Series 9

1. Localization for barrier options

a) Let $r \geq 0$ be the constant interest rate and let $\tau_B = \inf\{t \geq 0 \mid S_t = B\}$ be the first hitting time of B by the process S_t, or equivalently in log-price, the first hitting time of $\log(B)$ by the process $X_t = \log(S_t)$. In log-price, the value of a down-and-out option V is then given by

$$v_{do}(t,x) = \mathbb{E}\left[e^{r(t-T)} g(e^{X_T}) 1_{\{T<\tau_B\}} \mid X_t = x\right].$$

As for plain vanilla options, we can localize the problem to a bounded domain $G = (\log(B), R)$. Suppose the payoff function $g\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfies a *polynomial growth condition*: There exist $C_1 > 0, q \geq 1$ such that

$$g(s) \leq C_1(s+1)^q\,, \quad \text{for all } s \in \mathbb{R}_+.$$

Show that there exists $C(T,\sigma,r), \gamma_1, \gamma_2 > 0$, such that

$$|v_{do}(t,x) - v_R(t,x)| \leq C(T,\sigma,r) e^{-\gamma_1 R + \gamma_2 |x|}.$$

Hint: Follow the proof of Theorem 4.3.1 in the textbook.

2. Barrier options in the Black–Scholes market

Consider the European up-and-out and up-and-in barrier options in the Black–Scholes market,

$$V_{\mathrm{uo}}^{\mathrm{Eur}}(t,s) = \mathbb{E}\big[e^{-r(T-t)} g(S_T) 1_{\{T\leq\tau_B\}} \mid S_t = s\big], \tag{1}$$
$$V_{\mathrm{ui}}^{\mathrm{Eur}}(t,s) = \mathbb{E}\big[e^{-r(T-t)} g(S_T) 1_{\{T>\tau_B\}} \mid S_t = s\big], \tag{2}$$

respectively, with $g(s) = (s-K)_+$ and $\tau_B = \inf\{t \geq 0 \mid S_t = B\}$ being the first hitting time of B by the process S. Switching to time-to-maturity and log-moneyness and truncating the domain to $G = \left(-R, \log\left(\frac{B}{K}\right)\right)$ we obtain the following PDE for $v(t,x) = \frac{1}{K} V_{\mathrm{uo}}^{\mathrm{Eu}}(T-t, Ke^x)$

$$\begin{aligned} \partial_t v - \frac{1}{2}\sigma^2 \partial_{xx} v - \big(r - \frac{1}{2}\sigma^2\big)\partial_x v + rv = &\quad 0, \quad \text{in } J \times G \\ v(0,x) = &\quad \tilde{g}(x) \quad \text{in } G \\ v(t,-R) = v(t, \log(B/K)) = &\quad 0, \quad \text{in } J \end{aligned} \tag{3}$$

where $J = (0,T]$ and $\tilde{g}(x) = (e^x - 1)_+$.

a) Prove the following in-out parity for the European barrier options,

$$V^{\mathrm{Eur}}(t,s) = V_{\mathrm{ui}}^{\mathrm{Eur}}(t,s) + V_{\mathrm{uo}}^{\mathrm{Eur}}(t,s), \tag{4}$$

where V^{Eur} is the value of the plain vanilla European option.

b) State the variational formulation of the system (3) in log-moneyness.

c) We discretize in space using a uniform mesh, $-R = x_0 < x_1 < \cdots < x_N < x_{N+1} = \log(\frac{B}{K})$, mesh width $h := \frac{R+\log(B/K)}{N+1}$. We also discretize in time with time points $t_m := mk$, $m = 0,\ldots,M$, and time step $k := \frac{T}{M}$. The ϑ-scheme is used to discretize the time variable and Finite Elements is used to discretize the space variable. The used Finite Element space is spanned by the continuous hat-functions. Derive the matrix formulation and state precisely all parts of it.

d) Modify the template `bs_barrier.py` to compute the fair price of the European up-and-in barrier option. Use the parameters $K = 60$, $T = 1$, $B = 80$, $\sigma = 0.3$, $r = 0.01$, $\vartheta = 0.5$ and plot the price on the area of interest $\tilde{G} = \{x \in G \mid |x| \leq .75\}$ in spot price for $N = 2^8 - 1, M = 2^8$.

3. Caplet in the CIR model

We consider the CIR model for the interest rate r_t, i.e. the short rate process satisfies the following SDE

$$dr_t = (\alpha - \beta r_t)dt + \sigma\sqrt{r_t}dW_t, \quad r_0 = \tilde{r} > 0,$$

where α, β, $\sigma > 0$ and W is a one-dimensional standard Brownian motion and $\alpha \geq \sigma^2$.

Fix $T_1 > 0$. The price $B(t, T_1, r)$ of a zero coupon bond of maturity T_1, at time $t < T_1$ and interest rate r at time t is defined by

$$B(t, T_1, r) \coloneqq \mathbb{E}\left[e^{-\int_t^{T_1} r_s\, ds} \mid r_t = r\right] .$$

Let $V_0(t, r) = B(t, T_1, r)$. Then V_0 solves the PDE

$$\begin{cases} \partial_t V_0 + \frac{1}{2}\sigma^2 r \partial_{rr} V_0 + (\alpha - \beta r)\partial_r V_0 - rV_0 &= 0 \qquad \text{in } (0, T_1) \times G\,, \\ V_0 &= 0 \qquad \text{in } (0, T_1) \times \{R\}\,, \\ V_0(T_1, \cdot) &= g_0 \qquad \text{in } G\,, \end{cases} \tag{5}$$

where $G \coloneqq (0, R)$ for some $R > 0$ and $g_0 \equiv 1$ is the constant function equal to 1 on G.

Let $\mathcal{H}_\mu \coloneqq L^2(G; r^{2\mu}\, dr)$ be the Hilbert space with inner-product

$$(w, v)_\mu \coloneqq \int_G w(r)v(r)r^{2\mu}\, dr\,,$$

and let $W_{1/2,\mu} \coloneqq \overline{C_0^\infty(G)}^{\|\cdot\|_{1/2,\mu}}$ which is the completion of $C_0^\infty(G)$ with respect to the norm

$$\|w\|_{1/2,\mu}^2 \coloneqq \int_G r^{1+2\mu}|\partial_r w|^2 + r^{2\mu}|w|^2\, dr\,, \quad w \in W_{1/2,\mu}\,.$$

Let $J_0 \coloneqq (0, T_1)$ and $\mu \in (-1/2, 0)$. Then the value $u_0(t, r) = V_0(T_1 - t, r)$ satisfies a variational problem of the form

$$\text{Find } u_0 \in L^2(J_0; W_{1/2,\mu}) \cap H^1(J_0; \mathcal{H}_\mu) \text{ such that, for all } v \in W_{1/2,\mu}\,,$$

$$\begin{cases} (\partial_t u_0, v)_\mu + a_{1/2,\mu}^{\mathrm{CIR}}(u_0, v) &= 0 \qquad \text{a.e. in } J_0\,, \\ u_0(0, \cdot) &= g_0 \qquad \text{a.e. in } G\,, \end{cases} \tag{6}$$

Let L be the simply compounded interest rate, i.e.

$$\forall 0 < t < T_1\,, \quad L(t, T_1, r) = \frac{1 - B(t, T_1, r)}{(T_1 - t)B(t, T_1, r)}\,.$$

A caplet is an option on the simply compounded interest rate. Its value can be expressed as

$$V_1(t, r) = \mathbb{E}\left[e^{-\int_t^{T_1} r_s\, ds}(T_1 - T)\left(L(T, T_1, r_T) - K\right)_+ \mid r_t = r\right]\,,$$

where $T < T_1$ is fixed, $t < T$, and $K > 0$ is a strike interest rate. In this formula, we have denoted $x_+ \coloneqq \max(x, 0)$.

a) Derive the bilinear form $a^{\mathrm{CIR}}_{1/2,\mu}$ in the variational problem (6). Put it in the form

$$a^{\mathrm{CIR}}_{1/2,\mu}(\phi,\psi) = (a(\cdot)\phi',\psi')_{L^2(G)} + (b(\cdot)\phi',\psi)_{L^2(G)} + (c(\cdot)\phi,\psi)_{L^2(G)}$$

for real valued functions $a, b, c : G \to \mathbb{R}$.

b) Derive the function $\tilde{g}_1$ such that

$$V_1(t,r) = \mathbb{E}\left[e^{-\int_t^{T_1} r_s\,ds}\tilde{g}_1(V_0(T,r_T)) \mid r_t = r\right] .$$

c) Let $\mathcal{G}_t = \sigma(r_s : s \le t)$. Using the tower property of conditional expectations, one can write

$$V_1(t,r) = \mathbb{E}\left[\mathbb{E}\left[e^{-\int_t^{T_1} r_s\,ds}\tilde{g}_1(V_0(T,r_T)) \mid \mathcal{G}_T\right] \mid r_t = r\right] .$$

Deduce that

$$V_1(t,r) = \mathbb{E}\left[e^{-\int_t^{T} r_s\,ds}g_1(V_0(T,r_T)) \mid r_t = r\right] ,$$

where $g_1(x) = x\tilde{g}_1(x)$.

d) Deduce a variational problem satisfied by $u_1(t,r) = V_1(T-t,r)$.

Due: Wednesday, May 8th, at 2pm.