Exam: Solutions

1. **a)** A sufficient condition is that b and σ are uniformly Lipschitz functions on $\mathbb{R}$ and that they satisfy a linear growth condition. More precisely, there exists a constant $C > 0$ such that

- for all $t \in \mathbb{R}^+$ and $(x, y) \in \mathbb{R}^2$,

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq C|x - y|\,,$$

- for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$,

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|)\,.$$

In the Black-Scholes model, we have $b(t,s) = rs$ where $r \in \mathbb{R}$ and $\sigma(t,s) = \sigma s$ for some $\sigma > 0$. Those functions satisfy the hypothesis with $C = \sigma + |r|$. In the Constant Elasticity of Variance model, we have $\sigma(t,s) = \sigma s^{\varrho}$ for $\varrho \in (0,1)$. This function is not uniformly Lipschitz, since it has an unbounded derivative at $s = 0$ (while differentiable uniformly Lipschitz functions have bounded derivatives).

b) Applying $v'(t)$ to some $\varphi \in \mathcal{V}$, we thus get, for almost all $t \in J$,

$$\begin{aligned}\langle \frac{dv}{dt}, \varphi \rangle_{\mathcal{V}^*,\mathcal{V}} &= -\lambda \langle v, \varphi \rangle_{\mathcal{V}^*,\mathcal{V}} + e^{-\lambda t} \langle \frac{du}{dt}, \varphi \rangle_{\mathcal{V}^*,\mathcal{V}} \\ &= -\lambda (v, \varphi)_{\mathcal{H}} - e^{-\lambda t} a(u, \varphi)\,,\end{aligned}$$

i.e.

$$\langle \frac{dv}{dt}, \varphi \rangle_{\mathcal{V}^*,\mathcal{V}} + \tilde{a}(v, \varphi) = 0\,,$$

where $\tilde{a}$ is defined by

$$\tilde{a}(\varphi, \psi) := a(\varphi, \psi) + \lambda(\varphi, \psi)_{\mathcal{H}}\,.$$

We deduce that v satisfies the variational problem:

Find $v \in L^2(J; \mathcal{V}) \cap H^1(J; \mathcal{V}^*)$ such that, for all $\varphi \in \mathcal{V}$,

$$\begin{aligned}\frac{d}{dt} \langle v, \varphi \rangle_{\mathcal{V}^*,\mathcal{V}} + \tilde{a}(u, \varphi) = 0\,, &\quad \text{a. e. in } J \\ v(0) = u_0\,.&\end{aligned}$$

That the bilinear form a satisfies a Gårding inequality means that there exists $C_1 > 0$ and $C_2 \geq 0$ such that

$$\forall \varphi \in \mathcal{V}\,, \quad a(\varphi, \varphi) \geq C_1 \|\varphi\|_{\mathcal{V}}^2 - C_2 \|\varphi\|_{\mathcal{H}}^2\,.$$

One can write

$$\tilde{a}(\varphi, \varphi) = a(\varphi, \varphi) + \lambda \|\varphi\|_{\mathcal{H}}^2 \geq C_1 \|\varphi\|_{\mathcal{V}}^2 + (\lambda - C_2) \|\varphi\|_{\mathcal{H}}^2$$

hence, for $\lambda \geq C_2$, $\tilde{a}$ is strongly coercive.

c) For all $\varphi, \psi \in H^1(G)$, we have by definition

$$a^{\text{BS}}(\varphi, \psi) = \int_{-R}^{R} \frac{1}{2}\sigma^2 \varphi' \psi' + \left(\frac{1}{2}\sigma^2 - r\right) \varphi \psi' + r\varphi\psi \, dx \, .$$

We apply this equation for $\psi = g$ (which is indeed in $H^1(G)$), and $\varphi \in C_0^1(G)$. Since $g(x) = (K - e^x)_+$, its weak derivative g' is given by

$$g'(x) = \begin{cases} -e^x & \text{if } x < \ln K \, , \\ 0 & \text{otherwise.} \end{cases}$$

We can assume that $\ln K \in (-R, R)$ since otherwise, $g \equiv 0$ and the result is trivial. In this case, we have

$$\begin{aligned} a^{\text{BS}}(\varphi, g) &= \int_{-R}^{\ln K} \frac{1}{2}\sigma^2 (-e^x)\varphi' + \left(\frac{1}{2}\sigma^2 - r\right)(-e^x)\varphi + r(K - e^x)\varphi \, dx \\ &= \int_{-R}^{\ln K} rK\varphi - \frac{1}{2}\sigma^2 e^x \left(\varphi' + \varphi\right) \, dx. \end{aligned}$$

By integration by parts, we have

$$\int_{-R}^{\ln K} \varphi' e^x \, dx = -\int_{-R}^{\ln K} \varphi e^x + \varphi(\ln K) e^{\ln K} \, dx = -\int_{-R}^{\ln K} \varphi e^x + K\varphi(\ln K) \, dx.$$

Thus,

$$a^{\text{BS}}(\varphi, g) = -\frac{1}{2}\sigma^2 K \varphi(\ln K) + rK \int_{-R}^{\ln K} \varphi \, dx.$$

d) The short rate is modeled by the equation

$$dr_t = b(t, r_t)dt + \sigma(t, r_t) dW_t \, , \quad r_0 = r$$

where

- $b(t, r) = \alpha - \beta r$, where $\alpha, \beta > 0$
- $\sigma(t, r) = \sigma$ for the Vasicek model and $\sigma(t, r) = \sigma\sqrt{r}$ for the CIR model, where $\sigma > 0$ is a constant.

In the CIR model, r_t remains non-negative at all times, while r_t can be negative in the Vasicek model.

e) The infinitesimal generator $\mathcal{A}$ is defined for $f \in C^2(G)$ with bounded derivatives by

$$\mathcal{A}f = \frac{1}{2}\text{tr}[\mathcal{Q}(x) D^2 f(x)] + b(x)^T \nabla f(x) \, ,$$

where $\mathcal{Q}(x) := \Sigma(x)\Sigma(x)^T$ is the covariance matrix of the process X. The B-S model corresponds to

$$\Sigma_{ij}(s) = \boldsymbol{\Sigma}_{ij} s_i$$

for some consant matrix $\boldsymbol{\Sigma}$, and $b_i(s) = rs_i$ where $r \in \mathbb{R}$ is the riskless interest rate.

f) The bilinear form reads

$$a^{\mathrm{BS}}(\varphi,\psi) = \int_{\mathbb{R}^d} \frac{1}{2}\nabla\varphi \cdot (\mathcal{Q}\nabla\psi) + (\mu^T\nabla\varphi)\psi + r\varphi\psi \, dx \, .$$

A sufficient condition such that a satisfies a Gårding inequality is that $\mathcal{Q}$ be positive definite, i.e. there exists a constant $\alpha > 0$ such that

$$\forall x \in \mathbb{R}^d \, , \quad x^T \mathcal{Q} x > \alpha \, x^T x \, .$$

g) Let $V : [0,T] \times \mathbb{R}_+$ denote the value of the American option. The smooth pasting condition states that the mapping $t \mapsto V(t, s^*(t))$ is continuously differentiable, where s^* is the exercise boundary. This holds for Lévy models if $\sigma^2 > 0$, so in particular in the Black-Scholes case.

h) Advantages (one of the following is sufficient):

- The Galerkin matrices are expressed only in terms of the Galerkin matrices of 1d problems.
- Generic formulas for arbitrary dimension d (while it is non-trivial to consider simplicial meshes of arbitrary space dimensions).

Limitations:

- Already for $d = 2$, the formulas for stochastic volatility models are quite cumbersome (more so than the FEM on triangular meshes)
- Can't handle non-cubic geometries (e.g. knockout barrier options on non square domains),
- In stochastic volatility models, need to assume tensorized model parameters b and Σ,

2. **a)** [3 pts.] Since we assume a Black-Scholes market, the price process S of the asset underlying the European option satisfies the SDE

$$dS_t = rS_t dt + \sigma S_t dW_t\,,$$

where r is the risk-free interest rate and $\sigma > 0$ is the volatility. Hence, the infinitesimal generator $\mathcal{A}$ takes the form

$$\mathcal{A} = \frac{1}{2}\sigma^2 s^2 \partial_{ss} + rs\partial_s\,.$$

The value V_1 is by definition,

$$V_1(t,s) = \mathbb{E}\left[e^{-r(T_1-t)} g_1(S_{T_1})\ \middle| S_t = s\right]\,.$$

Since V_1 is regular and g_1 is a Lipschitz function, we can apply the Feynman-Kac Theorem, which tells that V_1 satisfies the PDE

$$\begin{cases} \partial_t V_1 + \mathcal{A}V_1 - rV_1 &= 0 \qquad \text{in } (0,T)\times\mathbb{R}^+\,,\\ V_1(T,s) &= g_1(s) \qquad \text{for } s\in\mathbb{R}^+\,, \end{cases}$$

where $\mathcal{A}$ is defined, for $f \in C^2(\mathbb{R}^+)$ by

$$\mathcal{A}f = \frac{1}{2}\sigma^2 s^2 \partial_{ss} f + rs\partial_s f\,.$$

Let $h : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by

$$h(s) := g(V_1(T_c, s))\,.$$

By definition V_c satisfies

$$V_c(t,s) = \mathbb{E}\left[e^{-r(T-t)} h(S_T)\ \middle|\ S_t = s\right]\,.$$

One has

$$\begin{aligned} |h(s_1) - h(s_2)| &\le \kappa_{g_c} |V_1(T,s_1) - V_1(T,s_2)| \\ &\le \kappa_{g_c} \max_{s\in G} |\partial_s V_1(T,s)|(s_1 - s_2)\,, \end{aligned}$$

since by assumption, $\partial_s V_1$ is bounded on G. Here, κ_{g_c} is the Lipschitz constant of g_c. Hence h is Lipschitz with constant

$$\kappa_h \le \kappa_{g_c} \max_{s\in G} |\partial_s V_1(T,s)|$$

and by the Feynman-Kac theorem, we conclude that

$$\begin{cases} \partial_t V_c + \mathcal{A}V_c - rV_c &= 0 \qquad \text{in } (0,T_c)\times\mathbb{R}^+\,,\\ V_c(T_c,s) &= h(s) \qquad \text{for } s\in\mathbb{R}^+\,. \end{cases}$$

b) [2 pts.]

Let $v(t,x) = V(T-t, e^x)$ such that $x = \log(s)$. Then

$$\begin{aligned}
\partial_x v(t,x) &= [\partial_s V](T-t, e^x)e^x\,, \\
\partial_{xx} v(t,x) &= [\partial_s V](T-t, e^x)e^x + e^{2x}[\partial_{ss} V](T-t, e^x), \\
\partial_t v(t,x) &= -(\partial_t V)(T-t, e^x),
\end{aligned}$$

hence

$$\begin{aligned}
&[\partial_t V](T-t,e^x) + (\mathcal{A}V)(T-t,e^x) - rV(T-t,e^x) = \\
&\qquad -\partial_t v(t,x) + \frac{1}{2}\sigma^2 \partial_{xx} v(t,x) + \left(r - \frac{1}{2}\sigma^2\right)\partial_x v(t,x) - rv(t,x)\,.
\end{aligned}$$

Let $\mathcal{A}^{\mathrm{BS}} := \frac{1}{2}\sigma^2\partial_{xx} + \left(r - \frac{1}{2}\sigma^2\right)\partial_x$. Then we deduce that

$$\left\{\begin{aligned}
\partial_t v_1 - \mathcal{A}^{\mathrm{BS}} v_1 + r v_1 &= 0 && \text{in } (0,T_1)\times\mathbb{R}\,, \\
v_1(0,x) &= g_1(e^x) && \text{for } x\in\mathbb{R}\,,
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\partial_t v_c - \mathcal{A}^{\mathrm{BS}} v_c + r v_c &= 0 && \text{in } (0,T)\times\mathbb{R}\,, \\
v_c(0,x) &= h(e^x) && \text{for } x\in\mathbb{R}\,.
\end{aligned}\right.$$

There holds, for all $(t,x)\in(0,T_1)\times\mathbb{R}$,

$$V_1(t,e^x) = v_1(T_1 - t, x)$$

thus

$$h(e^x) = g_c(V_1(T,e^x)) = g_c(v_1(T_1 - T, x))\,.$$

c) [1 pts.]

Let $\tau_G := \inf_{t\geq 0}\{S_t \notin G\}$. Then, $v_{1,R}$ is the price of a knockout barrier option given by

$$v_{1,R}(t,x) = \mathbb{E}\left[e^{-r(T_1-t)} g_1(\exp(X_{T_1}))\mathbb{1}_{T<\tau_G}\,\middle|\,X_t = x\right]\,,$$

where $X_t = \ln(S_t)$, and similarly,

$$v_R(t,x) = \mathbb{E}\left[e^{-r(T-t)} g(v_{1,R}(T,X_T))\mathbb{1}_{T<\tau_G}\,\middle|\,X_t = x\right]\,.$$

One can show that, under suitable assumptions, v_R and $v_{1,R}$ converge to v and v_1 respectively, and the truncation of the domain makes it possible to compute approximations of $v_{1,R}$ and v_R by the finite element methods.

d) [5 pts.]

We apply the parabolic variational framework from the lecture. We shall show the following facts:

(i) The function $\tilde{g}_1 := g_1 \circ \exp$ is in $L^2(G)$

(ii) a^{BS} is continuous on $H_0^1(G) \times H_0^1(G)$.

(iii) a^{BS} is coercive, i.e. there exist a constant $c > 0$ such that

$$\forall v \in H_0^1(G)\,, \quad a^{\mathrm{BS}}(v,v) \geq c\|v\|^2_{H_0^1(G)}\,.$$

(this is a particular case of the Garding inequality)

For (i), since g_1 is Lipschitz, it is continuous, hence $\tilde{g}_1$ is continuous on the bounded set G, thus $\tilde{g}_1 \in L^2(G)$.

For (ii), applying the triangle and Cauchy-Schwarz inequalities, we can write

$$|a^{\mathrm{BS}}(u,v)| \leq \frac{1}{2}\sigma^2\|\partial_x u\|_{L^2(G)}\|\partial_x v\|_{L^2(G)} + \left|r - \frac{1}{2}\sigma^2\right|\|\partial_x u\|_{L^2(G)}\|v\|_{L^2(G)} + r\|u\|_{L^2(G)}\|v\|_{L^2(G)}\,.$$

We choose to define $\|u\|_{H_0^1(G)} := \|\partial_x u\|_{L^2(G)}$. This indeed defines a norm which is equivalent to the H^1 norm on $H_0^1(G)$, since by Poincaré's inequality, there exists a constant C_P such that, for all $u \in H_0^1(G)$,

$$\|u\|_{L^2(G)} \leq C_P\|u\|_{H_0^1(G)}\,.$$

Hence,

$$a^{\mathrm{BS}}(u,v) \leq \left[\frac{1}{2}\sigma^2 + C_P|r - \frac{1}{2}\sigma^2| + rC_P^2\right]\|u\|_{H_0^1(G)}\|v\|_{H_0^1(G)}\,.$$

Let us now prove (iii). Let $u \in C_0^1(G)$. We remark that integrating by parts yields

$$\int_{-R}^{R} u(x)\,\partial_x u(x)\,dx = -\int_{-R}^{R} u(x)\,\partial_x u(x)\,dx$$

hence $\int_{-R}^{R} u(x)\,\partial_x u(x) = 0$, and therefore

$$a^{\mathrm{BS}}(u,u) = \frac{1}{2}\sigma^2\|u\|^2_{H_0^1(G)} + r\|u\|^2_{L^2(G)} \geq \frac{1}{2}\sigma^2\|u\|^2_{H_0^1(G)}\,.$$

The variational problem and the PDE (1) have the following link: when the PDE admits a sufficiently regular solution, this solution is also a solution of the variational problem.

e) [2 pts.]

Let $\underline{v}_N \in \mathbb{R}^N$ be an arbitrary column vector and let $v_N(x) := \sum_{i=1}^N \underline{v}_{N,i} b_i(x)$. Injecting this and the expression

$$u_N(t,x) = \sum_{i=1}^{N} \underline{u}_{N,i}(t) b_i(x)$$

in Eq. (4), we find

$$\sum_{1\leq i,j\leq N} \underline{v}_{N,i}\,(b_j, b_i)_{L^2(G)}\,\underline{u}_{N,j}(t) + \underline{v}_{N,i}\,a^{\mathrm{BS}}(b_j,b_i)\,\underline{u}_{N,j}(t) = 0\,,$$

which can be written in matrix form

$$\underline{v}^T \left[\mathbf{M}\partial_t \underline{u}_N(t) + \mathbf{A}\underline{u}_N(t)\right] = 0\,,$$

where

$$\mathbf{M}_{i,j} = (b_j, b_i)_{L^2(G)}\,, \quad \mathbf{A}_{i,j} = a^{\mathrm{BS}}(b_j, b_i)\,, \quad 1 \le i,j \le N\,.$$

Since this holds for all $V \in \mathbb{R}^N$, we deduce that

$$\mathbf{M}\partial_t \underline{u}_N(t) + \mathbf{A}\underline{u}_N(t) = 0\,.$$

f) [5 pts.] Let $i \in \{1\ \dots\ N\}$. One has

$$\varphi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h} & \text{if } x \in (x_{i-1}, x_i) \\ \frac{x_{i+1}-x}{h} & \text{if } x \in (x_i, x_{i+1}) \\ 0 & \text{if } x \notin (x_{i-1}, x_{i+1})\,. \end{cases}$$

Thus, if $i < N$, we have

$$\int_G b_i\, b'_{i+1}\, dx = \int_{x_i}^{x_{i+1}} \frac{x_{i+1}-x}{h}\left(\frac{x-x_i}{h}\right)' dx = \frac{1}{h^2}\int_0^h u\, du = \frac{1}{2}\,,$$

where we used the change of variables $u \leftarrow x_{i+1} - x$, which satisfies

$$dx = -du\,, \quad 0 \le u \le h\,.$$

Thus $\mathbf{W}_{i,i+1} = \frac{1}{2}$. On the other hand,

$$\int_G bi' b_{i+1}\, dx = \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1}-x}{h}\right)' \frac{x-x_i}{h}\, dx = \frac{-1}{h^2}\int_0^h u\, du = -\frac{1}{2}\,.$$

Hence $W_{i+1,i} = -\frac{1}{2}$. Finally,

$$\begin{aligned}\int_G bi' b_i\, dx &= \int_{x_{i-1}}^{x_i} \frac{x-x_{i-1}}{h}\left(\frac{x-x_{i-1}}{h}\right)' dx + \int_{x_i}^{x_{i+1}} \frac{x_{i+1}-x}{h}\left(\frac{x_{i+1}-x}{h}\right)' dx \\ &= \frac{1}{h^2}\int_0^h u\, du + \frac{-1}{h^2}\int_0^h u\, du \\ &= 0\,.\end{aligned}$$

Hence $\mathbf{W}_{ii} = 0$. Other possibility: argue that for $u, v \in H_0^1(G)$,

$$\int_G u'v = -\int_G uv'$$

so that $\mathbf{W}$ is anti-symmetric. Since b_i are in $H_0^1(G)$, deduce that $\mathbf{W}_{ii} = 0$, that $\mathbf{W}_{i+1,i} = -\mathbf{W}_{i,i+1}$, and do only the first computation.

For $\mathbf{A}$, we write

$$\mathbf{A} = \frac{\sigma^2}{2}\mathbf{S} + \left(\frac{1}{2}\sigma^2 - r\right)\mathbf{W} + r\mathbf{M}\,.$$

g) [2 pts.]The matrices $\mathbf{B}_\vartheta$ and $\mathbf{C}_\vartheta$ are given by

$$\mathbf{B}_\vartheta = \mathbf{M} + k\vartheta\mathbf{A}\,, \quad \mathbf{C}_\vartheta = \mathbf{M} - k(1-\vartheta)\mathbf{A}\,.$$

h) [7 pts.] See code for correction.

i) [2 pts.] See code for correction

j) [3 pts.]

The FEM-ϑ-scheme is of order $O(h^2) + O(k)$ for $\vartheta > 1/2$, of order $O(h^2) + O(k^2)$ for $\vartheta = \frac{1}{2}$ and unstable for $\vartheta < 1/2$ unless the CFL condition

$$k \leq C\frac{h^2}{1-2\vartheta}$$

is fulfilled. Since the time step k is $\frac{T}{M}$ and $M \approx \frac{T}{h}$, we have $h \approx k$. Hence, for h small enough, the CFL condition is violated, so the theoretical behavior is

- Instability for $\vartheta < 1/2$
- Order 2 for $\vartheta = \frac{1}{2}$
- Order 1 for $\vartheta > \frac{1}{2}$.

k) [5 pts.] See code for correction. We observe that the compound option value is smaller than the underlying call option value. Between the following two contracts:

(i) At time T_1, you will have the right to buy the asset S at price K_1

(ii) At time $T < T_1$, you will have the right to buy contract (i) (with updated maturity $T_1 - T$ instead of T) at the price K.

clearly, the contract (ii) is less advantageous than contract (i). Indeed, if someone has the contract (i), they will be already in possession of the contract (i) at time T, while the owner of the contract (ii) would have to exercise their contract (and hence pay K) to get contract (i) at T.

3. **a)** The SDE reads

$$S_t = rS_t dt + \sigma S_t^{\varrho} dW_t \,.$$

This corresponds to the volatility function

$$\sigma(s) = \sigma s^{\varrho}$$

which is not Lipschitz continuous at $s = 0$ if $\varrho < 1$. Furthermore, the change of variables $x = \ln(s)$ doesn't remove the singularity, unless $\varrho = 1$ (which corresponds to the B-S model). Thus, we need consider the more involved weighted Sobolev spaces to ensure well-posedness of the problem.

b) Let us fix u and v in $C_0^\infty(0, R)$. We can then extend the reasoning to $W_{\varrho,\mu}$ by density (since by definition, $C_0^\infty(0, R)$ is dense in $W_{\varrho,\mu}$).

First term. By the Cauchy-Schwarz inequality, one has

$$|(s^{2\varrho}u', v')_\mu| \leq ||s^{\varrho}u'||_\mu \, ||s^{\varrho}v'||_\mu \leq \|u\|_{\varrho,\mu}\|v\|_{\varrho,\mu} \,.$$

Second term. Again by Cauchy-Schwarz inequality, we have

$$|(s^{2\varrho-1}u', v)_\mu| \leq \|s^{\varrho}u'\|_\mu \, \|s^{\varrho-1}v\|_\mu \,.$$

The assumptions on ϱ and μ ensure that $2\varrho + 2\mu \geq 0$ and $2\varrho + 2\mu \neq 1$, so we can apply Hardy's inequality:

$$\|s^{\varrho-1}v\|_\mu^2 \leq \left(\frac{2}{2\varrho + 2\mu - 1}\right)^2 \|s^{\varrho}v'\|_\mu^2 \leq C\|v\|_{\varrho,\mu}^2 \,.$$

Hence,

$$|(s^{\varrho-1}u', v)_\mu| \leq C\|s^{\varrho}u'\|_\mu \, \|s^{\varrho}v'\|_\mu \leq C\|u\|_{\varrho,\mu} \, \|v\|_{\varrho,\mu} \,.$$

Third tem. We write

$$(su', v)_\mu = (s^{\varrho}u', s^{1-\varrho}v)_\mu \,.$$

By the Cauchy-Schwarz inequality, and using the fact that $s^{1-\varrho} \leq R^{1-\varrho}$ on G, we find

$$|(su', v)_\mu| \leq R^{1-\varrho}\|u\|_{\varrho,\mu}\|v\|_\mu \leq C\|u\|_{\varrho,\mu}\|v\|_{\varrho,\mu} \,.$$

Fourth term. By Cauchy-Schwarz inequality,

$$|(u, v)_\mu| \leq \|u\|_\mu\|v\|_\mu \leq \|u\|_{\varrho,\mu}\|v\|_{\varrho,\mu} \,.$$

Grouping the four inequalities, we conclude that $a_{\varrho,\mu}^{\mathrm{CEV}}$ is continuous on $W_{\varrho,\mu} \times W_{\varrho,\mu}$.

c) Rewriting the variational inequality in the finite dimensional space V_N leads to the following discrete variational problem:

Find $u_N \in C^1(J; V_N)$ such that $u_N(t,\cdot) \in V_N \cap \mathcal{K}_{0,R}$ and for all $\varphi \in V_N \cap \mathcal{K}_{0,R}$,

$$\left\{ \begin{array}{rclr} (\partial_t u_N, \varphi - u_N)_\mu + a^{\mathrm{CEV}}(u_N, \varphi - u_N) & \geq & -a^{\mathrm{CEV}}(g, \varphi - u_N) & \text{a.e. in } J \\ u(0,x) & = & 0 & \text{a.e. in } G\,, \end{array} \right. \tag{1}$$

Let u_N be the solution of this variational inequality, and let

$$\underline{u}_N(t) = (u_N(t,x_1), \ldots, u_N(t,x_N))^T .$$

Let $\underline{\varphi} \in \mathbb{R}^N_{\geq 0}$ and

$$\varphi := \sum_{i=1}^N \underline{\varphi}_i b_i$$

Then $\varphi \in V_N \cap \mathcal{K}_{0,R}$, so by the first line in the discrete variational inequality,

$$(\underline{\varphi} - \underline{u}_N(t))^T (\mathbf{M}^{\mathrm{CEV}} \underline{u}_N'(t) + \mathbf{A}^{\mathrm{CEV}} \underline{u}_N(t)) \geq (\underline{\varphi} - \underline{u}_N(t))^T \underline{G}\,,$$

where $\mathbf{A}^{\mathrm{CEV}}$ and $\mathbf{M}^{\mathrm{CEV}}$ are the $N \times N$ matrices defined by

$$\mathbf{A}^{\mathrm{CEV}}_{i,j} := a^{\mathrm{CEV}}(b_j, b_i)\,, \quad \mathbf{M}^{\mathrm{CEV}}_{i,j} := (b_i, b_j)_\mu\,, \quad 1 \leq i,j \leq N\,,$$

and $\underline{G}$ is the $N \times 1$ column vector defined by

$$\underline{G}_i := -a^{\mathrm{CEV}}(g, b_i)\,.$$

We now apply an implicit Euler scheme ϑ-scheme, i.e. we rewrite the previous inequality at the times $t_m := km$ where $m = 0, \ldots, M$ and $k = \frac{T}{M}$. We formally replace $\underline{u}_N(t_m)$ by $\underline{u}_N^{m+1}$ and $\underline{u}_N'(t_m)$ by $\frac{\underline{u}_N^{m+1} - \underline{u}_N^m}{k}$. This leads to

$$(\underline{\varphi} - \underline{u}_N^{m+1})^T \left(\mathbf{M}^{\mathrm{CEV}} \frac{\underline{u}_N^{m+1} - \underline{u}_N^m}{k} + \mathbf{A}^{\mathrm{CEV}} \left[\underline{u}_N^{m+1} + \right] \right) \geq$$
$$\geq (\underline{\varphi} - \underline{u}_N^{m+1})^T \underline{G}\,.$$

Rearranging the terms leads to

$$(\underline{\varphi} - \underline{u}_N^{m+1})^T \mathbf{B} \underline{u}_N^{m+1} \geq (\underline{\varphi} - \underline{u}_N^{m+1})^T \underline{F}^m\,,$$

where

$$\mathbf{B} := \mathbf{M}^{\mathrm{CEV}} + k\mathbf{A}^{\mathrm{CEV}}\,,$$
$$\underline{F}^m := k\underline{G} + \mathbf{M}^{\mathrm{CEV}} \underline{u}_N^m\,.$$

d) See code

e) See code

f) The European option can only be exercised at maturity while the American option can be exercised at any time including maturity. Thus, the holder of an American option can always ensure a benefit at least equal to that of a European option, by applying the strategy "exercise at maturity". Hence, the american option can yield higher or equal profits, so it is worth more. In the particular case of call options, the inequality is in fact an equality: the American call contract has the same value as the European call contract. This can be checked numerically using the code of this exercise and changing the payoff functions to call payoffs.