Lecture 33 - Numerical calculus of observations with errors

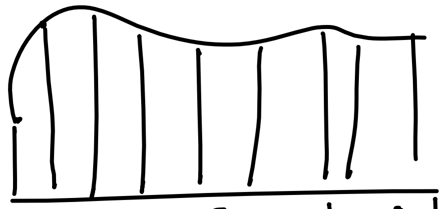
True value

$$y_i = \tilde{y}_i + \varepsilon_i$$

error

observation/ data

$$I_f = \frac{f'(x_i) + f'(x_{i+1})}{2}\,(x_{i+1} - x_i)$$



Integral operator data Integrated values

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} \quad \\ \quad \end{bmatrix} =$$

Numerical Differential / Integral operators

$$\frac{y_i + y_{i+1}}{2} = \hat{\tilde{y}}_{i+1/2} \Rightarrow \frac{1}{N}\sum_{i=1}^{N} y_i = \hat{y}$$

$$\hat{y} \rightarrow \sigma_{\hat{y}} = \frac{\sigma_y}{\sqrt{N}}$$

$$y_i = \tilde{y}_i + \varepsilon_i \rightarrow \pm \sigma_{y_i}$$

$$y_{i+1} = \tilde{y}_{i+1} + \varepsilon_{i+1} \rightarrow \pm \sigma_{y_{i+1}}$$

Independent → important assumption

Identically distributed

$$I = \frac{y_i + y_{i+1}}{2} (\Delta_i - \Delta_{i+1}) \qquad E\left\{ (y - \bar{y})^2 \right\}$$

$$\sigma_I^2 = \left(\sigma_{y_i}^2 + \sigma_{y_{i+1}}^2 \right) \left(\frac{\Delta_i - \Delta_{i+1}}{2} \right)^2$$

↓

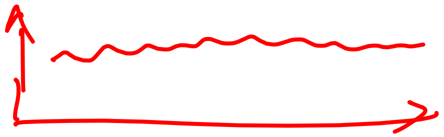
C

$$\sigma_I^2 = C^2 \left(\sigma_{y_i}^2 + \sigma_{y_{i+1}}^2 \right)$$

$$C^2 \left(\sigma_y^2 + \sigma_y^2 \right) \qquad \text{IID}$$

$$y_j = \tilde{y}_j + \varepsilon_j$$

Fourier $\{\varepsilon_i\}$

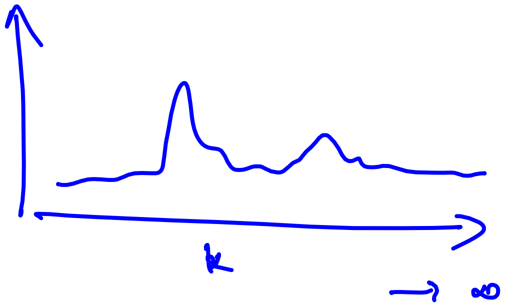


Fourier $\left\{\frac{d\varepsilon}{dx}\right\}$

$$I = \mathcal{F}\, y_j = \mathcal{F}(\tilde{y}_j + \varepsilon_j)$$

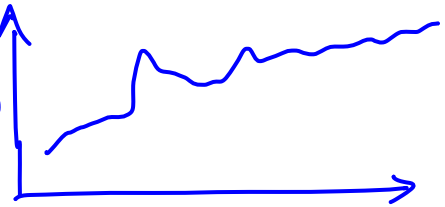
D

Fourier $\{y\}$



$$y(x) = \sum_{k=-\infty}^{\infty} a_k\, e^{ikx}$$

Fourier $\left\{\frac{dy}{dx}\right\}$



$$\frac{dy}{dx} = \sum_{k=-\infty}^{\infty} \boxed{ik}\, a_k\, e^{ikx}$$

$$\int y(x)\, dx = \sum_{k=-\infty}^{\infty} \frac{a_k}{ik}\, e^{ikx}$$

Solving Ordinary Differential Equations

ODE Solver

State equations

linear $\quad \dfrac{d^n y}{dx^n} + \dfrac{d^{n-1} y}{dx^{n-1}} + \dfrac{d^{n-2} y}{dx^{n-2}} + \dots + y = \begin{cases} 0 & \rightarrow \text{Homogeneous} \\ \text{value} & \rightarrow \text{Inhomogeneous} \end{cases}$

non-linear $\quad \dfrac{d^n y}{dx^n} f(x, y) + \dots + g(x, y) = \text{value}$

$$\frac{dv}{dt} = g - \frac{c}{m} v$$

drag coefficient

velocity

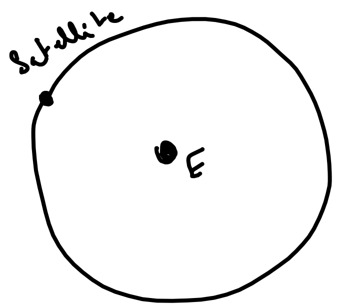
mass of the parachutist

$\frac{dv}{dt}$: acceleration of the parachutist

g: acceleration due to gravity

$$\frac{dv}{dt} + \frac{c}{m} v = g$$

Orbit integration



$$\int \ddot{\vec{r}} = \int -\frac{GM}{r^3}\vec{r}$$

$$\ddot{\vec{r}} =$$

Second order

$$\underbrace{m\frac{d^2x}{dt^2}}_{\text{acceleration/ Force}} + \underbrace{c\frac{dx}{dt}}_{\text{damping}} + \underbrace{k}_{\text{spring constant}}x = 0$$

equation of a damped spring

$$y = \frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{d^2x}{dt^2}$$

$$\frac{dy}{dt} = -\left(\frac{cy + kx}{m}\right)$$

First order equation

Integration of the ODE solver

$$\int \frac{dv}{dt}\, dt = \int \left(g - \frac{c}{m} v\right) dt$$

$$\underline{v} = \int \left(g - \frac{c}{m} v\right) dt$$

$t = 0, \quad v = 0$

Analytical form

$$v = \frac{gm}{c}\left(1 - e^{(c/m)t}\right)$$

Initial value
Boundary value

$\left.\begin{matrix}\text{Initial value} \\ \text{Boundary value}\end{matrix}\right\}$ problem

$\underbrace{\text{Initial value / Boundary value}}_{\text{auxilliary information}}$

Numerical part

$$v_{i+1} = v_i + \left(g - \frac{c}{m} v_i\right) \Delta t$$

Euler integration

$$y_{i+1} = y_i + f(x_i, y_i)\, h \longrightarrow h = x_{i+1} - x_i$$

$$E_t = O(h^2)$$