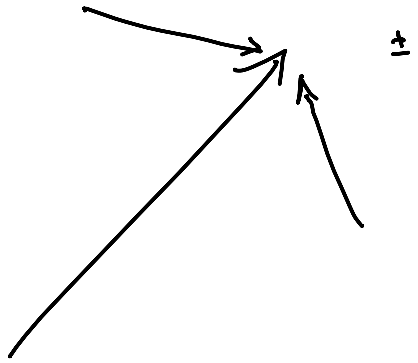
Householder algorithm

$$\left(I - 2\vec{u}\vec{u}^T\right)\vec{x} = c\,\vec{e}_1 \qquad c = \|\vec{x}\|_2$$



$$\vec{u} = \frac{1}{2(\vec{u}^T\vec{x})}\left(\vec{x} - c\,\vec{e}_1\right)$$

$$\vec{u} = \text{scale factor}\,\left(\vec{x} \underbrace{\pm \text{ Reduction in the first coordinate of } \vec{x}}_{\text{use it to form } \vec{u}}\right)$$

$$\vec{v} = \vec{x} \pm c\,\vec{e}_1$$

$\pm$ → addition or subtraction

↳ dependent on the sign of $\vec{x}(1)$

$$\vec{v} = \vec{x} - \text{sign}(\vec{x}(1))\,\underline{c}\,\vec{e}_1$$

↳ inner product

don't do this step → $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}$$

Wilkinson diagram

$$P_1 A = A_1 \qquad A_1 = \begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{bmatrix}$$

$$\underset{5\times5}{P_1}\,\underset{5\times4}{A} = \underset{5\times4}{A_1}$$

$$\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}$$

$$\underset{5\times5}{P_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & P_2' & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix}$$

$$\underset{4\times4}{P_2'}\,A_1' = A_2' = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix}$$

$$A_2 = \begin{bmatrix} A_1(:,1) & \\ 0 & \\ 0 & A_2' \\ 0 & \\ 0 & \end{bmatrix}$$

$$A_1' = \begin{bmatrix} \cancel{x} & \cancel{x} & \cancel{x} & \cancel{x} \\ \cancel{0} & x & x & x \\ \cancel{0} & x & x & x \\ \cancel{0} & x & x & x \\ \cancel{0} & x & x & x \end{bmatrix}$$

omit

3 iterations

$$PA = \underbrace{P_3 P_2 P_1}_{Q^T} A = R \rightarrow \text{upper triangular matrix}$$

$$Q^T A = R$$

$$A = QR$$

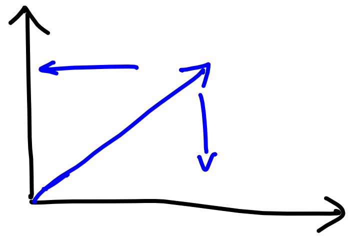
Givens rotation

↳ rotate

unknown

$$R \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ 0 \end{bmatrix}$$

Rotation matrix



$$\left.\begin{aligned} R^T R &= I \\ R R^T &= I \end{aligned}\right\} \det(R) = 1$$

$$\cos\theta x - \sin\theta y = x' \quad - (1)$$

$$\sin\theta x + \cos\theta y = 0 \quad - (2)$$

$$\cos\theta = \frac{-\sin\theta x}{y} \quad - (3)$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$ → hypotenuse

$$-\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

Sub (3) in (1)

$$-\sin\theta x^2 - \sin\theta y^2 = y x' \rightarrow \sqrt{x^2+y^2}$$

$$-\sin\theta (x^2+y^2) = y\sqrt{x^2+y^2}$$

$$-\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$G_1 \; G_2 \; G_3 \; G_4$

Parallel computations

$$G_1 \begin{bmatrix} c & -s & & & & & 0 \\ s & c & & & & & \\ & & 1 & & & & \\ & 0 & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = A_1 \begin{bmatrix} x & x & x & \cdots & \\ 0 & x & x & \cdots & \\ x & & \ddots & & \\ x & & & \ddots & \\ x & & & & \ddots \\ \vdots & & & & \end{bmatrix}$$

$$G_2 \begin{bmatrix} 1 & & & & & & 0 \\ & c & -s & & & & \\ & s & c & & & & \\ & & & 1 & & & \\ & 0 & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} A_1$$

$$\underbrace{G_n G_{n-1} \dots G_2 G_1}_{Q^T} A = R$$

$$A = R$$

$$A = QR$$

$$Ax = b$$

$$QRx = b$$

$$Rx = Q^T b$$

Find x by backward substitution