Networks and Markets
Homework 3

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Part 1: Matching Markets and Exchange Networks

1. Consider two sellers, a and b, each offering a distinct house for sale, and a set of two buyers, x and y. x has a value of 2 for a's house and 4 for b's house, while y has a value of 3 for a's house and 6 for b's house, summarized in the following table.

Buyer	a's house	b's house
value for x	2	4
value for y	3	6

   (a) Suppose that a charges a price of 0 for her house, and b charges a price of 1. Is this set of prices market-clearing? As part of your answer, say what the preferred choice graph is with this given set of prices, and use this to justify your answer.

   (b) If the above prices are not market-clearing, compute a set of market-clearing prices p with an associated matching M. Explain how you found them and why they are market-clearing (you don't necessarily need to use Theorem 8.2 in the notes for this, but it could be helpful).

   (c) For the market equilibrium computed above (corresponding to a set of market-clearing prices), compute the social value of the buyers and the social welfare of the buyers and sellers. Briefly explain why these are the maximum possible in this matching market context.

2. Suppose now we have a set of three sellers a, b, and c, and a three buyers labeled, x, y, and z. The valuations of the buyers are summarized in the following table.

Buyer	a's house	b's house	c's house
value for x	5	7	1
value for y	2	3	1
value for z	5	4	4

   (a) Using the algorithm from Theorem 8.2 in the notes, compute a market equilibrium (consisting of a matching and set of prices). Show your work.

   (b) Recall that in chapter 9, we showed that every matching market context can be encoded as an exchange network, and furthermore, exchange networks are "symmetric" in the sense that the buyers and sellers have the same role. As a result, we can flip the script such that the sellers are now the "buyers" and the buyers are now the "sellers." The corresponding matching market context now has the following set of values.

Seller	if x buys	if y buys	if z buys
value possible for a	5	2	5
value possible for b	7	3	4
value possible for c	1	1	4

You can think of the value of a seller being the maximum value it can charge for a home. Now the prices for the buyer correspond to discounts on the house (increasing value for the buyers).

Using the algorithm from Theorem 8.2 in the notes, compute a market equilibrium (consisting of a matching and set of "prices") for this new matching market context. Show your work.

(c) Interpret the matching and "prices" computed in part (b) as a matching and actual prices charged when selling the houses. How do these values compare to those computed in part (a)? Give a 1-3 sentence intuitive explanation of why the values might be different. In particular, explain what the consequence is that the algorithm of Theorem 8.2 always has a "free" item in the resulting market equilibrium.

3. (a) We observed (Claim 9.2) that an exchange network consisting of a cyclic graph of 3 nodes has no stable outcome. Does this generalize to every cyclic graph? If not, can we characterize for which values of n the n-node cyclic graph has a stable outcome? Justify your answers.

   (b) Show by constructing an example that an exchange network that contains a 3-node cycle (but doesn't necessarily entirely consist of one) can still have a stable outcome.

Part 2: Auctions and Mechanism Design

4. Consider an auction where a seller wants to sell one unit of a good to a group of bidders. The seller runs a sealed-bid, second-price auction. Your firm will bid in the auction, but it does not know for sure how many other bidders will participate in the auction. There will be either two or three other bidders in addition to your firm. All bidders have independent, private values for the good. Your firm's value for the good is c. What bid should your firm submit, and how does it depend on the number of other bidders who show up? Give a brief (1-3 sentence) explanation for your answer.

5. Consider a second-price, sealed-bid auction with two bidders who have independent, private values v_i which are either 1 or 3. For each bidder, suppose the probabilities of v_i being 1 or 3 are independent (from other bidders) and both 1/2. (If there is a tie at a bid of x for the highest bid the winner is selected at random from among the highest bidders and the price is x.)

   (a) What is the seller's expected revenue from the auction? Justify your answer.

   (b) Assume that we add a third bidder that behaves identically to the first two (whose value v_i is also chosen independently of those for the first two). What is the seller's expected revenue? Justify your answer.

   (c) Explain the trend you noticed between parts (a) and (b)—that is, why changing the number of bidders affects (or doesn't affect) the seller's expected revenue.

* **Bonus Question 1.** Let's say we define a "third-price auction" in the same manner that we defined the first-price and second-price auctions. Is this mechanism DST, NT, or neither? Does it DST-implement, NT-implement, Nash-implement, or fail to implement social value maximization? Justify your answers, by proof or by counterexample.

Part 3: Sponsored Search

6. Suppose a search engine has two ad slots that it can sell. Slot a has a clickthrough rate of 10 and slot b has a clickthrough rate of 5. There are three advertisers who are interested in these slots. Advertiser x values clicks at 3 per click, advertiser y values clicks at 2 per click, and advertiser z values clicks at 1 per click.

   Compute the socially optimal allocation and the VCG prices for it (using the Clarke pivot rule). Show your work in full.

Part 4: Implementing Matching Market Pricing

The goal of this exercise is to implement an algorithm for finding market-clearing and VCG prices in a bipartite matching market. Include your code in `hw3_matchingmarket.py`.

7. Recall the procedure constructed in Theorem 8.2 of the notes to find a market equilibrium in a matching market.

   (a) The first step of this procedure involves either finding a perfect matching or a constricted set. Recall that this can be done using maximum flow. Now, using your maximum-flow implementation from assignment 2, implement an algorithm `matching_or_cset` that finds either a perfect matching M or a constricted set S in a bipartite graph.

   (b) Now, given a bipartite matching frame with n players, m items, and values of each player for each item, implement the full procedure `market_eq` to find a market equilibrium. Make sure you test your algorithm on the matching market frame with values as described in Lecture 5 Page 7.

   The outcome found by `market_eq` should be socially optimal. (Make sure you understand why this is true.) Remember to thoroughly test your code on a variety of inputs.

8. We now switch gears, and examine another way of assigning prices in a matching market, but this time in the context of mechanism design. In this problem we implement VCG pricing for a matching market.

   (a) Implement the VCG mechanism and Clarke pivot rule to construct an algorithm that produces a positive set of VCG prices in any matching market frame. Make sure you test your algorithm on the matching market frame with values as described in Lecture 5 Page 7, and on your own choice of test cases.

   (b) How do the prices output by the VCG mechanism compare with the ones output by the algorithm from Problem 7? Include your observations in your pdf write-up, and be sure to include multiple concrete examples.

* **Bonus Question 2.** (Please note that each part of this question is intended to be harder than the previous parts; each part will be graded separately, and you are not required to do the whole thing.)

(a) We now examine bundles of identical goods, where "item" i is actually a bundle of i identical goods (meaning the same good across all bundles, so e.g. item i is a bundle of i apples and item j is a bundle of j apples). Implement a routine `random_bundles_valuations` that takes parameters n, m, and generates a matching market context for m bundles of identical goods, where each of n buyers has a random value for an *individual good* (between 1 to 50; ties are allowed). Now, run your VCG pricing algorithm on such a randomly chosen context, for $n = m = 20$. Analyze the results in your write-up pdf – do they make sense in the bundles context?

(b) Implement an algorithm `gsp()` for GSP pricing in a matching-market context. Compare your VCG prices from part (a) to the GSP prices when run in the same contexts (with the same randomness). Also, try to find and characterize some contexts where VCG and GSP prices are similar, and where they are wildly different. Write-up your analysis and include a few concrete examples.

(c) GSP isn't truthful, so interesting things might happen if we run BRD on a GSP matching market auction (where a player's "strategy" is their valuation report). Implement an algorithm that picks a random starting point and runs BRD to attempt to find a GSP equilibrium state. What happens? Does BRD converge; if so, how quickly? Write-up your analysis and include a few concrete examples.

(d) *(Very difficult.)* Either prove that BRD will converge in a GSP context, or disprove it by counterexample.

Part 5: Exchange Networks for Uber

We will construct a simplified market scenario for a ridesharing app like Uber. Our world will consist of an $\ell \times \ell$ grid (think of 100×100 as a test example), and there will be two types of participants, riders and drivers.

- A rider R is specified by a current location $(x_0, y_0) \in [\ell] \times [\ell]$, a desired destination $(x_1, y_1) \in [\ell] \times [\ell]$, and a value for reaching that destination.
- A driver D is specified by a current location $(x_0, y_0) \in [\ell] \times [\ell]$.

We define the cost of a matching between a rider R and a driver D, $c(R, D)$, to be the distance from the driver to the rider and then to the destination (measured via Manhattan distance (i.e., L^1 distance), so the distance from $(0, 0)$ to $(5, 2)$ is 7). Include your code in `hw3_uber.py`.

9. Encode the above example as an exchange network and implement it in your code (in `exchange_network_from_uber`). Namely, define a graph $G = (V, E)$ where the vertices are the riders and drivers and there is an edge between every rider and driver. What value should we associate with each edge in the graph? Justify your answer in your pdf write-up.

10. Using your algorithm for matching markets above, implement a procedure (`stable_outcome`) that computes a stable outcome in this context. Note that there may not be the same number of riders and drivers.

(a) Construct at least two test examples with at least 5 riders and drivers (filling in `rider_driver_example_1` and `rider_driver_example_2`). Compute the stable outcome and convince yourself that the results – i.e. driver profit and price the riders pay – make sense.

(b) Implement a procedure (`random_riders_drivers_stable_outcomes`) that on a 100×100 grid (1) generates n riders (located randomly in the grid with random destinations) each with a value of 100, m drivers (also located randomly in the grid) and (2) computes a stable outcome given this context. Run your procedure many times when $n = m$ (say, 10 each), n is much less than m (say, 5 vs. 20), and when n is much greater than m (say, 20 vs. 5). Discuss your results in detail. Specifically, analyze prices and profits for different ranges of n and m and include your analysis in the pdf write-up.

11. With ridesharing apps, drivers often have preferences for where they want to drive. For example, they know they are more likely to get a high value ride from the airport. Describe in a few sentences how you could modify the existing setup to include driver's preferences.

* **Bonus question 3** Suppose that the city implements a public transportation system. The cost to take the public transportation system is $a + b \cdot dist(\text{initial location}, \text{destination})$, where a is a fixed base fare and b is some constant factor multiplier. Furthermore, anyone can take this public transportation system from any location.

(a) Implement a procedure that includes the public transportation option in when computing the stable outcome above. Does a stable outcome always exist in this case?

(b) Choose different values for a and b and discuss how it affects the prices charged to riders/ profits of drivers.