MATH1502 - C Group Calculus II Final Exam
Wednesday May 4th, 2011, 11:30 - 2:20

No calculators or notes are allowed. There are 133 marks on the final exam, 125=100%.

Name____

Student Number________________________________

Group C ________________________________

TA________________________ ______________

Question	**Grade**	**Out of**
1		14
2		15
3		19
4		40
5		45
Total		133

Question 1

(a) Find the radius of convergence and interval of convergence of the power series

$$\sum_{k=2}^{\infty} \frac{1+(-1)^k/5}{k(\ln k)^2}(3x)^k. \qquad \text{(8 marks)}$$

Solution

We use the root test applied to

$$a_k = \left|\frac{1+(-1)^k/5}{k(\ln k)^2}(3x)^k\right| = \frac{1+(-1)^k/5}{k(\ln k)^2}(3|x|)^k.$$

We see that

$$\begin{aligned}
\lim_{k\to\infty} a_k^{1/k} &= \lim_{k\to\infty}\left(\frac{1+(-1)^k/5}{k(\ln k)^2}|x|^k\right)^{1/k} \\
&= \lim_{k\to\infty}\frac{\left(1+(-1)^k/5\right)^{1/k} 3|x|}{k^{1/k}(\ln k)^{2/k}} \\
&= 3|x|.
\end{aligned}$$

(**Justification of this, not really required from students:**
Recall $\lim_{k\to\infty} k^{1/k} = 1$. Moreover, $\lim_{k\to\infty}(\ln k)^{1/k} = 1$. They can assume the second limit, but give them extra credit if they give some proof. For example, $\lim_{k\to\infty}\ln k = \infty$ and

$$\lim_{k\to\infty}\frac{\ln k}{k} = 0,$$

so

$$1 \le \ln k \le k \text{ for large enough } k$$

and then

$$1^{1/k} \le (\ln k)^{1/k} \le k^{1/k}.$$

By the pinching/ sandwich rule,

$$\lim_{k\to\infty}(\ln k)^{1/k} = 1.$$

Finally,

$$\left(\frac{4}{5}\right)^{1/k} \le \left(1+(-1)^k/5\right)^{1/k} \le \left(\frac{6}{5}\right)^{1/k},$$

so by the pinching rule,

$$1=\lim_{k\to\infty}\left(\frac{4}{5}\right)^{1/k}\le\lim_{k\to\infty}\left(1+(-1)^k/5\right)^{1/k}\le\lim_{k\to\infty}\left(\frac{6}{5}\right)^{1/k}=1.)$$

By the root test, we have convergence for

$$3|x|<1\Leftrightarrow|x|<1/3.$$

And we have divergence for $|x|>1/3$. So the **radius of convergence is** 1/3.

(4 marks)

Next, we test the endpoints $x=\pm1/3$.
$\mathbf{x=-1/3}$
Here

$$\sum_{k=2}^{\infty}\frac{1+(-1)^k/5}{k(\ln k)^2}(3x)^k=\sum_{k=2}^{\infty}\frac{1+(-1)^k/5}{k(\ln k)^2}(-1)^k.$$

Although this looks like an alternating series, we CANNOT use the alternating series test, since the terms do not decrease in size. (They could split it up). However, we can take the absolute values,

$$\left|\frac{1+(-1)^k/5}{k(\ln k)^2}\right|\le\frac{2}{k(\ln k)^2}$$

and use that the series $\sum\frac{1}{k(\ln k)^2}$ converges. This they have seen, and may assume, or justify via the integral tests. The comparison test shows that

$$\sum_{k=2}^{\infty}\frac{1+(-1)^k/5}{k(\ln k)^2}(-1)^k$$

converges absolutely, and hence converges.
$\mathbf{x=1/3}$
Here again

$$\sum_{k=2}^{\infty}\frac{1+(-1)^k/5}{k(\ln k)^2}(3x)^k=\sum_{k=2}^{\infty}\frac{1+(-1)^k/5}{k(\ln k)^2}$$

and the argument used above shows that the series converges absolutely, and hence converges. So the **interval of convergence** is $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

(4 marks)

(b) Find the radius of convergence and interval of convergence of the power series

$$\sum_{k=0}^{\infty} 2^{-k^2} k! x^k. \qquad \text{(6 marks)}$$

Solution

Because of the factorials, we use the ratio test, applied to

$$a_k = \left|2^{-k^2} k! x^k\right| = 2^{-k^2} k! \left|x\right|^k,$$

with $x \neq 0$. We see that

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_{k+1}}{a_k} &= \lim_{k\to\infty} \frac{2^{-(k+1)^2}(k+1)! \left|x\right|^k}{2^{-k^2} k! \left|x\right|^k} \qquad \text{(2 marks)} \\
&= \lim_{k\to\infty} 2^{-k^2-2k-1+k^2} \frac{(k+1)!}{k!} \left|x\right| \\
&= \lim_{k\to\infty} 2^{-2k-1}(k+1)\left|x\right| \\
&= \frac{1}{2} \lim_{k\to\infty} \frac{k+1}{4^k} \left|x\right| = 0.
\end{aligned}$$

(We are using the class limit that $\lim_{k\to\infty} k/4^k = 0$). By the ratio test, this converges, no matter is what $|x|$. That is, the radius of convergence is ∞, and hence the interval of convergence is $(-\infty, \infty)$.

(2 marks)

Question 2

(a)

$$\frac{y'}{\cos x \sin x} + 2y = 4$$

that satisfies

$$y(\pi) = -1. \qquad \text{(8 marks)}$$

Solution

We want the form

$$y' + p(x)\,y = q(x),$$

so multiply by $\cos x \sin x$:

$$y' + (2\cos x \sin x)\,y = 4\cos x \sin x.$$

Step 1 Integrating Factor

Then

$$p(x) = 2\cos x \sin x$$

and

$$H(x) = \int p(x)\,dx = \int 2\cos x \sin x \; dx = \sin^2 x$$

so the integrating factor is

$$e^{H(x)} = e^{\sin^2 x}. \qquad \text{(2 marks)}$$

(Alternative:

$$H(x) = -\cos^2 x \text{ and } e^{H(x)} = e^{-\cos^2 x}.)$$

Step 2 Multiply by integrating factor

$$e^{\sin^2 x} y' + e^{\sin^2 x}(2\cos x \sin x)\,y = e^{\sin^2 x}(4\cos x \sin x),$$

$$\Rightarrow \frac{d}{dx}\left(e^{\sin^2 x} y\right) = e^{\sin^2 x}(4\cos x \sin x).$$

Step 3 Integrate

$$\begin{aligned}
e^{\sin^2 x} y &= \int e^{\sin^2 x} \left(4 \cos x \sin x\right) dx + C \\
&= 2 \int e^{\sin^2 x} \left(\frac{d}{dx} \sin^2 x \right) dx + C \\
&= 2 \int e^t dt + C \\
(\text{substitution } t &= \sin^2 x) \\
&= 2e^t + C = 2e^{\sin^2 x} + C.
\end{aligned}$$

Then

$$y = 2 + Ce^{-\sin^2 x}. \qquad \text{(4 marks)}$$

Step 4 Find C

$$\begin{gathered}
-1 = y(\pi) = 2 + Ce^0 = 2 + C \\
\Rightarrow C = -3.
\end{gathered}$$

So the solution is

$$y(x) = 2 - 3e^{-\sin^2 x}. \qquad \text{(2 marks)}$$

(If integrating factor was chosen as $e^{-\cos^2 x}$, solution is

$$y(x) = 2 - 3e^{-1+\cos^2 x}).$$

(b) Find the solution of the equation

$$x^2 (\cos y) y' = 1 - \left(y^2 + y^3\right) x^2 y',$$

with

$$y(1) = 0. \qquad \text{(7 marks)}$$

Solution
We need to separate x's and y's. First bring together terms in y' :

$$y' \left(x^2 \cos y + \left(y^2 + y^3\right) x^2\right) = 1.$$

Next, divide by x^2:

$$y' \left(\cos y + y^2 + y^3\right) = \frac{1}{x^2}.$$

Now bring all terms to left (or, integrate, if you like):

$$y' \left(\cos y + y^2 + y^3\right) - \frac{1}{x^2} = 0. \qquad \text{(3 marks)}$$

Now integrate with respect to x :

$$\int \left(\cos y + y^2 + y^3\right) \frac{dy}{dx} dx - \int \frac{1}{x^2} dx = \int 0 dx$$

$$\Rightarrow \int \left(\cos y + y^2 + y^3\right) dy + \frac{1}{x} = C$$

$$\Rightarrow \sin y + \frac{1}{3} y^3 + \frac{1}{4} y^4 + \frac{1}{x} = C. \qquad \text{(2 marks)}$$

We now solve for C by setting $x = 1$:

$$\sin 0 + 0 + 0 + 1 = C.$$

So the solution is

$$\sin y + \frac{1}{3} y^3 + \frac{1}{4} y^4 + \frac{1}{x} = 1. \qquad \text{(2 marks)}$$

Question 3

$$A = \begin{bmatrix} -1 & 2 & 1 & 2 \\ -2 & 5 & 6 & 5 \\ -2 & 2 & -5 & 2 \\ -1 & 4 & 8 & 7 \end{bmatrix}.$$

(i) Find a lower triangular matrix L and an upper triangular matrix U such that

$$A = LU. \qquad \text{(11 marks)}$$

(ii) Use your working from (a) to find A^{-1}.

(8 marks)

Solution

(i) We apply row reduction to

$$[A|I] = \left[\begin{array}{cccccccc} -1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ -2 & 5 & 6 & 5 & 0 & 1 & 0 & 0 \\ -2 & 2 & -5 & 2 & 0 & 0 & 1 & 0 \\ -1 & 4 & 8 & 7 & 0 & 0 & 0 & 1 \end{array}\right].$$

Step 1 Row 2 - 2× Row 1; Row 3 - 2× Row 1; Row 4 - Row 1

$$\left[\begin{array}{cccccccc} -1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & -2 & 1 & 0 & 0 \\ 0 & -2 & -7 & -2 & -2 & 0 & 1 & 0 \\ 0 & 2 & 7 & 5 & -1 & 0 & 0 & 1 \end{array}\right].$$

Step 2 Row 3 + 2× Row 2; Row 4 - 2× Row 2

$$\left[\begin{array}{cccccccc} -1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -6 & 2 & 1 & 0 \\ 0 & 0 & -1 & 3 & 3 & -2 & 0 & 1 \end{array}\right].$$

Step 3 Row 4 + Row 3

$$\left[\begin{array}{cccccccc} -1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -6 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & -3 & 0 & 1 & 1 \end{array}\right] = [U|R].$$

So

$$U=\begin{bmatrix}-1&2&1&2\\0&1&4&1\\0&0&1&0\\0&0&0&3\end{bmatrix};R=\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-6&2&1&0\\-3&0&1&1\end{bmatrix}. \qquad \text{(6 marks)}$$

Next, we find $L=R^{-1}$ by applying row reduction to

$$[R|I]=\left[\begin{array}{cccc cccc}1&0&0&0&1&0&0&0\\-2&1&0&0&0&1&0&0\\-6&2&1&0&0&0&1&0\\-3&0&1&1&0&0&0&1\end{array}\right].$$

Step 1 Row 2 + 2× Row 1; Row 3 + 6× Row 1; Row 4 + 3× Row 1

$$\left[\begin{array}{cccc cccc}1&0&0&0&1&0&0&0\\0&1&0&0&2&1&0&0\\0&2&1&0&6&0&1&0\\0&0&1&1&3&0&0&1\end{array}\right].$$

Step 2 Row 3 - 2× Row 2

$$\left[\begin{array}{cccc cccc}1&0&0&0&1&0&0&0\\0&1&0&0&2&1&0&0\\0&0&1&0&2&-2&1&0\\0&0&1&1&3&0&0&1\end{array}\right].$$

Step 3 Row 4 - Row 3

$$\left[\begin{array}{cccc cccc}1&0&0&0&1&0&0&0\\0&1&0&0&2&1&0&0\\0&0&1&0&2&-2&1&0\\0&0&0&1&1&2&-1&1\end{array}\right]=\left[I|R^{-1}\right]=[I|L]. \qquad \text{(5 marks)}$$

So

$$A=LU=\begin{bmatrix}1&0&0&0\\2&1&0&0\\2&-2&1&0\\1&2&-1&1\end{bmatrix}\begin{bmatrix}-1&2&1&2\\0&1&4&1\\0&0&1&0\\0&0&0&3\end{bmatrix}.$$

(ii) We applied row reduction to $[A|I]$, as above, giving

$$\begin{bmatrix} -1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -6 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & -3 & 0 & 1 & 1 \end{bmatrix} = [U|R]\,.$$

Step A Divide rows by diagonal entries

$$\begin{bmatrix} 1 & -2 & -1 & -2 & -1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -6 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Step B Row 2 - Row 4; Row 1 + 2× Row 4

$$\begin{bmatrix} 1 & -2 & -1 & 0 & -3 & 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & 1 & 4 & 0 & -1 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & -6 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Step C Row 2 - 4× Row 3; Row 1 + Row 3

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -9 & 2 & \frac{5}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 0 & 23 & -7 & -4\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & -6 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Step D Row 1 + 2 × Row 2

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 37 & -12 & -7 & 0 \\ 0 & 1 & 0 & 0 & 23 & -7 & -4\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & -6 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 37 & -12 & -7 & 0 \\ 23 & -7 & -4\frac{1}{3} & -\frac{1}{3} \\ -6 & 2 & 1 & 0 \\ -1 & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

(8 marks)

(Check, not required of students:

$$
A\begin{bmatrix} 37 & -12 & -7 & 0 \\ 23 & -7 & -4\frac{1}{3} & -\frac{1}{3} \\ -6 & 2 & 1 & 0 \\ -1 & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}
$$

$$
= \begin{bmatrix} -1 & 2 & 1 & 2 \\ -2 & 5 & 6 & 5 \\ -2 & 2 & -5 & 2 \\ -1 & 4 & 8 & 7 \end{bmatrix} \begin{bmatrix} 37 & -12 & -7 & 0 \\ 23 & -7 & -4\frac{1}{3} & -\frac{1}{3} \\ -6 & 2 & 1 & 0 \\ -1 & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix})
$$

Question 4

Let

$$A = \begin{bmatrix} 1 & 1 & 4 & 6 \\ 2 & 5 & 5 & 12 \\ 0 & 1 & -1 & 0 \\ 3 & 1 & 0 & 4 \end{bmatrix}.$$

(a) Find an orthonormal basis for $Img\,(A)$.

(13 marks)

(b) (i) What is the rank of A?

(ii) Are the columns of A linearly independent? Why?

(iii) Is A invertible? Why?

(3 marks)

(c) Compute the QR factorisation of A.

(12 marks)

(d) Which of the following is true?

(i) $Q^TQ = I$;

(ii) $QQ^T = I$.

Briefly explain your answer.

(2 marks)

(e) Find the least squares solution of

$$A\mathbf{x} = \mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 4 \end{bmatrix}. \qquad \text{(10 marks)}$$

Is this also a solution to $A\underline{x} = \underline{b}$? If not, find the error $|A\underline{x} - \underline{b}|^2$.

Solution

(a) We apply Gram-Schmidt to

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 12 \\ 0 \\ 4 \end{bmatrix} \right\}.$$

Step 1 $\mathbf{u}_1$
We form
$$|\mathbf{v}_1| = \sqrt{1+4+0+9} = \sqrt{14}.$$
Then
$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}. \qquad \text{(2 marks)}$$

Step 2 $\mathbf{u}_2$
We form
$$\mathbf{w}_2 = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{u}_1)\, \mathbf{u}_1.$$
Here
$$\mathbf{v}_2 \cdot \mathbf{u}_1 = \begin{bmatrix} 1 \\ 5 \\ 1 \\ 1 \end{bmatrix} \cdot \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} \right) = \frac{1}{\sqrt{14}} (1+10+0+3) = \frac{14}{\sqrt{14}} = \sqrt{14}.$$
Then
$$\begin{aligned} \mathbf{w}_2 &= \begin{bmatrix} 1 \\ 5 \\ 1 \\ 1 \end{bmatrix} - \left(\sqrt{14}\right) \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix}. \end{aligned}$$
Then
$$|\mathbf{w}_2| = \sqrt{0+9+1+4} = \sqrt{14},$$
so
$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{|\mathbf{w}_2|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix}. \qquad \text{(3 marks)}$$

Step 3 $\mathbf{u}_3$
We form
$$\mathbf{w}_3 = \mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{u}_1)\, \mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2)\, \mathbf{u}_2.$$

Here

$$\mathbf{v}_3\cdot\mathbf{u}_1 = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 0 \end{bmatrix} \cdot \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} \right) = \frac{1}{\sqrt{14}} (4 + 10 + 0 + 0) = \frac{14}{\sqrt{14}} = \sqrt{14}.$$

$$\mathbf{v}_3\cdot\mathbf{u}_2 = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 0 \end{bmatrix} \cdot \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} \right) = \frac{1}{\sqrt{14}} (0 + 15 - 1 + 0) = \frac{14}{\sqrt{14}} = \sqrt{14}.$$

Then

$$\begin{aligned} \mathbf{w}_3 &= \begin{bmatrix} 4 \\ 5 \\ -1 \\ 0 \end{bmatrix} - \left(\sqrt{14}\right) \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} \right) - \left(\sqrt{14}\right) \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} \right) \\ &= \begin{bmatrix} 4 \\ 5 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \\ -1 \end{bmatrix}. \end{aligned}$$

Then

$$|\mathbf{w}_3| = \sqrt{9 + 0 + 4 + 1} = \sqrt{14},$$

so

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{|\mathbf{w}_3|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ 0 \\ -2 \\ -1 \end{bmatrix}. \qquad \text{(4 marks)}$$

Step 4

$$\mathbf{w}_4 = \mathbf{v}_4 - (\mathbf{v}_4\cdot\mathbf{u}_1)\,\mathbf{u}_1 - (\mathbf{v}_4\cdot\mathbf{u}_2)\,\mathbf{u}_2 - (\mathbf{v}_4\cdot\mathbf{u}_3)\,\mathbf{u}_3.$$

Here

$$\mathbf{v}_4\cdot\mathbf{u}_1 = \begin{bmatrix} 6 \\ 12 \\ 0 \\ 4 \end{bmatrix} \cdot \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} \right) = \frac{1}{\sqrt{14}} (6 + 24 + 0 + 12) = \frac{42}{\sqrt{14}} = 3\sqrt{14};$$

$$\mathbf{v}_4\cdot\mathbf{u}_2 = \begin{bmatrix} 6 \\ 12 \\ 0 \\ 4 \end{bmatrix} \cdot \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} \right) = \frac{1}{\sqrt{14}} (0 + 36 + 0 - 8) = \frac{28}{\sqrt{14}} = 2\sqrt{14};$$

$$\mathbf{v}_4\cdot\mathbf{u}_3 = \begin{bmatrix} 6 \\ 12 \\ 0 \\ 4 \end{bmatrix} \cdot \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ 0 \\ -2 \\ -1 \end{bmatrix} \right) = \frac{1}{\sqrt{14}} (18 + 0 + 0 - 4) = \frac{14}{\sqrt{14}} = \sqrt{14};$$

Then

$$\begin{aligned} \mathbf{w}_4 &= \begin{bmatrix} 6 \\ 12 \\ 0 \\ 4 \end{bmatrix} - \left(3\sqrt{14}\right) \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} \right) - \left(2\sqrt{14}\right) \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} \right) \\ &\quad - \left(\sqrt{14}\right) \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ 0 \\ -2 \\ -1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 6 \\ 12 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ -6 \\ 0 \\ -9 \end{bmatrix} + \begin{bmatrix} 0 \\ -6 \\ -2 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad \text{(4 marks)} \end{aligned}$$

So, there is no $\mathbf{u}_4$. Thus an orthonormal basis for $\mathrm{Img}(A)$ is

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \frac{1}{\sqrt{14}} \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ 0 \\ -2 \\ -1 \end{bmatrix} \right\}.$$

(b) (i) The rank is 3 as there are three vectors in the orthonormal basis.
(ii) No, the columns are not linearly independent: the 4th column is a linear combination of the earlier ones/ we obtained $\mathbf{w}_4 = \mathbf{0}$.
(iii) No, A is not invertible, its rank is 3, less than its number of columns.

(3 marks)

(c) We form

$$Q = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = \begin{bmatrix} 1/\sqrt{14} & 0 & 3/\sqrt{14} \\ 2\ /\sqrt{14} & 3/\sqrt{14} & 0 \\ 0 & 1/\sqrt{14} & -2/\sqrt{14} \\ 3/\sqrt{14} & -2/\sqrt{14} & -1/\sqrt{14} \end{bmatrix}. \qquad \text{(2 marks)}$$

Then

$$\begin{aligned} R &= \begin{bmatrix} \mathbf{u}_1{\cdot}\mathbf{v}_1 & \mathbf{u}_1{\cdot}\mathbf{v}_2 & \mathbf{u}_1{\cdot}\mathbf{v}_3 & \mathbf{u}_1{\cdot}\mathbf{v}_4 \\ 0 & \mathbf{u}_2{\cdot}\mathbf{v}_2 & \mathbf{u}_2{\cdot}\mathbf{v}_3 & \mathbf{u}_2{\cdot}\mathbf{v}_4 \\ 0 & 0 & \mathbf{u}_3{\cdot}\mathbf{v}_3 & \mathbf{u}_3{\cdot}\mathbf{v}_4 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{u}_1{\cdot}\mathbf{v}_1 & \sqrt{14} & \sqrt{14} & 3\sqrt{14} \\ 0 & \mathbf{u}_2{\cdot}\mathbf{v}_2 & \sqrt{14} & 2\sqrt{14} \\ 0 & 0 & \mathbf{u}_3{\cdot}\mathbf{v}_3 & \sqrt{14} \end{bmatrix}. \qquad \text{(4 marks)} \end{aligned}$$

Here

$$\mathbf{u}_1{\cdot}\mathbf{v}_1 = \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}} (1 + 4 + 0 + 9) = \sqrt{14}.$$

(Alternatively, $\mathbf{u}_1{\cdot}\mathbf{v}_1 = |\mathbf{v}_1| = \sqrt{3}$). Next,

$$\mathbf{u}_2{\cdot}\mathbf{v}_2 = \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 5 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{14}} (0 + 15 + 1 - 2) = \sqrt{14}.$$

$$\mathbf{u}_3{\cdot}\mathbf{v}_3 = \left(\frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ 0 \\ -2 \\ -1 \end{bmatrix} \right) \cdot \begin{bmatrix} 4 \\ 5 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{14}} (12 + 0 + 2 + 0) = \sqrt{14}.$$

So

$$R = \begin{bmatrix} \sqrt{14} & \sqrt{14} & \sqrt{14} & 3\sqrt{14} \\ 0 & \sqrt{14} & \sqrt{14} & 2\sqrt{14} \\ 0 & 0 & \sqrt{14} & 1\sqrt{14} \end{bmatrix}. \qquad \text{(6 marks)}$$

Then the QR factorization $A = QR$ is

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 2 & 5 & 5 & 12 \\ 0 & 1 & -1 & 0 \\ 3 & 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} & 0 & 3/\sqrt{14} \\ 2\ /\sqrt{14} & 3/\sqrt{14} & 0 \\ 0 & 1/\sqrt{14} & -2/\sqrt{14} \\ 3/\sqrt{14} & -2/\sqrt{14} & -1/\sqrt{14} \end{bmatrix} \begin{bmatrix} \sqrt{14} & \sqrt{14} & \sqrt{14} & 3\sqrt{14} \\ 0 & \sqrt{14} & \sqrt{14} & 2\sqrt{14} \\ 0 & 0 & \sqrt{14} & 1\sqrt{14} \end{bmatrix}.$$

(c) (i) $Q^TQ = I$ is true as the columns of Q are orthonormal (guaranteed by Gram-Schmidt).
(ii) $QQ^T = I$ is not true as Q is not square (alternatively, not of full rank, not invertible).

(2 marks)

(d) We must solve

$$R\mathbf{x} = Q^T\mathbf{b}. \qquad \text{(2 marks)}$$

Here

$$Q^T\mathbf{b} = \begin{bmatrix} 1/\sqrt{14} & 2/\sqrt{14} & 0 & 3/\sqrt{14} \\ 0 & 3/\sqrt{14} & 1/\sqrt{14} & -2/\sqrt{14} \\ 3/\sqrt{14} & 0 & -2/\sqrt{14} & -1/\sqrt{14} \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{14}{\sqrt{14}} \\ 0 \\ -\frac{14}{\sqrt{14}} \end{bmatrix}.$$

We must solve

$$\begin{bmatrix} \sqrt{14} & \sqrt{14} & \sqrt{14} & 3\sqrt{14} \\ 0 & \sqrt{14} & \sqrt{14} & 2\sqrt{14} \\ 0 & 0 & \sqrt{14} & 1\sqrt{14} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \sqrt{14} \\ 0 \\ -\sqrt{14} \end{bmatrix}. \qquad \text{(2 marks)}$$

The first, second, third columns are pivotal, and the fourth is not, so we set $x_4 = t$. Now substitute back:

$$\begin{array}{rrrrr} \sqrt{14}x_1+ & \sqrt{14}x_2 & +\sqrt{14}x_3+ & 3\sqrt{14}x_4 = & \sqrt{14} \\ & \sqrt{14}x_2 & +\sqrt{14}x_3+ & 2\sqrt{14}x_4 = & 0 \\ & & \sqrt{14}x_3+ & \sqrt{14}x_4 = & -\sqrt{14} \end{array}.$$

We can simplify the arithmetic by dividing by $\sqrt{14}$ in the first, second, third rows

$$\begin{array}{rrrrr} x_1+ & x_2 & +x_3+ & 3x_4 = & 1 \\ & x_2 & +x_3+ & 2x_4 = & 0 \\ & & x_3+ & x_4 = & -1 \end{array}.$$

Then

$$x_3 = -1 - x_4 = -1 - t.$$

Next,

$$x_2 = -x_3 - 2x_4 = 1 + t - 2t = 1 - t.$$

Then

$$\begin{aligned} x_1 &= 1 - x_2 - x_3 - 3x_4 \\ &= 1 - (1-t) - (-1-t) - 3t \\ &\Rightarrow x_1 = 1 - t. \end{aligned}$$

Then the least squares solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-t \\ 1-t \\ -1-t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}. \qquad \text{(4 marks)}$$

We see that

$$A\mathbf{x} = A\left(\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \text{(2 marks)}$$

so it is also a solution to $A\mathbf{x} = \mathbf{b}$.

Question 5

Let

$$A = \begin{bmatrix} 7 & 9 & -9 \\ 9 & 7 & -9 \\ -9 & -9 & -5 \end{bmatrix}.$$

(a) Compute the characteristic polynomial $p_A(t)$ of A. Find the eigenvalues of A, and their algebraic multiplicities.

(14 marks)

(b) For each eigenvalue give a basis for the set of corresponding eigenvectors. Also describe the geometric multiplicity of each eigenvalue.

(15 marks)

(c) Find an invertible matrix V and a diagonal matrix D such that

$$A = VDV^{-1}. \qquad \text{(2 marks)}$$

(You need not compute V^{-1}).

(d) Find an orthogonal matrix U such that

$$A = UDU^T. \qquad \text{(4 marks)}$$

(e) Write down an expression for A^{-5} in terms of U and D. What are the eigenvalues of A^{-5}?

(4 marks)

(f) Write down $\det(A)$ using the result of (c) or (d).

(3 marks)

(g) Write down the singular values of A.

(3 marks)

Solution

(a) We calculate

$$p_A(t) = \det(A - tI) = \det \begin{bmatrix} 7-t & 9 & -9 \\ 9 & 7-t & -9 \\ -9 & -9 & -5-t \end{bmatrix}. \qquad \text{(2 marks)}$$

Rather than perform row operations with $7-t$, we swap:
Step 1: Swap rows 1 and 2

$$p_A(t) = -\det\begin{bmatrix} 9 & 7-t & -9 \\ 7-t & 9 & -9 \\ -9 & -9 & -5-t \end{bmatrix}$$

Step 2: Row 2 - $\left(\frac{7-t}{9}\right)\times$ Row 1; Row 3 + Row 1

$$\begin{aligned}
p_A(t) &= -\det\begin{bmatrix} 9 & 7-t & -9 \\ 0 & 9-\left(\frac{7-t}{9}\right)(7-t) & -9-\left(\frac{7-t}{9}\right)(-9) \\ 0 & -9+(7-t) & -5-t-9 \end{bmatrix} \\
&= -\det\begin{bmatrix} 9 & 7-t & -9 \\ 0 & \frac{1}{9}\left(9^2-(7-t)^2\right) & (-1)(9-(7-t)) \\ 0 & -2-t & -14-t \end{bmatrix} \\
&= -\det\begin{bmatrix} 9 & 7-t & -9 \\ 0 & \frac{1}{9}(9-(7-t))(9+(7-t)) & (-1)(9-(7-t)) \\ 0 & -2-t & -14-t \end{bmatrix} \\
&= -(9-(7-t))\det\begin{bmatrix} 9 & 7-t & -9 \\ 0 & \frac{1}{9}(9+(7-t)) & -1 \\ 0 & -2-t & -14-t \end{bmatrix} \\
&= -(t+2)\det\begin{bmatrix} 9 & 7-t & -9 \\ 0 & \frac{1}{9}(16-t) & -1 \\ 0 & -2-t & -14-t \end{bmatrix}
\end{aligned}$$

We could expand by first column; in case you don't know that
Step 3: Swap columns 2 and 3

$$p_A(t) = (t+2)\det\begin{bmatrix} 9 & -9 & 7-t \\ 0 & -1 & \frac{1}{9}(16-t) \\ 0 & -(14+t) & -2-t \end{bmatrix}$$

Step 4: Row 3 - $(14+t)\times$ **Row 2**

$$\begin{aligned}
p_A(t) &= (t+2)\det\begin{bmatrix} 9 & -9 & 7-t \\ 0 & -1 & \frac{1}{9}(16-t) \\ 0 & 0 & -2-t-(14+t)\frac{1}{9}(16-t) \end{bmatrix} \\
&= (t+2)(9)(-1)\left(-2-t-(14+t)\frac{1}{9}(16-t)\right) \\
&= -(t+2)(-18-9t-(14+t)(16-t)) \\
&= -(t+2)\left(-18-9t-\left(-t^2+2t+14(16)\right)\right) \\
&= -(t+2)\left(t^2-11t-242\right) \\
&= -(t+2)(t+11)(t-22).
\end{aligned}$$

This vanishes when $t=-2,-11,22$. So the eigenvalues are $-2,-11,22$. Each has algebraic multiplicity 1.

(12 marks)

(b) Next we find the eigenvectors. For each eigenvalue μ, we must find a vector $\mathbf{v}\neq\mathbf{0}$ such that

$$(A-\mu I)\mathbf{v}=\mathbf{0},$$

so we apply row reduction to

$$[A-\mu I|\mathbf{0}] = \begin{bmatrix} 7-\mu & 9 & -9 & 0 \\ 9 & 7-\mu & -9 & 0 \\ -9 & -9 & -5-\mu & 0 \end{bmatrix}.$$

Eigenvectors for $\mu=-11$

We apply row reduction to

$$\begin{bmatrix} 7-\mu & 9 & -9 \\ 9 & 7-\mu & -9 \\ -9 & -9 & -5-\mu \end{bmatrix} = \begin{bmatrix} 18 & 9 & -9 & 0 \\ 9 & 18 & -9 & 0 \\ -9 & -9 & 6 & 0 \end{bmatrix}$$

Step 1: Row 2 -$\frac{1}{2}\times$ **Row 1; Row 3 +** $\frac{1}{2}\times$ **Row 1**

$$\begin{bmatrix} 18 & 9 & -9 & 0 \\ 0 & 13\frac{1}{2} & -4\frac{1}{2} & 0 \\ 0 & -4\frac{1}{2} & 1\frac{1}{2} & 0 \end{bmatrix}$$

Step 2: Divide Row 1 by 9; Row 2 by $4\frac{1}{2}$; Row 3 by $1\frac{1}{2}$

$$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -3 & 1 & 0 \end{bmatrix}$$

Step 3: Row 3 + Row 2

$$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then x_1, x_2 are pivotal, x_3 is not. So set $x_3 = t$.

Step 3: Back Substitution

$$\begin{array}{rrrl} 2x_1 & +x_2 & -x_3 = & 0 \\ & 3x_2 & -x_3 = & 0 \end{array}$$

Then

$$\begin{aligned} 3x_2 &= x_3 = t \\ &\Rightarrow x_2 = \frac{1}{3}t. \end{aligned}$$

Then

$$\begin{aligned} 2x_1 &= -x_2 + x_3 = -\frac{1}{3}t + t = \frac{2}{3}t \\ &\Rightarrow x_1 = \frac{1}{3}t. \end{aligned}$$

So

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t \\ \frac{1}{3}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}.$$

There is one linearly independent eigenvector and we can take $t = 3$ to get

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}. \qquad \text{(4 marks)}$$

A basis for the corresponding eigenspace is $\{\mathbf{v}_1\}$. As there is one linearly independent eigenvector, $\mu = -11$ has geometric multiplicity 1.

(1 mark)

Eigenvectors for $\mu = -2$
We apply row reduction to

$$\begin{bmatrix} 9 & 9 & -9 & 0 \\ 9 & 9 & -9 & 0 \\ -9 & -9 & 3 & 0 \end{bmatrix}$$

Step 1: Row 2 - Row 1; **Row 3 + Row 1**

$$\begin{bmatrix} 9 & 9 & -9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix}$$

Step 2 Swap Rows 2, 3

$$\begin{bmatrix} 9 & 9 & -9 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 3 Back Substitution

$$\begin{array}{rrrl} 9x_1 & +9x_2 & -9x_3 = & 0 \\ & & -6x_3 = & 0 \end{array}$$

We see that x_1, x_3 are pivotal, while x_2 is non-pivotal. So set $x_2 = t$. Then

$$-6x_3 = 0 \Rightarrow x_3 = 0.$$

Next,

$$9x_1 = -9x_2 + 9x_3 = -9t - 0 \Rightarrow x_1 = -t.$$

So

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

There is one linearly independent eigenvector and we can take

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}. \qquad \text{(4 marks)}$$

A basis for the corresponding eigenspace is $\{\mathbf{v}_2\}$. As there is one linearly independent eigenvector, $\mu = -2$ has geometric multiplicity 1.

(1 mark)

Eigenvectors for $\mu = 22$
We apply row reduction to

$$\left[\begin{array}{cccc} 7-\mu & 9 & -9 & 0 \\ 9 & 7-\mu & -9 & 0 \\ -9 & -9 & -5-\mu & 0 \end{array}\right] = \left[\begin{array}{cccc} -15 & 9 & -9 & 0 \\ 9 & -15 & -9 & 0 \\ -9 & -9 & -27 & 0 \end{array}\right]$$

Step 1: Swap Rows 1, 3 to simplify arithmetic

$$\left[\begin{array}{cccc} -9 & -9 & -27 & 0 \\ 9 & -15 & -9 & 0 \\ -15 & 9 & -9 & 0 \end{array}\right]$$

Step 2: Row 2 + Row 1; Row 3 -$\frac{15}{9}\times$ Row 1

$$\left[\begin{array}{cccc} -9 & -9 & -27 & 0 \\ 0 & -24 & -36 & 0 \\ 0 & 24 & 36 & 0 \end{array}\right]$$

Step 2: Row 3 + Row 2

$$\left[\begin{array}{cccc} -9 & -9 & -27 & 0 \\ 0 & -24 & -36 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then x_1, x_2 are pivotal, x_3 is not. So set $x_3 = t$.
Step 3: Back Substitution

$$\begin{array}{rrrl} -9x_1 & -9x_2 & -27x_3 = & 0 \\ & -24x_2 & -36x_3 = & 0 \end{array}$$

We see that x_1, x_2 are pivotal, while x_3 is non-pivotal. So set $x_3 = t$. Then

$$-24x_2 = 36x_3 \Rightarrow x_2 = -1\frac{1}{2}t.$$

Next,

$$\begin{aligned}
-9x_1 &= 9x_2 + 27x_3 = -13\frac{1}{2}t + 27t = 13\frac{1}{2}t \\
&\Rightarrow\ x_1 = -1\frac{1}{2}t
\end{aligned}$$

So

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1\frac{1}{2}t \\ -1\frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} -1\frac{1}{2} \\ -1\frac{1}{2} \\ 1 \end{bmatrix}.$$

There is one linearly independent eigenvector and we can take $t = 2$ to get

$$\mathbf{v}_3 = \begin{bmatrix} -3 \\ -3 \\ 2 \end{bmatrix}. \qquad \text{(4 marks)}$$

A basis for the corresponding eigenspace is $\{\mathbf{v}_3\}$. As there is one linearly independent eigenvector, $\mu = 22$ has geometric multiplicity 1.

(1 mark)

(c) We let

$$V = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{bmatrix} 1 & -1 & -3 \\ 1 & 1 & -3 \\ 3 & 0 & 2 \end{bmatrix}$$

and

$$D = \begin{bmatrix} -11 & & \\ & -2 & \\ & & 22 \end{bmatrix}$$

and then

$$\begin{aligned}
A &= VDV^{-1} \\
&= \begin{bmatrix} 1 & -1 & -3 \\ 1 & 1 & -3 \\ 3 & 0 & 2 \end{bmatrix}\begin{bmatrix} -11 & & \\ & -2 & \\ & & 22 \end{bmatrix}\begin{bmatrix} 1 & -1 & -3 \\ 1 & 1 & -3 \\ 3 & 0 & 2 \end{bmatrix}^{-1} \text{(2.marks)}
\end{aligned}$$

(d) We apply Gram-Schmidt to $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$. But, since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ come from distinct eigenvalues, they are already orthogonal to each other. (The matrix is symmetric). So we can just let

$$\mathbf{u}_1 = \frac{1}{|\mathbf{v}_1|}\mathbf{v}_1 = \frac{1}{\sqrt{11}}\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix};$$

$$\mathbf{u}_2 = \frac{1}{|\mathbf{v}_2|}\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix};$$

$$\mathbf{u}_3 = \frac{1}{|\mathbf{v}_3|}\mathbf{v}_3 = \frac{1}{\sqrt{22}}\begin{bmatrix} -3 \\ -3 \\ 2 \end{bmatrix}.$$

Then we can take

$$\begin{aligned} U &= [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3] \\ &= \begin{bmatrix} 1/\sqrt{11} & -1/\sqrt{2} & -3/\sqrt{22} \\ 1/\sqrt{11} & 1/\sqrt{2} & -3/\sqrt{22} \\ 3/\sqrt{11} & 0 & 2/\sqrt{22} \end{bmatrix} \end{aligned}$$

(6 marks)

and U is orthogonal so

$$A = UDU^{-1} = UDU^T.$$

(e) As all eigenvalues are non-zero, A is invertible. So

$$\begin{aligned} A^{-5} &= UD^{-5}U^T \\ &= U\begin{bmatrix} (-11)^{-5} & & \\ & (-2)^{-5} & \\ & & 22^{-5} \end{bmatrix}U^T. \end{aligned}$$

(2 marks)

The eigenvalues are just $\left\{(-11)^{-5}, (-2)^{-5}, (22)^{-5}\right\}$.

(2 marks)

(f)

$$\begin{aligned}
\det(A) &= \det\left(VDV^{-1}\right) = \det(V)\det(D)\det\left(V^{-1}\right) \\
&= \det\left(VV^{-1}\right)\det(D) \\
&= \det(I)\det(D) \\
&= (-11)(-2)(22) \\
&= 484. && \text{(3 marks)}
\end{aligned}$$

(g) The singular values of A are just the square roots of the eigenvalues of $A^T A = A^2$, since A is symmetric. But the eigenvalues of A^2 are $(-11)^2, (-2)^2, 22^2$, so the singular values are

$$2, 11, 22$$

(3 marks)