Solutions: Series 10

1. Feynman Kac links

a) Set $Z_t := e^{-\int_0^t r(X_s)\,ds}$. Apply the Itô formula and we have

$$
\begin{aligned}
d(Z_t f(t,X_t)) &= -r(X_t)Z_t\,dt + Z_t\,df(t,X_t)\\
&= -r(X_t)Z_t\,dt + Z_t\Bigg(b\left(X_t\right)^\top \nabla f\left(X_t\right)\,dt + \sum_{i=1}^d \partial_{x_i} f\left(X_t\right) \sum_{j=1}^n \Sigma_{ij}\left(X_t\right)\,dW_t^j\\
&\quad + \frac{1}{2}\sum_{i,j=1}^d \left(D^2 f\right)_{ij}\left(X_t\right)\mathcal{Q}_{ij}\left(X_t\right)\,dt\Bigg)\\
&= Z_t\Bigg(-r(X_t)Z_t + b\left(X_t\right)^\top \nabla f\left(X_t\right)\,dt + \frac{1}{2}\sum_{i,j=1}^d \left(D^2 f\right)_{ij}\left(X_t\right)\mathcal{Q}_{ij}\left(X_t\right)\Bigg)\,dt\\
&\quad + Z_t\sum_{i=1}^d \partial_{x_i} f\left(X_t\right)\sum_{j=1}^n \Sigma_{ij}\left(X_t\right)\,dW_t^j\,.
\end{aligned}
$$

It remains to show that the process

$$
Z_t\sum_{i=1}^d \partial_{x_i} f\left(X_t\right)\sum_{j=1}^n \Sigma_{ij}\left(X_t\right)\,dW_t^j
$$

is a martingale. According to Proposition 1.2.7 in the textbook, it is sufficient to show that

$$
\mathbb{E}\left[\int_0^T \Big(Z_t\partial_{x_i} f(X_t)\Sigma_{ij}(X_t)\Big)^2\,ds\right] < \infty\,.
$$

This holds because, for any i,j, by using the Lipschitz continuity of Σ and the boundedness of derivatives of f, there exists a constant C such that,

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T \Big(Z_t\partial_{x_i} f(X_t)\Sigma_{ij}(X_t)\Big)^2\,ds\right] &\le C\left(\sup_{\tau\in[0,T]}|Z_\tau|^2\right)\left(\sup_{x\in\mathbb{R}^d}|\partial_{x_i} f(x)|^2\right)\mathbb{E}\left[\int_0^T 1+|X_s|^2\,ds\right]\\
&\le C\,\mathbb{E}\left[\int_0^T 1+|X_s|^2\,ds\right]\\
&\le TC\left(1+\mathbb{E}\left[\sup_{0\le s\le T}|X_s|^2\right]\right) < \infty\,,
\end{aligned}
$$

where the last step follows by Theorem 1.2.6 in the textbook.

b) Here we only verify the case $t=0$. Since by our assumptions $\partial_t V + \mathcal{A}V - rV = 0$, we have, by **a)**, that the process $M_t := e^{-\int_0^t r(X_s)\,ds}V(t,X_t)$ is a martingale. Thus,

$$
\begin{aligned}
V(0,x) &= \mathbb{E}\left[M_t \mid X_0 = x\right]\\
&= \mathbb{E}\left[M_T \mid X_0 = x\right]\\
&= \mathbb{E}\left[e^{-\int_0^T r(X_s)\,ds}V(T,X_T) \mid X_0 = x\right]\\
&= \mathbb{E}\left[e^{-\int_0^T r(X_s)\,ds}g(T,X_T) \mid X_0 = x\right]\,.
\end{aligned}
$$

2. Basic properties of the Kronecker product

a) Assume that $\mathbf{A}_{ij} \neq 0$ and $\mathbf{B}_{kl} \neq 0$, then the entry of $\mathbf{A}\otimes\mathbf{B}$ located at the cross of $\big((i-1)s+k\big)$-th row and $\big((j-1)t+l\big)$-th column equals to $\mathbf{A}_{ij}\cdot\mathbf{B}_{kl} \neq 0$. This contradicts $\mathbf{A}\otimes\mathbf{B} = \mathbf{0}$. Therefore, at least one of the two matrices $\mathbf{A}$ and $\mathbf{B}$ must be a zero matrix.

b)

$$\begin{aligned}
\mathbf{A}\otimes\mathbf{B} + \mathbf{A}\otimes\mathbf{C} &= \begin{pmatrix} \mathbf{A}_{11}\mathbf{B} & \mathbf{A}_{12}\mathbf{B} & \cdots & \mathbf{A}_{1m}\mathbf{B} \\ \mathbf{A}_{21}\mathbf{B} & \mathbf{A}_{22}\mathbf{B} & \cdots & \mathbf{A}_{2m}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{n1}\mathbf{B} & \mathbf{A}_{n2}\mathbf{B} & \cdots & \mathbf{A}_{nm}\mathbf{B} \end{pmatrix} + \begin{pmatrix} \mathbf{A}_{11}\mathbf{C} & \mathbf{A}_{12}\mathbf{C} & \cdots & \mathbf{A}_{1m}\mathbf{C} \\ \mathbf{A}_{21}\mathbf{C} & \mathbf{A}_{22}\mathbf{C} & \cdots & \mathbf{A}_{2m}\mathbf{C} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{n1}\mathbf{C} & \mathbf{A}_{n2}\mathbf{C} & \cdots & \mathbf{A}_{nm}\mathbf{C} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{A}_{11}(\mathbf{B}+\mathbf{C}) & \mathbf{A}_{12}(\mathbf{B}+\mathbf{C}) & \cdots & \mathbf{A}_{1m}(\mathbf{B}+\mathbf{C}) \\ \mathbf{A}_{21}(\mathbf{B}+\mathbf{C}) & \mathbf{A}_{22}(\mathbf{B}+\mathbf{C}) & \cdots & \mathbf{A}_{2m}(\mathbf{B}+\mathbf{C}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{n1}(\mathbf{B}+\mathbf{C}) & \mathbf{A}_{n2}(\mathbf{B}+\mathbf{C}) & \cdots & \mathbf{A}_{nm}(\mathbf{B}+\mathbf{C}) \end{pmatrix} \\
&= \mathbf{A}\otimes(\mathbf{B}+\mathbf{C}).
\end{aligned}$$

c)

$$\begin{aligned}
(\mathbf{A}\otimes\mathbf{B})\otimes\mathbf{C} &= \begin{pmatrix} \mathbf{A}_{11}\mathbf{B} & \mathbf{A}_{12}\mathbf{B} & \cdots & \mathbf{A}_{1m}\mathbf{B} \\ \mathbf{A}_{21}\mathbf{B} & \mathbf{A}_{22}\mathbf{B} & \cdots & \mathbf{A}_{2m}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{n1}\mathbf{B} & \mathbf{A}_{n2}\mathbf{B} & \cdots & \mathbf{A}_{nm}\mathbf{B} \end{pmatrix} \otimes \mathbf{C} \\
&= \begin{pmatrix} \mathbf{A}_{11}\mathbf{B}_{11}\mathbf{C} & \mathbf{A}_{11}\mathbf{B}_{12}\mathbf{C} & \cdots & \mathbf{A}_{1m}\mathbf{B}_{1t}\mathbf{C} \\ \mathbf{A}_{11}\mathbf{B}_{21}\mathbf{C} & \mathbf{A}_{11}\mathbf{B}_{22}\mathbf{C} & \cdots & \mathbf{A}_{2m}\mathbf{B}_{2t}\mathbf{C} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{n1}\mathbf{B}_{s1}\mathbf{C} & \mathbf{A}_{n2}\mathbf{B}_{s2}\mathbf{C} & \cdots & \mathbf{A}_{nm}\mathbf{B}_{st}\mathbf{C} \end{pmatrix} \\
&= \mathbf{A}\otimes \begin{pmatrix} \mathbf{B}_{11}\mathbf{C} & \mathbf{B}_{12}\mathbf{C} & \cdots & \mathbf{B}_{1t}\mathbf{C} \\ \mathbf{B}_{21}\mathbf{C} & \mathbf{B}_{22}\mathbf{C} & \cdots & \mathbf{B}_{2t}\mathbf{C} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{s1}\mathbf{C} & \mathbf{B}_{s2}\mathbf{C} & \cdots & \mathbf{B}_{st}\mathbf{C} \end{pmatrix} \\
&= \mathbf{A}\otimes(\mathbf{B}\otimes\mathbf{C}).
\end{aligned}$$

d) Consider, for example,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\mathbf{A}\otimes\mathbf{B} = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{B}\otimes\mathbf{A}$$

e) For any $1 \leq j \leq j' \leq ns$, assume that

$$j = s(i_1 - 1) + i_2, \quad j' = s(i_1' - 1) + i_2'$$

with $i_1, i_1' \in \{1, 2, \ldots n\}$ and $i_2, i_2' \in \{1, 2, \ldots s\}$. Then $(\mathbf{A}\otimes\mathbf{B})_{j,j'} = \mathbf{A}_{i_1,i_1'}\cdot\mathbf{B}_{i_2,i_2'}$. Since $\mathbf{A}$ and $\mathbf{B}$ are symmetric,

$$(\mathbf{A}\otimes\mathbf{B})_{j,j'} = \mathbf{A}_{i_1,i_1'}\cdot\mathbf{B}_{i_2,i_2'} = \mathbf{A}_{i_1',i_1}\cdot\mathbf{B}_{i_2',i_2} = (\mathbf{A}\otimes\mathbf{B})_{j',j}.$$

3. The pricing equation for multi-asset options

a) Let us fix a time $t \in J$. We multiply with a test function $v \in C^\infty(G)$ and integrate over G. We obtain

$$(\partial_t u(t,\cdot), v) - \frac{1}{2}\int_G v(\mathbf{x})\, \nabla \cdot (\boldsymbol{\mathcal{Q}} \nabla u(t,\mathbf{x}))\, d\mathbf{x} + \int_G \mu^\top \nabla u(t,\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x} + r\int_G u(t,\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x} = 0$$

and by partial integration, this is equivalent to, using Green's formula,

$$\begin{aligned}(\partial_t u(t,\cdot), v) + \frac{1}{2}\int_G \nabla v(\mathbf{x})^\top\, \boldsymbol{\mathcal{Q}} \nabla u(t,\mathbf{x})\, d\mathbf{x} + \int_G \mu^\top \nabla u(t,\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x}& \\ + r\int_G u(t,\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x} = \frac{1}{2}\int_{\partial G} (\boldsymbol{\mathcal{Q}} \nabla u) \cdot \mathbf{n} v\, ds\,.&\end{aligned}$$

We define the bilinear form $a(w,v) := \frac{1}{2}\int_G \nabla w(\mathbf{x})^\top\, \boldsymbol{\mathcal{Q}} \nabla v(\mathbf{x})\, d\mathbf{x} + \int_G \mu^\top \nabla w(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x} + r\int_G w(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x}$. The right-hand side vanishes if $u, v \in V$ as the integrand vanishes on both $\partial_1 G$ and $\partial_2 G$ due to the boundary conditions. This leads to the weak formulation.

b) We firstly show that $\boldsymbol{\mathcal{Q}}$ is positive definite. This requires us to verify that for any nonzero $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{x}^\top \boldsymbol{\mathcal{Q}} \mathbf{x} > 0$. This claim is correct because

$$\mathbf{x}^\top \boldsymbol{\mathcal{Q}} \mathbf{x} = (\boldsymbol{\Sigma}^\top \mathbf{x})^\top (\boldsymbol{\Sigma}^\top \mathbf{x}) > 0.$$

Here the last inequality is valid because $\det(\boldsymbol{\Sigma}) \neq 0$ implies that $\boldsymbol{\Sigma}^\top \mathbf{x}$ is not a zero vector.

The positive definiteness of $\boldsymbol{\mathcal{Q}}$ implies that there exists $\gamma > 0$ such that for any $\mathbf{x} \in \mathbb{R}^d$ we have $\mathbf{x}^\top \boldsymbol{\mathcal{Q}} \mathbf{x} \geq \gamma \mathbf{x}^\top \mathbf{x}$. So, for $v \in V$ and any constant $C_2 > 0$,

$$\begin{aligned}a(v,v) + C_2\|v\|^2_{L^2(G)} &= \frac{1}{2}\int_G \nabla v(\mathbf{x})^\top\, \boldsymbol{\mathcal{Q}} \nabla v(\mathbf{x})\, d\mathbf{x} + \int_G \mu^\top \nabla v(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x} + (r + C_2)\int_G v(\mathbf{x})^2\, d\mathbf{x} \\ &\geq \frac{\gamma}{2}\|\nabla v\|^2_{L^2(G)} - \|\mu^\top \nabla v\|_{L^2(G)} \cdot \|v\|_{L^2(G)} + (r + C_2)\|v\|^2_{L^2(G)} \\ &\geq \frac{\gamma}{2}\|\nabla v\|^2_{L^2(G)} - \sqrt{2}\|\mu\|_{l^\infty} \cdot \|\nabla v\|_{L^2(G)} \cdot \|v\|_{L^2(G)} + (r + C_2)\|v\|^2_{L^2(G)}.\end{aligned}$$

Apply the inequality $ab \leq \epsilon/2 \cdot a^2 + 1/(2\epsilon) \cdot b^2$ with $\epsilon = \gamma/(2\sqrt{2}\|\mu\|_\infty)$, $a = \|\nabla v\|_{L^2(G)}$ and $b = \|v\|_{L^2(G)}$ to bound $\sqrt{2}\|\mu\|_\infty \|\nabla u\|_{L^2(G)} \|v\|_{L^2(G)}$, we have

$$\begin{aligned}a(v,v) + C_2\|v\|^2_{L^2(G)} &\geq (\frac{\gamma}{2} - \frac{\gamma}{4})\|\nabla v\|^2_{L^2(G)} + (r + C_2 - \frac{\sqrt{2}\|\mu\|_\infty}{2\epsilon})\|v\|^2_{L^2(G)} \\ &\geq \frac{\gamma}{4}\|\nabla v\|^2_{L^2(G)} + \big(C_2 - (\frac{\sqrt{2}\|\mu\|_\infty}{2\epsilon} - r)\big)\|v\|^2_{L^2(G)}.\end{aligned}$$

This means that we could set $C_1 = \frac{\gamma}{4}$ and $C_2 = |\frac{\sqrt{2}\|\mu\|_\infty}{2\epsilon} - r|$.

c) We proved before that $\boldsymbol{\mathcal{Q}}$ is positive definite. By Theorem 8.3.1 in the textbook, we know that for $\widehat{G} := (-\frac{\sqrt{2}}{2}R, \frac{\sqrt{2}}{2}R)^2 \subset G$, there exists $C, \widehat{\gamma}_1, \gamma_2 > 0$ depending only on $\boldsymbol{\mathcal{Q}}, \mu, T, g$ such that

$$\begin{aligned}|V_{\widehat{G}}(t, s_1, s_2) - V(t, s_1, s_2)| &= \mathbb{E}\left[e^{-r(T-t)} g(S^1_T, S^2_T)\mathbf{1}_{\{T \geq \tau_{\widehat{G}}\}} \mid S^i_t = s_i,\ i = 1,2\right] \\ &\leq C\exp(-\widehat{\gamma}_1 \cdot \frac{\sqrt{2}}{2}R + \gamma_2 \max(s_1, s_2)).\end{aligned}$$

Here,

$$V_{\widehat{G}}(t, s_1, s_2) = \mathbb{E}\left[e^{-r(T-t)} g(S^1_T, S^2_T)\mathbf{1}_{\{T < \tau_{\widehat{G}}\}} \mid S^i_t = s_i,\ i = 1,2\right].$$

Moreover, it is easy to observe that $\{T \geq \tau_{\widehat{G}}\} \supset \{T \geq \tau_G\}$. Therefore, by setting $\gamma_1 = \widehat{\gamma}_1 \cdot \frac{\sqrt{2}}{2}$,

$$\begin{aligned}
|V_G(t, s_1, s_2) - V(t, s_1, s_2)| &= \mathbb{E}\left[e^{-r(T-t)} g(S_T^1, S_T^2) \mathbf{1}_{\{T \geq \tau_G\}} \mid S_t^i = s_i,\ i = 1, 2\right] \\
&\leq \mathbb{E}\left[e^{-r(T-t)} g(S_T^1, S_T^2) \mathbf{1}_{\{T \geq \tau_{\widehat{G}}\}} \mid S_t^i = s_i,\ i = 1, 2\right] \\
&\leq C \exp(-\widehat{\gamma}_1 \cdot \frac{\sqrt{2}}{2} R + \gamma_2 \max(s_1, s_2)) \\
&= C \exp(-\gamma_1 \cdot R + \gamma_2 \max(s_1, s_2)).
\end{aligned}$$