A PHYSICAL MODEL OF THE NONLINEAR SITAR STRING

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ABSTRACT

The acoustic properties of the sitar string are studied with the aid of a physical model. The nonlinearity of the string movement, caused by the bridge acting as an obstacle to the vibrating string, is of special interest. Comparison of the model's audio output to recordings of the instrument shows interesting similarities. The effects dispersion and bridge have on the sound of the instrument are demonstrated in the model.

1. INTRODUCTION

The sitar is a stringed instrument from South Asia known for its very distinctive timbre, which is quite different from the sound of most western plucked instruments. Here a physical model of the sitar's string is designed to verify common theses about the instrument's sound formation as they are found in literature. Sympathetic strings and the resonating body, although beyond doubt important for its sound because of their filtering effect and because of interactions between them and the vibrating string, are neglected. The physical model could also be the first step in developing a sitar synthesizer, since with all it's sympathetic strings the sitar is rather difficult to synthesize with other techniques such as sampling.

In section 2 the sound of the sitar is analyzed and important theses concerning the string movement are summarized. In section 3 the physical model is presented. In section 4 the model is analyzed by comparing its output to recordings of the sitar. It is investigated if the theses presented earlier can be applied to the modelled string as well.

Sample sounds of the model and the model itself can be downloaded from `www.talaash.at/sitar`.

2. CHARACTERISTICS OF THE SITAR

2.1. Sound Analysis

The sitar sound analyzed was recorded in an anechoic room. All strings except the plucked string (c with $f_0 \approx 131$Hz) where damped. Comparison to western plucked string instruments, such as the guitar, shows four major differences:

1. Descending formants are one striking feature of the sitar sound which can not only be heard easily, but is also very obvious in spectrograms (see fig. 1).
2. The envelopes of the partials of the sitar sound are very complex (see fig. 2) in comparison to most western plucked instruments. Since energy is fed to the system only when the string is plucked, the fluctuations in the partials' envelopes must be caused by energy flow between partials or between the string and the body.
3. On the ideal string, when plucked at one fifth of its length, the fifth, tenth, fifteenth etc. modes remain silent, because the string is plucked at one of their nodes [1]. This rule, known as the Young-Helmholtz law, is not valid for the sitar [2], as figure 3 shows.

2.2. The Sound Formation of the Sitar

2.3. The Effect of the flat Bridges

Raman identified the flat bridges[1] (see fig. 4) of the sitar as the main reason for its distinctive sound [2]. The bridge disturbs the free vibration of the string which is resting on it, varying the length of the vibrating part periodically. Laws which assume that the string has constant length are thus not applicable.

2.4. The Importance of Dispersion

Although commonly neglected in literature describing string vibration, dispersion plays a key role in the effects described above, as noted by Bertrand [3] and Valette and Cuesta [4]. Looking at one wavefront travelling towards the bridge the importance of dispersion becomes clear: When the lower frequency waves of the wavefront, which usually let the string vibrate with higher amplitude, cause the string to collide with the bridge, a part of the higher frequency waves will already have passed the point of impact and thus not die away when string and bridge collide [4]. Furthermore, dispersion also alters the shape of the vibrating string [5], so that the contact of string and bridge is affected.

[1] It should be noted that the bridges of the sitar are not really flat. Their surface is rather a curved plane. In default of a prevalent term the adjective "flat" was chosen to emphazise that a string segment of a certain length rests on that curved surface of the bridge, or collides with it when vibrating.

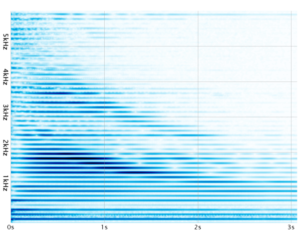


Figure 1: Spectrogram of the sitar

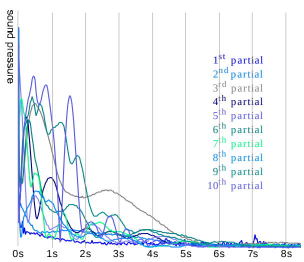


Figure 2: Envelopes of sitar partials

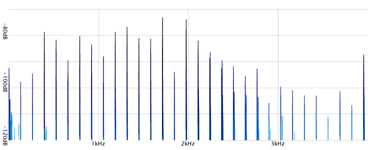


Figure 3: Spectrum of the sitar string, plucked at one fifth of its length

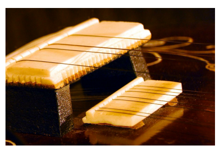


Figure 4: The "flat" bridges of the sitar

3. PHYSICAL MODELLING

3.1. Simplifications of the String Vibration

In the proposed model the string is broken down into a finite number of points, of which only vertical displacement in a plane perpendicular to the bridge is considered. Longitudinal waves, torsional waves and transversal waves in a plane parallel to the bridge are neglected, although especially the latter might be of great importance for the sitar string. Such simplifications are, however, acceptable, because the coupling between transversal waves perpendicular to the bridge and the body is more effective than the coupling between longitudinal waves or transversal waves parallel to the bridge and the body [1]. Additionally the main interaction between bridge and string takes place in the plane modelled.

3.2. Difficulties in Modelling the Sitar String

As the bridges damping effect on the strings movement increases with amplitude of string vibration, the amplitude of the input and output signal are not related in a linear way. The string can therefore not be described as a linear system.

Dispersion is no trivial problem either and most works about vibrating strings with constraints tend to neglect it. Burridge, Kappraff and Morshedi [6] as well as Vyasarayani, Birkett and McPhee [7] describe the movement of the sitar string with sets of equations, while Taguti [8] describes the vibration of a biwa or shamisen string (with similar vibrational constraints as caused by the sitar's bridge) with finite differences. In all these works dispersion is neglected, which causes a huge difference as we shall see below. Valette and Cuesta [4], who consider dispersion and vibrational constraints (for the tanpura, which is slightly different), mention the simulation of the string vibration, but do not tell the reader how they do it.

3.3. Design of the Model

Waveguides are a fast and efficient way to model string vibration and will be used in the sitar string model, but the string's interaction with the bridge is difficult to model in waveguides, since they are based on the wave equation of the ideal string and nonlinearities can't be inserted into the model directly. The part of the string colliding with the bridge is therefore modelled using finite differences and coupled to the model as described in [9].

3.4. Waveguide Modelling

In waveguides [10] the travelling waves on a string are used to model string vibration. One right travelling and one left travelling wave determine the string's shape (see fig. 6) according to the simple relation

$$y(x,t) = y_l(x,t) + y_r(x,t), \tag{1}$$

which is based on d'Alembert's solution of the wave equation [10]

$$y(x,t) = y_r(x-ct) + y_l(x+ct). \tag{2}$$

The pitch of the modelled string is determined by the length of the waveguide L according to the formula [10]

$$f_0 = \frac{f_S}{2 \cdot L}, \tag{3}$$

where f_S is the sampling rate.

After the initial shapes of the travelling waves are derived from the strings initial displacement according to the relation above (1), they are shifted by one sample into their direction each time step. To model the reflection at the fixed ends of the string, samples shifted out of the waveguide are appended to the end of the wave travelling in the opposite direction with inverted sign.

Damping can be modelled by integrating filters into the travelling waves' loops. Using filters with low-pass characteristics can account for stronger damping of higher partials, as it can be observed in real strings [10].

Dispersion means that the propagation speed of waves changes with frequency. The travelling waves' higher frequency components travel faster than the lower frequency components. The effect of dispersion can thus be imitated by inserting all pass filters, which cause delay for higher partials, into the travelling waves' loop [10]. As in [11] a cascade of several first order all-pass filters is used.

3.5. Finite Difference Modelling (FDM)

The displacement y of a vibrating string with linear density μ and tension T at time t can be calculated with the well known wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{4}$$

where where $c^2 = T/\mu$. In FDM [12] this equation is solved approximately with finite differences. The first derivation of a function $f(x)$ is approximated as

$$f'(x) \approx \frac{f(x+\Delta x/2) - f(x-\Delta x/2)}{\Delta x}. \tag{5}$$

The second derivation is therefore

$$f''(x) \approx \frac{f'(x+\Delta x/2) - f'(x-\Delta x/2)}{\Delta x} \tag{6}$$

or

$$f''(x) \approx \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2}. \tag{7}$$

Neglecting inexactness application on the wave equation yields

$$\begin{aligned} &\frac{y(x,t+\Delta t) - 2y(x,t) + y(x,t-\Delta t)}{\Delta t^2} = \\ &c^2 \frac{y(x+\Delta x,t) - 2y(x,t) + y(x-\Delta x,t)}{\Delta x^2}, \end{aligned} \tag{8}$$

where $y(x,t)$ is the displacement of the string at position x and time t.

If we now define $\Delta x = c \cdot \Delta t$ the vertical displacement y at position x of the string in the next time step $t + \Delta t$ can be calculated from the current displacement of the neighbouring points and the previous displacement at x [12]:

$$y(x,t+\Delta t) = y(x+\Delta x,t) + y(x-\Delta x,t) - y(x,t-\Delta t) \tag{9}$$

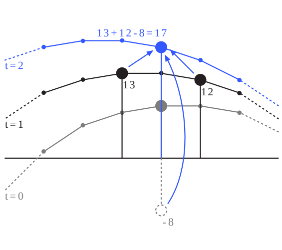


Figure 5: Calculating one point in the FDM

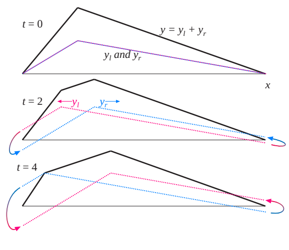


Figure 6: Two travelling waves determine the shape of the string in a waveguide model

Figure 5 shows this relation schematically. If outermost points are fixed, the movement of a whole string can be calculated.

By applying finite differences to the enhanced wave equation describing the movement of the dispersive string [1]

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \frac{E \cdot S \cdot K^2}{\mu} \frac{\partial^4 y}{\partial x^4} \tag{10}$$

where E is the Young modulus of the string, S the string's cross-sectional area, K the radius of gyration and μ as before linear density, we can calculate the movement of each point of the dispersive string as

$$\begin{aligned} y(x, t + \Delta t) &= y(x + \Delta x, t) + y(x - \Delta x, t) \\ -y(x, t - \Delta t) &- Q[y(x + 2\Delta x, t) - 4y(x + \Delta x, t) \\ &+ 6y(x, t) - 4y(x - \Delta x, t) + y(x - 2\Delta x, t)]. \end{aligned} \tag{11}$$

where

$$Q = \frac{E \cdot S \cdot K^2}{c^4 \cdot \mu \cdot \Delta t^2}. \tag{12}$$

By introducing constraints for the movement of some points at one end of the string, the effect of the sitar's bridge can be imitated in FDM. Since those points loose their kinetic energy when colliding with the bridge, the string is damped. Further damping is achieved by introducing additional coefficients in equation 11 as proposed by Karjalainen for the non-dispersive equation [12]:

$$\begin{aligned} y(x, t + \Delta t) &= g \cdot [y(x + \Delta x, t) + y(x - \Delta x, t)] \\ -a \cdot y(x, t - \Delta t) &- Q[y(x + 2\Delta x, t) - 4y(x + \Delta x, t) \\ &+ 6y(x, t) - 4y(x - \Delta x, t) + y(x - 2\Delta x, t)]. \end{aligned} \tag{13}$$

Karjalainen calculates g and a from the two constants b and d by

$$a = 2bd - 1 \tag{14}$$

$$g = 1 - d \tag{15}$$

where $0 \leq d \leq 1$ and $0 \leq b \leq 1$. Proposed values for the sitar string's model are $b = 0.97$ and $d = 0.00005$ to avoid too fast decay.

3.6. Sound output

In acoustic string instruments the resonator, which is coupled to the strings through the bridge, serves as an amplifier and radiates sound waves into the air. The audio output of the string models is therefore the force exercised by the string on the bridge.

4. ANALYSIS OF THE MODEL

4.1. Graphical Output

The effects of dispersion and the flat bridge are best seen in stroboscopic images of the string movement. Figure 7, depicting the vibration of an ideal string and a dispersive string, both without flat bridges, shows how dispersion alters the form of the vibrating string over time. The effect this deformation would have on the string's interaction with the bridge is obvious, especially when closely examining the snapshots taken at $t = 84$ms and $t = 126$ms, in which the form at the left end of the dispersive string, where we would insert the flat bridge, is quite different.

The importance of the flat bridge can be seen in figure 8. Since each snapshot is taken several periods after the preceding one, even the first snapshot after the excitation in the right image already shows the effect of the string's periodic collision with the bridge.

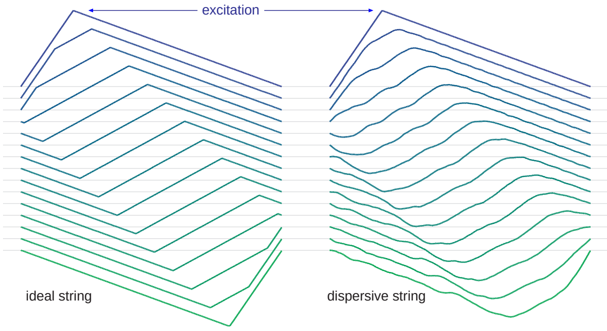


Figure 7: Stroboscopic images of the ideal string and the dispersive string; time interval between the snapshots $\Delta t = 14$ms

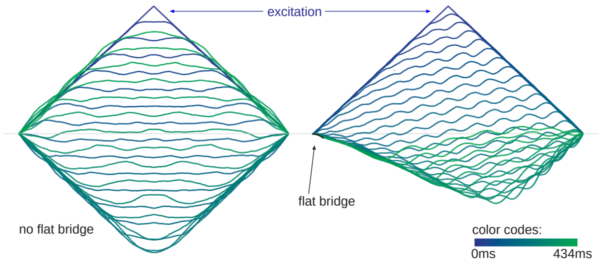


Figure 8: Stroboscopic images of the dispersive string without vibrational constraints (left) and with flat bridge (right), $\Delta t = 14$ms

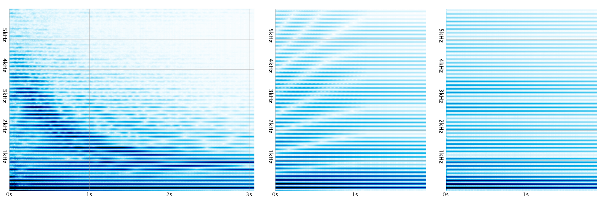
4.2. Sound analysis

When the string's interaction with the bridge and dispersion are modelled, the same phenomena which were observed in the sitar can be observed in the model.

1. Descending Formants
   The importance of dispersion can be verified in the model: Descending formants, as in the sound of the real instrument, can only be observed in dispersive models. The flat bridge's effect on the sound spectrum is much weaker if dispersion is neglected (see fig. 9).

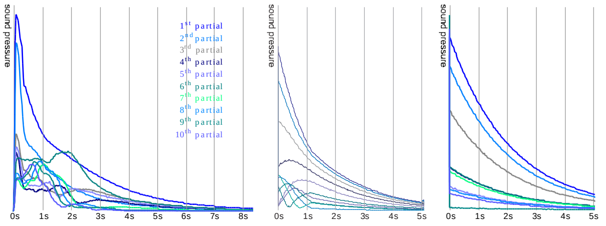
2. Energy flow between partials
   In a non-dispersive model with flat bridge energy flow between the harmonics can only be observed at the very beginning of the sound. Similar complexity in the structure of the harmonics' envelopes as observed in the sitar sound can only be found in the model with flat bridge and dispersion (see fig. 10).

3. Young-Helmholtz law not applicable
   Different modelled strings have been "plucked" at one fifth of their length. Only in the case of ideal strings the Young-Helmholtz law is fully applicable. Even dispersion introduces some limitations, because the ratio between the fundamental frequency and the frequency of



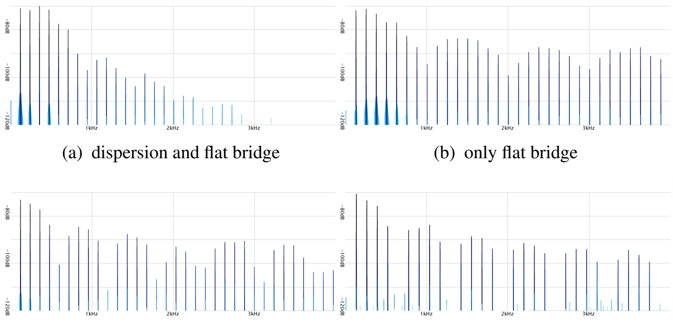
(a) dispersion and flat bridge (b) only flat bridge (c) only dispersion

Figure 9: Spectrograms of different models



(a) dispersion and flat bridge (b) only flat bridge (c) only dispersion

Figure 10: The partial's envelopes of different models



(a) dispersion and flat bridge (b) only flat bridge

(c) only dispersion (d) ideal string

Figure 11: Spectrums of different models

the higher partials is not an integer. In string models with flat bridges, the Young-Helmholtz law is not valid, as in the case of the sitar (see fig. 11).

5. CONCLUSIONS

A real time physical model of the sitar string has been proposed. Audio output of the model shows important characteristic features also found in the sound of the sitar. Not only the flat bridge, but also the dispersion on the string plays an important role in the sound formation of the instrument. Neglecting dispersion when describing the vibration of strings periodically colliding with objects is an oversimplification and abstraction with very restricted validity.

6. ACKNOWLEDGMENTS

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