Networks and Markets
Homework 2

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Part 1: Best-Response Dynamics

1. Let G be a game and assume that every node in its BRD graph has a path from it to a PNE. Does BRD converge in the following cases:

   (a) G has two players.

   (b) For every player i, vector of actions for other players a_{-i}, and two actions $X \neq Y$ for player i we have $u_i(a_{-i}, X) \neq u_i(a_{-i}, Y)$, where $u_i(b)$ is the utility for i under action profile b. (In other words, given the choices of all other players, every choice of player i gives him a different utility.)

   (c) The assumptions from (a) and (b) both hold.

2. (a) Let G_1 and G_2 be games with the same set of players and action sets for each player, and *assume that BRD converges to a PNE* in both of these games. Consider a game $G = (G_1 + G_2)$ where each player plays a single action for both G_1 and G_2 (i.e. plays the same action in both games at the same time) and receives utility equal to the *sum* of the utilities they would earn from G_1 and G_2. *Will BRD converge in this game?* If so, prove it; if not, find a counterexample and argue why it doesn't converge.

   (b) Assume in a particular game $G = G_1 + G_2$ that BRD does converge. Will it necessarily converge to a state that is also a PNE in either G_1 or G_2? If so, prove it; if not, find a counterexample.

3. Consider the process of "better-response dynamics" rather than BRD, where instead of choosing a player's best response (i.e. the response that maximizes their utility given others' actions), a player chooses any response that *strictly* increases their utility given other players' actions.

   (a) Prove that if better-response dynamics converges, then BRD must also converge.

   (b) Can you think of a game for which BRD converges but "better-response dynamics" might not? Show an example or justify that one doesn't exist. (*Hint:* Remember that the better-response dynamics graph can have edges that the BRD graph might not.)

* **Bonus Question 1.** Recall the Traveler's Dilemma that we studied in chapter 1. We have already illustrated why BRD converges in this game; find a **weakly** ordinal potential function Φ over states of this game and use it to definitively prove the convergence of BRD. Make sure to justify that the potential function you find is weakly ordinal (you can do this via computer if you wish).

* **Bonus Question 1.5.** (*This may be very difficult!*) Determine whether *better-response dynamics* converge for the Traveler's Dilemma. No formal proof is necessary, and you need not come up with a potential function. You are free to either use simulations, show (experimentally) that the graph contains no cycle, or find an ordinal potential function and either prove it or experimentally verify.

Part 2: Networked Coordination Games

4. Consider the following simple coordination game between two players:

	$(*, X)$	$(*, Y)$
$(X, *)$	(x, x)	(0, 0)
$(Y, *)$	(0, 0)	(y, y)

   Show how we can pick x and y and then modify this payoff matrix by adding an intrinsic utility for a single player and a single choice (e.g. give player 1 some intrinsic utility for choice Y) such that the socially optimal state of the game is no longer an equilibrium.

5. Consider a networked coordination game on a complete graph (i.e. a clique) of five nodes. Assume that all nodes have the same intrinsic values for X and Y (that is, $R_i(X) = R_j(X)$ for any nodes i, j, and respectively for Y).

   (a) Can it be the case that every pure-strategy Nash equilibrium is socially optimal? If not, prove it. Otherwise, give a possible assignment of intrinsic and coordination utilities such that every pure-strategy Nash equilibrium of the resulting game is socially optimal (this requires finding all PNE and arguing that they give the maximum social welfare).

   (b) Let $\epsilon > 0$. Is there a possible assignment of intrinsic and coordination utilities such that there exists a pure-strategy Nash equilibrium with social welfare less than or equal $\epsilon \cdot M$, where M is the maximum attainable social welfare? Your answer may or may not depend on the value of ϵ. Explain what can be deduced for upper bounds on the price of anarchy.

Part 3: Cascading Behavior in Networks

6. Consider the network in Figure 1. Suppose that each node starts with the behavior Y, and has a $t = 2/5$ threshold for adopting behavior X. (That is, if at least 2/5 of a node's neighbors have adopted X, that node will as well.)

   (a) Let e and f form a set S of initial adopters of X. Which nodes will eventually switch to X? (Assume that S will not participate in BRD.)

   (b) Find a cluster of density $1 - t = 3/5$ in the part of the graph outside S which blocks X from spreading to all nodes starting from S.

   (c) Add one node to S such that a complete cascade will occur at the threshold $t = 2/5$. Demonstrate how the complete cascade could occur (i.e. in what order nodes will switch).

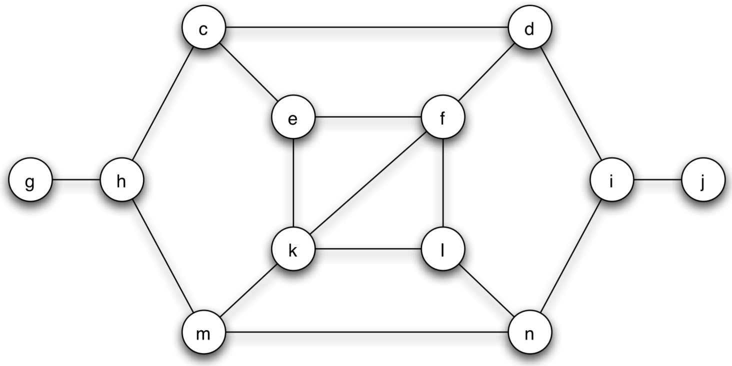

Figure 1: Question 6.

7. (a) Formulate a graph G, threshold t, and set S of initial adopters such that, assuming we start with S choosing X (and, importantly, able to participate in BRD) and other nodes choosing Y, we can either end up with every node in G playing X or with every node in G playing Y, depending on the order of switches in the BRD process.
   (b) What must be true of the set density of S for the above property to hold? Be as precise as possible. Additionally, justify why this holds in your example above (although your answer should hold for any such example).
   (c) What must be true of the set density in any subset of $V \setminus S$ for the above property to hold? Be as precise as possible. Again, justify why this holds in your example above.

Part 4: Traffic Networks

8. There are two cities, A and B, joined by two routes which pass through towns C and D respectively. There are 120 travelers who begin in city A and must travel to city B, and may take the following roads:
   - a local street from A to C with travel time $15 + x$, where x is the number of travelers using it,
   - a highway from C to B with travel time 90,
   - a highway from A to D with travel time 90, and
   - a local street from D to B with travel time $15 + y$, where y is the number of travelers using it.

   (a) Draw the network described above and label the edges with their respective travel times. The network should be a directed graph (assume that all roads are one-way). Note: you may just describe the graph if you are typing the assignment up. You may also include an image of the graph if that is easier for you.
   (b) Find the Nash equilibrium values of x and y. (Show that this is the equilibrium.)
   (c) The government now adds a new *two-way* road connecting the nodes where local streets and highways meet. This new road is extremely efficient and requires no travel time. Find the new Nash equilibrium.

(d) What happens to the total travel time as a result of the availability of the new road? (You don't need to explain, a calculation is fine.)

(e) Now suppose that the government, instead of closing the new road, decides to assign routes to travelers to shorten the total travel time. Find the assignment that minimizes the total travel time, and determine the total travel time using this assignment. (*Hint:* It is possible to achieve a total travel time less than the original equilibrium. Remember that, with the new road, there are now four possible routes that each traveler can take.)

Part 5: Experimental Evaluations

9. This problem will make use of the "facebook_combined.txt" data set.

(a) Consider the contagion examples that we observed in chapter 5 of the notes. Given an *undirected* graph, a set of early adopters S, and a threshold t (such that a certain choice X will spread to a node if more than (or exactly) t fraction of its neighbors are playing it). Let $k = |S|$ be the number of early adopters. Produce an algorithm that permanently infects the set S of early adopters with X and then runs BRD on the remaining nodes to determine whether, and to what extent, the choice will cascade through the network. (*Note:* "BRD" in this case is simply the process of iteratively deciding whether there is a node that will switch its choice and performing this switch.)

To test your code, construct undirected graphs corresponding to the two examples from Figure 4.1 in the notes (page 44 in slides of lecture 3). Let S be the set of nodes choosing X in each figure, and for each of the graphs, find a value of t that causes a complete cascade, and also find a value of t that does not cause a complete cascade. Implement the corresponding methods in `hw2_p9.py`.

(b) Run your algorithm several (100) times on a fairly small random set of early adopters ($k = 10$) with a low threshold ($t = 0.1$) on the Facebook data set. What happens? Is there a complete cascade? If not, how many nodes end up being "infected" on average? Please include your code in the main method of `hw2_p9.py`, and also include the results in your pdf write-up.

(c) Run your algorithm several (10) times with different values of t (try increments of 0.05 from 0 to 0.5), and with different values of k (try increments of 10 from 0 to 250). Observe and record the rates of "infection" under various conditions. What conditions on k and t are likely to produce a complete cascade in this particular graph, given your observations? Please include your code in the main method of `hw2_p9.py`, and also include your analysis in your pdf submission.

* **Bonus Question 2.** (Optional, extra credit awarded depending on quality of solution.) Design an algorithm that, given a graph and a threshold t, finds (an approximation of) the smallest possible set of early adopters that will cause a complete cascade. Try running it on the Facebook data with different values of t and seeing how large a set we need.

10. Consider the following problem: There are n Uber drivers and m potential riders. At a fixed point in time, each driver has a list of compatible riders that she can pick up. Our goal will be to match drivers to riders such that the most riders at this time are picked up. We will use the maximum-flow algorithm, described in Chapter 6 of the notes to do this.

    (a) First, implement an algorithm that, given a *directed* graph, a source s, a sink t, and edge capacities over each edge in E, computes the maximum flow from s to t (you must implement this algorithm yourself). Turn in your code and verify that your algorithm works on a few simple test cases. In particular, make sure to test your algorithm on the graphs in Figures 6.1 and 6.3 from the lecture notes.

    (b) Given a set of n drivers, m riders, and sets of possible riders that each driver can pick up,

        i. explain how we can use this maximum-flow algorithm to determine the the maximum *number* of matches, and

        ii. explain how we can additionally extend this to actually find the matchings.

        Include your explanations in your write-up. (*Hint:* See the notes if you are confused about how to do this.)

    (c) Implement a maximal matching algorithm for Uber drivers and riders. Specifically, given n drivers with constraints specified on m riders, computed the assignments of drivers to riders. Test your algorithm on at least 2 examples (with at least 5 riders and drivers each). Briefly explain your examples and your results.

    (d) Now consider the case where there are n drivers and n riders, and each driver is connected to each rider with probability p. Fix $n = 1000$ (or maybe 100 if that's too much), and estimate the probability that all n riders will get matched for varying values of p. Plot your results and include your analysis in your pdf write-up.

    * **Bonus Question 3:** In the above example, let $p^*(n)$ be the smallest value of p where all n riders get matched with at least 99% probability. Prove bounds on what $p^*(n)$ is as a function of n, e.g. is $p^*(n) \geq 1/n$ or $p^*(n) \leq 1/2$. You will get partial credit for providing experimental evidence towards a proposed idea.