# Chapter-2

## System of Linear Equations (SLE)

In mathematics, linear systems are the basis and a fundamental part of linear algebra, a subject which is used in most parts of modern mathematics. Computational algorithms for finding the solutions are an important part of numerical linear algebra, and play a prominent role in engineering, physics, chemistry, computer science, and economics. A system of linear equations is a group of two or more linear equations containing the same variables. In a system of equations there is more than one unknown since the equations contain more than one variable. We will explore applications that involve systems of linear equations and look at how to set up a system of equations with given information. Systems of linear equations arise in a wide variety of applications. System of linear equations arises in the problem of polynomial curve fitting, network analysis and analysis of an electric circuit and the linear programming problem etc. System of linear equations also arises when we want to solve mixture problems and distance-rate-time problem. One of the most frequent occasions where linear systems of $n$ equations in $n$ unknowns arise is in least-squares optimization problems. Least squares problems lead to square (i.e. $n \times n$) linear systems of equations. Also systems of linear equations arise in the problem of graph theory and cryptography**.** In cryptanalysis (breaking codes mathematically) we use linear in solving systems of equations related to both a grammar and language in cipher text.

**Linear equation:**
An equation in two or more variables (unknowns) is linear if it contains no products of unknowns or exponent of each unknown is 1.

**Example:**

1. $2x + 3y = 8 \text{ (linear)}$
2. $x_1 + x_2 + .... + x_n = 1 \text{ (linear)}$
3. $x^2 + 4x = 8 \text{ (non} - \text{linear)}$

**Solution:**
A solution of linear equation $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ is a sequence of $n$ numbers $\alpha_1, \alpha_2 ..., \alpha_n$ such that the equation is satisfied when we substitute $x_1 = \alpha_1, x_2 = \alpha_2, ..., x_n = \alpha_n$ . The set of all such solutions of the linear equation is called a solution set.

$$\left.\begin{array}{l} x + y = 2 \\ x + y = 0 \end{array}\right\} \text{has no solution.}$$

**System of linear equations:**

A group of $m$ linear equations of $n$ variables $x_1, x_2, ..., x_n$ are of the form

$$\left.\begin{array}{ccccc} a_{11}x_1 + & a_{12}x_2 + & . \ . \ . & +a_{1n}x_n = b_1 \\ a_{21}x_1 + & a_{22}x_2 + & . \ . \ . & +a_{2n}x_n = b_2 \\ . & . & . \ . \ . & . \\ . & . & . \ . \ . & . \\ . & . & . \ . \ . & . \\ a_{m1}x_1 + & a_{m2}x_2 + & . \ . \ . & +a_{mn}x_n = b_m \end{array}\right\} \cdots\cdots\cdots (1)$$

is known as system of linear equation. Here the co-efficient $a_{ij}$, $i=1,2,...m$, $j=1,2,...,n$ of the variable and the free term $b_i$, $i=1,2,...,m$ are real numbers.
By a solution (set) of a system means such a set of real numbers that satisfies each equation in a system.

**Matrices and system of linear equations:**
The system of linear equations (1) can be written in the matrix form.

$$\begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{n2} & . & . & . & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ . \\ b_m \end{pmatrix}$$

or simply $AX = B$ ……………………………………….(2)
where co-efficient matrix, $A=(a_{ij})$, variable matrix, $X=(x_i)$ and constant matrix, $B=(b_i)$

The associated homogeneous system of (1) is $AX = 0$.

The system (1) also can be written in augmented matrix form $(A|B)$ or $(A \vdots B)$.

There are three commonly used methods to solve system of linear equations:

1. Using inverse matrix,
2. Using elementary row operations (Gaussian elimination and Gauss-Jordan elimination),
3. Cramer’s rule.

**Solution of a system of linear equations:**
A sequence of numbers $\alpha_1, \alpha_2 ..., \alpha_n$ is called solution of the system of linear equations given by (1) if $\alpha_1, \alpha_2 ..., \alpha_n$ is a solution of every equation in the system.

**Degenerate and non-degenerate linear equation:**
A linear equation is said to be **degenerate** if it has the form $0x_1 + 0x_2 + ... + 0x_n = b$. That is, if every coefficient of the variable is equal to zero. The solution of such a generate linear equation is as follows:

(i) If the constant $b \neq 0$, then the above equation has no solution.

(ii) If the constant $b = 0$, then every vector $u = (\alpha_1, \alpha_2, ......, \alpha_n)$ is a solution of the above equation.

The general linear equation $a_1x_1 + a_2x_2 + ..... + a_nx_n = b$ is called **non-degenerate** linear equation.

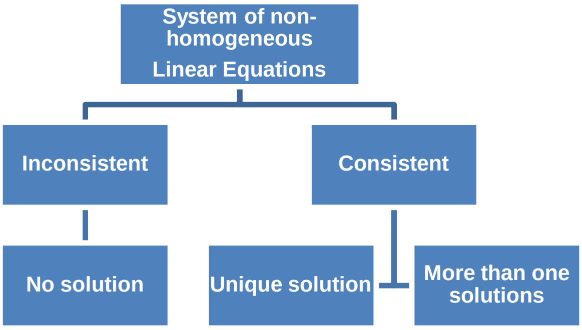
**Consistent and inconsistent equations:**
A system of linear equations is called consistent if it has at least one set of solution. A system of linear equations is called inconsistent if it has no solution.

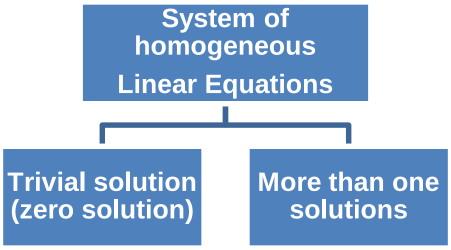
**Consistency theorem:** The system of linear equations $AX = B$ ($m$ equations and $n$ unknowns) is consistent (i.e. there is at least one solution of the system) if the coefficient matrix $A$ and the augmented matrix $(A|B)$ have the same rank.

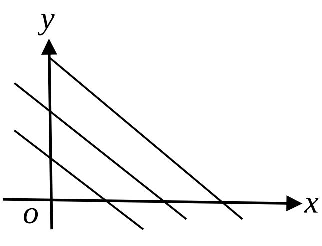
**Determinate and Indeterminate:**
A consistent system is called determinate if it has a unique solution and indeterminate if it has more than one solution.
An indeterminate system of linear equation always has an infinite number of solutions.

**Then 3 cases arise:**

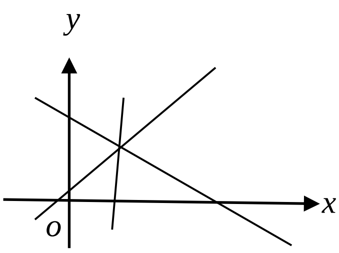
- ► SLE is inconsistent $\Rightarrow$ straight lines do not intersect (i.e., parallel);
- ► SLE has a unique solution $\Rightarrow$ all straight lines pass through a single point;
- ► SLE is redundant $\Rightarrow$ actually one straight line, with which others coincide, exists.



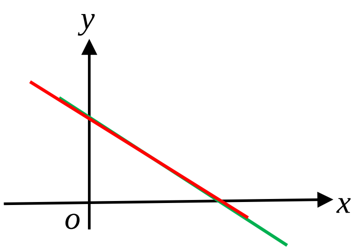




An inconsistent system
(no common point)




A unique system
(only 1 common point)




Infinitely many solution system
(overlapping lines)

**Example:** Following augmented matrices illustrate the consistency of the linear system.

**(i)** $(\mathrm{A}|\mathrm{B}) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right)$,
$\operatorname{rank}(A) = \operatorname{rank}(A|B) = 3$
So, this system is consistent.

**(ii)** $(\mathrm{A}|\mathrm{B}) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 2 \end{array}\right)$,
$\operatorname{rank}(A) = 2$; $\operatorname{rank}(A|B) = 3$
So, this system is inconsistent. There is no solution for this system.

**(iii)** $(\mathrm{A}|\mathrm{B}) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$.
$\operatorname{rank}(A) = 2$;
$\operatorname{rank}(A|B) = 2$
So, this system is consistent but infinitely many solutions.

**Note:** For additional details on Rank, see lecture note of Chapter–1 in page no 18.

**Example:** Test the consistency of the following system of linear equations with the help of the rank of the matrix

$$\begin{aligned} 3x_1 + 4x_2 - x_3 + 2x_4 &= 1 \\ x_1 - 2x_2 + 3x_3 + x_4 &= 2 \\ 3x_1 + 14x_2 - 11x_3 + x_4 &= 3 \end{aligned} \quad .$$

**Solution:** The corresponding augmented matrix is

$$\left(\begin{array}{cccc|c} 3 & 4 & -1 & 2 & 1 \\ 1 & -2 & 3 & 1 & 2 \\ 3 & 14 & -11 & 1 & 3 \end{array}\right) \widetilde{r_1} \leftrightarrow r_2 \left(\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 2 \\ 3 & 4 & -1 & 2 & 1 \\ 3 & 14 & -11 & 1 & 3 \end{array}\right)$$

$$\begin{array}{l} \widetilde{r_2} \rightarrow r_2 - 3r_1 \\ \widetilde{r_3} \rightarrow r_3 - 3r_1 \end{array} \left(\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 2 \\ 0 & 10 & -10 & -1 & -4 \\ 0 & 20 & -20 & -2 & -3 \end{array}\right)$$

$$\widetilde{r_3} \rightarrow r_3 - 2r_2 \left(\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 2 \\ 0 & 10 & -10 & -1 & -4 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right) \widetilde{r_2} \rightarrow \frac{1}{10} r_2 \left(\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 2 \\ 0 & 1 & -1 & -\frac{1}{10} & -\frac{2}{5} \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Now $\operatorname{rank}(A) = 2, \operatorname{rank}(A|b) = 3$. Since $\operatorname{rank}(A) \neq \operatorname{rank}(A|b)$ , the system is inconsistent. It has no solution.

**Example:** Test the consistency of the following system of linear equations with the help of the rank of the matrix, If consistent solve the system.

$$\begin{aligned} 2x + 2y + 3z &= 2 \\ -x + 3y + z &= 1 \, . \\ x - y + z &= 3 \end{aligned}$$

**Solution:** The corresponding augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 2 & 3 & 2 \\ -1 & 3 & 1 & 1 \\ 1 & -1 & 1 & 3 \end{array}\right) \begin{array}{l} \widetilde{r_2} \rightarrow 2r_2 + r_1 \\ \widetilde{r_3} \rightarrow 2r_3 - r_1 \end{array} \left(\begin{array}{ccc|c} 2 & 2 & 3 & 2 \\ 0 & 8 & 5 & 4 \\ 0 & -4 & -1 & 4 \end{array}\right)$$

$$\widetilde{r_3} \rightarrow 2r_3 + r_2 \left(\begin{array}{ccc|c} 2 & 2 & 3 & 2 \\ 0 & 8 & 5 & 4 \\ 0 & 0 & 3 & 12 \end{array}\right)$$

Now $\operatorname{rank}(A) = 3, \operatorname{rank}(A|b) = 3$. Since $\operatorname{rank}(A) = \operatorname{rank}(A|b)$ , the system is consistent.

Echelon matrix can be written to system of linear equations

$$2x + 2y + 3z = 2, \quad 8y + 5z = 4, 3z = 12$$
$$=> z = 4, \quad y = -2, \qquad x = -3 \quad \mathit{Solved}.$$

**Example: A system of linear equations with exactly one solution**

Consider the system

$2x - y = 1$

$3x + 2y = 12$

Solving the first equation for $y$ in terms of $x$, we obtain the equation

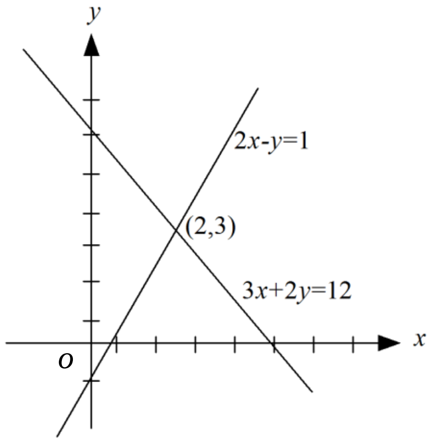
$y = 2x - 1$

Substituting this expression for $y$ into the second equation yields

$$3x+2(2x-1)=12$$
$$\Rightarrow 3x+4x-2=12$$
$$\Rightarrow 7x=14$$
$$\therefore x=2$$

Finally, substituting this value of $x$ into the expression for $y$ gives $y=2(2)-1=3$

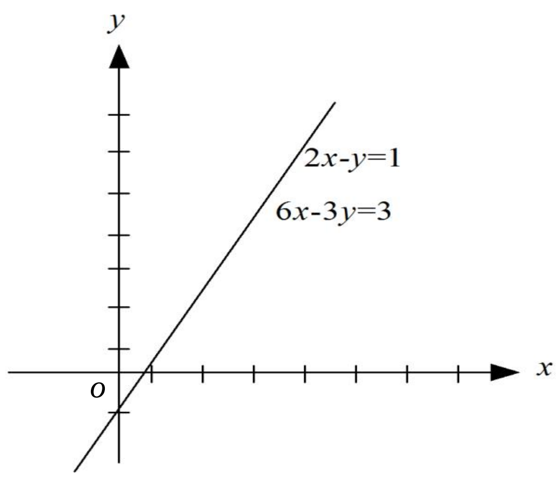
**Graphical representation of the system of linear equations for unique solution (One solution):**




Therefore, the unique solution of the system is given by $x=2$ and $y=3$. Geometrically, the two lines represented by the two equations that make up the system intersect at the point $(2,3)$. So, the solutions are $x=2$ and $y=3$.

**Example: A system of linear equations which are consistent has infinitely many solutions:**
(Graphical representation of the system for infinitely many solutions)

Consider the system $2x-y=1$; $6x-3y=3$.




Solving the first equation for $y$ in terms of $x$, we obtain the equation

$$y=2x-1$$

Substituting this expression for $y$ into the second equation yields

$$6x-3(2x-1)=3$$

$\Rightarrow 6x - 6x + 3 = 3$

$\therefore 0 = 0$

Which is a true statement. This result follows from the fact that the second equation is equivalent to the first. Our computations have revealed that the system of two equations is equivalent to the single equation $2x - y = 1$. Thus, any ordered pair of numbers $(x, y)$ satisfying the equations $2x - y = 1$ or $y = 2x - 1$ constitutes a solution to the system.

In particular, by assigning the value $t$ to $x$, where $t$ is any real number, we find that $y = 2t - 1$ and so the ordered pair $(t, 2t - 1)$ is a solution of the system. The variable $t$ is called a parameter. For example, setting $t = 0$ gives the point $(0, -1)$ as a solution of the system, and setting $t = 1$ gives the point $(1, 1)$ as another solution. Since $t$ represents any real number, there are infinitely many solutions of the system. Geometrically, the two equations in the system represent the same line, and all solutions of the system are points lying on the line (Figure). Such a system is said be dependent.

**Example: A system of linear equations that has no solution:**

Consider the system

$2x - y = 1$

$6x - 3y = 12$

Solving the first equation for $y$ in terms of $x$, we obtain the equation

$y = 2x - 1$

Substituting this expression for $y$ into the second equation yields

$6x - 3(2x - 1) = 12$

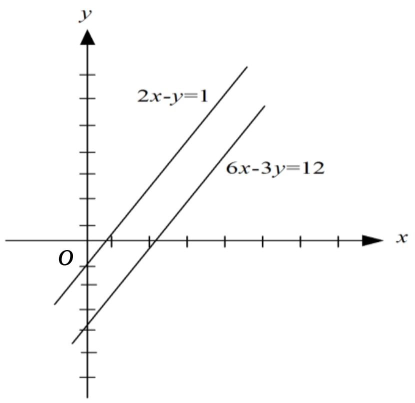
$\Rightarrow 6x - 6x + 3 = 12 \qquad \therefore 0 = 9$

which is clearly untrue. Thus, there is no solution to the system of equations.

$y = 2x - 1$

$y = 2x - 4$

**Graphical representation of the system of linear equations for no solution:**




It has been observed that these two lines are parallel to each other.

**Homogeneous and nonhomogeneous linear equation:**

A system of linear equations is called homogeneous if all the constant terms $b_1, b_2, \ldots, b_n$ of the Non-homogeneous system are zero such as the system has the form:

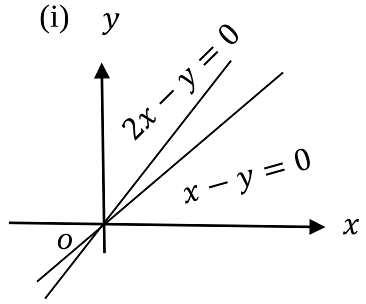
$$\begin{array}{cccccc} a_{11}x_1 + & a_{12}x_2 + & . \; . \; . & + a_{1n}x_n & = 0 \\ a_{21}x_1 + & a_{22}x_2 + & . \; . \; . & + a_{2n}x_n & = 0 \\ . & . & . \; . \; . & . & \\ . & . & . \; . \; . & . & \\ . & . & . \; . \; . & . & \\ a_{m1}x_1 + & a_{m2}x_2 + & . \; . \; . & + a_{mn}x_n & = 0 \end{array}$$

Homogeneous system of linear equations has two types of solutions. They are

(i) Trivial (zero) solution (all $x_i = 0$)

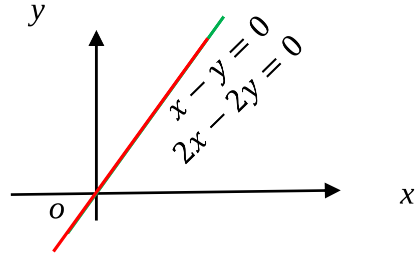
(ii) More than one solutions

(i)




Trivial (zero solution)

(ii)




More than one solutions

**Example:** Write the system of linear equations in augmented matrix form

$$\left.\begin{aligned} 2x + 3y - 4z &= 7 \\ x - 2y - 5z &= 3 \end{aligned}\right\}$$

**Solution:**

The augmented matrix of the above system of linear equations is

$$\begin{pmatrix} 2 & 3 & -4 \vdots 7 \\ 1 & -2 & -5 \vdots 3 \end{pmatrix}.$$

## Solution of linear equation by applying matrices:

**$m$ (no. of linear equations) $= n$(no. of variables) for the system of linear equations:**

Consider, $m$ (no. of linear equations) $= n$(no. of variables) for the system of linear equations $AX = B$.

Let, $D$ be the determinant of the matrix $A$. we have to evaluate the determinant. If $\det(A) = D = 0$, $A$ is singular. So $A^{-1}$ doesn't exist and hence the system has no solution. If $D \neq 0$, A is nonsingular. So, $A^{-1}$ exists and hence the system has a solution. Now multiplying both sides of $AX = B$ by $A^{-1}$, we have

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B$

$\therefore X = A^{-1}B$ since $A^{-1}A = I$, $IX = X$ and $B = (l_i)$.

That is,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_n \end{pmatrix} = \frac{1}{|A|} \begin{pmatrix} A_{11} & A_{12} & . & . & . & A_{1n} \\ A_{21} & A_{22} & . & . & . & A_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ A_{n1} & A_{n2} & . & . & . & A_{nn} \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ . \\ . \\ l_n \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ . \\ . \\ m_n \end{pmatrix} \text{ (say)}$$

Where determinant of the matrix, $A$ is $|A|$. Then, $x_1 = m_1, x_2 = m_2, x_3 = m_3, \ldots\ldots\ldots\ldots x_n = m_n$ (say) is a solution of the given system of $n$ linear equations.
It is to be noted that the solution of the system of equation can also be found by reducing the augmented matrix of the given system to reduced echelon form.

**Note:** For additional details of cofactor method, see lecture note of Chapter–1.

**$m$ (no. of linear equations) $<$ $n$ (no. of unknowns or variables) of the following system of linear equations:**

After reduced the system of linear equations (1) into echelon form,

(i) Number of variable(s) is equal to the number of equation(s) gives the unique solution
(ii) Number of variable(s) is greater than the number of equation(s) gives more than one solution.

**Example of the algorithm**
Suppose the goal is to find and describe the set of solutions to the following system of linear equations:

$2x + y - z = 8 \qquad (r_1)$
$-3x - y + 2z = -11 \qquad (r_2)$
$-2x + y + 2z = -3 \qquad (r_3)$

The table below is the row reduction process applied simultaneously to the system of equations, and its associated augmented matrix. The row reduction procedure may be summarized as follows: eliminate $x$ from all equations below $r_1$, and then eliminate $y$ from all equations below $r_2$. This will put the system into triangular form. Then, using back-substitution, each unknown can be solved.

| System of equations | row operations | augmented matrix |
|---|---|---|
| $2x + y - z = 8$<br>$-3x - y + 2z = -11$<br>$-2x + y + 2z = -3$ | | $\left(\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right)$ |
| $2x + y - z = 8$<br>$y + z = 2$<br>$2y + z = 5$ | $r_2 \to 2r_2 + 3r_1$<br>$r_3 \to r_3 + r_1$ | $\left(\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 5 \end{array}\right)$ |
| $2x + y - z = 8$<br>$y + z = 2$<br>$-z = 1$ | $r_3 \to r_3 - 2r_2$ | $\left(\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{array}\right)$ |

The matrix is now in echelon form

| | | |
|---|---|---|
| $2x + y \quad = 7$<br>$y \quad = 3$<br>$-z = 1$ | $r_1 \to r_1 - r_3$<br>$r_2 \to r_2 + r_3$ | $\left(\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 1 \end{array}\right)$ |
| $2x \quad = 4$<br>$y \quad = 3$<br>$-z = 1$ | $r_1 \to r_1 - r_2$ | $\left(\begin{array}{ccc\|c} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 1 \end{array}\right)$ |
| $x \quad = 2$<br>$y \quad = 3$<br>$z = -1$ | $r_1 \to \frac{1}{2} r_1$<br>$r_2 \to -r_2$ | $\left(\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right)$ |

The solution is $z = -1$, $y = 3$, and $x = 2$. So, there is a unique solution to the original system of equations.

**Example:** Solve the following system of equations using Gaussian elimination method

$$
\begin{aligned}
2x + y + 3z &= 1 \qquad (r_1) \\
2x + 6y + 8z &= 3 \qquad (r_2) \\
6x + 8y + 18z &= 5 \qquad (r_3)
\end{aligned}
$$

$r_2 \to r_2 - r_1$
$r_3 \to r_3 - 3r_1$

$$\left(\begin{array}{ccc\vdots c} 2 & 1 & 3 & 1 \\ 0 & 5 & 5 & 2 \\ 0 & 5 & 9 & 2 \end{array}\right)$$

$r_3 \to r_3 - r_2$

$$\left(\begin{array}{ccc\vdots c} 2 & 1 & 3 & 1 \\ 0 & 5 & 5 & 2 \\ 0 & 0 & 4 & 0 \end{array}\right)$$

From this stage, we can get the solution by back solving

$z = 0$
$5y + 5(0) = 2$
$\Rightarrow y = \dfrac{2}{5}$
and $2x + \frac{2}{5} + 3(0) = 1 \qquad \Rightarrow x = \frac{3}{10}$

So, the solution is:

$$(x, y, z) = \left(\frac{3}{10}, \frac{2}{5}, 0\right)$$

**Example:** Solve the following system of equations using elementary row operations

$$
\begin{aligned}
3x + y - 6z &= -10 \qquad (r_1) \\
2x + y - 5z &= -8 \qquad (r_2) \\
6x - 3y + 3z &= 0 \qquad (r_3)
\end{aligned}
$$

$r_2 \to 3r_2 - 2r_1$
$r_3 \to r_3 - 2r_1$

$$\left(\begin{array}{ccc\vdots c} 3 & 1 & -6 & -10 \\ 0 & 1 & -3 & -4 \\ 0 & 1 & -3 & -4 \end{array}\right)$$

$r_3 \to r_3 - r_2$

$$\left(\begin{array}{ccc\vdots c} 3 & 1 & -6 & -10 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let, $z = a$, where $a$ is a free variable.

We have,
$y = 3a - 4$
$\therefore x + 3z - 4 - 6z = -10$
$\Rightarrow x = a - 2$
So, the general solution of the system is $(x, y, z) = (a - 2, 3a - 4, a)$.
For particular solution, putting $a = 1$ (putting any suitable value)

$$(x, y, z) = (-1, -1, 1)$$

**Example:** Solve the following system of equations using Gaussian elimination method

$x \quad + z = 1 \qquad (r_1)$
$x + y + z = 2 \qquad (r_2)$
$x - y + z = 1 \qquad (r_3)$

$r_2 \to r_2 - r_1$
$r_3 \to r_3 - r_1$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 \vdots & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$r_3 \to r_3 + r_2$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 \vdots & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The third row $'0 = 1'$ it does not exist. So, the system is inconsistent. That means the system has no solution.

**Example:** Solve the following system using Gauss-Jordan elimination method.

$$2x - y + z = 1$$
$$x + 4y - 3z = -2$$
$$3x + 2y - z = 0$$

**<u>Solution</u>**

$$\begin{pmatrix} 2 & -1 & 1 & 1 \\ 1 & 4 & -3 \vdots & -2 \\ 3 & 2 & -1 & 0 \end{pmatrix} \underset{r_1 \leftrightarrow r_2}{\sim} \begin{pmatrix} 1 & 4 & -3 & -2 \\ 2 & -1 & 1 \vdots & 1 \\ 3 & 2 & -1 & 0 \end{pmatrix} \begin{matrix} \widetilde{r_2} \to r_2 - 2r_1 \\ \\ \widetilde{r_3} \to r_3 - 3r_1 \end{matrix} \begin{pmatrix} 1 & 4 & -3 & -2 \\ 0 & -9 & 7 \vdots & 5 \\ 0 & -10 & 8 & 6 \end{pmatrix}$$

$$\underset{r_2}{\sim} \to r_2 - r_3 \begin{pmatrix} 1 & 4 & -3 & -2 \\ 0 & 1 & -1 \vdots & -1 \\ 0 & -10 & 8 & 6 \end{pmatrix} \underset{r_3}{\sim} \to r_3 + 10r_2 \begin{pmatrix} 1 & 4 & -3 & -2 \\ 0 & 1 & -1 \vdots & -1 \\ 0 & 0 & -2 & -4 \end{pmatrix}$$

$$\underset{r_3}{\sim} \to -\tfrac{1}{2}r_3 \begin{pmatrix} 1 & 4 & -3 & -2 \\ 0 & 1 & -1 \vdots & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \underset{r_1}{\sim} \to r_1 - 4r_2 \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 \vdots & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{matrix} \widetilde{r_1} \to r_1 - r_3 \\ \widetilde{r_2} \to r_2 + r_3 \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 \vdots & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Hence the solution is $x = 0, y = 1, z = 2$.

**Example:** Solve the following system of equations using matrix inversion and justify your answer.

$$\left.\begin{aligned} 2x + y &= 1 \ \ (r_1) \\ x - 2y &= 3 \ \ (r_2) \end{aligned}\right\} \ (1)$$

**Solution:** System (1) is written in the matrix form

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$AX = B$

$X = A^{-1}B$

Where, $A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, and $B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

The determinant of the matrix $A$ is $\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -6 \neq 0$

So, the matrix $A$ is non-singular and $A^{-1}$ exists.

Now $A^{-1} = \frac{1}{-5}\begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix}$

And $X = A^{-1}B$

$$= \frac{1}{-5}\begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$= \frac{1}{-5}\begin{pmatrix} -5 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore x = 1$ and $y = -1$

Verification:

$$r_1: L.H.S = 2x + y = 2.1 - 1 = 1 = R.H.S$$
$$r_1: L.H.S = x - 2y = 1 - 2(-1) = 3 = R.H.S$$

**Example:** Solve the following system of equations using matrix inversion.

$$2x - y + 3z = 53$$
$$4x - z = -53$$
$$3x + 3y + 2z = 106$$

**Solution:** We write down the given system as

$$\begin{pmatrix} 2 & -1 & 3 \\ 4 & 0 & -1 \\ 3 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 53 \\ -53 \\ 106 \end{pmatrix}$$

$\Rightarrow \quad A\,X = B \qquad$ [ say]

$\Rightarrow \quad X = A^{-1}\,B \qquad$ [ since $A^{-1}$exists]

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{53}\begin{pmatrix} 3 & 11 & 1 \\ -11 & -5 & 14 \\ 12 & -9 & 4 \end{pmatrix}\begin{pmatrix} 53 \\ -53 \\ 106 \end{pmatrix} = \begin{pmatrix} -6 \\ 22 \\ 29 \end{pmatrix} \qquad \text{[ after finding } A^{-1}\text{]}$$

$\Rightarrow \quad x = -6, y = 22, z = 29$ is the required solution.

**Example**: Using matrix inversion solve the system of linear equations

$$x_1 + 2x_2 + 3x_3 = 5$$
$$2x_1 + 5x_2 + 3x_3 = 3 \; .$$
$$x_1 + 8x_3 = 17$$

**Solution:**

The system of equations can be written in the matrix form as $AX = B$ , where $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $B = \begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix}$. The solution can be written as $X = A^{-1}B$. Let us find $A^{-1}$ using elementary row operations.

$$[A \vdots I] = \left(\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \underset{\substack{r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - r_1}}{\sim} \left(\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right) \underset{r_3 \to r_3 + 2r_2}{\sim}$$

$$\left(\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \underset{r_1 \to r_1 - 2r_2}{\sim} \left(\begin{array}{ccc:ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \underset{r_1 \to r_1 + 9r_3}{\sim}$$

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \underset{\substack{r_2 \to r_2 - 3r_3 \\ r_3 \to -r_3}}{\sim} \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

Now, $X = A^{-1}B = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 17 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ $x_1 = 1, x_2 = -1$ and $x_3 = 2$.

## Cramer's Rule:

Let a system of linear equations is given $AX = B$ and $\det(A) = D$. Then the solution is given by

$$x_i = \frac{D_i}{D} \; (i = 1, 2, \ldots\ldots, n),$$

$D_i$ can be obtained by replacing i[th] column by right hand side.

**Explicit formulas for small systems**

Consider the linear system $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$

which in matrix format is $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Assume $a_1b_2 - b_1a_2$ nonzero. Then, with help of determinants $x$ and $y$ can be found with Cramer's rule as

$$x=\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}=\frac{c_1b_2-b_1c_2}{a_1b_2-b_1a_2} \quad \text{and} \quad y=\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}=\frac{a_1c_2-c_1a_2}{a_1b_2-b_1a_2}.$$

The rules for $3 \times 3$ matrices are similar. Given

$$\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{cases}$$

which in matrix format is

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Then the values of $x$, $y$ and $z$ can be found as follows:

$$x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, y=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \text{ and } z=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Similar idea can be extended for $n \times n$ systems

**Example:**

Let us demonstrate Cramer's rule with the following system:

$$\begin{aligned} x+2y+3z&=1 \\ -x \qquad +2z&=2 \\ -2y+z&=-2 \end{aligned}$$

**Step 1:**

The coefficient matrix of this system is $A=\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 0 & -2 & 1 \end{pmatrix}$

Note that the matrix is square (it has 3 rows and 3 columns), and so we may proceed with the next step of Cramer's rule.

**Step 2:**

Now find the determinant of the coefficient matrix $A$; use the matrix manipulator in the tools box if you would like help in this computation. You should get $|A|=12$. This is not zero, so Cramer's rule may be applied here.

**Step 3:**

$A_x=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ -2 & -2 & 1 \end{pmatrix}$ and its determinant is $|A_x|=-20$. Therefore $x=\frac{|A_x|}{|A|}=-\frac{20}{12}=-\frac{5}{3}$

**Step 4:**

Using the same method, the values for the remaining 2 variables, $x$ and $y$, are computed below:

$A_y = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 2 & 2 \\ 0 & -2 & 1 \end{pmatrix}$ and its determinant is $|A_y| = 13$. Therefore $y = \frac{|A_y|}{|A|} = -\frac{13}{12}$

$A_z = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 0 & -2 & -2 \end{pmatrix}$ and its determinant is $|A_z| = 2$. Therefore $z = \frac{|A_z|}{|A|} = \frac{2}{12} = \frac{1}{6}$

**Example:** Verify whether the following system of linear equations is consistent. If consistent then solve the system using Cramer's rule. Also check your answer.

$2x - 3y - 5z = 40$
$17x + 14y - 22\,z = 22$
$15x + 17y - 17z = -18$ .

**Solution:**

$D = \begin{vmatrix} 2 & -3 & -5 \\ 17 & 14 & -22 \\ 15 & 17 & -17 \end{vmatrix} = 0$ and $\neq \mathbf{0}$ . Therefore, this SLE is either inconsistent or more than one solutions.

**Example:**

$2x - 5y + 6z = -27$
$10x - 11y - 9z = 0$
$-3x \qquad + 2z = 16$

**Solution:**

$D = \begin{vmatrix} 2 & -5 & 6 \\ 10 & -11 & -9 \\ -3 & 0 & 2 \end{vmatrix} = -277$ and $B \neq 0$ and, therefore, this SLE has a unique solution.

Now $D_1 = \begin{vmatrix} -27 & -5 & 6 \\ 0 & -11 & -9 \\ 16 & 0 & 2 \end{vmatrix}$ ; $D_2 = \begin{vmatrix} 2 & -27 & 6 \\ 10 & 0 & -9 \\ -3 & 16 & 2 \end{vmatrix}$ ; $D_3 = \begin{vmatrix} 2 & -5 & -27 \\ 10 & -11 & 0 \\ -3 & 0 & 16 \end{vmatrix}$ .

$= 2370$ ; $= 1059$ ; $= 1339$.

Thus $x = -\dfrac{2370}{277}$ ; $y = -\dfrac{1059}{277}$ ; $z = -\dfrac{1339}{277}$ .

**Example:**

Solve the following system of linear equations using Cramer's rule

$2x + y + z = 3$
$x - y - z = 0$
$x + 2y + z = 0$

We have the left-hand side of the system with the variables (the "coefficient matrix") and the right-hand side with the answer values. Let $D$ be the determinant of the coefficient matrix of the above system, and let $D_x$ be the determinant formed by replacing the $x$-column values with the answer-column values: Evaluating each determinant, we get:

$$D = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3$$

$$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix} = 3, \quad D_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = -6, \quad D_z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 9$$

So, according to Cramer's rule:

$$x = \frac{D_x}{D} = \frac{3}{3} = 1, \quad y = \frac{D_y}{D} = \frac{-6}{3} = -2 \text{ and } z = \frac{D_z}{D} = \frac{9}{3} = 3$$

**Example:**
Solve the following system of linear equations using Crammer's rule
$2x + 3y + z = 10$
$x - y + z = 4$
$4x - y - 5z = -8$ .

**Solution:** Each unknown will be the quotient of the determinant obtained by substituting the answers in the right sides of the equations for the coefficients of the unknown divided by the determinant formed by taking the coefficients on the left sides of the equations.

$$x = \frac{\begin{vmatrix} 10 & 3 & 1 \\ 4 & -1 & 1 \\ -8 & -1 & -5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & -5 \end{vmatrix}} = \frac{84}{42} = 2, \; y = \frac{\begin{vmatrix} 2 & 10 & 1 \\ 1 & 4 & 1 \\ 4 & -8 & -5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & -5 \end{vmatrix}} = \frac{42}{42} = 1, \; z = \frac{\begin{vmatrix} 2 & 3 & 10 \\ 1 & -1 & 4 \\ 4 & -1 & -8 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 1 \\ 4 & -1 & -5 \end{vmatrix}} = \frac{126}{42} = 3$$

**Example:**
Determine the value(s) of $\lambda$ and $\mu$ such that the following system of linear equations has (i) no solution, (ii) more than one solution, and (iii) a unique solution.

$$\begin{cases} x + y + z = 6 \\ x + 2y + 3z = 10 \\ x + 2y + \lambda z = \mu \end{cases}$$

**Solution:** The corresponding augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right) \quad \begin{array}{c} \sim \\ r_2 \to r_2 - r_1 \\ r_3 \to r_3 - r_2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 6 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right)$$

The above system is in echelon form. Now we consider the following three cases:

(i) If $\lambda = 3$ and $\mu \neq 10$ then third equation of (1) is of the form $0 = a$, where $a = \mu - 10 \neq 0$ which is not true. So, the system is inconsistent. Thus, the system has no solution for $\lambda = 3$ and $\mu \neq 10$.

(ii) If $\lambda = 3$ and $\mu = 10$ then third equation of (1) is vanishes and the system will be in echelon form having two equations in three variables. So, it has $3 - 2 = 1$ free variables which is $z$. Hence the given system has more than one solution for $\lambda = 3$ and $\mu = 10$.

(iii) For a unique solution, the coefficient of $z$ in the 3rd equation must be non-zero i.e., $\lambda \neq 3$ and $\mu$ may have any value. Therefore, the given system has unique solution for $\lambda \neq 3$ and arbitrary values of $\mu$.

**Example:**
Determine the value(s) of $\lambda$ and $\mu$ such that the following system of linear equations has (i) no solution (ii) more than one solution and (iii) a unique solution.

$$\begin{cases} x+y-z=1 \\ 2x+3y+\lambda z=3 \\ x+\lambda y+3z=2 \end{cases}.$$

**Solution:** The given system of liner equations is $\begin{cases} x+y-z=1 \\ 2x+3y+\lambda z=3 \\ x+\lambda y+3z=2 \end{cases}$

The corresponding augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & \lambda & 3 \\ 1 & \lambda & 3 & 2 \end{array}\right) \underset{\substack{r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 - r_1}}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & \lambda-1 & 4 & 1 \end{array}\right)$$

$$\underset{r_3 \to r_3 + r_2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & 0 & 4-(\lambda-1)(\lambda+2) & 2-\lambda \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & 0 & 6-\lambda-\lambda^2 & 2-\lambda \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & 0 & (\lambda+3)(2-\lambda) & 2-\lambda \end{array}\right)$$

The above system is in echelon form. Now we consider the following three cases:

(i) From third equation of $(1)$, we see that if $\lambda+3=0$ or $\lambda=-3$ then the equation becomes $0=5$, which is contradiction. Therefore, the system is inconsistent if $\lambda=-3$. Thus, the system has no solution for $\lambda=-3$.

(ii) We know, if the number of variables is greater than the number of equations, then the system has more than one solution. From third equation of $(1)$, we see that if $\lambda=2$ then it becomes $0=0$. In this case the system has three variables with two equations. So, the given system has more than one solution for $\lambda=2$.

(iii) We know, if the number of variables and the number of equations be equal, then the system has unique solution. The system $(1)$ has a unique solution
$(\lambda+3)(2-\lambda) \neq 0 \Rightarrow \lambda \neq -3, \lambda \neq 2.$

**Example:**
Determine the value(s) of $\lambda$ and $\mu$ such that the following system of linear equations has (i) no solution, (ii) more than one solution, and (iii) a unique solution.

$$\begin{cases} 2x+3y+z=5 \\ 3x-y+\lambda z=2 \\ x+7y-6z=\mu \end{cases}$$

**Solution:** The given system of liner equations is

$$\begin{cases} 2x+3y+z=5 \\ 3x-y+\lambda z=2 \\ x+7y-6z=\mu \end{cases}$$

The corresponding augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 5 \\ 3 & -1 & \lambda & 2 \\ 1 & 7 & -6 & \mu \end{array}\right) \underset{\substack{r_2 \to 2r_2 - 3r_1 \\ r_3 \to 2r_3 - r_1}}{\sim} \left(\begin{array}{ccc|c} 2 & 3 & 1 & 5 \\ 0 & -11 & 2\lambda-3 & -11 \\ 0 & 11 & -13 & 2\mu-5 \end{array}\right)$$

$$\underset{r_3 \to r_3 + r_2}{\sim} \left(\begin{array}{ccc|c} 2 & 3 & 1 & 5 \\ 0 & -11 & 2\lambda-3 & -11 \\ 0 & 0 & 2(\lambda-8) & 2(\mu-8) \end{array}\right)$$

The above system is in echelon form. Now we consider the following three cases:

(i) For a unique solution, the coefficient of $z$ in the 3rd equation of $(1)$ must be non-zero i.e., $\lambda \neq 8$ and $\mu$ may have many values. Therefore, the given system has unique solution for $\lambda \neq 8$ and arbitrary values of $\mu$.

(ii) If $\lambda = 8$ and $\mu = 8$ then third equation of $(1)$ is vanishes and the system will be in echelon form having two equations in three variables. So, it has $3 - 2 = 1$ free variables which is $z$. Hence the given system has more than one solution for $\lambda = 8$ and $\mu = 8$..

**(iii)** If $\lambda = 8$ and $\mu \neq 8$ then third equation of $(1)$ is of the form $0 = a$, where $a = \mu - 8 \neq 0$ which is not true. So, the system is inconsistent. Thus, the system has no solution for $\lambda = 8$ and $\mu \neq 8$.

**Example:**
Find the values of $k$ such that the following system of linear equations has non-zero solution.
$x + ky + 3z = 0$
$4x + 3y + kz = 0.$
$2x + y + 2z = 0$

**Solution:**
The augmented matrix $C = (A{:}\,B)$

$$\sim \begin{pmatrix} 1 & k & 3 & : & 0 \\ 4 & 3 & k & : & 0 \\ 2 & 1 & 2 & : & 0 \end{pmatrix}$$

On interchanging first row and third row, we have

$$\sim \begin{pmatrix} 2 & 1 & 2 & : & 0 \\ 4 & 3 & k & : & 0 \\ 1 & k & 3 & : & 0 \end{pmatrix}$$

Reducing the system to row echelon form by the elementary row operations …

$$\sim \begin{pmatrix} 2 & 1 & 2 & : & 0 \\ 0 & 1 & k-4 & : & 0 \\ 0 & k-\frac{1}{2} & 2 & : & 0 \end{pmatrix} \quad \begin{bmatrix} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - \frac{1}{2}R_1 \end{bmatrix}$$

$$\sim \begin{pmatrix} 2 & 1 & 2 & : & 0 \\ 0 & 1 & k-4 & : & 0 \\ 0 & 0 & 2-\left(k-\frac{1}{2}\right)(k-4) & : & 0 \end{pmatrix} \quad \left[R_3' = R_3 - (k-\tfrac{1}{2})R_2\right]$$

So,

$$2-\left(k-\frac{1}{2}\right)(k-4) = 0$$

$$\Rightarrow -k^2 + \frac{9}{2}k = 0$$

$$\Rightarrow k\left(-k+\frac{9}{2}\right) = 0$$

$$\therefore \quad k = 0, \frac{9}{2}$$

**Example:** A medicine company wishes to produce three types of medicine : type $P, Q$ and $R$. To manufacture a type $P$ medicine requires 2 minutes each on machine $I$ and $II$ and 3 minutes on machine $III$. A type of $Q$ medicine requires 2 minutes on machine $I$, 3 minutes on machine $II$ and 4 minutes of machine $III$. A type $R$ medicine requires 3 minutes on machine $I$, 4 minutes on machine $II$ and 3 minutes on machine $III$. There are 3.5 hours available on machine 1, 4.5 hours available on machine $II$ and 5 hours available on machine $III$. How many medicine of each type should company make in order to use all the available time?

**Solution:**

Here, 3.5 hours $= 210$ minutes, 4.5 hours $= 270$ minutes and 5 hours $= 300$ minutes.

Let $x$, $y$ and $z$ be the number of medicines of types $P$, $Q$ and $R$ respectively. Then we have the following system of linear equations: $\begin{cases} 2x+2y+3z = 210 \\ 2x+3y+4z = 270 \\ 3x+4y+3z = 300 \end{cases}$

The augmented matrix of the above system is

$$\begin{pmatrix} 2 & 2 & 3 & : & 210 \\ 2 & 3 & 4 & : & 270 \\ 3 & 4 & 3 & : & 300 \end{pmatrix}$$

Reducing the system to echelon form by the elementary row operations

$$\sim \begin{pmatrix} 2 & 2 & 3 & : & 210 \\ 0 & 1 & 1 & : & 60 \\ 0 & 2 & -3 & : & -30 \end{pmatrix} \begin{bmatrix} r_2' = r_2 - r_1 \\ r_3' = 2r_3 - 3r_1 \end{bmatrix}$$

$$\sim \begin{pmatrix} 2 & 2 & 3 & : & 210 \\ 0 & 1 & 1 & : & 60 \\ 0 & 0 & -5 & : & -150 \end{pmatrix} [r_3' = r_3 - 2r_1]$$

Hence the solution of the above system is $x = 30, y = 30, z = 30$

Thus, the number of each type of medicine is $30$.

**Example:** Determine the polynomial $p(x) = a_0 + a_1x + a_2x^2$ whose graph passes through the points $(1,4)$, $(2,0)$ and $(3,12)$.

**Solution:**
Given polynomial $p(x) = a_0 + a_1x + a_2x^2$
Substituting $x = 1, 2$ and $3$ into $p(x)$ and the corresponding $y-$values produces the system of linear equations in the variables $a_0, a_1$ and $a_2$ shown below:

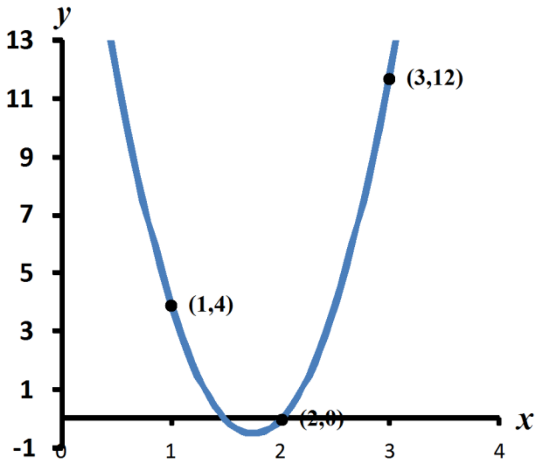
$$\begin{cases} p(1) = a_0 + a_1(1) + a_2(1)^2 = a_0 + a_1 + a_2 = 4 \\ p(2) = a_0 + a_1(2) + a_2(2)^2 = a_0 + 2a_1 + 4a_2 = 0 \\ p(3) = a_0 + a_1(3) + a_2(3)^2 = a_0 + 3a_1 + 9a_2 = 12 \end{cases}$$

Reducing this system to echelon form by the elementary operations,

$$\begin{cases} a_0 + a_1 + a_2 = 4 \\ a_1 + 3a_2 = -4 \\ a_1 + 5a_2 = 12 \end{cases} \quad \begin{bmatrix} L_2' = L_2 - L_1 \\ L3' = L_3 - L_2 \end{bmatrix}$$

$$\begin{cases} a_0 + a_1 + a_2 = 4 \\ a_1 + 3a_2 = -4 \\ 2a_2 = 16 \end{cases} \quad \begin{bmatrix} L_3' = L_3 - L_2 \end{bmatrix}$$

By back substitution method from 3rd equation, we have $a_2 = 8$
From the 2nd equation, we get $a_1 + 24 = -4 \therefore a_1 = -28$
and from 1st equation, we get $a_0 - 28 + 8 = 4 \therefore a_0 = 24$
Hence, the solution of this system is $a_0 = 24$, $a_1 = -28$ and $a_2 = 8$.
So, the polynomial function is $p(x) = 24 - 28x + 8x^2$
The graph $p$ is shown in the following figure:




**Example:** Find the polynomial that fits the points $(-2, 3), (-1, 5), (0, 1), (1, 4)$ and $(2, 10)$.

**Solution:** We have provided five points, so we chose a fourth-degree polynomial function

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

Substitution the given points into $p(x)$ products the system of linear equations listed below:

$$\begin{aligned} a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4 &= 3 \\ a_0 - a_1 + a_2 - a_3 + a_4 &= 5 \\ a_0 &= 1 \\ a_0 + a_1 + a_2 + a_3 + a_4 &= 4 \\ a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 &= 10 \end{aligned}$$

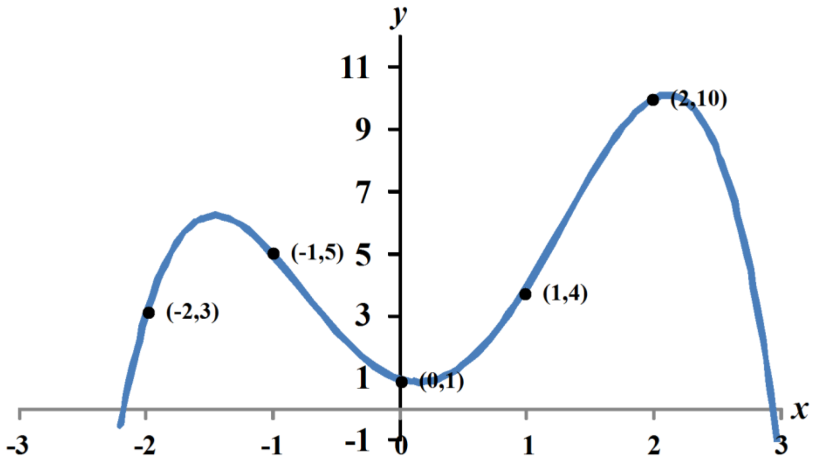
The solution of the above system is

$$a_0 = 1, \quad a_1 = -\frac{5}{24}, \quad a_2 = \frac{101}{24}, a_3 = \frac{18}{24}, a_4 = -\frac{17}{24}$$

Which means the polynomial function is

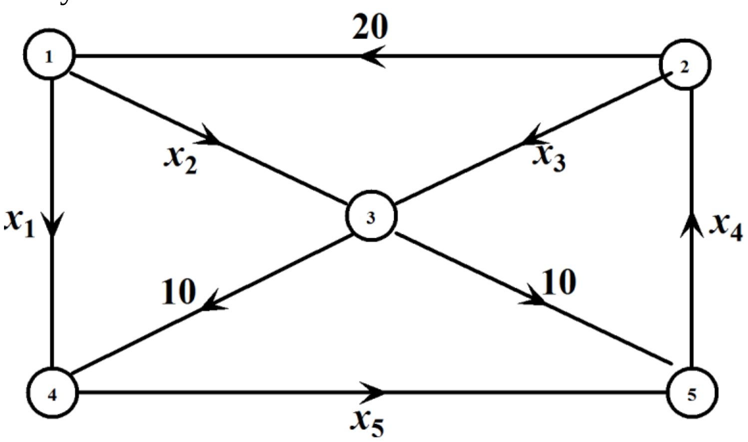
$$p(x) = 1 - \frac{5}{24}x + \frac{101}{24}x^2 + \frac{18}{24}x^3 - \frac{17}{24}x^4 = \frac{1}{24}(24 - 5x + 101x^2 + 18x^3 - 17x^4)$$

The graph of $p(x)$ is shown in the following figure:




**Example:**

Set up a system of linear equations to represent the network shown in the following figure and solve the system.




**Solution:**

Each of the network's five junctions gives rise to a linear equation, as shown below:

$x_1 + x_2 = 20$ junction 1
$x_3 + 20 = x_4$ junction 2
$x_2 + x_3 = 10 + 10$ junction 3
$x_1 + 10 = x_5$ junction 4
$x_5 + 10 = x_4$ junction 5

The augmented matrix is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & : & 20 \\ 0 & 0 & 1 & -1 & 0 & : & -20 \\ 0 & 1 & 1 & 0 & 0 & : & 20 \\ 1 & 0 & 0 & 0 & -1 & : & -10 \\ 0 & 0 & 0 & -1 & 1 & : & -10 \end{pmatrix}$$

Reduce the system to echelon form by the elementary row operations

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & : & 20 \\ 0 & 1 & 1 & 0 & 0 & : & 20 \\ 0 & 0 & 1 & -1 & 0 & : & -20 \\ 0 & 0 & 0 & -1 & 1 & : & -10 \\ 1 & 0 & 0 & 0 & -1 & : & -10 \end{pmatrix} \begin{bmatrix} R_2 \leftrightarrow R_3 \\ R_4 \leftrightarrow R_5 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & : & 20 \\ 0 & 1 & 1 & 0 & 0 & : & 20 \\ 0 & 0 & 1 & -1 & 0 & : & -20 \\ 0 & 0 & 0 & -1 & 1 & : & -10 \\ 0 & -1 & 0 & 0 & -1 & : & -30 \end{pmatrix} \begin{bmatrix} R_5' = R_5 - R_1 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & : & 20 \\ 0 & 1 & 1 & 0 & 0 & : & 20 \\ 0 & 0 & 1 & -1 & 0 & : & -20 \\ 0 & 0 & 0 & -1 & 1 & : & -10 \\ 0 & 0 & 1 & 0 & -1 & : & -10 \end{pmatrix} \begin{bmatrix} R_5' = R_5 + R_2 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & : & 20 \\ 0 & 1 & 1 & 0 & 0 & : & 20 \\ 0 & 0 & 1 & -1 & 0 & : & -20 \\ 0 & 0 & 0 & 1 & -1 & : & 10 \\ 0 & 0 & 0 & 1 & -1 & : & 10 \end{pmatrix} \begin{bmatrix} R_4' = (-1)R_4 \\ R_5' = R_5 - R_3 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & : & 20 \\ 0 & 1 & 1 & 0 & 0 & : & 20 \\ 0 & 0 & 1 & 0 & -1 & : & -10 \\ 0 & 0 & 0 & 1 & -1 & : & 10 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{pmatrix} \begin{bmatrix} R_3' = R_3 + R_4 \\ R_5' = R_5 - R_4 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & : & 20 \\ 0 & 1 & 0 & 0 & 1 & : & 30 \\ 0 & 0 & 1 & 0 & -1 & : & -10(\ ) \\ 0 & 0 & 0 & 1 & -1 & : & 10 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{pmatrix} \left[R_2^{/} = R_2 - R_3\right]$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & : & -10 \\ 0 & 1 & 0 & 0 & 1 & : & 30 \\ 0 & 0 & 1 & 0 & -1 & : & -10 \\ 0 & 0 & 0 & 1 & -1 & : & 10 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{pmatrix} \left[R_1^{/} = R_1 - R_2\right]$$

The corresponding system of equations are as follows:

$$\begin{cases} x_1 \qquad\qquad\qquad - x_5 = -10 \\ \qquad x_2 \qquad\qquad\quad x_5 = 30 \\ \qquad\qquad x_3 \qquad - x_5 = -10 \\ \qquad\qquad\qquad x_4 - x_5 = 10 \end{cases}$$

The above system is in echelon form having 4 equations in 5 unknowns. So, it has $(5-4)=1$ free variable, which is $x_5$.

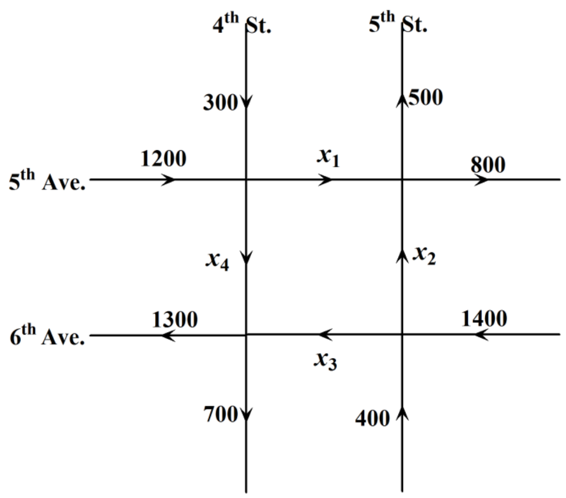
Let $x_5 = t$, then by back substitution method, we have

$x_4 = t+10,\ x_3 = t-10,\ x_2 = 30-t,\ x_1 = t-10,$ where $t$ is a real number.

So, this system has an infinite number of solutions.

**Example:**

The following figure shows the flow of downtown traffic in a current city during the rush hours on a typical weekday. The arrows indicate the direction of traffic flow on each-way road, and the average number of vehicles per hour entering and leaving each intersection appears beside each road. 5th Avenue and 6th Avenue can each handle up to 2000 vehicles per hour without causing congestion, whereas the maximum capacity of both 4th street and 5th street is 1000 vehicles per hour. The flow of traffic is controlled by traffic lights installed at each of the four intersections.



(a) Write a general expression involving the rates of flow $x_1, x_2, x_3, x_4$ and suggest two possible flow patterns that will ensure no traffic congestion.
(b) Suppose the part of 4th street between 5th Avenue and 6th Avenue is to be resurfaced and that traffic flow between the two junctions must therefore be reduced to at most 300 vehicles per hour. Find two possible flow patterns that will result in a smooth flow of traffic.

**Solution:**
(a) To avoid congestion, all traffic entering an intersection must also leave that intersection. Applying this condition to each of the four intersections in a clockwise direction beginning with the 5th Avenue and 4th Street intersection, we obtain the following equations:

$$\begin{aligned} 1500 &= x_1 + x_4 \\ 1300 &= x_1 + x_2 \\ 1800 &= x_2 + x_3 \\ 2000 &= x_3 + x_4 \end{aligned}$$

This system of four linear equations in the four variables $x_1, x_2, x_3, x_4$ may be written in the more standard form

$$\begin{array}{ccccccccc} x_1 & & & & & + x_4 & = & 1500 \\ x_1 & + x_2 & & & & & = & 1300 \\ & x_2 & & + x_3 & & & = & 1800 \\ & & & x_3 & & + x_4 & = & 2000 \end{array}$$

Using Gauss-Jordan elimination method, we obtain

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 1 & 1 & 0 & 0 & 1300 \\ 0 & 1 & 1 & 0 & 1800 \\ 0 & 0 & 1 & 1 & 2000 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 0 & 1 & 0 & -1 & -200 \\ 0 & 1 & 1 & 0 & 1800 \\ 0 & 0 & 1 & 1 & 2000 \end{array}\right) \left[R_2' = R_2 - R_1\right]$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 0 & 1 & 0 & -1 & -200 \\ 0 & 0 & 1 & 1 & 2000 \\ 0 & 0 & 1 & 1 & 2000 \end{array}\right) \left[R_3' = R_3 - R_2\right]$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 0 & 1 & 0 & -1 & -200 \\ 0 & 0 & 1 & 1 & 2000 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \left[R_4' = R_4 - R_3\right]$$

The last augmented matrix is in row-reduced form and is equivalent to a system of three linear equations in the four variables $x_1, x_2, x_3, x_4$. This we may express three of the variables-say, $x_1, x_2, x_3$ in terms of $x_4$. Setting $x_4 = t$ ($t$ a parameter), we may write the infinitely many solutions of the system as

$$x_1 = 1500 - t$$
$$x_2 = -200 + t$$
$$x_3 = 2000 - t$$
$$x_4 = t$$

Observe that for a meaningful solution we must have $200 \le t \le 1500$ since $x_1, x_2, x_3$ and $x_4$ must all be nonnegative and the maximum capacity of a street is $1500$.
For example, picking $t = 300$ gives the flow pattern

$$x_1 = 1200,\ x_2 = 100,\ x_3 = 1700,\ x_4 = 300$$

Selecting $t = 500$ gives the flow pattern

$$x_1 = 1000,\ x_2 = 300,\ x_3 = 1500,\ x_4 = 500$$

(b) In this case, $x_4$ must not exceed $300$. Again, using results of part$(a)$, we find, upon setting $x_4 = t = 300$, the flow pattern $x_1 = 1200, x_2 = 100, x_3 = 1700, x_4 = 300$ obtained earlier.

(c) Picking $t = 250$ gives the flow pattern $x_1 = 1250, x_2 = 50, x_3 = 1750, x_4 = 250.$

# Linear Programming Problem:

The linear programming is the modern method of mathematics to solve the system of linear inequalities. The solution makes the objective linear function a minimum (or maximum) and which satisfies the constraints and non-negative conditions.
General linear programming problems:
Let be $Z$ a linear function by $Z = \sum_{i=1}^{n} c_i x_i \dots \dots \dots \dots \dots \dots \dots (i)$
where $c_i$ is set of $n$ constants.
Let $a_{ij}$ be $mn$ constants and $b_i$ be a set of $m$ constants such that

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \dots \dots \dots \dots \dots \dots + a_{1n}x_n (\leq, =, \geq)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots \dots \dots \dots \dots \dots + a_{2n}x_n (\leq, =, \geq)\ b_2 \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots \dots \dots \dots \dots \dots + a_{mn}x_n (\leq, =, \geq)\ b_m \end{gathered}$$

and finally let $x_1 \geq 0, x_2 \geq 0, \dots \dots \dots \dots, x_n \geq 0 \dots \dots \dots \dots \dots .. (3)$

(i) The problem of solving the values of $x_1, x_2, \dots, x_n$ which make $Z$ a minimum (or maximum) and which satisfies equations (2) and (3) is called general linear programming.
(ii) $Z = c_1x_1 + c_2x_2 + \dots \dots \dots + c_nx_n$ is called objective function.
(iii) System of linear inequalities eq$^n$(2) is called constrains and in eq$^n$(3) $x_i \geq 0$ is called non negative restriction.
(iv) Solutions: Values of unknowns $x_1, x_2, \dots, x_n$ which the constraints eq$^n$(2) of a general linear programming problem are called general solutions.
(v) Feasible solution: Any solution if GLPP which satisfies the non-negative restrictions of the problem is called feasible solution of GLPP.
(vi) Optimum solution: Any feasible solution which optimizes (minimizes, maximizes) the objective function is called optimum solution.

**Linear programming problem** can be solved by (i) Graphical method, (ii) Simplex method (Pivoting method).

**Example:** A company manufactures and sells two models of lamps, L1 and L2. To manufacture each lamp, the manual work involved in model L1 is 20 minutes and for L2, 30 minutes. The mechanical (machine) work involved for L1 is 20 minutes and for L2, 10 minutes. The manual work available per month is 100 hours and the machine is limited to only 80 hours per month. Knowing that the profit per unit is 15 and 10 for L1 and L2, respectively, determine the quantities of each lamp that should be manufactured to obtain the maximum benefit.

Solution:

Let
$x$ = number of lamps L1
$y$ = number of lamps L2
Objective function $f(x, y) = 15\,x + 10\,y$

Convert the time from minutes to hours.

20 min = 1/3 h  30 min = 1/2 h  10 min = 1/6 h

| | L1 | L2 | Time |
|---|---|---|---|
| Manual | 1/3 | 1/2 | 100 |
| Machine | 1/3 | 1/6 | 80 |

Writing the constraints as a system of inequalities we get

$$\frac{1}{3}x + \frac{1}{2}y \leq 100$$

$$\frac{1}{3}x + \frac{1}{6}y \leq 80$$

As the numbers of lamps are natural numbers, we have $x \geq 0$ & $y \geq 0$

Represent the constraints graphically.

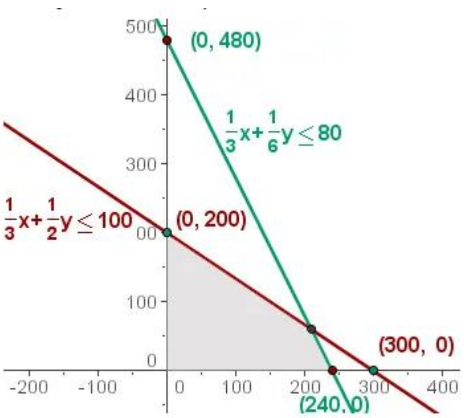
As $x \geq 0$ & $y \geq 0$, work in the first quadrant.

Solve the inequation graphically: $\frac{1}{3}x + \frac{1}{2}y \leq 100$; and take a point on the plane, for example (0,0).

$$\frac{1}{3} \cdot 0 + \frac{1}{2} \cdot 0 \leq 100$$

$$\frac{1}{3} \cdot 0 + \frac{1}{6} \cdot 0 \leq 80$$

The area of intersection of the solutions of the inequalities would be the solution to the system of inequalities, which is the set of feasible solutions.



The optimal solution, if unique, is a vertex. These are the solutions to systems:

$$\frac{1}{3}x + \frac{1}{2}y = 100;\ x = 0\ (0, 200)$$

$$\frac{1}{3}x + \frac{1}{6}y = 80;\ y = 0(240, 0)$$

$$\frac{1}{3}x + \frac{1}{2}y = 100;\ \frac{1}{3}x + \frac{1}{6}y = 80(210, 60)$$

To determine which of them has the maximum or minimum values.

In the objective function, place each of the vertices that were determined in the previous step.

$$f(x, y) = 15x + 10y$$

$$f(0, 200) = 15 \cdot 0 + 10 \cdot 200 = 2000$$

$$f(240{,}0) = 15240 + 100 = 3{,}600$$

$$f(210, 60) = 15 \cdot 210 + 10 \cdot 60 = 3750$$

So, (210,60) is our required answer.

**Example:** A calculator company manufactures two types of calculator: a handheld calculator and a scientific calculator. Statistical data projects that there is an expected demand of at least 100 scientific and 80 handheld calculators each day. Since the company has certain limitations on the production capacity, the company can only manufacture 200 scientific and 170 handheld calculators per day. The company has received a contract to deliver a minimum of 200 calculators per day. If there is a loss of 2 taka on each scientific calculator that you sold and a profit of 5 taka on each handheld calculator, then how many calculators of each type the company should manufacture daily to maximize the net profit?

**Solution:** To solve this problem, let's first formulate it properly by following the steps.
**Step 1: Identify the number of decision variables.**
In this problem, since we have to calculate how many calculators of each type should be manufactured daily to maximize the net profit, the number of scientific and handheld calculators each are our decision variables.
Consider,
number of scientific calculators manufactured = $x$
number of handheld calculators manufactured = $y$

**Step 2: Identify the constraints on the decision variables**.

The lower bound, as mentioned in the problem (there is an expected demand of at least 100 scientific and 80 handheld calculators each day) are as follows.

Hence, $x \geq 100$ and $y \geq 80$.

The upper bound owing to the limitations mentioned the problem statement (the company can only manufacture 200 scientific and 170 handheld calculators per day) are as follows:

Hence, $x \leq 200$ and $y \leq 170$.

In the problem statement, we can also see that there is a joint constraint on the values of $x$ and $y$ due to the minimum order on a shipping consignment that can be written as:

$$x + y \geq 200$$

**Step 3: Write the objective function in the form of a linear equation.**

In this problem, it is clearly stated that we have to optimize the net profit. As stated in the problem(If there is a loss of 2 taka on each scientific calculator that you sold and a profit of 5 taka on each handheld calculator), the net profit function can be written as:

**Profit (P) = $-2x + 5y$**

**Step 4: Explicitly state the non-negativity restriction.**

Since the calculator company cannot manufacture a negative number of calculators.

**Hence, $x \geq 0$ and $y \geq 0$**

Since we have formulated the problem, let's convert the problem into a mathematical form to solve it further.

Maximization of P = $-2x + 5y$

subject to:

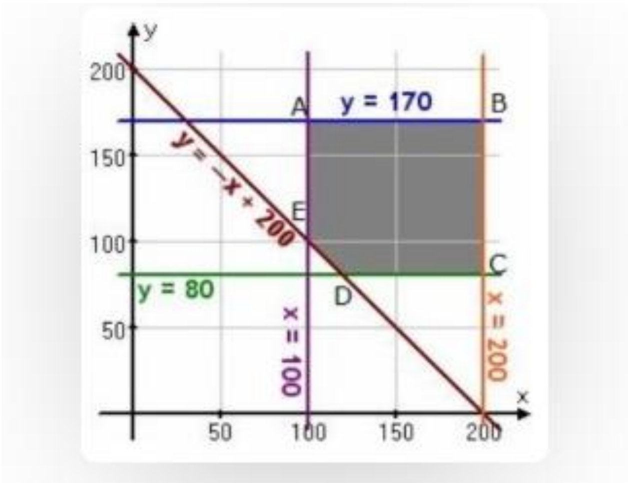
$$100 \leq x \leq 200$$
$$80 \leq y \leq 170$$
$$x + y \geq 200$$

**Step 5: Plot the constraints on the graph.**

Let's plot all the constraints defined in step 2 on a graph in a similar manner as we plot an equation.

**Step 6: Highlight the feasible region on the graph.**

After plotting the coordinates on the graph, shade the area that is outside the constraint limits (which is not possible). The highlighted feasible area will look like this:



**Step 7: Find the coordinates of the optimum point.**

To find the coordinates of the optimum point, we will solve the simultaneous pair of linear equations.

| **Corner Points** | **Equation, P = $-2x + 5y$** |
|---|---|
| **A (100, 170)** | **P = 650** |
| **B (200, 170)** | **P = 450** |
| **C (200, 80)** | **P = 0** |
| **D (120, 80)** | **P = 160** |
| **E (100, 100)** | **P = 300** |

**Step 8: Find the optimum point.**

The above table shows that the maximum value of P is **650** that is obtained at
**$(x, y)$ = A (100, 170).**

# Cryptographically Problem:

The process to write (encoded) and read (decoded) any secret messages by using matrices is known as Cryptography.

**Specific Aims:** We will

- **be able to encode a message using matrix multiplication;**
- **decode a coded message using the matrix inverse and matrix multiplication.**

**Algorithm to Encode a Message:**

- Assign the numbers 1-26 to the letters (capital/small) in the alphabet given below and assign the number 0 to a blank to provide for space between words.

| Blank | A | B | C | D | E | F | G | H | I | J | K | L | M | N |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **0** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

| O | P | Q | R | S | T | U | V | W | X | Y | Z |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |

- Write the provided message corresponds to the sequence of numbers.
- Matrix, "A" (say) can be used as an encoding matrix, if elements are positive integer of the considered matrix and inverse matrix exists.
- Divide the numbers in the sequence into groups of the order of matrix, "A" (or size of the matrix) and use these groups as the columns of a matrix, "B" (say). Proceed down the columns not across the rows.
- Write the provided message corresponds to the sequence. Then, multiply this matrix, "B" on the left by matrix, "A".
- Coded message will be written by picking the elements in each column from left of the matrix "AB".

**Algorithm to Decode a Message:**

- Find the inverse of encoding matrix, "A", if exists.
- Divide the numbers in the sequence into groups of the order of matrix, "A" (or size of the matrix) and use these groups as the columns of a matrix, "B" (say). Proceed down the columns not across the rows.
- Multiply this matrix, "B" on the left by inverse matrix, "$A^{-1}$".
- Writing the numbers in the columns of this matrix "$A^{-1}B$" in sequence and using the letters to correspondence numbers given below.

| O | P | Q | R | S | T | U | V | W | X | Y | Z | Blank |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | **0** |

| A | B | C | D | E | F | G | H | I | J | K | L | M | N |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

- These letters give decoded message.

If an element of the matrix $AB$ or $A^{-1}B$ is greater than 27 or negative number, then consider the element of the matrix $AB$ or $A^{-1}B$ as '$n$'

- ***n* is a positive number and *n* < 27**

  Then, write the number corresponding to the sequence of the letter.

- ***n* is a positive number and *n* > 27**

  Then, use the modulus operation to bring it within the range [0, 26]. The modulus operation can be defined as:

  $n \bmod 27$

  This operation provides the remainder when $n$ is divided by 27.

  For example:

  If $n=30$, $30 \bmod 27=3$

  Thus, 30 is replaced by 3.

- ***n* is a negative number and $|n| < 27$**

  We can add the modulus until the number is non-negative:

  $n \equiv (n+27) \bmod 27$

  For example:

  If $n=-3$, $-3 \bmod 27=24$

  Since $-3+27=24$, $-3$ is equivalent to 24 modulo 27.

- ***n* is a negative number and $|n| > 27$**

  $n \equiv (n \bmod 27+27) \bmod 27$

  For example:

  If $n=-42$,

  $-42 \bmod 27=(-15+27) \bmod 27$

  $-42 \bmod 27=12 \bmod 27$

  So, $-15+27=12$, $-42$ is equivalent to 12 modulo 27.

**Example 1:** Encode the message **SECRET CODE** by using matrix $A = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$.

**Solution:**

**Step 1:** The provided message "**SECRET CODE** " corresponds to the sequence

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **S** | **E** | **C** | **R** | **E** | **T** | **space** | **C** | **O** | **D** | **E** |
| **19** | **5** | **3** | **18** | **5** | **20** | **0** | **3** | **15** | **4** | **5** |

**Step 2:** Divide these numbers in the sequence into groups of 2 (based on the size of given matrix) and use these groups as the columns (proceed down the columns) of a matrix, B of two rows. Thus,

$$B = \begin{pmatrix} 19 & 3 & 5 & 0 & 15 & 5 \\ 5 & 18 & 20 & 3 & 4 & 0 \end{pmatrix}$$

**Step 3:** Now,

$$AB = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 19 & 3 & 5 & 0 & 15 & 5 \\ 5 & 18 & 20 & 3 & 4 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} 4(19)+3(5) & 4(3)+3(18) & 4(5)+3(20) & 4(0)+3(3) & 4(15)+3(4) & 4(5)+3(0) \\ 1(19)+1(5) & 1(3)+1(18) & 1(5)+1(20) & 1(0)+1(3) & 1(15)+1(4) & 1(5)+1(0) \end{bmatrix}$$

$$AB = \begin{bmatrix} 91 & 66 & 80 & 9 & 72 & 20 \\ 24 & 21 & 25 & 3 & 19 & 5 \end{bmatrix}$$

**Step 4:** Therefore, the coded message is **91 24 66 21 80 25 9 3 72 19 20 5.**

**Example 2:** The encoded message is 7 **6 28 20 23 5**. Decode this message by using matrix,

$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

**Solution:**

**Step 1:** When elements of the encoding matrix, $A$ are positive and find inverse matrix of $A$.

$$A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

**Step 2:** The encoded message is 7 **6 28 20 23 5.**

Since the encoding matrix, $A$ is $2 \times 2$, make a matrix "C" having two rows by picking two numbers from the left of encoded message as columns of matrix "C". We have,

$$C = \begin{pmatrix} 7 & 28 & 23 \\ 6 & 20 & 5 \end{pmatrix}$$

**Step 3:** Now,

$$A^{-1}C = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 7 & 28 & 23 \\ 6 & 20 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 20 & 5 \\ 1 & 8 & 18 \end{pmatrix}$$

**Step 4:** Writing the numbers in the columns of this matrix in sequence and using the letters to correspondence numbers. Thus,

**6 1 20 8 5 18**

**F A T H E R**

**Example 3:** Encode the message **DO OR DIE** by using key matrix $A = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$. Also decode your encrypted message using the same matrix**.**

**Solution:**

**Step 1:** The provided message "**DO OR DIE**" corresponds to the sequence

**D O space O R space D I E**

**4 15 0 15 18 0 4 9 5**

**Step 2:** Divide these numbers in the sequence into groups of 2 (based on the size of given matrix) and use these groups as the columns (proceed down the columns) of a matrix, B of two rows. Thus,

$$B = \begin{pmatrix} 4 & 0 & 18 & 4 & 5 \\ 15 & 15 & 0 & 9 & 0 \end{pmatrix}$$

**Step 3:** Now,

$$AB = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 18 & 4 & 5 \\ 15 & 15 & 0 & 9 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -18 & -30 & 54 & -6 & 15 \\ -7 & -15 & 36 & -1 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 24 & 0 & 21 & 15 \\ 20 & 12 & 9 & 26 & 10 \end{pmatrix}$$

**Step 4:** Therefore, the coded message is **9 20 24 12 0 9 21 26 15 10.**

Now, Decode the above message by using matrix,

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}.$$

**Step 1:** Find the inverse matrix of $A$.

$$A^{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

**Step 2:** The encoded message is **9 20 24 12 0 9 21 26 15 10.**

Since the key matrix, $A$ is $2 \times 2$, make a matrix "C" having two rows by picking two numbers from the left of encoded message as columns of matrix "C". We have,

$$C = \begin{pmatrix} 9 & 24 & 0 & 21 & 15 \\ 20 & 12 & 9 & 26 & 10 \end{pmatrix}$$

**Step 3:** Now,

$$A^{-1}C = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 9 & 24 & 0 & 21 & 15 \\ 20 & 12 & 9 & 26 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 31 & 0 & 18 & 31 & 5 \\ 42 & -12 & 27 & 36 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 18 & 4 & 5 \\ 15 & 15 & 0 & 9 & 0 \end{pmatrix}$$

**Step 4:** Writing the numbers in the columns of this matrix in sequence and using the letters to correspondence numbers. Thus,

**4 15 0 15 18 0 4 9 5**

**D O O R D I E**

# Exercise 2

1. Find the rank of the matrices corresponding to each of the following systems of linear equations. Hence test whether the systems of linear equations are consistent. If consistent find the solutions of the system. Also check your answer by direct substitution.

**a.**
$$\begin{aligned} 2x_1 - 3x_2 &= -2 \\ 2x_1 + x_2 &= 1 \\ 3x_1 + 2x_2 &= 1 \end{aligned}$$
Ans: Inconsistent

**b.**
$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$
Ans: Inconsistent

**c.**
$$\begin{aligned} x - 2y + z &= 0 \\ 2y - 8z &= 8 \\ -4x + 5y + 9z &= -9 \end{aligned}$$
Ans. $(29,\ 16,\ 3)$

**d.**
$$\begin{aligned} x + y + 2z &= 0 \\ y + z &= 0 \\ -2x + 3y + z &= 0 \end{aligned}$$
Ans. $(-a, -a, a)$

**e.**
$$\begin{aligned} x + 2y + 4z &= -13 \\ 3x - y + z &= 5 \\ x + 3y &= 3 \end{aligned}$$
Ans. $(3,\ 0,\ -4)$

**f.**
$$\begin{aligned} x_1 - x_2 + 2x_3 &= 5 \\ 2x_1 + x_2 - x_3 &= 2 \\ 2x_1 - x_2 - x_3 &= 4 \end{aligned}$$
Ans: $(2, -1, 1)$

**g.**
$$\begin{aligned} 2x - y - 5z &= 4 \\ x + y + z &= -3 \\ -x - 4y + z &= 4 \end{aligned}$$
Ans: $(-1, -1, -1)$

**h.**
$$\begin{aligned} -x + 2y + 2z &= -2 \\ 3x + 2y - z &= 9 \\ x + 4y + z &= 5 \end{aligned}$$
Ans: $(2, 1, -1)$

**i.**
$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 2 \\ -x_1 + x_2 - 2x_3 &= -1 \\ x_1 - 4x_2 - 2x_3 &= 8 \end{aligned}$$
Ans: $(2, -1, -1)$

**j.**
$$\begin{aligned} 2x - 3y + 4z &= 8 \\ 3x + 4y - 5z &= -4 \\ 5x - 7y + 6z &= 9 \end{aligned}$$
Ans: $(1, 2, 3)$

**k.**
$$\begin{aligned} 3x + y + 2z &= 14 \\ 2y + 5z &= 22 \\ 2x + 5y - z &= -22 \end{aligned}$$
Ans: $(2, -4, 6)$

**l.**
$$\begin{aligned} x + 2y - 3z &= 6 \\ 2x - y + 4z &= 2 \\ 4x + 3y - 2z &= 14 \end{aligned}$$
Ans: $(2 - a, 2 + 2a, a)$

**m.**
$$\begin{aligned} -x + y - z &= 0 \\ 3x - y + 2z &= -2 \\ 2x + 4y + 3z &= 2 \end{aligned}$$
Ans: $(-2, 0, 2)$

**n.**
$$\begin{aligned} x + 2y + 3z &= 1 \\ x + 3y + 6z &= 3 \\ 2x + 6y + 13z &= 5 \end{aligned}$$
Ans: $(-6, 5, -1)$

**o.**
$$\begin{aligned} x + y + z &= -1 \\ x - y + z &= -5 \\ 2x + y - z &= 5 \end{aligned}$$
Ans: $(0, 2, -3)$

**p.**
$$\begin{aligned} x + 2y + 3z &= 5 \\ 2x + 5y + 3z &= 3 \\ x + 8z &= 17 \end{aligned}$$
Ans: $(1, -1, 2)$

**q.**
$$\begin{aligned} 3x + 2y - z &= -15 \\ 5x + 3y + 2z &= 0 \\ 3x + y + 3z &= 11 \\ -6x - 4y + 2z &= 30 \end{aligned}$$
Ans: $(-4, 2, 7)$

2. Solve the following system of linear equations by using Cramer's rule and Matrix inversion methods (If Applicable).

a.
$$\begin{aligned} -3x + 2y - 3z &= -8 \\ 2x - y + z &= 4 \\ x + 2y - 4z &= -2 \end{aligned}$$
Ans: $(2, 2, 2)$

b.
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$
Ans: $(3, 1, 2)$

c.
$$\begin{aligned} 3x - y + z &= -2 \\ x + 5y + 2z &= 6 \\ 2x + 3y + z &= 0 \end{aligned}$$
Ans: $(-2, 0, 4)$

d.
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 - x_2 + 2x_3 &= -1 \\ 3x_1 - 4x_2 - 3x_3 &= -16 \end{aligned}$$
Ans: $(-2, 1, 2)$

e.
$$\begin{aligned} x + y + 2z &= 1 \\ y + z &= 1 \\ -2x + 3y + z &= 3 \end{aligned}$$
Ans: Determinant zero, not applicable

f.
$$\begin{aligned} 2x_1 - x_2 - x_3 &= 6 \\ x_1 + 3x_2 + 2x_3 &= 1 \\ 3x_1 - x_2 - 5x_3 &= 1 \end{aligned}$$
Ans: $(3, -2, 2)$

3. Determine the value(s) of $\lambda$ such that the following system of linear equations has (i) no solution, (ii) more than one solution, and (iii) a unique solution.
$$\begin{cases} \lambda x + y + z = 1 \\ x + \lambda y + z = 1 \\ x + y + \lambda z = 1 \end{cases}.$$
Ans: $(i)\ \lambda = -2, (ii)\ \lambda = 1, (iii)\ \lambda \neq 1, \lambda \neq -2.$

4. Determine the value(s) of $\lambda$ such that the following system of linear equations has (i) no solution, (ii) more than one solutions, and (iii) a unique solution.

$$\begin{cases} x \qquad -3z = -3 \\ 2x + \lambda y - z \ = -2 \\ x + 2y + \lambda z = 1 \end{cases}.$$

Ans: $(i)\ \lambda = -5, (ii)\ \lambda = 2, (iii)\ \lambda \neq 2, \lambda \neq -5.$

5. Determine the value(s) of $\lambda$ and $\mu$ such that the following system of linear equations has (i) no solution, (ii) more than one solutions, and (iii) a unique solution.

$$\begin{aligned} x + y \ \ + z \ \ &= 2 \\ x + 3y + \lambda z &= 6. \\ x + 2y + 3z &= \mu \end{aligned}$$

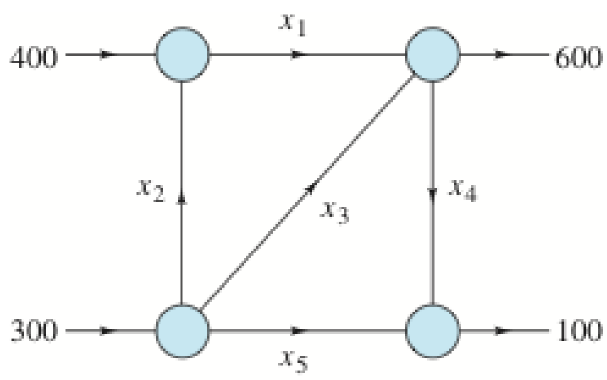
Ans: $(i)\ \lambda = 5,\ \ \mu \neq 4, (ii)\ \lambda = 5,\ \ \mu = 4, (iii)\ \lambda \neq 5.$

6. Determine the value(s) of $\lambda$ such that the following system of linear equations has (i) no solution, (ii) more than one solutions, and (iii) a unique solution.

$$\begin{cases} x + y + \lambda z = 1 \\ x + \lambda y + z = \lambda \\ \lambda x + y + z = \lambda^2 \end{cases}.$$

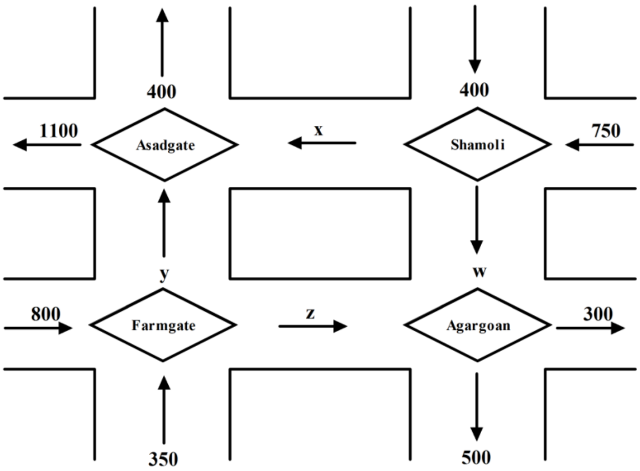
Ans: $(i)\ \lambda = -2, (ii)\ \lambda = 1, (iii)\ \lambda \neq 1, \lambda \neq -2.$

7. The network in the figure below shows the traffic flow (in vehicles per hour) over the several one-way streets. Determine the general flow pattern for the network.



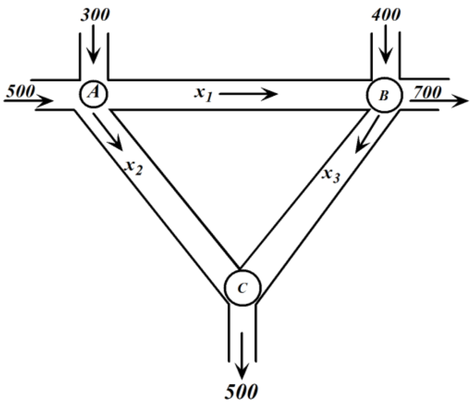

Ans: Here $x_3$ and $x_5$ are free variables.

8. The network in the figure below shows the traffic flow (in vehicles per hour) over the several one-way streets. Determine the general flow pattern for the network.

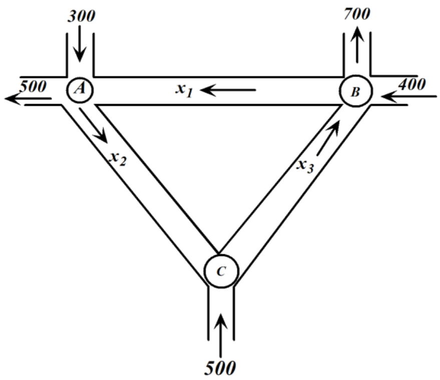



9. The network in the figure below shows the traffic flow (in vehicles per hour) over the several one-way streets. Determine the general flow pattern for the network.

**a.**



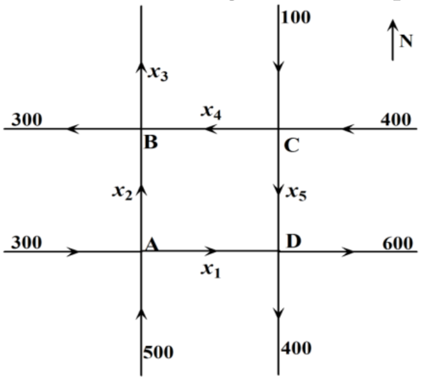

**b.**




Ans: (a) Here $x_3$ is the free variable. The system has an infinite number of solutions. But to remove negativity $x_3$ must be between 0 to 500.

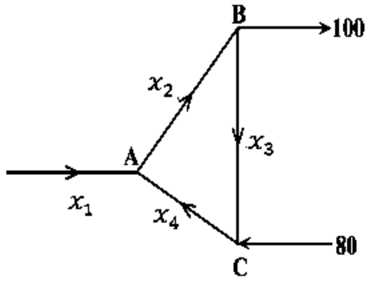
(b) Here $x_3$ is the free variable. The system has an infinite number of solutions. But to remove negativity $x_3$ must be greater than 500.

10. The network in the figure below shows the traffic flow (in vehicles per hour) over the several one-way streets. Determine the general flow pattern for the network.




Ans: Here $x_5$ is the free variable. The system has an infinite number of solutions. But to remove negativity $x_5$ must start from 200.

11. Find the general flow pattern of the network system in the figure. Assuming that the flows are all nonnegative, what is the smallest possible value for $x_4$.

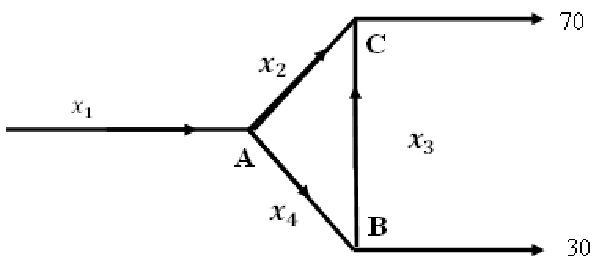



12. A company produces three products $x, y, z$ every day. Their total production on a certain day is $45$ tons. It is found that the production of $z$ exceeds the production of $x$ by $8$ tons while the total production of '$x$' and '$z$' is twice the production of '$y$'. Determine the production level of each product.

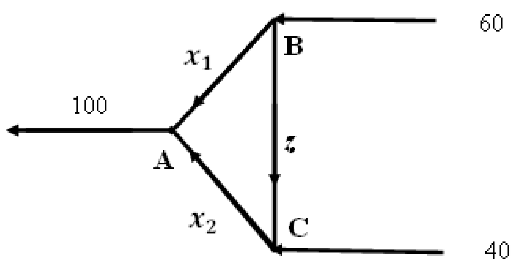
    Ans: $x = 11;\ y = 15;\ z = 19.$

13. Construct the system of linear equations from the following diagrams, reduced the system to echelon form and finally find the general flow pattern, where $x_i$ is the number of cars.

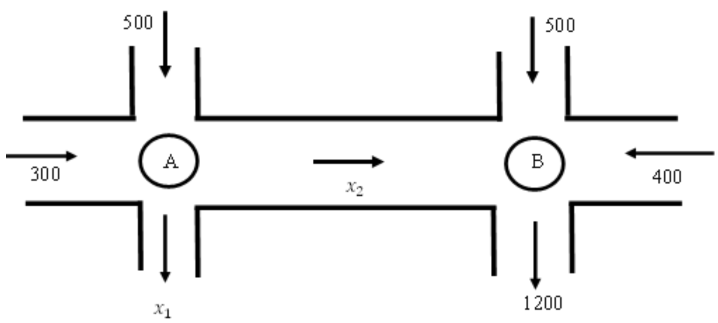
(i)



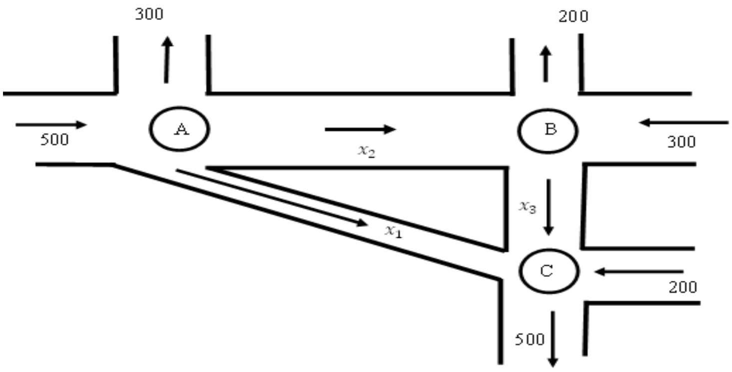

(ii)




(iii)

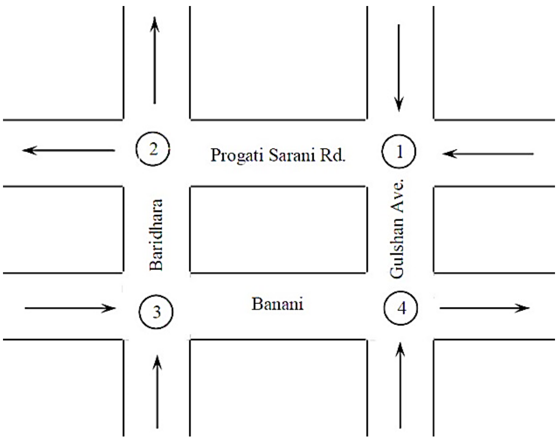


(iv)




14. Traffic congestion is encountered at the intersections shown in the figure. All the streets are one-way and in the directions shown. In order for effective movement of traffic, it is necessary that for every car that arrives at a given comer, another car must leave so that the number of cars arriving per unit time must equal the number of cars leaving per unit time. Traffic engineers gather the following information:

- 600 cars per hour come down Progati Sarani Rd. to intersection #1 and 300 cars per hour enter intersection #1 on Gulshan Ave.
- 650 cars per hour leave intersection #2 along Progati Sarani Rd. and 50 cars per hour leave intersection #2 along Baridhara.
- 350 cars per hour come up Banani to intersection #3 and 50 cars per hour enter intersection #3 along Baridhara.
- 900 cars per hour leave intersection #4 along Banani and 300 cars per hour enter intersection #4 from Gulshan Ave.




Find $n_1$, $n_2$, $n_3$, and $n_4$, where $n_1$ denotes the number of cars traveling per hour along Progati Sarani Rd. from intersection #1 to intersection #2, $n_2$ denotes the number of cars traveling per hour along Baridhara from intersection #3 to intersection #2, $n_3$ denotes the number of cars traveling per hour along Banani from intersection #3 to intersection #4, and $n_4$ denotes the number of cars traveling per hour along Gulshan Ave. from intersection #1 to intersection #4.

**Ref: Numerical Methods for Engineers and Scientists by Amos Gilat and Vish Subramaniam, 3rd edition (Page-164).**

15. Encode the message **TRY YOUR BEST** by using matrix, $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Also decode your encrypted message using the same matrix**.**

16. Encode the message **HONESTY IS THE BEST POLICY** by using matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ given above.

17. Encrypt the message **EFFORT NEVER DIES** by the provided matrix, $B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.

18. The encoded message is **28 13 28 20 23 5**. Decode this message by using matrix, $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

19. Encrypt the message **CRYPTOGRAPHY** by the provided matrix, $A = \begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}$.

## Linear Programming:

**Reference: Operation Research by Hamdy A. Taha, 10th edition.**
**Problem-2.1-1 and 2.1-2 (Page-45-52), problem 2.7 , 2.8, 2.9, 2.13 (page 77, 78).**

## Supplementary:

MATLab command for finding unique solution (if exists) of a system of equation:

$$\begin{aligned} 3x + 2y - z &= 20 \\ 2x + 3y - 3z &= 7 \\ x - y + 6z &= 41 \end{aligned}$$

Ans: (5,6,7)

| For | Input Command | Output |
|---|---|---|
| Coefficient matrix: $A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 3 & -3 \\ 1 & -1 & 6 \end{pmatrix}$ | `>> A = [3 2 -1;2 3 -3;1 -1 6]` | A = <br> 3 2 -1 <br> 2 3 -3 <br> 1 -1 6 |
| Right hand side matrix: $B = \begin{pmatrix} 20 \\ 7 \\ 41 \end{pmatrix}$ | `>> B= [20;7;41]` | B = <br> 20 <br> 7 <br> 41 |

| | | |
|---|---|---|
| checking whether there exists a unique solution or not! | `>> if det(A)~=0`<br>`    disp ('There exists a unique solution for the given system.')`<br>`else`<br>`    disp ('There is no unique solution for the given system.')`<br>`end` | There exists a unique solution for the given system. |
| Solution set, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$<br>where, $X = A^{-1}B$ | `>> X=inv(A)*B` | X =<br>5.0000<br>6.0000<br>7.0000 |

**References:**


- Linear Programming by Thomas S. Ferguson
- Linear Programming by George B. Dantzig, Mukund N. Thapa
- Operations Research by Ravindran, Phillips & Solberg
- Operations Research by H. Taha