**Practice problems for midterm 2, Math 225 R1**

Fall 2012

- You must not communicate with other students during this test.
- No books, notes, or written materials of any kind allowed.
- No phones, calculators, iPods or electronic devices of any kind are allowed for ANY reason, including checking the time (you may use a simple wristwatch).
- Do not turn this page until instructed to.

Name : Key

1. Compute the determinants using a cofactor expansion

(a) $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix} = 3 \begin{vmatrix} 3 & 2 \\ 5 & -1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} + 4 \cdot \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix}$

$\underline{\underline{= 1}}$

$\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix} + 4 \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}$

$\underline{\underline{= -5}}$

2. Answer the following questions. Give reasons for your answer.

(a) What is the determinant of an elementary row replacement matrix?

ex) $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ det $= 1$.

(b) What is the determinant of an elementary scaling matrix with $k$ on diagonal?

ex) $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ det $= k$.

3. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $k$ be a scalar. Find a formula that relates $\det kA$ and $\det A$.

$$\det kA = \begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = ka \cdot kd - kb \cdot kc = k^2(ad-bc)$$

4. Each equation illustrates a property of determinants. State the property.

$$\begin{vmatrix} 0 & 5 & -2 \\ 1 & -3 & 6 \\ 4 & -1 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & -3 & 6 \\ 0 & 5 & -2 \\ 4 & -1 & 8 \end{vmatrix}$$

Row 1 & row 2 are interchanged. So the determinant changes sign.

$$\begin{vmatrix} 1 & 3 & -4 \\ 2 & 0 & -3 \\ 5 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -4 \\ 0 & -6 & 5 \\ 5 & -4 & 7 \end{vmatrix}$$

The row replacement operation does not change the determinant

5. Find the determinants by row reduction to echelon form.

(a) $\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{vmatrix} = 3$

(b) $\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$

6. Find the determinants followings, where $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$

(a) $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}$ $5 \times 7$

(b) $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$ $-7$

7. Find a formula for $\det(rA)$, where $A$ is an $n \times n$ matrix.

$\det(rA) = r^n \det(A)$

8. Use Cramer's rule to compute the solutions of the systems.

(a) $\begin{cases} 5x_1 + 7x_2 = 3 \\ 2x_1 + 4x_2 = 1 \end{cases}$

$A = \begin{pmatrix} 5 & 7 \\ 2 & 4 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$A_1(\vec{b}) = \begin{pmatrix} 3 & 7 \\ 1 & 4 \end{pmatrix}$ $A_2(\vec{b}) = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}$

$x_1 = \dfrac{\det A_1(\vec{b})}{\det A} = \dfrac{5}{6}$, $x_2 = \dfrac{\det A_2(\vec{b})}{\det A} = -1/6$.

(b) $\begin{cases} 3x_1 - 2x_2 = 7 \\ -5x_1 + 6x_2 = -5 \end{cases}$

$A = \begin{pmatrix} 3 & -2 \\ -5 & 6 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$

$x_1 = \dfrac{\det A_1(\vec{b})}{\det A} = 4$, $x_2 = \dfrac{\det A_2(\vec{b})}{\det A} = 5/2$.

(c) $\begin{cases} 2x_1 + x_2 = 7 \\ -3x_1 + x_3 = -8 \\ x_2 + 2x_3 = -3 \end{cases}$

$A = \begin{pmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 7 \\ -8 \\ -3 \end{pmatrix}$

$x_1 = \dfrac{\det A_1(\vec{b})}{\det A} = 3/2$

$x_2 = \dfrac{\det A_2(\vec{b})}{\det A} = 4$, $x_3 = \dfrac{\det A_3(\vec{b})}{\det A} = -7/2$.

9. Determine the values of the parameter $s$ for which the system has a unique solution, and describe the solution.

(a) $\begin{cases} 6sx_1 + 4x_2 = 5 \\ 9x_1 + 2sx_2 = -2 \end{cases}$

$A = \begin{pmatrix} 6s & 4 \\ 9 & 2s \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

$\det A = 12s^2 - 36 = 12(s^2-3)$. $\det A \neq 0 \iff s \neq \pm\sqrt{3}$.

∘ For $s \neq \pm\sqrt{3}$, the system has a unique soln and

$x_1 = \frac{\det A_1(\vec{b})}{\det A} = \frac{\begin{vmatrix} 5 & 4 \\ -2 & 2s \end{vmatrix}}{\begin{vmatrix} 6s & 4 \\ 9 & 2s \end{vmatrix}} = \frac{5s+4}{6(s^2-3)}$, $x_2 = \frac{\det A_2(\vec{b})}{\det A} = \frac{-4s-15}{4(s^2-3)}$.

(b) $\begin{cases} sx_1 - 2sx_2 = -1 \\ 3x_1 + 6sx_2 = 4 \end{cases}$

$A = \begin{pmatrix} s & -2s \\ 3 & 6s \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ $\Rightarrow \det A = 6s^2 + 6s = 6s(s+1)$.

$\Rightarrow \det A \neq 0 \iff s \neq 0$ & $s \neq -1$.

For $s \neq 0$ & $-1$,

$x_1 = \frac{\det A_1(\vec{b})}{\det A} = \frac{1}{3(s+1)}$, $x_2 = \frac{\det A_2(\vec{b})}{\det A} = \frac{4s+3}{6s(s+1)}$.

10. Compute the adjugate of the given matrix, and then give the inverse of the matrix.

(a) $\begin{pmatrix} 0 & -2 & -1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}$

$C_{11} = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0, \quad C_{12} = -\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = -3, \quad C_{13} = \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = 3$

...

$\operatorname{adj} A = \begin{pmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{pmatrix}$ and $A^{-1} = \dfrac{\operatorname{adj} A}{\det A} = \begin{pmatrix} 0 & 1/3 & 0 \\ -1 & -1/3 & -1 \\ 1 & 2/3 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

$\operatorname{adj} A = \begin{pmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{pmatrix}$ and $A^{-1} = \frac{1}{6}\begin{pmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{pmatrix}$

11. Let $V$ be the first quadrant in the $xy$-plane; that is let $V = \{(x, y) : x \geq 0, y \geq 0\}$.

(a) If $u$ and $v$ are in $V$, is $u + v$ in $V$?

Yes

( $u = (a,b)$, $v = (c,d)$ , $a,b,c,d \geq 0$ then

$u+v = (a+c, b+d)$ and $a+c \geq 0$ & $b+d \geq 0$

(b) Find a specific vector $u$ and specific scalar $c$ such that $cu$ is *not* in $V$. (This shows that $V$ is not a vector space.)

$-1 \cdot (1.0) = (-1.0) \notin V$

12. Let $H$ be the set of points inside and on the unit circle in the $xy$-plane. That is, let $H = \{(x, y) : x^2 + y^2 \leq 1\}$. Find a specific example - two vectors or a vector and a scalar - to show that $H$ is not a subspace of $\mathbb{R}^2$. For $(0,1)$ $(1,0) \in H$ but

1) $(1,0) + (0,1) = (1.1) \notin H$ $(1^2 + 1^2 = 2 > 1)$

2) $3 \cdot (1,0) = (3,0) \notin H$

13. Determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate value of $n$.

(a) All polynomials in $\mathbb{P}_n$ such that $p(0) = 0$.

Yes.

for $f, g \in \mathbb{P}_n$ with $f(0) = g(0) = 0$.

① $0 \in \{ f \in \mathbb{P}_n \mid f(0) = 0 \} = A$

② $(f+g)(0) = f(0) + g(0) = 0 + 0 = 0 \Rightarrow f+g \in A$.

(closed under addition)

③ $(c \cdot f)(0) = c \cdot f(0) = c \cdot 0 = 0 \Rightarrow cf \in A$.

By ①, ②, and ③, $A$ is a subspace of $\mathbb{P}_n$.

(b) All polynomials of the form $p(t) = a + t^2$, where $a \in \mathbb{R}$.

No!

$2 \cdot (a + t^2) = 2a + \underline{\underline{2t^2}}$

14. Let $H$ be the set of all vectors of the form $\begin{pmatrix} -2t \\ 5t \\ 3t \end{pmatrix}$. Find a vector $v \in \mathbb{R}^3$ such that $H = Span\{v\}$. Explain why this shows $H$ is a subspace of $\mathbb{R}^3$.

$H = \text{span}\left\{\begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix}\right\}$ and $H$ is a subspace (of dim 1) of $\mathbb{R}^3$

15. Let $W$ be the set of all vectors of the form $\begin{pmatrix} 2b+3c \\ -b \\ 2c \end{pmatrix}$ where $b, c$ are arbitrary real numbers. Find vectors $u, v$ such that $W = Span\{u, v\}$. Explain why this shows $W$ is a subspace of $\mathbb{R}^3$.

$W = \text{span}\left\{\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}\right\}$

16. The set of all continuous real-valued functions defined on a closed interval $[a, b]$ in $\mathbb{R}$ is denoted by $C[a, b]$. This set is a subspace of the vector space of all real-valued functions defined on $[a, b]$.

(a) What facts about continuous functions should be proved in order to demonstrate that $C[a, b]$ is indeed a subspace as claimed?

① addition of two continuous functions is continuous.

② Scalar multiple of continuous function is continuous.

(b) Show that $\underbrace{\{f \in C[a, b] : f(a) = f(b)\}}_{A''}$ is a subspace of $C[a, b]$.

① $0 \in A$ $\quad ( 0(a) = 0(b) = 0 )$

② For $f, g \in A$, $\quad (f+g)(a) = f(a) + g(a) = f(b) + g(b) = (f+g)(b)$

Hence $f+g \in A$

③ For $f \in A$, $\quad (cf)(a) = c \cdot f(a) = cf(b) = (cf)(b) \Rightarrow cf \in A$.

17. Find an explicit description of Nul $A$, by listing vectors that span the null space.

(a) $A = \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{pmatrix}$ $\qquad (A, 0) \sim \begin{pmatrix} 1 & 0 & -2 & 4 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{pmatrix}$

$x_1 = 2x_3 - 4x_4$

$x_2 = -3x_3 + 2x_4$ $\qquad \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

$x_3\ x_4$ free

(b) $A = \begin{pmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

$x_1 = 4x_2 - 2x_4$

$x_3 = 5x_4$ $\qquad \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 5 \\ 1 \\ 0 \end{pmatrix}$

$x_5 = 0$

18. Either use an appropriate theorem to show that the given set $W$, is a vector space, or find a specific example to the contrary.

(a) $\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} : a+b+c=2 \right\}$

No! 0 is not in this set.

(b) $W = \left\{ \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} : p-3q=4s, 2p=s+5r \right\}$

$W = \{ p-3q-4s=0, \; 2p-s-5r=0 \}$

$= \mathrm{Nul}(A)$, $A = \begin{pmatrix} 1 & -3 & -4 & 0 \\ 2 & 0 & -1 & -5 \end{pmatrix}$.

Hence W is a vector space (since it is a nul-space of a matrix).

19. Find $A$ such that the given set is Col $A$.

(a) $\left\{ \begin{pmatrix} 2s+t \\ r-s+2t \\ 3r+s \\ 2r-s-t \end{pmatrix} : r,s,t \in \mathbb{R} \right\}$

$$\Rightarrow r\begin{pmatrix} 0 \\ 1 \\ 3 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} r \\ s \\ t \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

(b) $\left\{ \begin{pmatrix} b-c \\ 2b+3d \\ b+3c-3d \\ c+d \end{pmatrix} : b,c,d \in \mathbb{R} \right\}$

$$b\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} -1 \\ 0 \\ 3 \\ 1 \end{pmatrix} + d\begin{pmatrix} 0 \\ 3 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & -3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} b \\ c \\ d \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & -3 \\ 1 & 1 & 1 \end{pmatrix}$$

ee

20. Let $A = \begin{pmatrix} 6 & -4 \\ -3 & 2 \\ -9 & 6 \\ 9 & -6 \end{pmatrix}$. Find a nonzero vector in Nul $A$ and a nonzero vector in Col $A$.

21. Find a basis for the set of vectors in $\mathbb{R}^3$ in the place $x - 3y + 2z = 0$. [*Hint* : Think of the equation as a "system" of homogeneous equations.]

$A = (1, -3, 2)$

$\Rightarrow \quad x = 3y - 2z \quad \&\ y, z$ free.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3y - 2z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Basis for Nul $A = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

22. Assume that $A$ is row equivalent to $B$. Find bases for Nul $A$ and Col $A$.

$$A = \begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

① First & second columns are pivot columns

$\Rightarrow$ Basis for Col $A$ = $\left\{ \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ -6 \\ 8 \end{pmatrix} \right\}$

② If we solve $B\vec{x} = 0$, $x_1 = -6x_3 - 5x_4$, $x_2 = -\frac{5}{2}x_3 - \frac{3}{2}x_4$

$x_3$ $x_4$ free

$\Rightarrow \vec{x} = x_3 \begin{pmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{pmatrix}$ $\Rightarrow$ basis for Nul $A$ = $\left\{ \begin{pmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

23. Let $v_1 = \begin{pmatrix} 4 \\ -3 \\ 7 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 9 \\ -2 \end{pmatrix}$, $v_3 = \begin{pmatrix} 7 \\ 11 \\ 6 \end{pmatrix}$, and also let $H = \text{Span}\{v_1, v_2, v_3\}$. It can be verified that $4v_1 + 5v_2 - 3v_3 = 0$. Use this information to find a basis for $H$.

$V_3$ is a linear combination of $V_1$ & $V_2$ and it is easy to see $V_1$ & $V_2$ are linearly independent.

Hence $\{V_1, V_2\}$ is a basis for $H$.

( Exactly the same reasoning shows that $\{V_1, V_3\}$ or $\{V_2, V_3\}$ are basis for $H$, too. )

24. Mark each statement True or False.

(a) A single vector by itself is linearly dependent.

False

(b) If $H = \text{Span}\{b_1, b_2, \cdots, b_p\}$, then $\{b_1, \cdots b_p\}$ is a basis for $H$.

False

(c) The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

True

(d) In some cases, the linear dependence relations among the columns of a matrix can be affected by certain elementary row operations on the matrix.

False

25. Consider the polynomials $p_1(t) = 1+t^2$ and $p_2(t) = 1-t^2$. Is $\{p_1(t), p_2(t)\}$ a linearly independent set in $\mathbb{P}_3$?

If $a(1+t^2) + b(1-t^2) = 0$ then

$\Rightarrow (a+b) + (a-b)t^2 = 0 \quad \Rightarrow a+b = a-b = 0$

$\Rightarrow a = b = 0$.

Hence any linear combination of $P_1(t)$ & $P_2(t)$ that reduces to 0 must be a trivial linear combination.

$\Rightarrow$ $P_1(t)$ & $P_2(t)$ are linearly independent.