Series 7

1. Implementation of the CEV model in Python

We continue to study the CEV model from Exercise Sheet 6. Recall that on the truncated domain $G = (0, R)$, the variational formulation of this model reads as follows

$$\begin{aligned}&\text{Find } v \in L^2(J; W_{\rho,\mu}) \cap H^1(J; (W_{\rho,\mu})^*) \text{ such that for a.e. } t \in J\,,\\ &(\partial_t v, w)_\mu + a^{\mathrm{CEV}}_{\rho,\mu}(v, w) = 0\,, \quad \forall\, w \in W_{\rho,\mu}\,,\\ &v(0,\cdot) = g(\cdot)\,, \quad \text{in } \mathcal{H}_\mu\,,\end{aligned} \tag{1}$$

where the bilinear form $a^{\mathrm{CEV}} : W_{\rho,\mu} \times W_{\rho,\mu} \to \mathbb{R}$, is given by

$$a^{\mathrm{CEV}}_{\rho,\mu}(u, v) = \frac{\sigma^2}{2}\left(s^{2\rho}u', v'\right)_\mu + \sigma^2(\rho+\mu)\left(s^{2\rho-1}u', v\right)_\mu - r\,(su', v)_\mu + r\,(u, v)_\mu\,.$$

The inner product $(\cdot,\cdot)_\mu$ and the function space $W_{\rho,\mu}$ were already defined in Exercise Sheet 6. We assume from now on that r, ρ, μ are selected such that $r > 0$, $\mu > -\frac{1}{2}$ and $0 \le \rho + \mu < \frac{1}{2}$.

We introduce, for any $N \in \mathbb{N}$, the uniform grid

$$x_i = ih\,, \quad 0 \le i \le N+1\,,$$

where $h = \frac{R}{N+1}$, and let

$$V_{N+1} = \left\{u \in C^0(G) \;\middle|\; u(R) = 0 \text{ and } \forall i \in \{0,\dots,N\},\ u_{|s\in(x_i,x_{i+1})} \text{ is a linear function}\right\}.$$

Let b_i be the element of V_{N+1} defined by

$$b_i(x_j) = \delta_{i,j}\,.$$

a) Given $M \in \mathbb{N}$, set the discrete points in time $0 = t_0 < t_1 < \cdots < t_M = T$ with $t_j = j \cdot \frac{T}{M}$, for $j = 0, 1, \dots, M$, and set $k = \frac{T}{M}$. Write down the Galerkin approximation of (1) in the finite element space V_{N+1} and the corresponding θ-scheme in time. Give the expressions of the matrices appearing in this new context.

b) Complete in the template `CEVmodel.py`, the Python functions `assembleMatrixA(N,alpha,R)`, `assembleMatrixB(N,beta,R)`, `assembleMatrixC(N,gamma,R)` and `assembleMatrix(N,alpha,beta,gamma,R)`. They should output the sparse matrices $\mathbf{A}(\alpha)$, $\mathbf{B}(\beta)$, $\mathbf{C}(\gamma)$ and $\mathbf{A}(\alpha) + \mathbf{B}(\beta) + \mathbf{C}(\gamma)$. These matrices $\mathbf{A}(\alpha)$, $\mathbf{B}(\beta)$ and $\mathbf{C}(\gamma)$ are defined by, for $1 \le i, j \le N+1$,

$$\mathbf{A}_{i,j}(\alpha) = (\alpha b'_{j-1}, b'_{i-1}), \quad \mathbf{B}_{i,j}(\beta) = (\beta b'_{j-1}, b_{i-1}), \quad \mathbf{C}_{i,j}(\gamma) = (\gamma b_{j-1}, b_{i-1}).$$

Here $(\cdot,\cdot)$ denotes the L^2-inner product on G, `N` is the discretization parameter, `R` determines the right-side endpoint of the interval G and `alpha,beta,gamma` are functions. These matrices have to be approximated using the *two-point Gaussian quadrature* formula

$$\int_a^b F(x)dx \approx \frac{b-a}{2}\left(F\left(\frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) + F\left(\frac{a+b}{2} + \frac{b-a}{2\sqrt{3}}\right)\right),$$

for $F : [a, b] \to \mathbb{R}$. As an example, applying this quadrature rule on the support of the integrands we have

$$(\alpha b'_i, b'_i) \approx \frac{1}{2h}\left(\alpha\left(x_{i-1} + p_1\right) + \alpha\left(x_{i-1} + p_2\right) + \alpha\left(x_i + p_1\right) + \alpha\left(x_i + p_2\right)\right),$$

for $i \in \{1, \dots, N\}$ and

$$(\alpha b_0', b_0') \approx \frac{1}{2h} \left(\alpha\left(p_1\right) + \alpha\left(p_2\right) \right) ,$$

while

$$(\alpha b_i', b_{i+1}') \approx (\alpha b_{i+1}', b_i') = -\frac{1}{2h} \left(\alpha\left(x_i + p_1\right) + \alpha\left(x_i + p_2\right) \right) .$$

Here, the quadrature points are given by

$$p_1 = w_1 h \qquad p_2 = w_2 h \quad w_1 = \frac{1 - \frac{1}{\sqrt{3}}}{2}, \quad w_2 = \frac{1 + \frac{1}{\sqrt{3}}}{2}.$$

One then proceeds similarly to approximate $(\beta b_{j-1}', b_{i-1})$ and $(\gamma b_{j-1}, b_{i-1})$.

c) Using your code from the previous question, complete the functions `buildMassCEV(N,mu,R)` and `buildACEV(N,sigma,rho,mu,r,R)`. Those functions should compute the matrices $\mathbf{M}$ and $\mathbf{A}$, defined by

$$\mathbf{M}_{i,j}^{\mathrm{CEV}} := \left(b_{j-1}, b_{i-1}\right)_\mu ,$$
$$\mathbf{A}_{i,j}^{\mathrm{CEV}} := a_{\rho,\mu}^{\mathrm{CEV}}(b_{j-1}, b_{i-1})_\mu ,$$

respectively. Here `N` is the discretization parameter and `rho,mu` stand for ρ and μ in the CEV model. To finish this subquestion, you also need to define the functions `alpha(s),beta(s),gamma(s)` *inside the function bodies* of `buildMassCEV(N,mu)` and `buildACEV(N,rho,mu)`, so that functions defined in the previous subquestion could be implemented with different parameters.

d) Check the validity of your code by verifying that for $R = 4$, $\rho = 0.5$ and $\mu = -0.25$,

$$v(r = 0.05, \sigma = 0.3, s = 1, T = 1, K = 1) \approx 0.093650 .$$

The function `FEM_theta(N,sigma,rho,mu,r,R,M,T,K,theta)` implemented for you realizes the θ-scheme and returns a column vector of the numerical solution at time level T. Set $N = 999, M = 1000$ and complete the function `test()`, which prints the numerical solution at $(s,T) = (1,1)$.

2. FEM for American put in a BS market

Consider v_R, the solution of the truncated American put problem, in log price and time to maturity, for $x \in G = (-R, R)$ and $J = (0, T]$

$$\begin{array}{rcll} \partial_t v_R - \mathcal{A}^{\mathrm{BS}} v_R + r v_R & \geq & 0 & \text{in } J \times G \\ v_R & \geq & \tilde{g} & \text{in } J \times G \\ (\partial_t v_R - \mathcal{A}^{\mathrm{BS}} v_R + r v_R)(\tilde{g} - v_R) & = & 0 & \text{in } J \times G \\ v_R(0, \cdot) & = & \tilde{g} & \text{in } G \\ v_R(t, \pm R) & = & \tilde{g}(\pm R) & \text{in } J , \end{array} \tag{2}$$

with the Black-Scholes infinitesimal generator

$$\mathcal{A}^{\mathrm{BS}} = \frac{1}{2}\sigma^2 \partial_{xx} + \left(r - \frac{1}{2}\sigma^2 \right) \partial_x ,$$

and the payoff function for a put contract in spot/log-price given by

$$g(s) := (K - s)_+ , \quad \tilde{g}(x) := g(e^x) .$$

The variational formulation of the truncated problem for *the excess to payoff* $u_R = v_R - \tilde{g}$ reads

$$\begin{array}{l} \text{Find } u_R \in L^2(J; H_0^1(G)) \cap H^1(J; H^{-1}(G)) \text{ such that } u_R(t, \cdot) \in \mathcal{K}_{0,R} \text{ and} \\ \langle \partial_t u_R, v - u_R \rangle + a^{\mathrm{BS}}(u_R, v - u_R) \geq -a^{\mathrm{BS}}(\tilde{g}, v - u_R) , \quad \forall v \in \mathcal{K}_{0,R} , \\ u_R(0) = 0 , \end{array} \tag{3}$$

with

$$\mathcal{K}_{0,R} := \{ v \in H_0^1(G) : \ v \geq 0 \text{ a.e. } x \in G \}.$$

a) Assume that for all $t > 0$, $v_R(t,\cdot) \in C^2(G)$ and satisfies (2) in the strong sense. Show that in this case, $u_R = v_R - \tilde{g}$ satisfies problem (3).

Remark: unlike v_R, $\tilde{g} \notin C^2(G)$ (hence neither is u_R).

Given $N, M \in \mathbb{N}$, set the spatial mesh $\mathcal{T} := \{-R = x_0 < x_1 < x_2 < \cdots < x_{N+1} = R\}$ such that $x_i = -R + i \cdot \frac{2R}{N+1}$ for $i = 0, 1, \ldots, N+1$ and the discrete points in time $0 = t_0 < t_1 < \cdots < t_M = T$ with $t_j = j \cdot \frac{T}{M}$ for $j = 0, 1, \ldots, M$. Set also the spatial mesh size $h = \frac{2R}{N+1}$ and time step $k = \frac{T}{M}$.

b) Use Lemma 5.3.1 in the textbook to verify that approximating the solution of (3) with finite elements in the spatial domain and the backward Euler scheme in the temporal domain formally leads to solving the following sequence of problems : given $\underline{u}_N^0 = \underline{0}$, find $\underline{u}_N^{m+1} \in \mathbb{R}^N$ such that for $m = 0, \ldots, M-1$

$$\begin{aligned} &\mathbf{B}\underline{u}_N^{m+1} \geq \underline{F}^m, \\ &\underline{u}_N^{m+1} \geq \underline{0}, \\ &(\underline{u}_N^{m+1})^\top \left(\mathbf{B}\underline{u}_N^{m+1} - \underline{F}^m\right) = 0. \end{aligned} \tag{4}$$

Derive the expressions for $\mathbf{B}$ and $\underline{F}^m$.

Hint: On p. 86 in the textbook, it is shown that for all $\varphi \in H_0^1(G)$, there holds

$$-a^{\mathrm{BS}}(\tilde{g}, \varphi) = \frac{1}{2}K\sigma^2\varphi(\ln K) - rK\int_{-R}^{\ln K} \varphi(x)\,dx\,.$$

c) In the template `PSOR.py`, complete the function `PSOR(A,b,c,x0)` which uses PSOR algorithm to solve the following Linear Complementarity Problem (LCP): Given $\mathbf{A} \in \mathbb{R}^N$ and $b, c \in \mathbb{R}^N$, find $x \in \mathbb{R}^N$ such that

$$\begin{aligned} &\mathbf{A}x \geq b, \\ &x \geq c, \\ &(x-c)^\top \left(\mathbf{A}x - b\right) = 0. \end{aligned} \tag{5}$$

In the routine `PSOR(A,b,c,x0)`, `A,b,c` are the matrices and vectors from (5), and `x0` serves as the initial guess for the PSOR algorithm.

Use the function `criterion(A,b,c,x,tol)` as a stopping criterion for your algorithm. This function has input `A` standing for $\mathbf{A} \in \mathbb{R}^{N\times N}$, `b,c,x` representing $b, c, x \in \mathbb{R}^N$ and `tol`> 0 as the tolerance. It returns a boolean variable indicating whether the following relations

$$\begin{aligned} &\mathbf{A}x \geq b - \mathtt{tol}, \\ &x \geq c - \mathtt{tol}, \\ &\left|(x-c)^\top \left(\mathbf{A}x - b\right)\right| < \mathtt{tol} \end{aligned} \tag{6}$$

are fully satisfied or not. Test your algorithm using the function `testPSOR()` provided. This function randomly generates a LCP as well as its exact solution x, computes the numerical solution $xguess$ via PSOR and prints the L^∞-norm of the error vector $x - xguess$.

d) In the template `BSAmericanPut.py`, complete `computeOptionValue(r,sigma,K,T,N,M)` which computes the price of the American put option using FEM and your implementation of the PSOR algorithm. This function shall have inputs `r,sigma,K,T` as the model parameters and `N,M` as discretization parameters. It should also output the vector of the numerical solution at time step T. You are allowed to add additional functions for the assembly of matrices and vectors used in PSOR. We have implemented the function `gput(x)` for you, which computes the payoff function.

Plot the result for the parameters

$$r = 0.05\,, \quad \sigma = 0.3\,, \quad K = 1\,, \quad T = 1\,, \quad N = 401\,, \quad M = 400.$$

In the FEM settings, set $R = 3$. Don't forget that $\underline{u}_N$ is the excess-to-payoff. We have also implemented functions plotting the result of the European option value using BS-formula for you. Compare the values of the American option with that of the European option.

Hint: You may firstly test your code with lower resolution/higher tolerance for faster debugging.

Due: Wednesday, April 24th, at 2pm.

a) Given $M \in \mathbb{N}$, set the discrete points in time $0 = t_0 < t_1 < \cdots < t_M = T$ with $t_j = j \cdot \frac{T}{M}$, for $j = 0, 1, \ldots, M$, and set $k = \frac{T}{M}$. Write down the Galerkin approximation of (1) in the finite element space V_{N+1} and the corresponding θ-scheme in time. Give the expressions of the matrices appearing in this new context.

$m = 0, 1, \ldots, M$ with $t_m = m \cdot \frac{T}{M}$ $\quad k = \frac{T}{M}$ = time step

We continue to study the CEV model from Exercise Sheet 6. Recall that on the truncated domain $G = (0, R)$, the variational formulation of this model reads as follows

$$
\begin{aligned}
&\text{Find } v \in L^2(J; W_{\rho,\mu}) \cap H^1(J; (W_{\rho,\mu})^*) \text{ such that for a.e. } t \in J\,,\\
&(\partial_t v, w)_\mu + a^{\mathrm{CEV}}_{\rho,\mu}(v, w) = 0\,, \quad \forall w \in W_{\rho,\mu}\,,\\
&v(0,\cdot) = g(\cdot)\,, \quad \text{in } \mathcal{H}_\mu\,,
\end{aligned} \tag{1}
$$

where the bilinear form $a^{\mathrm{CEV}} : W_{\rho,\mu} \times W_{\rho,\mu} \to \mathbb{R}$, is given by

$$a^{\mathrm{CEV}}_{\rho,\mu}(u, v) = \frac{\sigma^2}{2}\left(s^{2\rho} u', v'\right)_\mu + \sigma^2(\rho + \mu)\left(s^{2\rho - 1} u', v\right)_\mu - r\,(s u', v)_\mu + r\,(u, v)_\mu\,.$$

The inner product $(\cdot,\cdot)_\mu$ and the function space $W_{\rho,\mu}$ were already defined in Exercise Sheet 6. We assume from now on that r, ρ, μ are selected such that $r > 0$, $\mu > -\frac{1}{2}$ and $0 \le \rho + \mu < \frac{1}{2}$.

We introduce, for any $N \in \mathbb{N}$, the uniform grid

$$x_i = ih\,, \quad 0 \le i \le N+1\,,$$

where $h = \frac{R}{N+1}$, and let

$$V_{N+1} = \left\{ u \in C^0(G) \mid u(R) = 0 \text{ and } \forall i \in \{0, \ldots, N\},\ u_{|s \in (x_i, x_{i+1})} \text{ is a linear function} \right\}.$$

Let b_i be the element of V_{N+1} defined by

$$b_i(x_j) = \delta_{i,j}\,.$$

The Galerkin formulation for CEV is then approximated by considering $v_N \in V_{N+1}$

$$\frac{d}{dt}\left(v_N(t), w_N\right)_\mu + a^{\mathrm{CEV}}_{\rho,\mu}(v_N(t), w_N) = 0$$

with basis $\{b_i\}_{i = 0, \ldots, N}$ $\Rightarrow$ $v_N(t, s) = \sum_{i=0}^{N} v_{N,i}(t)\, b_i(s)$ & $w_N(s) = \sum_{i=0}^{N} w_{N,i}\, b_i(s)$

$$\therefore \frac{d}{dt}\left(\sum_{i=0}^{N} v_{N,i}(t)\, b_i, b_j\right)_\mu + \frac{\sigma^2}{2}\left(s^{2\rho} \sum_{i=0}^{N} v_{N,i}(t)\, b_i', b_j'\right)_\mu + \sigma^2(\rho+\mu)\left(s^{2\rho-1} \sum_{i=0}^{N} v_{N,i}(t)\, b_i', b_j\right)_\mu$$

$$- r\left(s \sum_{i=0}^{N} v_{N,i}(t)\, b_i', b_j\right)_\mu + r\left(\sum_{i=0}^{N} v_{N,i}(t)\, b_i, b_j\right)_\mu = 0$$

$$\Rightarrow \sum_{i=0}^{N} \frac{d}{dt} v_{N,i}(t)\,(b_i, b_j)_\mu + \frac{\sigma^2}{2} v_{N,i}(t)\,(s^{2\rho} b_i', b_j')_\mu + \sigma^2(\rho+\mu)\, v_{N,i}(t)\,(s^{2\rho-1} b_i', b_j)_\mu$$

$$- r\, v_{N,i}(t)\,(s b_i', b_j)_\mu + r\, v_{N,i}(t)\,(b_i, b_j)_\mu = 0$$

$$M \frac{d}{dt}\underline{v}_N + A\, \underline{v}_N = 0$$

$$M_{i,j} = (b_i, b_j)_\mu$$

$$A_{i,j} = \frac{\sigma^2}{2}(S^{2\rho} b_i', b_j')_\mu + \sigma^2(\rho+\mu)(S^{2\rho-1} b_i', b_j)_\mu - r(S b_i', b_j)_\mu + r(b_i, b_j)_\mu$$

$$= a^{CEV}(b_i, b_j)$$

Thus,

mix θ back. $(1-\theta)$ frwrd.

$$M\left(\frac{\underline{v}_N^{m+1} - \underline{v}_N^{m}}{k}\right) + A\left(\theta \underline{v}_N^{m+1} + (1-\theta)\underline{v}_N^{m}\right) = 0$$

And finally,

$$(M + \theta k A)\underline{v}_N^{m+1} = (M - k(1-\theta)A)\underline{v}_N^{m}$$

Consider v_R, the solution of the truncated American put problem, in log price and time to maturity, for $x \in G = (-R, R)$ and $J = (0, T]$

$$
\begin{aligned}
\partial_t v_R - \mathcal{A}^{\mathrm{BS}} v_R + r v_R &\geq 0 && \text{in } J \times G \\
v_R &\geq \tilde{g} && \text{in } J \times G \\
(\partial_t v_R - \mathcal{A}^{\mathrm{BS}} v_R + r v_R)(\tilde{g} - v_R) &= 0 && \text{in } J \times G \qquad (2)\\
v_R(0, \cdot) &= \tilde{g} && \text{in } G \\
v_R(t, \pm R) &= \tilde{g}(\pm R) && \text{in } J\,,
\end{aligned}
$$

$\Rightarrow$ $\langle \partial_t v_R, v \rangle + a^{BS}(u,v) \geq 0$

$\langle \partial_t v_R, \tilde{g} - v_R \rangle + a^{BS}(v_R, \tilde{g} - v_R) = 0$

with the Black-Scholes infinitesimal generator

$$\mathcal{A}^{\mathrm{BS}} = \frac{1}{2}\sigma^2 \partial_{xx} + \left(r - \frac{1}{2}\sigma^2\right)\partial_x\,,$$

$a^{BS}(u,v) = \frac{1}{2}\sigma^2 (u', v') + (\sigma^2/2 - r)(u', v) + r(u,v)$

$=$

and the payoff function for a put contract in spot/log-price given by

$$g(s) := (K - s)_+\,, \quad \tilde{g}(x) := g(e^x)\,.$$

The variational formulation of the truncated problem for *the excess to payoff* $u_R = v_R^{(t,x)} - \tilde{g}^{(x)}$ reads

$$
\begin{aligned}
&\text{Find } u_R \in L^2(J; H_0^1(G)) \cap H^1(J; H^{-1}(G)) \text{ such that } u_R(t, \cdot) \in \mathcal{K}_{0,R} \text{ and} \\
&\langle \partial_t u_R, v - u_R \rangle + a^{\mathrm{BS}}(u_R, v - u_R) \geq -a^{\mathrm{BS}}(\tilde{g}, v - u_R)\,, \quad \forall v \in \mathcal{K}_{0,R}\,, \qquad (3)\\
&u_R(0) = 0\,,
\end{aligned}
$$

with

$$\mathcal{K}_{0,R} := \{v \in H_0^1(G) : v \geq 0 \text{ a.e. } x \in G\}.$$

a) Assume that for all $t > 0$, $v_R(t, \cdot) \in C^2(G)$ and satisfies (2) in the strong sense. Show that in this case, $u_R = v_R - \tilde{g}$ satisfies problem (3).

Remark: unlike v_R, $\tilde{g} \notin C^2(G)$ *(hence neither is* u_R*).*

Given $N, M \in \mathbb{N}$, set the spatial mesh $\mathcal{T} := \{-R = x_0 < x_1 < x_2 < \cdots < x_{N+1} = R\}$ such that $x_i = -R + i \cdot \frac{2R}{N+1}$ for $i = 0, 1, \ldots, N+1$ and the discrete points in time $0 = t_0 < t_1 < \cdots < t_M = T$ with $t_j = j \cdot \frac{T}{M}$ for $j = 0, 1, \ldots, M$. Set also the spatial mesh size $h = \frac{2R}{N+1}$ and time step $k = \frac{T}{M}$.

$u_R = v_R - \tilde{g}$ $\qquad \langle \partial_t (v_R - \tilde{g}), v - v_R + \tilde{g} \rangle + a^{BS}(v_R - \tilde{g}, v - v_R + \tilde{g}) \geq -a^{BS}(\tilde{g}, v - v_R + \tilde{g})$

$\Rightarrow \langle \partial_t v_R, v \rangle - \langle \partial_t v_R, v_R \rangle + \langle \partial_t v_R, \tilde{g} \rangle + a^{BS}(v_R, v) - a^{BS}(v_R, v_R) + a^{BS}(v_R, \tilde{g}) - \cancel{a^{BS}(\tilde{g}, v)} + \cancel{a^{BS}(\tilde{g}, v_R)}$

$- \cancel{a^{BS}(\tilde{g}, \tilde{g})} \geq - \cancel{a^{BS}(\tilde{g}, v)} + \cancel{a^{BS}(\tilde{g}, v_R)} - \cancel{a^{BS}(\tilde{g}, \tilde{g})}$

$\Rightarrow \langle \partial_t v_R, v \rangle - \langle \partial_t v_R, v_R \rangle + \langle \partial_t v_R, \tilde{g} \rangle + a^{BS}(v_R, v) - a^{BS}(v_R, v_R) + a^{BS}(v_R, \tilde{g}) \geq 0$

$\Rightarrow \underbrace{\langle \partial_t v_R, v \rangle + a^{BS}(v_R, v)}_{\geq 0} + \underbrace{\langle \partial_t v_R, \tilde{g} - v_R \rangle + a^{BS}(v_R, \tilde{g} - v_R)}_{= 0} \geq 0$ ■

$u_R(0, \cdot) = v_R(0, \cdot) - \tilde{g} = \tilde{g} - \tilde{g} = 0$ ■

b) Use Lemma 5.3.1 in the textbook to verify that approximating the solution of (3) with finite elements in the spatial domain and the backward Euler scheme in the temporal domain formally leads to solving the following sequence of problems : given $\underline{u}_N^0 = \underline{0}$, find $\underline{u}_N^{m+1} \in \mathbb{R}^N$ such that for $m = 0, \ldots, M-1$

$$
\begin{aligned}
&\mathbf{B}\underline{u}_N^{m+1} \geq \underline{F}^m, \\
&\underline{u}_N^{m+1} \geq \underline{0}, \qquad (4)\\
&(\underline{u}_N^{m+1})^\top (\mathbf{B}\underline{u}_N^{m+1} - \underline{F}^m) = 0.
\end{aligned}
$$

Derive the expressions for $\mathbf{B}$ and $\underline{F}^m$.

Hint: On p. 86 in the textbook, it is shown that for all $\varphi \in H_0^1(G)$, there holds

$$-a^{\mathrm{BS}}(\tilde{g}, \varphi) = \frac{1}{2}K\sigma^2\varphi(\ln K) - rK\int_{-R}^{\ln K} \varphi(x)\,dx\,.$$

Find $\underline{u}_N^{m+1} \in \mathbb{R}_{\geq 0}^N$ such that for $m = 0, \ldots, M-1$,

$$(\underline{v} - \underline{u}_N^{m+1})^\top (\mathbf{M} + k\mathbf{A}^{\mathrm{BS}})\underline{u}_N^{m+1} \geq (\underline{v} - \underline{u}_N^{m+1})^\top (k\underline{f} + \mathbf{M}\underline{u}_N^m), \quad \forall \underline{v} \in \mathbb{R}_{\geq 0}^N, \quad (5.9)$$

$\underline{u}_N^0 = \underline{0}$,

Discretizing ($\mathsf{VI}_R^{\mathrm{BS}}$) with implicit Euler time stepping ($\theta = 1$) and piecewise linear finite elements $S_{\mathcal{T},0}^1$ in space, we obtain

must be solved for each time step.

$m = \boxed{0}, 1, \ldots, M-1, \boxed{M}$

Find $\underline{u}_N^{m+1} \in \mathbb{R}_{\geq 0}^N$ s.t. for $m = 1, \ldots, M-1$

$$(\underline{v} - \underline{u}_N^{m+1})^\top (\mathbf{M} + k\mathbf{A}^{\mathrm{BS}})\underline{u}_N^{m+1} \boxed{\geq} (\underline{v} - \underline{u}_N^{m+1})^\top (k\underline{f} + \mathbf{M}\underline{u}_N^m) \quad \forall \underline{v} \in \mathbb{R}_{\geq 0}^N,$$

cannot use Gaussian Elim (inequalities)

$$\underline{u}_N^0 = \underline{0}, \qquad (\mathsf{VI}_{N,M}^{\mathrm{BS}})$$

where

$$\Phi_N(t,x) = v_N(x) - u_N(t,x)$$

$$v_N(x) = \sum_{i=0}^{N} v_{N,i}\, b_i(x)$$

$$u_N(t,x) = \sum_{i=0}^{N} u_{N,i}(t)\, b_i(x)$$

A variational formulation of the localized problem ($\mathrm{PDI}_R^{\mathrm{BS}}$) for the excess to payoff is then given by

$$a^{BS}(u,v) = \int_G \frac{\sigma^2}{2} u_x v_x + \left(\frac{\sigma^2}{2} - r\right) v_x u \, dx$$

$\in L^2(G)$!!

Find $u_R \in L^2(J; H_0^1(G)) \cap H^1(J; H^{-1}(G))$ s.t. $u(t,\cdot) \in \mathcal{K}_{0,R}$ and

$$\langle \partial_t u_R, v - u_R \rangle + a^{\mathrm{BS}}(u_R, v - u_R) \geq -a^{\mathrm{BS}}(\tilde{g}, v - u_R)$$

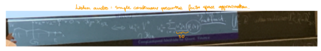
$\forall v \in \mathcal{K}_{0,R},$ a.e. in J,

$$u_R(0) = 0. \qquad (\mathrm{VI}_R^{\mathrm{BS}})$$

weak formulation does not require $\partial^2 \tilde{g}$ => avoid dirac measure that pops up in standard BS (cannot solve)

parabolic variational inequality (audio)

Listen audio: simple continuous piecewise finite space approximation.



$$(\partial_t u_N, v_N - u_N) + a^{BS}(u_N, v_N - u_N) \geq -a^{BS}(\tilde{g}, v_N - u_N)$$

$$\downarrow$$

$$\left(\partial_t \sum_{i=0}^{N} u_{N,i}(t)\, b_i(x), \sum_{j=0}^{N} v_{N,j} b_j - \sum_{\ell=0}^{N} u_{N,\ell}(t) b_\ell\right) + a^{BS}\left(\sum_{i=0}^{N} u_{N,i}(t) b_i, \sum_{j=0}^{N} v_{N,j} b_j - \sum_{\ell=0}^{N} u_{N,\ell}(t) b_\ell\right)$$

$$\geq -a^{BS}\left(\tilde{g}, \sum_{j=0}^{N} v_{N,j} b_j - \sum_{i=0}^{N} u_{N,i}(t) b_i\right)$$

$$\downarrow$$

$$\sum_{i=0}^{N} \frac{d}{dt} u_{N,i}(t) \left(b_i, \sum_{j=0}^{N} v_{N,j} b_j\right) - \sum_{i=0}^{N}\sum_{\ell=0}^{N} \frac{d}{dt}(u_{N,i}(t))\, u_{N,\ell}(t) (b_i, b_\ell)$$

$$+ \sum_{i=0}^{N} u_{N,i}(t)\, a^{BS}\left(b_i, \sum_{j=0}^{N} v_{N,j} b_j\right) - \sum_{i=0}^{N}\sum_{\ell=0}^{N} u_{N,i}(t)\, u_{N,\ell}(t)\, a^{BS}(b_i, b_\ell)$$

$$\geq -a^{BS}\left(\tilde{g}, \sum_{j=0}^{N} v_{N,j} b_j\right) + \underbrace{\sum_{i=0}^{N} u_{N,i}(t)\, a^{BS}(\tilde{g}, b_i)}_{\sum_{i=0}^{N} u_{N,i}(t) \left\{ \frac{1}{2} K \sigma^2 b_i(\ln K) - rK \int_{-R}^{\ln K} b_i(x)\, dx \right\}}$$

$$M_{ij} = (b_i, b_j) \qquad f_j := -a^{\mathrm{BS}}(\tilde{g}, b_j)$$

$$A_{ij} = a^{BS}(b_i, b_j)$$

$$\downarrow$$

$$-a^{\mathrm{BS}}(\tilde{g}, \varphi) = \frac{1}{2} K \sigma^2 \varphi(\ln K) - rK \int_{-R}^{\ln K} \varphi(x)\, dx\,.$$

$$\sum_{i=0}^{N} \frac{u_{N,i}^{m+1} - u_{N,i}^{m}}{k} \left(b_i, \sum_{j=0}^{N} v_{N,j} b_j\right) - \sum_{i=0}^{N}\sum_{\ell=0}^{N} \frac{u_{N,i}^{m+1} - u_{N,i}^{m}}{k} u_{N,\ell}^{m+1} (b_i, b_\ell)$$

$$+ \sum_{i=0}^{N} u_{N,i}^{m+1}\, a^{BS}\left(b_i, \sum_{j=0}^{N} v_{N,j} b_j\right) - \sum_{i=0}^{N}\sum_{\ell=0}^{N} u_{N,i}^{m+1} u_{N,\ell}^{m+1}\, a^{BS}(b_i, b_\ell)$$

$$\geq -a^{BS}\left(\tilde{g}, \sum_{j=0}^{N} v_{N,j} b_j\right) + \sum_{i=0}^{N} u_{N,i}^{m+1}\, a^{BS}(\tilde{g}, b_i)$$

$$\downarrow$$

$$\sum_{j=0}^{N} v_{N,j} \sum_{i=0}^{N} u_{N,i}^{m+1} (b_i, b_j) - \sum_{i=0}^{N}\sum_{j=0}^{N} u_{N,i}^{m+1} \cdot u_{N,j}^{m+1} (b_i, b_j) + k \sum_{j=0}^{N} v_{N,j} \sum_{i=0}^{N} u_{N,i}^{m+1}\, a^{BS}(b_i, b_j)$$

$$- k \sum_{i=0}^{N}\sum_{j=0}^{N} u_{N,i}^{m+1} u_{N,j}^{m+1}\, a^{BS}(b_i, b_j) \geq k \sum_{j=0}^{N} v_{N,j} \cdot \left(-a^{BS}(\tilde{g}, b_j)\right) - k \sum_{i=0}^{N} u_{N,i}(t) \left(-a^{BS}(\tilde{g}, b_j)\right)$$

$$+ \sum_{i=0}^{N} u_{N,i}^{m} \sum_{j=0}^{N} v_{N,j} (b_i, b_j) - \sum_{i=0}^{N}\sum_{j=0}^{N} u_{N,i}^{m} u_{N,j}^{m+1} (b_i, b_j)$$

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then let $M_{ij} = (b_i, b_j)$, $A_{ij} = a^{BS}(b_i, b_j)$ $f_j = -a^{BS}(\tilde{g}, b_j)$

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thus, $\displaystyle \sum_{j=0}^{N} v_{N,j} \sum_{i=0}^{N} u_{N,i}^{m+1} (b_i, b_j) + k \sum_{j=0}^{N} v_{N,j} \sum_{i=0}^{N} u_{N,i}^{m+1}\, a^{BS}(b_i, b_j) - \left(\sum_{i=0}^{N} u_{N,i}^{m+1} \sum_{j=0}^{N} u_{N,j}^{m+1} (b_i, b_j) + \right.$

$$\left. k \sum_{i=0}^{N}\sum_{j=0}^{N} u_{N,i}^{m+1} u_{N,j}^{m+1}\, a^{BS}(b_i, b_j) \right) \geq k \sum_{j=0}^{N} v_{N,j} \cdot \left(-a^{BS}(\tilde{g}, b_j)\right) - k \sum_{i=0}^{N} u_{N,i}(t) \left(-a^{BS}(\tilde{g}, b_j)\right)$$

$f_j = -a^{BS}(\tilde{g}, b_j)$ $+ \sum_{i=0}^{N} u^m_{N,i} \sum_{j=0}^{N} v_{N_j} (b_i, b_j) - \sum_{i=0}^{N}\sum_{j=0}^{N} u^m_{N,i} u^{m+1}_{N,j} (b_i, b_j)$

$$\Rightarrow \boxed{(\underline{v} - \underline{u}_N^{m+1})^T (M + kA^{BS}) \underline{u}^{m+1} \geq (\underline{v} - \underline{u}_N^{m+1})^T (k\underline{f} + M\underline{u}_N^m)}$$

$\underline{u}_N^0 = \underline{0}$ by IC

Then, $\boxed{\begin{aligned} B &= M + kA^{BS} \\ \underline{F}^m &= k\underline{f} + M\underline{u}_N^m \end{aligned}}$ $\Rightarrow$

$$\left[\begin{aligned} &B\underline{u}_N^{m+1} \geq \underline{F}^m \\ &\underline{u}_N^{m+1} \geq \underline{0} \quad \text{as } u_R(t,\cdot) \in K_{0,R} \\ &(\underline{u}_N^{m+1})^T (B\underline{u}_N^{m+1} - \underline{F}^m) = 0 \end{aligned}\right.$$

Lemma 5.3.1 *Denote by* $\mathbf{B} := \mathbf{M} + k\mathbf{A}^{\mathrm{BS}}$, $\underline{F}^m := k\underline{f} + \mathbf{M}\underline{u}_N^m$. *Then, problem* (5.9) *is equivalent to: given* $\underline{u}_N^0 = \underline{0}$, *find* $\underline{u}_N^{m+1} \in \mathbb{R}^N$ *such that for* $m = 0, \ldots, M-1$,

$$\begin{aligned} &\mathbf{B}\underline{u}_N^{m+1} \geq \underline{F}^m, \\ &\underline{u}_N^{m+1} \geq \underline{0}, \qquad (5.10) \\ &(\underline{u}_N^{m+1})^\top \left(\mathbf{B}\underline{u}_N^{m+1} - \underline{F}^m\right) = 0. \end{aligned}$$

Find $u_R \in L^2(J; H_0^1(G)) \cap H^1(J; H^{-1}(G))$ such that $u_R(t,\cdot) \in \mathcal{K}_{0,R}$ and

$$\begin{aligned} &\langle \partial_t u_R, v - u_R \rangle + a^{\mathrm{BS}}(u_R, v - u_R) \geq -a^{\mathrm{BS}}(\tilde{g}, v - u_R), \quad \forall v \in \mathcal{K}_{0,R}, \qquad (3) \\ &u_R(0) = 0, \end{aligned}$$