q is the matrix of the one particle process. Q is the matrix of the two particle process. f and g are eigenvectors of q.

$$\begin{aligned}
q_i^j f_j &= \phi f_i \\
q_i^j g_j &= \gamma g_i \\
f^i g_i &= 0 \\
\\
Q_{ij}^{kl} &= q_j^l \delta_i^k + q_i^k \delta_j^l + q_i^k \delta_i^l \delta_j^k + q_i^l \delta_i^k \delta_j^l - q_j^l \delta_i^k \delta^{kl} - q_i^k \delta_j^l \delta^{kl}
\end{aligned}$$

A is an anti-symmetrix eigenfunction of Q.

$$\begin{aligned}
A_{ij} &= f_i g_j - g_i f_j \\
Q_{ij}^{kl} A_{kl} &= \left(q_j^l \delta_i^k + q_i^k \delta_j^l + q_i^k \delta_i^l \delta_j^k + q_i^l \delta_i^k \delta_j^l - q_j^l \delta_i^k \delta^{kl} - q_i^k \delta_j^l \delta^{kl}\right) \cdot (f_k g_l - g_k f_l) \\
\\
&= q_j^l \delta_i^k f_k g_l - q_j^l \delta_i^k g_k f_l + q_i^k \delta_j^l f_k g_l - q_i^k \delta_j^l g_k f_l \\
&\quad + q_i^k \delta_i^l \delta_j^k f_k g_l - q_i^k \delta_i^l \delta_j^k g_k f_l + q_i^l \delta_i^k \delta_j^l f_k g_l - q_i^l \delta_i^k \delta_j^l g_k f_l \\
&\quad - \left(q_j^l \delta_i^k + q_i^k \delta_j^l\right) \delta^{kl} \left(f_k g_l - g_k f_l\right) \\
&= q_j^l f_i g_l - q_j^l g_i f_l + q_i^k f_k g_j - q_i^k g_k f_j \\
&\quad + q_i^j f_j g_i - q_i^j f_i g_j + q_i^j f_i g_j - q_i^j f_j g_i \\
&= \gamma f_i g_j - \phi g_i f_j + \phi f_i g_j - \gamma g_i f_j \\
&= \gamma (f_i g_j - g_i f_j) + \phi (f_i g_j - g_i f_j) \\
&= (\gamma + \phi) A_{ij}
\end{aligned}$$

$S^{(1)}$ is a symmetric eigenvector of Q.

$$\begin{aligned}
S_{ij}^{(1)} &= f_i + f_j \\
Q_{ij}^{kl} S_{kl}^{(1)} &= \left(q_j^l \delta_i^k + q_i^k \delta_j^l + q_i^k \delta_i^l \delta_j^k + q_i^l \delta_i^k \delta_j^l - q_j^l \delta_i^k \delta^{kl} - q_i^k \delta_j^l \delta^{kl}\right) \cdot (f_k + f_l) \\
&= q_j^l \delta_i^k f_k + q_j^l \delta_i^k f_l + q_i^k \delta_j^l f_k + q_i^k \delta_j^l f_l + q_i^k \delta_i^l \delta_j^k f_k + q_i^k \delta_i^l \delta_j^k f_l + q_i^l \delta_i^k \delta_j^l f_k + q_i^l \delta_i^k \delta_j^l f_l \\
&\quad - q_j^l \delta_i^k \delta^{kl} f_k - q_j^l \delta_i^k \delta^{kl} f_l - q_i^k \delta_j^l \delta^{kl} f_k - q_i^k \delta_j^l \delta^{kl} f_l \\
&= \phi f_j + \phi f_i + q_i^j f_j + q_i^j f_i + q_i^j f_i + q_i^j f_j - q_j^i f_i - q_j^i f_i - q_i^j f_j - q_i^j f_j \\
&= \phi (f_i + f_j)
\end{aligned}$$

$S^{(2)}$ is **not** a symmetric eigenvector of Q.

$$\begin{aligned}
S_{ij}^{(2)} &= (n-2)(f_i g_j + g_i f_j) + 2(f_i g_i + f_j g_j) \\
Q_{ij}^{kl} S_{kl}^{(2)} &= \left(q_j^l \delta_i^k + q_i^k \delta_j^l + q_i^k \delta_i^l \delta_j^k + q_i^l \delta_i^k \delta_j^l - q_j^l \delta_i^k \delta^{kl} - q_i^k \delta_j^l \delta^{kl}\right) \cdot ((n-2)(f_k g_l + g_k f_l) + 2(f_k g_k + f_l g_l)) \\
&= (n-2)(q_j^l \delta_i^k f_k g_l + q_i^k \delta_j^l f_k g_l + q_i^k \delta_i^l \delta_j^k f_k g_l + q_i^l \delta_i^k \delta_j^l f_k g_l - q_j^l \delta_i^k \delta^{kl} f_k g_l - q_i^k \delta_j^l \delta^{kl} f_k g_l) +
\end{aligned}$$

$$
\begin{aligned}
& (n-2)(q_j^l\delta_i^k g_k f_l + q_i^k\delta_j^l g_k f_l + q_i^k\delta_i^l\delta_j^k g_k f_l + q_i^l\delta_i^k\delta_j^l g_k f_l - q_j^l\delta_i^k\delta^{kl} g_k f_l - q_i^k\delta_j^l\delta^{kl} g_k f_l) + \\
& 2(q_j^l\delta_i^k f_k g_k + q_i^k\delta_j^l f_k g_k + q_i^k\delta_i^l\delta_j^k f_k g_k + q_i^l\delta_i^k\delta_j^l f_k g_k - q_j^l\delta_i^k\delta^{kl} f_k g_k - q_i^k\delta_j^l\delta^{kl} f_k g_k) + \\
& 2(q_j^l\delta_i^k g_l f_l + q_i^k\delta_j^l g_l f_l + q_i^k\delta_i^l\delta_j^k g_l f_l + q_i^l\delta_i^k\delta_j^l g_l f_l - q_j^l\delta_i^k\delta^{kl} g_l f_l - q_i^k\delta_j^l\delta^{kl} g_l f_l) \\
= \; & (n-2)(q_j^l f_i g_l + q_i^k f_k g_j + q_i^j f_j g_i + q_i^j f_i g_j - q_j^i f_i g_i - q_i^j f_j g_j) + \\
& (n-2)(q_j^l g_i f_l + q_i^k g_k f_j + q_i^j g_j f_i + q_i^j g_i f_j - q_j^i g_i f_i - q_i^j g_j f_j) + \\
& 2(q_j^l f_i g_i + q_i^k f_k g_k + q_i^j f_j g_j + q_i^j f_i g_i - q_j^l f_i g_i - q_i^j f_j g_j) + \\
& 2(q_j^l g_l f_l + q_i^k g_j f_j + q_i^j g_i f_i + q_i^j g_j f_j - q_j^i g_i f_i - q_i^j g_j f_j) \\
= \; & (n-2)(\gamma f_i g_j + \phi f_i g_j + q_i^j f_j g_i + q_i^j f_i g_j - q_i^j f_i g_i - q_i^j f_j g_j) + \\
& (n-2)(\phi f_j g_i + \gamma f_j g_i + q_i^j f_i g_j + q_i^j f_j g_i - q_i^j f_i g_i - q_i^j f_j g_j) + \\
& 2\left(q_i^k f_k g_k + q_j^l g_l f_l\right) \\
= \; & (n-2)\Big((\gamma+\phi)(f_i g_j + f_j g_i) + 2q_i^j(f_i g_j + f_j g_i - f_i g_i - f_j g_j)\Big) + 2(q_i^k + q_j^k) f_k g_k \\
= \; & (\gamma+\phi)(n-2)(f_i g_j + f_j g_i) + 2(n-2)q_i^j(f_i g_j + f_j g_i - f_i g_i - f_j g_j) + 2(q_i^k + q_j^k) f_k g_k \\
= \; & (\gamma+\phi)S_{ij}^{(2)} - (\gamma+\phi)2(f_i g_i + f_j g_j) + 2(n-2)q_i^j(f_i g_j + f_j g_i - f_i g_i - f_j g_j) + 2(q_i^k + q_j^k) f_k g_k
\end{aligned}
$$

P is a symmetric eigenvector of Q, orthogonal to all $S^{(1)}$'s ($S = S^{(1)}$).

$$
\begin{aligned}
P_{ij} &= P_{ij} \\
P_{ii} &= 0 \\
0 &= P_{ij} S^{kl} \\
0 &= P_{ij}(f^i + f^j) \\
0 &= P_{ij} f^i 1^j + P_{ij} f^j 1^i \\
0 &= P_{ij} f^j 1^i
\end{aligned}
$$

because the f's are a complete basis:

$$
\sum_i P_{ij} = \sum_j P_{ij} = 0
$$

$$
\begin{aligned}
Q_{ij}^{kl} P_{kl} &= \left(q_j^l\delta_i^k + q_i^k\delta_j^l + q_i^k\delta_i^l\delta_j^k + q_i^l\delta_i^k\delta_j^l\right) P_{kl} \\
&= q_j^l\delta_i^k P_{kl} + q_i^k\delta_j^l P_{kl} + q_i^k\delta_i^l\delta_j^k P_{kl} + q_i^l\delta_i^k\delta_j^l P_{kl} \\
&= q_j^k P_{ik} + q_i^k P_{jk} + 2q_i^j P_{ij}
\end{aligned}
$$

P is an eigenvector of Q.

$$
q_j^k P_{ik} + q_i^k P_{jk} + 2q_i^j P_{ij} = \lambda P_{ij}
$$