Lecture 37 - Multi-step methods

General idea of Multi-step methods

predictor improve $f(x, y)$

Corrector

Heun's method $\rightarrow$ Self-starting method

predictor Euler's method $y_{i+1}^{0} = y_i + f(x_i, y_i)h$

Corrector Trapezoidal rule

$$y_{i+1} = y_i + h\frac{f(x_i, y_i) + f(x_i, y_{i+1}^{0})}{2}$$

RK

Single-step Multi-step

Adaptive

Fixed-step

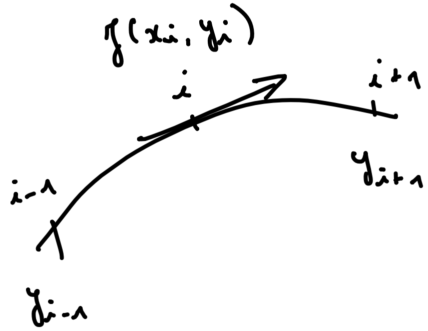
ODE problems

Stiff non-stiff

Single

System

Heun's method → Non-self-starting method



non-self-starting predictor

$$y^0_{i+1} = y^m_{i-1} + f(x_i, y^m_i)\,2h \qquad \text{Mid-point formula}$$

$$y^1_{i+1} = y^m_i + \frac{f(x_i, y^m_i) + f(x_i, y^0_{i+1})}{2}\,h \qquad \text{Trapezoidal formula}$$

Iterative corrector

$$y^2_{i+1} = y_i + \frac{f(x_i, y^m_i) + f(x_i, y^1_{i+1})}{2}\,h$$

$$\vdots$$

$$y^j_{i+1} = y_i + \frac{f(x_i, y^m_i) + f(x_i, y^{j-1}_{i+1})}{2}\,h, \quad j \in [0, m]$$

Improve the Heun's method

- Modifying predictor $\longrightarrow$ adding error estimates
- Modifying corrector $\longrightarrow$

$$E_p = \tfrac{1}{3} h^3 y^{(3)}(\xi_p)$$ Predictor truncation error

$$\text{True value} = y^{0}_{i+1} + E_p \quad - \quad 1$$

$$E_c = -\tfrac{1}{12} h^3 y^{(3)}(\xi_c)$$

$$\text{True value} = y^{m}_{i+1} - \underline{\tfrac{1}{12} h^3 y^{(3)}(\xi_c)} \quad - \quad 2$$

1 - 2

Assuming $\xi_p = \xi_c$

$$0 = y^{m}_{i+1} - y^{0}_{i+1} - \tfrac{1}{3} h^3 y^{(3)}(\xi_p) - \tfrac{1}{12} h^3 y^{(3)}(\xi_c)$$

$$= y^{m}_{i+1} - y^{0}_{i+1} - \tfrac{1}{3} h^3 y^{(3)}(\xi) - \tfrac{1}{12} h^3 y^{(3)}(\xi)$$

$$0 = y^{m}_{i+1} - y^{0}_{i+1} - \tfrac{5}{12} h^3 y^{(3)}(\xi) \Rightarrow \quad -\tfrac{1}{12} h^3 y^{(3)}(\xi) = \frac{-y^{m}_{i+1} + y^{0}_{i+1}}{5}$$

Modifier for the corrector

$$E_p = \frac{1}{3} h^3 y^{(3)}(\xi)$$

$$h^3 y^{(3)}(\xi) = -\frac{12}{5}\left(\underline{y_i^o} - \underline{y_i^m}\right)$$

Modifier for predictor

Strength of Multi-step methods

- Estimate errors
- Leeway in the use of integration formulas

Adams integration → Multi-step method

Predictor → Modifier of predictor

Open formula

Taylor series $y_{i+1} = y_i + h \{ \underline{f_i} + \dots \}$

Corrector → Modifier of corrector

Backward Closed formula

Taylor series $y_{i+1} = y_i - h \{ \underline{f_{i+1}} + \dots \}$

Taylors series

- Truncate it at a very high order
  - ↳ Numerical differentiation
- Integrate using Newton-Cotes integration
  - ↳ Open formula for predictor
  - ↳ closed formula for corrector