Exam: Solutions

1. a) Let $P : L^2(G) \to S^1_{\mathcal{T}}$ be the orthogonal projection onto $S^1_{\mathcal{T}}$. Specifically, for every $v \in L^2(G)$, we define $P(v) \in S^1_{\mathcal{T}}$ to be the unique element in $S^1_{\mathcal{T}}$ such that

$$(v, v_N)_{L^2(G)} = (P(v), v_N)_{L^2(G)}, \quad \forall v_N \in S^1_{\mathcal{T}}.$$

This can be restated as

$$\sum_{j=1}^{N} (v, b_j) v_N^j = \sum_{i,j=1}^{N} P(v)_i (b_i, b_j) v_N^j, \quad \forall (v_N^0, \ldots, v_N^{N+1})^\top \in \mathbb{R}^{N+2}.$$

We define the vectors $\underline{v}$ and $\underline{P(v)}$ and the matrix $\mathbf{M}$ by

$$\underline{v}_j := (v, b_j)_{L^2(G)}, \quad \forall j = 0, \ldots, N+1, \quad \underline{P(v)} := (P(v)_0, \ldots, P(v)_{N+1})^\top,$$

and

$$\mathbf{M}_{i,j} := (b_j, b_i)_{L^2(G)}, \quad \forall i, j = 0, \ldots, N+1.$$

The vector $\underline{P(v)}$ is the solution to the linear equation $\mathbf{M}\underline{P(v)} = \underline{v}$. Thus, the coefficients of $\underline{P(v)}$ are given by $\underline{P(v)}_i = (\mathbf{M}^{-1}\underline{v})_i$, $i = 0, \ldots, N+1$.

If $v = \sum_{i=0}^{N+1} v_i b_i \in S^1_{\mathcal{T}}$, where the coordinate vector is denoted by $\overline{v} \in \mathbb{R}^{N+2}$, i.e., $\overline{v}_i = v_i$ then $\underline{v} = \mathbf{M}\overline{v}$. Hence, $P(v) = v$ for every $v \in S^1_{\mathcal{T}}$.

$P : L^2(G) \to S^1_{\mathcal{T}}$ is well-defined, since the matrix $\mathbf{M}$ has full rank.

b) The θ-scheme is unconditionally stable for some $\theta \in [0, 1]$, if there exists a constant $C > 0$ such that for every number of FE degrees of freedom N, and for every number of timesteps M,

$$\|u_N^M\|_{L^2(G)} \leq C, \tag{1}$$

where G is an open interval.

The θ-scheme is conditionally stable for some $\theta \in [0, 1]$, if (1) only holds under a certain condition on the time steps and FE mesh width.

For $\theta \in [1/2, 1]$, the θ-scheme is unconditionally stable. For $\theta \in [0, 1/2)$, θ-scheme is conditionally stable.

Then, stability can be recovered provided there exists a constant $C > 0$ such that for time step size k and minimal FE meshwidth $h_{\min}$,

$$k \leq C \frac{h_{\min}^2}{1 - 2\theta}, \qquad 0 \leq \theta < \frac{1}{2}.$$

c) Assume there exists a constant $C > 0$ such that, for all $t \geq 0$,

$$\begin{aligned} |b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| &\leq C|x-y| \qquad && \forall x, y \in \mathbb{R}, \\ |b(t,x)| + |\sigma(t,x)| &\leq C(1+|x|) \qquad && \forall x \in \mathbb{R}. \end{aligned}$$

Then, for every $T \geq 0$, there exists a $\mathbb{P}$-a.s. unique solution on the time horizon $t \in [0,T]$.

Example satisfying these conditions: Black–Scholes model, where $b(t,s) = rs$ and $\sigma(t,s) = \sigma s$ are globally Lipschitz and of linear growth.

Example which does *not* fulfill these conditions: CEV model with $\varrho = 1/2$, where $b(t,s) = rs$ and $\sigma(t,s) = \sigma\sqrt{s}$. Note that $\sigma(t,s)$ is not Lipschitz (another possible popular example here CIR model, same argument).

d) i) In the case $\beta_+ > \beta_-$, the Lévy density has stronger tails for negative z. This means that downward jumps occur more frequently and more intense.

ii) The case $c = 0$ means that the Lévy measure is zero. This implies that the stiffness matrix is sparse, because the infinitesimal generator of X is just a differential operator of second order. Numerically, linear systems are fast to solve.
In the case $c > 0$ the infinitesimal generator of X has an integration part, which implies that the stiffness matrix has in general no sparsity. Numerically, this means that linear systems are expensive to solve, which limits the space discretization, where speed and needed memory to store the matrix becomes challenging.

iii) A possible modification could be

$$k(z) = c\left(\frac{e^{-\beta_+|z|}}{|z|^{1+\alpha}}\mathbb{1}_{\{z>0\}}\right).$$

iv) The value $\alpha = 2$ is not admissible, since

$$\int_{-1}^{1} z^2 k(z)dz \geq c\min\{e^{-\beta_+}, e^{-\beta_-}\}\int_{-1}^{1} z^{1-\alpha} = \infty.$$

But any Lévy measure satisfies that $\int_{-1}^{1} z^2\nu(dz) < \infty$.

e) The log-moneyness transformation yields that obtained values for spot and option prices can be multiplied by K to achieve the corresponding result for the desired strike $K \neq 1$.

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2. **a)** The price V satisfies the following PDE

$$\begin{aligned} \partial_t V + \mathcal{A}V - rV &= 0 && \text{in } [0,T) \times (0,\infty), \\ V(T,s) &= (s-K)_+ && \text{for every } s \in (0,\infty). \end{aligned}$$

where $\mathcal{A} = \frac{1}{2}\sigma^2 s^2 \partial_{ss} + (r-\delta)s\partial_s$, i.e.,

$$\begin{aligned} \partial_t V + \frac{1}{2}\sigma^2 s^2 \partial_{ss} V + (r-\delta)s\partial_s V - rV &= 0 && \text{in } [0,T) \times (0,\infty), \\ V(T,s) &= (s-K)_+ && \text{for every } s \in (0,\infty). \end{aligned}$$

b) Define $v(t,x) := V(T-t, e^x)$ for every $t \in [0,T]$ and every $x \in \mathbb{R}$. We obtain for every $(t,x) \in (0,T) \times \mathbb{R}$ that

$$\begin{aligned} \partial_t v(t,x) &= -\partial_t V(T-t, e^x), \\ \partial_x v(t,x) &= \partial_s V(T-t, e^x)e^x, \\ \partial_{xx} v(t,x) &= \partial_x(\partial_s V(T-t,e^x)e^x) = \partial_s V(T-t,e^x)e^x + \partial_{ss} V(T-t,e^x)e^{2x}. \end{aligned}$$

This implies that

$$s^2 \partial_{ss} V = \partial_{xx} v - \partial_x v.$$

Inserting into the result from the solution of **a)** implies that

$$\begin{aligned} \partial_t v - \frac{1}{2}\sigma^2 \partial_{xx} v + \left(\frac{1}{2}\sigma^2 - (r-\delta)\right)\partial_x v + rv &= 0 && \text{in } (0,T] \times (0,\infty), \\ v(0,x) &= (e^x - K)_+ && \text{for every } x \in \mathbb{R}. \end{aligned}$$

We define v_R to be solution of the following PDE on the truncated domain $(-R,R)$, where we assume that as minimal condition on R that $|\log(K)| < R$:

$$\begin{aligned} \partial_t v_R - \frac{1}{2}\sigma^2 \partial_{xx} v_R + \left(\frac{1}{2}\sigma^2 - (r-\delta)\right)\partial_x v_R + rv_R &= 0 && \text{in } (0,T] \times (-R,R), \\ v(0,x) &= (e^x - K)_+ && \forall x \in (-R,R), \\ v(t,\pm R) &= 0 && \forall t \in (0,T]. \end{aligned}$$

c) We have the following representation for u and v: $u(t,x) = \sum_{i=1}^N u_i(t)b_i(x)$ and $v(x) = \sum_{i=1}^N v_i b_i(x)$. Then, the discrete variational formulation is given for $\underline{\mathbf{u}}(t) =$

Please turn over.

$(u_1(t),\dots,u_N(t))^T$ and every $\underline{\mathbf{v}} = (v_1,\dots,v_N)^T \in \mathbb{R}^N$ by

$$\begin{aligned}
\sum_{i,j}(\partial_t u_i(t) b_i, v_j\, b_j)_{L^2(G)} + \frac{1}{2}\sigma^2 \sum_{i,j}(u_i(t) b_i', v_j\, b_j')_{L^2(G)} & \\
+ (\frac{\sigma^2}{2} - r)\sum_{i,j}(u_i(t) b_i', v_j\, b_j)_{L^2(G)} & \\
+ r\sum_{i,j}(u_i(t) b_i, v_j\, b_j)_{L^2(G)} &= 0 \\
\sum_{i,j}\partial_t u_i(t)(b_i\, b_j)_{L^2(G)} v_j + \frac{1}{2}\sigma^2 \sum_{i,j} u_i(t)(b_i'\, b_j')_{L^2(G)} v_j & \\
+ (\frac{\sigma^2}{2} - r)\sum_{i,j} u_i(t)(b_i'\, b_j)_{L^2(G)} v_j & \\
+ r\sum_{i,j} u_i(t)(b_i\, b_j)_{L^2(G)} v_j &= 0
\end{aligned}$$

Therefore, setting $\mathbf{M}_{i,j} = (b_j, b_i)_{L^2(G)}$ and $\mathbf{S}_{i,j} = (b_j', b_i')_{L^2(G)}$ and $\mathbf{B}_{i,j} = (b_j', b_i)_{L^2(G)}$ and $\mathbf{A} = \frac{1}{2}\sigma^2\mathbf{S} + (\frac{\sigma^2}{2} - r)\mathbf{B} + r\mathbf{M}$, we get

$$\mathbf{M}\partial_t\underline{\mathbf{u}}(t) + \mathbf{A}\underline{\mathbf{u}}(t) = 0.$$

We discretize in time with the θ-scheme and get ($\theta \in [0,1]$)

$$\mathbf{M}\frac{\underline{\mathbf{u}}^m - \underline{\mathbf{u}}^{m-1}}{k} + (\theta\underline{\mathbf{u}}^m + (1-\theta)\mathbf{A}\underline{\mathbf{u}}^{m-1}) = 0$$

which leads to

$$(\mathbf{M} + k\theta\mathbf{A})\underline{\mathbf{u}}^m = (\mathbf{M} - k(1-\theta)\mathbf{A})\underline{\mathbf{u}}^{m-1}.$$

Here, $\underline{\mathbf{u}}^m = \underline{\mathbf{u}}(t_m)$. So we have for the variational form in matrix formulation: Find $\underline{\mathbf{u}}^m \in \mathbb{R}^N$ such that for all $m = 1,\dots,M$

$$(\mathbf{M} + k\theta\mathbf{A})\underline{\mathbf{u}}^m = (\mathbf{M} - k(1-\theta)\mathbf{A})\underline{\mathbf{u}}^{m-1}$$

and $\underline{\mathbf{u}}^0 = \underline{\mathbf{u}}(0)$.

d) see code below

e) We assume that the solution u for a given constant interest rate is differentiable with respect to r and apply ∂_r to the pricing equation. Define $w := \partial_r v$. From Equation (2) on the Problem Sheet we obtain by formally differentiating that for all $u \in V$ and almost all $t \in (0,T)$,

$$\langle \partial_r\partial_t v, u\rangle_{V^*,V} + \frac{\sigma^2}{2}(\partial_r v', u') + (\frac{\sigma^2}{2} - r)(\partial_r v', u') - (v', u) + r(\partial_r v, u) + (v, u) = 0\ .$$

We emphasize that by *formally differentiating* we mean that we assume that taking derivatives with respect to r make sense, that changing the order of differentiating

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is allowed, and that the usual chain and product rule hold, where we only relied on the product rule. Hence,

$$\langle \partial_t w, u\rangle_{V^*\times V} + a^{\mathrm{BS}}(w,u) = -(v,u) + (v',u) \quad \forall w \in V.$$
$$w(0) = 0 \quad \text{in } L^2(G),$$

is a parabolic PDE for the sensitivity Rho that is denoted by $w = \partial_r v$. The homogeneous initial condition results from the independence of the payoff with respect to r.

f) We are given a FE space V_N with hat basis functions $\{b_i\}_{i=1,\dots,N}$. We reformulate the variational formulation using V_N instead of V. Since V_N is finite-dimensional and we are given a linear equation, it is sufficient to test with the given basis functions. We known from the lecture that the discretization of the equation using the θ scheme and the finite element space $S^1_{\mathcal{T},0}$ leads to the following linear system

$$\mathbf{M}\frac{\underline{w}^{m+1} - \underline{w}^m}{k} + \mathbf{A}(\theta \underline{w}^{m+1} + (1-\theta)\underline{w}^m) = \theta \underline{f}^{m+1} + (1-\theta)\underline{f}^m, \; m = 0,\dots,M-1,$$

where here for every $m = 0,\dots,M$ the coefficient vector $\underline{f}^m$ of the right hand side is given by

$$\underline{f}_j^m = -\sum_{i=1}^N \underline{v}_i^m(b_i,b_j) + \sum_{i=1}^N \underline{v}_i^m(b_i',b_j), \quad j = 1,\dots,N.$$

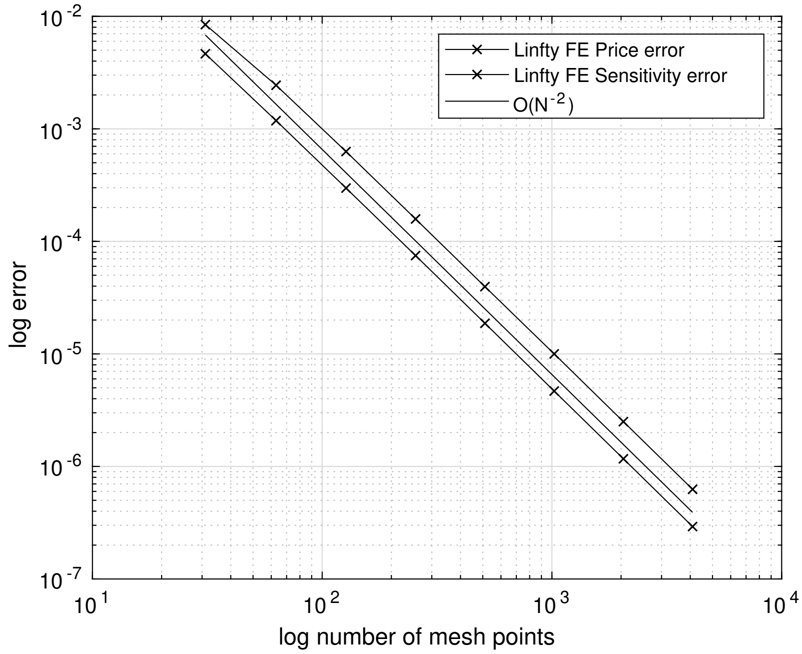
Thus,

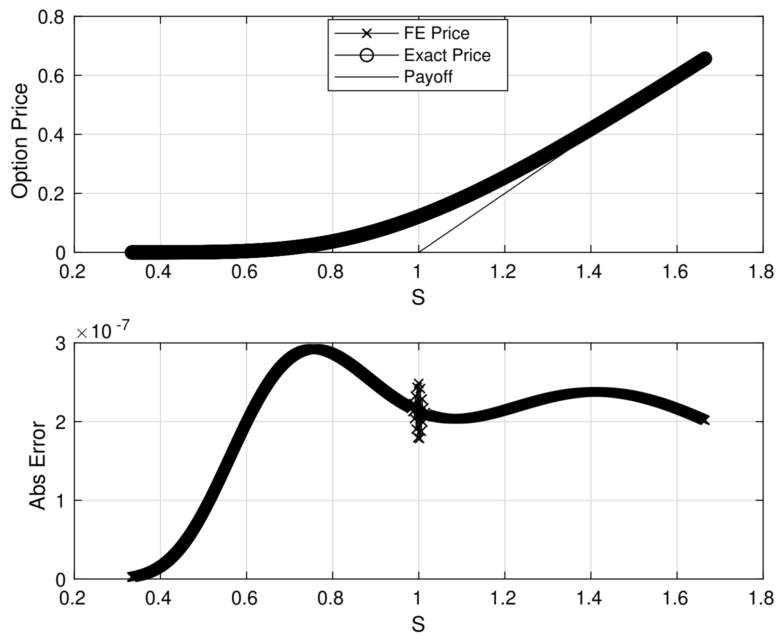
$$\underline{f}^m = \widetilde{\mathbf{A}}\underline{v}^m, \quad \text{for every } m = 0,\dots,M,$$

where $\widetilde{\mathbf{A}}_{ji} = -(b_i,b_j) + (b_i',b_j)$, $j,i = 1,\dots,N$, as defined on the problem sheet. Therefore, the desired equations follow.

g)
```
% MAIN solves the Black-Scholes PDE for a European call option with
% dividend yield delta

clear all;
close all;

%--------------------------------------------------------------------------
%  Set Parameters
%--------------------------------------------------------------------------

L = 5:12;                                    % Levels of discretization
N = 2.^(L')-1;                               % number of nodes
R = 4;                                       % domain (-R,R)

% Model Parameters
sigma = 0.3;
```

Please turn over.






```
r = 0.07;
delta = 0.05;

% Contract Parameters
K = 1;
T = 1;

% Numerical Scheme Parameters
```

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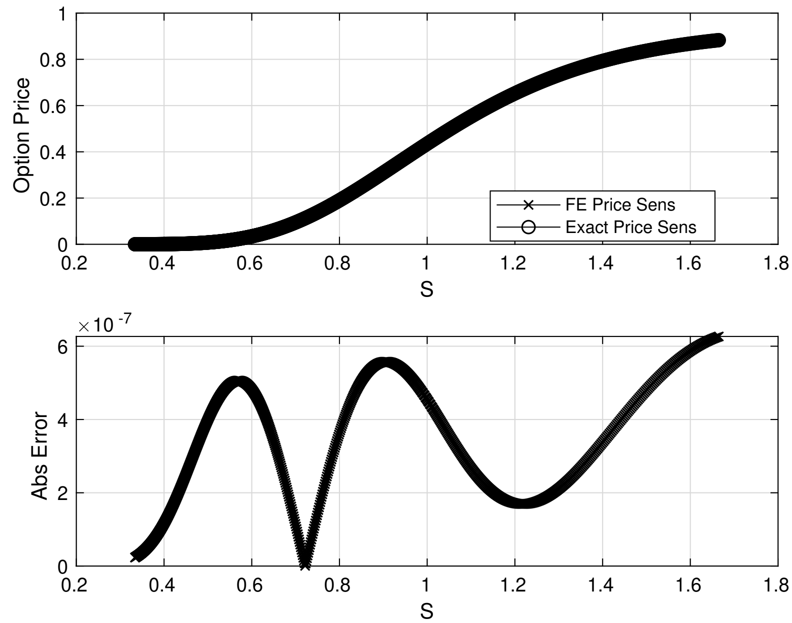


```
theta = 0.5;

% Function handles
u0_HANDLE    = @(x)(max(exp(x)-1,0));       % [1 pt]
SOL_HANDLE = @(t,s) bseucalldiv(s,t,K,r,sigma,delta); % [1 pt]

% coefficients of generator
alpha_HANDLE = @(x) sigma^2/2;
beta_HANDLE = @(x) sigma^2/2-(r-delta);
gamma_HANDLE = @(x) r;

% loop over mesh points
errorLinf = zeros(length(N),1);
errorLinf_sensitivity = zeros(length(N),1);
%
for i = 1:length(N)

    %-----------------------------------------------------------------------
    %  Discretization
    %-----------------------------------------------------------------------

    n = N(i);                           % number of inner spatial mesh nodes
    h = (2*R)/(n+1);                    % spatial mesh size
    x = linspace(-R,R,n+2)';            % spatial mesh nodes
    M = ceil(T/h);                      % number of time steps
    k = T/M;                           % time step
```

Please turn over.

```
%-----------------------------------------------------------------------
%  Compute stiffness matrix and load vector
%-----------------------------------------------------------------------

% Generator
A = stiff(x,alpha_HANDLE,beta_HANDLE,gamma_HANDLE);    % [2 pt]
Am = stiff(x,@(x) 0,@(x) 0,@(x) 1);         % [1 pt]

dof=2:n+1;                                            % [1 pt]

%-----------------------------------------------------------------------
%  Solver
%-----------------------------------------------------------------------

% prealocate memory to Dirichelt Data
u = zeros(n+2,M+1);     % [1 pt]

%initial solution vector
u(:,1) = u0_HANDLE(x);      % [1 pt]
%loop over timesteps
for m=1:M   % [1 pt together with the line to solve for one step ]

     B= Am+k*theta*A;   % [1 pt together with next line]
     C=  Am-k*(1-theta)*A;
     u(dof,m+1) = B(dof,dof)\(C(dof,dof)*u(dof,m));

end


% transformation from log-moneyness
u = K*u;
S = K*exp(x);

%domain of interest
I = logical((abs(exp(x) - 1) < 2/3)); % [1 pt]

%-----------------------------------------------------------------------
%  Error Price
%-----------------------------------------------------------------------

%L_infinity error
errorLinf(i) = max(abs(u(I,M+1) - SOL_HANDLE(T,S(I)))); % [1 pt]
```

Continued overleaf.

```
    %compute sensitivity

    SOL_HANDLE_SENS = @(t,s) bseurho(s,t,K,r,sigma,delta); % [1 pt]

    Ac = stiff(x,@(x) 0,@(x) 1,@(x) 0);     % [1 pt]
    w = zeros(n+2,1);       % [1 pt]

    % loop over time-steps
    for m=1:M % [4 pt for the whole loop]
    w(dof,1) = B(dof,dof) \ (C(dof,dof) * w(dof,1) ...
        - k * theta * Am(dof,dof) * u(dof,m+1) ...
        - k * (1-theta) * Am(dof,dof) * u(dof,m) ...
        + k * theta * Ac(dof,dof) * u(dof,m+1) ...
        + k * (1-theta) * Ac(dof,dof) * u(dof,m) );
    end
    % [1 pt for next line]
    errorLinf_sensitivity(i) = max(abs(w(I,1) - SOL_HANDLE_SENS(T,S(I))));


end

%--------------------------------------------------------------------------
%  Postprocessing
%--------------------------------------------------------------------------

% compute convergence rate
pLinf = polyfit(log(N),log(errorLinf),1);
fprintf('Price: Convergence rate in L infinity at maturity s = %2.1f\n',pLinf(1));
pLinf_sens = polyfit(log(N),log(errorLinf_sensitivity),1);
fprintf('Rho: Convergence rate in L infinity at maturity s = %2.1f\n',pLinf_sens(1));

% plot convergence rate
gcf1=figure(1); clf;
loglog(N,errorLinf,'bx-')
hold on;
loglog(N,errorLinf_sensitivity,'gx-')
hold on
loglog(N,1.5*exp(pLinf(2))*N.^-2,'k-')
grid on
xlabel('log number of mesh points')
ylabel('log error')
legend('Linfty FE Price error','Linfty FE Sensitivity error','O(N^-^2)')
```

Please turn over.

```
% plot option price
gcf2=figure(2); clf;
subplot(2,1,1)
plot(S(I),u(I,end),'-rx'); hold on
plot(S(I),SOL_HANDLE(T,S(I)),'-bo'); hold on
plot(S(I),max(S(I) - K,0),'-k'); hold on
legend('FE Price','Exact Price','Payoff','Location','Best')
xlabel('S')
ylabel('Option Price')
grid on
axis on
subplot(2,1,2)
plot(S(I),abs(u(I,end) - SOL_HANDLE(T,S(I))),'-rx'); hold on
xlabel('S')
ylabel('Abs Error')
grid on
axis on

% plot option price
gcf3=figure(3); clf;
subplot(2,1,1)
plot(S(I),w(I,end),'-rx'); hold on
plot(S(I),SOL_HANDLE_SENS(T,S(I)),'-bo'); hold on
legend('FE Price Sens','Exact Price Sens','Location','Best')
xlabel('S')
ylabel('Option Price')
grid on
axis on
subplot(2,1,2)
plot(S(I),abs(w(I,end) - SOL_HANDLE_SENS(T,S(I))),'-rx'); hold on
xlabel('S')
ylabel('Abs Error')
grid on
axis on


%% Save the plot (do not change) %%
saveas(gcf1, 'rate.eps', 'eps')
saveas(gcf2, 'price.eps', 'eps')
saveas(gcf3, 'greeks.eps', 'eps')
```

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3. **a)** By the multi-dimensional Itô formula we have

$$\begin{pmatrix} \mathrm{d}X_t \\ \mathrm{d}Y_t \end{pmatrix} = \begin{pmatrix} r - \frac{1}{2}\xi(Y_t)^2 \\ \alpha(m - Y_t) \end{pmatrix} + \begin{pmatrix} \xi(Y_t) & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \mathrm{d}W_t^1 \\ \mathrm{d}W_t^2 \end{pmatrix}$$

Thus, the generator is given by

$$\begin{aligned} \mathcal{A}^{\mathrm{MS}} &= \frac{1}{2}\operatorname{tr}\left(\begin{pmatrix} \xi(y) & 0 \\ 0 & \beta \end{pmatrix}^2 \begin{pmatrix} \partial_{xx} & \partial_{yx} \\ \partial_{xy} & \partial_{xx} \end{pmatrix}\right) + \begin{pmatrix} r - \frac{1}{2}\xi(y)^2 \\ \alpha(m-y) \end{pmatrix}^\top \nabla \\ &= \frac{1}{2}\xi(y)^2\partial_{xx} + \frac{1}{2}\beta^2\partial_{yy} + \left(r - \frac{1}{2}\xi(y)^2\right)\partial_x + \alpha(m-y)\partial_y. \end{aligned}$$

b) The bilinear form for $v(x,y) = b_i(x)b_j(y)$, $w(x,y) = b_{i'}(x)b_{j'}(y)$ is

$$\begin{aligned} a^{\mathrm{St}}(b_i(x)b_j(y), b_{i'}(x)b_{j'}(y)) = {} & \frac{1}{2}\int_{-R_1}^{R_1} b_i'(x)b_{i'}'(x)\mathrm{d}x \int_0^{R_2} y^2 b_j(y)b_{j'}(y)\mathrm{d}y \\ & + \frac{1}{2}\beta^2 \int_{-R_1}^{R_1} b_i(x)b_{i'}(x)\mathrm{d}x \int_0^{R_2} b_j'(y)b_{j'}'(y)\mathrm{d}y \\ & + \frac{1}{2}\int_{-R_1}^{R_1} b_i'(x)b_{i'}(x)\mathrm{d}x \int_0^{R_2} y^2 b_j(y)b_{j'}(y)\mathrm{d}y \\ & - \alpha m \int_{-R_1}^{R_1} b_i(x)b_{i'}(x)\mathrm{d}x \int_0^{R_2} b_j'(y)b_{j'}(y)\mathrm{d}y \\ & + \alpha \int_{-R_1}^{R_1} b_i(x)b_{i'}(x)\mathrm{d}x \int_0^{R_2} y b_j'(y)b_{j'}(y)\mathrm{d}y\,. \end{aligned}$$

Therefore, the stiffness matrix has the tensor product structure

$$\begin{aligned} \mathbf{A} &= \frac{1}{2}\mathbf{S}^1 \otimes \mathbf{M}^{y^2} + \frac{1}{2}\beta^2\mathbf{M}^1 \otimes \mathbf{S}^2 \\ &\quad + \frac{1}{2}\mathbf{B}^1 \otimes \mathbf{M}^{y^2} - \alpha m\mathbf{M}^1 \otimes \mathbf{B}^2 + \alpha\mathbf{M}^1 \otimes \mathbf{B}^y \\ &= \frac{1}{2}\left(\mathbf{S}^1 + \mathbf{B}^1\right) \otimes \mathbf{M}^{y^2} + \mathbf{M}^1 \otimes \left(\frac{1}{2}\beta^2\mathbf{S}^2 - \alpha m\mathbf{B}^2 + \alpha\mathbf{B}^y\right). \end{aligned}$$

c)
```
%  MAIN_P3 computes European call wit the Stein-Stein model using finite
%          elements
%

clear all;
close all;

% -------------------------------------------------------------------------
%  Set parameters
% -------------------------------------------------------------------------
```

Please turn over.

```
Nx = 51;                % number of nodes in x-ccordinate
Ny = 51;                % number of nodes in y-ccordinate
M = 50;                 % number of time steps
R_1 = 4;                % domain (-R_1,R_1)
R_2 = 3;              % domain (0,R_2)
T = 1;                % maturity
K = 1;                  % strike

%%%%
%%%% !!! NOTE: On the exam problem sheet it says
%%%% !!! alpha = 2.5 and beta = 0.5 instead
%%%% !!! Please mark both sets of parameters as correct.
%%%%

alpha = 1.5;            % rate of mean reversion
mt = 0.06;              % level of mean reversion
beta = 0.7;             % volatility of volatility

% ---------------------------------------------------------------------------
%  Discretization
% ---------------------------------------------------------------------------
% mesh size in x-coordinate
hx = (2*R_1)/(Nx+1);
% mesh size in y-coordinate
hy = (R_2)/(Ny+1);
% mesh nodes in x-coordinate
x = linspace(-R_1,R_1,Nx+2)';
% mesh nodes in y-coordinate
y = linspace(0,R_2,Ny+2)';
% time steps
k = T/M;

% ---------------------------------------------------------------------------
%  Compute Generator/ Source Term/ Initial Data
% ---------------------------------------------------------------------------

%compute system matrices x-coordinate [3 pt in total, 1 pt each]
M1 = stiff(x,@(x) 0, @(x) 0, @(x) 1);
B1 = stiff(x,@(x) 0, @(x) 1, @(x) 0);
S1 = stiff(x,@(x) 1, @(x) 0, @(x) 0);

%compute system matrices y-coordinate [5 pt in total, 1 pt each]
S2  = stiff(y,@(y) 1, @(y) 0, @(y) 0);
B2  = stiff(y,@(y) 0, @(y) 1, @(y) 0);
M2  = stiff(y,@(y) 0, @(y) 0, @(y) 1);
```

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```
By  = stiff(y,@(y) 0, @(y) y, @(y) 0);
My2 = stiff(y,@(y) 0, @(y) 0, @(y) y.^2);

% incorporate boundary conditions
dofx = 2:Nx+1;          % [1 pt]
dofy = 1:Ny+2;
M1 = M1(dofx, dofx);    % [1 pt for these three lines]
B1 = B1(dofx, dofx);
S1 = S1(dofx, dofx);

% define matrices Y1, Y2 [points together with points for A_full]
Y1 = 1/2 * (S1 + B1);
Y2 = 1/2*beta^2*S2 - alpha*mt*B2 + alpha*By;

% tensor product
M_full = kron(M1,M2);                % [1 pt]
A_full = kron(Y1,My2)+kron(M1,Y2);  % [1 pt]

% initial data
u0x = max(0,exp(x(2:end-1))-K);
u0y = ones(length(y),1);
u0 = kron(u0x,u0y);                  % [1 pt for these 3 lines]

% -------------------------------------------------------------------------
%  Solver
% -------------------------------------------------------------------------

theta = 0.5;
B = M_full+k*theta*A_full;
C = M_full-(1-theta)*k*A_full;

% lopp over time points
u = u0;
for m = 1:M
    u = B\(C*u);
end

% -------------------------------------------------------------------------
%  Postprocessing
% -------------------------------------------------------------------------

% area of interest
idxd = find(x <= -1,1,'last');
idxu = find(x >= 1,1);
idyd = find(y <= 0.1,1,'last');
```

Please turn over.

```
idyu = find(y >= 1.2,1);

% plot option price
u = reshape(u,Ny+2,Nx);
u = [zeros(Ny+2,1),u,zeros(Ny+2,1)];
u = u(idyd:idyu,idxd:idxu);
u0 = reshape(u0,Ny+2,Nx);
u0 = [zeros(Ny+2,1),u0,zeros(Ny+2,1)];
u0 = u0(idyd:idyu,idxd:idxu);
S = exp(x(idxd:idxu)); y = y(idyd:idyu);
[X,Y] = meshgrid(S,y);

gcf1=figure(1); clf;
surf(X,Y,u),
hold on
mesh(X,Y,u0)
title('European Call option in Stein-Stein model')
xlabel('S'), ylabel('y'), zlabel('u')

%% Save the plot (do not change) %%
saveas(gcf1, 'option_price.eps', 'eps')
```

d) Let $v, w \in C^\infty(G_R)$ with $v = w = 0$ on Γ_{D}. Then, by the Cauchy–Schwarz inequality

$$
\begin{aligned}
\left|a^{\mathrm{St}}(v,w)\right| &\le \frac{1}{2}\|y\partial_x v\|_{L^2(G_R)}\|y\partial_x w\|_{L^2(G_R)} + \frac{1}{2}\beta^2\|\partial_y v\|_{L^2(G_R)}\|\partial_y w\|_{L^2(G_R)} \\
&\quad + \frac{1}{2}\|y\partial_x v\|_{L^2(G_R)}\|yw\|_{L^2(G_R)} + \alpha m\|\partial_y v\|_{L^2(G_R)}\|w\|_{L^2(G_R)} \\
&\quad + \alpha\|\partial_y v\|_{L^2(G_R)}\|yw\|_{L^2(G_R)} \\
&\le \frac{1}{2}\|v\|_V\|w\|_V + \frac{1}{2}\beta^2\|v\|_V\|w\|_V + \frac{1}{2}R_2\|y\partial_x v\|_{L^2(G_R)}\|w\|_{L^2(G_R)} \\
&\quad + \alpha m\|v\|_V\|w\|_V + \alpha R_2\|\partial_y v\|_{L^2(G_R)}\|w\|_{L^2(G_R)} \\
&\le \frac{1}{2}\left(1 + \beta^2 + R_2 + 2\alpha m + 2\alpha R_2\right)\|v\|_V\|w\|_V,
\end{aligned}
$$

and continuity follows. Furthermore, we find

$$
\begin{aligned}
a^{\mathrm{St}}(v,v) = \frac{1}{2}\|y\partial_x v\|^2_{L^2(G_R)} + \frac{1}{2}\beta^2\|\partial_y v\|^2_{L^2(G_R)} + \frac{1}{2}(y^2\partial_x v, v)_{L^2(G_R)} \\
- \alpha m(\partial_y v, v)_{L^2(G_R)} + \alpha(y\partial_y v, v)_{L^2(G_R)}
\end{aligned}
$$

Continued overleaf.

We first note that $(y^2\partial_x v, v)_{L^2(G_R)} = 0$ since

$$
\begin{aligned}
2(y^2\partial_x v, v)_{L^2(G_R)} = \left(y^2, \partial_x\left(v^2\right)\right)_{L^2(G_R)} &= \int_0^{R_2} y^2 \int_{-R_1}^{R_1} \partial_x\left(v(x,y)^2\right)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{R_2} y^2\left(v(R_1,y)^2 - v(-R_1,y)^2\right)\,\mathrm{d}y = 0.
\end{aligned}
$$

Furthermore, by the weighted Young inequality, for any $\varepsilon, > 0$, we have

$$
\begin{aligned}
-\alpha m(\partial_y v, v)_{L^2(G_R)} &\geq -\varepsilon\|\partial_y v\|^2_{L^2(G_R)} - \frac{\alpha^2 m^2}{4\varepsilon}\|v\|^2_{L^2(G_R)} \\
\alpha(y\partial_y v, v)_{L^2(G_R)} &\geq -\alpha R_2\|\partial_y v\|_{L^2(G_R)}\|v\|_{L^2(G_R)} \\
&\geq -\varepsilon\|\partial_y v\|^2_{L^2(G_R)} - \frac{\alpha^2 R_2^2}{4\varepsilon}\|v\|^2_{L^2(G_R)}.
\end{aligned}
$$

We choose $\varepsilon = \beta^2/8 > 0$ and set $C_2 := \min\{1/2, \beta^2/4\} > 0$. Thus,

$$
\begin{aligned}
a^{\mathrm{St}}(v,v) &\geq C_2\left(\|y\partial_x v\|^2_{L^2(G_R)} + \|\partial_y v\|^2_{L^2(G_R)}\right) - 2\alpha^2\beta^{-2}\left(m^2 + R_2^2\right)\|v\|^2_{L^2(G_R)} \\
&\geq C_2\|v\|_V^2 - C_3\|v\|^2_{L^2(G_R)},
\end{aligned}
$$

where $C_3 := C_2 + 2\alpha^2\beta^{-2}\left(m^2 + R_2^2\right)$.