Exam [maximal number of points: 80 pt]

time: 120 minutes

1. General questions [in total 14 pt]

a) [4 pt] Let the function $\varphi : [-1,1] \to \mathbb{R}$ be piecewise differentiable such that $\varphi|_{[-1,0)} \in C^1([-1,0))$ and $\varphi|_{(0,1]} \in C^1((0,1])$ such that $\varphi_- := \lim_{0<h\to 0} \varphi(-h)$ and $\varphi_+ := \lim_{0<h\to 0} \varphi(h)$ exist.

i) Prove that if $\varphi_- = \varphi_+$, then $\varphi \in H^1((-1,1))$. State the weak derivative of φ.

ii) Prove that if $\varphi_- \neq \varphi_+$, then $\varphi \notin H^1((-1,1))$.
Hint: You may use that the Dirac functional δ_0 is **not** in $L^2((-1,1))$.

b) [1 pt] What is a major difference in the qualitative behavior of the short rate in the Vasicek model in comparison to the short rate in the Cox-Ingersoll-Ross (CIR) model?

c) [4 pt] Let X be a Lévy process with characteristic triplet (σ^2, ν, γ) to be used as log-price process. Let $\sigma > 0$. Suppose that $\nu(\mathrm{d}z) = k(z)\mathrm{d}z$ and that for $\alpha \in (0,2)$, $\beta_+ > 1$, $\beta_- > 0$, $C > 0$

$$k(z) \leq Ce^{-\beta_-|z|}, \qquad z < -1, \tag{1}$$

$$k(z) \leq Ce^{-\beta_+ z}, \qquad z > 1, \tag{2}$$

$$k(z) \leq C\frac{1}{|z|^{1+\alpha}}, \qquad 0 < |z| < 1. \tag{3}$$

i) Explain why in the context of European plain vanilla option pricing the condition $\beta_+ > 1$ is required.

ii) Describe from a numerical perspective the two cases $k(z) = 0$ for every $z \in \mathbb{R}$ and $k(z) > 0$ for every $z \in \mathbb{R}$.

iii) Why is the value $\alpha = 2$ not admissible?

d) [1 pt] Let $T > 0$ and $x \in \mathbb{R}$ be fixed. How does the localization $v_R(t,x)$ converge in terms of R (where $R > |x|$) to the option price $v(t,x)$, for $t \in (0,T)$, in the one dimensional Black–Scholes model for this fixed value x.

e) [1 pt] Consider a Black–Scholes model in one dimension. Assume you have access to a software that prices European call options for strike $K = 1$. The software returns spot prices and corresponding option values. How can this be applied for strike values $K \neq 1$?

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f) [3 pt] Consider a three dimensional basket option with sufficiently smooth payoff in the Black-Scholes model. We use a piecewise linear continuous Finite Element discretization in space and the θ-scheme in time for the discretization of the arising pricing equation. Which convergence rate in the $L^2(D)$-norm at maturity on an appropriately chosen domain of interest do you expect in terms of the mesh width h and of the number of degrees of freedom N? What is the relationship between h and N?

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2. Options with butterfly payoff in the Black–Scholes market [in total 41 pt]

Consider the Black–Scholes model, i.e., the stock price satisfies the following SDE

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = s_0 > 0,$$

where $r \in \mathbb{R}$, $\sigma > 0$. The **butterfly payoff** is given by

$$g^{\mathrm{BF}}(s) := g^{\mathrm{C}}_{K_0}(s) - 2g^{\mathrm{C}}_{K_1}(s) + g^{\mathrm{C}}_{K_2}(s), \quad s \in (0,\infty), \quad \text{such that} \quad K_1 = \frac{K_0 + K_2}{2},$$

where for every $K > 0$, $g^{\mathrm{C}}_K(\cdot)$ denotes the payoff of a call option with strike $K > 0$.

a) [2 pt] **State** the definition of the price $V(t,s)$ of an European option with butterfly payoff on the stock $(S_t : t \geq 0)$ with maturity $T > 0$.

State the PDE satisfied by the price $V(t,s)$ in this market model.

b) [3 pt] To get a numerically tractable problem, we change to time-to-maturity $t \to T - t$ and log-price coordinates $x = \log(s)$, restrict the PDE to a bounded domain $G := (-R, R)$, $R > 0$ sufficiently large, and impose homogeneous Dirichlet boundary conditions. Let $J := (0, T]$. **Derive** the corresponding localized PDE.

Let $H = L^2(G)$ and $V = H^1_0(G)$. The variational formulation corresponding to the localized PDE in **b)** reads

$$\begin{aligned}
&\text{Find } v_R \in L^2(J;V) \cap H^1(J;V^*) \text{ such that for a.e. } t \in J \\
&\langle \partial_t v_R, u \rangle_{V^*,V} + a(v_R, u) = 0 \qquad \forall\, u \in V \\
&v_R(0,x) = g^{\mathrm{BF}}(e^x) \text{ for every } x \in G
\end{aligned}$$

The bilinear form $a(\cdot,\cdot)$ is defined by

$$a(w,u) := \frac{1}{2}\sigma^2 \int_G \partial_x w\, \partial_x u \mathrm{d}x - (r - \frac{1}{2}\sigma^2) \int_G \partial_x w\, u \mathrm{d}x + r \int_G w\, u \mathrm{d}x.$$

c) [6 pt] We discretize in space using the mesh, $-R = x_0 < x_1 < \cdots < x_N < x_{N+1} = R$, with equidistant points $x_n := nh - R$, $n = 0, \ldots, N+1$ and mesh width $h := \frac{2R}{N+1}$. We discretize in time with time points $t_m := mk$, $m = 0, \ldots, M$ and time step $k := \frac{T}{M}$. The θ-scheme is used to discretize the time variable and Finite Elements is used to discretize the space variable. The used Finite Element space is spanned by the continuous hat-functions $b_j \in V$ that correspond to the mesh. **Derive and state** the matrix formulation and define precisely all parts of it. You **don't** have to compute the specific entries of the matrices.

d) [14 pt] **Modify** the template `P2_main.m` to solve the problem obtained in **c)** numerically and report on the convergence rate in the $L^2(J; L^2(\widetilde{G}))$-norm using the exact solution of a European call option `bs_formula_C.m`, where $\widetilde{G}$ is the domain of interest. Choose the parameters $K_0 = 0.5$, $K_2 = 1.5$, $\sigma = 0.2$, $r = 0.005$, $T = 1$, $R = 5$, $\theta = 0.5$, $k = h$, and $\widetilde{G} = \{x \in G \mid |e^x - K_1| < \frac{3}{2}\}$. You **can** use the

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routine `stiff.m` for the computation of the stiffness and mass matrices. You can also compute the stiffness and mass matrices directly.

Explain the observed behavior of the $L^2(J; L^2(\widetilde{G}))$-error in your plot, how can it be improved?

Hint: The $L^2(J, L^2(\widetilde{G}))$-norm of a function $u : J \times \widetilde{G} \to \mathbb{R}$ can be approximated by

$$\sqrt{\sum_{m=1}^{M} k\, h \sum_{i \text{ s.t. } x_i \in \widetilde{G}} |u(t_m, x_i)|^2}.$$

Let $V^{\mathrm{Am}}(t,s)$ denote the price of an American option with butterfly payoff $g^{\mathrm{BF}}(\cdot)$. Define $\widetilde{v}_R(t,x)$ to be price of the American option with butterfly payoff in time-to-maturity, in log-price, and on the truncated domain G. Define the excess-to-payoff variable

$$w_R(t,x) := \widetilde{v}_R(t,x) - g^{\mathrm{BF}}(e^x), \quad t \in J, x \in G.$$

We approximate $w_R(t,x)$ with Finite Elements (as in **c)**) and backward Euler time stepping, i.e., for a given $\underline{w}^0$: find $\underline{w}^{m+1}$ such that for $m = 0, \ldots, M-1$

$$\begin{aligned} &\mathbf{B}\underline{w}^{m+1} \geq \underline{F}^m, \\ &\underline{w}^{m+1} \geq 0, \\ &(\underline{w}^{m+1})^\top (\mathbf{B}\underline{w}^{m+1} - \underline{F}^m) = 0, \end{aligned} \tag{4}$$

where

$$\mathbf{B} = \mathbf{M} + k\mathbf{A} \quad \text{and} \quad \underline{F}^m = k\underline{f} + \mathbf{M}\underline{w}^m, \quad \forall m = 0, \ldots, M-1,$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{A}$ is the stiffness matrix, $k = \frac{T}{M}$, and

$$\underline{f}_j = -a(g^{\mathrm{BF}}(e^x), b_j), \quad \text{for every } j \text{ such that } b_j \in V.$$

e) [1 pt] **State** the definition of the price $V^{\mathrm{Am}}(t,s)$.

f) [5 pt] **Show** that for a test function $\varphi \in H_0^1(G)$ and $e^{-R} < K < e^R$,

$$-a(g_K^{\mathrm{C}}(e^x), \varphi) = \frac{\sigma^2 K}{2}\varphi(\log(K)) + rK \int_{\log(K)}^{R} \varphi \mathrm{d}x$$

Use this to express $\underline{f}_j$ in terms of b_j and K_i for every j such that $b_j \in V$ and $i = 0, 1, 2$. **Derive** a formula for

$$b_j(\log(K_i)), \qquad i = 0, 1, 2.$$

Be aware that $\log(K_i)$, $i = 0, 1, 2$, may not be on the Finite Element mesh.

g) [7 pt] **Modify** the template `P2_main.m` to approximate $\widetilde{v}_R$, i.e., solve (4) with the PSOR algorithm to compute $\widetilde{v}_R$ at final time T. Plot your approximation to V^{Am} at the start of the contract in spot-price. Choose the parameters $K_0 = 0.5$, $K_2 = 1.5$, $\sigma = 0.2$, $r = 0.005$, $T = 1$, $R = 5$, $\theta = 0.5$, $k = h$, and $\widetilde{G} = \left\{x \in G \mid |e^x - K_1| < \frac{3}{2}\right\}$. You can use the routines `stiff.m`, `load_vec.m`, and `psor.m`.

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h) [3 pt] Compute and plot the boundaries between the exercise and continuation region. Explain according to your plot when the holder should continue to hold and when the holder should exercise the option.

Hint:

- You may use that $V^{\mathrm{Am}}(t, K_1) = g^{\mathrm{BF}}(K_1)$ for every $t \in J$.
- You may also use that there exists i such that $x_i = \log(K_1)$.

Please turn over.

3. Option pricing in a stochastic volatility model [in total 25 pt]

Consider the risk neutral dynamics

$$\begin{aligned} \mathrm{d}X_t &= (r - \frac{1}{2}Y_t)\mathrm{d}t + \sqrt{Y_t}\mathrm{d}W_t^1, \\ \mathrm{d}Y_t &= \alpha(\overline{m} - Y_t)\mathrm{d}t + \beta\sqrt{Y_t}(\rho \mathrm{d}W_t^1 + \sqrt{1-\rho^2}\mathrm{d}W_t^2), \end{aligned} \tag{5}$$

where W_t^1, W_t^2 are two independent standard Brownian motions. We set $r = 0$ for simplicity and want to find the option price with payoff $g(\cdot)$ and maturity $T > 0$

$$u(t, x, y) = \mathbb{E}\big[g(e^{X_T}) \mid X_t = x, Y_t = y\big].$$

Let $J = (0, T)$ and let G_R be a bounded domain

$$G_R = (-R_1, R_1) \times (0, R_2)$$

and consider the pricing equation for (5) in *time-to-maturity*

$$\begin{aligned} \partial_t u - \mathcal{A}u = 0 \quad &\text{in } J \times G_R, \\ u(0, x, y) = g(e^x) \quad &\text{in } G_R\,, \end{aligned}$$

where the generator of (5) is given by

$$\mathcal{A} = \frac{1}{2}y\partial_{xx} + \beta\rho y\partial_{xy} + \frac{1}{2}\beta^2 y\partial_{yy} - \frac{1}{2}y\partial_x - \alpha(y - \overline{m})\partial_y\,.$$

We enforce the following **mixed boundary conditions**:

$$u(t, -R_1, y) = 0 = u(t, R_1, y), \quad \text{for every } t \in J, y \in (0, R_2),$$

and

$$\partial_y u(t, x, 0) = 0 = \partial_y u(t, x, R_2), \quad \text{for every } t \in J, x \in (-R_1, R_1).$$

a) [2 pt] **Explain** the meaning of the parameters ρ and $\overline{m}$ in (5).

b) [4 pt] The variational formulation reads as follows: we do **not** introduce the decay factor $e^{-\kappa y^2}$ from the lecture, i.e.,

Find $u \in L^2(J; V) \cap H^1(J; V^*)$ such that $u(0, x, y) = g(e^x)$ and
$(\partial_t u, v) + a(u, v) = 0 \quad \forall v \in V,$

where $V = \{v \in H^1(G_R) : v|_{\{-R_1, R_1\}\times(0,R_2)} = 0\}$. **Derive** the bilinear form $a(u, v)$.

We discretize in time by the θ-scheme with time step $k = \frac{T}{M}$ and in space by using the finite element space

$$V_{N_x,N_y} = \text{span}\,\{b_i(x)b_j(y) \mid i = 1, \ldots, N_x;\ j = 0, \ldots, N_y + 1\}\,,$$

of continuous, piecewise bilinears with respect to the two-dimensional grid (x_i, y_j), $x_i = -R_1 + ih_x$, $y_j = jh_y$, with mesh width $h_x = \frac{2R_1}{N_x+1}$, $h_y = \frac{R_2}{N_y+1}$.

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c) [5 pt] The matrix formulation reads as

$$\begin{aligned}
&\text{Given } \underline{u}_N^0 \in \mathbb{R}^N, \text{ find } \underline{u}_N^m \in \mathbb{R}^N \text{ such for } m = 0, \dots, M-1 \\
&\frac{1}{k}\mathbf{M}\big(\underline{u}_N^{m+1} - \underline{u}_N^m\big) + \theta \mathbf{A} \underline{u}_N^{m+1} + (1-\theta)\mathbf{A}\underline{u}_N^m = 0\,, \\
&\text{where } \underline{u}_N^0 = \underline{g} \otimes \underline{1}.
\end{aligned} \tag{6}$$

State what $\mathbf{M}$ is in (6) and **state** N in terms of N_x and N_y. **Show** that the stiffness matrix has the tensor product structure

$$\begin{aligned}
\mathbf{A} = &\frac{1}{2}\mathbf{S}^1 \otimes \mathbf{M}^y - \beta\rho \mathbf{B}^1 \otimes \mathbf{B}^y + \frac{\beta^2}{2}\mathbf{M}^1 \otimes \mathbf{S}^y \\
&+ (\frac{\beta^2}{2} - \alpha\overline{m})\mathbf{M}^1 \otimes \mathbf{B}^2 + \frac{1}{2}\mathbf{B}^1 \otimes \mathbf{M}^y + \alpha \mathbf{M}^1 \otimes \mathbf{B}^y
\end{aligned}$$

where the one-dimensional matrices above are given by

$$\begin{aligned}
\{\mathbf{M}^1\}_{i',i} &= (b_i, b_{i'})_{L^2(-R_1,R_1)} & \{\mathbf{M}^y\}_{j',j} &= (y b_j, b_{j'})_{L^2(0,R_2)} \\
\{\mathbf{B}^1\}_{i',i} &= (b_i', b_{i'})_{L^2(-R_1,R_1)} & \{\mathbf{B}^y\}_{j',j} &= (y b_j', b_{j'})_{L^2(0,R_2)} \\
\{\mathbf{S}^1\}_{i',i} &= (b_i', b_{i'}')_{L^2(-R_1,R_1)} & \{\mathbf{S}^y\}_{j',j} &= (y b_j', b_{j'}')_{L^2(0,R_2)} \\
\{\mathbf{B}^2\}_{j',j} &= (b_j', b_{j'})_{L^2(0,R_2)} & &
\end{aligned}$$

d) [2 pt] **Show** that the stiffness matrix $\mathbf{A}$ can be simplified to

$$\mathbf{A} = \frac{1}{2}\mathbf{S}^1 \otimes \mathbf{M}^y + \mathbf{B}^1 \otimes \mathbf{Y}^{1,y} + \mathbf{M}^1 \otimes \mathbf{Y}^{2,y}\,,$$

and find the matrices $\mathbf{Y}^{1,y}$, $\mathbf{Y}^{2,y}$.

e) [12 pt] **Modify** the MATLAB routine `P3_main.m` to compute a European call option price with strike $K > 0$ in this stochastic volatility model. Choose $T = 0.5$, $K = 1$, $\rho = -0.5$, $\alpha = 1.5$, $\overline{m} = 0.06$, $\beta = 0.7$, $N_x = N_y = 51$, $M = 50$, and compare your solution with the exact solution obtained by `stochvol_exact.m`. To implement the weighted one-dimensional matrices you can either use the routine `stiff.m` or compute these by hand.