Solutions: Series 4

1. Spaces $H_0^1(0,1)$ and $H^{-1}(0,1)$

a) The Sobolev embedding theorem implies that $u \in H_0^1(0,1) \subset C^0(0,1)$ and we denote by $\tilde{u}$ the continuous modification of u. Therefore $\tilde{u}(0.5)$ is well-defined.

Clearly $\delta_{0.5}$ is a linear functional on $H_0^1(0,1)$, it remains to show that this functional is bounded. We obtain for any $u \in H_0^1(0,1)$ with Hölder's inequality

$$\begin{aligned}|\delta_{0.5}(u)| = |\tilde{u}(0.5)| = \left|\int_0^{0.5} u'(x)\,dx + u(0)\right| &\leq \int_0^{0.5} |u'(x)|\,dx \\ &\leq \frac{\sqrt{2}}{2}\left(\int_0^{0.5} |u'(x)|^2\,dx\right)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{2}}{2}\|u\|_{H^1(0,1)}.\end{aligned}$$

Therefore $\delta_{0.5}$ is a bounded linear functional on $H_0^1(0,1)$ and $\delta_{0.5} \in H^{-1}(0,1)$.

b) We define the bilinear form $a(u,v) = \int_0^1 u'(x)v'(x)\,dx$. The Poincaré inequality shows that there exists a constant $C_{poin} > 0$ such that, for any $u \in H_0^1(0,1)$,

$$a(u,u) \geq C_{poin}\|u\|^2_{L^2(0,1)}\,.$$

Therefore,

$$a(u,u) \geq \frac{1}{2}\|u'\|^2_{L^2(0,1)} + \frac{C_{poin}}{2}\|u\|^2_{L^2(0,1)} \geq \frac{\min(1, C_{poin})}{2}\|u\|^2_{H^1(0,1)}\,.$$

This shows that $a(\cdot,\cdot)$ is *coercive* (See Definition A.3.1 in Appendix A of the textbook).

Moreover, we could show that $a(\cdot,\cdot)$ is continuous with respect to $H^1(0,1)$-norm: By the Cauchy-Schwarz inequality,

$$|a(u,v)| \leq \|u'\|_{L^2(0,1)} \cdot \|v'\|_{L^2(0,1)} \leq \|u\|_{H^1(0,1)} \cdot \|v\|_{H^1(0,1)}.$$

As $F \in H^{-1}(0,1)$, the Lax-Milgram lemma applies here and there exists a unique $u \in H_0^1(0,1)$ for the variational problem.

c) Set

$$w(x) = \begin{cases} x/2 & x \in (0, 0.5)\,, \\ 1/2 - x/2 & x \in [0.5, 1)\,. \end{cases}$$

Then $w \in H_0^1(0,1)$. We shall verify that for any $v \in H_0^1(0,1)$,

$$\int_0^1 w'(x)v'(x)\,dx = \tilde{v}(0.5)\,.$$

That is, we show that w is the unique (due to **b)**) solution to the variational problem: Find $u \in H_0^1(0,1)$ such that for any $v \in H_0^1(0,1)$,

$$\int_{(0,1)} u'v'dx = \delta_{0.5}(v)\,. \tag{1}$$

We have,

$$\begin{aligned}\int_0^1 w'(x)v'(x)\,dx &= \int_0^{0.5} w'(x)v'(x)\,dx + \int_{0.5}^1 w'(x)v'(x)\,dx \\ &= 0.5\left(\int_0^{0.5} v'(x)\,dx - \int_{0.5}^1 v'(x)\,dx\right) \\ &= 0.5\left(\tilde{v}(0.5) - \tilde{v}(0) - \tilde{v}(1) + \tilde{v}(0.5)\right) = \tilde{v}(0.5)\,.\end{aligned}$$

This shows that w is a solution to (1) and hence it follows that $w = (-\Delta)^{-1}\delta_{0.5}$.

d) $(-\Delta^{-1}) : H^{-1}(0,1) \to H_0^1(0,1)$ is a well-defined continuous, linear mapping due to 2b). This mapping is injective as the variational problem has a unique solution. It is surjective due to the observation that for any $u \in H_0^1(0,1)$, the linear functional $(-\Delta)u$ defined by $((-\Delta)u)(v) = \int_{(0,1)} u'v'\,dx$, satisfies $(-\Delta)^{-1}((-\Delta)u) = u$.

e) For any $f, g, h \in H^{-1}(0,1)$ and any $\alpha \in \mathbb{R}$, we shall verify the following:

1. $((f,f)) \geq 0$ and $((f,f)) = 0 \iff f = 0$. Here the latter "0" shall be understood as the zero functional, e.g. it maps any function in $H_0^1(0,1)$ to the function which is identically zero.
2. $((f,g)) = ((g,f))\,, \quad \forall f,g \in H^{-1}(0,1)$.
3. $((\alpha f + g, h)) = \alpha \cdot ((f,h)) + ((g,h))\,, \quad \forall f,g \in H^{-1}(0,1),\ \alpha \in \mathbb{R}$.

We set, for any linear functional $f \in H^{-1}(0,1)$, $\widehat{f} = (-\Delta)^{-1}f \in H_0^1(0,1)$ (then for $g,h \in H^{-1}(0,1)$, we write $\widehat{g} = (-\Delta)^{-1}g \in H_0^1(0,1)$ and $\widehat{h} = (-\Delta)^{-1}h \in H_0^1(0,1)$).

To check the first item here, note that $(-\Delta)^{-1}$ is defined via a variational problem, we take $\widehat{f}$ as the test function,

$$((f,f)) = f((-\Delta)^{-1}f) = f(\widehat{f}) = \int_{(0,1)} (\widehat{f}')^2\,dx \geq C_{poin}\|\widehat{f}\|_{L^2(0,1)}\,,$$

due to Poincaré's inequality . So $((f,f)) \geq 0$. Note that $((f,f)) = 0$ implies $\|\widehat{f}\|_{L^2(0,1)} = 0$, which leads to $\widehat{f} = 0$ and $f = 0$.

We turn to the second item. We have,

$$((f,g)) = f((-\Delta)^{-1}g) = f(\widehat{g}) = \int_{(0,1)} \widehat{f}'\widehat{g}'\,dx\,.$$

The last step is derived by taking $\widehat{g}$ as the test function in the variational problem with linear functional f. Similarly we have

$$((g,f)) = g((-\Delta)^{-1}f) = g(\widehat{f}) = \int_{(0,1)} \widehat{g}'\widehat{f}'\,dx\,.$$

Therefore $((f,g)) = ((g,f))$. Finally we study the third item, we have,

$$((\alpha f + g, h)) = ((h, \alpha f + g)) = h((-\Delta)^{-1}(\alpha f + g)) = \alpha h(\widehat{f}) + h(\widehat{g}) = \alpha((h,f)) + ((g,f))\,,$$

because $(-\Delta)^{-1}$ is a linear mapping.

2. FEM for parabolic PDEs: Implementation

a) Fix some column vector $\underline{V} = (V_1, \ldots, V_N)^\top \in \mathbb{R}^N$ and let $V_N = \mathrm{span}\{\phi_{N,i}\}_{i=1}^N$. Then

$$v_N(x) = \sum_{i=1}^N V_i \phi_{N,i}(x)\,, \quad u_N(t,x) = \sum_{i=1}^N u_{N,i}(t)\phi_{N,i}(x)\,.$$

Then we have,

$$\frac{d}{dt}\left(\sum_{j=1}^N u_{N,j}(t)\phi_{N,j}, \sum_{i=1}^N V_i\phi_{N,i}\right)_{L^2(G)} + a\left(\sum_{j=1}^N u_{N,j}(t)\phi_{N,j}, \sum_{i=1}^N V_i\phi_{N,i}\right) = \left(f(t), \sum_{i=1}^N V_i\phi_{N,i}\right)_{L^2(G)}.$$

Therefore, for each $t \in J$,

$$\sum_{i=1}^N\sum_{j=1}^N (V_i\ (\phi_{N,j}, \phi_{N,i})_{L^2(G)} \frac{d\,u_{N,j}}{dt}(t) + V_i\,a\,(\phi_{N,j}, \phi_{N,i})\,u_{N,j}(t)) = \sum_{i=1}^N V_i\ (f(t), \phi_{N,i})_{L^2(G)}\ , \quad (2)$$

or we could rewrite it as

$$\underline{V}^\top\left(\mathbf{M}\frac{d\,\underline{u}_N}{dt}(t) + \mathbf{A}\underline{u}_N(t)\right) = \underline{V}^\top \underline{f}(t)\,,$$

where $\mathbf{M}$ and $\mathbf{A}$ are the matrices given by

$$\mathbf{M}_{i,j} = (\phi_{N,j}, \phi_{N,i})_{L^2(G)} = \begin{cases} \frac{2h}{3}, & i = j, \\ \frac{h}{6}, & |i-j| = 1, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and}$$

$$\mathbf{A}_{i,j} = a\,(\phi_{N,j}, \phi_{N,i}) = \int_G \alpha(x)\partial_x\phi_{N,i}\partial_x\phi_{N,j} + \beta(x)(\partial_x\phi_{N,j})\phi_{N,i} + \gamma(x)\phi_{N,i}\phi_{N,j}\,dx\,, \quad 1 \le i,j \le N\,.$$

Moreover, $\underline{f}(t)$ is the column vector given by

$$f_i(t) = (f(t), \phi_{N,i})_{L^2(G)}\ , \quad 1 \le i \le N\,.$$

Since Eq. (2) holds for any choice of column vector $\underline{V} \in \mathbb{R}^N$, we deduce that

$$\mathbf{M}\frac{d\,\underline{u}_N}{dt}(t) + \mathbf{A}\underline{u}_N(t) = \underline{f}(t)\,, \quad \forall t \in J\,.$$

By a time-discretization with the θ-scheme this is transformed to the fully-discretized form in Question 2. Note that $\mathbf{M}$ is symmetric but $\mathbf{A}$ is not (necessarily) symmetric.

b) We have,

$$\begin{aligned} f &= \partial_t u - \partial_x((1+x^2)\partial_x u) + 2x\partial_x u + (\pi^2x^2)u \\ &= -e^{-t}\sin(\pi x) + (1+x^2)\pi^2e^{-t}\sin(\pi x) - 2x\partial_x u + 2x\partial_x u + (\pi^2x^2)e^{-t}\sin(\pi x) \\ &= (2\pi^2x^2 + \pi^2 - 1)e^{-t}\sin(\pi x). \end{aligned}$$

c) It holds for the entries of the stiffness matrix $\mathbf{A}$ for $1 \le i \le N$:

$$
\begin{aligned}
A_{i,i} &= a(\phi_{N,i}, \phi_{N,i}) \\
&= \int_{x_{i-1}}^{x_i} \alpha(x)\phi'_{N,i}(x)\phi'_{N,i}(x) + \beta(x)\phi_{N,i}(x)\phi'_{N,i}(x) + \gamma(x)\phi_{N,i}(x)\phi_{N,i}(x)dx \\
&\quad + \int_{x_i}^{x_{i+1}} \alpha(x)\phi'_{N,i}(x)\phi'_{N,i}(x) + \beta(x)\phi_{N,i}(x)\phi'_{N,i}(x) + \gamma(x)\phi_{N,i}(x)\phi_{N,i}(x)dx \\
&= \frac{1}{h_i^2}\int_{x_{i-1}}^{x_i} \alpha(x) + \beta(x)(x - x_{i-1}) + \gamma(x)(x - x_{i-1})^2 dx \\
&\quad + \frac{1}{h_{i+1}^2}\int_{x_i}^{x_{i+1}} \alpha(x) - \beta(x)(x_{i+1} - x) + \gamma(x)(x_{i+1} - x)^2 dx
\end{aligned}
$$

$$
\begin{aligned}
A_{i,i+1} &= a(\phi_{N,i+1}, \phi_{N,i}) \\
&= \int_{x_i}^{x_{i+1}} \alpha(x)\phi'_{N,i+1}(x)\phi'_{N,i}(x) + \beta(x)\phi'_{N,i+1}(x)\phi_{N,i}(x) + \gamma(x)\phi_{N,i+1}(x)\phi_{N,i}(x)dx \\
&= \frac{1}{h_{i+1}^2}\int_{x_i}^{x_{i+1}} -\alpha(x) + \beta(x)(x_{i+1} - x) + \gamma(x)(x_{i+1} - x)(x - x_i)dx
\end{aligned}
$$

$$
\begin{aligned}
A_{i+1,i} &= a(\phi_{N,i}, \phi_{N,i+1}) \\
&= \int_{x_i}^{x_{i+1}} \alpha(x)\phi'_{N,i}(x)\phi'_{N,i+1}(x) + \beta(x)\phi'_{N,i}(x)\phi_{N,i+1}(x) + \gamma(x)\phi_{N,i}(x)\phi_{N,i+1}(x)dx \\
&= \frac{1}{h_{i+1}^2}\int_{x_i}^{x_{i+1}} -\alpha(x) - \beta(x)(x - x_i) + \gamma(x)(x - x_i)(x_{i+1} - x)dx
\end{aligned}
$$

Using the Simpson rule for $1 \le i \le N$:

$$
\begin{aligned}
\tilde{A}_{i,i} = &\frac{1}{6h_i}\left[\alpha\left(x_{i-1}\right) + 4\alpha\left(\frac{x_{i-1}+x_i}{2}\right) + 2h_i\beta\left(\frac{x_{i-1}+x_i}{2}\right) + h_i^2\gamma\left(\frac{x_{i-1}+x_i}{2}\right) + \alpha(x_i) + h_i\beta(x_i) + h_i^2\gamma(x_i)\right] \\
&+ \frac{1}{6h_{i+1}}\left[\alpha(x_i) - h_{i+1}\beta(x_i) + h_{i+1}^2\gamma(x_i) + 4\alpha\left(\frac{x_i+x_{i+1}}{2}\right) - 2h_{i+1}\beta\left(\frac{x_i+x_{i+1}}{2}\right)\right. \\
&\left. + h_{i+1}^2\gamma\left(\frac{x_i+x_{i+1}}{2}\right) + \alpha(x_{i+1})\right]
\end{aligned}
$$

$$
\tilde{A}_{i,i+1} = \frac{1}{3h_{i+1}}\left[-\alpha\left(x_i\right) - 4\alpha\left(\frac{x_i+x_{i+1}}{2}\right) - \alpha(x_{i+1}) + h_{i+1}\beta\left(x_i\right) + 2h_{i+1}\beta\left(\frac{x_i+x_{i+1}}{2}\right) + h_{i+1}^2\gamma\left(\frac{x_i+x_{i+1}}{2}\right)\right]
$$

$$
\tilde{A}_{i+1,i} = \frac{1}{3h_{i+1}}\left[-\alpha\left(x_i\right) - 4\alpha\left(\frac{x_i+x_{i+1}}{2}\right) - \alpha(x_{i+1}) - 2h_{i+1}\beta\left(\frac{x_i+x_{i+1}}{2}\right) - h_{i+1}\beta\left(x_{i+1}\right) + h_{i+1}^2\gamma\left(\frac{x_i+x_{i+1}}{2}\right)\right]
$$

d) We observe convergence for $\theta = 0.5, 1$ (the experimental convergence rates are 1.008 and 1.006, respectively, with respect to k). Since $\frac{h^2}{k}$ is a constant, we verify the convergence order $O(h^2 + k)$ for $\theta = 1$. For $\theta = 0.5$, the error from spatial discretization is now dominating. The method is not stable with $\theta = 0.3$.

e) The experimental convergence rates observed for $\theta = 1$ is approximately 1.042 with respect to k while the rate for $\theta = 0.5$ is 2.000. With $\theta = 0.3$ the numerical scheme does not converge. Since $\frac{h}{k}$ is a constant, we verify the convergence order $O(h^2 + k^2)$ for $\theta = 0.5$. In the case that $\theta = 1$, the error from time discretization will be dominating. $\theta = 0.3$ will lead to instability.

Failures with $\theta = 0.3$ for both questions remind us that for $\theta \in [0, \frac{1}{2})$ we require that $\frac{h^2}{k}$ shall not be too small to guarantee the stability. In fact, if you test with $\theta = 0.3$, $N = 2^l - 1$ and $M = 6 \times 4^l$(so that $\frac{h^2}{k}$ is increased by six times) with $l = \{2, 3, 4, 5, 6\}$, then the numerical scheme converges with convergence rate 1.009.