Series 11

1. Heston model

In the lecture we studied the following SDE describing a Heston model for the process $\mathbf{Z}_t = (X_t, Y_t)^\top \in \mathbb{R}^2$:

$$\mathrm{d}\mathbf{Z}_t = \mathbf{b}(\mathbf{Z}_t)\mathrm{d}t + \mathbf{\Sigma}(\mathbf{Z}_t)\mathrm{d}\mathbf{W}_t, \quad \mathbf{Z}_0 = \mathbf{z}_0 \in \mathbb{R}^2. \tag{1}$$

Here $\mathbf{W}_t$ is a $\mathbb{R}^2$-valued standard Brownian motion and for $\mathbf{z} = (s, y)^\top$, $\alpha, \beta > 0$ and $\rho \in [-1, 1]$,

$$\mathbf{b}(\mathbf{z}) = \begin{pmatrix} rs \\ \alpha(m-y) - \lambda(s,y) \end{pmatrix},$$

$$\mathbf{\Sigma}(\mathbf{z}) = \begin{pmatrix} s\sqrt{y} & 0 \\ \beta\rho\sqrt{y} & \beta\sqrt{1-\rho^2}\sqrt{y} \end{pmatrix}.$$

We consider some problems related to this model.

a) Verify that the operator $\mathcal{A}$ in (9.11) in the textbook could be written as

$$\mathcal{A} = \frac{1}{2}\left(s^2 y \partial_s^2 + 2s\beta\rho y \partial_s \partial_y + \beta^2 y \partial_y^2\right) + rs\partial_s + \alpha(m-y)\partial_y\,. \tag{2}$$

b) Verify that by introducing the new variable $x = \log(s)$, one could rewrite (2) as

$$\mathcal{A} = \frac{1}{2}\left(y\partial_x^2 + 2\beta\rho y \partial_x \partial_y + \beta^2 y \partial_y^2\right) + (r - \frac{y}{2})\partial_x + \alpha(m-y)\partial_y\,. \tag{3}$$

Weighted Sobolev spaces have to be used in the variational formulation of stochastic volatility models to recover well-posedness. This is due to the fact that the pricing equations in log-price and time-to-maturity are **degenerate** parabolic equations. We take the log-Heston model as example and verify rigorously that a Gårding-type inequality does not hold on a standard Sobolev space.

Set $J = (0, T)$ and fix a given payoff $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Consider the following log-Heston model localized to $G_R = (-R_1, R_1) \times (0, R_2)$ with $R_1, R_2 > 0$ with *time-to-maturity* and with solution $v_R : J \times G_R \to [0, \infty)$

$$\begin{aligned} \partial_t v_R - \left(\frac{1}{2}y\partial_{xx} + y\partial_{xy} + 2y\partial_{yy} - \frac{1}{2}y\partial_x - 2(y-1)\partial_y\right)v_R = 0 \quad &\text{in } J \times G_R, \\ v_R(0,x,y) = g(e^x) \;\; &\text{in } G_R\,, \\ v_R(t,x,y) = 0 \;\; &\text{on } J \times \partial G_R. \end{aligned} \tag{4}$$

This equation is a degenerate parabolic PDE. Its variational formulation using the standard Sobolev space $H_0^1(G_R)$ reads:

$$\begin{aligned} &\text{Find } u \in L^2(J; H_0^1(G_R)) \cap H^1(J; H^{-1}(G_R)) \text{ such that } u(0,x,y) = g(e^x) \text{ and} \\ &(\partial_t u, v) + a(u,v) = 0 \quad \forall v \in H_0^1(G_R). \end{aligned} \tag{5}$$

Here, $a(u,v) : H_0^1(G_R) \times H_0^1(G_R) \to \mathbb{R}$ is given by

$$\begin{aligned} a(u,v) = \frac{1}{2}\int_{G_R} y\,\partial_x u\,\partial_x v\,dx\,dy + \int_{G_R} y\,\partial_y u\,\partial_x v\,dx\,dy + 2\int_{G_R} y\,\partial_y u\,\partial_y v\,dx\,dy \\ + \frac{1}{2}\int_{G_R} y\,\partial_x u\,v\,dx\,dy + 2\int_{G_R} y\partial_y u\,v\,dx\,dy\,. \end{aligned}$$

In order to prove this formulation is ill-posed, fix a sequence of functions $H_0^1(G_R) \ni u_n(x,y) = \psi(x) \cdot \phi_n(y)$ for $n \in \mathbb{N}$ such that

1. $\psi(x) \in C_0^\infty(-R_1, R_1), \phi_n(y) \in C_0^\infty(0, R_2)$ are nonzero,
2. $\|\psi\|_{H^1(-R_1,R_1)} = \|\phi_n\|_{H^1(0,R_2)} = 1$ for any $n \in \mathbb{N}$ and
3. $\operatorname{supp} \phi_n \subset (0, \frac{1}{n})$.

c) Show that $\|\phi_n\|^2_{L^2(0,R_2)} \to 0$ and $\|\partial_y \phi_n\|^2_{L^2(0,R_2)} \to 1$ as $n \to +\infty$.
Hint: You may use without proof the following *Poincaré inequality*: for any $a, b \in \mathbb{R}$ and $v \in H_0^1(a,b)$, there exists $C_{poin} > 0$ such that

$$\|v\|_{L^2(a,b)} \leq C_{poin}(b-a)\|v'\|_{L^2(a,b)}.$$

d) Show that $\|u_n\|_{L^2(G_R)} \to 0$ as $n \to +\infty$.

e) Show that there exists a constant $C_K > 0$ such that $\|u_n\|^2_{H^1(G_R)} \geq C_K$ for all $n \in \mathbb{N}$.

f) Show that for any $C_2 > 0$ and any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that for all $n \geq N_\epsilon$,

$$a(u_n, u_n) + C_2\|u_n\|^2_{L^2(G_R)} < \epsilon.$$

g) Show that there could not exist $C_1, C_2 > 0$ such that the Gårding-type inequality $a(u,u) \geq C_1\|u\|^2_{H^1(G_R)} - C_2\|u\|^2_{L^2(G_R)}$ holds for any $u \in H_0^1(G_R)$.

2. Discretization for the Heston model

We consider in this exercise the tensor-product FEM approximation to Equation (4).

a) Rewrite (4) in the following form

$$\begin{aligned} \partial_t v_R - (\frac{1}{2}\operatorname{tr}(\boldsymbol{\mathcal{Q}} D^2 v_R) + \mu(x,y)^\top \nabla v_R + c(x,y) v_R) = 0 \quad \text{in } J \times G_R \, , \\ v_R(t,x,y) = 0 \quad \text{in } J \times \partial G_R \, , \\ v_R(0,x,y) = g(e^x) \quad \text{in } G_R \, . \end{aligned} \tag{6}$$

Specify $\boldsymbol{\mathcal{Q}} \in \mathbb{R}^{2\times 2}$, $\mu(x,y) : G_R \to \mathbb{R}^2$ and $c(x,y) : G_R \to \mathbb{R}$. Verify that these coefficients $\boldsymbol{\mathcal{Q}}$, $\mu(x,y)$ and $c(x,y)$ satisfy Assumption 9.5.1 in the textbook.

For the discretization in space, let $N, M \in \mathbb{N}$, $h = \frac{2R_1}{N+1}$, $k = \frac{R_2}{M+1}$ and let $x_i = -R_1 + ih$, $y_j = jk$ for $i = 0, 1, \ldots, N+1$, $j = 0, 1, \ldots, M+1$. Set $b_i(x) = \max\{0, 1 - h^{-1}|x_i - x|\}$ on $(-R_1, R_1)$ and $b_j(y) = \max\{0, 1 - k^{-1}|y_j - y|\}$. The finite element space $V_{NM} \subset V$ with dimension NM is then given by

$$V_{NM} := \operatorname{span}\{b_i(x) \cdot b_j(y) : 1 \leq i \leq N, \ 1 \leq j \leq M\}.$$

b) Specify the matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ corresponding to the bilinear form $a(\cdot,\cdot)$ given in the previous exercise.

3. Inverse Gamma processes

Let X be a Lévy process with state space $\mathbb{R}$. We assume that the associated Lévy measure $\nu(\mathrm{d}z) = k(z)\mathrm{d}z$ satisfies

$$k(z) \leq C \begin{cases} e^{-\beta_-|z|}, & z < -1\,, \\ e^{-\beta_+|z|}, & z > 1\,, \end{cases}$$

with $\beta_- > 0$ and $\beta_+ > 1$. Then, the associated integrodifferential operator $\mathcal{A}^{\mathrm{J}}$ is given by

$$\mathcal{A}^{\mathrm{J}} f(x) = \int_{\mathbb{R}} \left(f(x+z) - f(x) - z f'(x)\right) k(z)\, dz\,.$$

a) Show by integration-by-parts that $\mathcal{A}^{\mathrm{J}}$ can be written for $f \in C_0^2(\mathbb{R})$ as

$$\mathcal{A}^{\mathrm{J}} f(x) = \int_{\mathbb{R}} f''(x+z) k^{(-2)}(z)\, dz\,,$$

where the functions $k^{(i)} : \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{N}$, are defined by

$$k^{(0)} = k\,, \quad k^{(-i)}(z) = \begin{cases} \int_{-\infty}^{z} k^{(-i+1)}(s)\, ds & \text{if } z < 0\,, \\ -\int_{z}^{\infty} k^{(-i+1)}(s)\, ds & \text{if } z > 0\,. \end{cases}$$

b) Let $k(z) = e^{-\beta|z|}\,|z|^{-1}$. Show that for $z > 0$, there holds

$$k^{(-3)}(z) = -\mathrm{Ei}(\beta z)\,\frac{z^2}{2} + e^{-\beta z}\left(\frac{z}{2}\beta^{-1} - \frac{1}{2}\beta^{-2}\right),$$
$$k^{(-4)}(z) = -\mathrm{Ei}(\beta z)\,\frac{z^3}{6} + e^{-\beta z}\left(\frac{z^2}{6}\beta^{-1} - \frac{z}{6}\beta^{-2} + \frac{1}{3}\beta^{-3}\right),$$

where

$$\mathrm{Ei}(x) = \int_x^{\infty} e^{-s} s^{-1}\, ds\,, \quad x > 0$$

is the exponential integral.

Suppose now that Y is a Gamma process with parameters a and b. This is the unique Levy process Y_t such that its characteristic function is given by

$$\phi(u) = \mathbb{E}\left[e^{iuY_t}\right] = \left(1 - \frac{iu}{b}\right)^{-a}. \tag{7}$$

Remark: This implies that Y_t follows a Gamma(at, b) distribution. The density $f_{\tilde{a},b} : \mathbb{R}^+ \to \mathbb{R}$ of a Gamma$(\tilde{a}, b)$ law is given by

$$f_{\tilde{a},b}(x) = \frac{b^{\tilde{a}}}{\Gamma(\tilde{a})} x^{\tilde{a}-1} \exp\left(-xb\right), \quad x > 0\,. \tag{8}$$

Also note that the Levy triplet of the Gamma process is given by

$$(\sigma^2, \nu, \gamma) = \left(0\,, \frac{ae^{-bx}\, dx}{x} 1_{\{x>0\}}\,, \frac{a\left(1 - e^{-b}\right)}{b}\right).$$

c) Use ϕ from Equation (7) to verify that for a random variable Y following a Gamma(a, b) distribution, there holds

(i) Mean: $\mathbb{E}[Y] = a/b$,

(ii) Variance: $\sigma_Y^2 = \text{Var}(Y) = a/b^2$,

(iii) Skewness: $\mathbb{E}[(\frac{Y-E[Y]}{\sigma_Y})^3] = 2a^{-\frac{1}{2}}$,

(iv) Kurtosis: $\mathbb{E}[(\frac{Y-E[Y]}{\sigma_Y})^4] = 3(1+2a^{-1})$,

(v) Scaling Property: $cY \sim \text{Gamma}(a, \frac{b}{c})$ for any $c > 0$.

4. Inverse Gaussian processes

Recall that for a Lévy process X with characteristic triplet (σ^2, ν, γ), the characteristic function of X_t is given as $\mathbb{E}[e^{iuX_t}] = e^{-t\psi(u)}$ where

$$\psi(u) = -i\gamma u + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (1 - e^{iuz} + iuz 1_{\{|z| \leq 1\}}) \nu(dz).$$

The inverse Gaussian process with parameters $a, b > 0$ is the Lévy process with

$$\psi(u) = a(\sqrt{-2iu + b^2} - b)\,,$$

and Lévy measure

$$\nu(dx) = k(x)dx = \frac{a}{\sqrt{2\pi}} x^{-\frac{3}{2}} e^{-\frac{b^2}{2}x} 1_{(0,\infty)}(x)dx.$$

a) Calculate γ for the Lévy triplet $(0, \nu, \gamma)$ of the inverse Gaussian process.

Hint: determine the value of the following integral

$$\int_0^\infty \frac{b}{\sqrt{2}\Gamma(1/2)} x^{-1/2} e^{-\frac{b^2 x}{2}} e^{iux} dx$$

using eqs. (7) *and* (8).

b) Let X be an inverse Gaussian process with parameters $a, b > 0$ and consider the normal-inverse Gaussian (NIG) process Y defined by

$$Y_t := \gamma_0 t + \sigma W_{X_t} + \theta X_t\,,$$

for given parameters $\gamma_0, \theta \in \mathbb{R}$ and $\sigma > 0$. Derive the characteristic function of Y and deduce that it is a pure jump process, i.e. $\sigma = 0$ in the Lévy triplet of Y.

Due: Wednesday, May 22nd, at 2pm.