Lecture 27 – Discrete Fourier Transform & Fast Fourier Transform algorithm

Fourier series $\rightarrow$ Transforms $\rightarrow$ Spectral domain

$\downarrow$

Amplitudes of frequencies

Fourier Integral

discrete $\rightarrow$ finite domain & curvilinear

Fourier series $\Rightarrow$ $$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx} \qquad x \in [0, 2\pi)$$

continuous

$$a_k = \frac{1}{2\pi} \underline{\int_0^{2\pi} f(x)\, e^{-ikx}\, dx}$$

Transform pair

Inverse

Fourier Integral $\Rightarrow$ $$f(x) = \int_{-\infty}^{\infty} a(f)\, e^{ifx}\, df \qquad x \in (-\infty, \infty)$$

forward

$$a(f) = \frac{1}{2\pi} \underline{\int_{-\infty}^{\infty} f(x)\, e^{-ifx}\, dx} \qquad f = 2\pi \; ?$$

Transform pair

$f(x) \longrightarrow$ discrete $\rightarrow f[x]$

Fourier Integral

Synthesis $f[\underline{x}] = \sum_{f=-N/2}^{N/2} a[f]\, e^{i \underline{f} \underline{x}}$

Analysis $a[\underline{f}] = \sum_{x=0}^{M-1} f[x]\, e^{-i \underline{f} \underline{x}}$

We are allowed to do this as the orthogonality is retained after discretization

Bandlimited frequency domain

$\begin{cases} \text{limited domain} \\ \text{Sampling} \end{cases}$

Equispaced sampling

saviour of orthogonality

8-data points

8-frequencies

-4
-3
-2
-1
0
1
2
3

spectra

base function.

$f[x_1] = a[f_{-4}]\, e^{i f_{-4} x_1} \quad a[f_{-3}]\, e^{i f_{-3} x_1} \quad a[f_{-2}]\, e^{i f_{-2} x_1} \quad \cdots \quad a[f_3]\, e^{i f_3 x_1}$

$\vdots$

$f[x_8] =$

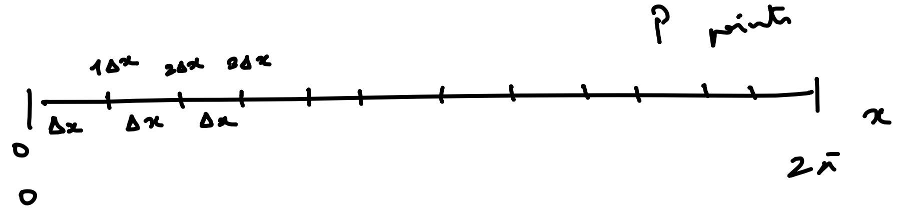
Observations

Hermitian matrix

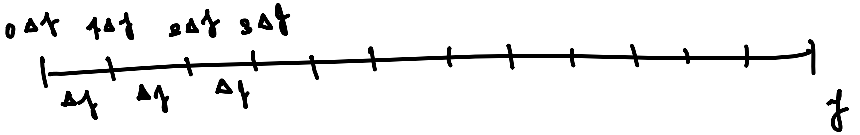
$\underset{8\times 1}{\underline{f}} = \underset{8\times 8}{\underline{F}}\ \underset{8\times 1}{\underline{a}} \Rightarrow \underline{a} = \underline{F}^{-1 \to +} \underline{f} \Rightarrow \underline{F}\underline{F}^{+} = \underline{I}$

DFT

$$f[x] = \sum_{f=-N/2}^{N/2} a[f]\, e^{ifx}$$



$$f[n\Delta x] = \sum_{f=-N/2}^{N/2} a[f]\, e^{in\Delta x\, f}$$



$$f[n\Delta x] = \sum_{k=-N/2}^{N/2} a[k\Delta f]\, e^{in\Delta x\, k\boxed{\Delta f}}$$

$\Delta f = \frac{2\pi}{P}$ ↪ in our case $P = N$

$\Delta f = \frac{2\pi}{N}$

$$f[n] = \sum_{k=-N/2}^{N/2} a[k]\, e^{in\Delta x\, k\frac{2\pi}{N}}$$

$$= \sum_{k=-N/2}^{N/2} a[k]\, e^{i2\pi\frac{kn}{N}\Delta x}$$

$e^{ik\theta}$ ↪ angle $[0, 2\pi)$

P → arbitrary period $[0, P)$

$\theta = \boxed{\frac{2\pi}{P}} x$ ↪ linear

fundamental frequency

$e^{i\boxed{k\frac{2\pi}{P}}x}$

$k\frac{2\pi}{P}$

f_0

$\boxed{kf_0} = k$ ↪ frequency number

$$f[n] = \sum_{k=-N/2}^{N/2} a[k]\, e^{i 2\pi \frac{kn}{N} \Delta x}$$

Δx should not be there in the final formula!

Missing a trick here!

$$f[n] = \sum_{k=-N/2}^{N/2} a[k] \boxed{e^{i\frac{2\pi}{N}}}^{kn\Delta x}$$

↳ exponent of fundamental frequency

$$f[n] = \sum_{k=-N/2}^{N/2} a[k] \boxed{W}^{kn\Delta x}$$

$$\begin{array}{c} \\ n \downarrow \end{array} \begin{bmatrix} f[1] \\ f[2] \\ f[3] \\ \vdots \\ f[8] \end{bmatrix} = \begin{array}{c} \xrightarrow{\quad k \quad} \\ \begin{bmatrix} W^{-4\cdot 1\Delta x} & W^{-3\cdot 1\Delta x} & W^{-2\cdot 1\Delta x} & W^{-1\cdot 1\Delta x} & \cdots & W^{3\cdot 1\Delta x} \\ \vdots & \ddots & & & & \vdots \\ \vdots & & & & \ddots & \vdots \\ \vdots & \cdots & & & \cdots & \end{bmatrix} \end{array} \begin{bmatrix} a[-4] \\ a[-3] \\ \vdots \\ a[3] \end{bmatrix}$$

FFT $\longrightarrow$ To utilize the symmetries of the complex exponential function

$W = e^{i 2\pi / N}$

$i^0 \quad i^1$

$i^2 \quad i^3 \quad i^4$

i

-1

$-i$

1

Sande - Tukey Algorithm

$N = 2^M$

$\downarrow$

$$a[k] = \sum_{n=0}^{N-1} f[n] W^{nk}$$

$$a[k] = \sum_{n=0}^{N/2-1} f[n] W^{nk} + \sum_{n=N/2}^{N-1} f[n] W^{nk}$$

Introduce $m = n - N/2$

$$a[k] = \sum_{n=0}^{N/2-1} f[n] W^{nk} + \sum_{m=0}^{N/2-1} f[m+N/2] W^{(m+N/2)k}$$

$$= \quad '' \quad + \sum_{n=0}^{N/2-1} f[n+N/2] W^{nk} W^{Nk/2}$$

change $m \to n$ without loss of generality