★ $N(A^TA) = N(A) \xrightarrow{\text{using}} \text{Rank}(A^TA) = \text{Rank}(A)$ → A^TA is invertible exactly if $N(A)$ only got the zero vector or A has independent cols

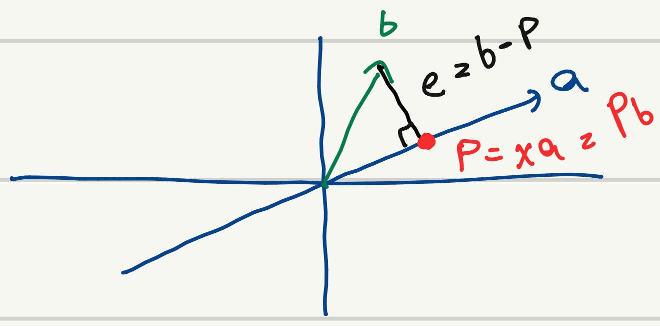
↳ $Ax = 0 \to x \in N(A)$ ↳ $n - r_{A^TA} = n - r_A$

$\xrightarrow{\times A^T} A^TAx = A^T0 = 0 \to x \in N(A^TA)$ → $r_{A^TA} = r_A$

Projection

: L15

Project b on a



→ Find the point that is closest to b on line a → it's the P point

e: Error → the is prependicular to a , P is some multiple (x) of a → Find it!

we know $a \perp e$: $a^T(b - xa) = 0 \to x\underbrace{a^Ta}_{\text{number}} = a^Tb \to x = \dfrac{a^Tb}{a^Ta}$ $\quad P = \underbrace{ax}_{\text{number} \leftarrow xa \rightarrow \text{number}} = a\dfrac{a^Tb}{a^Ta}$ [if $b \to 2b$, $P \to 2P$ | if $a \to 2a$, $P \to P$:)]

Projection matrix

Proj $p = Pb$ $\quad \underbrace{P}_{\text{Matrix}} = \dfrac{\overbrace{aa^T}^{\text{col x row = matrix}}}{\underbrace{a^Ta}_{\text{number}}}$

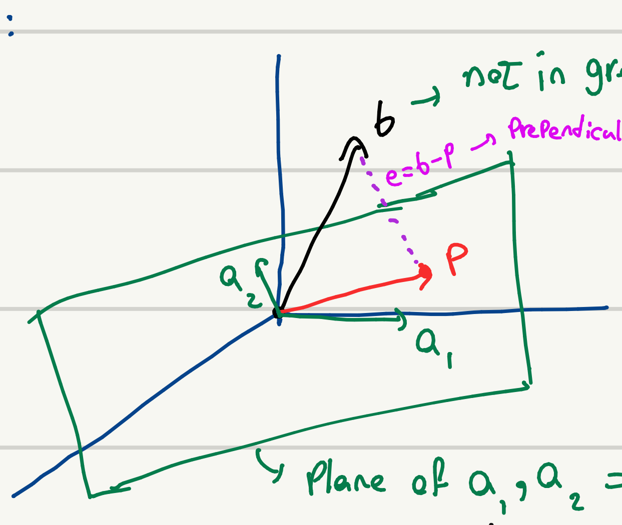
★ $C(P)$ = line through a , rank$(P) = 1$ $\quad$ ★ P is symmetric → $\left(\dfrac{aa^T}{\underbrace{a^Ta}_{\text{number}}}\right)^T = \dfrac{aa^T}{a^Ta}$

★ if we project twice, we get the same answer

★ Two Properties of Projection matrix → $\boxed{P^T = P \ , \ P^2 = P}$

Why Project? Because $Ax = b$ may have no sol → solve $A\hat{x} = P$ instead (P is Proj of b onto colspace → closest valid b that we could get)

Let's move to more dims:



→ not in green plane (if b is in colspace, Proj of b onto colspace = b)

→ Prependicular to the Plane

↳ Plane of $\underbrace{a_1, a_2}_{\text{basis}}$ = colspace of $A = \begin{bmatrix} \vdots & \vdots \\ a_1 & a_2 \\ \vdots & \vdots \end{bmatrix}$ → could have n cols

$n \times 1$

$P = \hat{x}_1 a_1 + \hat{x}_2 a_2 = \underbrace{A}_{m\times 2}\underbrace{\hat{x}}_{2\times 1}$ → Find $\hat{x}$! $\quad$ key: $b - A\hat{x}$ is perpendicular to the plane (and a_1, a_2 ofc)

→ $a_1^T(b - A\hat{x}) = 0 \quad a_2^T(b - A\hat{x}) = 0 \quad \xrightarrow{\text{in matrix form}} \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to A^T\underbrace{(b - A\hat{x})}_{} = 0$

e is in $N(A^T)$ → $e \perp C(A)$!!

→ $A^TA\hat{x} = A^Tb \to \hat{x} = (A^TA)^{-1}A^Tb$, $P = A\hat{x} = A(A^TA)^{-1}A^Tb$ [in 1-D it was $\dfrac{aa^T}{a^Ta}$, here we can't just divide, we use inverse]

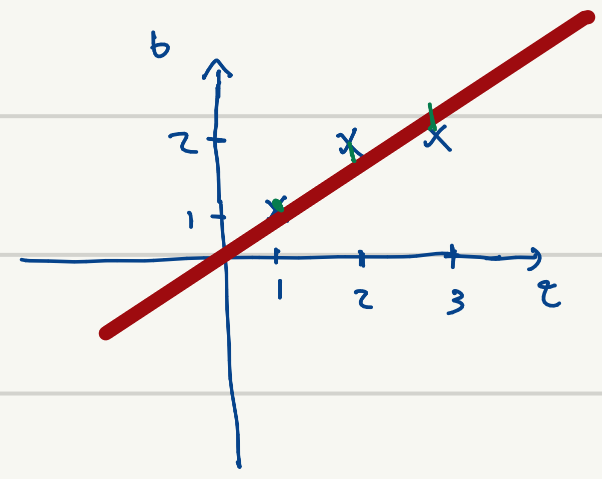
Proj matrix $P = A(A^TA)^{-1}A^T$ [you can't use the rule $(ab)^{-1} = b^{-1}a^{-1}$ here because A is not a square matrix so A^{-1} does not exist —

if A is an invertible square matrix, then colspace is whole $\mathbb{R}^n$ and b is already in colspace and $P = I$

which is correct]

☆ $P^T = P$, $P^2 = P \to [A(A^TA)^{-1}\cancel{A^T A (A^TA)^{-1}}A^T \to A(A^TA)^{-1}A^T \checkmark]$

Least squares fitting by a line : Fit the points $(1,1), (2,2), (3,2)$ by a line



• error $\to$ Make it small as it could get

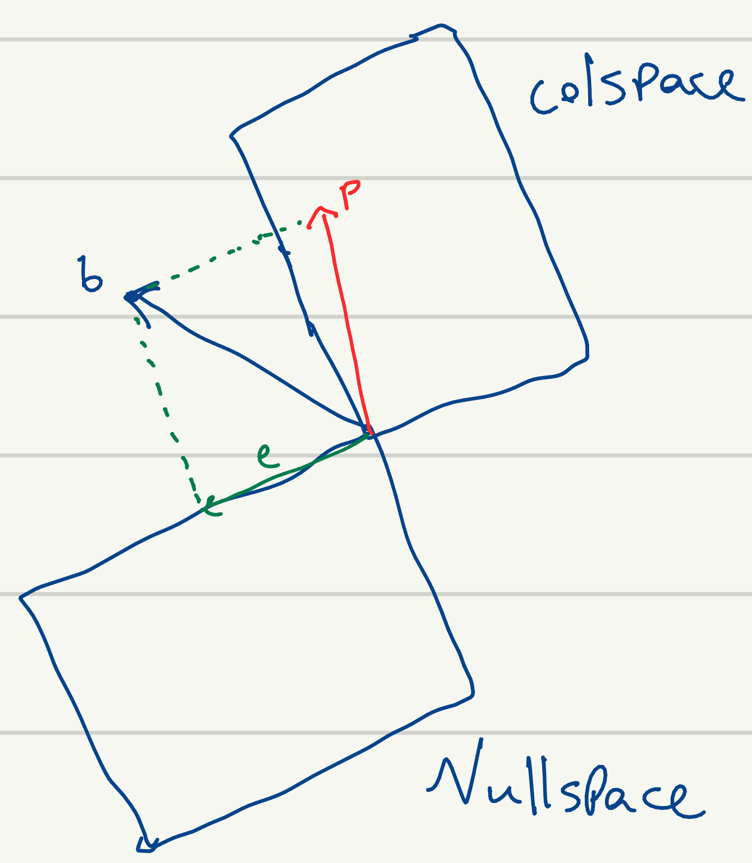
Find the best line : $b = C + Dt$ $\to$ $C + D = 1$, $C + 2D = 2$, $C + 3D = 2$ $\to$ Can't solve

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{b}$$

: L16

☆ If b in colspace $\to Pb = b \to b = Ax$ cause it's in colspace $\to$ $Pb = A\underbrace{(A^TA)^{-1}(A^TA)}_{I}x = Ax = b$

☆ If $b \perp$ colspace $\to Pb = 0 \to b$ is in nullspace of A^T $\quad Pb = A(A^TA)^{-1}\underbrace{A^Tb}_{0} = 0$



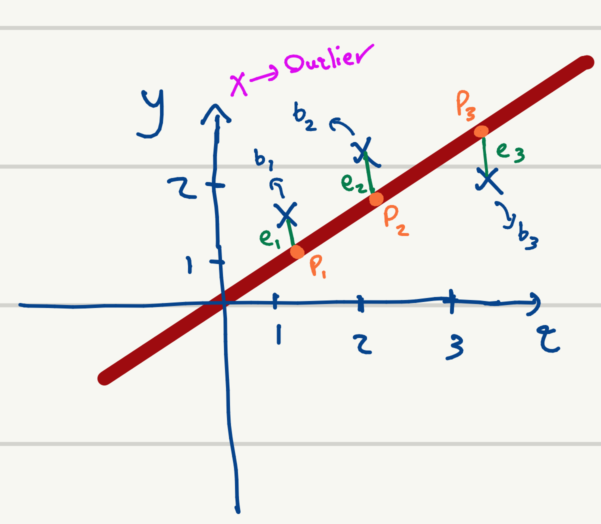
Pb , $(I-P)b \to$ Proj onto $\perp$ space

$P + e = b$

why? $\quad b = P + e \to e = b - P = b - Pb$
$= (I-P)b$

☆ $(I-P)$ is a projection matrix as well

Find the best straight line



$y = C + Dt \to$ Find a line with min error

$C + D = 1$

$C + 2D = 2$ $\longrightarrow$ $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \to$ Minimize $\|Ax - b\|^2 = \|e\|^2 = e_1^2 + e_2^2 + e_3^2$ $\to$ We are doing Linear Regression here

$C + 3D = 2$

Basis for colspace

☆ If we have Outliers, sum of squares would not be a good measure of error

Find $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, P

$A^TA\hat{x} = A^Tb$ $\to$ $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 6 & 5 \\ 6 & 14 & 11 \end{array}\right] \to$ $3C + 6D = 5$, $6C + 14D = 11$

normal eqs

Let's solve it by Calculus ①

Minimize: $(C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2 \to$ Take partial derivative wrt to $C, D \to$ set them to zero

$\frac{\partial e}{\partial c} = 3C + 6D = 5$

$\frac{\partial e}{\partial d} = 6C + 14D = 11$

Solve eqs: $2D = 1 \to D = \frac{1}{2}$ $\quad 3C = 2 \to C = \frac{2}{3}$ $\to$ Best line: $\frac{2}{3} + \frac{1}{2}t$

$p_1 = \frac{7}{6}$ $\quad p_2 = \frac{5}{3}$ $\quad p_3 = \frac{13}{6}$

$e_1 = -\frac{1}{6}$ $\quad e_2 = \frac{1}{3}$ $\quad e_3 = -\frac{1}{6}$ $\qquad P = A\hat{x}$

because $p + e = b$

So:

$$\begin{array}{c} b \\ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \end{array} = \begin{array}{c} p \\ \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} \end{array} + \begin{array}{c} e \\ \begin{bmatrix} -\frac{1}{6} \\ \frac{2}{6} \\ -\frac{1}{6} \end{bmatrix} \end{array} \to$$

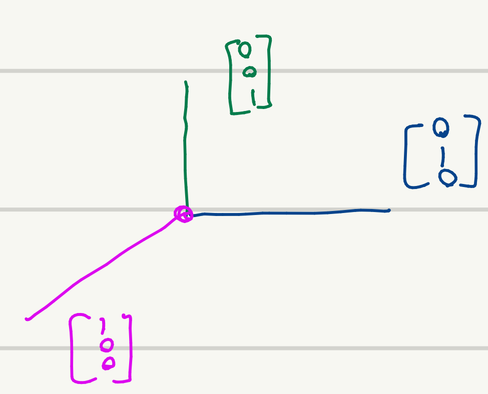
★ $p \cdot e = 0 \to$ P and e are perpendicular

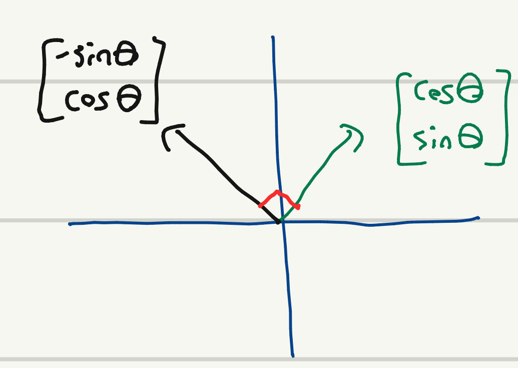
★ e is perpendicular to vectors in colspace

Key Requirement here: If $\overbrace{A \text{ has independent cols}}^{①}$ then A^TA is invertible $\to$ To Prove: suppose $A^TAx = 0$ $\to$ we must show $x = 0$

IDEA: Multiply both sides by x^T: $x^TA^TAx = 0 = (Ax)^T(Ax) = 0 \implies Ax$ has to be zero cause $y^Ty =$ square $\xrightarrow{\text{using ①}} x = 0$

★ Cols are definitely independent if they're $\underbrace{\text{perpendicular unit vector}}_{\text{orthonormal vectors}}$





: L17

★ Orthogonal basis in this lecture $q_1, \dots, q_n$ $\qquad$ orthogonal matrix Q (when it's square)

Orthonormal:

$$q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$Q = \begin{bmatrix} \vdots & & \vdots \\ q_1 & \cdots & q_n \\ \vdots & & \vdots \end{bmatrix}_{m \times n} \qquad Q^TQ = \begin{array}{c} Q^T \\ \begin{bmatrix} \cdots q_1^T \cdots \\ \vdots \\ \cdots q_n^T \cdots \end{bmatrix}_{n \times m} \end{array} \begin{array}{c} Q \\ \begin{bmatrix} \vdots & & \vdots \\ q_1 & \cdots & q_n \\ \vdots & & \vdots \end{bmatrix}_{m \times n} \end{array} = \begin{array}{c} I \\ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & \cdots & 1 \end{bmatrix}_{n \times n} \end{array}$$

★ If Q is square, then $Q^TQ = I$ tells us $Q^T = Q^{-1}$

Examples:

Permutation $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\qquad QQ^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}^{\text{orthonormal}} = I$

Another examples

$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\qquad Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\qquad Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ $\qquad$ orthonormal basis $Q = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$

$Q = \frac{1}{3} \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$

Suppose: Q has orthonormal columns, we want to project onto its colspace $\to$ what is the proj matrix

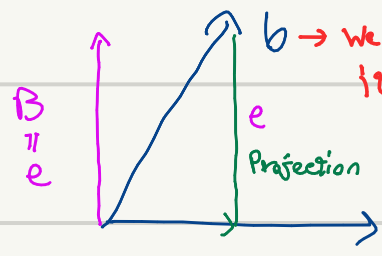
$P = Q\underbrace{(Q^TQ)^{-1}}_{I}Q^T = QQ^T$ if Q is square $\to = I$

if Q isn't square $\to$ Properties $\begin{bmatrix} 1- \text{Symmetric matrix} \\ 2- (QQ^T)(QQ^T) = QQ^T \end{bmatrix}$ (with $\underbrace{Q^TQ}_{I}$ in the middle) $\xrightarrow{\text{Example}}$ $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}_{4 \times 2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}_{2 \times 4} = \frac{1}{4}\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$

★ Benefits of having orthonormal matrices: In normal eq: $A^TA\hat{x} = A^Tb$ → Now A is Q → $\hat{x} = Q^Tb$ → $\hat{x}_i = q_i^T b$

Gram-Schmidt:

Independent vectors a, b → we want to produce A, B (orthogonal vectors) → Make them orthonormal $q_1 = \frac{A}{\|A\|}$ $q_2 = \frac{B}{\|B\|}$

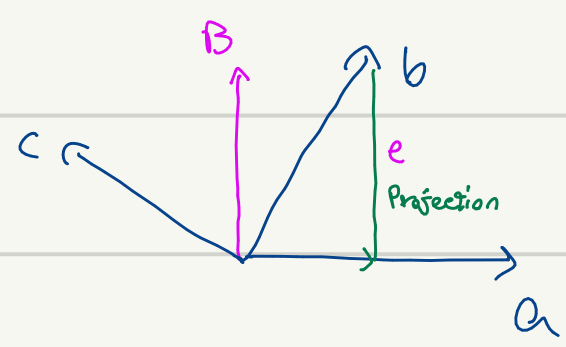


b → we want B that start from b but make it orthogonal to A

a → can be A

$$e = B = b - \frac{A^Tb}{A^TA}A \;\rightarrow\; A \perp B \Rightarrow A^T.B = 0 \Rightarrow A^T\Big(b - \underbrace{\overbrace{\frac{A^Tb}{A^TA}}^{\text{number}} A}_{= A\frac{A^Tb}{A^TA}}\Big) = A^Tb - \frac{\cancel{A^TA}\,A^Tb}{\cancel{A^TA}} = 0$$

What if we had 3 vectors? Let's add vector c (independent from a, b) → Find C (perpendicular to A, B) → we know A, B



$$C = c - \frac{A^Tc}{A^TA}A - \frac{B^Tc}{B^TB}B$$

$c \perp A$
$c \perp B$

Example: $a = \begin{bmatrix}1\\1\\1\end{bmatrix}$ $b = \begin{bmatrix}1\\0\\2\end{bmatrix}$ ① $A = a$ $B = \begin{bmatrix}1\\0\\2\end{bmatrix} - \frac{\begin{bmatrix}1&1&1\end{bmatrix}\begin{bmatrix}1\\0\\2\end{bmatrix}}{\begin{bmatrix}1&1&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\2\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$ $A.B = 0 \to A \perp B$

$Q = \begin{bmatrix}q_1 & q_2\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{3}} & 0\\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\end{bmatrix}$

Just a random name

$\hat{D} = \begin{bmatrix}1&1\\1&0\\1&2\end{bmatrix}$ → $C(Q) = C(D)$ because $A = a$, $B = b$ - comb of a so colspace stays the same

the old one

$q_i^T = 1\times m$ $a_j = m \times 1$ → $q_i a_j$ is scalar

★ $D = QR$ or $\hat{A} = QR$ → upper triangular   (not ①)

($Q^TQ = I$ → $R = Q^TA$ → $R = \begin{bmatrix}q_1^Ta_1 & q_1^Ta_2\\ q_2^Ta_1 & q_2^Ta_2\end{bmatrix} = \begin{bmatrix}a_1^Tq_1 & a_2^Tq_1\\ a_1^Tq_2 & a_2^Tq_2\end{bmatrix}$

$A = \begin{bmatrix}a_1 & a_2\end{bmatrix} = \underbrace{\begin{bmatrix}q_1 & q_2\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}a_1^Tq_1 & a_2^Tq_1\\ \underbrace{a_1^Tq_2}_{=0} & a_2^Tq_2\end{bmatrix}}_{R}$

$q_1 = a_1$, $q_1 \perp q_2$ ←
→ $a_1 \perp q_2$ → $a_1^Tq_2 = 0$

: L18

Determinant: A number associated with every square matrix det A or |A|

Properties of determinant:

① det I = 1   ① $\begin{vmatrix}1&0\\0&1\end{vmatrix} = 1$ ② $\begin{vmatrix}0&1\\1&0\end{vmatrix} = -1$

② Exchange rows: reverse the sign of the det → det $P = \begin{cases}1 & \text{even number of permutation}\\ -1 & \text{odd number of permutation}\end{cases}$ Permutation matrix

③ A = For any matrix, if I multiply one row by t and leave the other rows alone → $\begin{vmatrix}ta & tb\\ c & d\end{vmatrix} = t\begin{vmatrix}a & b\\ c & d\end{vmatrix}$

③ B = If we have $\begin{vmatrix}a+a' & b+b'\\ c & d\end{vmatrix} = \begin{vmatrix}a & b\\ c & d\end{vmatrix} + \begin{vmatrix}a' & b'\\ c & d\end{vmatrix}$ → determinant behaves like a linear func of the first row if other rows stay the same

★ 3-B is not saying: X det(A+B) = det A + det B X → we have linearity for each row seperately!

④ 2 equal rows → det = 0 : let's prove for $n\times n$: Exchange those rows → same matrix det didn't change but property 2 says sign changed → so det = 0

⑤ Subtract $l \times$ row i from row k → DET doesn't change!

Proof for 2x2: $\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{=0 \text{ (Property 4)}}$

6) Row of zeros $\rightarrow$ det $A = 0$ Proof: use 3-A with $t=0$

7) $U = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & - & 0 & d_n \end{bmatrix}$ (elim would get us here) det $u = (d_1)(d_2)\cdots(d_n)$ [elim then prod of pivots $\rightarrow$ Matlab use this to find det]

* If there was row exchanges during elim, we need to watch out for sign as it might be "− Prod of Pivots"

Proof: Suppose $d_i \neq 0$ $\rightarrow$ use pivots (from d_n to d_1) to make $*$'s into zeros $\rightarrow$ Now it's in diagonal form (rule 5)

$\begin{bmatrix} d_1 & 0 & - & 0 \\ 0 & \ddots & & \vdots \\ 0 & \cdots & & d_n \end{bmatrix} \rightarrow$ Factor d_i's $(d_1)(d_2)\cdots(d_n) \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & & 1 \end{bmatrix}$ (rule 3-A) $\rightarrow$ det $I = 1$ (rule 1) so det $= (d_1)\cdots(d_n)$ ✓

if some d_i is zero, with the help of elim we make that row to all zeros $\rightarrow$ (rule 6) det $= 0$ ✓

8) -A: det $A = 0$ when A is singular Proof: By elim go from A to U $\rightarrow$ We would get a row of zeros $\rightarrow$ det $A = 0$

8) -B: det $A \neq 0$ = = = non-singular or invertible

* Det $A_{2\times2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow$ elim $\begin{bmatrix} a & b \\ 0 & d-\frac{c}{a}b \end{bmatrix} \rightarrow$ det $A = a(d - \frac{c}{a}b) = ad - bc$ $\rightarrow$ if a was zero we have to exchange and if we couldn't $\rightarrow$ singular

9) det $AB = (\det A)(\det B)$ $\quad$ det $A^{-1} =$) $AA^{-1} = I \xrightarrow{\det} (\det A)(\det A^{-1}) = 1 \rightarrow \det A^{-1} = \frac{1}{\det A}$

* det $A^2 = (\det A)^2$ $\quad$ det $2A = 2^n$ det A ($A_{n\times n}$) (factor 2s of out of each row)

10) det $A^T =$ det A $\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$ ✓

* If a col is all zero $\rightarrow$ det $= 0$ (cause you can transpose it and have a row of zeros which has det $= 0$)

* Exchanging two cols $\rightarrow$ reverses the sign

Proof of 10: $|A^T| \overset{?}{=} |A| \rightarrow |U^T L^T| \overset{?}{=} |LU| \xrightarrow{r9} |U^T|\,|L^T| \overset{?}{=} |L|\,|U| \rightarrow |U^T| = |U|$ ($|L^T|$, $|L|$: ones on diag $\rightarrow$ det $= 1$)

:L19

Det of 2x2 using Properties: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \underbrace{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}_{0} + \underbrace{\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}}_{ad} + \underbrace{\begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}}_{-cb} + \underbrace{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}_{0} = ad - bc$

($\begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix} \rightarrow -cb$)

* We can use this method for $n\times n$ $\rightarrow n^n$

For 3x3:

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \rightarrow$ 27 det which a lot of them are zero $\rightarrow$ survivors: $\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix}$ --- the survivors have one entry from each row & each col

$\rightarrow a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$

(exchange r2, r3) $\quad \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix}$

Big Formula

$$\det A = \sum_{n! \text{ terms}} \pm\, a_{1\alpha}\, a_{2\beta}\, a_{3\gamma} \cdots a_{n\omega} \quad (\alpha, \beta, \gamma, \ldots, \omega) = \text{Perm of } (1, \ldots, n)$$

→ From this formula we can see $\det I = 1$ → only $a_{11}a_{22}a_{33}$ term is 1 and the rest = 0

Example

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$(4,3,2,1) \to +1$

$(3,2,1,4) \to -1$

Det = +1 − 1 = 0

Cofactors: 3x3

$$\det = \underbrace{a_{11}(a_{22}a_{33} - a_{23}a_{32})}_{\begin{vmatrix} a_{11} & \cancel{0} & \cancel{0} \\ \cancel{0} & a_{22} & a_{23} \\ \cancel{0} & a_{32} & a_{33} \end{vmatrix}} + \underbrace{a_{12}(-a_{21}a_{33} + a_{23}a_{31})}_{\begin{vmatrix} \cancel{0} & a_{12} & \cancel{0} \\ a_{21} & \cancel{0} & a_{23} \\ a_{31} & \cancel{0} & a_{33} \end{vmatrix}} + a_{13}(\cdots)$$

Smaller matrix

Cofactor of a_{ij} = C_{ij} = $(-1)^{i+j}\det$ (n−1 matrix with row i, col j erased)

Cofactor formula:

$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$ (along row 1, can change to row i)

Example

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example

$$A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$|A_1| = 1$ $\quad |A_2| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$ $\quad |A_3| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1$ $\quad |A_4| = 1 \cdot |A_3| - 1|A_2|$

$|A_4| = -1$ $\quad |A_5| = -1-(-1) = 0$ $\quad |A_6| = 0-(-1) = 1$ $\quad |A_7| = 1$ $\quad |A_8| = 0$ $\quad |A_9| = -1$

We can go row-wise as well

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

~~1, 0, −1, −1, 0,~~ 1, 0, −1, −1, 0, 1, … Period = 6 → $|A_{61}| = |A_1| = 1$

★ $|A_n| = |A_{n-1}| - |A_{n-2}|$

A^{-1} Formula:

for 2x2: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \underbrace{\frac{1}{ad-bc}}_{\text{Det}} \overbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}^{\text{Matrix of cofactors}}$

$d = C_{11}$

$-b = C_{21}$

$-c = C_{12}$

$a = C_{22}$

→ big formula $A^{-1} = \frac{1}{\det A} C^T$

Product of n−1 entries

Products of n entries

: L 20

Check $AC^T = (\det A)\, I$:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}C_{11} + \dots + a_{1n}C_{1n} & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \det A \end{bmatrix}$$

why?

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ & \vdots & & \\ & a_{n2} & & \end{bmatrix}$$

Imagine $A_s : \begin{bmatrix} a & b \\ a & b \end{bmatrix}$ → $\det A_s = ab + b(-a) = 0$

Now in $A_{2\times2}$: $a_{11}C_{21} + a_{12}C_{22} = a(-b) + ba = 0$ } Just like this → So we generalize

$$\sum_{k=1}^{n} a_{ik}C_{jk} = \begin{cases} \det A & i=j \\ 0 & i \neq j \end{cases}$$

★ With this formula: $A^{-1} = \frac{1}{\det A} C^T$ we can understand how the inverse changes when the matrix changes (like adding 1 to an entry → check what happened to $\det A$ and C^T)

$Ax = b \to x = A^{-1}b = \frac{1}{\det A} C^T b$

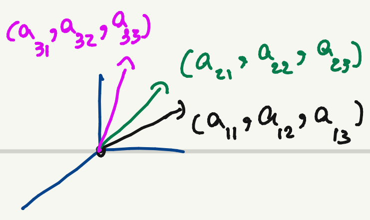
Cramer's rule:

$x_1 = \frac{\det B_1}{\det A}$ $\quad x_2 = \frac{\det B_2}{\det A}$ $\quad B_1 = \begin{bmatrix} b & \begin{matrix} n-1 \text{ cols} \\ \text{of } A \end{matrix} \end{bmatrix}$ → Matrix A with col 1 replaced by the right-hand side b

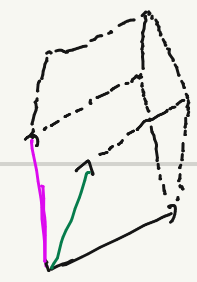
↳ $\det B_1$ = cofactor formula down col 1 = $C_{11}b_1 + C_{21}b_2 + \cdots$

B_j = A with col j replaced by b $\quad x_j = \frac{\det B_j}{\det A}$

3x3
①
$|\det A| =$ Volume of box



$\rightarrow$

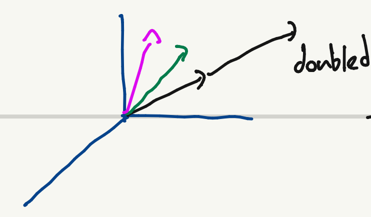


$\rightarrow$ Volume of this!

★ If $A = I$ $\rightarrow$ box = Cube $\rightarrow$ has Property 1 $\rightarrow$ formula ① is ok ★ This also follows Property 2 cause box wouldn't change by reversing rows and absolute value doesn't care about sign

★ If $A = Q$ (orthogonal matrix) $\rightarrow$ box = cube $\rightarrow$ dif from I? it's just rotated det Q: we know $Q^TQ = I \rightarrow |Q^TQ| = |I| = 1 \rightarrow |Q^T||Q| = 1 \rightarrow |Q|^2 = 1 \rightarrow |Q| = \pm 1 \rightarrow$ formula ① is ok

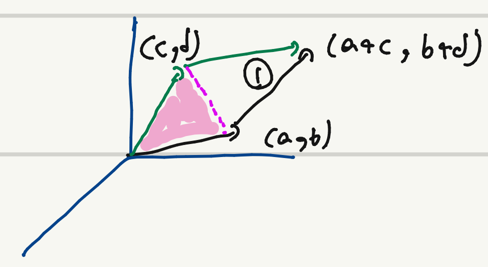
Lets move from cubes to rectangular boxes



$\rightarrow$ Volume doubles, determinant doubles $\rightarrow$ follows 3-a $\rightarrow$ formula ① is ok

check Property 3-b:

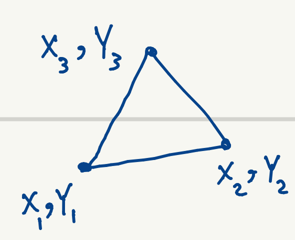
$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$



area of Parallelogram ①: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ $\rightarrow$ If you know the Coordinates of a box (of the corners) $\rightarrow$ you can find area

area of Triangle : $\frac{1}{2}(ad - bc)$

★ What if our Triangle didn't start at $(0,0)$:



area $= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ (we can do elim $r_2 = r_2 - r_1$, $r_3 = r_3 - r_1$ $\rightarrow$ Moved back to origin)

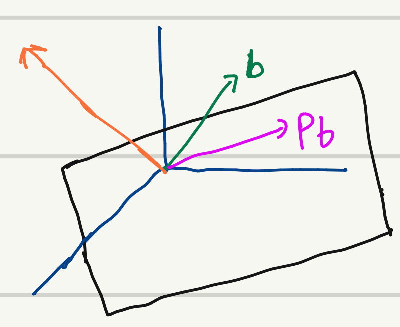
Eigen Vectors: we matrix A and vector $x \rightarrow A$ acts on vector x $\rightarrow$ input: Vector x output: vector Ax like a function : L21

★ There is certain vectors where Ax comes out Parallel to x $\rightarrow$ Eigenvectors $\equiv$ Parallel to $x \Rightarrow Ax = \lambda x$ (lambda (Eigen Value) is allowed to be Negative or Zero or positive

↳ Eigenvector

★ If A is singular, then $\lambda = 0$ is eigenvalue

What are the x's and λs for a Projection matrix? $A = P$



$\rightarrow$ b is not eigenvector because Pb is in different direction (Not Parallel)

Any x in the plane would be eigenvector $\rightarrow$ $Px = x$ $\rightarrow$ x is eigenvector & eigenvalue = 1 Two-dim eigenvector cause plane is 2-dim

Any $x \perp$ Plane: $Px = 0x \rightarrow$ eigenvalue $= 0$ 1-dim eigenvector cause dim Nullspace of the Plane = 3-2 = 1 $\rightarrow$ ⊕ $\rightarrow$ 3 independent eigenvectors

$\longrightarrow$ Eigenvalues for Projection matrices $\Rightarrow$ $\lambda_1 = 1$ $\lambda_2 = 1$ $\lambda_3 = 0$

Another matrix $\rightarrow$ $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (Permutation matrix)

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \lambda = 1$

$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow Ax = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \lambda = -1$

★ $n \times n$ matrices $\rightarrow$ n eigenvalues

★ $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$ (Trace)

How to solve $Ax = \lambda x$

Rewrite: $(A - \lambda I)x = 0$ $\rightarrow$ if there is a x, matrix '$A - \lambda I$' has to be singular $\rightarrow |A - \lambda I| = 0$ $\rightarrow$ Find λ('s) first.

Example:

$A_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ (Symmetric: $\underset{\mathbb{R}}{\text{Real}}$ eigenvalues, eigenvectors are Perpendicular)

$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0 \rightarrow \lambda^2 - \underbrace{6}_{\text{Trace}}\lambda + \overbrace{8}^{\det A} = 0$ $\quad (\lambda - 2)(\lambda - 4) = 0$ $\rightarrow \lambda_1 = 4$, $\lambda_2 = 2$

(★ in 2x2 : $\lambda^2 - \text{Trace} \times \lambda + \det A = 0$)

Find eigenvectors for $\lambda=4$: $A-4I=\overbrace{\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}}^{\text{Singular}} \to x_1=\begin{bmatrix}1\\1\end{bmatrix}$

Find eigenvectors for $\lambda=2$: $A-2I=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \to x_2=\begin{bmatrix}-1\\1\end{bmatrix}$

★In the example befor this one, we had $A_2=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ $\lambda_1=1$ $\begin{bmatrix}1\\1\end{bmatrix}$, $\lambda_2=-1$ $\begin{bmatrix}-1\\1\end{bmatrix}$ $\to A_1=A_2+3I$, $\text{eigenvalues}_1=\text{eigenvalues}_2+3$ $\text{eigenvectors}_1=\text{eigenvectors}_2$

why? suppose we have $Ax=\lambda x \xrightarrow{+3I} (A+3I)x=\lambda x+3x=(\lambda+3)x$

Another example

IF $Ax=\lambda x$, $Bx=\alpha x$ (we know eigenvalues of A, B) $\to$ Then $(A+B)x=(\lambda+\alpha)x$? No, x might not be an eigenvector of B $\to$ so $By=\alpha y$ is more accurate

★ Normally eigenvalues of $A+B$ or AB are not $\text{eigenvalues}_A+\text{eigenvalues}_B$ or $\text{eigenvalues}_A \times \text{eigenvalues}_B$

★ IF B is multiple of I that's great (like in the last example)

Example

rotation matrix, rotate every vec by 90°: $Q=\begin{bmatrix}\cos 90^\circ & -\sin 90^\circ\\ \sin 90^\circ & \cos 90^\circ\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ Trace $=0=\lambda_1+\lambda_2$ $\det Q=1=\lambda_1\lambda_2$

★What matrix come out parallel to itself after 90° rotation | eigenvalues have problem as well (add to zero, multiply to one)

$\to$ A way One: $\det(Q-\lambda I)=\begin{vmatrix}-\lambda & -1\\ 1 & -\lambda\end{vmatrix}=\lambda^2+1=0$ $\lambda_1=i$ $\lambda_2=-i$

★IF we had a symmetric matrix (or close to it), eigenvalues would be real

★ Our Q was anti-symmetric $Q^T=-Q$ $\to$ these have pure imaginary eigenvalues (extreme case)

Example

$A=\begin{bmatrix}3 & 1\\ 0 & 3\end{bmatrix}$ $\to$ matrix is triangular: eigenvalues are on diagonal $\to 3, 3$

$\det(A-\lambda I)=\begin{vmatrix}3-\lambda & 1\\ 0 & 3-\lambda\end{vmatrix}=(3-\lambda)(3-\lambda)-0=(3-\lambda)(3-\lambda)$ $\to$ eigenvalues are on diag $\lambda_1=\lambda_2=3$

eigenvectors: $(A-\lambda I)x=0 \to \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\end{bmatrix}=\begin{bmatrix}0\end{bmatrix}$ (Free, Pivot) $\to$ basis for nullspace $=x_1=\begin{bmatrix}1\\0\end{bmatrix}$ $x_2=$ Need it to be independent from x_1 $\to$ there isn't one! $\to$ only 1 eigenvector (shortage of eigenvector)

★These are the matrices that eigenvectors don't give the complete story

: L22

Suppose we have n $\overbrace{\text{linearly independent}}^{S \text{ is invertible}}$ eigenvectors of A — Put them in the cols of S (eigenvector matrix)

$$AS=A\begin{bmatrix}\vdots & \vdots & & \vdots\\ x_1 & x_2 & \cdots & x_n\\ \vdots & \vdots & & \vdots\end{bmatrix}=\underbrace{\begin{bmatrix}\lambda_1x_1 & \lambda_2x_2 & \cdots & \lambda_nx_n\end{bmatrix}}_{Ax=\lambda x}=\begin{bmatrix}x_1 & x_2 & \cdots & x_n\end{bmatrix}\underbrace{\begin{bmatrix}\lambda_1 & & 0\\ 0 & \cdots & \vdots\\ \vdots & & 0\\ 0 & & \lambda_n\end{bmatrix}}_{\text{Diagonal eigenvalue Matrix } \Lambda}=S\Lambda$$

Capital lambda

$AS=S\Lambda \to S^{-1}AS=\Lambda \to$ diagonalization

$\to A=S\Lambda S^{-1}$

What are the eigenvectors / vals of A^2? IF $Ax\overset{①}{=}\lambda x \to A^2x=\lambda Ax\overset{①}{=}\lambda^2x$ $\to$ eigenvals of A^2 are λ^2, eigenvecs of A^2 are the same as A

another way to do it:

$A^2=S\Lambda \underbrace{S^{-1}S}_{I}\Lambda S^{-1}=S\Lambda^2S^{-1}$ $\to$ eigenvecs are the same (S), eigenvals are squared also $A^k = S\Lambda^k S^{-1}$ → eigenvals are powered by k, eigenvecs are the same (s)

☆ eigenvals tells you about powers of a matrix

Theorem: $A^k \to 0$ as $k \to \infty$ if all $|\lambda_i| < 1$ (if we have n independent eigenvecs)

☆ If we don't have n independent eigenvecs, we can't diagonalize the matrix

→ A is sure to have n independent eigenvecs (and be diagonalizable) if all the λ's are different → No repeated λ's

☆ Repeated eigenvals case → may or may not have n independent eigenvecs ($I_{10\times 10} \to \lambda_i = 1$, every vector is an eigenvec so take 10 independent vecs)

☆ If A was already diagonal (like I) then $\Lambda = A$ ($S^{-1} I S = I = A = \Lambda$)

Suppose $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ (Triangular) $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda)$ $\lambda_1 = \lambda_2 = 2$

eigenvecs: $A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \to x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → A is not diagonalizable

Example: Solve this equation: (start with a given vector u_0) → $u_{k+1} = A u_k$: $u_1 = A u_0$, $u_2 = A^2 u_0 \to u_k = A^k u_0 \to u_{100} = A^{100} u_0$

To really solve: $u_0 = \underbrace{c_1 x_1 + \dots + c_n x_n}_{Sc} \xrightarrow{\times A^{100}} A^{100} u_0 = c_1 \lambda_1^{100} x_1 + \dots + c_n \lambda_n^{100} x_n = \Lambda^{100} S c = u_{100}$

because we have n independent vectors in n-dim space → They are basis in that space → Their linear comb can create any matrix like u_0

Example: Fibonacci: 0, 1, 1, 2, 3, 5, 8, 13, ... $\boxed{F_{100} = ?}$, How fast are these fib numbers are growing?

eq1 $F_{k+2} = F_{k+1} + F_k \to u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

eq2 $F_{k+1} = F_{k+1} \to u_{k+1} = \underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} \underset{u_k}{\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}} \to u_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{Symmetric}} u_k$ $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (\lambda-1)\lambda - 1 = \lambda^2 - \lambda - 1 = 0$ $\lambda = \frac{1 \pm \sqrt{1+4}}{2}$ → $\lambda_1 = \frac{1}{2}(1+\sqrt{5}) \simeq 1.618$ → bigger one → Controls the growth; $\lambda_2 = \frac{1}{2}(1-\sqrt{5}) \simeq -.618$

$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \underbrace{\begin{bmatrix} \lambda \\ 1 \end{bmatrix}}_{\text{Trick for } 2\times 2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

$F_{100} \approx c_1 \left(\frac{1+\sqrt{5}}{2}\right)^{100}$

$u_0 = c_1 x_1 + c_2 x_2 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → we can find c_1, c_2

$(-0.618)^{100} \approx 0$

why? $u_{100} = \underbrace{c_1 \lambda_1^{100} x_1}_{\text{the dominant one is the bigger } \lambda} + \dots + c_n \lambda_n^{100} x_n$ $(1.618)^{100}$

:L23

Example: Solve

$\frac{du_1}{dt} = -u_1 + 2u_2$ $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\frac{du_2}{dt} = u_1 - 2u_2$ $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$ → singular → $\lambda_1 = 0$, $\lambda_2 = -3$ (using trace)

$\lambda_1 = 0 \to x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $Ax_1 = 0x_1$

Sol: $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$ (Pure exp sols) → Check: $\frac{du}{dt} = Au$ → Plug in $e^{\lambda_1 t} x_1$: $\lambda_1 \cancel{e^{\lambda_1 t}} x_1 = A \cancel{e^{\lambda_1 t}} x_1$ ✓

$\lambda_2 = -3 \to \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $Ax_2 = -3x_2$

$\overset{S}{\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}} \overset{C}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = u(0)$

Find $c_1, c_2 \to u(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \to u(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to c_1 = c_2 = \frac{1}{3}$ → $u(t) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xrightarrow{\text{as } t \to \infty}$ Steady state $= u(\infty) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

① Stability $u(t) \to 0$ / need $e^{\lambda t} \to 0$ / $\lambda < 0$ → if λ is a complex number like $-3+6i$ → $\underset{\text{abs}}{\left| e^{(-3+6i)t} \right|} = e^{-3t}$ cause $|e^{6it}| = 1$ → so the real part of λ has to be < 0

② Steady state: $\lambda_1 = 0$ and the other eigenvalue $Re\{\lambda\} < 0$

③ Blow up: if any $Re\{\lambda\} > 0$

☆ 2x2 stability: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to Re\,\lambda_1 < 0$ $Re\,\lambda_2 < 0$ → trace $= a+d = \lambda_1 + \lambda_2 < 0$

☆ Trace could be negative but still blow up: $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$

☆ If we know two λ_1, λ_2 are negative $\rightarrow$ det >0

☆☆ So condition for 2x2 stability: 1- Trace <0 2- det >0

$\frac{du}{dt} = Au \rightarrow$ Set $u = \underset{\text{eigenvec matrix}}{S}v \rightarrow S\frac{dv}{dt} = ASv \rightarrow \frac{dv}{dt} = S^{-1}ASv = \Lambda v$ $\Big| \frac{dv_1}{dt} = \lambda v_1 \; \vdots$ $\rightarrow v(t) = e^{\Lambda t}v(0) \rightarrow e^{At} = Se^{\Lambda t}S^{-1}$

$u(t) = Se^{\Lambda t}S^{-1}u(0) = e^{At}u(0)$

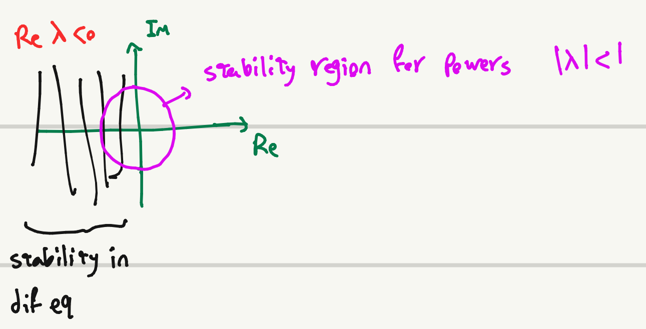
Matrix exponential e^{At} $\left(e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right) \rightarrow I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots + \frac{(At)^n}{n!} \rightarrow$ Finite ①

$\left(\frac{1}{1-x} = \sum_{0}^{\infty} x^n\right) \rightarrow (I-At)^{-1} = I + At + (At)^2 + \cdots \rightarrow$ could blow up (unless all the |eigenvalues of $At| < 1$)

①: $e^{At} = SS^{-1} + \underbrace{S\Lambda S^{-1}}_{A}t + \frac{S\Lambda^2S^{-1}}{2}t^2 + \frac{S\Lambda^3S^{-1}}{6}t^3 + \cdots = Se^{\Lambda t}S^{-1}$ (only works if A can be diagonalized)

$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \; \lambda_n \end{bmatrix}$ $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$

Complex plain



Example: $y'' + by' + ky = 0$ How to change 1 2nd order eq into 2x2 1st order system?

$u = \begin{bmatrix} y' \\ y \end{bmatrix} \rightarrow u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y' \\ y \end{bmatrix}$ $\begin{bmatrix} - & - & - & - & - \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{bmatrix}$: 5th order to 5x5 1st order

$y' = y'$

: L24

Applications of eigenvalues

Markov Matrix:

$A = \begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$ Properties: $\begin{cases} \text{1- All entries} \geq 0 \\ \text{2- All cols add to 1} \end{cases}$ $\rightarrow$ Markov matrix

key points: 1- $\lambda = 1$ is an eigenvalue in Markov matrices 2- All other $|\lambda_i| < 1$

$u_k = A^k u_0 = c_1\lambda_1^k x_1 + c_2\lambda_2^k x_2 + \cdots \rightarrow c_1x_1 =$ Steady state (x_1 part of u_0)

($k \rightarrow \infty$; $\lambda_1 = 1$; $\lambda_2 < 1$, $\lambda_2^k \rightarrow 0$)

Proof for 1:

$A - 1I = \begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{bmatrix} \rightarrow$ Prove that this is singular $\rightarrow$ all cols of $A-I$ add to zero $\rightarrow$ $A-I$ is singular (row3 + row2 + row1 = 0 $\rightarrow$ rows are dependent $\rightarrow$ (1,1,1) is in $N(A^T)$ $\rightarrow$ Then x_1 is in $N(A)$)

☆ eigenvalues of A = eigenvalues of A^T $\rightarrow$ why? $\det(A - \lambda I) = 0 \xrightarrow{\text{Transpose}} \det(A^T - \lambda I) = 0$

$\begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{bmatrix}\begin{bmatrix} 0.6 \\ 33 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

☆ u_{cal} = Population of people in California u_{mass} = Population of people in Massachusetts

Applications of Markov Matrices: $u_{k+1} = Au_k$ A is Markov
(Modeling movement of people with total count conserved)

$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=k+1} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_k$ $\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$

(0.9: stayed in cal; 0.2: moved to cal; 0.1: moved to mass; 0.8: stayed in mass)

$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=1} = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$ Find eigenvals/vecs: $\lambda_1 = 1$ $\lambda_2 = 0.7$ (using Trace) $x_1 \rightarrow \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix}\underset{x_1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_2 \rightarrow \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}\underset{x_2}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2(0.7)^k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow c_1 = \frac{1000}{3}$ $c_2 = \frac{2000}{3}$

Projections with orthonormal basis $(q_1, q_2, \ldots, q_n)$ (we are in n-dim)
(expansion)

any $V = x_1q_1 + x_2q_2 + \cdots + x_nq_n \rightarrow q_1^TV = x_1\underbrace{q_1^Tq_1}_{1} + 0 + \cdots + 0 \rightarrow q_1^TV = x_1$ ① (in matrix form $\begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = V \rightarrow Qx = V \rightarrow x = Q^{-1}V \xrightarrow[Q^{-1} = Q^T]{Q\text{'s cols are orthonormal}} x = Q^TV$

$x_1 = q_1^TV$ like ①

Fourier Series: $f(x) = a_0 1 + a_1\cos x + b_1\sin x + a_2\cos 2x + b_2\sin 2x + \cdots$ • basis $\rightarrow$ orthogonal

$f(x+2\pi) = $

Dot (inner) Product for functions: $f^Tg = \int_0^{2\pi} f(x)g(x)\,dx$ $\left(\int_0^{2\pi}\sin x\cos x\,dx = 0\right)$

$\int_0^{2\pi} f(x)\cos x\,dx = a_1\underbrace{\int_0^{2\pi}(\cos x)^2\,dx}_{\pi} = a_1\pi \rightarrow a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$

Solving Linear Systems with Eigenvalues and Eigenvectors

We want to solve the system of differential equations:

```
du/dt = A u
```

Step 1: Change of Variables
Let u = S v, where S is the matrix of eigenvectors of A.

```
dv/dt = S^(-1) A S v
```

Since S^(-1) A S = Λ (a diagonal matrix of eigenvalues):

```
dv/dt = Λ v
```

Step 2: Solve the Diagonal System
Λ is diagonal:

```
Λ = diag(lambda_1, lambda_2, ..., lambda_n)
```

So the system decouples into n independent equations:

```
dv_i/dt = lambda_i v_i
```

Step 3: Solve Each Component
This is a simple first-order ODE:

```
dv_i/dt = lambda_i v_i
```

Separating variables:

```
dv_i / v_i = lambda_i dt
```

Integrate:

```
ln|v_i| = lambda_i t + C
```

Exponentiate:

```
v_i(t) = C' * e^(lambda_i t)
```

Using the initial condition v_i(0) = C', we find C' = v_i(0).

```
So: v_i(t) = e^(lambda_i t) * v_i(0)
```

Step 4: Vector Form
Putting all components together:

```
v(t) = e^(Λ t) v(0)
```

where e^(Λ t) is a diagonal matrix with entries e^(lambda_i t).

Step 5: Transform Back to u

Since u = S v:

```
u(t) = S v(t)
```

```
u(t) = S e^(Λ t) v(0)
```

But v(0) = S^(-1) u(0). So:

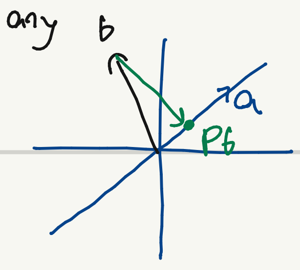
```
u(t) = S e^(Λ t) S^(-1) u(0)
```

Final Result

```
u(t) = S e^(Λ t) S^(-1) u(0)
```

This is the general solution of du/dt = A u. It shows the system evolves according to the eigenvalues (Λ) and eigenvectors (S) of A.

Q1) $a = \begin{bmatrix}2\\1\\2\end{bmatrix}$ → Find Projection matrix P that project onto the line through a.



$P = A(A^TA)^{-1}A^T = \frac{aa^T}{a^Ta} = \frac{1}{9}\begin{bmatrix}2\\1\\2\end{bmatrix}\begin{bmatrix}2&1&2\end{bmatrix} = \frac{1}{9}\begin{bmatrix}4&2&4\\2&1&2\\4&2&4\end{bmatrix}$: L24b

Rank of P: 1, Colspace = line through $\begin{bmatrix}2\\1\\2\end{bmatrix}$

- What are the eigenvalues of P ⇒ $\lambda = 0$ (singular), 0 (cause nullspace is 2-dim so there would be 2 independent eigenvecs with $\lambda=0$), 1 (using trace)

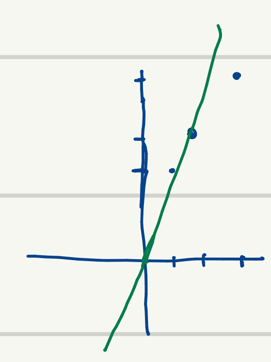
- What is the eigenvector for $\lambda = 1$: $Px = x$ → it's a cause $Pa = a$ $\left[\frac{aa^T}{a^Ta}a = a\right]$

- Solve $u_{k+1} = Pu_k$, $u_0 = \begin{bmatrix}9\\9\\0\end{bmatrix}$, find u_k

$u_1 = \frac{aa^T}{a^Ta}u_0 = a\,\frac{27}{9} = 3a = \begin{bmatrix}6\\3\\6\end{bmatrix}$ ($\hat{x}$) $u_2 = u_k = P^ku_0 = Pu_0 = \begin{bmatrix}6\\3\\6\end{bmatrix}$

Q2) Fitting a straight line to points, a straight line through the origin.

$t=1$ $y=4$
$t=2$ $y=5$
$t=3$ $y=8$



$y = Dt$

$\begin{bmatrix}1\\2\\3\end{bmatrix}$ $1D=4$, $2D=5$, $3D=8$ → $\begin{bmatrix}1\\2\\3\end{bmatrix} D = \begin{bmatrix}4\\5\\8\end{bmatrix}$ → Best D: $A^TA\hat{D} = A^Tb$
$A \quad x = b$ $14\hat{D} = 38$
$\hat{D} = \frac{19}{7}$

- What vector am I projecting onto what line or what subspace? Projecting b onto colspace of A (line)

Q3) $a_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $a_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ → these 2 vecs give a plane: colspace of A → Find two orthogonal vecs in that plane?

(Graham-Schmidt) start with a_1, $\underset{\text{Perp to } a_1}{B} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{a_1^Ta_2}{a_1^Ta_1}a_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{6}{14}\begin{bmatrix}1\\2\\3\end{bmatrix}$

Q4) a 4x4 matrix → eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$:

- What's the condition on the lambdas so that the matrix is invertible? Invertible ⟷ no zero eigenvalues

- What's the $\det(A^{-1})$? eigenvalues of $A^{-1} = \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \frac{1}{\lambda_4}$ → $\det(A^{-1}) = \frac{1}{\lambda_1}\times\frac{1}{\lambda_2}\times\frac{1}{\lambda_3}\times\frac{1}{\lambda_4}$ $\left[Ax = \lambda x \xrightarrow{\times A^{-1}} x = \lambda A^{-1}x \rightarrow A^{-1}x = \frac{1}{\lambda}x \rightarrow \text{eigenvals } A^{-1} = \frac{1}{\lambda}\right]$

- What's the trace of $A+I$? $\lambda_1+\lambda_2+\lambda_3+\lambda_4+4$

Q5) $\underset{4\times4}{A} = \begin{bmatrix}1&1&0&0\\1&1&1&0\\0&1&1&1\\0&0&1&1\end{bmatrix}$ $D_n = \det A_n$

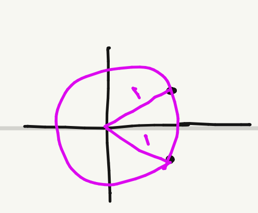
- use cofactors to show: $D_n = 1\,D_{n-1} + (-1)\,D_{n-2}$

$\begin{bmatrix}1&1&0&0\\1&1&1&0\\0&1&1&1\\0&0&1&1\end{bmatrix}$ $\begin{bmatrix}1&1&0&0\\1&1&1&0\\0&1&1&1\\0&0&1&1\end{bmatrix}$

- Solve the recurrence: $D_1 = 1$ $D_2 = 0$ $1D_{n-1}$ $-1\times1\times D_{n-2}$

$\begin{bmatrix}D_n\\D_{n-1}\end{bmatrix} = \begin{bmatrix}1&-1\\1&0\end{bmatrix}\begin{bmatrix}D_{n-1}\\D_{n-2}\end{bmatrix}$

- Find the eigenvalues: $\begin{vmatrix}1-\lambda&-1\\1&-\lambda\end{vmatrix} = \lambda^2-\lambda+1 = 0$ $\frac{1\pm\sqrt{1-4}}{2} = \frac{1+\sqrt{3}i}{2}$ ($e^{i\frac{\pi}{3}}$) $\frac{1-\sqrt{3}i}{2}$ ($e^{-i\frac{\pi}{3}}$) Magnitude $= (\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2 = 1$



Q6) $A_4 = \begin{bmatrix}0&1&0&0\\1&0&2&0\\0&2&0&3\\0&0&3&0\end{bmatrix} = A_4^T$ $A_3 = \begin{bmatrix}0&1&0\\1&0&2\\0&2&0\end{bmatrix}$

- Find P onto the colspace A_3 $A = \begin{bmatrix}0&1\\1&0\\0&2\end{bmatrix}$ → $P = A(A^TA)^{-1}A^T$

- Find eigenvals/vecs of A_3 $|A_3 - \lambda I| = -\lambda^3 + 5\lambda = 0$ → $\lambda(-\lambda^2+5) = 0$ $\lambda = 0, -\sqrt{5}, +\sqrt{5}$

- Find P onto the colspace A_4 $\begin{bmatrix}0&1&0&0\\1&0&2&0\\0&2&0&3\\0&0&3&0\end{bmatrix}$ → $\mathrm{Det}(A_4) = 9$ → $P = I$

★1 The eigenvalues of a Real Symmetric matrix are REAL : L25

★2 the eigenvectors " " " " " are (can be choosen) PERPENDICULAR → read Proof in book

★3 Usual case $A = S\Lambda S^{-1}$, Symmetric case $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$ → Spectral theorem (Breaking a matrix into pure eigenvals/vecs)

Proof for ★1: $Ax = \lambda x$ → $\underset{\text{conjugate}}{\bar{A}\bar{x}} = \bar{\lambda}\bar{x}$ $\xrightarrow[\text{Abt real } A]{\text{we are speaking}}$ $A\bar{x} = \bar{\lambda}\bar{x}$ $\xrightarrow{T}$ $\bar{x}^TA^T = \bar{x}^T\bar{\lambda}$ $\xrightarrow{A \text{ is sym}}$ $\bar{x}^TA = \bar{x}^T\bar{\lambda}$ $\xrightarrow{\times x}$ $\bar{x}^TAx = \bar{\lambda}\bar{x}^Tx$ ①

$\xrightarrow{\times\bar{x}^T}$ $\bar{x}^TAx = \lambda\bar{x}^Tx$ ② ①,②: $\lambda\bar{x}^Tx = \bar{\lambda}\bar{x}^Tx$ ★ → $\lambda = \bar{\lambda}$ → λ is Real

★ $[\bar{x}_1, \dots, \bar{x}_n]\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = \bar{x}_1x_1 + \dots + \bar{x}_nx_n$ → Pos number ★if a vector is complex $\bar{x}^Tx$ = length squared
$(a-ib)$ $(a+ib) = a^2+b^2$

Good matrices: Real λ's, Perpendicular x's $\rightarrow A = A^T$, if A is real $\quad A = \bar{A}^T$, if complex

So far: When $A = A^T$: $A = Q\Lambda Q^T = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} = \lambda_1 \underbrace{q_1 q_1^T}_{\text{Projection matrix}} + \lambda_2 q_2 q_2^T + \cdots$ → ☆ Every Symm matrix is a comb of Perp Projection matrices

(we don't need $\frac{1}{q_1^T q_1}$ cause it's already in unit vec form)

☆ For symm matrices, signs of the pivots are same as the signs of the λ's → # Pos Pivots = # Pos λ's

☆ For , Product = Product of the λ's = determinant

Positive definite Symmetric matrix

$\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$ → Pivots: $5, \frac{\det}{5} = \frac{11}{5}$

→ λ's: $\lambda^2 - 8\lambda + 11 = 0 \rightarrow \frac{8 \pm \sqrt{64-44}}{2} = 4 \pm \sqrt{5}$

- all the λ's are positive
- all the pivots are positive
- all subdeterminants are positive

: L 26

FFT: Reduced the # multiplications from n^2 to $n \log n$

$z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ z_i: complex numbers → z is in $\mathbb{C}^n$ (instead of $\mathbb{R}^n$)

☆ length: $z^T z$ is no good X → $\begin{bmatrix} 1 & i \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = 0$ X → $\bar{z}^T z = \text{length}^2$ $\quad \begin{bmatrix} 1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + 1 = 2$

☆ $z^H z = \bar{z}^T z$ $\quad$ H → Hermitian $\qquad$ ☆ Inner Product $= \bar{y}^T x = y^H x$

☆ Symmetric $A^T = A$ no good if A is complex $\quad \bar{A}^T = A = A^H$ $\quad \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$ → Hermitian matrices

☆ Perpendicular $q_1, q_2, \ldots, q_n \rightarrow \bar{q}_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

☆ Orthogonal word → Unitary

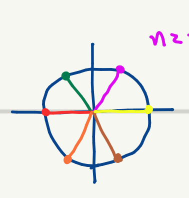
☆ $Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$ $\quad \bar{Q}^T Q = I = Q^H Q$

Fourier Matrix

$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & & w^{n-1} \\ 1 & w^2 & w^4 & & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & & w^{(n-1)^2} \end{bmatrix}$, $(F_n)_{ij} = w^{ij}$ $\quad i, j = 0, \ldots, n-1$

Full Matrix (no zero)

$w^n = 1 \rightarrow w = e^{i\frac{2\pi}{n}}$



$e^{i\frac{2\pi}{6}}$, $w^2 = e^{i\frac{4\pi}{6}}$, w^3, w^4, w^5, $w^6 = 1$

$n = 4$: $w = e^{i\frac{2\pi}{4}} = i$; $i, i^2 = -1, i^3 = -i, i^4 = 1$

$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$

☆ Cols of F_4 are orthogonal → inner product of cols = zero

☆ If we divide by length (=2) → cols are orthonormal $\quad F_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$

☆ $F^H F = I \rightarrow F^{-1} = F^H$

☆ $w_{64} = e^{i\frac{2\pi}{64}} \rightarrow w_{64}^2 = e^{i\frac{2\pi}{32}} = w_{32}$

$$\begin{bmatrix} F_{64} \end{bmatrix} = \underbrace{\overset{\text{diagonal matrix}}{\begin{bmatrix} I & D \\ I & -D \end{bmatrix}} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}}_{2(32)^2 + \text{Fix up (diagonal part)} = 2(32)^2 + 32}$$

→ Permutation Matrix: Takes the even number components first and then the odds

$D = \begin{bmatrix} 1 & & & \\ & w & & \\ & & w^2 & \\ & & & \ddots \\ & & & & w^{31} \end{bmatrix}$

Calculation: 64^2

Break down F_{32}'s $= \begin{bmatrix} I & D \\ I & -D \end{bmatrix}\begin{bmatrix} \begin{matrix} I & D \\ I & -D \end{matrix} & 0 \\ 0 & \begin{matrix} I & D \\ I & -D \end{matrix} \end{bmatrix}\begin{bmatrix} F_{16} & & & \\ & F_{16} & & \\ & & F_{16} & \\ & & & F_{16} \end{bmatrix}\begin{bmatrix} P & \\ & P \end{bmatrix}\begin{bmatrix} P_{64} \end{bmatrix}$ → $2\left[2[16]^2 + 16\right] + 32$ → Finally: counts $= \frac{1}{2} n \log_2 n$ (from n^2) → FFT

$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ → Symm → Is it Positive definite? TESTS

: L 27

Test 1: $\lambda_1 > 0 \quad \lambda_2 > 0$

Test 2: $a > 0$, $ad - b^2 > 0$

Test 3: Pivots $a > 0 \quad \overset{\det/a}{\frac{ac - b^2}{a}} > 0$

Test 4: $x^T A x > 0$ (except at $x = 0$)

Examples: $\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$ $\det = 0$ → Positive semidefinite $\quad \lambda_1 = 0 \quad \lambda_2 = 20$, Pivots: 2

Test 4: $\underset{x^T}{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = 2x_1^2 + 12x_1x_2 + 18x_2^2$ → Quadratic form

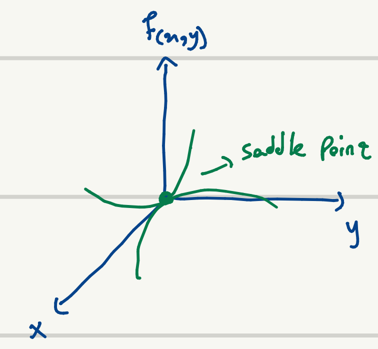
$ax^2 + 2bxy + cy^2 \overset{?}{>} 0$ for all x and y

Example 2: $\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$

Test 4: $\underset{x^T}{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = 2x_1^2 + 12x_1x_2 + 7x_2^2$

$ax^2 + 2bxy + cy^2 \overset{?}{>} 0$ for all x and y → No $\quad x = 1$, $y = -1$

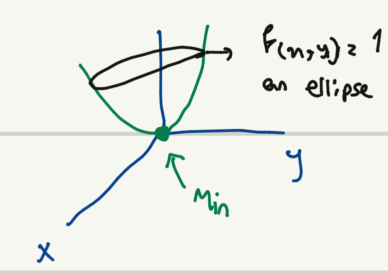
Graphs of $f(x,y) = \vec{x}^T A \vec{x} = ax^2 + 2bxy + cy^2$ $\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$



$2x^2 + 12xy + 7y^2$

Examples: $\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ det = 4 → positive definite $f(x,y) = 2x^2 + 12xy + 20y^2$ 1st derivs = 0 2nd derivs > 0 → min

Test 4: $\underset{x^T}{[x_1 \ x_2]} \underset{A}{\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = 2x^2 + 12x_1x_2 + 20x_2^2$
$ax^2 + 2bxy + cy^2 \overset{?}{>} 0$ for all x and y



Calculus: $\underset{f(x)}{\text{Min}} \sim \frac{du}{dx} = 0, \ \frac{d^2u}{dx^2} > 0$

$f(x,y) = 2x^2 + 12xy + 20y^2 = 2(x+3y)^2 + 2y^2$ → always positive

Pivots

Linear Algebra: $\underset{f(x_1,\dots,x_n)}{\text{Min}}$ ~ Matrix of 2nd derivs is pos definite

☆ Completing the square ~ Gaussian elim $\underset{A}{\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}} \to \underset{U}{\begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}}$ $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

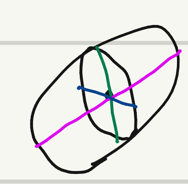
☆ Matrix of 2nd deriv: $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ $(f_{xy} = f_{yx})$ → has to be pos definite for min

3x3 example:

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ dets: 2, 3, 4 pivots: $2, \frac{3}{2}, \frac{4}{3}$ λ's: $2-\sqrt{2}, 2, 2+\sqrt{2}$

$x^T A x = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 = 2(x_1 - \frac{1}{2}x_2)^2 + \frac{3}{2}(x_2 - \frac{2}{3}x_3)^2 + \frac{4}{3}x_3^2$

Cut through at height 1 → $2(x_1 - \frac{1}{2}x_2) + \frac{3}{2}(x_2 - \frac{2}{3}x_3)^2 + \frac{4}{3}x^2 = 1$ → Eq of Ellipsoid



axis are in the direction of eigenvecs
length of axis = eigenvals

Principle axis theorem $A = Q\Lambda Q^T$
diagonalization for sym matrix

☆ Suppose A is pos definite, Is it's inverse pos definite? Yes, λ's for A^{-1}: $\lambda'_i = \frac{1}{\lambda_i}$ $\lambda'_i > 0$ ✓ $A = Q\Lambda Q^T \to A^{-1} = (Q\Lambda Q^T)^{-1} = Q\Lambda^{-1}Q^T$ ✓ : L28

$z = Q^T x$, $x^T A^{-1} x = x^T (Q\Lambda^{-1}Q^T) x = (Q^T x)^T \Lambda^{-1} (Q^T x) = z^T \Lambda^{-1} z = [z_1, \dots, z_n] \begin{bmatrix} d_1 z_1 \\ d_2 z_2 \\ \vdots \\ d_n z_n \end{bmatrix} = \sum_{i=1}^{n} d_i z_i^2 = \sum_{i=1}^{n} \frac{z_i^2}{\lambda_i} > 0$

($z \neq 0$ for $x \neq 0$, because Q is invertible and is not singular so there isn't any $Q^T x = 0$ for $x \neq 0$)

☆ If A, B are pos definite, what about A+B? $x^T A x > 0$, $x^T B x > 0 \xrightarrow{\text{add}} x^T (A+B) x$

☆ Suppose $A_{m \times n}$ → $A^T A$ is square, symmetric, is it pos definite? $x^T (A^T A) x$ → can never be negative: $(Ax)^T (Ax) = \|Ax\|^2 \geq 0$ (pos semi definite) → if rank A is n, then $\|Ax\|^2 > 0$ → Positive definite

☆ With pos definite matrices, you never do row exchange

Similar matrices: $A_{n \times n}$ and $B_{n \times n}$ are similar means: for some M, $B = M^{-1}AM$ (this topic is related to any square matrices, not just sym)

Example: A is similar to lambda → $S^{-1}AS = \Lambda$ (we can take any M and find B's similar to A)

Example: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$, $\underset{M^{-1}}{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}} \underset{M}{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix} = B$

☆ A, B (similar matrices) have the same eigenvalues $\lambda_A = 3, 1$ $\lambda_B = 3, 1$ C with $\lambda_C = 3, 1$ is a member of the A, B family as well $C = \begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix}$ (Random), $D = \begin{bmatrix} 1 & 7 \\ 0 & 3 \end{bmatrix}$

Why? Suppose $Ax = \lambda x$ and $B = M^{-1}AM$ → $AMM^{-1}x = \lambda x \to M^{-1}AMM^{-1}x = \lambda M^{-1}x = BM^{-1}x = \lambda M^{-1}x \to Bz = \lambda z$ → λ is eigenvalue of B (eigenvec didn't stay the same)

☆ Eigenvector of $B = M^{-1}$ (eigenvector of A)

Bad Case $\lambda_1 = \lambda_2 = 4$ (example)

One small family has $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ → Only member in its family $M^{-1} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} M = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

they are not in the same family

The big family includes $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ → is not diagonalizable (only 1 eigenvector)

→ Jordan Form:

More members of the family :

Best one : $\begin{bmatrix}4&1\\0&4\end{bmatrix}$ $\quad \begin{bmatrix}5&1\\-1&3\end{bmatrix}, \begin{bmatrix}4&0\\17&4\end{bmatrix}, \begin{bmatrix}a&-\\-&8-a\end{bmatrix}$ trace = 8, det = 16 $\rightarrow$ All not diagonalizable, same lambdas, same number of independent eigenvectors

Example: $\left[\begin{array}{cc|cc}0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{array}\right]$ $\lambda_{1,2,3,4} = 0$ | 2 independent eigenvectors $\rightarrow$ dim $N(A) = 2$, 2 are missing

$\left[\begin{array}{ccc|c}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{array}\right]$ $\lambda_{1,2,3,4} = 0$ | 2 independent eigenvectors $\rightarrow$ dim $N(A) = 2$, 2 are missing

These are not similar $\rightarrow$ blocks are different sizes $\rightarrow$ not similar

Jordan Block

Jordan Block $\quad J_i = \begin{bmatrix}\lambda_i&1&&0\\&\lambda_i&1&\\&&\ddots&1\\0&&&\lambda_i\end{bmatrix}$ $\rightarrow$ ones above $\quad \rightarrow$ has 1 eigenvector only

Jordan's theorem: Every square A is similar to a Jordan matrix J $\quad J = \begin{bmatrix}\boxed{J_1}&&\\&\boxed{J_2}&\\&&\ddots\\&&&\boxed{J_d}\end{bmatrix}$ $\quad$ #Blocks = #eigenvectors (1 eigenvector per block)

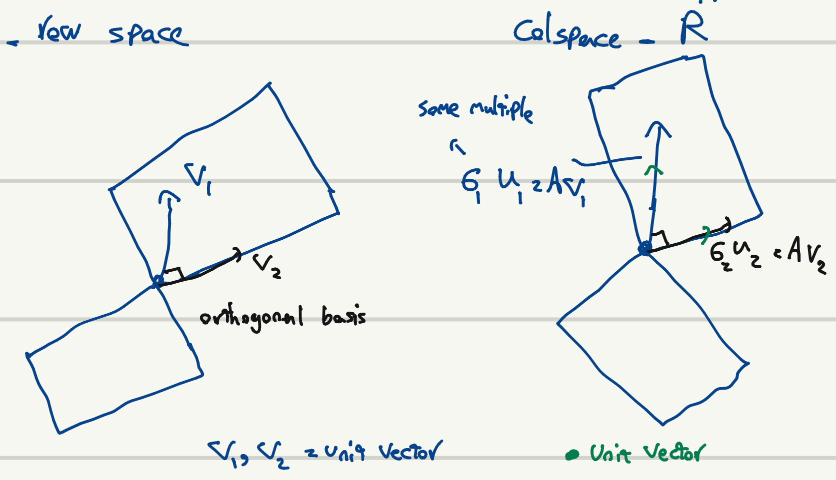
★ In good case (diff λ's) $\rightarrow J = \Lambda$

:L29

Singular Value Decomposition (SVD): $A = U\Sigma V^T$ $\quad \Sigma$: diagonal $\quad U, V$: Orthogonal

In positive definite case : $A = Q\Lambda Q^T$ $\quad \hookrightarrow \lambda_i(A)$

What does it mean? $\quad R^n$ - row space $\quad$ Colspace - R^m



$\rightarrow$ In SVD we are looking for: an orthogonal basis in row space $\xrightarrow{\text{TO}}$ an orthogonal basis in col space

$$A\,\underbrace{[v_1, v_2, \dots, v_r, \underbrace{v_{r+1}, \dots, v_n}_{\text{nullspace}}]}_{\text{basis vectors in row space}} = \underbrace{[u_1, \dots, u_r, \dots u_m]}_{\text{basis vectors in col space}} \underbrace{\begin{bmatrix}\sigma_1&&0\\&\sigma_2&\\0&&\ddots\ \sigma_n\end{bmatrix}}_{\text{multiplying vectors}}$$

Our Goal: $AV = U\Sigma$ ① (2 different orthogonal basis U, V) [in pos definite, they are the same]

$A = \begin{bmatrix}4&4\\-3&3\end{bmatrix}$ $\rightarrow$ looking for v_1, v_2 in rowspace R^2, u_1, u_2 in colspace R^2, $\sigma_1 > 0$ $\sigma_2 > 0$ $\xrightarrow{\text{so}}$ $Av_1 = \sigma_1 u_1$ $\quad Av_2 = \sigma_2 u_2$

①: $\xrightarrow{\times V^{-1}} A = U\Sigma V^{-1} = U\Sigma V^T$ $\xrightarrow{\text{disappear } U}$ $A^TA = V\Sigma^T \underbrace{U^TU}_{I}\Sigma V^T = V\begin{bmatrix}\sigma_1^2&&\\&\sigma_2^2&\\&&\ddots\end{bmatrix}V^T$ $\quad V$: Eigenvectors ($A = Q\Lambda Q^T$) $\quad \begin{bmatrix}\sigma_i^2&\\&\ddots\end{bmatrix}$: Eigenvalues $\rightarrow$ positive bcuz A^TA is pos definite

$\xrightarrow{\text{disappear } V}$ $AA^T = U\Sigma \underbrace{V^TV}_{I}\Sigma^TU^T = U\begin{bmatrix}\sigma_1^2&&\\&\sigma_2^2&\\&&\ddots\end{bmatrix}U^T$ $\quad U$: Eigenvectors ($A = Q\Lambda Q^T$) $\quad \begin{bmatrix}\sigma_i^2&\\&\ddots\end{bmatrix}$: Eignvalues $\rightarrow$ positive bcuz AA^T is pos definite

$A^TA = \begin{bmatrix}25&7\\7&25\end{bmatrix}$ $\rightarrow$ eigenvecs : $\begin{bmatrix}1\\1\end{bmatrix} \xrightarrow{\text{Normalize}} \frac{1}{\sqrt2}\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}1\\-1\end{bmatrix} \longrightarrow \frac{1}{\sqrt2}\begin{bmatrix}1\\-1\end{bmatrix}$ eigenvals: 32, 18 $\rightarrow$

$$\begin{bmatrix}4&4\\-3&3\end{bmatrix} = \underset{U}{\begin{bmatrix}\ \ &\ \ \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix}\sqrt{32}&0\\0&\sqrt{18}\end{bmatrix}} \underset{V}{\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\end{bmatrix}}$$

Find U's: u_1, u_2

$AA^T = \begin{bmatrix}32&0\\0&18\end{bmatrix}$ $\rightarrow$ eigenvecs : $\begin{bmatrix}1\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\end{bmatrix}$ $\quad$ eigenvals: 32, 18 $\rightarrow U = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\rightarrow$

$$\begin{bmatrix}4&4\\-3&3\end{bmatrix} = \underset{U}{\begin{bmatrix}1&0\\0&1\end{bmatrix}} \underset{\Sigma}{\begin{bmatrix}\sqrt{32}&0\\0&\sqrt{18}\end{bmatrix}} \underset{V}{\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\end{bmatrix}}$$ $\rightarrow$ a sign is wrong

Correction

$AV = \Sigma U$

$$\begin{bmatrix}4&4\\-3&3\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\end{bmatrix} = \begin{bmatrix}\sqrt{32}&0\\0&\sqrt{18}\end{bmatrix}\begin{bmatrix}u_1&u_2\\u_3&u_4\end{bmatrix}$$

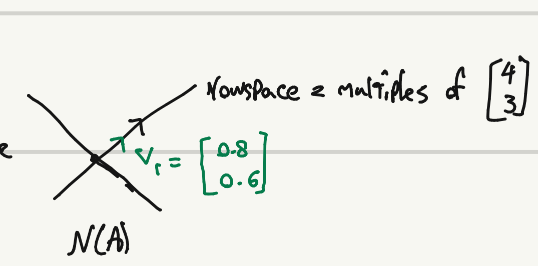
$\rightarrow$ This will get you the correct signs

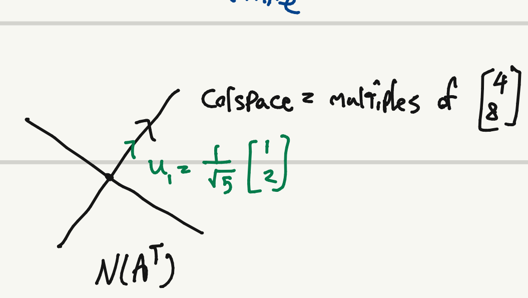
$\rightarrow U = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$

★ Eigenvalues of AB = Eigenvalues of BA $\qquad$ ★ $\sigma_i \geq 0$ bcuz $\sigma_i = \sqrt{\lambda_i(\underbrace{A^TA}_{\text{Symm, Pos semi definite}})}$

Example :

$A = \begin{bmatrix}4&3\\8&6\end{bmatrix}$ $\quad$ Rowspace





$$\begin{bmatrix}4&3\\8&6\end{bmatrix} = \underset{U}{\overset{①}{\begin{bmatrix}\frac{1}{\sqrt5}&\frac{2}{\sqrt5}\\\frac{2}{\sqrt5}&-\frac{1}{\sqrt5}\end{bmatrix}}} \overset{\Sigma}{\begin{bmatrix}\sqrt{125}&0\\0&0\end{bmatrix}} \underset{V}{\begin{bmatrix}0.8&0.6\\0.6&-0.8\end{bmatrix}}$$

★ don't use trace(A) here cause it's λ for A^TA or AA^T

① How to find v_2 or u_2? v_2 or u_2 is perp to v_1 or u_1 $\rightarrow v_2 = \begin{bmatrix}0.6\\-0.8\end{bmatrix}$

$A^TA = \begin{bmatrix}80&60\\60&45\end{bmatrix}$ $\rightarrow \lambda$'s: 125, 0

★ $v_1, \dots, v_r$: orthonormal basis for the rowspace $\qquad u_1, \dots, u_r$: orthonormal basis for colspace

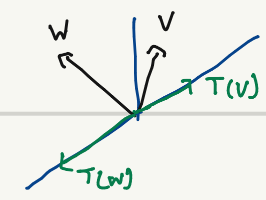
$v_{r+1} \dots v_n$: " " " " nullspace $\qquad u_{r+1}, \dots, u_m$: " " " $N(A^T)$

★ $Av_i = \sigma_i u_i$

Linear Transformations T without coordinates : no matrix

: L30

Example: Projection $T: R^2 \to R^2$ (Takes every vector in the plane, into a vector in the plane) : mapping

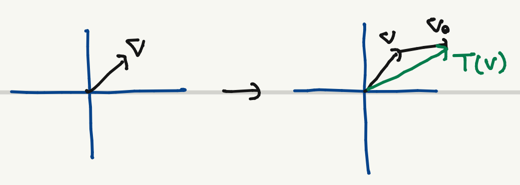


→ Projection is a linear transformation

Rules for Linear Transformation 1- $T(v+w) = T(v) + T(w)$ 2- $T(cv) = cT(v)$ ★ $T(0) = 0$ [cause when $v=0$, $T(c0) = T(0) = cT(0) \to T(0) = 0$]

★ Combining Two rules: $T(cv+dw) = cT(v) + dT(w)$

Example: Shift the whole plane by v_0

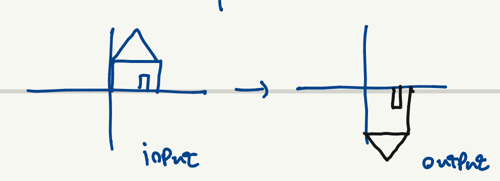


→ Not linear $c=2$, $T(2v) \neq 2T(v)$

Example: $T(v) = \|v\|$ $T: R^3 \to R^1$ $T(-2v) \neq -2\|v\|$
$= 2\|v\|$

Example: Rotation by $45°$ $T: R^2 \to R^2$ → linear Transformation

Example: Matrix A $T(v) = Av$ → Linear $A(v+w) = Av + Aw$ ✓ $A(cv) = cAv$ ✓ $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



Start: Suppose we have a (linear) $T: R^3 \to R^2$

Example: $\underbrace{T(v)}_{\text{output in } R^2} = \overset{2\times 3 \text{ matrix}}{A}\underset{\text{input}}{v}$

Information needed to know the T(v) for all input: $T(v_1), T(v_2)$ → We know what this transformation does to linear comb of v_1, v_2 [a plane with bases v_1, v_2 if v_1, v_2 are independent o.w you only know a line]

↳ We Need $T(v_1), \dots, T(v_n)$ for any input basis $v_1, \dots, v_n$ → Every $v = c_1v_1 + \dots + c_nv_n$ → We know $T(v) = c_1T(v_1) + \dots + c_nT(v_n)$

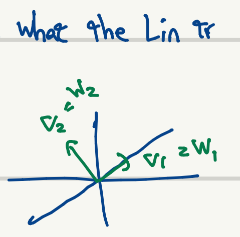
Coordinates: Come from a basis Coordinates of $v = \underline{c_1}v_1 + \dots + \underline{c_n}v_n$ → We always worked with standard bases: $v = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, but we are not entitled to do so

Construct the matrix A that represents lin tr T: Suppose $T: R^n \to R^m$

1- Choose a basis $v_1, \dots, v_n$ for inputs in R^n 2- Choose a basis $w_1, \dots, w_m$ for outputs in R^m 3- Take vector v, express it in its basis → Find coordinates (c's) 4- Multiply coordinates by A → coordinates of output

Want matrix A that does what the Lin tr does (w.r.t bases)

Example: Projection



• basis for input (v) and output (w)

Project every vec in the plane and project it onto that line → $v = c_1v_1 + c_2v_2$ $T(v) = c_1v_1$ $(c_1, c_2) \to (c_1, 0)$: matrix $= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\underset{\text{inp coor}}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = \underset{\text{outp coor}}{\begin{bmatrix} c_1 \\ 0 \end{bmatrix}}$

★ Eigenvector basis leads to the diag matrix Λ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, v_1, v_2 : eigenvecs basis → good coordinates

Example: Projecting onto $45°$ line. Use standard basis $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = w_1$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = w_2$ → $P = \frac{aa^T}{a^Ta} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Rule to find the Matrix A: Given inp / outp bases $v_1 — v_n$, $w_1 — w_m$

1st col of A: Write $T(v_1) = \underline{a_{11}}w_1 + \underline{a_{21}}w_2 + \dots + \underline{a_{m1}}w_m$

... We construct A

2nd col of A: = $T(v_2) = a_{12}w_1 + \dots + a_{m2}w_m$

Example: $T = \frac{d}{dx}$ ↙ linear Input: $c_1 + c_2x + c_3x^2$ basis: $1, x, x^2$ Output: $c_2 + 2c_3x$ basis: $1, x$

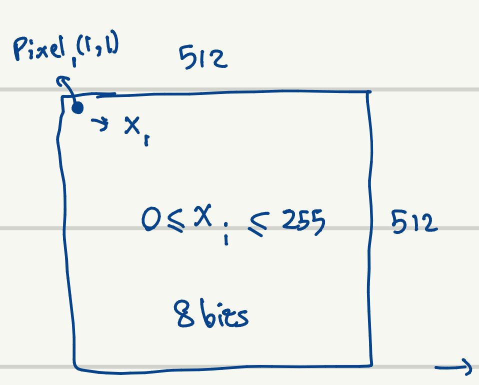
Matrix A: $\underset{2\times3}{A}\underset{3\times1}{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}} = \underset{2\times1}{\begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix}}$ $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Going by the rule:

$T(1) = 0 = 0\times1 + 0\times x \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $T(x) = 1 = 1\times1 + 0\times x$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $T(x^2) = 2x = 0\times1 + 2\times x$ $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ → $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

: L31

Image Compression



$x \in R^n$ $n = (512)^2$ [if it was colored, $3\times(512)^2$]

Standard Compression: JPEG → change of basis

→ $x = \begin{bmatrix} \vdots \\ 121 \\ 124 \\ \vdots \end{bmatrix}_{(512^2, 1)}$

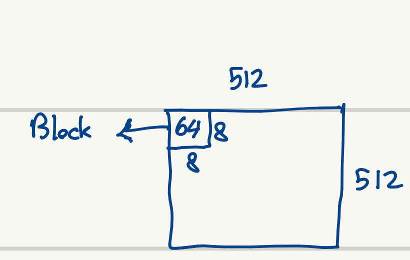
Standard basis $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$

Better Basis $\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}$ High freq

Best known basis (JPEG uses): Fourier basis 8×8 $\underset{\text{zero freq}}{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ w \\ w^2 \\ \vdots \\ w^{n-1} \end{bmatrix}$ ...

lossless: Signal P (64 Pixels)
↓ change basis
coeffs c → Linear Algebra: Find this

lossy: ↓ compression
$\hat{c}$ coef with many zeros)
↓
$\hat{x} = \sum \hat{c}_i v_i$ Reconstruct signal



☆Video: Sequence of images $\longrightarrow$ highly Correlated

Wavelets: Basis used in competition for Fourier

Wavelet w_1

$$\underbrace{\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\\-1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}1\\1\\-1\\-1\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\1\\1\\-1\\-1\end{bmatrix}\begin{bmatrix}1\\-1\\0\\0\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\-1\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\1\\-1\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\-1\end{bmatrix}}_{\text{Orthogonal}}$$

Standard basis (8 Gray-Scale Values)

$$P = \begin{bmatrix}P_1\\P_2\\\vdots\\P_8\end{bmatrix} = \underset{\text{wavelet}}{c_1 w_1} + \dots + c_8 w_8 = \underbrace{\begin{bmatrix}1 & 1 & \cdots\\1 & 1 & \\1 & 1 & \\1 & 1 & \\1 & -1 & \\1 & -1 & \\1 & -1 & \\1 & -1 & \end{bmatrix}}_{8 \text{ cols}}\begin{bmatrix}c_1\\\vdots\\c_8\end{bmatrix}$$

W

$\longrightarrow$ Transform: Solve $P = Wc$, $c = W^{-1}P$

☆Good basis: 1- Fast in mult, inverse, FFT or FWT (Wavelet)

2- Few is enough (Few basis vector should come close to the signal $\rightarrow$ Good compression) (To reproduce the img)

Change of basis: Cols of W = new basis vecs

$$\underset{\text{old basis}}{[x]} \xrightarrow{\text{Coordinates}} \underset{\text{New basis}}{[c]} \qquad x = Wc$$

Suppose (Lin Tr) $T: \mathbb{R}^8 \to \mathbb{R}^8$ w.r.t $v_1, \dots, v_8$ it has matrix A | w.r.t $w_1, \dots, w_8$ it has matrix B $\longrightarrow$ What's the connection between A and B? A, B are similar $B = M^{-1} A \underset{\text{change of basis matrix}}{M}$

What is A? (Reminder) using basis $v_1, \dots, v_8$: We know T completely from knowing $T(v_1), \dots, T(v_8)$ — Bcuz every $x = c_1 v_1 + \dots + c_8 v_8$, then $T(x) = c_1 T(v_1) + \dots + c_8 T(v_8)$

$$\left.\begin{aligned} \text{Write } T(v_1) &= a_{11} v_1 + a_{21} v_2 + \dots + a_{81} v_8 \\ T(v_2) &= a_{12} v_1 + a_{22} v_2 + \dots + a_{82} v_8 \\ &\vdots \end{aligned}\right\} \longrightarrow [A] = \begin{bmatrix} a_{11} & a_{12} & \\ \vdots & \vdots & \cdots \\ a_{81} & a_{82} & \end{bmatrix}$$

Example: Eigenvector basis $T(v_i) = \lambda_i v_i$, What is A? $T(v_1) = \lambda_1 v_1$ $T(v_2) = \lambda_2 v_2$

$$A = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \lambda_8 \end{bmatrix}$$

L32

☆If A is symm: 1- λ_i: real 2- x_i: Perpendicular 3- Diagonalizable $\rightarrow Q \Lambda Q^T$

☆If A, B are similar, B^k will look like A^k bcuz they have the same λ $B^k = M^{-1} A^k M$

Q1: $\frac{du}{dt} = Au = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} u$, general sol? $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3$ A is singular $\rightarrow \lambda_1 = 0$

$$\begin{vmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda^3 - 2\lambda = 0 \rightarrow \lambda(\lambda^2 + 2) = 0 \begin{cases} \lambda_1 = 0 \\ \lambda_2 = \sqrt{2}\, i \\ \lambda_3 = -\sqrt{2}\, i \end{cases}$$

$$u(t) = c_1 x_1 + c_2 \underbrace{\underbrace{e^{\sqrt{2} i t}}_{|e^{\sqrt{2} i}| = 1} x_2 + c_3 \underbrace{e^{-\sqrt{2} i t}}_{|e^{-\sqrt{2} i}| = 1}}_{\text{Wanders around Unit Circle}} x_3$$ $\rightarrow$ Doesn't blow up nor go to zero

- When does sol return to its initial value? Periodic $\rightarrow \sqrt{2}\, i T = 2\pi i \rightarrow T = \sqrt{2}\pi$

- Take Two eigenvecs: They are orthogonal (☆The eigenvecs of (Anti or Skew)-Symm are orthogonal

Orthogonal eigenvecs: When $AA^T = A^T A$ $\rightarrow$ Symm, Anti-Symm, Orthogonal Matrices ($A = Q$)

- How would I find e^{At} [Needed bcuz sol is $u(t) = e^{At} u(0)$]? (if A is diagonalizable $A = S \Lambda S^{-1}$) $\rightarrow e^{At} = S \underset{\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}}{e^{\Lambda t}} S^{-1}$

Q2: We're given a 3x3 matrix $\rightarrow \lambda_1 = 0$, $\lambda_2 = c$, $\lambda_3 = 2$ $x_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ $x_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$ $x_3 = \begin{bmatrix}1\\1\\-2\end{bmatrix}$:

- Is the matrix diagonalizable? (Conditions on c): We have 3 independent (actually orthogonal) eigenvecs $\rightarrow$ diagonalizable $\rightarrow$ all c

- For which Values of c is it Symm? All real c

- For which values of c is it pos definite? None, we have an eigenval = 0 $\rightarrow$ For pos def, all $\lambda_i > 0$ [For semi-def $\rightarrow$ all real $c \geq 0$]

- Is it a markov matrix? No, one λ can be 1 and the rest should be <1 but we have a 2 X

- Could $\frac{A}{2}$ be Projection matrix? Eigenval have to be real (cause P is Symm), and λ of P are $\lambda = 0, 1$ only $P^2 = P \rightarrow \lambda^2 = \lambda$ ($\lambda = 0, 1$) — c can be 0 or 2 (cause $\frac{A}{2} + \frac{c}{2} = 1$ ✓)

Q3: Reminder SVD $\rightarrow$ Every A = (orthogonal) (Diagonal)(orthogonal) $= U \Sigma V^T$, $\underset{\text{Symm}}{A^T A} = V(\Sigma^T \Sigma) V^T \rightarrow$ factorization for symm matrix ($A^T A$) $\rightarrow V$ = eigenvec matrix for $A^T A$ $\sigma_i^2 = \lambda(A^T A)$

(even rectangulars)

Find U with $A v_i = \sigma_i u_i$

- Suppose $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, $U = [u_1\ u_2]$, $V = [v_1\ v_2]$: A is non-singular (all $\lambda_i \neq 0$)

- Suppose $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}$, $U = [u_1\ u_2]$, $V = [v_1\ v_2]$: Can't be a SVD, in SVD, λ_i are not negative

- Suppose $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$, $U = [u_1\ u_2]$, $V = [v_1\ v_2]$: $[u_1\ u_2] \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} [v_1\ v_2]^T \rightarrow \text{Rank}(A) = 1$ $\dim N(A) = 1$

- Suppose $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$, $U = [u_1\ u_2]$, $V = [v_1\ v_2]$ $\rightarrow$ Give us a vec in nullspace: $v_2 \rightarrow$ it's the x for $\lambda_2 = 0$

Q4: Given A is Symm & Orthogonal: ① λ's can be: Symm $\rightarrow$ Real, Orth $\rightarrow |\lambda| = 1$ (new) why? $\underset{\text{orth}}{Q} x = \lambda x \rightarrow \|Qx\| = \|x\| \rightarrow \|x\| = |\lambda|\,\|x\| \rightarrow |\lambda| = 1$ $\rightarrow \lambda = 1, -1$

☆orth matrix doesn't change length

② A is sure to be pos definite? No, it can have $\lambda = -1$ so no pos def

③ It has no repeated λ? No – in fact if it's 3x3, it should have a repeated λ

④ Is it diagonalizable? Yes – it doesn't matter that we could have repeated λ, we can always find n independent eigenvec → diagonalizable

⑤ Is it non-singular? Yes – orthogonal matrices are always non-singular $Q^TQ = I$

⑥ Show $\frac{1}{2}(A+I)$ is a projection matrix? Prop of Proj matrix: 1. Symm ✓ 2. $P^2 = P$ $\frac{1}{4}(A^2 + 2AI + I) \stackrel{?}{=} \frac{1}{2}(A+I)$ $A^2 = AA^T = I \rightarrow \frac{1}{4}(2A + 2I) = \frac{1}{2}(A+I)$ ✓

⑦ What are the λ's of $\frac{1}{2}(A+I)$ $\lambda(A+I) = 0, 2 \xrightarrow{\frac{1}{2}} 0, 1$

: L33

2-sided inverse = Inverse $AA^{-1} = I = A^{-1}A$ $r = m = n$ full rank

Left inverse → Full col rank $r = n < m$ nullspace = $\{0\}$ independent cols / 0 or 1 sol to $Ax = b$ $\underset{n\times m \ m\times n}{A^TA}$ → Invertible A has one sided inverse: $\underbrace{(A^TA)^{-1}A^T}_{A^{-1}_{left}}$ Why? $(A^TA)^{-1}(A^TA)$ = I → Left inverse (multiply by A) ↓ $\underset{n\times m}{A^{-1}_{left}}\ \underset{m\times n}{A} = I_{n\times n}$

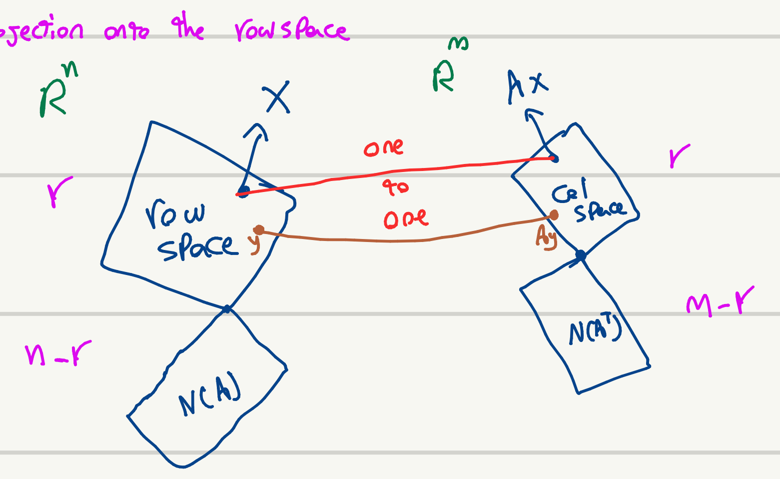
★ $AA^{-1}_{left} = \overbrace{A(A^TA)^{-1}A^T}^{\text{Projection matrix (onto colspace)}}$ → can't go any further bcuz $(A^TA)^{-1} = A^{-1}(A^T)^{-1}$ is true when A is an invertible square matrix

★ A rectangular matrix can't have 2-sided inverse

Right-inverse: Full row rank $r = m < n$ $N(A^T) = \{0\}$ independent rows / ∞ sols dim $N(A) = n - r$

$A\ \underline{A^T(AA^T)^{-1}} = I \rightarrow AA^{-1}_{right} = I$

★ $A^{-1}_{right}A = A^T(AA^T)^{-1}A$ → Projection onto the rowspace



Ax is in colspace

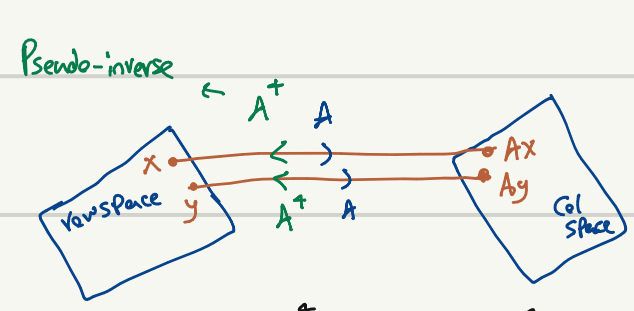
★ One-to-one: both have r dim

★ A is nice invertible mapping from rowspace & colspace

★ If x, y are different vecs in rowspace then $Ax \neq Ay$ (in colspace)

Proof: Suppose $Ax = Ay$ → $\underbrace{A(x-y)}_{\text{in the null space}} = 0$ but x in rowspace, y in rowspace → $x - y$ is in rowspace → $x - y$ → in rowspace, in nullspace → it's $\{0\}$ → $x = y$ ✗

Pseudo-Inverses:



$y = A^+(Ay)$ $x = A^+(Ax)$

Find the Pseudo-Inverse A^+:

① Start from SVD: $A = U\Sigma V^T$ $\underset{m\times n}{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \ 0 \end{bmatrix}}$ → rank r → $\Sigma^+ = \underset{n\times m}{\begin{bmatrix} \frac{1}{\sigma_1} & & 0 \\ & \ddots \ \frac{1}{\sigma_r} & \\ 0 & & 0 \end{bmatrix}}$ → rank r

$\Sigma\Sigma^+ = \underset{m\times m}{\begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix}}$ (P onto colspace) $\underset{n\times m\ \ m\times n}{\Sigma^+\Sigma} = \underset{n\times n}{\begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix}}$ (P onto rowspace)

★ Pseudo-Inverse: if you multiply on left/right you don't get I, but a P that brings you into the rowspace/colspace

★ $A^+ = V\Sigma^+U^T$
↳ doesn't have true inverse

★ When A has an inverse → there are no nullspaces and the mapping (from rowspace to colspace) is perfectly reversible; when it doesn't, information is lost into nullspaces ($Ax = 0$ → can't reverse), and the Pseudo-inverse works by projecting inputs (x) into rowspace (ignoring components in the nullspace) and projecting outputs onto the colspace (ignoring the components in the left nullspace).

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Q1: Given $Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ no sol $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has 1 sol.

– What can you tell me about m, n, r? no sol ⇒ $r < m$ $m = 3$ 1 sol $r = n$ → $r = n < m = 3$ $N(A) = \{0\}$

– Give us an example of such matrix: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

– $\det A^TA = \det AA^T$ (T/F) False, it would be true if A was square

- A^TA is invertible (T/F) invertible if $r = n$: independent cols & No Nullspace ✓

- $\underset{3\times n}{A}\,\underset{n\times 3}{A^T} \to 3\times 3$ is Pos definite (T/F) (AA^T is always symm & square) $AA^T \Rightarrow 3\times 3$ imagine $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $AA^T \to$ rank $= 2$ (in a 3×3 matrix) $\to$ No Pos definite ★ AA^T is always Positive semi definite

- Prove that $\underset{n\times m}{A^T} y = c$ at least one sol for every c & in fact ∞ sol for every c

  at least one sol bcuz $n = r$, $\dim N(A^T) = m - r > 0 \to$ infinite sol

Q2) Suppos $A = [v_1 \ v_2 \ v_3]$:

- Solve $Ax = v_1 - v_2 + v_3$: $x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

- Suppose $v_1 - v_2 + v_3 = 0$, Then sol is not unique (T/F): IF eq $= 0$ then $x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is in the nullspace of A so sols are never unique $\to c\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

- Suppose v_1, v_2, v_3 are orthonormal, what combination of v_1, v_2 is closest to v_3 ? $0v_1 + 0v_2$ is closest to v_3

Q3) $A = \begin{bmatrix} 0.2 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{bmatrix}$ (col1 + col2 = 2col3) $\lambda_1 = 0 \to$ singular $\lambda_2 = 1$ markov matrix $\lambda_3 =$ trace $- (0+1) = 0.8 - 1 = -0.2$

- Suppose I start the Markov process $u_k = A^k u(0)$ $u(0) = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$ $\to$ what does that approach? Sol after k steps $= c_1\lambda_1^k x_1 + c_2\lambda_2^k x_2 + c_3\lambda_3^k x_3 = c_2x_2 + c_3(-0.2)^k x_3$ $u_\infty = c_2x_2$

  Find x_2 : $\begin{bmatrix} -0.8 & 0.4 & 0.3 \\ 0.4 & -0.8 & 0.3 \\ 0.4 & 0.4 & -0.6 \end{bmatrix}\begin{bmatrix} \\ \\ \\ \end{bmatrix} = 0 \to x = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$ is in nullspace $u_\infty = c_2\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$ $c_2 = 1 \to$ bcuz total # of people stays the same $= 10$

Q4) Find 2x2 matrices

- Projection onto $a = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ $P = \dfrac{aa^T}{a^Ta} = \begin{bmatrix} & \\ & \end{bmatrix}$

- Matrix with $\lambda = 0, 3$ $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $A = \underset{S}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}} \underset{\Lambda}{\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}} \underset{S^{-1}}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}}$

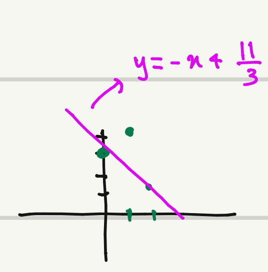
- A real matrix that can't be factored into B^TB any B No Symm $\to \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

- A matrix that has orthogonal eigenvecs but it's not symm. Skew-Symm $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, Orthogonal matrix $\begin{bmatrix} \cos & -\sin \\ \sin & \cos \end{bmatrix}$

Q5) $\underset{A}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}} \begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \leftarrow b$ $\hat{c} = \frac{11}{3}$ $\hat{d} = -1$:

- What is the Projection P of b onto cols of A ? $\frac{11}{3} \times$ col1 $-$ col2

- Draw the straight line Problem that corresponds to this system:



- Find a different vector $b \neq 0$ for which the least square sol would be zero? $b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ (needs to be orthogonal to cols)