Lecture 21 - Gauss-Seidel method

Iterative method

↳ Open methods for roots of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

System

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

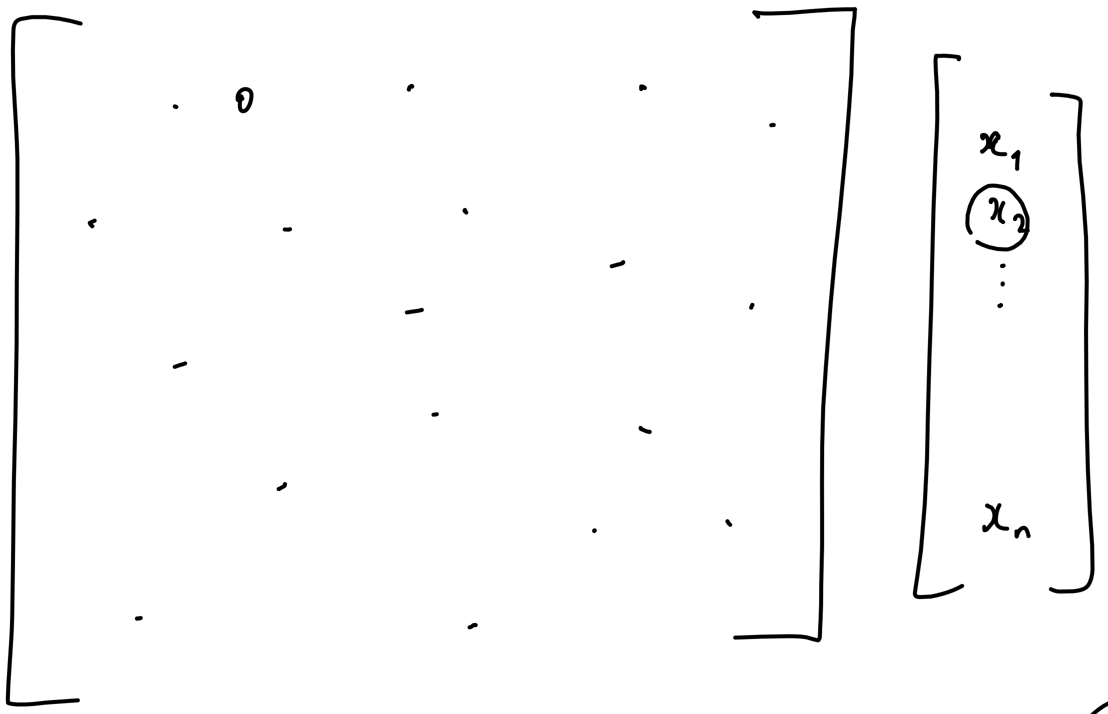
$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

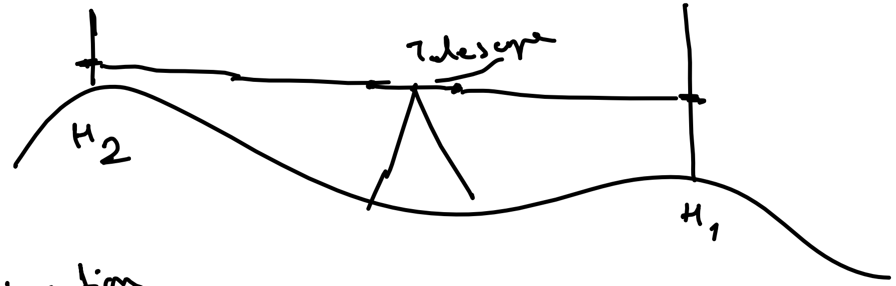
Convergence criteria / Stopping criteria

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

diagonally dominant matrix

Gauss-Seidel → Sparse linear system





Jacobi iteration

$H_2 - H_1 = \underline{\Delta H_{12}}$ observation

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}} \rightarrow x_1^1$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}} \quad x_2^1$$

$$x_3 = \frac{b_2 - a_{31}x_1 - a_{32}x_2}{a_{33}} \quad x_3^1$$

Iteration 0 $\quad x_1, x_2, x_3$ – Initial guess

Iteration 1 $\quad x_1^{(1)}, x_2^{(1)}, x_3^{(1)}$

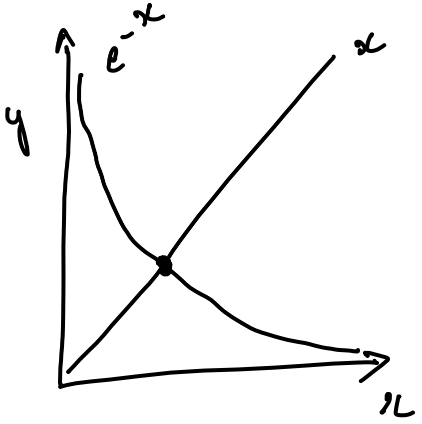
$x_1^{(2)}, x_2^{(2)}, x_3^{(2)}$

Open method -> Fixed-point method

$$f(x) = e^{-x} - x = 0$$

$$\Rightarrow \quad x = e^{-x}$$

initial guess $x = 0$



$$x^0 = e^{-0}$$

$$x^0 = 1$$

$$x^1 = e^{-1}$$

$$x^2 = e^{-(1/e)}$$

Convergence of the method

$$\left|\frac{\partial u_1}{\partial x_1}\right| + \left|\frac{\partial u_1}{\partial x_2}\right| + \left|\frac{\partial u_1}{\partial x_3}\right| < 1$$

absolute value

$$\left|\frac{\partial u_2}{\partial x_1}\right| \quad \dots$$

$$x_1 = \overbrace{\boxed{\frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}}}^{u_1}$$

$$0 + \left|\frac{a_{12}}{a_{11}}\right| + \left|\frac{a_{13}}{a_{11}}\right| < 1$$

not a necessary condition

sufficient condition

$$\frac{|a_{12}| + |a_{13}|}{|a_{11}|} < 1$$

$$\Rightarrow \quad |a_{12}| + |a_{13}| < |a_{11}|$$

Techniques for improving speed of convergence

$$x_i^{new} \longleftarrow \boxed{x_i^{old}}$$

Relaxation technique

$$\tilde{x}_i^{new} = \lambda x_i^{new} + (1-\lambda) x_i^{old} \qquad \lambda \in [0, 2]$$

↳ Oscillating systems

x^{new}

$\underline{0, 1}$

underrelaxation

$\underline{1, 2}$

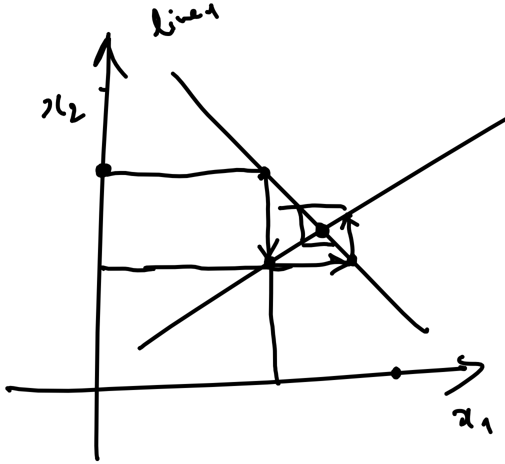
overrelaxation

Figure 11.5 of 6^{th} Edn. Chapra & Canale

$u(x_1, x_2)$ $\quad x_1, x_2$

line 1 $\underline{a_{11} x_1 + a_{12} x_2 = b_1}$ $\quad x_1$

line 2 $\underline{a_{21} x_1 + a_{22} x_2 = b_2}$



Limited applicability

↓

Gauss-Seidel

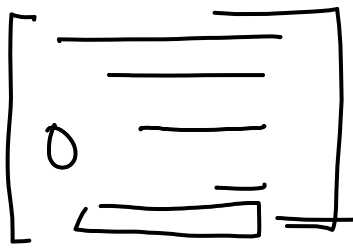
Factorizations

QR

SVD → Eigenvalue Decom.

Rank of a matrix

Gaussian elimination → Row echelon form



= row rank = column rank

→ row of zeros → rank of the matrix $\leq n$

$n \times n$

Gaussian elimination with pivoting