MATH1502 - C Group Calculus II Final Exam
Monday, April 30, 2012

May 2 Version

No calculators or notes are allowed. There are 139 marks on the final exam, 130=100%.

Name____

Student Number________________________________

Group C ______________________________________

TA_____________________________ ______________

Question	**Grade**	**Out of**
1		14
2		15
3		29
4		36
5		45
Total		139

Question 1

(a) Find the radius of convergence and interval of convergence of the power series

$$\sum_{k=2}^{\infty} \frac{e^{\sqrt{k}}}{(\ln k)^4} (11x)^k . \qquad \text{(8 marks)}$$

Solution

We use the root test applied to

$$a_k = \left| \frac{e^{\sqrt{k}}}{(\ln k)^4} (11x)^k \right| = \frac{e^{\sqrt{k}}}{(\ln k)^4} (11\,|x|)^k .$$

We see that

$$\begin{aligned} \lim_{k\to\infty} a_k^{1/k} &= \lim_{k\to\infty} \left(\frac{e^{\sqrt{k}}}{(\ln k)^4} (11\,|x|)^k \right)^{1/k} \\ &= \lim_{k\to\infty} \frac{e^{1/\sqrt{k}}}{(\ln k)^{4/k}} 11\,|x| \\ &= 11\,|x| . \end{aligned}$$

(**Justification of this, not really required from students:**

Recall $\lim_{k\to\infty} k^{1/k} = 1$. Moreover, $\lim_{k\to\infty} (\ln k)^{1/k} = 1$. They can assume the second limit, but give them extra credit if they give some proof. For example, $\lim_{k\to\infty} \ln k = \infty$ and

$$\lim_{k\to\infty} \frac{\ln k}{k} = 0,$$

so

$$1 \leq \ln k \leq k \text{ for large enough } k$$

and then

$$1^{1/k} \leq (\ln k)^{1/k} \leq k^{1/k}.$$

By the pinching/ sandwich rule,

$$\lim_{k\to\infty} (\ln k)^{1/k} = 1.)$$

By the root test, we have convergence for

$$11\,|x| < 1 \Leftrightarrow |x| < 1/11.$$

And we have divergence for $|x| > 1/11$. So the **radius of convergence is** 1/11.

(4 marks)

Next, we test the endpoints $x = \pm 1/11$.
$\mathbf{x = -1/11}$
Here

$$\sum_{k=2}^{\infty} \frac{e^{\sqrt{k}}}{(\ln k)^4}\,(11x)^k = \sum_{k=2}^{\infty} \frac{e^{\sqrt{k}}}{(\ln k)^4}\,(-1)^k\,.$$

Although this looks like an alternating series, we CANNOT use the alternating series test, since the terms do not decrease in size. In fact the terms do not approach 0, so the **basic divergence test** tells us the series diverges. (The students can state this without much justification). Indeed, using $e^x > x$,

$$\frac{e^{\sqrt{k}}}{(\ln k)^4} > \frac{\sqrt{k}}{(\ln k)^4} = \left(\frac{k}{(\ln k)^8}\right)^{1/2}.$$

As in class,

$$\lim_{k\to\infty} \frac{(\ln k)^8}{k} = 0, \text{ so } \lim_{k\to\infty} \frac{k}{(\ln k)^8} = \infty.$$

So

$$\sum_{k=2}^{\infty} \frac{e^{\sqrt{k}}}{(\ln k)^4}\,(-1)^k$$

diverges.
$\mathbf{x = 1/11}$
Here again

$$\sum_{k=2}^{\infty} \frac{e^{\sqrt{k}}}{(\ln k)^4}\,(11x)^k = \sum_{k=2}^{\infty} \frac{e^{\sqrt{k}}}{(\ln k)^4}$$

and the argument used above shows that the series diverges. So the **interval of convergence** is $\left(-\frac{1}{11}, \frac{1}{11}\right)$.

(4 marks)

(b) Find the radius of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{(4k)!}{k^k (k+1)^{3k}} x^k. \qquad \text{(6 marks)}$$

You may assume that

$$\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e \text{ and } \lim_{k\to\infty} \left(1 + \frac{1}{k+1}\right)^k = e.$$

Solution

Because of the factorials, we use the ratio test, applied to

$$a_k = \left| \frac{(4k)!}{k^k (k+1)^{3k}} x^k \right| = \frac{(4k)!}{k^k (k+1)^{3k}} |x|^k,$$

with $x \neq 0$. We see that

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_{k+1}}{a_k} &= \lim_{k\to\infty} \frac{\left(\frac{(4k+4)!}{(k+1)^{k+1}(k+2)^{3k+3}} |x|^k\right)}{\left(\frac{(4k)!}{k^k (k+1)^{3k}} |x|^k\right)} && \text{(1 mark)} \\
&= \lim_{k\to\infty} \frac{(4k+4)!}{(4k)!} \frac{k^k}{(k+1)^{k+1}} \frac{(k+1)^{3k}}{(k+2)^{3k+3}} |x| \\
&= \lim_{k\to\infty} \frac{(4k+4)(4k+3)(4k+2)(4k+1)}{(k+1)(k+2)^3} \left(\frac{k}{k+1}\right)^k \left(\frac{k+1}{k+2}\right)^{3k} |x| \\
&= \lim_{k\to\infty} \frac{k^4 (4+4/k)(4+3/k)(4+2/k)(4+1/k)}{k^4 (1+1/k)(1+2/k)^3} \frac{1}{\left(1+\frac{1}{k}\right)^k \left(1+\frac{1}{k+1}\right)^{3k}} |x| \\
&= 4(4)(4)(4) \frac{1}{ee^3} |x| = \left(\frac{4}{e}\right)^4 |x|. && \text{(3 marks)}
\end{aligned}$$

By the ratio test, this converges provided

$$\left(\frac{4}{e}\right)^4 |x| < 1 \Leftrightarrow |x| < \left(\frac{e}{4}\right)^4$$

So the radius of convergence is $\left(\frac{e}{4}\right)^4$.

(2 marks)

Question 2
(a) Find the solution of

$$\frac{y'}{(\ln x)^2} + \frac{y}{x} = \frac{7}{x}$$

that satisfies

$$y(1) = 6. \qquad \text{(8 marks)}$$

Solution
We want the form

$$y' + p(x)\, y = q(x),$$

so multiply by $(\ln x)^2$:

$$y' + \frac{(\ln x)^2}{x} y = 7\frac{(\ln x)^2}{x}.$$

Step 1 Integrating Factor
Then

$$p(x) = \frac{(\ln x)^2}{x}$$

and

$$\begin{aligned} H(x) &= \int p(x)\, dx = \int \frac{(\ln x)^2}{x} dx \\ &= \int t^2 dt \quad (\text{substitution } t = \ln x) \\ &= \frac{1}{3} t^3 = \frac{1}{3} (\ln x)^3 \end{aligned}$$

so the integrating factor is

$$e^{H(x)} = e^{\frac{1}{3}(\ln x)^3}. \qquad \text{(2 marks)}$$

Step 2 Multiply by integrating factor

$$e^{\frac{1}{3}(\ln x)^3} y' + e^{\frac{1}{3}(\ln x)^3} \frac{(\ln x)^2}{x} y = e^{\frac{1}{3}(\ln x)^3} \left(7\frac{(\ln x)^2}{x}\right),$$

$$\Rightarrow \frac{d}{dx}\left(e^{\frac{1}{3}(\ln x)^3} y\right) = e^{\frac{1}{3}(\ln x)^3} \left(7\frac{(\ln x)^2}{x}\right).$$

Step 3 Integrate

$$\begin{aligned}
e^{\frac{1}{3}(\ln x)^3} y &= \int e^{\frac{1}{3}(\ln x)^3} \left(7 \frac{(\ln x)^2}{x} \right) dx + C \\
&= 7 \int e^{\frac{1}{3}(\ln x)^3} \left(\frac{d}{dx} \left[\frac{1}{3} (\ln x)^3 \right] \right) dx + C \\
&= 7 \int e^t dt + C \\
(\text{substitution } t &= \frac{1}{3} (\ln x)^3) \\
&= 7e^t + C = 7e^{\frac{1}{3}(\ln x)^3} + C.
\end{aligned}$$

Then

$$y = 7 + Ce^{-\frac{1}{3}(\ln x)^3}. \qquad \text{(4 marks)}$$

Step 4 Find C

$$\begin{gathered}
6 = y(1) = 7 + Ce^0 = 7 + C \\
\Rightarrow C = -1.
\end{gathered}$$

So the solution is

$$y(x) = 7 - e^{-\frac{1}{3}(\ln x)^3}. \qquad \text{(2 marks)}$$

(b) Find the solution of the equation

$$y'(1-y) - e^{x+y} = x^2 e^y$$

with

$$y(0) = 0. \qquad \text{(7 marks)}$$

Solution
We must separate out the $x's$ and y's: rewrite as

$$y'(1-y) = e^{x+y} + x^2 e^y = e^y\left(e^x + x^2\right)$$
$$\Rightarrow e^{-y}(1-y)\,y' = e^x + x^2$$
$$\Rightarrow -e^x - x^2 + e^{-y}(1-y)\frac{dy}{dx} = 0. \qquad \text{(2 marks)}$$

Now integrate with respect to x :

$$\int\left(-e^x - x^2\right)dx + \int e^{-y}(1-y)\frac{dy}{dx}dx = \int 0\ dx = C.$$

Here

$$\int\left(-e^x - x^2\right)dx = -e^x - \frac{x^3}{3}.$$

Moreover, integrating by parts,

$$\begin{aligned} & \int e^{-y}(1-y)\frac{dy}{dx}dx \\ = & \int e^{-y}(1-y)\,dy \\ = & -e^{-y}(1-y) - \int\left(-e^{-y}\right)(-1)\,dy \\ = & -e^{-y}(1-y) + e^{-y} \\ = & ye^{-y}. \end{aligned}$$

So the solution is

$$-e^x - \frac{x^3}{3} + ye^{-y} = C. \qquad \text{(4 marks)}$$

Setting $x = 0$, we have

$$-1 - 0 - 0 = C$$

$$\Rightarrow C = -1.$$

So the solution is

$$-e^x - \frac{x^3}{3} + ye^{-y} = C = -1$$

$$\Rightarrow e^x + \frac{x^3}{3} - ye^{-y} = 1. \qquad \text{(1 mark)}$$

Question 3
Let

$$A = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 5 & 10 \\ -8 & 22 & 49 \end{bmatrix}.$$

(a) Find a lower triangular matrix L and an upper triangular matrix U such that

$$A = LU. \qquad \text{(12 marks)}$$

(b) Use part of your working in (a) to find A^{-1}.

(6 marks)

(c) Use your working above to find an upper triangular matrix U_1 and a lower triangular matrix L_1 such that

$$A^{-1} = U_1 L_1. \qquad \text{(7 marks)}$$

(d) Use your working above to find an upper triangular matrix U_2 and a lower triangular matrix L_2 such that

$$A^T = L_2 U_2. \qquad \text{(4 marks)}$$

Proof
(a) Form the augmented matrix

$$[A|I] = \left[\begin{array}{rrr|rrr} 1 & -2 & -3 & 1 & 0 & 0 \\ -2 & 5 & 10 & 0 & 1 & 0 \\ -8 & 22 & 49 & 0 & 0 & 1 \end{array}\right].$$

Copy down first column for L :

$$\begin{bmatrix} 1 \\ -2 \\ -8 \end{bmatrix}.$$

Now row reduce:
Step 1: Row 2 + 2 $\times$ Row 1; Row 3 + 8 $\times$ Row 1

$$\left[\begin{array}{rrr|rrr} 1 & -2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 1 & 0 \\ 0 & 6 & 25 & 8 & 0 & 1 \end{array}\right].$$

Copy down second column for L, starting at diagonal element:

$$\begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

Step 2: Row 3 - 6 $\times$ Row 2

$$\begin{aligned} & \begin{bmatrix} 1 & -2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 1 & 0 \\ 0 & 0 & 1 & -4 & -6 & 1 \end{bmatrix} \\ = \ & [U|E]. \end{aligned}$$

Copy down third column for L, starting at diagonal element:

$$[1].$$

So

$$U = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \text{(7 marks)}$$

Next, find L, by using copied columns above

$$\begin{bmatrix} 1 & & \\ -2 & 1 & \\ -8 & 6 & 1 \end{bmatrix}$$

Since all the diagonal elements are 1, we already have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -8 & 6 & 1 \end{bmatrix}. \qquad \text{(5 marks)}$$

(Alternatively, students can invert E to find L). Thus

$$A = LU.$$

(Check: not required of students:

$$\begin{aligned} & LU \\ = \ & \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -8 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \\ = \ & \begin{bmatrix} 1 & -2 & -3 \\ -2 & 5 & 10 \\ -8 & 22 & 49 \end{bmatrix}.) \end{aligned}$$

(b) We row reduced $[A|I]$, obtaining

$$[U|E] = \left[\begin{array}{rrrrrr} 1 & -2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 1 & 0 \\ 0 & 0 & 1 & -4 & -6 & 1 \end{array}\right].$$

Now we apply the algorithm for finding the inverse. We already have $1's$ along the diagonal, so need only clear entries above the diagonal:

Step 1: Row 2 - 4 $\times$ Row 3; Row 1 + 3 $\times$ Row 3

$$\left[\begin{array}{rrrrrr} 1 & -2 & 0 & -11 & -18 & 3 \\ 0 & 1 & 0 & 18 & 25 & -4 \\ 0 & 0 & 1 & -4 & -6 & 1 \end{array}\right].$$

Step 2: Row 1 + 2 $\times$ Row 2

$$\left[\begin{array}{rrrrrr} 1 & 0 & 0 & 25 & 32 & -5 \\ 0 & 1 & 0 & 18 & 25 & -4 \\ 0 & 0 & 1 & -4 & -6 & 1 \end{array}\right] = \left[I|A^{-1}\right].$$

So

$$A^{-1} = \left[\begin{array}{rrr} 25 & 32 & -5 \\ 18 & 25 & -4 \\ -4 & -6 & 1 \end{array}\right]. \qquad \text{(6 marks)}$$

(Check, not required of students:

$$\begin{aligned} & A\left[\begin{array}{rrr} 25 & 32 & -5 \\ 18 & 25 & -4 \\ -4 & -6 & 1 \end{array}\right] \\ = & \left[\begin{array}{rrr} 1 & -2 & -3 \\ -2 & 5 & 10 \\ -8 & 22 & 49 \end{array}\right]\left[\begin{array}{rrr} 25 & 32 & -5 \\ 18 & 25 & -4 \\ -4 & -6 & 1 \end{array}\right] = I.) \end{aligned}$$

(c) We know

$$A = LU$$

so

$$\begin{aligned} A^{-1} &= (LU)^{-1} \\ &= U^{-1}L^{-1} \\ &= U^{-1}E. \end{aligned}$$

We can take

$$U_1 = U^{-1}$$

and

$$L_1 = E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -6 & 1 \end{bmatrix}. \qquad \text{(2 marks)}$$

We find U^{-1} by applying the inverse algorithm to

$$[U|I] = \begin{bmatrix} 1 & -2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Since we already have $1's$ along the diagonal, we can clear entries above the diagonal:

Step 1: Row 2 - 4 $\times$ Row 3; Row 1 + 3 $\times$ Row 3

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Step 2: Row 1 + 2 $\times$ Row 2

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2 & -5 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \left[I|U^{-1}\right].$$

(Check, not required of students:

$$U \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = I.)$$

Thus we can take

$$U_1 = \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(5 marks)}$$

and then

$$A^{-1} = U_1 L_1,$$

or

$$\begin{bmatrix} 25 & 32 & -5 \\ 18 & 25 & -4 \\ -4 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -6 & 1 \end{bmatrix}.$$

(d)

$$\begin{aligned} A^T &= (LU)^T \\ &= U^T L^T \\ &= L_2 U_2, \end{aligned}$$

where

$$L_2 = U^T = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix}$$

$$U_2 = L^T = \begin{bmatrix} 1 & -2 & -8 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}.$$

(4 marks)

Question 4

Let

$$A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 1 & -1 & 7 \\ 1 & 1 & -2 & 1 \\ 2 & 2 & -4 & 2 \end{bmatrix}.$$

(a) (i) Apply the Gram-Schmidt process to the columns of A.
(ii) Give an orthonormal basis for Col$A$.

(13 marks)

(b) (i) What is the rank of A?
(ii) Are the columns of A linearly independent? Why?
(iii) Is A invertible? Why?

(3 marks)

(c) Compute the QR factorisation of A.

(8 marks)

(d) Which of the following is true?
(i) $Q^TQ = I$;
(ii) $QQ^T = I$.
Briefly explain your answer.

(2 marks)

(e) Find the least squares solution of

$$A\mathbf{x} = \mathbf{b} = \begin{bmatrix} -1 \\ -7 \\ -3 \\ -6 \end{bmatrix}.$$

(10 marks)

Is this also a solution to $A\mathbf{x} = \mathbf{b}$? If not, find the error $|A\mathbf{x} - \mathbf{b}|^2$.

Solution

(a) We apply Gram-Schmidt to

$$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

Step 1: $\mathbf{v}_1$
We just set

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}. \qquad \text{(1 mark)}$$

Step 2: $\mathbf{v}_2$
We form

$$\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1.$$

Here

$$\mathbf{x}_2 \cdot \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = 3\,(1) + 1(2 + 1(1) + 2(2) = 10.$$

Also

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = 1\,(1) + 2(2 + 1(1) + 2(2) = 10.$$

Then

$$\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{10}{10}\right) \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}. \qquad \text{(2 marks)}$$

Step 3: $\mathbf{v}_3$
We form

$$\mathbf{v}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right) \mathbf{v}_2.$$

Here

$$\mathbf{x}_3 \cdot \mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = 2\,(1) + (-1)\,(2) + (-2)\,(1) + (-4)\,2 = -10.$$

$$\mathbf{x}_3 \cdot \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 2\,(2) + (-1)\,(-1) + 0 + 0 = 5.$$

$$\mathbf{v}_2 \cdot \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 2\,(2) + (-1)\,(-1) + 0 + 0 = 5.$$

Then

$$\begin{aligned} \mathbf{v}_3 &= \begin{bmatrix} 2 \\ -1 \\ -2 \\ -4 \end{bmatrix} - \left(\frac{-10}{10}\right) \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{5}{5}\right) \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}. \end{aligned}$$

(3 marks)

Step 3: $\mathbf{v}_4$
We form

$$\mathbf{v}_4 = \mathbf{x}_4 - \left(\frac{\mathbf{x}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 - \left(\frac{\mathbf{x}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right) \mathbf{v}_2 - \left(\frac{\mathbf{x}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\right) \mathbf{v}_3.$$

Here

$$\mathbf{x}_4 \cdot \mathbf{v}_1 = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = 1(1) + 7(2) + 1(1) + 2(2) = 20;$$

$$\mathbf{x}_4 \cdot \mathbf{v}_2 = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 1(2) + 7(-1) + 1(0) + 2(0) = -5;$$

$$\mathbf{x}_4 \cdot \mathbf{v}_3 = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} = 1(1) + 7(2) + 1(-1) + 2(-2) = 10;$$

$$\mathbf{v}_3 \cdot \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} = 1\,(1) + 2(2) + (-1)\,(-1) + (-2)\,(-2) = 10.$$

So

$$\begin{aligned} \mathbf{v}_4 &= \begin{bmatrix} 1 \\ 7 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{20}{10}\right) \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{-5}{5}\right) \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{10}{10}\right) \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad \text{(3 marks)} \end{aligned}$$

So there is no $\mathbf{v}_4$. So we have three orthogonal vectors

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} \right\}.$$

Next, we normalize: set

$$\mathbf{u}_j = \frac{\mathbf{v}_j}{\|\mathbf{v}_j\|}, j = 1, 2, 3.$$

Here

$$\begin{aligned} \|\mathbf{v}_1\| &= \sqrt{1+4+1+4} = \sqrt{10}; \\ \|\mathbf{v}_2\| &= \sqrt{4+1+0+0} = \sqrt{5}; \\ \|\mathbf{v}_3\| &= \sqrt{1+4+1+4} = \sqrt{10}. \end{aligned}$$

So

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} \right\}. \qquad \text{(2 marks)}$$

(ii) The vectors above form an orthonormal basis for Col$A$. Indeed, they are linearly independent, as they are orthonormal, and span Col$A$, by one of our theorems.

(2 marks)

(b) (i) The rank is 3 as there are three vectors in the orthonormal basis.
(ii) No, the columns are not linearly independent: the 4th column is a linear combination of the earlier ones/ we obtained $\mathbf{v}_4 = \mathbf{0}$.
(iii) No, A is not invertible, its rank is 3, less than its number of columns.

(3 marks)

(c) We form

$$Q = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = \begin{bmatrix} 1/\sqrt{10} & 2/\sqrt{5} & 1/\sqrt{10} \\ 2/\sqrt{10} & -1/\sqrt{5} & 2/\sqrt{10} \\ 1/\sqrt{10} & 0 & -1/\sqrt{10} \\ 2/\sqrt{10} & 0 & -2/\sqrt{10} \end{bmatrix}. \qquad \text{(1 mark)}$$

Then

$$\begin{aligned} R &= Q^T A \\ &= \begin{bmatrix} 1/\sqrt{10} & 2/\sqrt{10} & 1/\sqrt{10} & 2/\sqrt{10} \\ 2/\sqrt{5} & -1/\sqrt{5} & 0 & 0 \\ 1/\sqrt{10} & 2/\sqrt{10} & -1/\sqrt{10} & -2/\sqrt{10} \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 1 & -1 & 7 \\ 1 & 1 & -2 & 1 \\ 2 & 2 & -4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 10/\sqrt{10} & 10/\sqrt{10} & -10/\sqrt{10} & 20/\sqrt{10} \\ 0 & 5/\sqrt{5} & 5/\sqrt{5} & -5/\sqrt{5} \\ 0 & 0 & 10/\sqrt{10} & 10/\sqrt{10} \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{10} & \sqrt{10} & -\sqrt{10} & 2\sqrt{10} \\ 0 & \sqrt{5} & \sqrt{5} & -\sqrt{5} \\ 0 & 0 & \sqrt{10} & \sqrt{10} \end{bmatrix}. \end{aligned}$$

Then $A = QR$, or,

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 1 & -1 & 7 \\ 1 & 1 & -2 & 1 \\ 2 & 2 & -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{10} & 2/\sqrt{5} & 1/\sqrt{10} \\ 2/\sqrt{10} & -1\,/\sqrt{5} & 2/\sqrt{10} \\ 1/\sqrt{10} & 0 & -1/\sqrt{10} \\ 2/\sqrt{10} & 0 & -2/\sqrt{10} \end{bmatrix} \begin{bmatrix} \sqrt{10} & \sqrt{10} & -\sqrt{10} & 2\sqrt{10} \\ 0 & \sqrt{5} & \sqrt{5} & -\sqrt{5} \\ 0 & 0 & \sqrt{10} & \sqrt{10} \end{bmatrix}.$$

(7 marks)

(c) (i) $Q^TQ = I$ is true as the columns of Q are orthonormal (guaranteed by Gram-Schmidt).
(ii) $QQ^T = I$ is not true as Q is not square (alternatively, not of full rank, not invertible).

(2 marks)

(d) We must solve

$$R\mathbf{x} = Q^T\mathbf{b}. \qquad \text{(2 marks)}$$

Here

$$Q^T\mathbf{b} = \begin{bmatrix} 1/\sqrt{10} & 2/\sqrt{10} & 1/\sqrt{10} & 2/\sqrt{10} \\ 2/\sqrt{5} & -1/\sqrt{5} & 0 & 0 \\ 1/\sqrt{10} & 2/\sqrt{10} & -1/\sqrt{10} & -2/\sqrt{10} \end{bmatrix} \begin{bmatrix} -1 \\ -7 \\ -3 \\ -6 \end{bmatrix} = \begin{bmatrix} -\frac{30}{\sqrt{10}} \\ \frac{5}{\sqrt{5}} \\ 0 \end{bmatrix}.$$

We must solve

$$\begin{bmatrix} \sqrt{10} & \sqrt{10} & -\sqrt{10} & 2\sqrt{10} \\ 0 & \sqrt{5} & \sqrt{5} & -\sqrt{5} \\ 0 & 0 & \sqrt{10} & \sqrt{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3\sqrt{10} \\ \sqrt{5} \\ 0 \end{bmatrix}. \qquad \text{(2 marks)}$$

The first, second, third columns are pivotal, and the fourth is not, so we set $x_4 = t$. Now substitute back:

$$\begin{array}{rrrrl} \sqrt{10}x_1+ & \sqrt{10}x_2 & -\sqrt{10}x_3+ & 2\sqrt{10}x_4 = & -3\sqrt{10} \\ & \sqrt{5}x_2 & +\sqrt{5}x_3- & \sqrt{5}x_4 = & \sqrt{5} \\ & & \sqrt{10}x_3+ & \sqrt{10}x_4 = & 0 \end{array}.$$

We can simplify the arithmetic by dividing by $\sqrt{10}$ in the first, third rows, and $\sqrt{5}$ in the second:

$$\begin{array}{rrrrr} x_1+ & x_2 & -x_3+ & 2x_4 = & -3 \\ & x_2 & +x_3- & x_4 = & 1 \\ & & x_3+ & x_4 = & 0 \end{array}.$$

Then

$$x_3 = -x_4 = -t.$$

Next,

$$x_2 = 1 - x_3 + x_4 = 1 + t + t = 1 + 2t.$$

Then

$$\begin{aligned} x_1 &= -3 - x_2 + x_3 - 2x_4 \\ &= -3 - 1 - 2t - t - 2t \\ &\Rightarrow x_1 = -4 - 5t \end{aligned}$$

Then the least squares solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4-5t \\ 1+2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 2 \\ -1 \\ 1 \end{bmatrix}. \qquad \text{(4 marks)}$$

We see that

$$A\mathbf{x} = A\left(\begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ -7 \\ -3 \\ -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \text{(2 marks)}$$

so it is also a solution to $A\mathbf{x} = \mathbf{b}$.

Question 5

Let

$$A = \begin{bmatrix} 5 & 1 & -5 \\ 1 & 1 & 1 \\ -5 & 1 & 5 \end{bmatrix}.$$

(a) Compute the characteristic polynomial $p(t)$ of A. Find the eigenvalues of A, and their algebraic multiplicities.

(14 marks)

(b) For each eigenvalue give a basis for the eigenspace of corresponding eigenvectors. Also describe the geometric multiplicity of each eigenvalue.

(15 marks)

(c) Find an invertible matrix V and a diagonal matrix D such that

$$A = VDV^{-1}. \qquad \text{(2 marks)}$$

(You need not compute V^{-1}).

(d) Find an orthogonal matrix U such that

$$A = UDU^T. \qquad \text{(4 marks)}$$

(e) Write down an expression for A^{-6} in terms of U and D. What are the eigenvalues of A^{-6}?

(4 marks)

(f) Write down $\det(A)$ using the result of (c) or (d).

(3 marks)

(g) Write down the singular values of A.

(3 marks)

Solution

(a) We calculate

$$p(t) = \det(A - tI) = \det \begin{bmatrix} 5-t & 1 & -5 \\ 1 & 1-t & 1 \\ -5 & 1 & 5-t \end{bmatrix}. \qquad \text{(2 marks)}$$

Step 1: Swap Rows 1 and 2

$$p(t) = -\det \begin{bmatrix} 1 & 1-t & 1 \\ 5-t & 1 & -5 \\ -5 & 1 & 5-t \end{bmatrix}$$

Step 2: Row 1 - $(5-t)$ $\times$**Row 1; Row 3 + 5** $\times$ **Row 1**

$$\begin{aligned} p(t) &= -\det \begin{bmatrix} 1 & 1-t & 1 \\ 0 & 1-(5-t)(1-t) & -5-(5-t) \\ 0 & 1+5(1-t) & 5-t+5 \end{bmatrix} \\ &= -\det \begin{bmatrix} 1 & 1-t & 1 \\ 0 & -t^2+6t-4 & t-10 \\ 0 & 6-5t & -(t-10) \end{bmatrix}. \end{aligned}$$

Step 3: Use cofactor expansion by first column

$$\begin{aligned} &= -\left\{\left[-t^2+6t-4\right][-(t-10)]-(t-10)[6-5t]\right\} \\ &= (t-10)\left\{-t^2+6t-4+6-5t\right\} \\ &= (t-10)\left\{-t^2+t+2\right\} \\ &= -(t-10)\left(t^2-t-2\right) = -(t-10)(t-2)(t+1). \end{aligned}$$

So the eigenvalues are $-1, 2, -10$, all distinct. They all have algebraic multiplicity 1.

(14 marks)

(b) Next we find the eigenvectors. For each eigenvalue λ, we must find a vector $\mathbf{v} \neq \mathbf{0}$ such that

$$(A-\lambda I)\mathbf{v} = \mathbf{0},$$

so we apply row reduction to

$$[A-\lambda I|\mathbf{0}] = \begin{bmatrix} 5-\lambda & 1 & -5 & 0 \\ 1 & 1-\lambda & 1 & 0 \\ -5 & 1 & 5-\lambda & 0 \end{bmatrix}.$$

Eigenvectors for $\lambda = -1$

We apply row reduction to

$$\begin{bmatrix} 6 & 1 & -5 & 0 \\ 1 & 2 & 1 & 0 \\ -5 & 1 & 6 & 0 \end{bmatrix}$$

Step 1: Swap Rows 1, 2

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 6 & 1 & -5 & 0 \\ -5 & 1 & 6 & 0 \end{bmatrix}$$

Step 2: Row 2 -$6\times$ **Row 1; Row 3** $+$ $5\times$ **Row 1**

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -11 & -11 & 0 \\ 0 & 11 & 11 & 0 \end{bmatrix}$$

Step 3: Divide Row 2 by -11; Row 3 by 11

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 4: Row 3 - Row 2

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Continue to reduced echelon form.
Step 5: Row 1 - 2 $\times$ **Row 2**

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here x_1, x_2 are basic, x_3 is free. So set $x_3 = t$.
Step 3: Back Substitution

$$\begin{array}{ccrl} x_1 & & -x_3 = & 0 \\ & x_2 & +x_3 = & 0 \end{array}$$

Then

$$x_1 = x_3 = t;$$
$$x_2 = -x_3 = -t.$$

So

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

There is one linearly independent eigenvector and we can take $t = 1$ to get

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}. \qquad \text{(4 marks)}$$

A basis for the corresponding eigenspace is $\{\mathbf{v}_1\}$. As there is one linearly independent eigenvector, $\lambda = -1$ has geometric multiplicity 1.

(1 mark)

Eigenvectors for $\lambda = 2$
We apply row reduction to

$$\begin{bmatrix} 3 & 1 & -5 & 0 \\ 1 & -1 & 1 & 0 \\ -5 & 1 & 3 & 0 \end{bmatrix}.$$

Step 1: Swap Rows 1 and Row 2

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & 1 & -5 & 0 \\ -5 & 1 & 3 & 0 \end{bmatrix}.$$

Step 2: Row 2 − 3×Row 1; Row 3 + 5× Row 1

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 4 & -8 & 0 \\ 0 & -4 & 8 & 0 \end{bmatrix}.$$

Step 3: Row 3 + Row 2

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Step 4: Divide Row 2 by 4

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Step 5: Row 1 + Row 2

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that x_1, x_2 are basic, while x_3 is free. So set $x_3 = t$.

Step 6: Back Substitution

$$\begin{array}{rrrl} x_1 & & -x_3 = & 0 \\ & x_2 & -2x_3 = & 0 \end{array}$$

Then,

$$x_1 = x_3 = t.$$

$$x_2 = 2x_3 = 2t.$$

So

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

There is one linearly independent eigenvector and we can take

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}. \qquad \text{(4 marks)}$$

A basis for the corresponding eigenspace is $\{\mathbf{v}_2\}$. As there is one linearly independent eigenvector, $\lambda = 2$ has geometric multiplicity 1.

(1 mark)

Eigenvectors for $\lambda = 10$

We apply row reduction to

$$\begin{bmatrix} -5 & 1 & -5 & 0 \\ 1 & -9 & 1 & 0 \\ -5 & 1 & -5 & 0 \end{bmatrix}$$

Step 1: Swap Rows 1 and Row 2

$$\begin{bmatrix} 1 & -9 & 1 & 0 \\ -5 & 1 & -5 & 0 \\ -5 & 1 & -5 & 0 \end{bmatrix}.$$

Step 2: Row 2 + 5×Row 1; Row 3 + 5× Row 1

$$\begin{bmatrix} 1 & -9 & 1 & 0 \\ 0 & -44 & 0 & 0 \\ 0 & -44 & 0 & 0 \end{bmatrix}.$$

Step 3: Divide Rows 2, 3 by -44

$$\begin{bmatrix} 1 & -9 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Step 4: Row 3 - Row 2

$$\begin{bmatrix} 1 & -9 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Step 5: Row 1 + 9 × Row 2

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then x_1, x_2 are basic, x_3 is free. So set $x_3 = t$.
Step 3: Back Substitution

$$\begin{array}{rcrcl} x_1 & & +x_3 & = & 0 \\ & x_2 & & = & 0 \end{array}$$

Then

$$x_1 = -x_3 = -t.$$

So

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

There is one linearly independent eigenvector and we can take $t = 1$ to get

$$\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \qquad \text{(4 marks)}$$

A basis for the corresponding eigenspace is $\{\mathbf{v}_3\}$. As there is one linearly independent eigenvector, $\lambda = 10$ has geometric multiplicity 1.

(1 mark)

(c) We let

$$V = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

and

$$D = \begin{bmatrix} -1 & & \\ & 2 & \\ & & 10 \end{bmatrix}.$$

and then

$$\begin{aligned} A &= VDV^{-1} \\ &= \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & & \\ & 2 & \\ & & 10 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \end{aligned} \qquad \text{(2 marks)}$$

(d) We apply Gram-Schmidt to $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$. But, since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ come from distinct eigenvalues, they are already orthogonal to each other. (The matrix is symmetric). So we can just let

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix};$$

$$\mathbf{u}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix};$$

$$\mathbf{u}_3 = \frac{1}{\|\mathbf{v}_3\|}\mathbf{v}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Then we can take

$$\begin{aligned} U &= [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3] \\ &= \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ -1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \end{aligned}$$

(4 marks)

and U is orthogonal so

$$A = UDU^{-1} = UDU^T.$$

(e) As all eigenvalues are non-zero, A is invertible. So

$$\begin{aligned} A^{-6} &= UD^{-6}U^T \\ &= U\begin{bmatrix} (-1)^{-6} & & \\ & (2)^{-6} & \\ & & 10^{-6} \end{bmatrix}U^T. \end{aligned}$$

(2 marks)

The eigenvalues are just $\left\{(-1)^{-6}, 2^{-6}, 10^{-6}\right\}$.

(2 marks)

(f)

$$\begin{aligned} \det(A) &= \det\left(VDV^{-1}\right) = \det(V)\det(D)\det\left(V^{-1}\right) \\ &= \det\left(VV^{-1}\right)\det(D) \\ &= \det(I)\det(D) \\ &= (-1)(2)(10) \\ &= -20. \end{aligned}$$

(3 marks)

(g) The singular values of A are just the square roots of the eigenvalues of $A^TA = A^2$, since A is symmetric. But the eigenvalues of A^2 are $(-1)^2, 2^2, 10^2$, so the singular values are

$$1, 2, 10$$

(3 marks)