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Assignment-1

Q1. Find the distance b/w $(5, \frac{3\pi}{2}, 0)$ and $(5, \frac{\pi}{2}, 10)$ in cylindrical coordinates.

$\rho = 5$
$\phi = \frac{3\pi}{2}$
$z = 0$

$\Rightarrow x = \rho\cos\phi \Rightarrow 5\cos\frac{3\pi}{2}$
$y = \rho\sin\phi \Rightarrow 5\sin\frac{3\pi}{2}$
$z = 10$

Point $P_1\left(-5\cos\frac{\pi}{2}, -5\times 1, 0\right) \Rightarrow (0, -5, 0)$

~~Given~~ $P_2\left(\rho\cos\frac{\pi}{2}, \rho\sin\frac{\pi}{2}, 10\right) \Rightarrow (0, 5, 10)$

$D = \sqrt{(0)^2 + (5+5)^2 + (10-0)^2}$

$= \sqrt{0 + 100 + 100}$

$= \sqrt{200} \Rightarrow 10\sqrt{2}.$

Q2. Show that $A = 4a_x - 2a_y$ and $B = 4a_x + 4a_y + 4a_z$ are perpendicular.

$\vec{A} \cdot \vec{B} = 4 - 8 + 4 = 0$

Hence, dot is 0 so both vectors are $\perp$.

Q3. Given $A = a_x + a_y$, $B = a_x + 2a_z$ and

$C = 2\hat{a_y} + a_z$, find $(A \times B) \times C$ and compare it with $A \times (B \times C)$.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= (2-0)\hat{a_x} - (2-0)\hat{a_y} + (0-1)\hat{a_y}$$

$$= 2\hat{a_x} - 2\hat{a_y} - \hat{a_z}$$

So,

$$(\vec{A} \times \vec{B}) \times \vec{C} = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ 2 & -2 & -1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= (-2+2)\hat{a_x} - (2-0)\hat{a_y} + (4-0)\hat{a_z}$$

$$= 0 - 2\hat{a_y} + 4\hat{a_z}$$

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= (0-4)\hat{a_x} - (1-0)\hat{a_y} + (2-0)\hat{a_z}$$

$$= -4\hat{a_x} - \hat{a_y} + 2\hat{a_z}$$

So,

$$\vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ 1 & 1 & 0 \\ -4 & -1 & 2 \end{vmatrix}$$

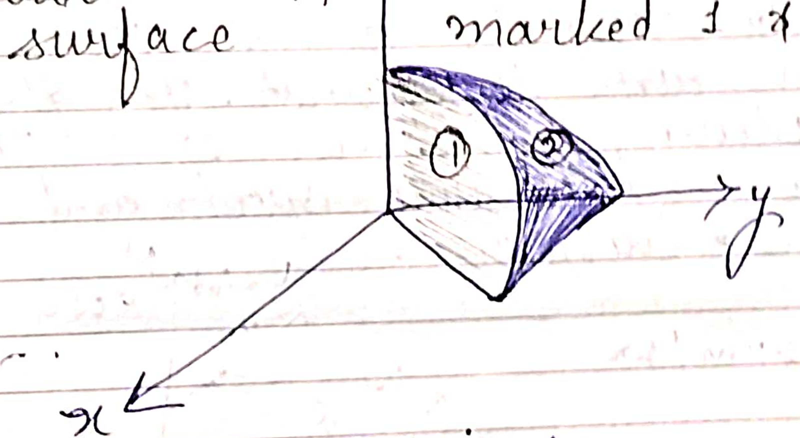
$= (2-0)\hat{a_x} - (2-0)\hat{a_y} + (-1+4)\hat{a_z}$

$= 2\hat{a_x} - 2\hat{a_y} + 3\hat{a_z}$

Comparing eqn :—

$(\vec{A} \times \vec{B}) \times \vec{C} \neq \vec{A} \times (\vec{B} \times \vec{C})$

Q4. Use spherical co-ordinates to write the differential surface areas dS_1 and dS_2 & integrate to obtain the area of the surface marked 1 & 2.



For surface ① r & θ are variable —

So, $dS_1 = r\,dr\,d\theta$

$$\int dS_1 = \int_0^1 \int_0^{\pi/2} r\,dr\,d\theta$$

$$= \int_0^1 r \left[\frac{\pi}{2}\right] dr \Rightarrow \frac{\pi}{2}\left[\frac{r^2}{2}\right]_0^1 \rightarrow \frac{\pi}{4}$$

For surface ②, φ & θ are variable

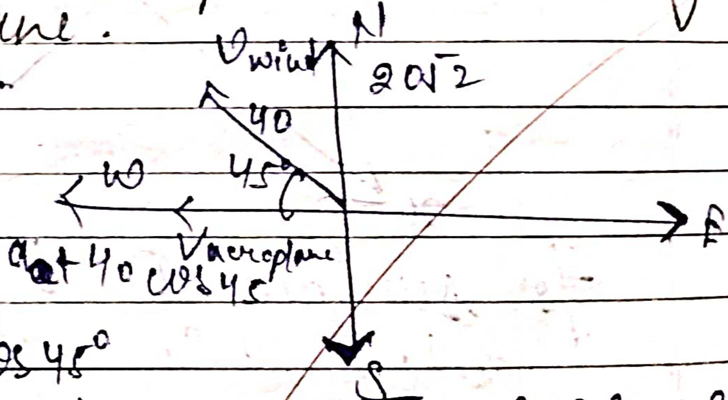
$$dS_2 = \int r^2 \sin\theta\, d\theta\, d\phi$$

$$= \int_{\pi/3}^{\pi/2}\int_{0}^{\pi/2} r^2 \sin\theta \, d\theta \, d\phi$$

$$= \int_{\pi/3}^{\pi/2} \left[-\cos\theta\right]_0^{\pi/2} d\phi$$

$$= \int_{\pi/3}^{\pi/2} d\phi = \left[\phi\right]_{\pi/3}^{\pi/2} = \pi/6.$$

Q5. An airplane has a ground speed of 350 km/hr in the dirn due west. If there is a wind blowing northwest at 40 km/hr, calculate the true speed and heading of the airplane.



$V_w = 40 \text{ km/hr}$

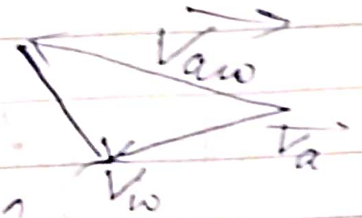
x component of wind $= 40 \cos 45°$

$= 40 \times \frac{1}{\sqrt{2}} \Rightarrow 20\sqrt{2} \Rightarrow 20.28$

$= \cancel{20 \times 1.414} \Rightarrow$

x component will going to add with the speed of aeroplane as aeroplane moving in the same dirn

True speed ~~of~~ = 37

True speed ($\vec{V_{aw}}$) is given by


$= \vec{V_a} + \vec{V_w}$

$= 350\hat{a_x} + \underline{26.28\hat{a_x}} + 26.28\hat{a_y}$

$\Rightarrow 376.28\hat{a_x} + 26.28\hat{a_y}$

$|\vec{V_{aw}}| = \sqrt{(376.28)^2 + (26.28)^2}$

$= \sqrt{143095.75 + 794.75}$

$= \sqrt{143895.5084}$

$= 379.33$

Heading is given by —

$\theta = \tan^{-1}\left(\frac{26.28}{376.28}\right) = \tan^{-1}(0.07475)$

$= 4.2754°$.

Q6. Let $\vec{E} = 3\hat{a_y} + 4\hat{a_z}$ & $F = 4\hat{a_x} - 10\hat{a_y} + 5\hat{a_z}$

a) Find component of $\vec{E}$ along $\vec{F}$

Projection $= |\vec{E}| \cos\theta$

$|\vec{E}| \times \frac{\vec{E} \cdot \vec{F}}{|\vec{E}||\vec{F}|} \Rightarrow \frac{\vec{E} \cdot \vec{F}}{|\vec{F}|}$

$= \frac{(3\hat{a_y} + 4\hat{a_z}) \cdot (4\hat{a_x} - 10\hat{a_y} + 5\hat{a_z})}{\sqrt{41}}$

$= \frac{30+80}{\sqrt{141}} = \frac{10}{\sqrt{141}}$

Vector form of component of $\vec{r}$ along $\vec{r}$ would be given as -

$= (\text{Projection}) \hat{r}$

$= \frac{10}{\sqrt{141}} \times \frac{\vec{r}}{|\vec{r}|}$

$= \frac{10 \times (4a_x - 10a_y + 5a_z)}{141}$

$= \frac{10}{141} (4a_x - 10a_y + 5a_z)$

(b) Unit vector $\perp$ to $\vec{r}$ & $\vec{r}$ is —

$= \frac{\vec{r} \times \vec{r}}{|\vec{r} \times \vec{r}|}$

$$\vec{r} \times \vec{r} = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ 0 & 3 & 4 \\ 4 & -10 & 5 \end{vmatrix}$$

$= (15+40)\hat{a_x} - (0-16)\hat{a_y} + (0-12)\hat{a_z}$

$= 65\hat{a_x} + 16\hat{a_y} - 12\hat{a_z}$

$|\vec{r} \times \vec{r}| = \sqrt{(65)^2 + (16)^2 + (12)^2}$

$= \sqrt{4225 + 256 + 144}$

$= 68$

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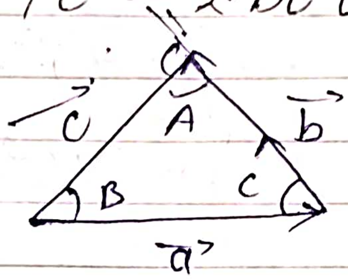
Unit vector $\perp$ to $\vec{E}$ & $\vec{F}$ is
$= 65\sqrt{a}\hat{x} + 16a\hat{y} - 12a\hat{z}$
$= 68$

Q7. Derive the cosine formula –
$a^2 = b^2 + c^2 - 2bc\cos A$ and sine formula –
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$



Let the Δ with angles A, B & C and angles A, B & C opposite to these sides respectively. Then,
$\vec{a} + \vec{b} + \vec{c} = 0$
$\Rightarrow \vec{b} + \vec{c} = -\vec{a}$ —— (1)
multiply (1) with $(-a)$ again –
$a^2 = (\vec{b} + \vec{c}) \cdot (\vec{b} + \vec{c})$
$= \vec{b}\cdot\vec{b} + \vec{c}\cdot\vec{c} + 2\vec{b}\cdot\vec{c}$
$= b^2 + c^2 - 2bc\cos A$
Since, we know that area of Δ is –
$= \left|\frac{1}{2} \times \text{base} \times \text{height}\right|$
$= \left|\frac{1}{2} \times \vec{a} \times \vec{b}\right| \Rightarrow \left|\frac{1}{2} \times \vec{b} \times \vec{c}\right| \Rightarrow \left|\frac{1}{2} \times \vec{c} \times \vec{a}\right|$
$\Rightarrow \vec{a} \times \vec{a} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$
$= ab\sin C = bc\sin A = ca\sin B$ — (2)
dividing eqn (2) by abc, –
$\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$

Q6. If P_1 is $(1, 2, -3)$ and $P_2(-4, 0, 5)$ find.

a) The distance P_1P_2

Given that $P_1(1, 2, -3)$

$P_2(-4, 0, 5)$

$$P_1P_2 = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$= \sqrt{(-4-1)^2 + (0-2)^2 + (5-(-3))^2}$$

$$= \sqrt{93}$$

(b) The vector eq^n of line P_1P_2

Considering origin $(0,0,0)$ as point O -

Position vector of line OP_1 is -

$$= a_x\hat{} + 2a_y\hat{} - 3a_z\hat{}$$

Position vector of line OP_2 is -

$$= -4a_x\hat{} + 5a_z\hat{}$$

So, $\overrightarrow{OP_1} = \overrightarrow{OP_2} + \overrightarrow{P_1P_2}$

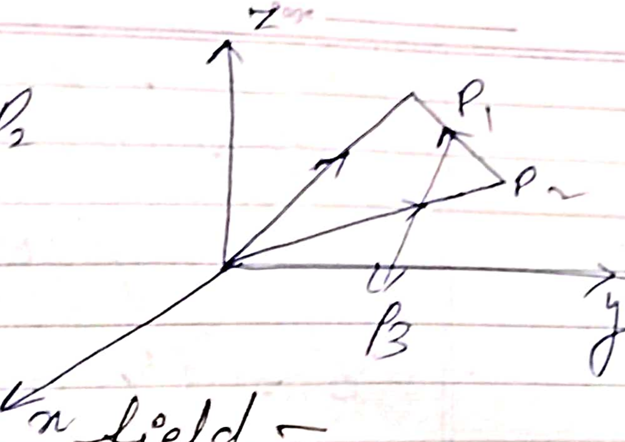
$$\overrightarrow{P_1P_2} = \overrightarrow{OP_1} - \overrightarrow{OP_2}$$

$$= (-a_x\hat{} + 2a_y\hat{} - 3a_z\hat{}) - (-4a_x\hat{} + 5a_z\hat{})$$

$$= 5a_x\hat{} + 2a_y\hat{} - 8a_z\hat{}$$

The final distance P_1P_2

$= 5a_{\hat{x}} + 2a_{\hat{y}} - 8a_{\hat{z}}$



Q9. a) Express the vector field –

$\vec{H} = xy^2z\, a_{\hat{x}} + x^2yz\, a_{\hat{y}} + xyz^2 a_{\hat{z}}$ in cylindrical & spherical coordinates.

$A_x = xy^2z$, $A_z = x^2yz$, $A_z = xyz^2$

So, in cylindrical co-ordinate system –

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} xy^2z \\ x^2yz \\ xyz^2 \end{bmatrix}$$

$$= \begin{bmatrix} xy^2z\cos\phi + x^2yz\sin\phi + 0 \\ -xy^2z\sin\phi + x^2yz\cos\phi + 0 \\ 0 + 0 + xyz^2 \end{bmatrix}$$

So, $A_\rho = xy^2z\cos\phi + x^2yz\sin\phi$

$A_\phi = -xy^2z\sin\phi + x^2yz\cos\phi$

$A_z = xyz^2$

Putting, $x = \rho\cos\phi$ & $y = \rho\sin\phi$ in above eq^n, we get —

$A_\rho = \rho^3 \cos\phi \sin^2\phi z + \rho^3 \cos^2\phi \sin^2\phi z$

$= 2\rho^3 \cos^2\phi \sin^2\phi z$

$A_\phi = \rho^3 \cos\phi \sin^3\phi z + \rho^3 \cos^3\phi \sin\phi z$

$= \rho^3 \cos\phi \sin\phi (\cos^2\phi - \sin^2\phi) z$

$= \rho^3 \cos\phi \cos 2\phi \sin\phi z$

$A_z = \rho^2 \cos\phi \sin\phi z^2$

So, $\vec{H} = [2\rho^3 \cos^2\phi \sin^2\phi z]\hat{a_\rho} + [\rho^3 \cos\phi \cos 2\phi \sin\phi z]\hat{a_\phi} + [\rho^2 \cos\phi \sin\phi z^2]\hat{a_z}$

Now, spherical co-ordinate system — (1)

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\phi\cos\theta & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} xy^2z \\ x^2yz \\ xyz^2 \end{bmatrix}$$

So,

$A_r = xy^2z \sin\theta\cos\phi + x^2yz \sin\theta\sin\phi + xyz^2\cos\theta$

$A_\theta = xy^2z \cos\theta\cos\phi + x^2yz\cos\theta\cos\phi - xyz^2\sin\theta$

$A_\phi = -xy^2z\sin\phi + x^2yz\cos\phi$

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

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in above eqn —

$A_r = r^4 \sin^4\theta \cos\theta \sin^2\phi \cos^2\phi + r^4 \sin^4\theta \cos\theta \sin^2\phi \cos^2\phi$
$+ r^4 \sin^2\theta \cos^3\theta \sin\phi \cos\phi$
$= 2r^4 \sin^4\theta \cos\theta \sin^2\phi \cos^2\phi + r^4 \sin^2\theta \cos^3\theta \sin\phi \cos\phi$

$A_\theta = r^4 \sin^3\theta \cos^2\theta \sin^2\phi \cos^2\phi + r^4 \sin^3\theta \cos^2\theta \cos^2\phi \sin^2\phi$
$- r^4 \sin^3\theta \cos^2\theta \cos\phi \sin\phi$
$= 2r^4 \sin^3\theta \cos^2\theta \sin^2\phi \cos^2\phi - r^4 \sin^3\theta \cos^2\theta \cos\phi \sin\phi$

$A_\phi = -r^4 \sin^3\theta \cos\theta \sin^3\phi \cos\phi + r^4 \sin^3\theta \cos\theta \cos^3\phi \sin\phi$
$= r^4 \sin^3\theta \cos\theta \cos\phi \sin\phi \cos 2\phi$

So,
$\vec{A} = [2r^4 \sin^4\theta \cos\theta \sin^2\phi \cos^2\phi + r^4 \sin^2\theta \cos^3\theta \sin\phi \cos\phi]\, \hat{a_r} + [2r^4 \sin^3\theta \cos^2\theta \sin^2\phi \cos^2\phi - r^4 \sin^3\theta \cos^2\theta \cos\phi \sin\phi]\, \hat{a_\theta} + [r^4 \sin^3\theta \cos\theta \cos\phi \sin\phi \cos 2\phi]\, \hat{a_\phi}$ —— (2)

(b) At point $(3, -4, 5)$
In Cartesian —

$\rho = \sqrt{9+16} = 5$
$\phi = \tan^{-1}\left(\frac{-4}{3}\right) = 126.86$
$z = 5$

$r = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2}$
$\theta = \tan^{-1}\sqrt{9+16} = \tan^{-1}(1) = 45^\circ$
$\phi = 126.86$

Put in (1) —

$$\vec{H} = [2\times125\times5\times\cos^2(126.86^\circ)\times\sin^\circ(126.86)]\,a_{\hat{\rho}}$$
$$+ [125\times8\times\cos(126.86^\circ)\times\cos(253.72)\times\sin(126.86)]$$
$$+ [25\times25\times\cos(126.86)\times\sin(126.86^\circ)]\,a_{\hat{z}}$$
$$= 280\,a_{\hat{\rho}} + 84\,a_{\hat{\phi}} - 300\,a_{\hat{z}}$$

Put in (2)

$$\vec{H} = [2\times50\times80\times\sin^4(45^\circ)\cos45^\circ\sin^2(126.86)\cos^2(126.86^\circ) + 50\times80\times\sin^2(45^\circ)\cos^3(45^\circ)\sin(126.86^\circ)\cos(126.86^\circ)]\,a_{\hat{r}}$$

$$+ [2\times50\times80\times\sin^3(45^\circ)\cos^2(45^\circ)\sin^2(126.86)\cos^2(126.86^\circ) - 50\times50\times\sin^3(45^\circ)\cos45^\circ\sin(126.86^\circ)\cos(126.86)]\,a_{\hat{\theta}} +$$
$$[80\times50\times\sin^3(45^\circ)\cos45^\circ)\cos(126.86)\times\sin(126.86)\times\cos(253.72^\circ)]\,a_{\hat{\phi}}$$

$$= [201.6 - 210]\,a_{\hat{r}} + [201.6 + 210]\,a_{\hat{\theta}}$$
$$= -8.4\,a_{\hat{r}} + 411.6\,a_{\hat{\theta}} + 84\,a_{\hat{\phi}}$$

Q10. Given vector field –
$$\vec{H} = \rho z\cos\phi\, a_\rho + e^{-2}\sin\frac{\phi}{2}\, a_\phi + \rho^2 a_{\hat{z}}$$
At point $(1, \pi/3, 0)$ find –

i) $H\times a_\theta$

$$\vec{H} = \rho z\cos\phi\, a_{\hat{\rho}} + e^{-2}\sin\frac{\phi}{2}\, a_{\hat{\rho}} + \rho^2 a_{\hat{z}}$$

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$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho z\cos\phi \\ e^{-2}\sin\phi/2 \\ \rho^2 \end{bmatrix}$$

$$= \begin{bmatrix} \rho z\cos^2\phi - e^{-2}\sin\phi\sin\phi/2 \\ \rho z\cos\phi\sin\phi + e^{-2}\cos\phi\sin\phi/2 \\ \rho^2 \end{bmatrix}$$

So, $A_x = \rho z\cos^2\phi - e^{-2}\sin\phi\sin\phi/2$

$A_y = \rho z\cos\phi\sin\phi + e^{-2}\cos\phi\sin\phi/2$

$A_z = \rho^2$

At $(1, \pi/3, 0)$ $\Rightarrow$ $A_x = -e^{-2}\frac{\sqrt{3}}{4}$.

$A_y = e^{-2}/4$, $A_z = 1$

In spherical co-ordinates —

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi \; \cancel{\sin\theta} & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} -e^{-2}\sqrt{3}/4 \\ e^{-2}/4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-2}\sqrt{3}/4\,\sin\theta\cos\phi + e^{-2}/4\,\sin\theta\sin\phi + \cos\theta \\ -e^{-2}\sqrt{3}/4\,\cos\theta\cos\phi + e^{-2}/4\,\cos\theta\sin\phi - \sin\theta \\ e^{-2}\sqrt{3}/4\cdot\sin\phi + e^{-2}/4\cos\phi \end{bmatrix}$$

Put $\phi = \pi/3$

$A_r = -\frac{e^{-2}\sqrt{3}}{8}\sin\theta + \frac{e^{-2}\sqrt{3}}{8}\sin\theta + \cos\theta$

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$$A_\theta = \frac{-e^{-2}\sqrt{3}}{8}\cos\theta + \frac{e^{-2}\sqrt{3}}{8}\cos\theta - \sin\theta$$

$$A_\phi = \frac{e^{-2}\cdot 3}{8} + \frac{e^{-2}}{8} = \frac{e^{-2}}{2}$$

$$\vec{H} = \cos\theta\, \hat{a_r} - \sin\theta\, \hat{a_\theta} + \frac{e^{-2}}{2}\hat{a_\phi}$$

Now $\vec{H} \times \hat{a_\theta} = \begin{vmatrix} \hat{a_r} & \hat{a_\theta} & \hat{a_\phi} \\ \cos\theta & -\sin\theta & e^{-2}/2 \\ 0 & 1 & 0 \end{vmatrix}$

$$= -\frac{e^{-2}}{2}\hat{a_r} + \cos\theta\, \hat{a_\phi}$$

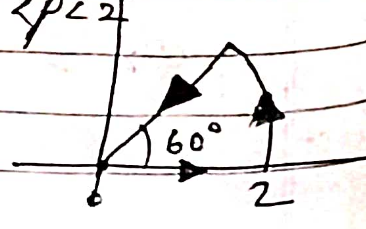
b) Vector component of $\vec{H}$ normal to surface $\rho = 1$ & —

$$\vec{H} = 1 \times z\cos\left(\frac{3}{2}\right)\hat{a_\rho} = 0\hat{a_\rho}$$

c) Scalar component of $\vec{H}$ tangential to the plane — $z=0$ $45°$ —

$$\frac{e^{-2}\cos\phi}{2} - \frac{e^{-2}}{2} \times 1 - \frac{e^{-2}}{2}$$

Q11. Calculate circulation — $A = \rho\cos\phi\, \hat{a_\rho} + z\sin\phi\, \hat{a_z}$ where edge L defined by $0<\rho<2$, $0<\phi<60°$, $z=0$ shown in figure



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$$\vec{A} = \rho\cos\phi\, \hat{a_\rho} + z\sin\phi\, \hat{a_z}$$

$$\oint \vec{A}\cdot\vec{dl} = \int_{\rho=2}^{2} \vec{A}\cdot\vec{dl} + \int_{\phi=0}^{\pi/3} \vec{A}\cdot dl + \int_{\rho=2}^{0} \vec{A}\cdot\vec{dl} \quad -(1)$$

$$\therefore \int_0^2 \vec{A}\cdot\vec{dl} = \int_0^2 (\rho\cos\phi\, \hat{a_\rho} + z\sin\phi\, \hat{a_z})\, d\rho\, \hat{a_\rho}$$

$$= \int_0^2 \rho\cos\phi\, d\rho = \left[\frac{\rho^2}{2}\right]_0^2 = 2$$

$$\therefore \int_0^{\pi/3} \vec{A}\cdot\vec{dl} = \int_0^{\pi/3} (\rho\cos\phi\, \hat{a_\rho} + z\sin\phi\, \hat{a_z})\, \rho\, d\phi\, \hat{a_\phi} = 0$$

$$\therefore \int_0^2 \vec{A}\cdot\vec{dl} = \int_2^0 (\rho\cos\phi\, \hat{a_\rho} + z\sin\phi\, \hat{a_z})\, d\rho\, \hat{a_\rho}$$

$$= \int_0^2 \rho\cos\phi\, d\rho = \frac{1}{2}\times\left[\frac{\rho^2}{2}\right]_2^0 = -\frac{1}{2}\times 2 = -$$

Put in (1) we get -

$$\oint \vec{A}\cdot\vec{dl} = 2+0-1 = 1$$

Q12. Given $A = xy + yz + xz$, find grad A at $(1,2,3)$ & dd of A at same point in direction toward point $(3,4,$

$$\vec{A} = xy + yz + xz$$

$$\nabla\vec{A} = \frac{\partial}{\partial x}(A)\hat{a_x} + \frac{\partial}{\partial y}(A)\hat{a_y} + \frac{\partial}{\partial z}(A)\hat{a_z}$$

$$= (y+z)\hat{a_x} + (x+z)\hat{a_y} + (y+x)\hat{a_z}$$

At $(1, 2, 3)$

$\vec{\nabla A} = 5\hat{a_x} + 4\hat{a_y} + 3\hat{a_z}$

Let $\vec{B} = 2\hat{a_x} + 2\hat{a_y} + \hat{a_z}$

Now $\vec{B}$ dd of $A = \dfrac{\nabla A \cdot \vec{B}}{|\vec{B}|}$

$= \dfrac{(5\hat{a_x} + 4\hat{a_y} + 3\hat{a_z}) \cdot (2\hat{a_x} + 2\hat{a_y} + \hat{a_z})}{\sqrt{4+4+1}}$

$= \dfrac{10+8+3}{\sqrt{9}} = \dfrac{21}{3} = 7$

Q13. Calculate the angle b/w normals to the surfaces $x^2y + z = 3$ and $x \log z - y^2 = -4$ at the point of intersection $(-1, 2, 1)$

$S_1 = x^2y + z - 3$

$S_2 = x \log z - y^2 + 4$

$\vec{\nabla S_1} = \dfrac{\partial}{\partial x}(x^2y + z - 3)\hat{a_x} + \dfrac{\partial}{\partial y}(x^2y + z - 3)\hat{a_y} + \dfrac{\partial}{\partial z}(x^2y + z - 3)\hat{a_z}$

$= 2xy\,\hat{a_x} + x^2\hat{a_y} + \hat{a_z}$

$\nabla S_2 = \log z\,\hat{a_x} + 2y\,\hat{a_y} + \dfrac{x}{z}\hat{a_z}$

$(\nabla S_1)_{(-1,2,1)} = -4\hat{a_x} + \hat{a_y} + \hat{a_z}$

$$(\overrightarrow{\nabla S_2})_{(-1,2,1)} = -4a\hat{y} - a\hat{z}$$

And $\overrightarrow{\nabla S_1}$ & ∇S_2 are normal to surface S_1 & S_2. So -

$$\theta = \cos^{-1}\left[\frac{(\overrightarrow{\nabla S_1})\cdot(\overrightarrow{\nabla S_2})}{|\overrightarrow{\nabla S_1}|\,|\overrightarrow{\nabla S_2}|}\right] = \cos^{-1}\left[\frac{-5}{\sqrt{18}\sqrt{17}}\right]$$

$$= \cos^{-1}[-0.23) = 106.55^\circ$$

Q14. Given $\vec{H} = x^2 a_x + y^2 a_{\hat{y}}$, $\int_L H\cdot dl = ?$ where L is along the curve $y = x^2$ from $(0,0)$ to $(1,1)$

$$\vec{H} = x^2 a_{\hat{x}} + y^2 a_{\hat{y}} \quad \&$$

$$\vec{dl} = dx\, a_{\hat{x}} + dy\, a_{\hat{y}}$$

$$\int \vec{H}\cdot\vec{dl} = \int x^2 dx + y^2 dy \quad \left\{ y = x^2,\ dy = 2x\,dx \right.$$

$$\int \vec{H}\cdot\vec{dl} = \int_0^1 (x^2 + x^4 \cdot 2x)\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + 2\left[\frac{x^6}{6}\right]_0^1$$

$$= \frac{1}{3} + \frac{2}{6} = \frac{2}{3} = 0.66.$$

Q15. Let $A = 2xy\,a_x + xz\,a_y - y\,a_z$. Evaluate $\int A\cdot dv$ over

a) a rectangular region $0 \le x \le 2$, $0 < y < 2$; $0 \le z < 2$

$$\int \vec{A} \cdot \vec{dv} = \int 2xy\, dx\, dy\, dz\, \hat{a_x} + \int xz\, dx\, dy\, dz\, \hat{a_y} - \int y\, dx\, dy\, dz\, \hat{a_z}$$

$$= 2\int_0^2 x\,dx \int_0^2 y\,dy \int_0^2 dz\, \hat{a_x} + \int_0^2 x\,dx \int_0^2 dy \int_0^2 z\,dz\, \hat{a_y} + \int_0^2 dx \int_0^2 y\,dy \cdot \int_0^2 dz\, \hat{a_z}$$

$$= 2\times2\times2\times2\, \hat{a_x} + 2\times2\times2\, \hat{a_y} - 2\times2\times2\, \hat{a_z}$$

$$= 16\hat{a_x} + 8\hat{a_y} - 8\hat{a_z}$$

b) a cylindrical region $\rho \le 3$, $0 \le z \le 5$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2xy \\ xz \\ -y \end{bmatrix}$$

$$A_\rho = 2xy\cos\phi + xz\sin\phi$$

$$= 2\rho^2\cos^2\phi\sin\phi + \rho z\sin\phi\cos\phi$$

$$A_\phi = -2xy\sin\phi + xz\cos\phi$$

$$= -2\rho^2\sin^2\phi\cos\phi + \rho z\cos^2\phi$$

$$A_z = -y = -\rho\cos\phi$$

and $dv = \rho\, d\rho\, d\phi\, dz$

$$\int A \cdot dv = \iiint \left[2\rho^2\cos\phi\, d(-\cos\phi)\, d\rho\, dz\, \hat{a_\rho} + \right.$$

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$\rho z \cos\phi \, d(-\cos\phi) d\rho \, dz \quad -2\rho^3 \sin^2\phi \, d(\sin\phi)$
$d\rho \, dz \, \hat{a_\phi} + \rho^2 z \cos^2\phi \, d\phi \, d\rho \, dz \, \hat{a_\rho}$ —
$\rho^2 \cos^2\phi \, d\phi \, d\rho \, dz \, \hat{a_z}]$

$\therefore \int_0^{2\pi} \cos\phi \, d\phi = 0$

$$\int_v A \cdot dv = -2\left[\frac{\rho^3}{3}\right](\cos\phi)_0^{2\pi}(z)_0^5 \hat{a_\rho} - \left(\frac{\rho^3}{3}\right)_0^3\left(\frac{z^2}{2}\right)_0^5$$
$$\left[\frac{\cos^2\phi}{2}\right]_0^{2\pi} \hat{a_\phi} - \left(\frac{2\rho}{4}\right)_0^3 (z)_0^3 \left[\frac{\sin^3\phi}{3}\right]_0^{2\pi} \hat{a_\phi}$$
$$+\left(\frac{\rho^3}{3}\right)_0^3\left(\frac{z^2}{2}\right)_0^5\left(\frac{1}{2}+\frac{\sin 2\phi}{4}\right)_0^{2\pi} \hat{a_\phi}$$
$$= 0 + 0 + \frac{22.5}{4}\hat{a_\phi}$$
$$= 56.25 \, \hat{a_\phi}$$

c) ϕ spherical region $r \le 4$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\theta \\ \cos\phi & -\sin\theta & 0 \end{bmatrix}\begin{bmatrix} 2xy \\ xz \\ -y \end{bmatrix}$$

$A_r = 2xy \sin\theta\cos\phi + xz\cos\phi\cos\theta + y\sin\theta$
$\quad + r^4 \sin^4\theta \cos\phi \sin\phi$

$A_\theta = \cancel{2xy \sin\theta \sin\phi + xz\cos\theta\sin\phi - y\cos\theta}$
$\quad = -r^3 \sin^2\theta \cos\phi \sin\phi$

$A_z = 2xy\cos\theta - xz\sin\theta$

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$= 2r^4 \sin^2\theta \cos\theta \sin\phi \cos\phi - r^4 \sin^3\theta \cos^2\theta \cos\phi$

$\int \vec{A} \cdot d\vec{v} = \left[\frac{r^5}{5}\right]_0^4 \left[\frac{1}{2} + \frac{\cos 2\phi}{4}\right]_0^{2\pi} \cos^2\theta(1-\cos^2\theta) \, \hat{a_r}$

$= 204.8 \times \frac{1}{2}\left[\int_0^{\pi} \cos^2\theta \, d\theta - \int_0^{\pi} \cos^4\theta \, d\theta\right] \hat{a_r}$

$= 204.8 \times \frac{1}{2}\left[\frac{\pi}{2} - \frac{\pi}{2} + \frac{\pi}{8}\right] \hat{a_r}$

$= 102.4 \times \pi/8 \, \hat{a_r}$

$= 40.2 \, \hat{a_r}$

Q16. The acc^n of particle $a = 2.4 a_z \, m/s^2$. u at $r|_{(0,0,0)}$ while initial $V = -2a_x + 5a_z$, a) Find position at $t=1$ b) Determine velocity

Let the velocity be V:

So, $a = \left[\frac{dV_x}{dt}, \frac{dV_y}{dt}, \frac{dV_z}{dt}\right] = 2.4\hat{a_z}$

here, $\frac{dV_x}{dt} = 0$, $\frac{dV_y}{dt} = 0$, $\frac{dV_z}{dt} = 2.4$

$V_x = A$, $V_y = B$, $V_z = 2.4t + C$

Initial velocity $(V_x, V_y, V_z) = (-2, 0, 5)$

$A = -2$, $B = 0$, $C = 5$ {at $t = 0$

$V_x = \frac{dx}{dt} = -2 \Rightarrow x = -2t + P$

$V_y = \frac{dy}{dt} = 0 \Rightarrow y = Q$

$V_z = \frac{dz}{dt} = 2.4t + 5 \Rightarrow z = 1.2t^2 + 5t + R$

At $t = 0$

$x = 0,\ y = 0$ and $x = 0$

$\therefore P = Q = R = 0$

Which gives —

$x = -2t,\ y = 0$ and $z = 1.2t^2 + 5t$

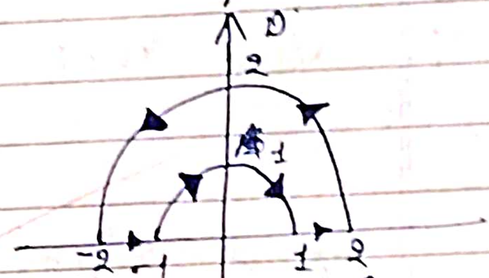
For $t = 1$

$x = -2,\ y = 0,\ z = 6.2$

and $v = (V_x, V_y, V_z)$

$= -2\hat{a_x} + (2.4t + 5)\hat{a_z}$

Q17. Let $A = \rho\sin\phi\, a_\rho + \rho^2 a_\phi$. Evaluate $\oint_L A \cdot dl$ given that L is the contour of below figure —



$A = \rho\sin\phi\,\hat{a_\rho} + \rho^2 a_\phi$

$\vec{dl} = d\rho\,\hat{a_\rho} + \rho\, d\phi\,\hat{a_\phi} + dz\,\hat{a_z}$

(1) From A to B $\rightarrow \rho$ 1 to 2

$\int \vec{A}\cdot\vec{dl} = \int \rho\sin\phi\, d\rho + \rho^3 \cancel{d\phi} = 0$

(2) from B to C $\rightarrow \rho = 2 \Rightarrow d\rho = 0$

$\int \vec{A}\cdot\vec{dl} = \int \rho\sin\phi\, \cancel{d\rho} + \rho^3 d\phi \Rightarrow \int \rho^3 d\phi \Rightarrow 8\int d\phi$

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$= 8[\phi]_0^{\pi} = 8\pi$

(3) from C to D $\rho = 2 \to 1$ $d\rho \neq 0$
$\phi = 0, d\phi = 0$

$\int \vec{A}\cdot d\vec{l} = \int \rho \sin\phi\, d\rho + \rho^3 d\phi = 0$

(4) from D to A, $\rho = 1 = 1$ $d\rho = 0$
$\phi = \pi$ to 0

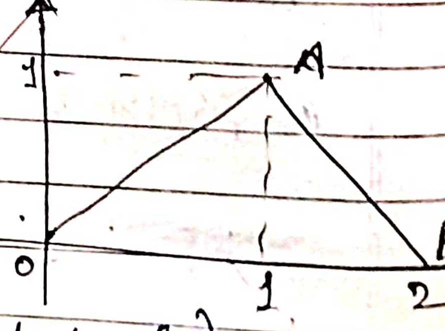
$\int \vec{A}\cdot d\vec{l} = \int \rho \sin\phi\, d\rho + \rho^3 d\phi$

$= \int \rho^3 d\phi \Rightarrow \int d\phi \Rightarrow [\phi]_{\pi}^{0} = -\pi$

So, $\oint \vec{A}\cdot d\vec{l} = 8\pi + 0 - \pi = 7\pi$ Ans

Q18. Given that $F = x^2 y a_x - y a_y$. find

$\vec{F} = x^2 y \hat{a_x} - y \hat{a_y}$



a) $\oint_L F\cdot dl$ where L is shown

$\vec{F}\cdot d\vec{l} = (x^2 y \hat{a_x} - y \hat{a_y})(dx\,\hat{a_x} + dy\,\hat{a_y})$

$F\cdot d\vec{l} = \int_{OA} + \int_{AB} + \int_{BO}$

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Along OA : $y = x$; $x \to 0$ to 1 , $dy = dx$

along AB : $y = -x+2$, $x \to 1$ to 2 $dy = -dx$

along BO : $y = 0$, $x \to 2$ to 1 $dy = 0$

$$\int_{OA} F \cdot \vec{dl} = \int_{OA} (x^2(x)dx - x\,dx)$$

$$= \int (x^3 dx - x\,dx)$$

$$= \int$$

$$\int_{AB} F \cdot dl = \int_{AB} \left[x^2(-x+2)\,dx - (-x+2)(-dx) \right]$$

$$= \int_{AB} \left[-x^3 + 2x^2 - x + 2 \right]$$

$$\int_{BO} F \cdot dl = 0 .$$

$$\therefore \int F \cdot dl = \int_0^1 (x^3 - x)\,dx + \int_1^2 (-x^3 + 2x^2 - x + 2)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_0^1 + \left[\frac{-x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{2} + 2x \right]_1^2$$

$$= \frac{-1}{4} + \left[\frac{-15}{4} + \frac{14}{3} - \frac{3}{2} + 2 \right]$$

$$= 7/6 .$$

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b) $\oint \int_s (\nabla \times F) \cdot ds$ where S is area bounded by L

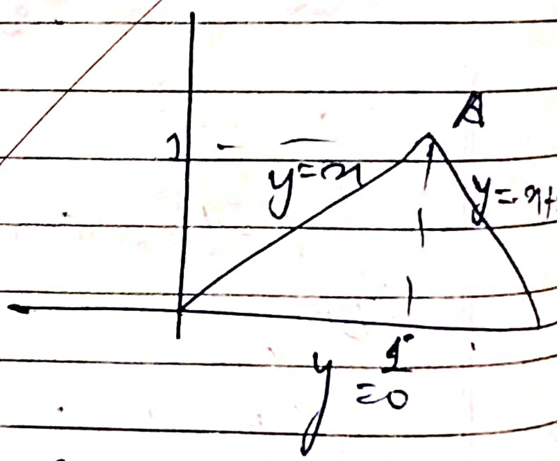
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -y & 0 \end{vmatrix}$$

$$= \hat{a_x}(0-0) - \hat{a_y}(0-0) + \hat{a_z}(-x^2)$$

$$\nabla \times \vec{F} = -x^2 \hat{a_z}$$

$$(\nabla \times F) ds = -x^2 a_z (-dx\,dy\,a_z)$$
$$= x^2 dx\,dy$$

$$(\nabla \times F) ds = \oint_{OCA} + \oint_{ABC}$$



In OCA : $y \to 0$ to x
$x \to 0$ to 1

In ABO : $y \to -x+2$ to 0
$x = 1$ to 2

$$(\nabla \times \vec{F}) ds = \int_0^1 \int_{y=0}^{x} x^2 dx\,dy + \int_{x=1}^{2} \int_{y=x+2}^{0} x^2 dx\,dy$$

$$= \int_0^1 [x^3] dx + \int_1^2 x^2 (x-2)$$

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$$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^4}{4} - \frac{2x^3}{3}\right]_1^2$$

$$= \frac{1}{4} + \frac{15}{4} - \frac{14}{3} \Rightarrow 7/6$$

$\therefore$ Yes, Stokes theorem is verified.