Series 2

Notation: 1_A denotes the indicator function of the set A.

1. Sobolev spaces and Banach-space valued functions

a) Given two functions $u_1(t) \in C^1([0,a])$ and $u_2(t) \in C^1([a,1])$ with $a \in (0,1)$. Prove that the function $u(t) : [0,1] \to \mathbb{R}$ which coincides with u_1 on $[0,a)$ and with u_2 on $[a,1]$ belongs to $H^1(0,1)$ if and only if $u_1(a) = u_2(a)$.

b) Let Y be a separable Banach space with norm $\|\cdot\|_Y$ and denote by Y^* the dual space of Y. We consider the Borel-σ-algebra $\mathcal{B}(Y)$ on Y, which is generated by all open sets in Y. For any open or closed interval $J \subset \mathbb{R}$, we say that a function $f : J \to Y$ is (strongly) measurable[1] if for every $E \in \mathcal{B}(Y)$, $f^{-1}(E) \in \mathcal{L}$, where $\mathcal{L}$ consists of all Lebesgue-measurable sets on J. To check the measurability of a function $f : J \to Y$, we may use *Pettis' Theorem*: For any *separable* Banach space Y (e.g. $L^2(0,1)$ and $H^1(0,1)$), the following two statements are equivalent:

i) $f : J \to Y$ is measurable.

ii) For any bounded linear functional $g \in Y^*$, the real function $J \ni t \mapsto g(f(t))$ is measurable.

Use Pettis' Theorem to prove that any function $f \in C([0,1];Y)$ where Y is a separable Banach space is measurable.

c) For any $A \in (-1,1)$, define $v_A(t,x) = (x-t)1_{\{t-x>A\}}$ on $(t,x) \in [0,1] \times [0,1]$. It could be regarded as a function $V_A(t)$ which maps from any $t \in [0,1]$ to $v_A(t,\cdot)$ as a function of x. Determine all possible values of A such that $V_A(t)$ belongs to

i) $C([0,1];L^2(0,1))$,

ii) $C([0,1];H^1(0,1))$.

2. Finite element method for the heat equation

Let $J = (0,T)$, $T > 0$, $G = (a,b) \subset \mathbb{R}$ and $f \in C(\overline{J};L^2(G))$. We wish to solve the heat equation with zero Dirichlet boundary conditions and with initial value $u_0 \in L^2(G)$,

$$\begin{cases} \partial_t u(t,x) - \partial_{xx} u(t,x) & = f(t,x) & \text{in } J \times G\,, \\ u(t,x) & = 0 & \text{on } J \times \partial G\,, \\ u(0,x) & = u_0(x) & \text{in } G. \end{cases} \tag{1}$$

[1] There are other definitions of measurability for Banach-space valued functions. For example, we say that $f : J \to Y$ is *weakly measurable* if for any $g \in Y^*$, $g(f(t)) : J \to \mathbb{R}$ is measurable. In this exercise sheet we will only consider strong measurability.

a) Show that if u is a smooth solution of this problem, then for all $v \in H_0^1(G)$, there holds

$$\frac{d}{dt}\left(u(t), v\right)_{L^2(G)} + a(u(t), v) = \left(f(t), v\right)_{L^2(G)} , \quad \forall t \in J \tag{2}$$

where

$$a(\varphi, \psi) = \int_G \partial_x \varphi(x) \partial_x \psi(x)\, dx , \quad (\varphi, \psi)_{L^2(G)} = \int_G \varphi(x)\psi(x)\, dx .$$

The variational formulation to problem (1) is: find a solution $u \in L^2(J; H_0^1(G)) \cap H^1(J; H^{-1}(G))$ to (2). It has been shown in Example 3.2.3 in the textbook that this problem exhibits a unique solution.

b) Let V_N be a subspace of $H_0^1(G)$ of finite dimension N. We seek an approximation u_N of the solution of system (1) as the element of $C^1([0,T]; V_N)$ satisfying the variational problem

$$\frac{d}{dt}\left(u_N(t), v_N\right)_{L^2(G)} + a(u_N(t), v_N) = \left(f(t), v_N\right)_{L^2(G)} , \quad \forall v_N \in V_N\, , \ \forall t \in J\, ,$$

with initial condition $u_N(0,x) = u_{0,N}$ where $u_{0,N}$ is some approximation of u_0 in V_N (this is sometimes called the "method of lines"). Given a basis $\{\phi_{N,j}\}_{1\le j\le N}$ of V_N, write

$$u_N(t,\cdot) = \sum_{j=1}^{N} u_{N,j}(t)\phi_{N,j}\, , \quad \underline{u}_N(t) = \begin{pmatrix} u_{N,1}(t) \\ \vdots \\ u_{N,N}(t) \end{pmatrix} .$$

Prove that the vector $\underline{u}_N(t)$ satisfies a system of coupled ordinary differential equations of the form:

$$\mathbf{M}\frac{d}{dt}\underline{u}_N + \mathbf{A}\underline{u}_N = \underline{\mathrm{F}}\, . \tag{3}$$

Give the expression of the matrices $\mathbf{M}$, $\mathbf{A}$ and the vector $\underline{\mathrm{F}}$. Why are $\mathbf{M}$, $\mathbf{A}$ nonsingular?

c) We seek approximations $u_{N,i}^m$ of the values of the coefficients $u_{N,i}(t_m)$ at each time $t_m = km$ where $k > 0$ is the time step and $m \in \mathbb{N}$. Let

$$\underline{u}_N^m := \begin{pmatrix} u_{N,1}^m \\ \vdots \\ u_{N,N}^m \end{pmatrix} .$$

Using a ϑ-scheme, propose a finite-difference discretization of Eq. (3), leading to an algorithm of the form

$$\mathbf{B}_\vartheta \underline{u}_N^{m+1} = \mathbf{C}_\vartheta \underline{u}_N^m + \underline{\mathrm{F}}_\vartheta^m\, ,$$

and specify $\mathbf{B}_\vartheta$, $\mathbf{C}_\vartheta$ and $\underline{\mathrm{F}}_\vartheta^m$.

3. Implementation in Python

From now on, we assume $T = 1$ so that $J = G = (0,1)$. For any $N, M \in \mathbb{N}$, we set $h = \frac{1}{N+1}, k = \frac{1}{M}$ and consider the spatial mesh points $x_i = hi, i = 1, 2, \ldots, N$. Let V_N be the vector space of continuous functions on G, vanishing at both ends of the interval, and which are linear on each (x_i, x_{i+1}). For each $i \in \{1, \ldots, N\}$, there is a unique element $\phi_{N,i}$ of V_N satisfying

$$\phi_{N,i}(x_j) = \delta_{i,j}\, , \quad \forall j \in \{1, \ldots, N\}$$

and $\{\phi_{N,i}\}_{1\le i\le N}$ is a basis of V_N.

a) For this choice of basis, compute the entries of $\mathbf{M}$ and $\mathbf{A}$ obtained in 2.b). In the script "`FEM_heat.py`" provided, fill the functions "`build_massMatrix(N)`" and "`build_rigidityMatrix(N)`" accordingly. The input for both functions are the discretization parameter N and the output should be the matrices $\mathbf{M}$ and $\mathbf{A}$ accordingly. You should compute the matrices A and G in closed form so that no numerical approximations are required.

For any $F \in C^4[a,b]$, one can evaluate approximately the integral $\int_a^b F(x)\,dx$ using the Simpson rule, which reads

$$\int_a^b F(x)\,dx \approx \frac{b-a}{6}\left(F(a) + 4F\left(\frac{a+b}{2}\right) + F(b)\right).$$

The order of approximation is $O(|b-a|^5)$.

b) Let $u(t,x) = e^{-t} x \sin(\pi x)$. Check that $u(t,x)$ is the solution of system (1) for

$$f(t,x) = e^{-t}\left((\pi^2 - 1)x\sin(\pi x) - 2\pi\cos(\pi x)\right), \quad u_0(x) = x\sin(\pi x).$$

Define the corresponding functions "`f(t,x)`", "`initial_value(x)`" and "`exact_solution_at_1(x)`" in "`FEM_heat.py`". Here "`f`" has the temporal and spatial variables (t,x) as the input and outputs the value of $f(t,x)$. "initial_value" and "exact_solution_at_1" shall receive a vector of grid points and computes a vector containing the value of $u(x,0)$ and $u(x,1)$ at these points respectively.

c) Show that

$$\int_G f(t,x)\phi_{N,i}(x)\,dx = h\frac{f(t,x_i - h/2) + f(t,x_i) + f(t,x_i + h/2)}{3} + O(h^5). \tag{4}$$

In the template "`FEM_heat.py`", complete the function "`build_F(t,N)`" accordingly. The parameters of this function are the time level t and the discretization parameter N. The output shall be the approximated value of the column vector $\underline{\mathrm{F}}$ at time t, using (4).

d) In the template "`FEM_heat.py`", complete the functions "`FEM_theta(N,M,theta)`" which shall implement the Finite Element Method with the θ-scheme derived in the previous exercise. Here $N, M, theta$ are the discretization parameters and the function returns the numerical solution on the spatial grid $x_i, i = 1, 2, \ldots, N$ at $t = 1$.

To finish the following two exercises, you first need to modify the block "error analysis" in the script. Follow the comments in this block.

e) Test your code with $\theta = 0.3, 0.5, 1$, $N = 2^l - 1$ and $M = 2^l$ with $l = \{2,3,4,5,6\}$. Do we get a convergent numerical solution for each θ? Use the template to obtain the convergence rates with respect to k and generate the plots describing the convergence rates if they converge. Comment on your results.

f) Test your code with $\theta = 0.3, 0.5, 1$, $N = 2^l - 1$ and $M = 4^l$ with $l = \{2,3,4,5,6\}$. As before, study if those numerical schemes converge and report the convergence rates if they converge. Comment on your results.

Due: Wednesday, March 13th, at 2pm.