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Linear Algebra

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: L1

Unknowns

$2x - y = 0$

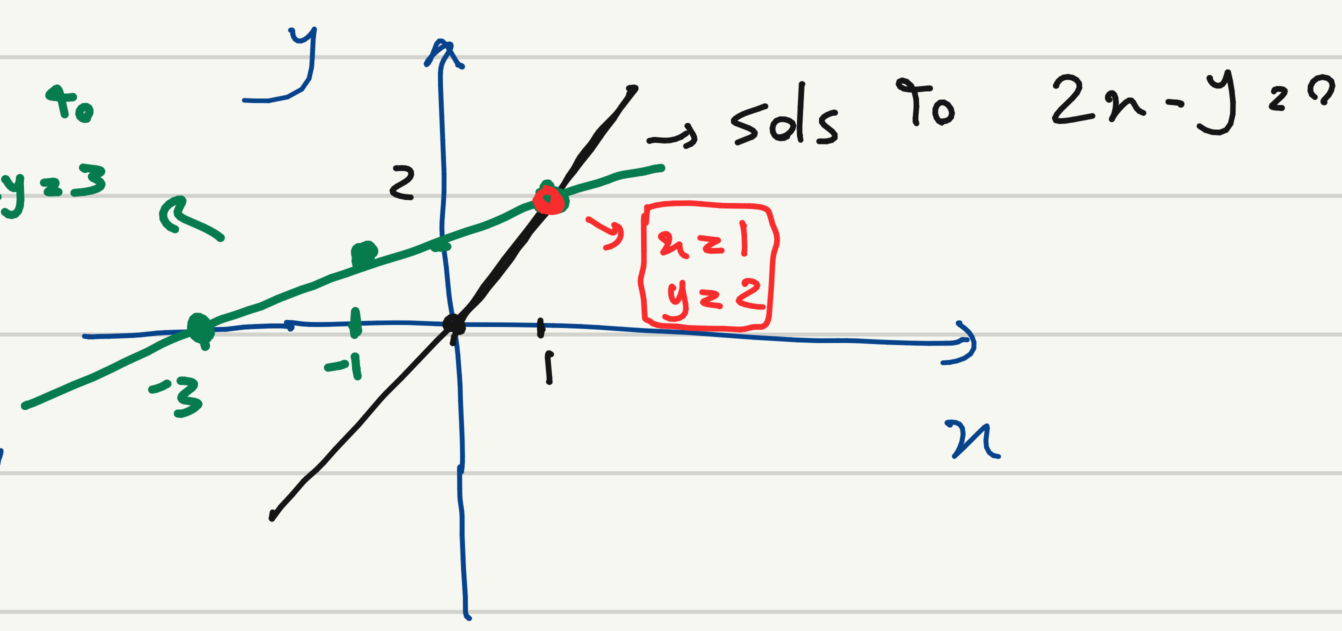
$-x + 2y = 3$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A \quad X = b$$

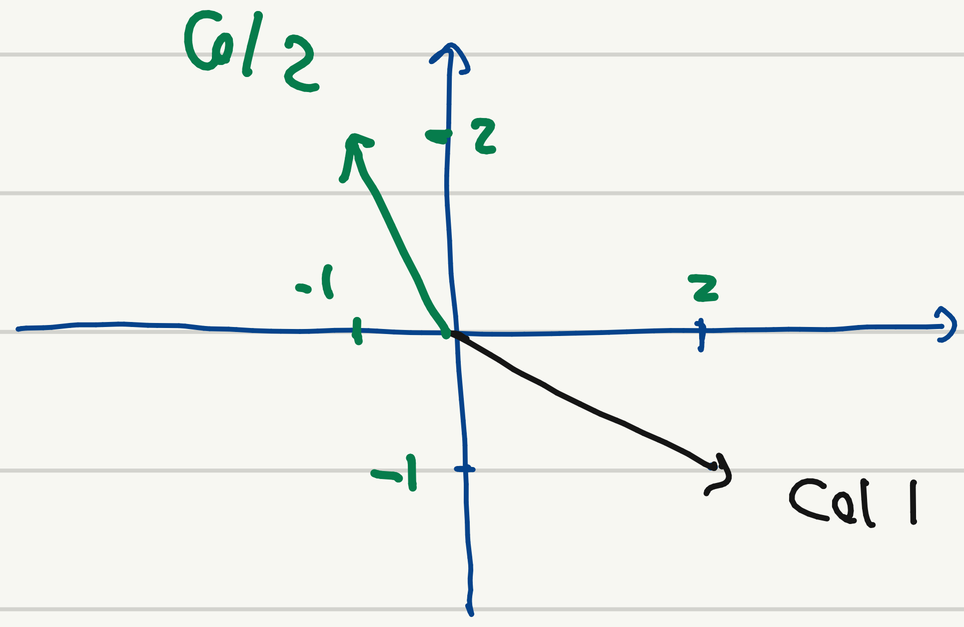
Row Picture:

Plot all the Points that satisfy first eq



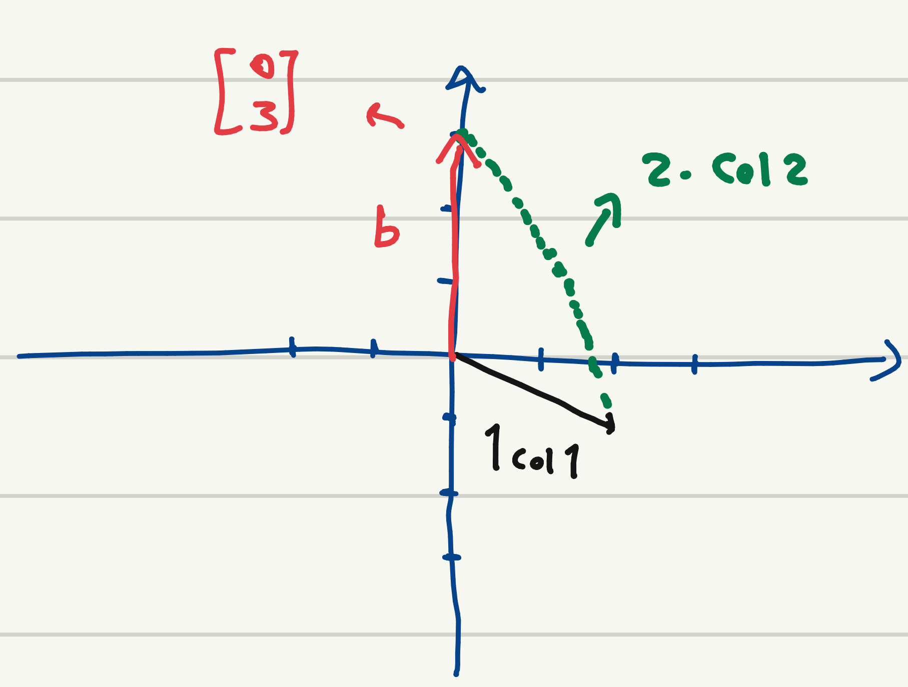
Col Picture:

Cols of matrix $x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ → linear combination of columns



if we take $X=1$ and $y=2$

$$1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$



$2x - y = 0$

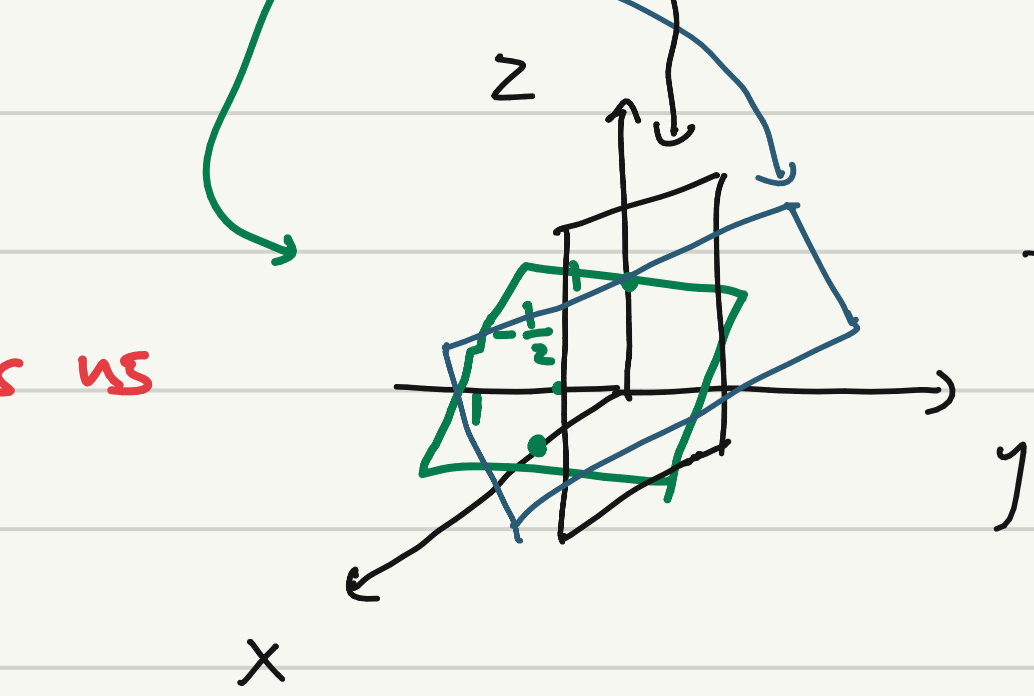
$-x + 2y - z = -1$

$-3y + 4z = 4$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Row Pic

each row gives us a Plane

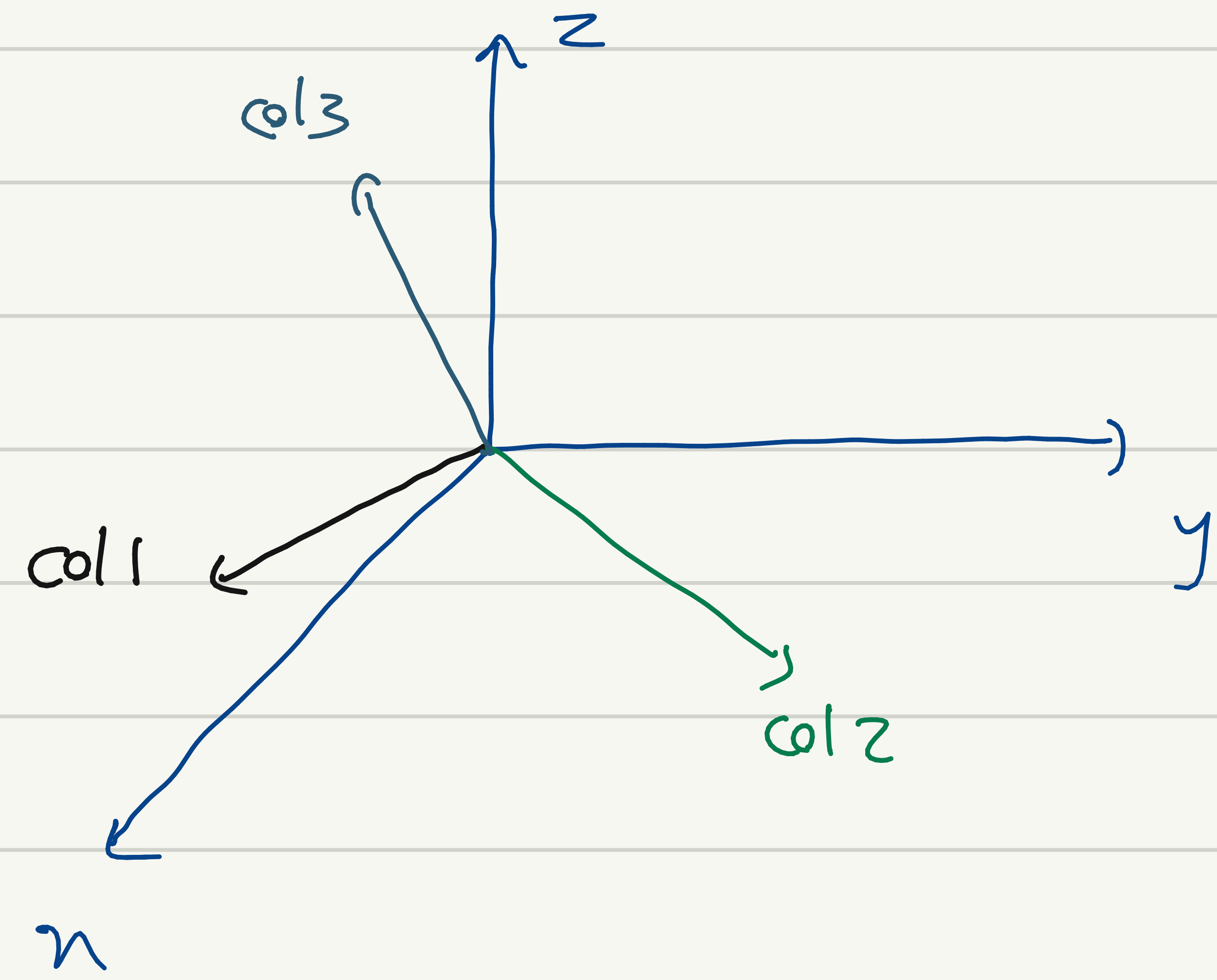


→ These Planes are not special so they do meet in 1 point

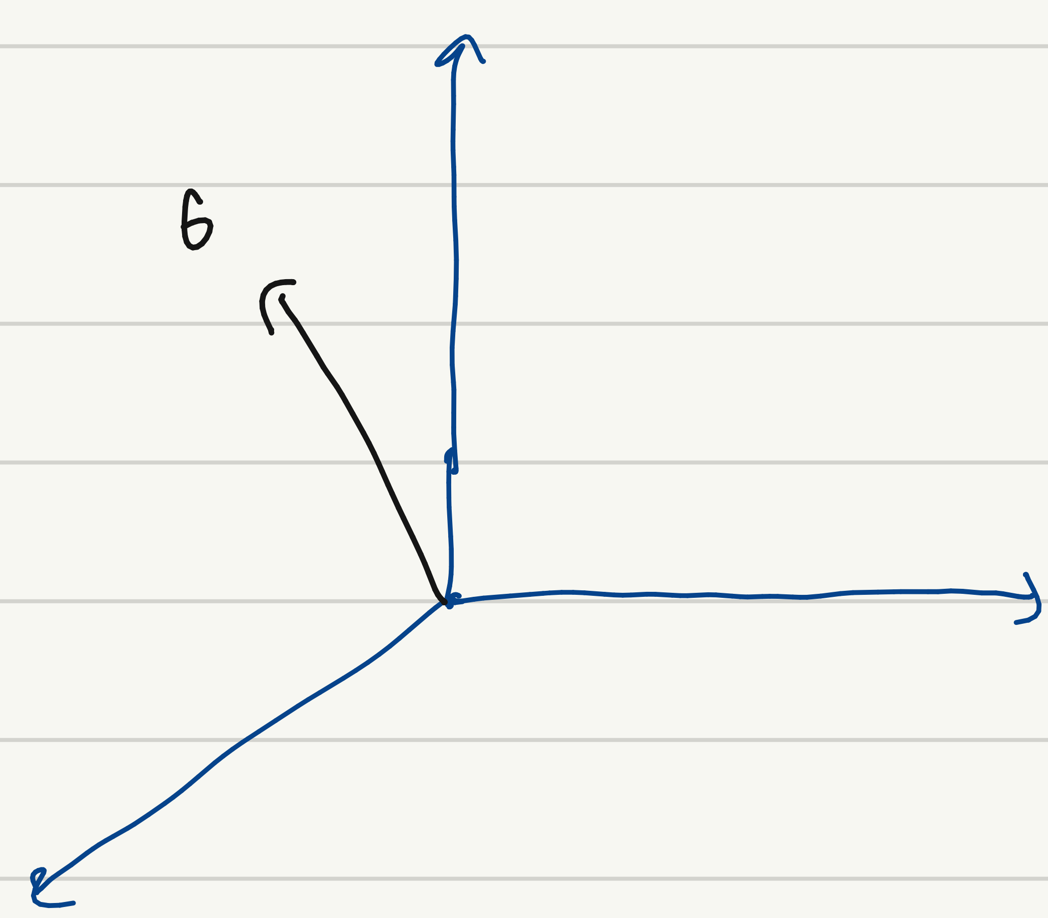
Col Pic

a linear combination ↑

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix}+y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}0\\-1\\4\end{bmatrix}$$



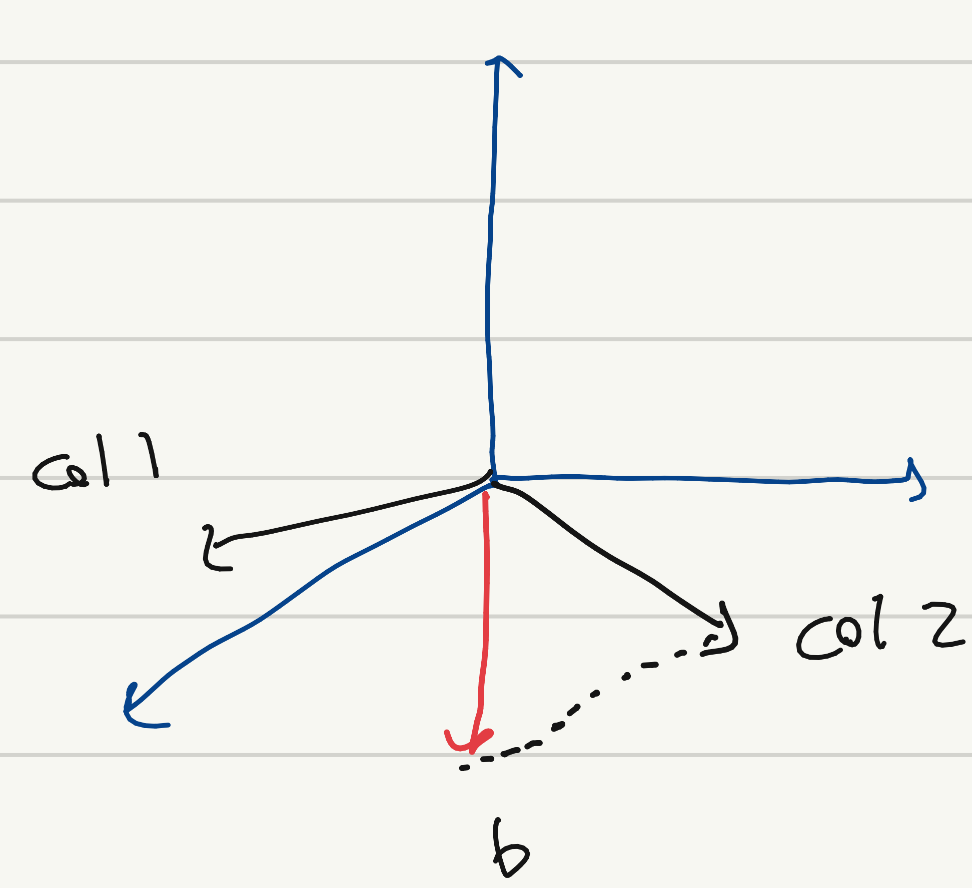
$x=0 \quad y=0 \quad z=1$



try with another b:

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix}+y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}1\\1\\-3\end{bmatrix}$$

$x=1 \quad y=1 \quad z=0$



Can I solve $Ax=b$ for every b? (Do the linear combinations of the cols fill 3D space?)

b

→ for this matrix, answer is yes

↳ This is a non-singular matrix

Cause their combination lies in the same Plane → Singular Case

If 3 cols lie in same Plane (e.g: col3 = col1 + col2) → we are in Trouble

$Ax = b$

How 2 multiply?

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{cases} \text{way 1: } 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix} \\ \text{way 2: } \begin{bmatrix} 2\times1 + 5\times2 = 12 \\ 1\times1 + 3\times2 = 7 \end{bmatrix} \end{cases}$$

Ax is a combination of cols of A

Elimination (Every software pkg use this) : L2

1 $x + 2y + z = 2$

2 $3x + 8y + z = 12$ → Solve using elimination → Success / → Failure

3 $4y + z = 2$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

first Pivot

To eliminate x (from eq 2), multiply eq 1 by 3 and subtract it from eq 2

Forward Elim

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{bmatrix}$$ → 2nd Pivot

row 2, col 1 is 0

row 3, col 1 is 0 as well (if it wasn't we should take another step to make it 0 [by multiplying eq 1 by some number and then subtract it from eq 3])

$$\xrightarrow{(3,2) \text{ elim}} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix}$$ → 3rd Pivot

U (for upper triangle)

★ Elimination gets us from A to U

★ Pivots can't be zero ★ Determinant of U is $1\times2\times5 = 10$!

★ if we had zero in a pivot (like first) we try to exchange rows

★ if exchange didn't help → failure [if $(3,3) = -4$ → Failure]

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

Augmented Matrix

Back-Substitution: Solving Eqs in reverse order cause system is triangular

New eqs $\quad Ux = c$

$$x + 2y + z = 2 \longrightarrow x + 2 - 2 = 2 \longrightarrow x = 2$$
$$2y - 2z = 6 \longrightarrow 2y + 4 = 6 \longrightarrow y = 1$$
$$5z = -10 \longrightarrow z = -2$$

back-Substitution

Elimination Matrices

⭐ Matrix × Column = Column

⟶ Put col on right

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{matrix} 3 \times col1 \\ + \\ 4 \times col2 \\ + \\ 5 \times col3 \end{matrix}$$

⭐ row × Matrix = row

Put row on left ⟵

$$\underset{1\times 3}{\begin{bmatrix} 1 & 2 & 7 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}} = \left.\begin{matrix} 1 \times row1 \\ + 2 \times row2 \\ + 7 \times row3 \end{matrix}\right\} \text{linear comb of rows}$$

Back to elim steps from last example

Step 1: Subtract 3×row1 from row2

[1 0 0] takes 1 of first row and 0 of others

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

with this, we get 0 in (2,1) pos ⟵ E_{21} ⟶ Elementry

0×row1 + 0×row2 + 1×row3 =

Step 2: Subtract 2×row2 from row3

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

What we did: $\quad E_{32}(E_{21}A) = U$

if we start with A how we get to U $\quad \bigcirc A = U$

$\longrightarrow (E_{32}E_{21})A = U$ $\quad$ we can change the Parenthesis (order of multiplication) ⟶ Associative law

★ Permutation matrix → Exchange rows

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

if we want to exchange i, j rows → Exchange i, j rows in identity matrix

★ Exchange cols → can't do it this way, because in order to multiply or matrix by a col, you need to put it on right

$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

$1 \times col1 + 0 \times col2$

$1 \times col1 + 0 \times col2 \to col2$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

↳ $0 \times col1 + 1 \times col2 \to col1$

★ $A \times B \neq B \times A$ → can't change the order of matrices
↳ Commutative law is false

How do we get from u back to A? Inverse matrix

Inverses :

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

→ if we subtract k times row j from row i, for inverse we add k x row j to row i

Multiply Matrices : L3

1st way → Regular way [rows of A x cols of B]

$$\underbrace{\begin{array}{c|c} \text{row}_i & \begin{bmatrix} \text{———} \end{bmatrix} \end{array}}_{A} \underbrace{\begin{array}{c} \text{col } j \\ \begin{bmatrix} | \end{bmatrix} \end{array}}_{B} = \underbrace{\begin{bmatrix} c_{ij} \end{bmatrix}}_{C = AB}$$

$c_{ij} = (\text{row } i \text{ of } A) \cdot (\text{col } j \text{ of } B)$

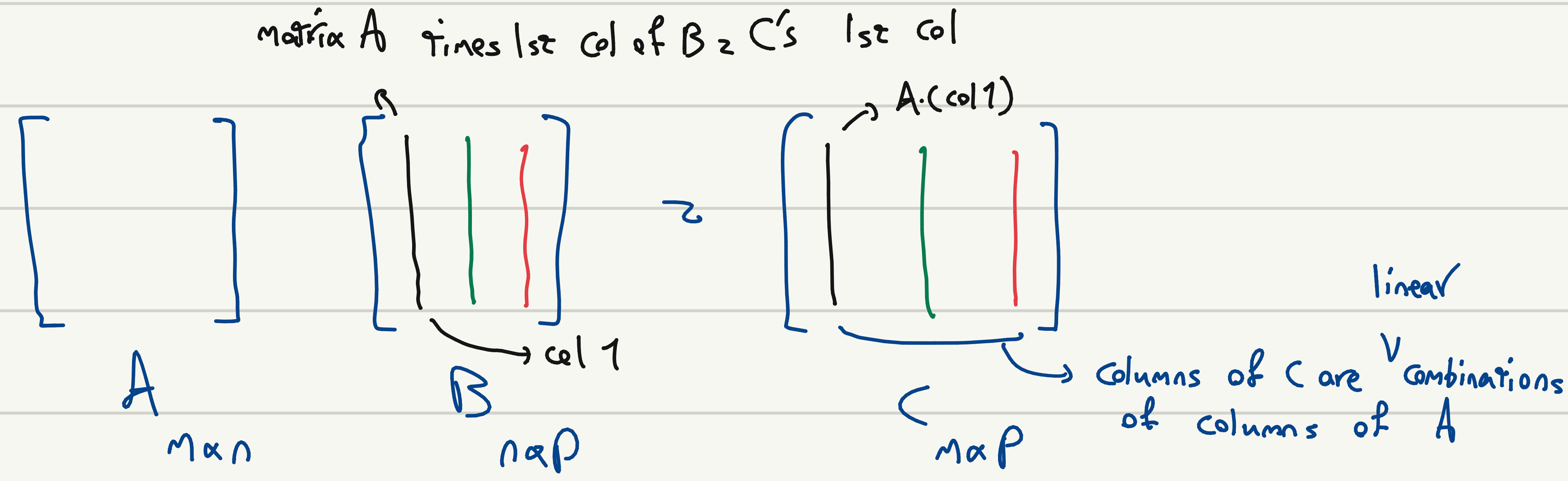
$a_{i1} b_{1j} + a_{i2} b_{2j} + \dots$

$= \sum_{k=1}^{n} a_{ik} b_{kj}$

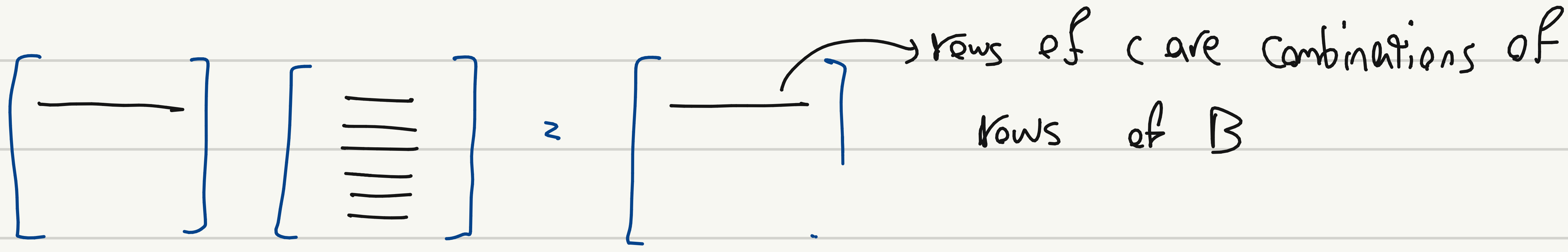
★ $A_{m \times n} \; B_{n \times p} = C_{m \times p}$

$\underbrace{m \times n \quad n \times p}_{\text{need to be same}}$

2nd way → col way



3rd way → row way



4th way → Column of A × row of B =

m×1 1×p m×p

m×1 1×

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} [1 \;\; 6] = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

① ② rows are multiples of ②

cols are multiples of ①

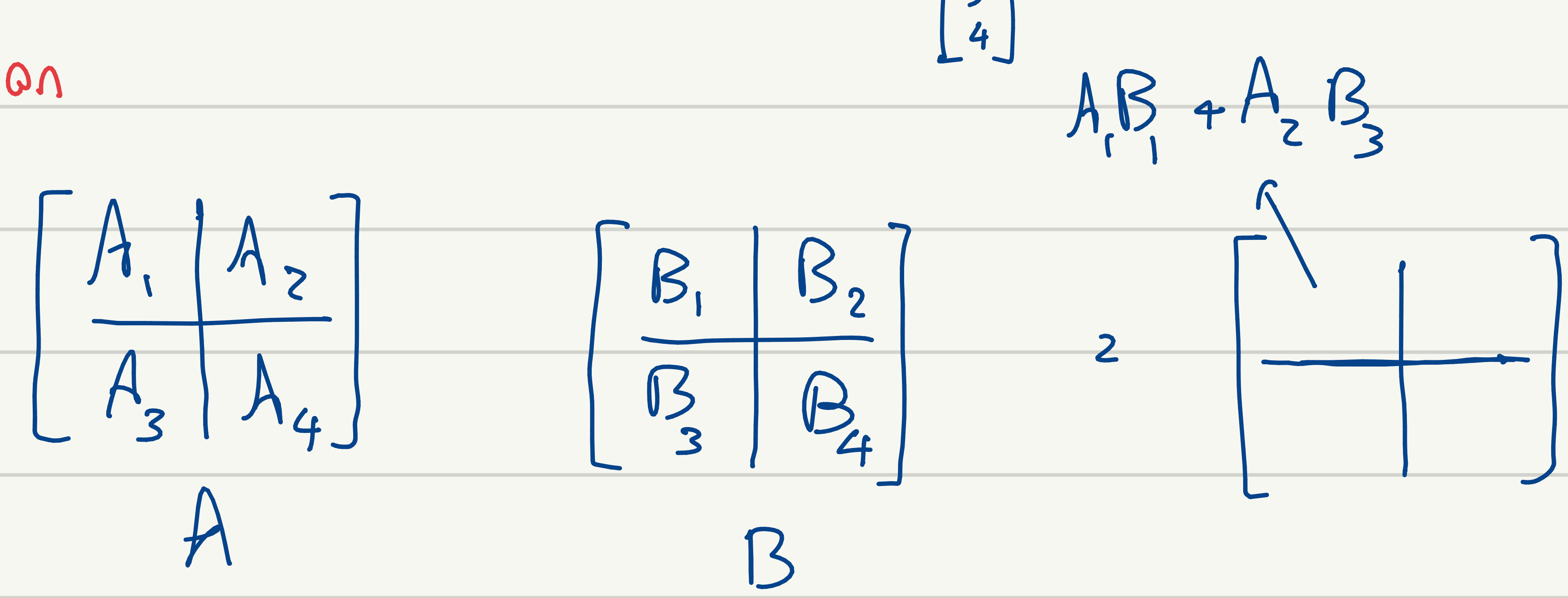
AB = sums of [(cols of A) × (rows of B)]

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}_{\text{1st col A}} \underbrace{[1 \;\; 6]}_{\text{1st row B}} + \underbrace{\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}}_{\text{2nd col A}} \underbrace{[0 \;\; 0]}_{\text{2nd row B}} = ①$$

☆ The row space of ① is just a line → lie on line through vector [1 6]

☆ The col space of ① is just a line → $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

Block Multiplication



Inverses (square matrices)

if A^{-1} exists → $A^{-1}A = I = AA^{-1}$

(invertible or non-singular)

Singular Case (No inverse): ☆ $Det(A) = 0$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

☆ if we multiply A by some matrix, the columns of result are are multiples of A's cols. For identity matrix we have $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ col but we can't get this col from A's cols [cause A's cols lie on same line and every combination of those cols lie in the same line aswell] which doesn't have $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

if A^{-1} exists,

→ We can find a vector X with $AX = 0$ $(X \neq 0)$ $X = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$AX = 0 \xrightarrow{\times A^{-1}} IX = 0 \rightarrow X = 0$ ✗ X is not $0 \rightarrow A^{-1}$ doesn't exist

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$A \times$ col j of A^{-1} = col j of I → solve 2 systems

Gauss - Jordan (solve 2 eqs at once)

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \xrightarrow{\times 2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{\times 3} \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

A I I A^{-1}

Continue elim until we get I here

Why this happens? $E\,[A \;\; I] = [I \;\; \underset{A^{-1}}{E}] = [I \;\; A^{-1}]$

Product of E_i's

$EA = I \rightarrow E = A^{-1}$ → A^{-1}

→ Elimination as solving n eqs at the same time

Result: n cols of A^{-1}

What is the inverse of AB? (we know A^{-1}, B^{-1})

$AB\,B^{-1}A^{-1} = I \rightarrow (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = (AI)A^{-1} = AA^{-1} = I$

$B^{-1}A^{-1}AB = I$

Transpose

$AA^{-1} = I \xrightarrow[\text{both side}]{\text{Transpose}} (A^{-1})^T A^T = I$ what is the inverse of A^T? $\rightarrow (A^{-1})^T$

☆ We can do Transposing & Inversing in either order : $(A^T)^{-1} = (A^{-1})^T$

Took away 4 row1 from row2 so the inverse should add 4row1 to row2

Calculated by solving $D - = U$ by our brain :)

$$\underset{E_{21}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{u}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}} \xrightarrow{\times E_{21}^{-1}} \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}} \xrightarrow[\text{to seperate Pivots}]{\text{Sometimes we may want}} \underset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}}$$

lower triangle (1's on diag)

Pivots on diag

more balanced but we stick to $A = LU$:)

Imagine A is 3×3 $\rightarrow$ $E_{32}E_{31}E_{21}A = u$ (no row exchanges) $\rightarrow A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$ L $= LU$

Why we Prefer E_i^{-1}'s ?

Suppose (E_{31} is I) $\rightarrow$ Arbitrary

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}$$

why? Cause we did row2 − 2row1 → row2 and row3 − 5row2 → row3 $\equiv$ row3 − 5(row2 − 2row1) = row3 − 5row2 + 10row1

row1 effects row3 so we have that 10

E (left of A) → EA = u

Now Inverses (Reverse order)

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = L \text{ (left of } u) \rightarrow A = LU$$

why? The order that matrices come for L is the right order row3 + 5row2 → row3, row2 + 2row1 → row2

2 and 5 don't interfere

we can achive L easily ← The multiplier(2,5) just sit in matrix L ←

A = LU

The point: If no row exchanges, the multipliers (like 2, 5) go directly into L → After we create L and U, store them and throw away A

How expensive is Elimination? OR How many operations on $n\times n$ matrix A? 1 Operation = Multiply + subtract

Say $n = 100$

100×100 → First Pivot, First stage 99×99 → about 100^2 operations → Second stage → about 99^2

Count $= n^2 + (n-1)^2 + \dots + 2^2 + 1^2 \approx \frac{1}{3}n^3$

[How? calculus. One way to look at it is $\int_1^n x^2 = \frac{1}{3}x^3$ → works for discrete Σ as well]

→ This was the cost on A, what about b? [A|b] Cost of b $= n^2$ [$n + (n-1) + \dots + 1 \sim n^2$]

→ The higher Price is always on turn A to LU, than the right-hand sides

Let allow row exchanges [use when there is a zero in pivot]

Permutations 3×3: 6 P's

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

★ If you multiply dif P's, the result would be one of the 6 P's

★ Inverse of a P is still one the 6 P's

★ In Permutation matrices, $P^{-1} = P^T$, $P^TP = PP^T = I$

★ For a 4x4 matrix → 24 Permutation matrices (4!)

★ Total number of reordering = $n!$

L5

Permutations P: execute row exchanges

$A = LU = \begin{bmatrix} 1 & & 0 \\ - & 1 & \\ - & - & 1 \end{bmatrix}\begin{bmatrix} \diagdown & & \\ & \diagdown & \\ 0 & & \diagdown \end{bmatrix}$ → assumes we don't have row exchanges (or $P = I$ in here)

with row exchanges becomes

→ get rows of A in a good order

$PA = LU$ // for any invertable A

[out of vid: $A = P^{-1}LU = P^TLU$]

P = Permutation matrix = Identity matrix with reordered rows

Transposes: $(A^T)_{ij} = A_{ji}$ $\quad \underset{3\times2}{\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T} = \underset{2\times3}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}}$

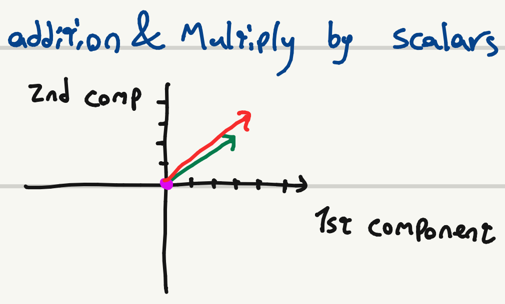
Symmetric matrices: $A^T = A$ e.g.: $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$

→ Imagine that we have a rectangular matrix R: R^TR is always semmetric

e.g.: $R = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$ $\quad R^T = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ $\quad R^TR = \underset{3\times2}{\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}} \underset{2\times3}{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 17 \end{bmatrix}$ → symmetric! also $(R^TR)^T = R^TR^{T^T} = R^TR$

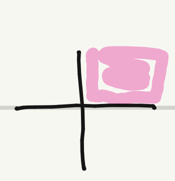
Vector Spaces (spaces: we got bunch of vectors that are closed under addition & multiply by scalars operations

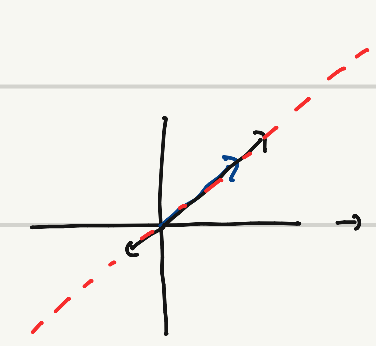
Examples: R^2 = all 2-dim real vectors such as $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}, \ldots$
= "X-Y Plane"

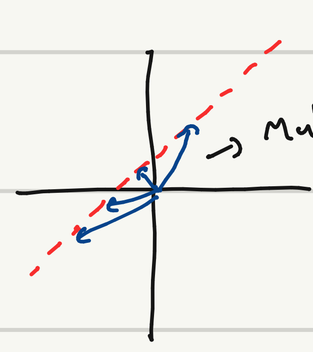


R^3 = all vectors with 3 real comp such as $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

R^n = all vectors with n real comp

→ Not a vector space e.g.:  [add ✓
multiplying by scalars (real nums) ✗

a vector space inside R^2 or subspace of R^2 e.g.:  → a line in R^2 that goes through zero vector

Not a subspace of R^2 e.g.: 

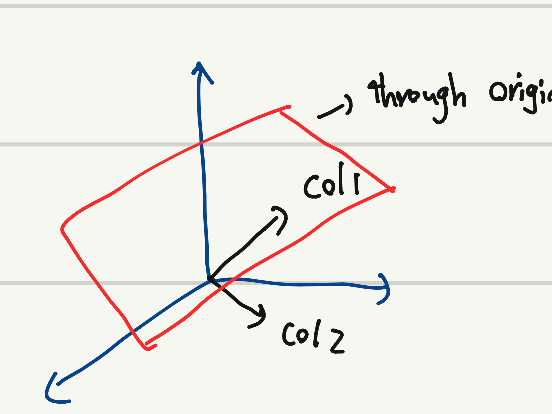
Subspaces of R^2:

1. all of R^2
2. any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or L
3. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or Z

Subspaces of R^3:

1. all of R^3
2. any line through origin
3. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or Z
4. any plane through origin

Create some subspaces of a matrix A



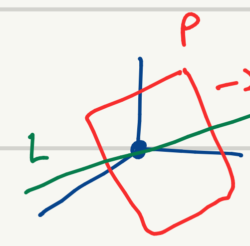
$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ → columns are in R^3 → one important subspace: all linear combinations of the columns called column space $C(A)$

★ if our columns were on a line, our column space would be a line

Vector space requirements: $V+W$ and CV are in the space, all combs $cv+dw$ are in the space

e.g. Is a vector space R^3

Subspace: a vector space inside a vector space



→ Plane through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is a subspace of R^3
↳ Line " " " " " " "

2 subspaces: P and L

① $P \cup L$ = all vectors in P or L or both → This is not a subspace (take v_1 from L and v_2 from P, v_1+v_2 might not be on plane or line)

② $P \cap L$ = all vectors in both P and L → This is a subspace [all subspace have at least origin as their intersection]

Column space of matrix $\underset{4\times 3}{A}$ (is a subspace of R^4)

$A = \begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}$ $C(A)$ = all linear combs of the cols

Does $Ax=b$ always have a sol for every b? No, 4 eqs and 3 unknowns

$Ax = \begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$ → there are going to be alot of vectors b that are not combs of these cols

> 🔥 **Core reason why it might *not* work:**
>
> Each equation is a plane (or hyperplane) in space. If you have **3 unknowns**, you're working in $\mathbb{R}^3$. Now you throw in **4 planes** and say: "Hey, find a point that lies on *all* 4."
>
> Well, that's only possible if all 4 planes **intersect at one point** — which is rare unless the 4th one is just a combination of the others (aka redundant).
> **Most of the time**, that extra equation is either:
>
> - **inconsistent** (doesn't intersect with the first 3), so the system has **no solution**, or
> - **redundant** (a combo of the first 3), and the system still has **one solution**.
>
> So yeah, having "more equations" isn't automatically helpful. It's only good if they're all consistent together. Otherwise, it breaks the system.

which b's allow this system to be solved?? one sol → $b = Z$ $(x = Z)$

another sol → $b = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ → $x = \begin{bmatrix}1\\0\\0\end{bmatrix}$ (b is one of the cols)

→ Can solve $Ax=b$ exactly when b is in $C(A)$ ($C(A)$ is in $R^4 \to m$)

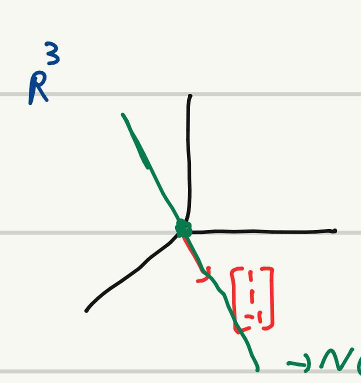
Nullspace of A = all sols $x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ to eq $Ax=0$ (x is in $R^3 \to n$)

$Ax = \begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

one sol → $x = \begin{bmatrix}0\\0\\0\end{bmatrix}$ (nullspace alway contains Z)

Another sol → $\underset{\text{can be zero}}{c}\begin{bmatrix}1\\1\\-1\end{bmatrix}$

Nullspace
$N(A)$ contains $c\begin{bmatrix}1\\1\\-1\end{bmatrix}$ → $N(A)$ is a line



→ $N(A)$ is a line in R^3

Check that sols to $Ax=0$ always gives a subspace?

if $Av=0$ and $Aw=0$ then $A(v+w)=0$ [$Av+Aw=0$ ✓]

if $Av=0$ then $A(\underset{\text{Just a scalar}}{12}v)=0$ [$12Av = 12\times 0 = 0$ ✓]

Do the sols of this following $Ax=b$ form a subspace? $Ax = \begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$

No! They don't even have $x=Z$ as their sol so no vector spaces hence no subspaces

3 sols → $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\-1\end{bmatrix}$

$A = \begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}$ → By doing elim, we are trying to solve $Ax=0$. Elim doesn't change $N(A)$

→ $\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix}$ → $\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix} = U$

No non-zero for row exchange so this is telling us that col2 is dependent on earlier cols - we move on from col2

↳ echelon form

2 Pivot cols (x_1, x_3)
2 Free cols (x_2, x_4) → can assign any number freely to vars x_2, x_4 like $X = \begin{bmatrix}x_1\\1\\x_3\\0\end{bmatrix}$
Rank of a matrix = number of pivots = 2 (for A)

Now solve for other x's: eqs → $x_1+2x_2+3x_3+2x_4=0 \rightarrow x_1=-2$

$2x_3+4x_4=0 \rightarrow x_3=0$

↳ $=0$ (we set it)

$\Big\}\ Ux=0$ ↑ Back substitution

$$\rightarrow X=\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$$ → a sol to $Ax=0$ ($-2\times$ col 1 of A + $1\times$ col 2 = 0 ✓), a bunch of other sols = $c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$

is that the whole N(A)? No, we can choose other numbers for x_2, x_4 like $x_2=0$ $x_4=1$

$$X=\begin{bmatrix}2\\0\\-2\\1\end{bmatrix} \text{ or } d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

$\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$ are our two special sols. why special? Cause we assign special numbers to x_2, x_4

so $$x=\overbrace{c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}}^{\text{all combs}}$$

★ # free vars for $A_{m\times n}$ = $n-\text{rank}(A)$ = n − # pivot vars

★ r (rank of a matrix) = # Independent eqs [in the last ex, we only had 2 eqs, cause eq3 = eq1 + eq2]

→ That is the complete algo for finding all sols to $Ax=0$

1: Do elim, make the matrix in echelon form & find Pivot & Free Vars

2: Back substitute and find all of special sols

3 Sols to $Ax=0$ → linear comb of all special sols

Take one more step to clean the matrix even more → Reduced row echelon form R (zero's above & below pivots and make pivots 1)

$$U=\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}$$ → btw row of 0's appeared because row3 is dependent on earlier rows ($r3=r1+r2$)

Reduced row echelon form

let's clean up further $\xrightarrow{\text{do elim upwards}}$ $\begin{bmatrix}1&2&0&-2\\0&0&2&4\\0&0&0&0\end{bmatrix} \rightarrow \begin{bmatrix}1&2&0&-2\\0&0&1&2\\0&0&0&0\end{bmatrix} = R = \text{rref}(A)$

↳ Notice $\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$ in pivot rows & cols

Back substitution: $x_1+2x_2-2x_4=0$

$x_3+2x_4=0$ $\Big\}\ Rx=0$

Another example:

before doing calcs, 2 pivots [col3 = col1 + col2] → col 1, col 2 = pivot cols

$$A=\begin{bmatrix}1&2&3\\2&4&6\\2&6&8\\2&8&10\end{bmatrix} \rightarrow \begin{bmatrix}1&2&3\\0&0&0\\0&2&2\\0&4&4\end{bmatrix} \xrightarrow{\text{row exchange}} \begin{bmatrix}1&2&3\\0&2&2\\0&0&0\\0&4&4\end{bmatrix} \rightarrow \underbrace{\begin{bmatrix}1&2&3\\0&2&2\\0&0&0\\0&0&0\end{bmatrix}}_{U}$$

2 pivot cols

1 free col → one special sol

$r=2$ (again)

$X=c\begin{bmatrix}-1\\-1\\1\end{bmatrix}$ → whole N(A)

☆ # Pivot cols for A = # Pivot cols for A^T

Push for R in the last example:

$$u = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \overbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}^{I} & \overbrace{\begin{matrix} 1 \\ 1 \end{matrix}}^{F\ (\text{Free Part})} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix} \rightarrow \begin{bmatrix} I & F \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$X = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = c \begin{bmatrix} -F \\ I \end{bmatrix} = N$ (null space matrix, whose cols are special sols)

Let's see what happend:

rref form

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

r Pivot cols

n−r Free cols

$RN = 0$ and get this

$\rightarrow$ N (nullspace matrix, cols = special sols) $= \begin{bmatrix} -F \\ I \end{bmatrix}$

we set this first

$Rx = 0$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0 \rightarrow x_{pivot} = -F x_{free}$$

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 2x_4 &= b_1 \\ 2x_1 + 4x_2 + 6x_3 + 8x_4 &= b_2 \\ 3x_1 + 6x_2 + 8x_3 + 10x_4 &= b_3 \end{aligned} \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \xrightarrow{\text{elim}} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right]$$

Augmented Matrix $= [A\ b]$

: L8

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right] \rightarrow 0 = b_3 - b_2 - b_1$$

Pivot cols

Imagine $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ $\rightarrow$ this an ok b cause it meet our condition ($b_3 = b_1 + b_2$)

Solvability Condition on b

☆ $Ax = b$ solvable when b is in $C(A)$

☆ If a comb of rows of A gives zero row, then same comb of the entries of b must give 0.

To find complete sol to $Ax = b$

① $X_{\text{Particular sol}}$: Set all free variables to zero, solve $Ax = b$ for pivot variables

② add on x in nullspace $X_{\text{nullspace}}$

③ Complete sol $X = x_p + X_n$

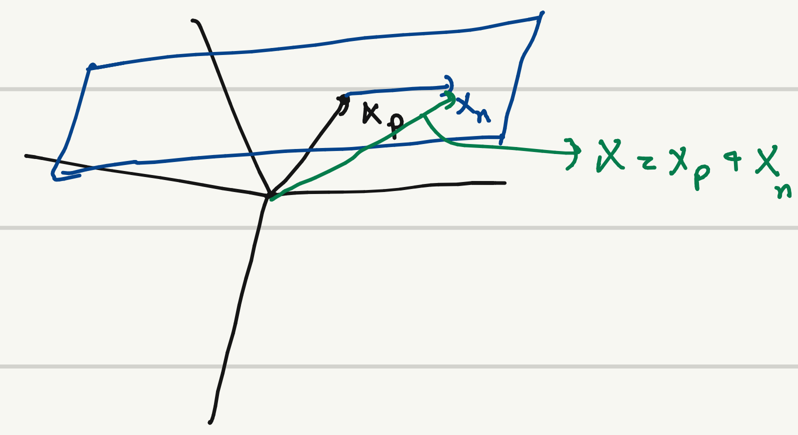
why? $\rightarrow$ $$\begin{aligned} Ax_p &= b \\ Ax_n &= 0 \\ \hline A(x_p + x_n) &= b \end{aligned}$$

In last example $x_2 = x_4 = 0$ (Free vars) $\rightarrow$ $x_1 + 2x_3 = 1$, $2x_3 = 3$ $\rightarrow$ $x_3 = \frac{3}{2}$ $\quad x_1 = -2$ $\rightarrow$ $X_{\text{Particular}} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$

$$X_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$\hookrightarrow$ can't multiply by constant cause those other sols wouldn't give us correct b

Plot all sols x in $\mathbb{R}^4$



m by n matrix A of rank r (we know $r \le m$, $r \le n$)

Full col rank $\rightarrow$ $r = n$: no free variables $\rightarrow$ $N(A) = \{\text{zero vector}\}$ sol to $Ax = b$: $x = \{x_p\}$ unique sol if it exists! (0 or 1 sol)

example : $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \rightarrow \text{rank}(A) = 2$ only $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives $Ax = 0$

For $Ax = b$ $\rightarrow$ we can have b's that gives no sol or b's that have 1 sol like $\begin{bmatrix} 4 \\ 3 \\ 7 \\ 6 \end{bmatrix}$ which gives $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

1 of first col + 1 of second col = b

RREF = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Full row rank $\rightarrow$ $r = m$: Can solve $Ax = b$ for every b $\rightarrow$ a sol exists

Left with $\underset{n-m}{\underline{n-r}}$ free variables

example :

$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \rightarrow \text{rank}(A) = 2$ RREF = $\begin{bmatrix} 1 & 0 & - & - \\ 0 & 1 & - & - \end{bmatrix}$ F

Full rank $r = m = n$

example :

$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ RREF = I $N(A) = \{\text{zero vector}\}$, can solve for every b and there is a unique sol

Invertible

Summary :

① $r = m = n$

RREF = I

1 sol to $Ax = b$

② $r = n < m$

RREF = $\begin{bmatrix} I \\ 0 \end{bmatrix}$

0 or 1 sol to $Ax = b$

③ $r = m < n$

RREF = $\begin{bmatrix} I & F \end{bmatrix}$

F could be mixed into the I $[\,I\!F\,]$

∞ sol (cause $N(A)$ have sols)

④ $r < m$ and $r < n$

RREF = $\begin{bmatrix} I\!F \\ 0 \\ 0 \end{bmatrix}$

0 sols or ∞ sols

☆ The rank tells you everything about the # of sols

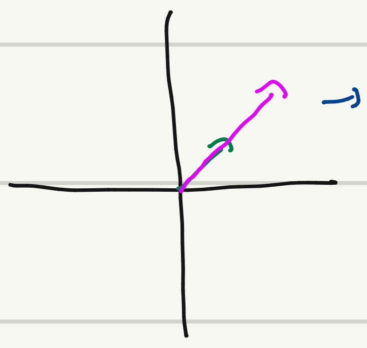
Suppose A is m by n with $m < n$. Then there are non-zero sols to $Ax = 0$ ① : L9

Reason : There will be at least one free variable

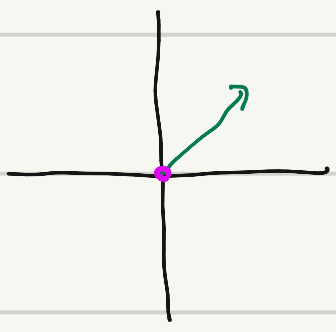
Independence: When vectors $x_1, x_2, \dots, x_n$ are independent? They are independent if no combination gives zero vector (except the zero comb [all c's are zero])

$$c_1x_1 + c_2x_2 + \dots + c_nx_n \neq 0$$

examples:

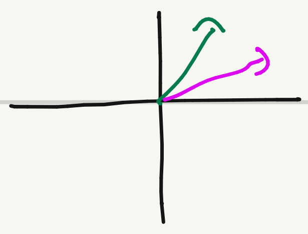


→ they are dependent $V1 = 2V2$



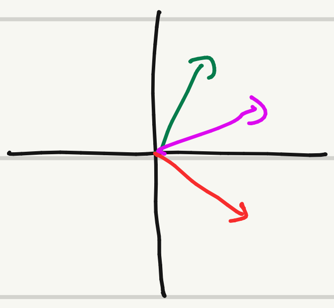
v_2 = zero vector

dependent $v_2 = 0 \times v_1$ or $\underline{0}\, v_1 + \underline{6}\, v_2 = 0$ (c ← 6)



Independent

☆ If one of the vectors is zero, independence is dead



$$A = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix}_{2\times3} \end{array} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

→ base on ① from prev page, these vectors are dependent cause there is smt in nullspace so the are some c's that would make a linear comb = 0

☆ Repeat: when $v_1, \dots, v_n$ are cols of A, they are independent if nullspace of A is the $\{$zero vector$\}$ → rank = n $N(A) = \{$zero vector$\}$ no free vars

☆ They are dependent if $Ac = 0$ for some non-zero c. → rank < n, some free vars

Span: Vectors $v_1, \dots, v_\ell$ span a space means: the space consists of all combs of those vectors.

→ cols of a matrix span columnspace

Basis for a space is a seq of vectors $v_1, v_2, \dots, v_d$ with 2 properties:

① They are independent

② They span the space

example: space is R^3

One basis: $\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}$ another basis: $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\4\\8\end{bmatrix}$

☆ $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}$: these two are independent but only span their comb

☆ n vectors give basis if the $n\times n$ matrix with those cols is Invertible

☆ The basis in not unique (in any invertible 3x3 matrix, it's cols are a basis for R^3)

☆ Given a space (R^n): Every basis has the same number of vectors in it → Dimention of the space

Example: space is $C(A)$

another basis for $C(A) = \begin{bmatrix}2\\2\\2\end{bmatrix}, \begin{bmatrix}7\\5\\7\end{bmatrix}$

one of vecs in $N(A) = \begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

Pivot cols

Matrix has a rank so becareful where u use the word rank

$\text{Rank}(A) = \#$ pivot cols = dimention of $C(A) = 2$

☆ If dim of $C(A)$ for matrix $A_{m\times n}$ is $= l$ and we have l independent vector (with shape $m\times 1$ of course) → Those vecs are basis for $C(A)$

☆ dim $C(A)$ = rank r

Dim of $N(A)$ = # free variables = $n-r$

first take the two free cols and assign them $\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} \rightarrow \begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$ → two special vecs → $N(A)$ is comb of these two

→ they are basis

: L 20

4 subspaces: $A_{m\times n}$

① Colspace $C(A)$ — in $\mathbb{R}^m$

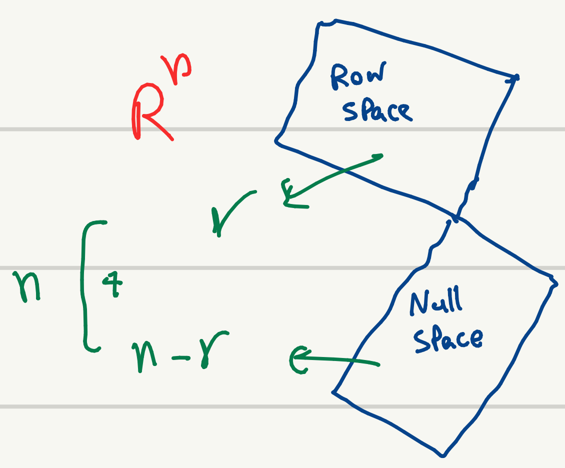
② Nullspace $N(A)$ — in $\mathbb{R}^n$

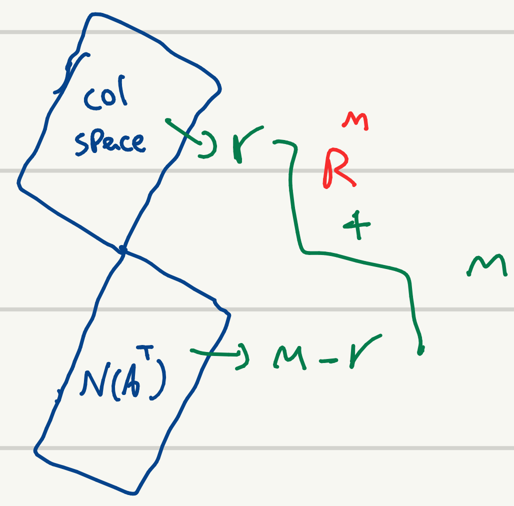
③ Rowspace: all combs of rows of A = all combs of cols of A^T = $C(A^T)$ — in $\mathbb{R}^n$

④ Nullspace of A^T = $N(A^T)$ aka the left nullspace of A — in $\mathbb{R}^m$

Drawings

• : dimension





Basis & dimensions:

$C(A)$ → basis = Pivot cols (of original A)

dimension = rank(A) r

Nullspace → basis = special solutions

dimension = $n-r$

Rowspace → dimension = dimension of $C(A^T)$ = rank(A) r

basis =

$$\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix} \rightarrow \begin{bmatrix}1&2&3&1\\0&-1&-1&0\\0&0&0&0\end{bmatrix} \rightarrow \begin{bmatrix}1&2&3&1\\0&1&1&0\\0&0&0&0\end{bmatrix} \rightarrow \begin{bmatrix}1&0&1&1\\0&1&1&0\\0&0&0&0\end{bmatrix} = R$$

I F 0

$C(R) \neq C(A)$

Basis for row space of R is the first r rows of R → Row space of R = Row space of A (because we did row operations that didn't change row space)

☆ This basis is also the natural basis for rowspace

Left Nullspace → dimension = m − r
$N(A^T)$

basis =

$A^T y = 0 \rightarrow \begin{bmatrix} & \end{bmatrix}\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \xrightarrow{\text{Transpose}} y^T A = 0^T \rightarrow \begin{bmatrix} & y^T & \end{bmatrix}\begin{bmatrix} & & \end{bmatrix} = \begin{bmatrix} & & \end{bmatrix}$ → Multiplying y from left → the reason for the name

$\text{rref}\,[A_{m\times n} \mid I_{m\times m}] \rightarrow [R_{m\times n} \mid E_{m\times m}]$ → Contain a record of what we did to get R from A

$EA = R$

lets apply the steps that took us from A to R on I

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = E \rightarrow \text{so} \quad \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ \boxed{-1 \quad 0 \quad 1} \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ what comb of rows of A gives 0? The last row of E → so the basis $[-1 \ 0 \ 1]$

New vector space! All 3x3 matrices!! : A+B, cA rules are followed → We call this matrix space M

subspaces of M : like all upper triangulars, all symmetric matrices, Diagonal matrices

★ Fun fact : Dim of diagonal matrix $D_{3\times 3} = 3$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → one basis for that → dim = 3

: L11

Basis for M = all 3x3's

Most obvious choice: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → dim = 9

Basis for Symmetric matrices $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ → dim = 6

Basis for upper Triangular $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ → dim = 6

Basis for $S \overset{\text{intersection}}{\cap} U$ = D diagonal matrices = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → dim = 3

★ $S \cup U$ → Not a subspace $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \notin S \cup U$

$S + U$: combination of things in S and things in U = any element of S + any element of U = all 3x3 matrices → dim = 9

★ $\overset{6}{\dim S} + \overset{6}{\dim U} = \overset{3}{\dim(S \cap U)} + \overset{9}{\dim(S+U)}$

Suppose: we have a dif eq like: $\frac{d^2y}{dx^2} + y = 0 \rightarrow$ sols $= y = \underbrace{\cos x, \sin x}_{\text{Basis}}$ complete sol $= y = c_1 \cos x + c_2 \sin x$

★Null space of a dif eq is the sol space   dim (sol space) = 2

Rank One Matrices

dim $C(A)$ = rank = dim $C(A^T)$ which is row space $= r = 1$

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$$

$\rightarrow$ ★ Every rank 1 matrix $A = u\,\underbrace{v^T}_{\text{col vecs}}$

$\rightarrow$ Imagine M = all 5x17 matrices, subset of rank 4 matrices, Is that a subspace? (If I add 2 rank 4 matrices, is the sum rank 4 ?)

①

No. rank could be 5 [This is the highest it could be because of ①]

If I add 2 rank 1 matrices, is the sum rank 1 ? No

$\rightarrow$ Suppose we are in R^4: $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$, S = all v in R^4 with $v_1 + v_2 + v_3 + v_4 = 0 \rightarrow$ Is it a subspace? Yes

What is the Dim = ? $\rightarrow$ S is nullspace of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \rightarrow$ rank = 1

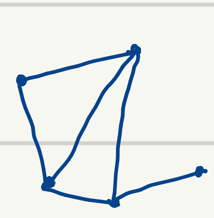
dim $N(A) = n - r = 4 - 1 = 3$ $\rightarrow$ Dim $S = 3$

Basis for S = ? $\rightarrow \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Col space of $A = C[1] = R^1$

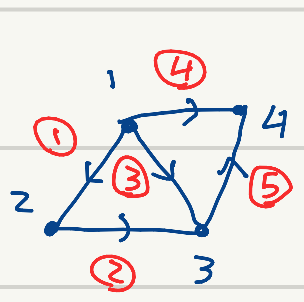
$N(A^T) = \{0\}$

Graph: {nodes, edges}



L12

Graph



$\rightarrow n = 4$ nodes

$\rightarrow m = 5$ edges

Incidence matrix:

$$A = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{cccc} \text{node 1} & 2 & 3 & 4 \end{array}$$

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{array}{l} \text{edge 1} \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

(edges 1–3) made a loop $\rightarrow$ edge 3 = edge 1 + edge 2 [loops correspond to linearly dependent rows]

Find nullspace $\rightarrow$ solve $Ax = 0 \rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$Ax = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow$ We have created a matrix that compute the dif across every edge

$X = x_1, x_2, x_3, x_4 \rightarrow$ Potentials at nodes

$\xrightarrow{\times A}$ $x_2 - x_1$, etc. $\rightarrow$ Potential differences across the edges

One vector in nullspace $= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow$ Nullspace $= c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ dim $N(A) = 1 \rightarrow$ rank = 3

$\rightarrow$ if all potentials are the same $\rightarrow$ No flow!

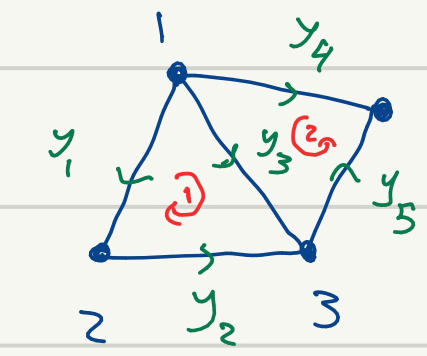
$A^T y = 0$ $\quad \dim N(A^T) = m - r = 5 - 3 = 2$

☆ why we care about $A^T y$ → Matrix C connects potential dif (e) to currents (y) → OHM's LAW $I = \frac{V}{R}$ and $A^T y = 0$ is Kirchoff's current law (KCL)

So $e = Ax \rightarrow y = Ce \rightarrow A^T y = 0$ (KCL) [if we had external current $f \rightarrow A^T y = f$] → $A^T C A x = 0$ or f → in equilibrium (without time)

$$A^T = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{aligned} -y_1 - y_3 - y_4 &= 0 \rightarrow \text{Node 1 balance} \\ y_1 - y_2 &= 0 \rightarrow \text{Node 2 balance} \\ y_2 + y_3 - y_5 &= 0 \rightarrow \text{Node 3 balance} \\ y_4 + y_5 &= 0 \rightarrow \text{Node 4 balance} \end{aligned}$$

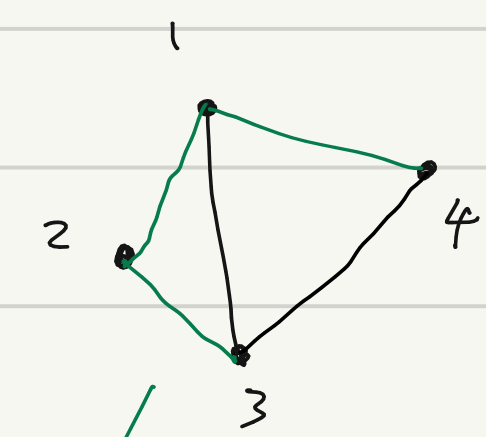
↑ ↑ ↑



Basis for $N(A^T)$: we need 2 vectors ($\dim N(A^T) = 2$) [without elim]

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Current around loop 1 $\quad$ Current around loop 2



Rowspace of A: $\dim (C(A^T)) = \dim$ row space $= 3$ $\quad$ Pivot $\underbrace{\text{cols}}_{\text{edges}}$ of A^T:

These 3 independent edge have no loops
↓
a smaller graph without a loop
↓
a Tree

☆ $\dim N(A^T) = m - r \rightarrow$ # loop = # edges − (# nodes − 1)

☆ rank = $n-1$ → the rows are linear dependent cause in incident matrix, A^T Σ row vectors $= 0$

so we choose $n-1$ of them = rank

→ # nodes − # edges + # loops = 1 → Euler's formula

: L13

Q1) Suppose u, v, w → non-zero vectors in $\mathbb{R}^7$. What are the possible dims of the subspace spanned by u, v, w?

↓ (without this, 0 dim would be possible)

1, 2, 3

Q2) $u_{5\times3}$, in rref and $r=3$: a) what's the nullspace? $\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

b) Imagine $B = \begin{bmatrix} u \\ 2u \end{bmatrix}$ → what is the rref? $\begin{bmatrix} u \\ 0 \end{bmatrix}$, $r = 3$

c) $C = \begin{bmatrix} u & u \\ u & 0 \end{bmatrix}$ → what is the rref? elim-step 1: $\begin{bmatrix} u & u \\ 0 & -u \end{bmatrix}$, step 2: $\begin{bmatrix} u & 0 \\ 0 & -u \end{bmatrix}$, step 3: $\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ → $r = 6$

$\dim N(C^T) = m - r = 10 - 6 = 4$

Q3) $Ax = \begin{bmatrix}2\\4\\2\end{bmatrix}$ $\quad x = \begin{bmatrix}2\\0\\0\end{bmatrix} + c\begin{bmatrix}1\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\0\\1\end{bmatrix}$: a) dim of rowspace? 1 $\quad$ b) what's A?

A is 3×3, $r = 3 - \dim N(A) = 3 - 2 = 1$

$$A = \begin{bmatrix}1 & -1 & 0\\2 & -2 & 0\\1 & -1 & 0\end{bmatrix}$$ (Think in colspace)

c) $Ax = b$ can be solved if? b has the form $= c\begin{bmatrix}1\\2\\1\end{bmatrix}$ (look at colspace) → Don't forget other cases $r = m$, $r = n$

Q4) Imagine A is a square matrix, if it's nullspace is just the zero vec, what about $N(A^T)$? zero vector

Q5) Look at the space of 5×5 matrices as a vector space, Do the invertible matrices form a subspace? No, if I add two invertible matrices, we don't know if the answer is invertible

Q6) If $B^2 = 0 \Rightarrow B = 0$? False → $B = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ → $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

Q7) A system of n eqs and n unknowns, is solvable for every right-hand side if the cols are independent? Yes

Q8) $B = \underbrace{\begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}1 & 0 & -1 & 2\\0 & 1 & 1 & -1\\0 & 0 & 0 & 0\end{bmatrix}}_{V}$ → without multiplying: a) Basis for $N(B) \subseteq \mathbb{R}^4$

↳ Invertible $\quad Bx = 0 \xrightarrow{\times Q^{-1}} Vx = 0$, $N(V) = N(B)$

* $N(CD) = N(D)$ if C is invertible.

basis for $N(B)$: $r = 2$ $\quad N(B) = c\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} + d\begin{bmatrix}-2\\1\\0\\1\end{bmatrix}$ (Think in colspace)

b) Complete sol to $Bx = \begin{bmatrix}1\\0\\1\end{bmatrix}$ $\quad x_p + x_n = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + c\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} + d\begin{bmatrix}-2\\1\\0\\1\end{bmatrix}$

↳ The first col of B is b

Q9) if $m = n$ then rowspace = colspace? No - only if the matrix was symmetric

Q10) The matrices A, $-A$ share the same four subspaces? Yes

Q11) If A, B have the same four subspaces the $A = cB$? False → A, B any invertible 6×6 → $N(A) = N(A^T) = N(B) = N(B^T) = \{0\}$

$r = 6$, rowspace = colspace $= \mathbb{R}^6$ $\quad (Ax = 0 \to x = A^{-1}0 = 0)$

Q12) If I exchange two rows of A, which subspaces stay the same? Rowspace, Nullspace

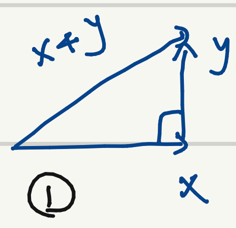
Q13) Why $v = \begin{bmatrix}1\\2\\3\end{bmatrix}$ can't be in Nullspace & be a row of A?

$\begin{bmatrix}1 & 2 & 3\\ - & - & -\\ - & - & -\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \rightarrow \times$

★ The intersection of Nullspace & rowspace = Zero Vector

:L14

Orthogonal Vectors



. Vectors X, Y are orthogonal if $x^T y = 0 \longrightarrow$ Why?

① : $\|x\|^2 + \|y\|^2 = \|x+y\|^2 \rightarrow \cancel{x^T x} + \cancel{y^T y} = (x+y)^T(x+y) = \cancel{x^T x} + \cancel{y^T y} + \underbrace{x^T y + y^T x}_{\text{Same [vectors]}} = \boxed{2x^T y = 0}$

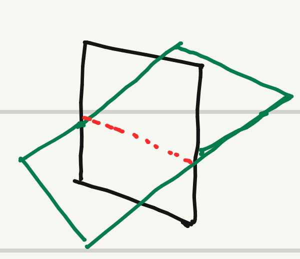
(going to be true when we have a right angle)

★ Zero vector is orthogonal to everybody

Subspace S is orthogonal to subspace T means every vector in S is orthogonal to every vector in T.

★ Two subspaces can only have zero vector as their intersection because other vector could not be orthogonal to themselves if they were in intersection.

★ These two planes are not orthogonal



★ Rowspace is orthogonal to Nullspace $\rightarrow$ why?

Nullspace has vectors X $\rightarrow Ax = 0 \rightarrow \begin{bmatrix}\text{row 1 of A}\\ \vdots \\ \text{row n of A}\end{bmatrix}\begin{bmatrix}x\end{bmatrix} = \begin{bmatrix}0\\0\\ \vdots\\ 0\end{bmatrix} \rightarrow$ X is orthogonal to all the rows

$(\text{row }1)^T x = 0 \quad \times c_1$
$(\text{row }2)^T x = 0 \quad \times c_2$
$\rightarrow c_1(\text{row }1)^T x + c_2(\text{row }2)^T x = [c_1 \text{ row }1 + c_2 \text{ row }2 + \dots]^T x = 0 \longrightarrow$ X is orthogonal to rowspace

★ Colspace is orthogonal to $N(A^T) \rightarrow$ We can prove by replacing A with A^T in the above proof

Imagine that we are in $\mathbb{R}^3 \rightarrow A = [1\ 2\ 5]$ dim rowspace = 1 dim Nullspace = 2 $\rightarrow$ a plane orthogonal to $[1\ 2\ 5]$

★ Nullspace and Rowspace are orthogonal complements in $\mathbb{R}^n \rightarrow$ Nullspace contains all vectors $\perp$ rowspace

NEW: Solve $Ax = b$ when there is no solution

$\underbrace{\underset{n\times m}{A^T}\ \underset{m\times n}{A}}_{n\times n} \rightarrow$ symmetric $[(A^TA)^T = A^TA]$

$\rightarrow$ We will see $A^TA\hat{x} = A^Tb$ hoping this will have a sol (This is my best sol)

Imagine: $A = \begin{bmatrix}1 & 1\\1 & 2\\1 & 5\end{bmatrix} \rightarrow \begin{bmatrix}1 & 1 & 1\\1 & 2 & 5\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 2\\1 & 5\end{bmatrix} = \begin{bmatrix}3 & 8\\8 & 30\end{bmatrix}$ invertible

$B = \begin{bmatrix}1 & 3\\1 & 3\\1 & 3\end{bmatrix} \rightarrow \begin{bmatrix}1 & 1 & 1\\3 & 3 & 3\end{bmatrix}\begin{bmatrix}1 & 3\\1 & 3\\1 & 3\end{bmatrix} = \begin{bmatrix}3 & 9\\9 & 27\end{bmatrix}$ Not invertible