a) We justify the first statement. Since $\beta(x) \in C^1(\overline{G})$ and $\alpha(x), \gamma(x) \in C(\overline{G})$, there exists a constant $K > 0$ such that $|\alpha(x)|, |\beta(x)|, |\gamma(x)| < K$. So, by the Cauchy-Schwarz inequality, for any $u, v \in H_0^1(G)$,

$$\begin{aligned}
|a(u,v)| &\le K(\|\partial_x u\|_{L^2(G)} \|\partial_x v\|_{L^2(G)} + \|\partial_x u\|_{L^2(G)} \|v\|_{L^2(G)} + \|u\|_{L^2(G)} \|v\|_{L^2(G)}) \\
&\le 3K \|u\|_{H^1(G)} \|v\|_{H^1(G)}.
\end{aligned}$$

Therefore the first statement is valid with $C_1 = 3K$.

Now we study the second statement. We have, for any $c > 0$ and any $u \in H_0^1(G)$,

$$\begin{aligned}
&a(u,u) + c\|u\|_{L^2(G)}^2 \\
=& \int_G \alpha(x)(\partial_x u)^2 + \beta(x)(\partial_x u)u + (k + \gamma(x))u^2 \, dx \\
=& \int_G \alpha(x)(\partial_x u)^2 + (c + \gamma(x) - \frac{1}{2}\partial_x \beta(x))u^2 \, dx \\
\ge& \underline{\alpha} \|\partial_x u\|_{L^2(G)}^2 + \min_{x \in \overline{G}} (c + \gamma(x) - \frac{1}{2}\partial_x \beta(x)) \|u\|_{L^2(G)}^2
\end{aligned}$$

Note that in the above derivation, we have used the following equality that for any $u \in H_0^1(G)$,

$$\int_G \beta(x) u \partial_x u \, dx = -\frac{1}{2} \int_G \partial_x \beta(x) u^2 \, dx.$$

This follows from partial integration.

Therefore, by choosing any $c \ge \max_{x \in \overline{G}} \frac{1}{2}\partial_x \beta(x) - \gamma(x)$, $C_3 = c$ and choosing $C_2 = \min(\underline{\alpha}, \min_{x \in \overline{G}}(c + \gamma(x) - \frac{1}{2}\partial_x \beta(x)))$, we verify the second statement.

b) We proved before the continuity and Gårding inequality for the bilinear form $a(\cdot,\cdot)$. The uniqueness of the solution could be verified by using Theorem 3.2.2 in the textbook or well-posedness theorem in lecture slides 2, with $V = H_0^1(G)$ and $H = L^2(G)$.

c) We will prove equivalently that $u(t,x)$ will not achieve its positive maximum value in $(J \times G) \cup \{t = 1, x \in G\}$.

We argue by contradiction: Assume first that $u(t,x)$ attains its positive maximum value at $(t_0, x_0) \in (J \times G)$. At this point, we have $\partial_t u = \partial_x u = 0$, $\partial_{xx} u \leq 0$ and $u > 0$. This leads to

$$\partial_t u - \partial_x(\alpha(x)\partial_x u) + \beta(x)\partial_x u + \gamma(x)u = -\alpha(x)\partial_{xx}u + \gamma(x)u > 0$$

at (t_0, x_0) because $\alpha(x) \geq \underline{\alpha} > 0$ and $\gamma(x) > 0$. However, this contradicts $f \equiv 0$.

We now assume that $u(t,x)$ attains its positive maximum value at $(t_0, x_0) \in \{t = 1, x \in G\}$. At this point, we have $\partial_t u \geq 0$, $\partial_x u = 0$, $\partial_{xx} u \leq 0$ and $u > 0$. This leads to

$$\partial_t u - \partial_x(\alpha(x)\partial_x u) + \beta(x)\partial_x u + \gamma(x)u = \partial_t u - \alpha(x)\partial_{xx}u + \gamma(x)u > 0$$

at (t_0, x_0). This contradicts $f \equiv 0$.

d) $v = -u$ solves

$$\begin{aligned} \partial_t v - \partial_x(\alpha(x)\partial_x v) + \beta(x)\partial_x v + \gamma(x)v &= 0 && \text{in } J \times G \\ v &= 0 && \text{on } \partial G \times J \\ v|_{t=0} &= -u_0 && \text{in } G. \end{aligned} \tag{3}$$

By using the result of **4c)**, we could prove that for any $(t,x) \in \overline{J \times G}$, $v(t,x) \leq \max(0, \max_{x \in \overline{G}} -u_0(x))$. Therefore, $u \geq \min(0, \min_{x \in \overline{G}} u_0(x))$. By combining this and the result of **4c)** we have

$$\min(0, \min_{x \in \overline{G}} u_0(x)) \leq u(t,x) \leq \max(0, \max_{x \in \overline{G}} u_0(x)) \quad \text{for any } (t,x) \in \overline{J \times G}.$$

This implies that for any $(t,x) \in \overline{J \times G}$, $|u(t,x)| \leq \max_{x \in \overline{G}} |u_0(x)|$.