Homework Assignment 3 - Coding Part Write-up
Networks and Markets

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August 6, 2024

Part 4: Implementing Matching Market Pricing

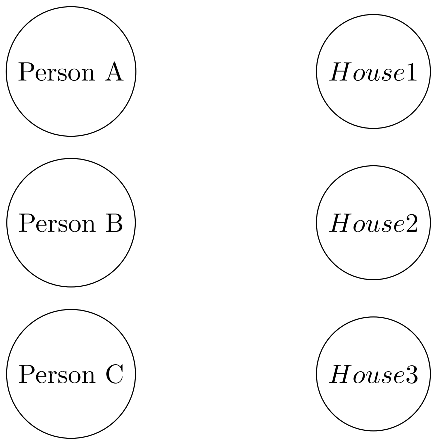
1 Question 7

(b) Consider the matching market example in Lecture 5 Page 7:

Values for Person A: $\{House1 : 4, House2 : 12, House3 : 5\}$

Values for Person B: $\{House1 : 7, House2 : 10, House3 : 9\}$

Values for Person C: $\{House1 : 7, House2 : 7, House3 : 10\}$



Formally, the matching market context is $\Gamma = (\{A, B, C\}, \{1, 2, 3\}, v)$, where v is the valuation function defined as follows:

$$v_A(1) = 4, v_A(2) = 12, v_A(3) = 5$$
$$v_B(1) = 7, v_B(2) = 10, v_B(3) = 9$$
$$v_C(1) = 7, v_C(2) = 7, v_C(3) = 10$$

We turn to run the algorithm of Theorem 8.8 to find a market equilibrium (p, M) to find the maximum social value, in order to validate out implementation's output. We begin by initializing the prices vector $\vec{p} \equiv 0$ to be the zero vector. We then proceed to run the algorithm, updating the prices vector until there is a perfect matching M in the induced preferred choice graph for $(\Gamma, \vec{p})$:

1. Observing the following *induced preferred-choice graph* from $(\Gamma, \vec{p})$:

Values for A: $\{1:4, 2:12, 3:5\}$

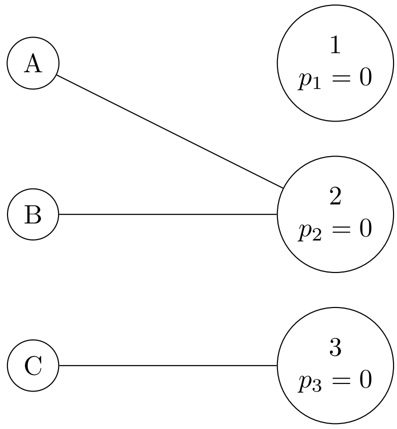
Utility for A: $\{1:4, 2:12, 3:5\}$

Values for B: $\{1:7, 2:10, 3:9\}$

Utility for B: $\{1:7, 2:10, 3:9\}$

Values for C: $\{1:7, 2:7, 3:10\}$

Utility for C: $\{1:7, 2:7, 3:10\}$



There obviously isn't a perfect matching as $S = \{A, B\}$ is a constricted set with $|N(S)| = |\{2\}| = 1 < 2 = |S|$ (which, by a theorem we've seen in class, implies that there isn't a perfect matching). Thus, we raise the prices for all items in $N(S)$ by 1, and update the prices vector $\vec{p}$ accordingly. The updated prices vector is $\vec{p} = (a:0, b:1, c:0)$. Not all prices are greater than zero, so we don't perform a shift operation, and we proceed to the next iteration.

2. Observing the following *induced preferred-choice graph* from $(\Gamma, \vec{p})$:

Values for A: $\{1:4, 2:12, 3:5\}$

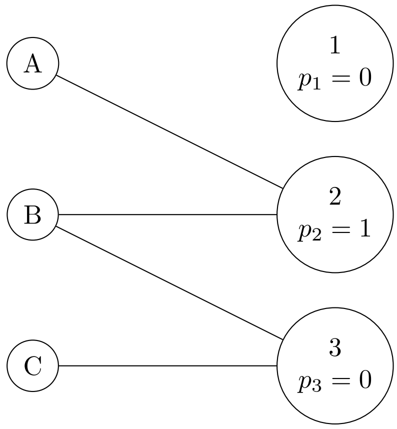
Utility for A: $\{1:4, 2:11, 3:5\}$

Values for B: $\{1:7, 2:10, 3:9\}$

Utility for B: $\{1:7, 2:9, 3:9\}$

Values for C: $\{1:7, 2:7, 3:10\}$

Utility for C: $\{1:7, 2:6, 3:10\}$



There obviously isn't a perfect matching as $S = \{A, B, C\}$ is a constricted set with $|N(S)| = |\{2, 3\}| = 2 < 3 = |S|$ (which, by a theorem we've seen in class, implies that there isn't a perfect matching). Thus, we raise the prices for all items in $N(S)$ by 1, and update the prices vector $\vec{p}$ accordingly. The updated prices vector is $\vec{p} = (a:0, b:2, c:1)$. Not all prices are greater than zero, so we don't perform a shift operation, and we proceed to the next iteration.

3. Observing the following *induced preferred-choice graph* from $(\Gamma, \vec{p})$:

Values for A: $\{1:4, 2:12, 3:5\}$

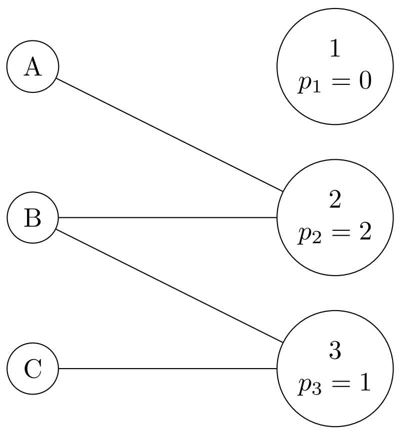
Utility for A: $\{1:4, 2:10, 3:4\}$

Values for B: $\{1:7, 2:10, 3:9\}$

Utility for B: $\{1:7, 2:8, 3:8\}$

Values for C: $\{1:7, 2:7, 3:10\}$

Utility for C: $\{1:7, 2:5, 3:9\}$



Similar to the previous iteration, we raise the prices for $\{2, 3\}$, and update the prices vector

$\vec{p}$ accordingly. The updated prices vector is $\vec{p} = (a : 0, b : 3, c : 2)$. Not all prices are greater than zero, so we don't perform a shift operation, and we proceed to the next iteration.

4. Observing the following *induced preferred-choice graph* from $(\Gamma, \vec{p})$:

Values for A: $\{1 : 4, 2 : 12, 3 : 5\}$

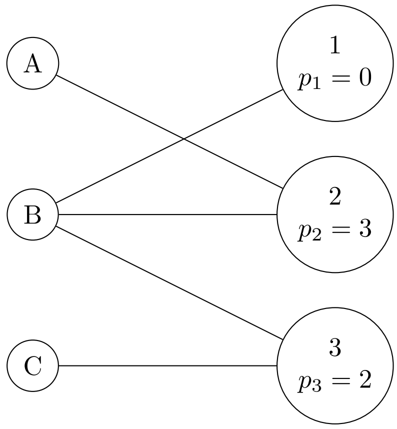
Utility for A: $\{1 : 4, 2 : 9, 3 : 3\}$

Values for B: $\{1 : 7, 2 : 10, 3 : 9\}$

Utility for B: $\{1 : 7, 2 : 7, 3 : 7\}$

Values for C: $\{1 : 7, 2 : 7, 3 : 10\}$

Utility for C: $\{1 : 7, 2 : 4, 3 : 8\}$



And there is a perfect matching in the induced preferred choice graph, which is $M = \{\{A, 2\}, \{B, 1\}, \{C, 3\}\}$. Thus, the market equilibrium is $(\vec{p}, M) = ((1 : 0, 2 : 3, 3 : 2), \{\{A, 2\}, \{B, 1\}, \{C, 3\}\})$, and we are done.

We found the market equilibrium to be $(\vec{p}, M) = ((1 : 0, 2 : 3, 3 : 2), \{\{A, 2\}, \{B, 1\}, \{C, 3\}\})$. The maximum social value is therefore $v(A, 2) + v(B, 1) + v(C, 3) = 12 + 7 + 10 = 29$.

Our algorithm found exactly this market equilibrium.

2 Question 8

(a) In this part we analyze how the prices output by the VCG mechanism compare with the ones output by the algorithm of Theorem 8.8 (finding a market equilibrium (p, M)). The following are the examples we analyze and their corresponding results for each mechanism:

1. Example 1:

Values for b_1: $\{i_1 : 3, i_2 : 0, i_3 : 15\}$

Values for b_2: $\{i_1 : 11, i_2 : 12, i_3 : 8\}$

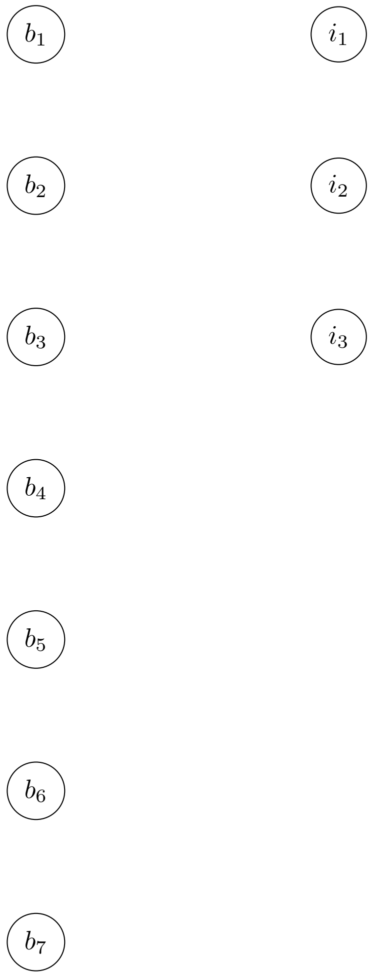
Values for b_3: $\{i_1 : 14, i_2 : 13, i_3 : 18\}$

Values for b_4: $\{i_1 : 3, i_2 : 18, i_3 : 5\}$

Values for b_5: $\{i_1 : 15, i_2 : 9, i_3 : 1\}$

Values for b_6: $\{i_1 : 14, i_2 : 17, i_3 : 9\}$

Values for b_7: $\{i_1 : 14, i_2 : 17, i_3 : 13\}$



And we observe that the prices output by the VCG mechanism and the algorithm of Theorem 8.8 are the same (the matching is also the same because we used the same algorithm to compute the socially optimal state as part of the VCG mechanism)

2. Example 2:

Values for b_1: $\{i_1 : 12, i_2 : 14, i_3 : 16, i_4 : 8, i_5 : 6, i_6 : 17\}$

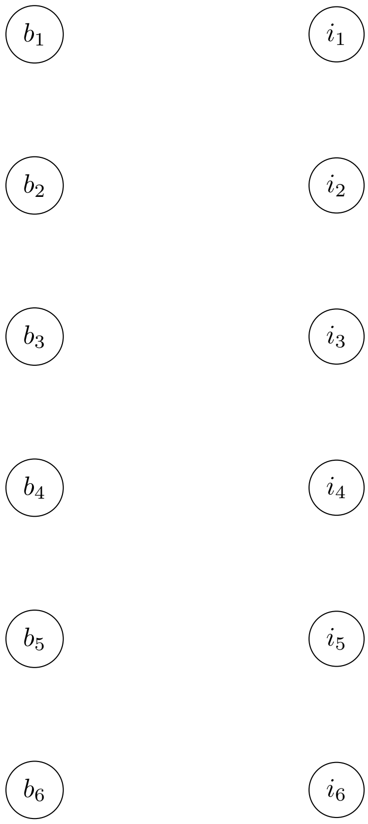
Values for b_2: $\{i_1 : 11, i_2 : 7, i_3 : 9, i_4 : 19, i_5 : 1, i_6 : 11\}$

Values for b_3: $\{i_1 : 18, i_2 : 13, i_3 : 17, i_4 : 17, i_5 : 2, i_6 : 16\}$

Values for b_4: $\{i_1 : 15, i_2 : 0, i_3 : 4, i_4 : 1, i_5 : 15, i_6 : 15\}$

Values for b_5: $\{i_1 : 7, i_2 : 8, i_3 : 5, i_4 : 12, i_5 : 18, i_6 : 13\}$

Values for b_6: $\{i_1 : 7, i_2 : 19, i_3 : 8, i_4 : 12, i_5 : 4, i_6 : 1\}$



And we observe that the prices output by the VCG mechanism and the algorithm of Theorem 8.8 are the same (the matching is also the same because we used the same algorithm to compute the socially optimal state as part of the VCG mechanism)

3. Example 3:

Values for b_1: $\{i_1 : 8, i_2 : 11, i_3 : 0, i_4 : 3, i_5 : 6, i_6 : 7\}$

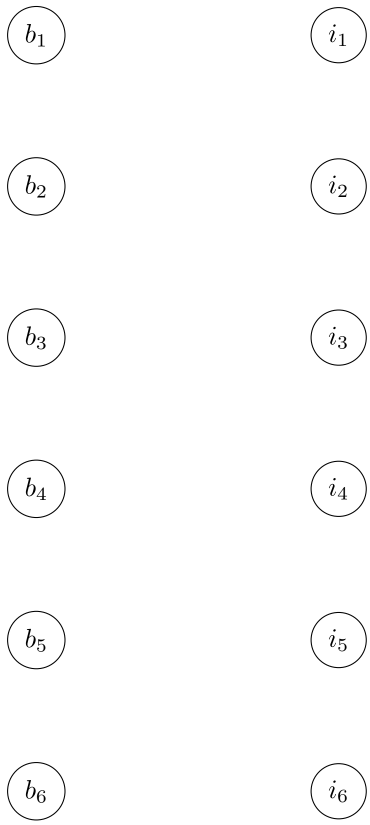
Values for b_2: $\{i_1 : 19, i_2 : 14, i_3 : 15, i_4 : 14, i_5 : 14, i_6 : 16\}$

Values for b_3: $\{i_1 : 17, i_2 : 19, i_3 : 19, i_4 : 13, i_5 : 8, i_6 : 17\}$

Values for b_4: $\{i_1 : 2, i_2 : 15, i_3 : 1, i_4 : 18, i_5 : 11, i_6 : 10\}$

Values for b_5: $\{i_1 : 8, i_2 : 9, i_3 : 7, i_4 : 15, i_5 : 6, i_6 : 10\}$

Values for b_6: $\{i_1 : 12, i_2 : 15, i_3 : 15, i_4 : 8, i_5 : 2, i_6 : 1\}$



And we observe that the prices output by the VCG mechanism and the algorithm of Theorem 8.8 are the same (the matching is also the same because we used the same algorithm to compute the socially optimal state as part of the VCG mechanism)

4. Example 4:

Values for b_1: $\{i_1 : 5, i_2 : 3, i_3 : 0, i_4 : 7, i_5 : 10, i_6 : 5, i_7 : 17, i_8 : 6, i_9 : 18, i_{10} : 8\}$

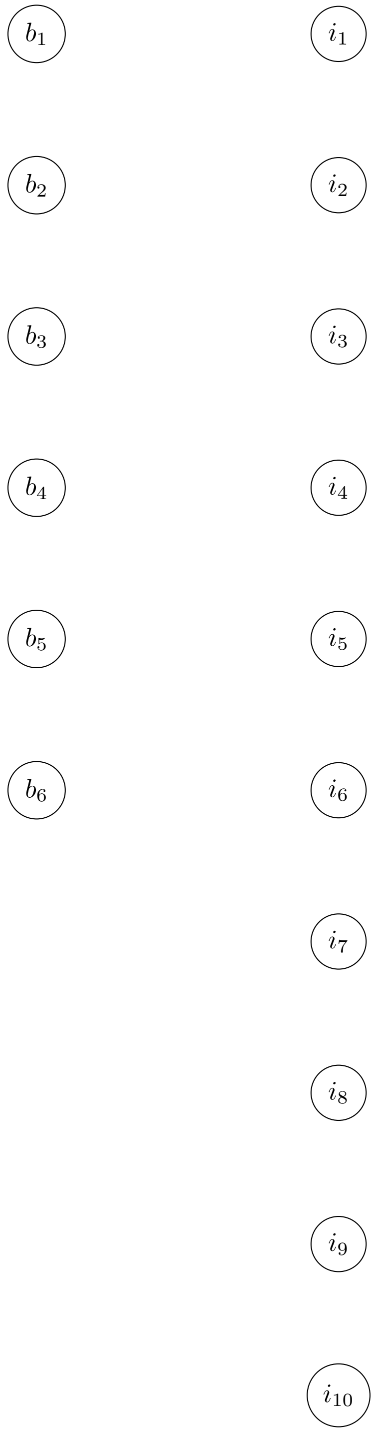
Values for b_2: $\{i_1 : 5, i_2 : 4, i_3 : 6, i_4 : 9, i_5 : 15, i_6 : 9, i_7 : 17, i_8 : 2, i_9 : 10, i_{10} : 14\}$

Values for b_3: $\{i_1 : 10, i_2 : 11, i_3 : 10, i_4 : 6, i_5 : 4, i_6 : 10, i_7 : 16, i_8 : 11, i_9 : 10, i_{10} : 6\}$

Values for b_4: $\{i_1 : 2, i_2 : 19, i_3 : 4, i_4 : 12, i_5 : 5, i_6 : 8, i_7 : 12, i_8 : 0, i_9 : 11, i_{10} : 11\}$

Values for b_5: $\{i_1 : 18, i_2 : 7, i_3 : 15, i_4 : 11, i_5 : 7, i_6 : 4, i_7 : 2, i_8 : 9, i_9 : 9, i_{10} : 8\}$

Values for b_6: $\{i_1 : 5, i_2 : 2, i_3 : 2, i_4 : 5, i_5 : 1, i_6 : 12, i_7 : 13, i_8 : 18, i_9 : 8, i_{10} : 1\}$



And we observe that the prices output by the VCG mechanism and the algorithm of Theorem 8.8 are the same (the matching is also the same because we used the same algorithm to compute the socially optimal state as part of the VCG mechanism)

5. Example 5:

Values for b_1: $\{i_1 : 15, i_2 : 3, i_3 : 1, i_4 : 5\}$

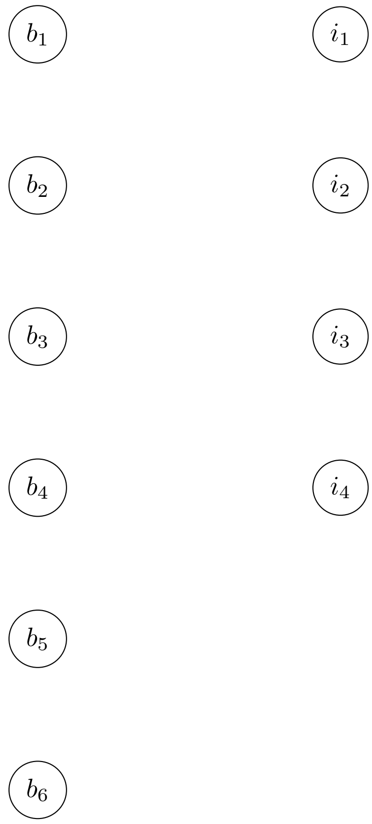
Values for b_2: $\{i_1 : 11, i_2 : 11, i_3 : 16, i_4 : 5\}$

Values for b_3: $\{i_1 : 9, i_2 : 15, i_3 : 13, i_4 : 17\}$

Values for b_4: $\{i_1 : 15, i_2 : 11, i_3 : 10, i_4 : 16\}$

Values for b_5: $\{i_1 : 19, i_2 : 0, i_3 : 12, i_4 : 7\}$

Values for b_6: $\{i_1 : 17, i_2 : 16, i_3 : 13, i_4 : 9\}$



And we observe that the prices output by the VCG mechanism and the algorithm of Theorem 8.8 are the same (the matching is also the same because we used the same algorithm to compute the socially optimal state as part of the VCG mechanism)

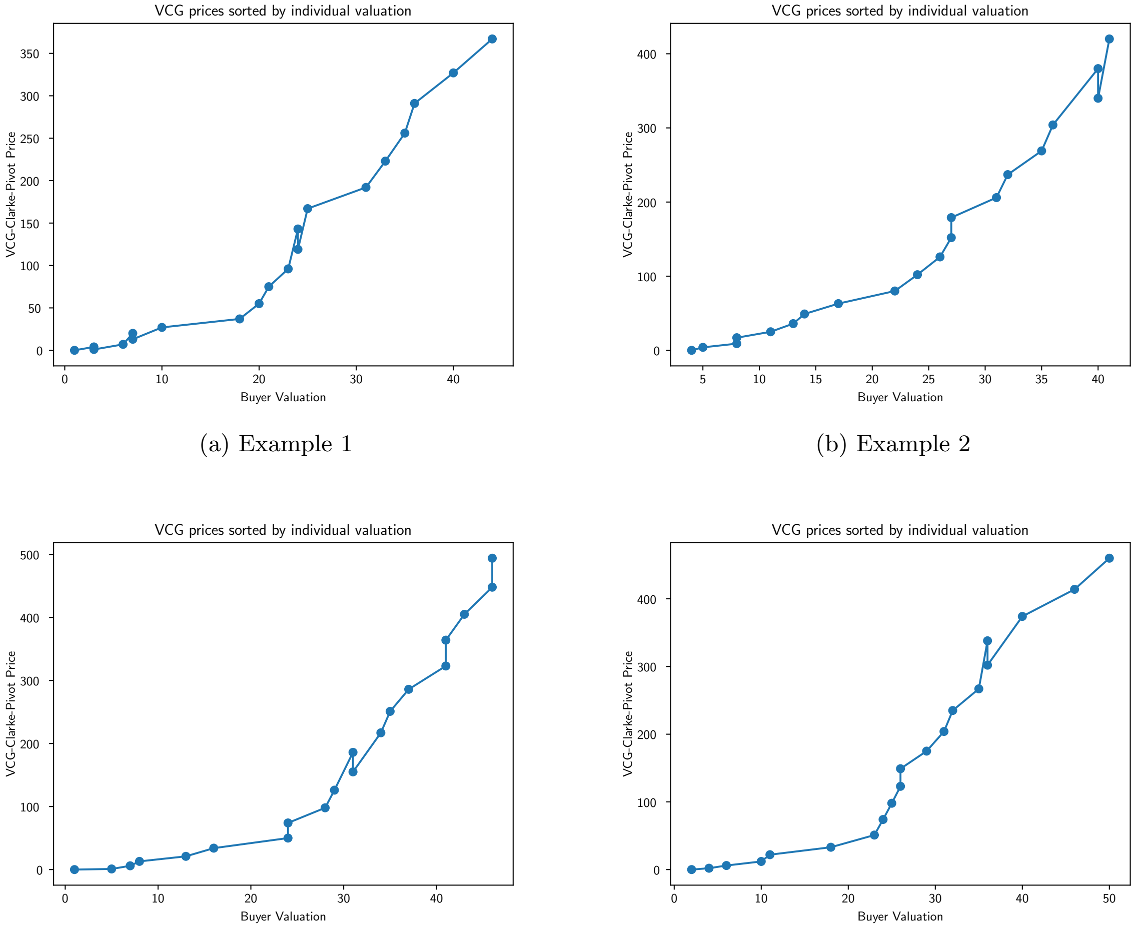
That is, in all examples we analyzed, the prices output by the VCG mechanism and the algorithm of Theorem 8.8 were the same, and the matching was also the same because we used the same algorithm to compute the socially optimal state as part of the VCG mechanism. We analyzed far more examples besides the ones presented here, and the results were consistent across all of them—the prices output by the VCG mechanism and the algorithm of Theorem 8.8 were the same (and the matching was also the same because we used the same algorithm to compute the socially optimal state as part of the VCG mechanism).

3 Bonus Question 2

(a) We structure a *markets-for-bundles* context of identical goods as a simple matching market context, where each bundle B_j's value for bidder b_i is the product of the value of b_i for the good and the amount of goods in the bundle, c_i. That is, $v_i(B_j) = c_j \cdot t_i$, where t_i is the value of b_i for the singular good, and c_j is the amount of goods in bundle B_j. We then run the VCG algorithm we implemented in the previous part on a few randomized examples of such a markets-for-bundles context, where there are $n = m = 20$ bundles and bidders, and the values t_i are randomized between 1 and 50, and where $c_j = j$ ($j \in \{1, 2, \ldots, 20\}$).

Figure 1 summarizes the results of the VCG algorithm on 4 such randomized examples of markets-for-bundles contexts (the results remained the same for other examples we ran). The x-axis represents the individual valuation t_i of the bidders for the singular good, and the y-axis represents the VCG price for the bidder—commonly referred to as the *externalities* of the bidder on the market. We observe a clear trend, where the VCG prices—i.e. *externalities* of bidders, are linearly increasing with the valuation of the bidders for the singular good, which is expected given the structure of the markets-for-bundles context.

As we've seen in class, the socially optimal assignment in a *markets-for-bundles* of identical goods, is that the larger bundles are assigned to the higher valuation of a singular good. Thus, as the valuation



(a) Example 1

(b) Example 2

(c) Example 3

(d) Example 4

Figure 1: VCG prices for bidders in markets-for-bundles contexts

of the singular good increases, the bidder b_i (w.l.o.g. the bidders are sorted by decreasing valuation of the singular goods) is assigned a larger bundle B_i (w.l.o.g. the bundles are sorted by decreasing sizes c_i). Without said bidder b_i, the bundles assignment shifts for the lower-bidding bidders. That is, $\forall j > i \,.\, b_j$ is changed to be assigned B_{j-1}. This sets the externality to be $\sum_{j=i+1}^{n}(c_{j-1} - c_j) \cdot t_j$ for bidder b_i. In our case, this is $\sum_{j=i+1}^{n} t_j$. In either case, it is easy to see that as the valuation of the singular good increases (i.e., i grows), the externality of the bidder (i.e., b_i) should increase for regularly distributed bundle sizes and valuations of the singular good (as in our randomization).

(b)

(c)

(d)

Part 5: Exchange Networks for Uber

1 Question 9

2 Question 10

(a)

(b)

3 Question 11

4 Bonus Question 3

(a)

(b)

References