Homework Assignment 3 - Coding Part Write-up
Networks and Markets

Omer Zohar Gil Aharoni Adam Tuby

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Part 4: Implementing Matching Market Pricing

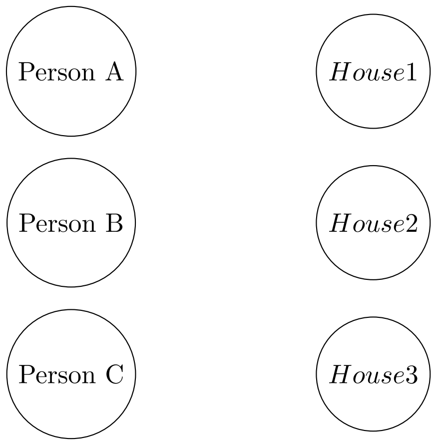
1 Question 7

(b) Consider the matching market example in Lecture 5 Page 7:

Values for Person A: $\{House1 : 4, House2 : 12, House3 : 5\}$

Values for Person B: $\{House1 : 7, House2 : 10, House3 : 9\}$

Values for Person C: $\{House1 : 7, House2 : 7, House3 : 10\}$



Formally, the matching market context is $\Gamma = (\{A, B, C\}, \{1, 2, 3\}, v)$, where v is the valuation function defined as follows:

$$v_A(1) = 4, v_A(2) = 12, v_A(3) = 5$$
$$v_B(1) = 7, v_B(2) = 10, v_B(3) = 9$$
$$v_C(1) = 7, v_C(2) = 7, v_C(3) = 10$$

We turn to run the algorithm of Theorem 8.8 to find a market equilibrium (p, M) to find the maximum social value, in order to validate out implementation's output. We begin by initializing the prices vector $\vec{p} \equiv 0$ to be the zero vector. We then proceed to run the algorithm, updating the prices vector until there is a perfect matching M in the induced preferred choice graph for $(\Gamma, \vec{p})$:

1. Observing the following *induced preferred-choice graph* from $(\Gamma, \vec{p})$:

Values for A: $\{1:4, 2:12, 3:5\}$

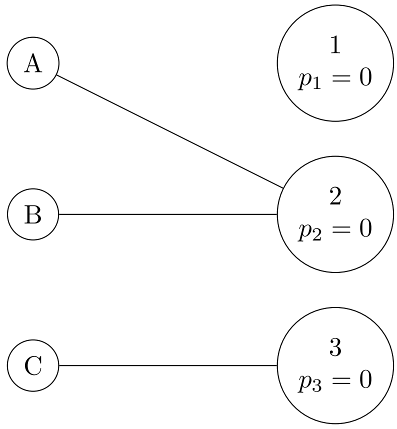
Utility for A: $\{1:4, 2:12, 3:5\}$

Values for B: $\{1:7, 2:10, 3:9\}$

Utility for B: $\{1:7, 2:10, 3:9\}$

Values for C: $\{1:7, 2:7, 3:10\}$

Utility for C: $\{1:7, 2:7, 3:10\}$



There obviously isn't a perfect matching as $S = \{A, B\}$ is a constricted set with $|N(S)| = |\{2\}| = 1 < 2 = |S|$ (which, by a theorem we've seen in class, implies that there isn't a perfect matching). Thus, we raise the prices for all items in $N(S)$ by 1, and update the prices vector $\vec{p}$ accordingly. The updated prices vector is $\vec{p} = (a:0, b:1, c:0)$. Not all prices are greater than zero, so we don't perform a shift operation, and we proceed to the next iteration.

2. Observing the following *induced preferred-choice graph* from $(\Gamma, \vec{p})$:

Values for A: $\{1:4, 2:12, 3:5\}$

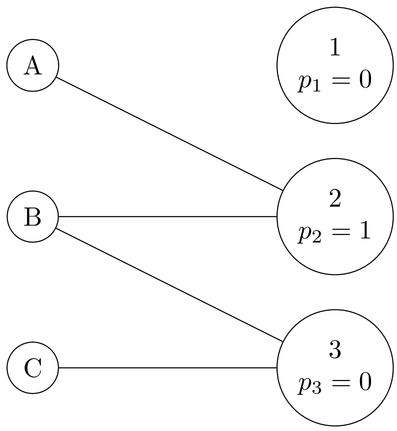
Utility for A: $\{1:4, 2:11, 3:5\}$

Values for B: $\{1:7, 2:10, 3:9\}$

Utility for B: $\{1:7, 2:9, 3:9\}$

Values for C: $\{1:7, 2:7, 3:10\}$

Utility for C: $\{1:7, 2:6, 3:10\}$



There obviously isn't a perfect matching as $S = \{A, B, C\}$ is a constricted set with $|N(S)| = |\{2, 3\}| = 2 < 3 = |S|$ (which, by a theorem we've seen in class, implies that there isn't a perfect matching). Thus, we raise the prices for all items in $N(S)$ by 1, and update the prices vector $\vec{p}$ accordingly. The updated prices vector is $\vec{p} = (a:0, b:2, c:1)$. Not all prices are greater than zero, so we don't perform a shift operation, and we proceed to the next iteration.

3. Observing the following *induced preferred-choice graph* from $(\Gamma, \vec{p})$:

Values for A: $\{1:4, 2:12, 3:5\}$

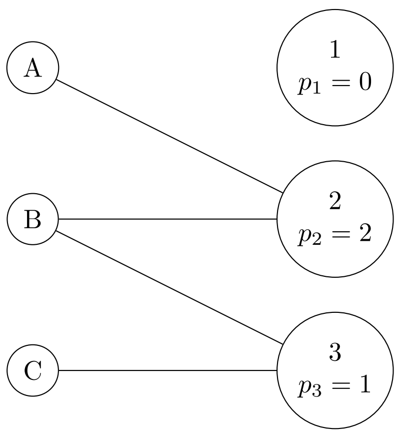
Utility for A: $\{1:4, 2:10, 3:4\}$

Values for B: $\{1:7, 2:10, 3:9\}$

Utility for B: $\{1:7, 2:8, 3:8\}$

Values for C: $\{1:7, 2:7, 3:10\}$

Utility for C: $\{1:7, 2:5, 3:9\}$



Similar to the previous iteration, we raise the prices for $\{2, 3\}$, and update the prices vector

$\vec{p}$ accordingly. The updated prices vector is $\vec{p} = (a : 0, b : 3, c : 2)$. Not all prices are greater than zero, so we don't perform a shift operation, and we proceed to the next iteration.

4. Observing the following *induced preferred-choice graph* from $(\Gamma, \vec{p})$:

Values for A: $\{1 : 4, 2 : 12, 3 : 5\}$

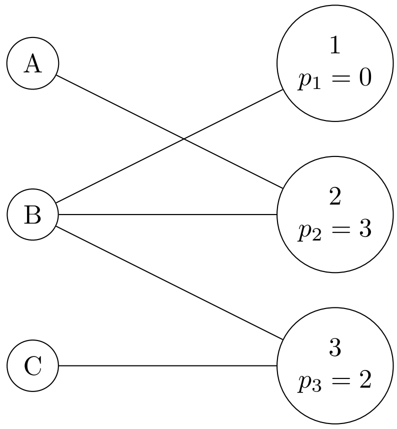
Utility for A: $\{1 : 4, 2 : 9, 3 : 3\}$

Values for B: $\{1 : 7, 2 : 10, 3 : 9\}$

Utility for B: $\{1 : 7, 2 : 7, 3 : 7\}$

Values for C: $\{1 : 7, 2 : 7, 3 : 10\}$

Utility for C: $\{1 : 7, 2 : 4, 3 : 8\}$



And there is a perfect matching in the induced preferred choice graph, which is $M = \{\{A, 2\}, \{B, 1\}, \{C, 3\}\}$. Thus, the market equilibrium is $(\vec{p}, M) = ((1 : 0, 2 : 3, 3 : 2), \{\{A, 2\}, \{B, 1\}, \{C, 3\}\})$, and we are done.

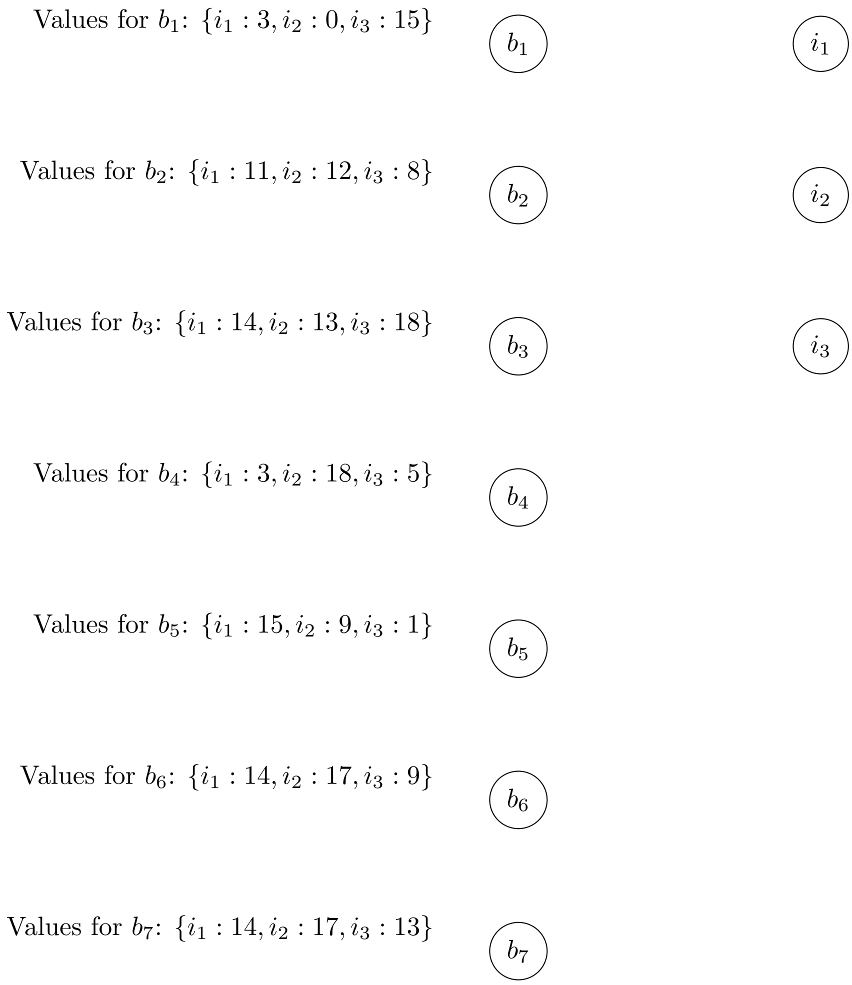
We found the market equilibrium to be $(\vec{p}, M) = ((1 : 0, 2 : 3, 3 : 2), \{\{A, 2\}, \{B, 1\}, \{C, 3\}\})$. The maximum social value is therefore $v(A, 2) + v(B, 1) + v(C, 3) = 12 + 7 + 10 = 29$.

Our algorithm found exactly this market equilibrium.

2 Question 8

(a) In this part we analyze how the prices output by the VCG mechanism compare with the ones output by the algorithm of Theorem 8.8 (finding a market equilibrium (p, M)). The following are the examples we analyze and their corresponding results for each mechanism:

1. Example 1:



And we observe that the prices output by the VCG mechanism and the algorithm of Theorem 8.8 are the same (the matching is also the same because we used the same algorithm to compute the socially optimal state as part of the VCG mechanism)

2. Example 2:

Values for b_1: $\{i_1 : 12, i_2 : 14, i_3 : 16, i_4 : 8, i_5 : 6, i_6 : 17\}$

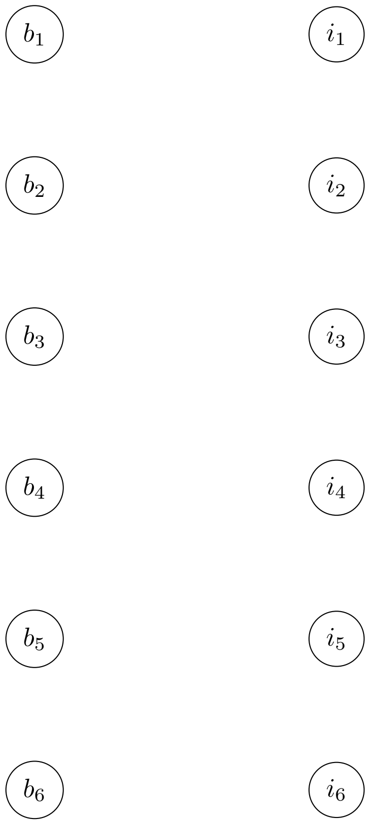
Values for b_2: $\{i_1 : 11, i_2 : 7, i_3 : 9, i_4 : 19, i_5 : 1, i_6 : 11\}$

Values for b_3: $\{i_1 : 18, i_2 : 13, i_3 : 17, i_4 : 17, i_5 : 2, i_6 : 16\}$

Values for b_4: $\{i_1 : 15, i_2 : 0, i_3 : 4, i_4 : 1, i_5 : 15, i_6 : 15\}$

Values for b_5: $\{i_1 : 7, i_2 : 8, i_3 : 5, i_4 : 12, i_5 : 18, i_6 : 13\}$

Values for b_6: $\{i_1 : 7, i_2 : 19, i_3 : 8, i_4 : 12, i_5 : 4, i_6 : 1\}$



And we observe that the prices output by the VCG mechanism and the algorithm of Theorem 8.8 are the same (the matching is also the same because we used the same algorithm to compute the socially optimal state as part of the VCG mechanism)

3. Example 3:

Values for b_1: $\{i_1 : 8, i_2 : 11, i_3 : 0, i_4 : 3, i_5 : 6, i_6 : 7\}$

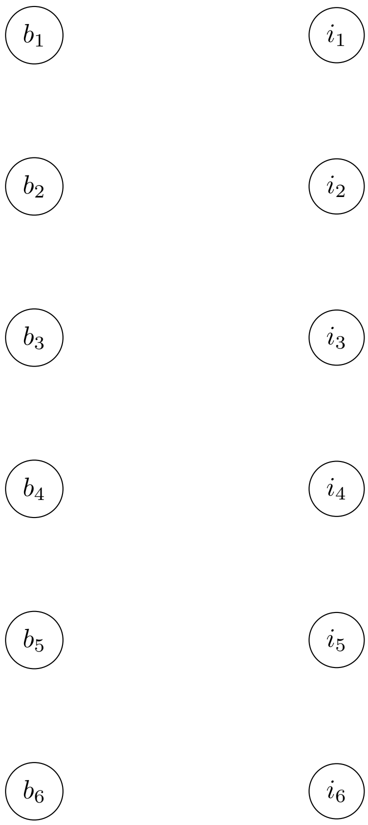
Values for b_2: $\{i_1 : 19, i_2 : 14, i_3 : 15, i_4 : 14, i_5 : 14, i_6 : 16\}$

Values for b_3: $\{i_1 : 17, i_2 : 19, i_3 : 19, i_4 : 13, i_5 : 8, i_6 : 17\}$

Values for b_4: $\{i_1 : 2, i_2 : 15, i_3 : 1, i_4 : 18, i_5 : 11, i_6 : 10\}$

Values for b_5: $\{i_1 : 8, i_2 : 9, i_3 : 7, i_4 : 15, i_5 : 6, i_6 : 10\}$

Values for b_6: $\{i_1 : 12, i_2 : 15, i_3 : 15, i_4 : 8, i_5 : 2, i_6 : 1\}$



And we observe that the prices output by the VCG mechanism and the algorithm of Theorem 8.8 are the same (the matching is also the same because we used the same algorithm to compute the socially optimal state as part of the VCG mechanism)

4. Example 4:

Values for b_1: $\{i_1 : 5, i_2 : 3, i_3 : 0, i_4 : 7, i_5 : 10, i_6 : 5, i_7 : 17, i_8 : 6, i_9 : 18, i_{10} : 8\}$

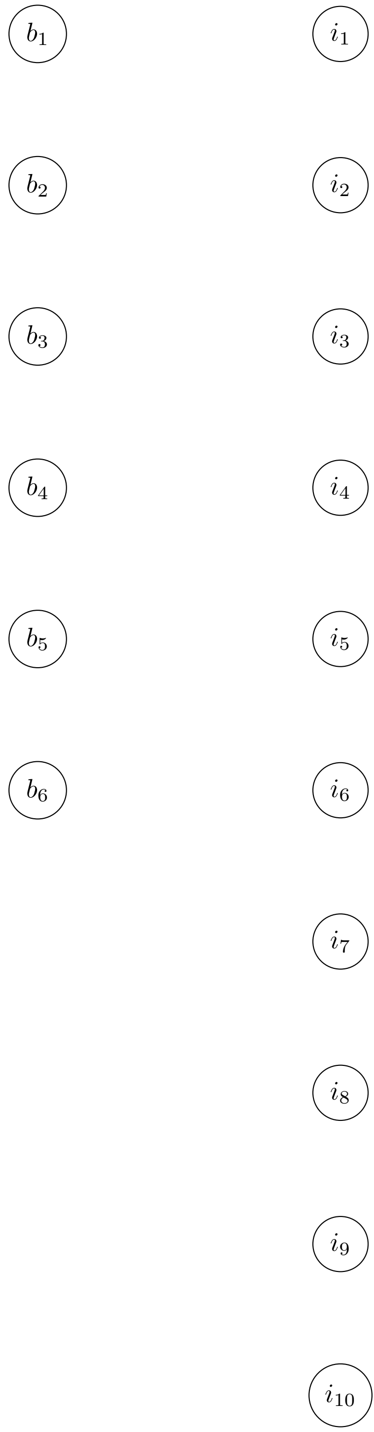
Values for b_2: $\{i_1 : 5, i_2 : 4, i_3 : 6, i_4 : 9, i_5 : 15, i_6 : 9, i_7 : 17, i_8 : 2, i_9 : 10, i_{10} : 14\}$

Values for b_3: $\{i_1 : 10, i_2 : 11, i_3 : 10, i_4 : 6, i_5 : 4, i_6 : 10, i_7 : 16, i_8 : 11, i_9 : 10, i_{10} : 6\}$

Values for b_4: $\{i_1 : 2, i_2 : 19, i_3 : 4, i_4 : 12, i_5 : 5, i_6 : 8, i_7 : 12, i_8 : 0, i_9 : 11, i_{10} : 11\}$

Values for b_5: $\{i_1 : 18, i_2 : 7, i_3 : 15, i_4 : 11, i_5 : 7, i_6 : 4, i_7 : 2, i_8 : 9, i_9 : 9, i_{10} : 8\}$

Values for b_6: $\{i_1 : 5, i_2 : 2, i_3 : 2, i_4 : 5, i_5 : 1, i_6 : 12, i_7 : 13, i_8 : 18, i_9 : 8, i_{10} : 1\}$



And we observe that the prices output by the VCG mechanism and the algorithm of Theorem 8.8 are the same (the matching is also the same because we used the same algorithm to compute the socially optimal state as part of the VCG mechanism)

5. Example 5:

Values for b_1: $\{i_1 : 15, i_2 : 3, i_3 : 1, i_4 : 5\}$

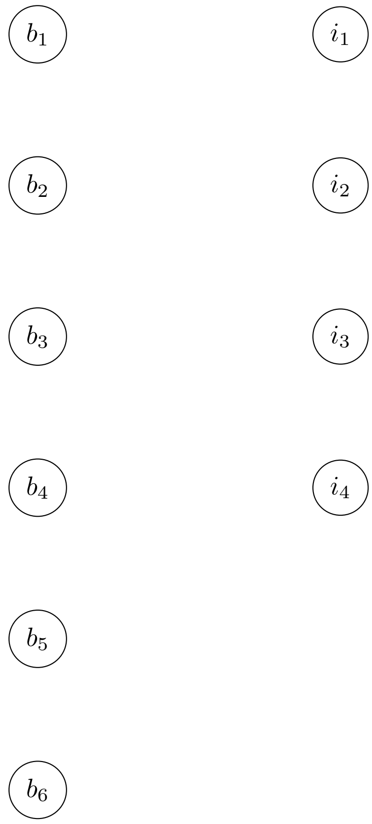
Values for b_2: $\{i_1 : 11, i_2 : 11, i_3 : 16, i_4 : 5\}$

Values for b_3: $\{i_1 : 9, i_2 : 15, i_3 : 13, i_4 : 17\}$

Values for b_4: $\{i_1 : 15, i_2 : 11, i_3 : 10, i_4 : 16\}$

Values for b_5: $\{i_1 : 19, i_2 : 0, i_3 : 12, i_4 : 7\}$

Values for b_6: $\{i_1 : 17, i_2 : 16, i_3 : 13, i_4 : 9\}$



And we observe that the prices output by the VCG mechanism and the algorithm of Theorem 8.8 are the same (the matching is also the same because we used the same algorithm to compute the socially optimal state as part of the VCG mechanism)

That is, in all examples we analyzed, the prices output by the VCG mechanism and the algorithm of Theorem 8.8 were the same, and the matching was also the same because we used the same algorithm to compute the socially optimal state as part of the VCG mechanism. We analyzed far more examples besides the ones presented here, and the results were consistent across all of them—the prices output by the VCG mechanism and the algorithm of Theorem 8.8 were the same (and the matching was also the same because we used the same algorithm to compute the socially optimal state as part of the VCG mechanism).

3 Bonus Question 2

(a)

(b)

(c)

(d)

Part 5: Exchange Networks for Uber

1 Question 9

2 Question 10

(a)

(b)

3 Question 11

4 Bonus Question 3

(a)

(b)

References