NUMERICAL INTEGRATION

What do you mean by integration?

Integration means to bring together as parts into a whole or unite.

Let us consider a function $f(x)$ and we want to find out the area under the curve $f(x)$ within the interval (a, b)

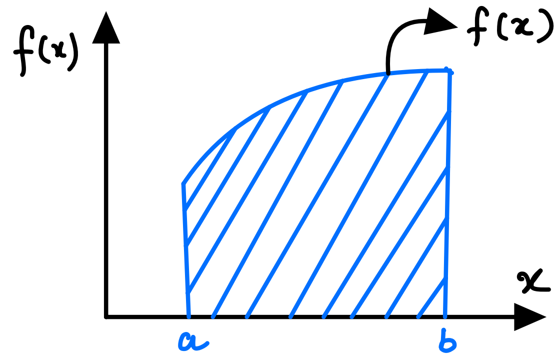
In order to find the area we must discretize the area into small elements and we will compute $f(x)\,dx$ and sum it up over the interval.

Mathematically it can be described as :

$$I = \int_a^b f(x)\,dx$$

This is a definite integral

$$= F(b) - F(a)$$



where $F(x)$ is an antiderivative of $f(x)$.

$$f(x) = \frac{d}{dx} F(x)$$

Why do we need numerical integration?

It is likely that sometimes function can be evaluated only at few discrete points. In such cases we do not have a continous function available for integration so that the required area or volume may be computed.

It is also so likely that the function is so complex that an analytical expression does not exists.

Therefore in those cases we have to move for numerical integration.

General approach to numerical integration

Approximate $f(x)$ with one or a piece wise continous set of polynomial's $p(x)$ and evaluate.

$$\int_a^b f(x)\,dx \approx \int_a^b p(x)\,dx$$

Remainder term

$$\int_a^b f(x)\,dx = \int_a^b [p(x) + r(x)]\,dx$$

Approximated polynomial

$$\int_a^b f(x) f(x)\,dx = I + \epsilon.$$

Error term

(a) Direct Fit Polynomials

- Can be used for both equally spaced & unequally spaced data.

$$f(x) = P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$P_n(x)$ is determined by one of the following methods

1. Given $N = n+1$ sets of discrete data points, determine the exact n^{th} degree polynomial that passes through these data points
2. If $N > n+1$ sets of discrete data points, determine least square. n^{th} degree polynomial that best fits this data
3. Given a known function $f(x)$ evaluate $f(x)$ at N discrete points and fit a polynomial by an exact fit or least square fit.

After the approximating polynomial has been fit the integral becomes

$$I = \int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx$$

Sub: $P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$$I \approx \left(a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \cdots \right)_a^b$$

Example : $I = \int_{3.1}^{3.9} 1/x \, dx$

x	$f(x)$
3.1	0.32258065
3.5	0.28571429
3.9	0.25641026

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$

$$0.32258065 = a_0 + a_1(3.1) + a_2(3.1^2)$$

$$0.28571429 = a_0 + a_1(3.5) + a_2(3.5^2)$$

$$0.25641026 = a_0 + a_1(3.9) + a_2(3.9^2)$$

Solving for a_0, a_1, a_2 by Gauss Elimination

$P_2 = 0.86470519 - 0.24813896x + 0.02363228x^2$

$I = \underline{0.22957974}$

$I_{true} = \underline{0.22957444}$

$\text{Error} = \underline{0.00000530}$.

Newton's Cotes formula for integration

Recall: Difference Table

x	$f(x)$	Δf	$\Delta^2 f$
x_0	f_0		
		$f_1 - f_0$	
x_1	f_1		$f_2 - 2f_1 + f_0$
		$f_2 - f_1$	
x_2	f_2		$f_3 - 2f_2 + f_1$
		$f_3 - f_2$	
x_3	f_3		

$$P_n(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \ldots + \frac{s(s-1)(s-2)\ldots[s-(n-1)]}{n!}\Delta^n f_x \quad \text{——} \; (1)$$

$$s = \frac{x - x_0}{\Delta x} = \frac{x - x_0}{h} \quad \text{——} \; (2)$$

* Eqn (1) is implicit function of x

$$I \approx \int_a^b P_n(x)\,dx \rightarrow \text{requires explicit } f(x).$$

$$hs = x - x_0$$

$$h\,ds = dx$$

$$\boxed{I = h\int_{s(a)}^{s(b)} P_n(s)\,ds}$$

Limits :-

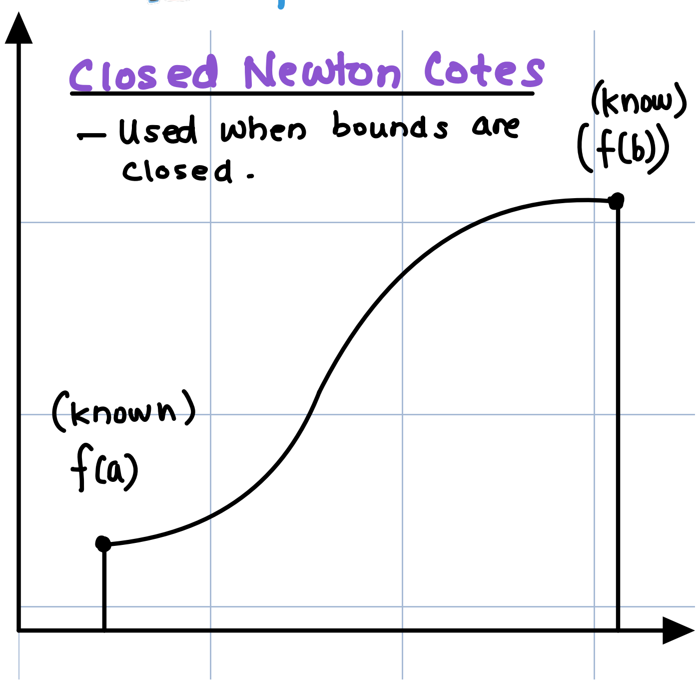
$x = a$ is chosen as base point

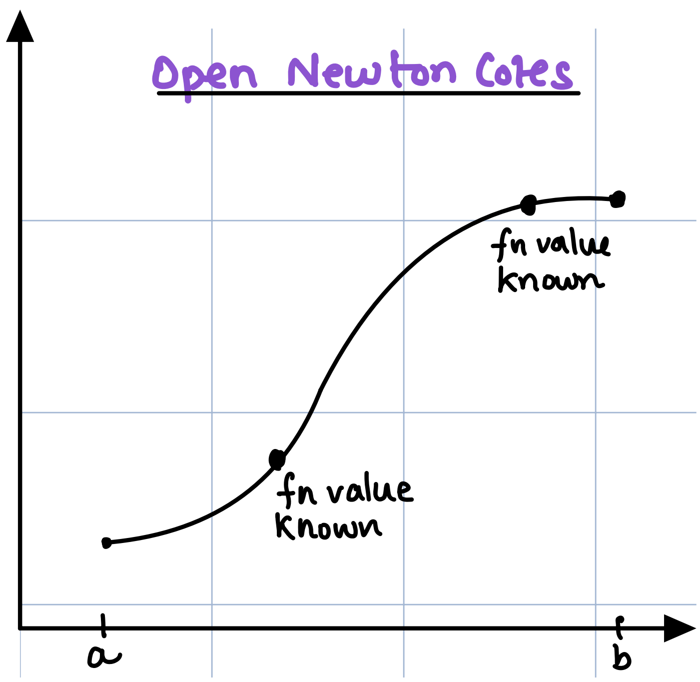
$x = 0 \rightarrow$ corresponds to $s = 0$

$x = b \longrightarrow$ corresponds to $s = s$

$$I = h\int_0^s P_n(s)\,ds$$

- Applicable when function values are available at equal intervals.
- There are 2 types of Newton's cotes formula.
  - Closed Newton's Cotes formula
  - Open Newton Cotes Formula





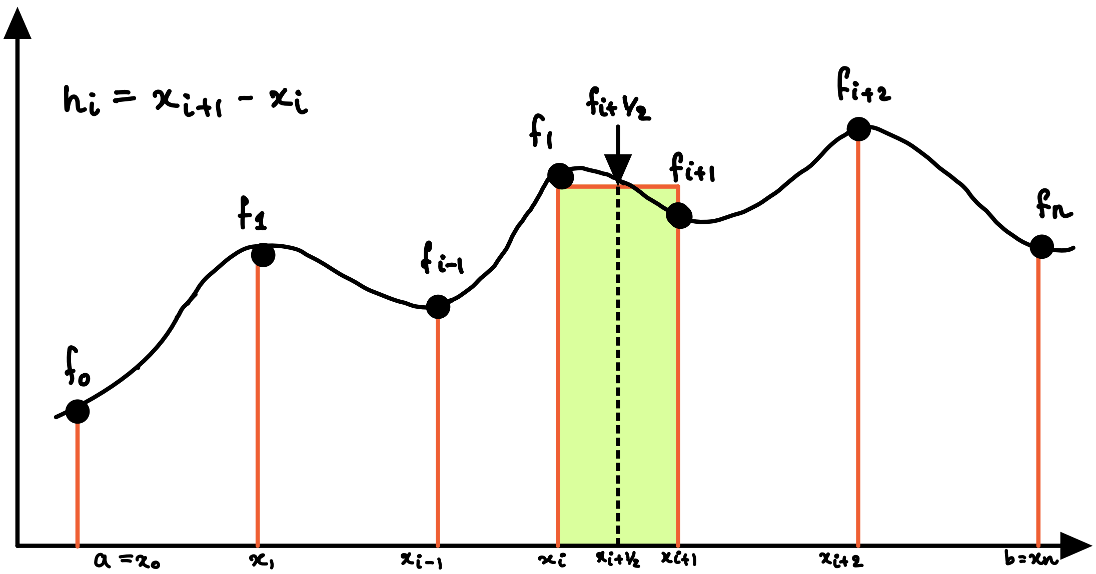
- Function value is known at a and b then Closed Newton cotes formula is used.
- Function value is known at some distance away from a and b in Open Newton cotes formula

CLOSED NEWTON COTE'S FORMULA

- Rectangular Rule
- Trapezoidal Rule
- Simpson's Rule

Newton Cotes Formula is applicable only when points are evenly spaced.

1. Rectangular rule



Suppose we have the function values at discrete points say $x_0, x_1, \ldots x_n$. We have to find the area within this curve.

The true nature of the curve between the function values is not known.

In Rectangular Rule we Approximate the curve by using a Constant Function.

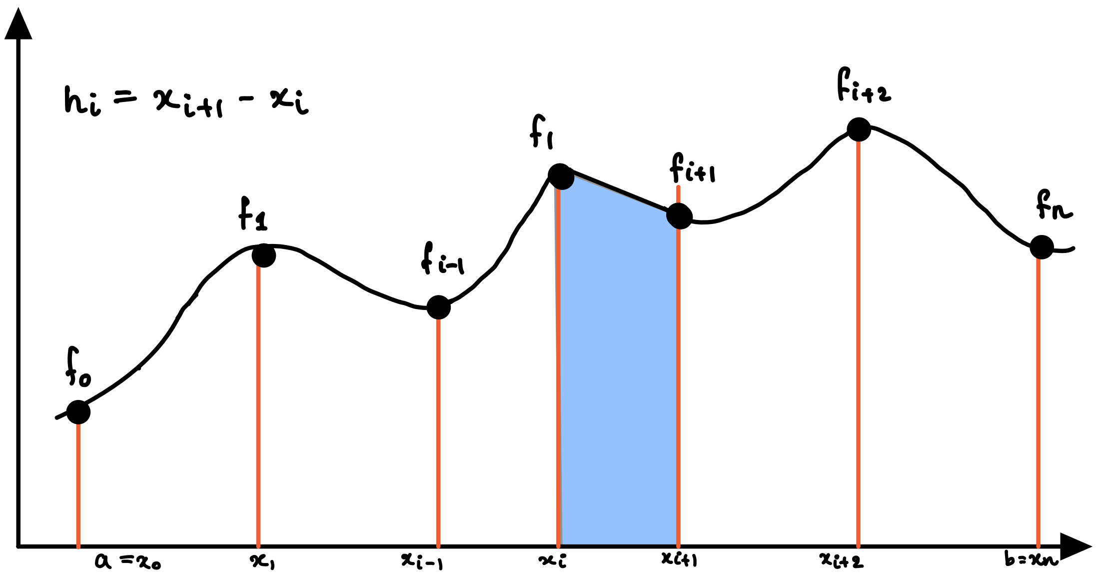
The constant function is at middle of an interval.

Polynomial $p(x)$ is peicewise constant function :- $p_i(x) = f_{i+1/2}$

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \int_{x_i}^{x_{i+1}} f_{i+1/2}\,dx = h_i f_{i+1/2}$$

Same has to be applied to each interval and then take its summation.

2. Trapezoidal rule



In this case we approximate the function by using a linear function. Polynomial $p(x)$ is piecewise linear function within every interval. Using Lagrange Interpolation Formula the piecewise linear function as

$$\boxed{f(x) = p(x) = \frac{x - x_i}{x_{i+1} - x_i} f_{i+1} + \frac{x - x_{i+1}}{x_i - x_{i+1}} f_i = \frac{f_{i+1}}{h_i}(x - x_i) - \frac{f_i}{h_i}(x - x_{i+1})}$$

Then we Integrate approximating Polynomial and get the result as:-

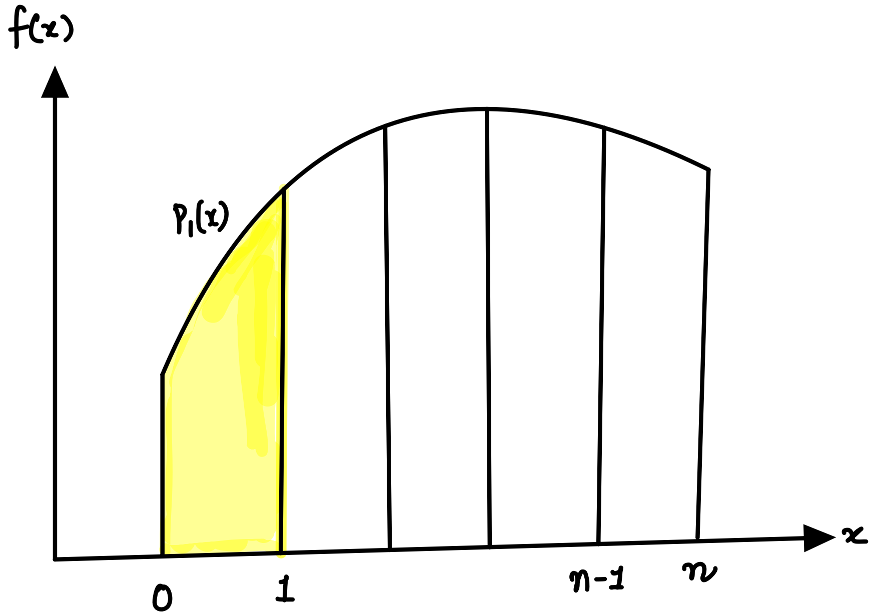
$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \int_{x_i}^{x_{i+1}} p(x)\,dx = \frac{f_{i+1}}{h_i}\int_{x_i}^{x_{i+1}} (x - x_i)\,dx - \frac{f_i}{h_i}\int_{x_i}^{x_{i+1}} (x - x_{i+1})\,dx$$

$$= \frac{f_{i+1}}{h_i}\left[\frac{h_i^2}{2}\right] - \frac{f_i}{h_i}\left[-\frac{h_i^2}{2}\right] = h_i\left(\frac{f_{i+1}}{2} + \frac{f_i}{2}\right)$$

$$I = \int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx \approx \sum_{i=0}^{n-1} h_i\left(\frac{f_{i+1}}{2} + \frac{f_i}{2}\right)$$

If the mesh size is uniform, $h_i = h \;\forall\, i$

$$\boxed{I = \int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx \approx h\left[\frac{f_0}{2} + \frac{f_n}{2} + \sum_{i=1}^{n-1} f_i\right]}$$



$$P_n(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \ldots + \frac{s(s-1)(s-2)\ldots[s-(n-1)]}{n!}\Delta^n f_x$$

If $n = 1$, and $s = 1$

$$\Delta I = h\int_0^1 (f_0 + s\Delta f_0)\, ds$$

$$= h\left(sf_0 + \frac{s^2}{2}\Delta f_0\right)_0^1$$

$$= h\left(f_0 + \frac{\Delta f_0}{2}\right) = h\left\{f_0 + \left(\frac{f_1 - f_0}{2}\right)\right\}$$

$$\Delta I = \frac{h}{2}(f_0 + f_1)$$

(Or) $\Delta I = \dfrac{h}{2}\left(f(x_i) + f(x_{i+1})\right)$

Composite Trapezoidal Rule:

$$I = \sum_{i=0}^{n-1} \Delta I = \sum_{i=0}^{n-1} \frac{h_i}{2}(f_i + f_{i+1})$$

If the data is equally spaced

$$h_i = x_{i+1} - x_i$$

$$I = \frac{h}{2} \sum_{i=0}^{n-1} (f_i + f_{i+1})$$

$$= \frac{h}{2} \left[f_0 + 2f_1 + 2f_2 + \dots + 2f_{n-1} + f_n \right]$$

Example $\int_{3.1}^{3.9} 1/x \, dx$

x	$f(x)$
3.1	0.3225
3.2	0.3125
3.3	0.3030
3.4	0.2941
3.5	0.2857
3.6	0.2778
3.7	0.2703
3.8	0.2631
3.9	0.2564

	Trapezoidal	Simpson's
h	I	I
0.8	0.2316	
0.4	0.2301	0.2296
0.2	0.2297	0.2297
0.1	0.2296	0.2296

3. Simpson's rule

In case of Rectangular or Trapezoidal Rule is we are trying to fit a constant or linear function within an interval.

How can we improve the estimates ?

- ✓ Intervals can be made very fine to get more accurate results and representation of the integral
- ✓ Also higher order polynomials can be used in order to get better representation of the polynomial within the interval.

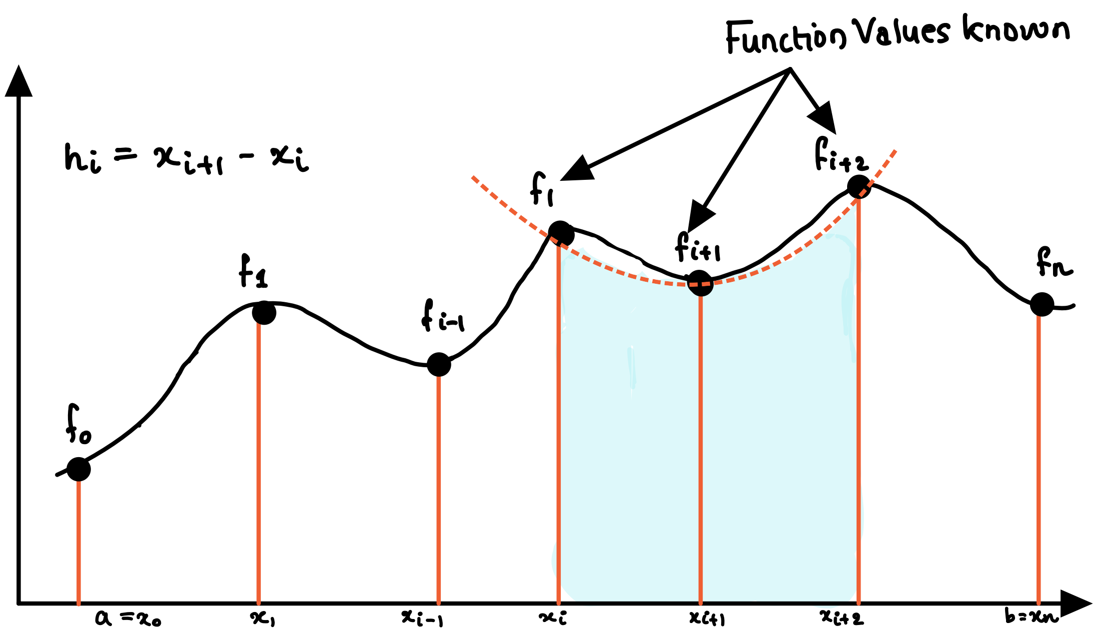
Let's say the integral within the interval x_i and x_{i+1} is to be evaluated. If we have another point in between the interval, at which we have the information of the function then we can fit a quadratic polynomial. Now if we have two extra points within the interval where the function value is known then we can fit a cubic polynomial to these 4 points.

The results of integral thus obtained by using fitting of higher order Polynomials is an impoved estimate of integral.

The formulas that utilize these higher order polynomials comes under Simpson's Rule.

(a) Simpson's 1/3 rd Rule

In this we try to fit a 2^{nd} Order Polynomial.



Polynomial $p(x)$ is piecewise quadratic function:

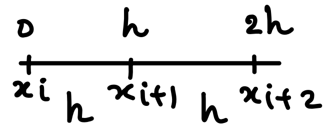
$$f(x) \approx p(x)$$

$$= \frac{(x-x_{i+2})(x-x_{i+1})}{(x_i-x_{i+2})(x_i-x_{i+1})} f_i + \frac{(x-x_{i+2})(x-x_i)}{(x_{i+1}-x_{i+2})(x_{i+1}-x_i)} f_{i+1} + \frac{(x-x_i)(x-x_{i+1})}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})} f_{i+2}$$

$$\int_{x_i}^{x_{i+2}} f(x)\,dx \approx \int_{x_i}^{x_{i+2}} p(x)\,dx$$

$$= f_i \int_{x_i}^{x_{i+2}} \frac{(x-x_{i+2})(x-x_{i+1})}{(x_i-x_{i+2})(x_i-x_{i+1})}\,dx + f_{i+1} \int_{x_i}^{x_{i+2}} \frac{(x-x_{i+2})(x-x_i)}{(x_{i+1}-x_{i+2})(x_{i+1}-x_i)}\,dx + f_{i+2} \int_{x_i}^{x_{i+2}} \frac{(x-x_i)(x-x_{i+1})}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})}$$

Assume $h_i = h_{i+1} = h$ and substitute $z = x - x_i$



$$x = z + x_i$$

$$x - x_{i+2} = z + x_i - x_{i+2} = z - 2h$$

$x_i - x_{i+2} = -2h.$ $\quad$ $x_{i+1} - x_{i+2} = -h$ $\quad$ $x - x_i = z + x_i - x_i = z$

$x_i - x_{i+1} = -h$ $\quad$ $x_{i+1} - x_i = h$ $\quad$ $x - x_{i+1} = z + x_i - x_{i+1} = z - h$

Eqn becomes:

$$\int_{x_i}^{x_{i+2}} p(x)\,dx = f_i \int_0^{2h} \frac{(z-2h)(z-h)\,dz}{(-2h)(-h)} + f_{i+1} \int_0^{2h} \frac{(z-2h)(z)\,dz}{(-h)(h)} + f_{i+2} \int_0^{2h} \frac{z(z-h)}{(2h)(h)}\,dz$$

$$= \frac{f_i}{2h^2} \int_0^{2h} (z-2h)(z-h)\,dz - \frac{f_{i+1}}{h^2} \int_0^{2h} (z-2h)\,z\,dz + \frac{f_{i+2}}{2h^2} \int_0^{2h} (z-h)z\,dz$$

Expanding, Integrating and applying limits:-

$$= \frac{f_i}{2h^2}\left[\frac{(2h)^3}{3} - 3h\frac{(2h)^2}{2} + 2h^2(2h)\right] - \frac{f_{i+1}}{2}\left[\frac{(2h)^3}{3} - 2h\frac{(2h)^2}{2}\right] + \frac{f_{i+2}}{2h^2}\left[\frac{(2h)^3}{3} - h\frac{(2h)^2}{2}\right]$$

$$\boxed{\int_{x_i}^{x_{i+2}} f(x)\,dx \approx \int_{x_i}^{x_{i+2}} p(x)\,dx = \frac{h}{3}\left[f_i + 4f_{i+1} + f_{i+2}\right]}$$

This is known as Simpson's $1/3^{rd}$ Rule.

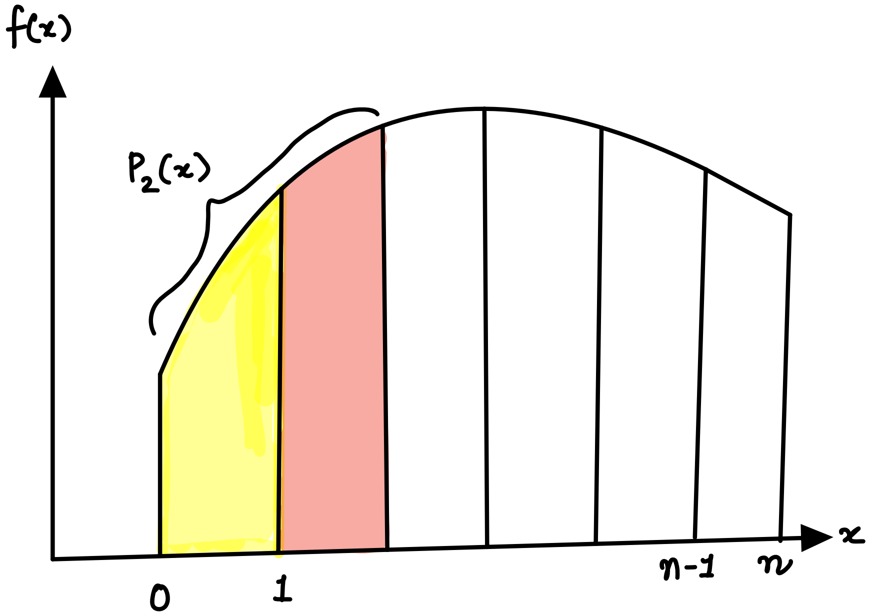
If the mesh is uniform, $h_i = h \ \forall \ i$

$$I = \int_a^b f(x)\,dx \approx \frac{h}{3}\left[f_0 + f_n + 4\sum_{\substack{i=1 \\ i=\text{odd}}}^{n-1} f_i + 2\sum_{\substack{i=1 \\ i=\text{even}}}^{n-2} f_i\right] = h\sum_{i=0}^{n} \omega_i f_i$$

ω_i is the weightage and depends on value of i.

NOTE: The number of intervals $(n) = 2m$ where m is an integer

$n = 2$, $s = 2$



$$P_n(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \ldots + \frac{s(s-1)(s-2)\ldots[s-(n-1)]}{n!}\Delta^n f_x$$

$$\Delta I = h\int_0^2 \left(f_0 + s\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0\right) ds$$

$$\Delta I = h\left[sf_0 + \frac{s^2}{2}\Delta f_0 + \frac{1}{2}\left(\frac{s^3}{3} - \frac{s^2}{2}\right)\Delta^2 f_0 \right]_0^2$$

$$= h\left[2f_0 + 2\Delta f_0 + \left(\frac{4}{3} - 1\right)\Delta^2 f_0 \right]$$

$$= h\left[2f_0 + 2(f_1 - f_0) + \frac{1}{3}\left[f_2 - 2f_1 + f_0\right]\right\}$$

$$= \frac{h}{3}\left[6f_0 + 6(f_1 - f_0) + f_2 - 2f_1 + f_0 \right]$$

$$= \frac{h}{3}\left[f_0 + 4f_1 + f_2 \right]$$

Applying the above eqn for entire range

$$I = \frac{1}{3}h\left[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 4f_{n-1} + f_n \right]$$

(B) Simpson's 3/8 th Rule

If the polynomial $p(x)$ is piecewise cubic function

$$f(x) \approx p(x) = \frac{(x-x_{i+3})(x-x_{i+2})(x-x_{i+1})}{(x_i-x_{i+3})(x_i-x_{i+2})(x_i-x_{i+1})} f_i + \frac{(x-x_{i+3})(x-x_{i+2})(x-x_i)}{(x_{i+1}-x_{i+3})(x_{i+1}-x_{i+2})(x_{i+1}-x_i)} f_{i+1} +$$

$$\frac{(x-x_i)(x-x_{i+1})(x-x_{i+3})}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})(x_{i+2}-x_{i+3})} f_{i+2} + \frac{(x-x_i)(x-x_{i+1})(x-x_{i+2})}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} f_{i+3}$$

Assume $h_i = h_{i+1} = h_{i+2} = h$ and substitute $z = (x-x_i)$

$$\int_{x_i}^{x_{i+2}} f(x)\,dx \approx \int_{x_i}^{x_{i+2}} p(x)\,dx$$

$$= \frac{-f}{6h^3}\int_0^{3h}(z-3h)(z-2h)(z-h)\,dz + \frac{f_{i+1}}{2h^3}\int_0^{3h}(z-3h)(z-2h)z\,dz - \frac{f_{i+2}}{2h^3}\int_0^{3h}(z-3h)(z-h)z\,dz + \frac{f_{i+2}}{6h^3}\int_0^{3h}(z-2h)(z-h)z\,dz$$

$$= -\frac{f_i}{6h^3}\left[\frac{(3h)^4}{4} - 6h\frac{(3h)^3}{3} + 11h^2\frac{(3h)^2}{2} - 6h^3(3h)\right] + \frac{f_{i+1}}{2h^3}\left[\frac{(3h)^4}{4} - 5h\frac{(3h)^3}{3} + 6h^2\frac{(3h)^2}{2}\right]$$

$$- \frac{f_{i+2}}{2h^3}\left[\frac{(3h)^4}{4} - 4h\frac{(3h)^3}{3} + 3h^2\frac{(3h)^2}{2}\right]$$

$$\int_{x_i}^{x_{i+2}} f(x)\,dx \approx \int_{x_i}^{x_{i+2}} p(x)\,dx = \frac{3h}{8}\left[f_i + 3f_{i+1} + 3f_{i+2} + f_{i+3}\right]$$

This is known as Simpson's $3/8^{th}$ Rule.

If the mesh is uniform, $h_i = h \ \forall i$

$$I = \int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f_0 + f_n + 3\sum_{i=1,4,7,10\ldots}^{n-1}(f_i + f_{i+1}) + 2\sum_{i=3,6,9\ldots}^{n-3} f_i\right] = h\sum_{i=0}^{n} w_i f_i$$

Note : In this case Number of Intervals has to be $n = 3m$ where m is an Integer.

Simpson's $3/8^{th}$ Rule is rarely used in Engineering Problems

$n = 3$, $S = 3$

$$\Delta I = h \int_0^3 \left(f_0 + s\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{6}\Delta^3 f_0 \right) ds$$

$$\Delta I = \frac{3}{8} h \left(f_0 + 3f_1 + 3f_2 + f_3 \right)$$

$$I = \frac{3h}{8}\left(f_0 + 3f_1 + 3f_2 + 2f_3 + 3f_4 + \dots + 3f_{n-1} + f_n \right).$$

Summary

- Rectangular rule – Uses Constant Function Approximation
- Trapezoidal rule – Uses Linear Function Approximation
- Simpson's rule. – Uses Quadratic/Cubic function Approximation.

Accuracy of numerical integration

How accurate are the numerical integration schemes with respect to the true integral?

When we are utilising these integration schemes, we are approximating the function $f(x)$ by function $p(x)$. When we approximate we are going to make **Local** as well as, **Global Errors**.

TRUNCATION ERROR ANALYSIS

Local Errors: If the error that we are making by utilising the approximating functions is within a particular integral, then it is called as Local Error.

Global Errors: When all the Local errors over the interval a to b are summed up, then the error obtained is known as Global Error.

> **Recall**: True Value (a) = Approximate Value $(\tilde{a})$ + Error (ε)

ERRORS IN NUMERICAL INTEGRATION

Multiple Segment Trapezoidal Rule

True Error

$$E_t = \frac{(b-a)^3}{12n^3} \frac{\sum_{i=1}^{n} f''(\alpha_i)}{n}$$

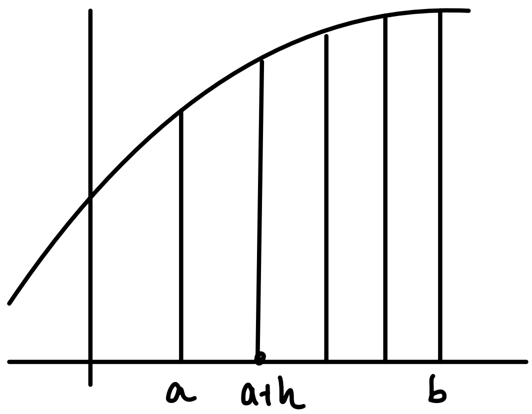
$\sum_{i=1}^{n} f''(\alpha_i) / n \simeq \bar{f}''$

n → no. of segments

$(b-a)^3$ → Cube of interval difference

where, α_i is some no. b/w lower & upper limit of segments. $a+(i-1)h < \alpha_i < a+ih$

n - segment



$$\int_a^b f(x)\,dx$$

$$h = \frac{b-a}{n}$$

$$\frac{\sum_{i=1}^{n} f''(\alpha_i)}{n}$$ → Avg. value of second derivative of function → its not a constant

As you ↑se n value this term is going to tend towards a constant value $\approxeq \bar{f}''$

$$\boxed{E_t \simeq \propto \frac{1}{n^2}}$$

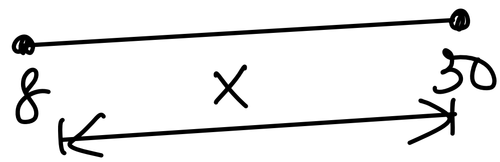
* Double no. of segments error is going to be quatered.

Example

Velocity of rocket is

$$v(t) = 2000 \ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t$$

Calculate the distance covered by the rocket b/w 8s to 30sec.



$$x = \int_{30}^{8} v(t)\,dt = 11061$$

$$f(t) = 2000 \ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t$$

$$f(8) = 177.27 \text{ m/s}$$

$$f(30) = 901.67 \text{ m/s}$$

$$I = \frac{h}{2}\left[f(8) + f(30)\right]$$

$$= 11868 \text{ m}$$

$$\int_{8}^{30} f(t)\,dt = \int_{8}^{19} f(t)\,dt + \int_{19}^{30} f(t)\,dt$$

$f(8) = 177.27$ m/s

$f(19) = 484.75$ m/s

$f(30) = 901.67$ m/s

$$\int_{8}^{30} f(t)\,dt = \frac{(19-8)}{2}\left(177.27 + 484.75\right) + \frac{(30-19)}{2}\left(901.67 + 484.75\right)$$

$$= \underline{11266 \text{ m}}$$

n	Approx value	E_t	$E_t/4$
2	11266	−205	−51.25
4	11113	−48	−12
8	11074	−13	3.25
16	11065	−4	

Richardson's extrapolation

- Proposed in 1927 for the numerical whether prediction.
- This method can be applied whenever it is known that an approximation technique has an error which has a predictable form and that error depends on a parameter which is usually the interval of size 'h'.

- True error in multiple segment Trapezoidal rule

$$E_t \simeq \propto \frac{1}{n^2} \qquad n : \text{no. of segment}$$

- We take advantage of this property and use it to come up with a better estimate of Integrals.

$$E \approx \frac{c}{n^2}$$

$$E_t = \text{True value} - \text{Aprox.value}$$

$$T.V = \text{Aprox. value} + E_t$$

$$T.V = (\text{Aprox Value})_n + (E_t)_n$$

$$\boxed{TV \approx (\text{Approx Value})_n + \frac{c}{n^2}} \quad \text{—— } (1)$$

So now if we double the no. of segments

$$\boxed{TV \approx (\text{Approx Value})_{2n} + \frac{c}{4n^2}} \quad \text{—— } (2)$$

① $-\ 4\times$②

$$TV \approx (\text{Approx Value})_n + \frac{c}{n^2} \quad \text{— ①}$$

$$4TV \approx 4(\text{Approx Value})_{2n} + \frac{c}{n^2} \quad \text{— ②} \times 4$$

$$-3TV \approx (\text{Approx Value})_n - 4(\text{Approx Value})_{2n}$$

$$TV \approx \frac{4}{3}(\underbrace{\text{Approx Value}}_{(A.V)})_{2n} - \frac{(\text{Approx Value})_n}{3}$$

$$\approx \frac{3(A.V)_{2n} + (AV)_{2n} - (A.V)_n}{3}$$

$$\approx (A.V)_{2n} + \frac{1}{3}\left\{AV_{2n} - (AV)_n\right\}$$

Example

$$X = \int_{8}^{30} 2000 \ln\left(\frac{140000}{140000 - 2100t}\right) - 9.8t \, dt$$

n	$(AV)_n$
1	11868
2	11266
4	11113
8	11074

Find a better estimate of the integral using Richardson's Extrapolation.

$$TV \simeq (A.V)_{2n} + \frac{1}{3}\left\{AV_{2n} - (AV)_n\right\}$$

$$\simeq (AV)_4 + \frac{1}{3}\left\{A.V_4 - (AV)_2\right\}$$

$$\simeq 11113 + \frac{1}{3}\left\{11113 - 11266\right\}$$

$$\simeq 11062$$

n	$(AV)_n$	Refinement
1	11868	
		11065 ftw
2	11266	
		11062
4	11113	
		11061 ftw
8	11074	

Exact 11061

ROMBERG INTEGRATION

In Romberg Integration : We combine 2 estimates of an Integral to compute the 3rd more accurate approximation.

Recall : In case of Trapezoidal Rule the global error is of $O(h^2)$

NOTE

Romberg Integration is much more efficient than the Simpson's 1/3 rd Rule.

Disadvantage: Both Romberg and Simpson's requires equally spaced points.

Romberg Integration can be derived for other methods as well. Here we will focus on Trapezoidal Rule.

Richardson Extrapolation formula

$$I = \int_a^b f(x)\,dx \quad \text{and} \quad E_t \simeq \alpha \frac{1}{n^2}$$

$$\left.\begin{array}{lll} n & - & I_n \\ 2n & - & I_{2n} \end{array}\right\rangle TV \simeq I_{2n} + \frac{I_{2n} - I_n}{3}$$

Continue to make such better approximations

$$\left\{ I_{2n} + \frac{I_{2n} - I_n}{3} \right\}$$

$$\begin{array}{llllllll}
n & I_n & & & & & & 3\text{rd order} \\
 & & TV_1 & & & & & \\
 & & & TV_4 & & & & \\
2n & I_{2n} & & & TV_6 & - & E_t \simeq \alpha\ ? & \\
 & & TV_2 & & & & & \\
4n & I_{4n} & & TV_5 & & & & \\
 & & TV_3 & & & & & \\
8n & I_{8n} & & & & & &
\end{array}$$

2^{nd} order

$E_t \simeq \alpha\ ?$

1^{st} order Extrapolation

$$E_t \propto \frac{1}{n^2}$$

Richardson's Extrapolation

$$n = \frac{x_{i+1} - x_i}{h}$$

$$E_t \simeq \alpha \frac{1}{n^2}$$

$$E_t \simeq \alpha\ h^2$$

The true error in trapezoidal rule is of the form

$$E_t = A_1 h^2 + A_2 h^4 + A_3 h^6 + \cdots$$

$A_1, A_2 \cdots A_n$ are not constants, they are based on derivatives of function which you integrating.

n	I_n
$2n$	I_{2n}
$4n$	I_{4n}
$8n$	I_{8n}

$$I_{2n} + \frac{I_{2n} - I_n}{3} \quad — J_n$$

$$I_{4n} + \frac{I_{4n} - I_{2n}}{3} \quad — J_{2n}$$

$$4^{(2-1)} - 1$$

$$J_{2n} + \frac{J_{2n} - J_n}{15} \rightarrow \text{why}$$

$E_t \propto h^4$ & not h^2

$$E_t \approx c h^4$$

$$TV \simeq J_n + C/n^2$$

$$TV \simeq J_n + \frac{C}{(4n)^2}$$

$$TV = J_{2n} + \frac{J_{2n} - J_n}{15}$$

$$15 \longrightarrow \underline{\underline{4^{2-1} - 1}}$$

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j-1} - I_{k-1,j}}{4^{k-1} - 1}$$

$k \geq 2$

k : Order of Extrapolation

$k = 1$ Trapezoidal Rule

$k = 2$ Richardson's Extrapolation

j : Less accurate estimate

$j+1$: Next accurate estimate.

$k, 1$: 1 and $2n$ segments

2 and $4n$ segments.

Example. Romberg Integration

Qn: Find $\int_0^2 e^x \, dx$. Using Romberg Integration upto an error of $e_a = 0.01\%$.

Soln: True value $= 6.389$.

Integration for different step sizes using the trapezoidal Rule.

$$\boxed{I = \int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx \approx h\left[\frac{f_0}{2} + \frac{f_n}{2} + \sum_{i=1}^{n-1} f_i\right]}$$

$$I = 2 \times \left[\frac{e^0}{2} + \frac{e^2}{2}\right] = 8.3891$$

$$I = 1 \times \left[\frac{e^0}{2} + \frac{e^2}{2} + e^1\right] = 6.9128$$

$$I = 0.5 \times \left[\frac{e^0}{2} + \frac{e^2}{2} + e^{0.5} + e^1 + e^{1.5}\right] = 6.5216$$

$$I = 0.25 \times \left[\frac{e^0}{2} + \frac{e^2}{2} + e^{0.25} + e^{0.5} + e^{0.75} + e^1 + e^{1.25} + e^{1.5} + e^{1.75}\right]$$

$$= 6.422298$$

n	Interval (h)	$I(h)$
1	2	8.3891
2	1	6.9128
4	0.5	6.5216
8	0.25	6.4223

$$I(h) = \frac{4}{3} I(h) - \frac{1}{3}(I(2h))$$

$$\boxed{I_{k+1}(h) = \frac{4^k I_k(h/2) - I_k(h)}{4^k - 1}}$$

$$\boxed{e_a = \left| \frac{I_{1,k}(h) - I_{2,k-1}(h/2)}{I_{1,k}(h)} \right| \times 100}$$

<u>Case 0</u> : e_t : $\left| \frac{6.389 - 8.3891}{6.389} \right| \times 100 = \underline{\underline{31.3\%}}$

<u>Case 1</u> $e_a = \left| \frac{6.4207 - 6.9128}{6.4207} \right| \times 100 = \underline{\underline{7.664\%}}$

<u>Case -2</u>

$$e_a = \left| \frac{6.3892 - 6.3912}{6.3892} \right| \times 100 = 0.0313\%$$

INTERVAL (h)		K = 1	K = 2	K = 3
1	8.3891			
2	6.9128	6.4207		
4	6.5216	6.3912	6.3892	
8	6.4223	6.3892	6.3891	$\boxed{6.3891}$
Error (e_t)	31.3%	0.49%		
Error (e_a)		7.664%	0.031%	0%

Improvement of numerical integration results.

For improvement of Numerical Integration results :-

(a) Simplest way is to increase the number of intervals.

(b) Increase order of Polynomial - Wiggling at ends

(c) Use error Information - Romberg Integration

(d) Optimally select the points for function evaluation - Adaptive quadrature rules (Gauss - Legendre Quadrature).

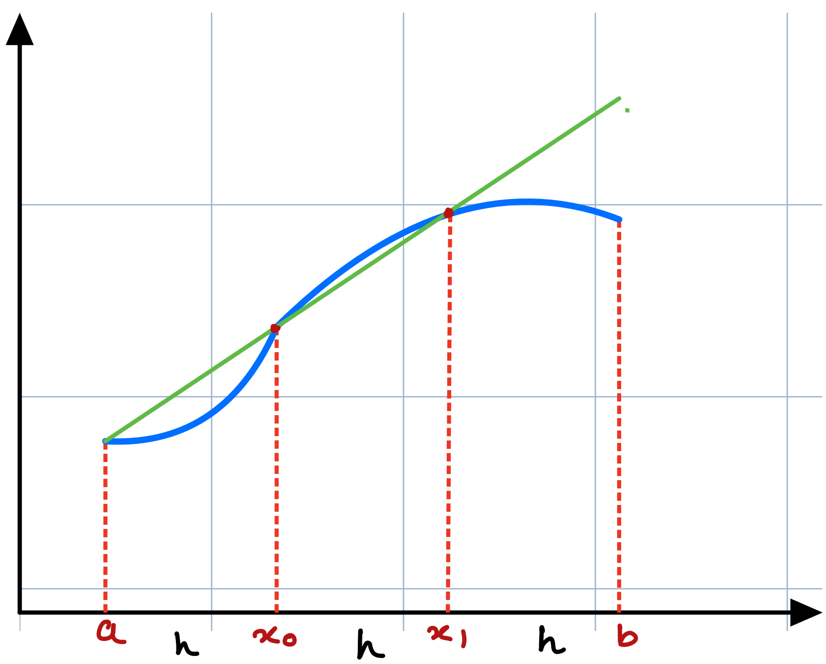
OPEN NEWTON-COTES FORMULA

[Refer Chapra and Cannayle]

When do you need Open Newton-cote's Formula?

Let us consider a function $f(x)$. and $\int_a^b f(x)dx$ is to be evaluated.

But the function values at x_0 and x_1 are available where $x_0 > a$ and $x_1 < b$.



- In such cases the function has to be extrapolated.
- Between x_0 and x_1 we can use an approximating function which is linear in nature. So we will basically go for the Trapezoidal Rule.
- We extend this approximating function towards the ends as well. (i.e. till a and b).

Trapezoidal Rule

- $$\int_a^b f(x)\,dx = I(h) + E(h).$$

$$I(h) = \int_{x_0-h}^{x_0+h} P_1(x)\,dx = \int_{x_0-h}^{x_0+h} \left[\frac{x-x_1}{x_1-x_0} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)\right] dx$$

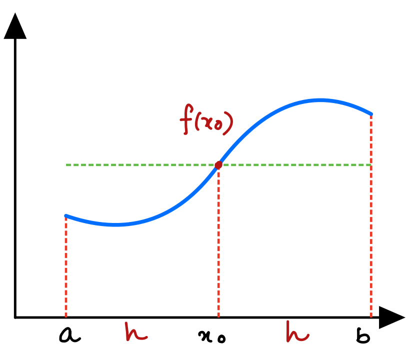
$$= \frac{3h}{2}\left[f(x_0 + f(x_1)\right] = \frac{b-a}{2}\left[f(x_0) + f(x_1)\right]$$

The error in this case would be :-

$$E = \int_{x_0-h}^{x_1+h} \frac{1}{2!} f''(\xi)\,(x-x_0)(x-x_1)\,dx = \frac{3}{4} f(\xi)\,h^3$$

Mid-Point Formula :-

$$I = 2h\, f(x_0)$$
$$= (b-a)\, f(x_0)$$

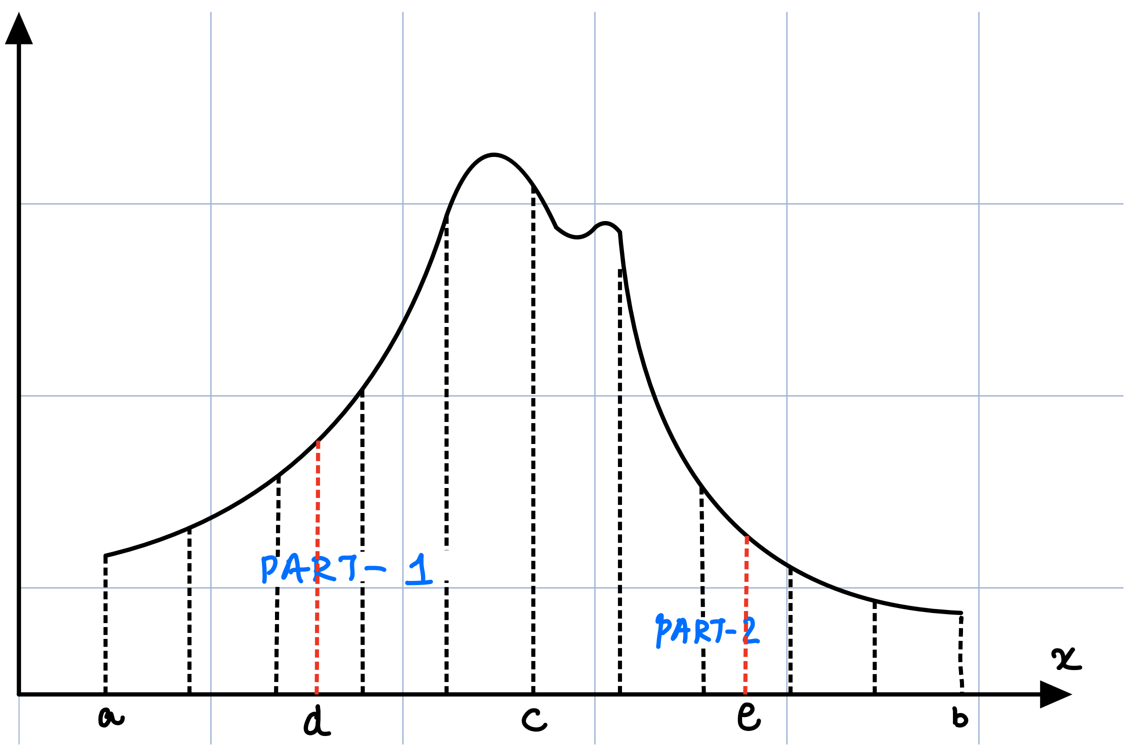


Summary.

- Error analysis
  - Local truncation error
  - Global truncation error – 1 order less than the LTE

ADAPTIVE QUADRATURE

- Romberg Integration is much more efficient than the Simpson's $1/3^{rd}$ Rule.
- **Disadvantage :** Both Simpson's & Romberg Integration requires equally spaced points.
- However in real world conditions it is likely to encounter functions which has region of high variability along with sections where the changes are gradual in the function. So such functions can be integrated using Adaptive Quadrature method.
- Adaptive Quadrature Method allows to adjust the step size.
- Facilitates the usage of small intervals in situations of Rapid variations and larger intervals where the function changes gradually.
- Based on Simpson's $1/3^{rd}$ Rule



1. Start by defining a point c such that $c = \frac{a+b}{2}$
2. For the first part ie. from a to c apply Simpson's $1/3^{rd}$ Rule to find the integral.

3. For the first part compute the integral for Interval of size h and $h/2$ [denoted as $I(h)$ and $I(h/2)$].

4. Find error b/w these 2 Integrals $I(h)$ and $I(h/2)$

$$e = \left| I(h/2) - I(h) \right|$$

5. Check whether this error is less than a particular Threshold value.

If yes: Define the Integration as :-

$$I = \frac{16}{15} I(h/2) - \frac{1}{15} I(h)$$

Else: Again introduce d such that $d = \frac{a+c}{2}$ and find out function values $f(c)$, $f(a)$ and $f(d)$.

6. Repeat Step 1 to 5 for other parts as well.

This is a recursive programming - Automatically changes the step size if there is a rapid change in the function value.

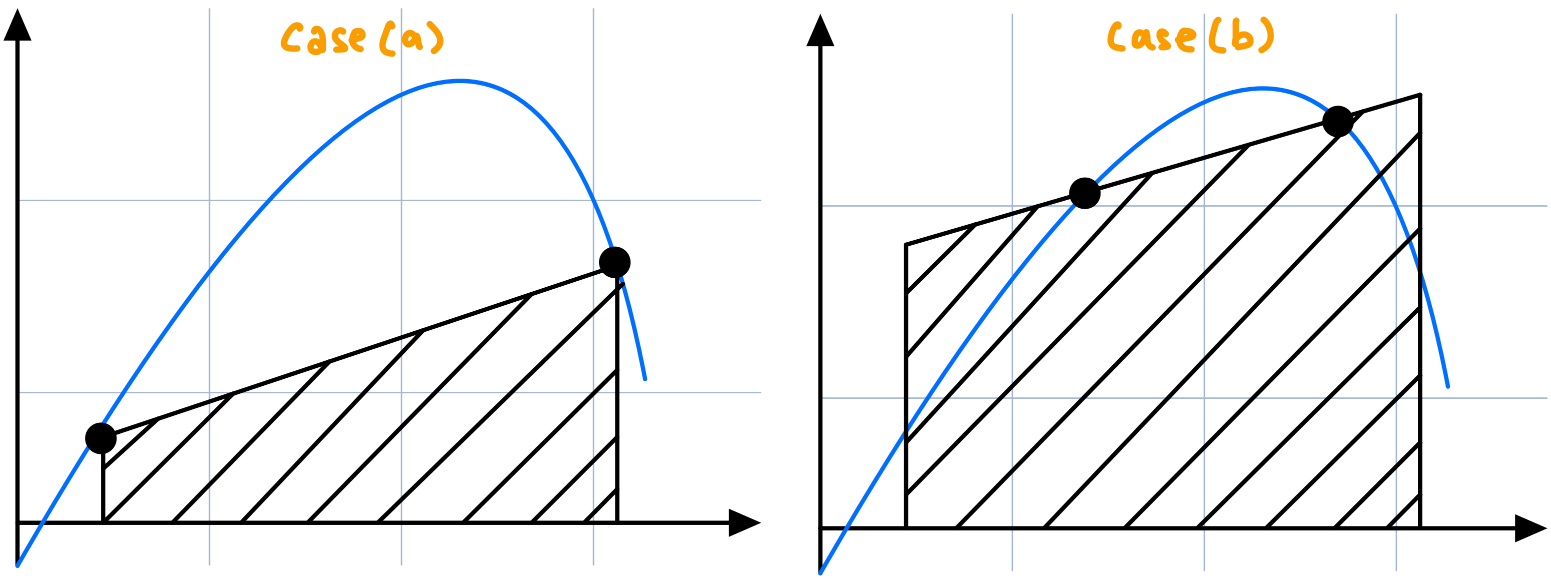
GAUSS-QUADRATURE METHOD

- Whatever Numerical Integration Rules that are discussed so far requires the data points at which we have the function values are evenly spaced.
- The location of these points is fixed.

In the figure given below :-

Case (a) : Fitting a straight line to the given data points we are making large amount of error in the integration at a local scale

Case (b) : Fitting a straight line through these alternately chosen points gives a better estimate.



Case (a) : Graphical Representation of Trapezoidal Rule

Case (b) : Improved integral estimate by taking the area under the straight line passing through two intermediate points. By positioning these points wisely the positive and negative errors are balanced, and an improved integral estimate results.
This is what is done in case of Gauss-Quadrature Method.

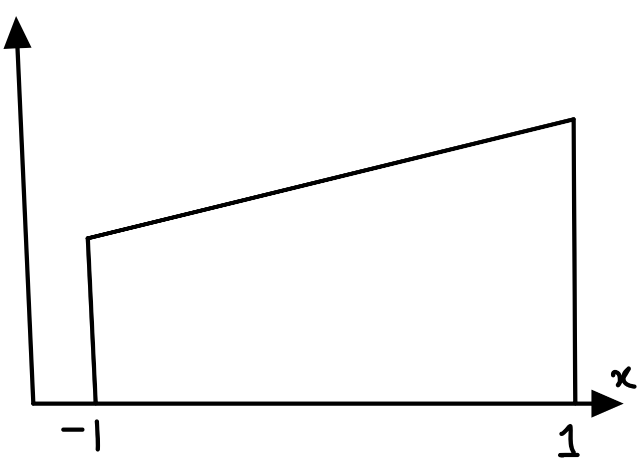
In Gauss-Quadrature Method there is an option of chosing the points in the x-direction and accordingly the interval is evaluated.

Gauss Quadrature Method is based on the method of undetermined coefficients.

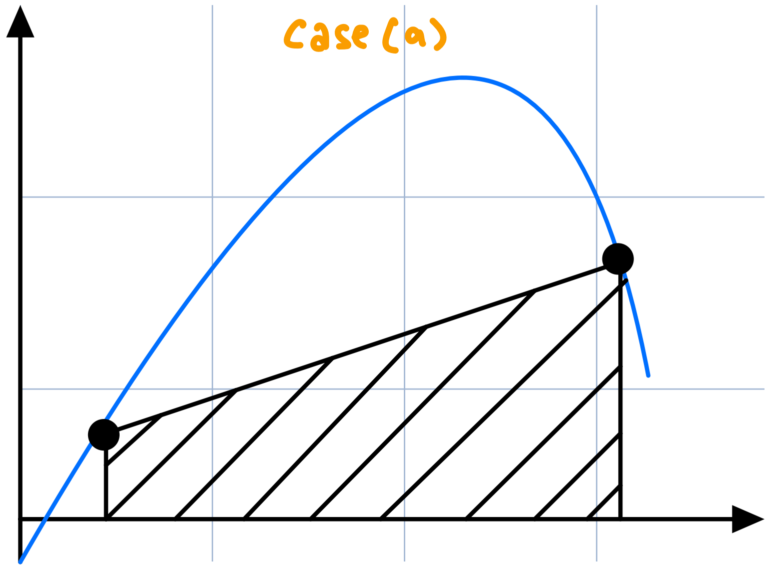
METHOD OF UNDETERMINED COEFFICIENTS.

$$\int_{-1}^{1} f(y)\,dy = \sum_{i=0}^{n} c_i f(y_i)$$

y_i : Arbitary/free parameters which can be chosen.



If a function $f(x)$ which is dependent on 'x' is to be approx. by a straight line [So that trapezoidal rule can be used to find the area).

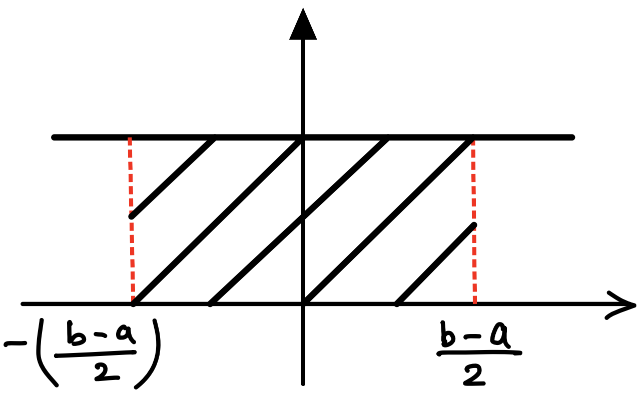


$$I = (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

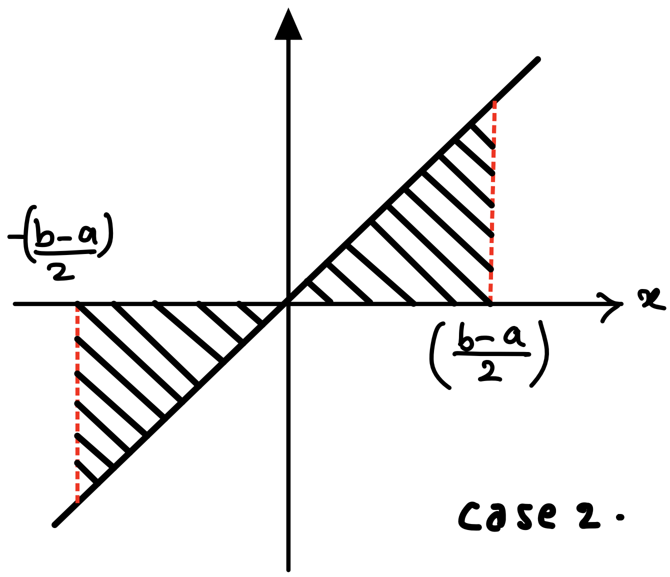
$$I \approx c_0 f(a) + c_1 f(b).$$

The trapezoidal Rule should give the exact Results when the function being integrated is a constant or a straight line. Two simple equations that represent these 2 cases are as follows:-

1. $y = 1$ and 2. $y = x$



case 1.



case 2.

Case - 1.

$$\int_{-\frac{(b-a)}{2}}^{\frac{b+a}{2}} 1 \cdot dx = c_0 + c_1$$

Case - 2

$$\int_{-\frac{(b-a)}{2}}^{\frac{b+a}{2}} x\,dx = -c_0\left(\frac{b-a}{2}\right) + c_1\left(\frac{b-a}{2}\right)$$

In case of Gauss, we approximate the Integral I

$$I \cong c_0 f(x_0) + c_1 f(x_1)$$

Here x_0, c_0, x_1, c_1 are unknowns.

$\therefore$ 4 unknowns requires 4 eqns to determine them exactly.

For the trapezoidal Rule, the two conditions were obtained by assuming that the equation fits the integrals of a constant and a linear function exactly.

So we can extend the same reasoning by assuming that it also fits the integral of a parabolic and a cubic function. By doing this we have 4 unknowns and 4 equations and the unknowns c_0, x_0, x_1, c_1 can be determined.

$y = 1, x, x^2$ or x^3.

If we Generalize the Gauss Quadrature formula:-

Given $(n+1)$ points $(i = 0, 1, 2 \cdots n)$ ie $2(n+1)$ unknowns and we can fit $(2n+1)$ order polynomial. which will estimate Integrals exactly for polynomials of order $\leq 2n+1$.

(a) Approximation by a Linear Function

Order of linear function $(n) = 1$

Integrating function that can be approximated can be upto the order of $2n+1 = 3$.

Number of unknowns $= 2(n+1) = 2(1+1) = \underline{\underline{4}}$.

4 equations:-

- $$\int_{-1}^{1} 1\, dy = 2 = c_0 f(y_0) + c_1 f(y_1)$$

$$c_0 + c_1 = 2 \qquad \text{———} \qquad (1)$$

- $\int_{-1}^{1} y\,dy = 0 = c_0 f(y_0) + c_1 f(y_1)$

$$y_0 c_0 + y_1 c_1 = 0 \quad \text{———————} \quad (2)$$

- $\int_{-1}^{1} y^2 dy = \frac{2}{3} = c_0 y_0^2 + c_1 y_1^2$

$$c_0 y_0^2 + c_1 y_1^2 = \frac{2}{3} \quad \text{———————} \quad (3)$$

- $\int_{-1}^{1} y^3 dy = 0 = c_0 y_0^3 + c_1 y_1^3$

$$c_0 y_0^3 + c_1 y_1^3 = 0 \quad \text{———————} \quad (4)$$

Solving :- $c_0 = 1$, $c_1 = 1$, $y_0 = \frac{-1}{\sqrt{3}}$ & $y_1 = \frac{1}{\sqrt{3}}$.

(b) Approximations by a Quadratic Function ($n = 2$)

Highest order :- $(2n+1) = 5$

unknowns :- 6

$c_0 = 5/9$

$c_2 = 5/9$

$c_1 = 8/9$

$y_0 = \sqrt{-3/5}$

$y_1 = 0$

$y_2 = \sqrt{\frac{3}{5}}$.

It turns out for a given $(n+1)$ points the y's are the roots of $(n+1)^{th}$ order Legendre Polynomials.

$$P_{n+1}(x) = \frac{2n+1}{n+1} x \, P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \longrightarrow x = \pm\sqrt{\frac{1}{3}}$$ (Linear Approx. Roots)

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x = 0 \longrightarrow x = 0, \pm\sqrt{\frac{3}{5}}$$ (Quadratic)

Hence Gauss. Quadrature is also called as Gauss Legendre Qudrature.

One-Point Integration

$$P_1(x) = x = 0 \quad ; \quad \omega \text{ (or) } c = 1$$

Two-Point Integration

$$P_2(x) = \frac{1}{2}(3x^2 - 1) = 0$$

$$x = \pm\frac{1}{\sqrt{3}} \qquad x_0 = -\frac{1}{\sqrt{3}} \text{ and } x_1 = \frac{1}{\sqrt{3}}$$

$$\omega_0 = \int_{-1}^{1} \frac{x - x_1}{x_0 - x_1}\, dx = \frac{-\sqrt{3}}{2}\int_{-1}^{1}\left(x - \frac{1}{\sqrt{3}}\right) dx = 1$$

$$\omega_1 = \int_{-1}^{1} \frac{x - x_1}{x_0 - x_1}\, dx = \frac{\sqrt{3}}{2}\int_{-1}^{1}\left(x + \frac{1}{\sqrt{3}}\right) dx = 1$$

Three-Point Integration

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right) = 0$$

$$x = 0, \pm\sqrt{\frac{3}{5}}.$$

$$x_0 = -\sqrt{\frac{3}{5}}, \quad x_1 = 0, \quad x_2 = \sqrt{\frac{3}{5}}$$

$$w_0 = \int_{-1}^{1} \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}\,dx = \int_{-1}^{1} \frac{(x - 0)\left(x - \sqrt{\frac{3}{5}}\right)}{-\sqrt{\frac{3}{5}}\left(-\sqrt{\frac{3}{5}} - \sqrt{\frac{3}{5}}\right)}\,dx$$

$$= \frac{5}{6}\int_{-1}^{1} x\left(x - \sqrt{\frac{3}{5}}\right)dx = \underline{\underline{\frac{5}{9}}}$$

$$w_1 = \int_{-1}^{1} \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}\,dx = -\frac{5}{3}\int_{-1}^{1}\left(x + \sqrt{\frac{3}{5}}\right)\left(x - \sqrt{\frac{3}{5}}\right)dx = \frac{8}{9}$$

$$w_2 = \int_{-1}^{1} \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}\,dx = \frac{5}{6}\int_{-1}^{1} x\left(x + \sqrt{\frac{3}{5}}\right)dx = \underline{\underline{\frac{5}{9}}}$$

TABLE 22.1 Weighting factors c and function arguments x used in Gauss-Legendre formulas.

Points	Weighting Factors	Function Arguments	Truncation Error
2	$c_0 = 1.0000000$	$x_0 = -0.577350269$	$\cong f^{(4)}(\xi)$
	$c_1 = 1.0000000$	$x_1 = 0.577350269$	
3	$c_0 = 0.5555556$	$x_0 = -0.774596669$	$\cong f^{(6)}(\xi)$
	$c_1 = 0.8888889$	$x_1 = 0.0$	
	$c_2 = 0.5555556$	$x_2 = 0.774596669$	
4	$c_0 = 0.3478548$	$x_0 = -0.861136312$	$\cong f^{(8)}(\xi)$
	$c_1 = 0.6521452$	$x_1 = -0.339981044$	
	$c_2 = 0.6521452$	$x_2 = 0.339981044$	
	$c_3 = 0.3478548$	$x_3 = 0.861136312$	
5	$c_0 = 0.2369269$	$x_0 = -0.906179846$	$\cong f^{(10)}(\xi)$
	$c_1 = 0.4786287$	$x_1 = -0.538469310$	
	$c_2 = 0.5688889$	$x_2 = 0.0$	
	$c_3 = 0.4786287$	$x_3 = 0.538469310$	
	$c_4 = 0.2369269$	$x_4 = 0.906179846$	
6	$c_0 = 0.1713245$	$x_0 = -0.932469514$	$\cong f^{(12)}(\xi)$
	$c_1 = 0.3607616$	$x_1 = -0.661209386$	
	$c_2 = 0.4679139$	$x_2 = -0.238619186$	
	$c_3 = 0.4679139$	$x_3 = 0.238619186$	
	$c_4 = 0.3607616$	$x_4 = 0.661209386$	
	$c_5 = 0.1713245$	$x_5 = 0.932469514$	

How to perform integration in an arbitrary range?

To perform integration in an arbitary range (a, b)

$$\int_a^b f(x)\,dx \xrightarrow[\text{Transformed}]{} \int_{-1}^{1} f(y)\,dy.$$

$$x = \alpha + \beta y$$
$$dx = \beta\,dy$$

$x = a$; $y = -1$ $\longrightarrow$ $a = \alpha - \beta$.

$x = b$; $y = 1$ $\longrightarrow$ $b = \alpha + \beta$.

$\alpha = \dfrac{a+b}{2}$ and $\beta = \dfrac{b-a}{2}$

$$\boxed{\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{b-a}{2}y\right)\frac{b-a}{2}\,dy}$$

Example - 2 .

Qn) Find $\int_0^2 e^x dx$ using Legendre Quadrature Method.

Soln : $\int_0^2 e^x dx \xrightarrow[\text{Transformed}]{} \int_{-1}^{1} f(x')dx'$

True value = 6.3891

$$= \int_{-1}^{1} e^{(1+y)} dy .$$

Approximation using 2-point formula.

$C_0 = 1 \qquad C_1 = 1$

$x_0 = \frac{-1}{\sqrt{3}} \qquad x_1 = \frac{1}{\sqrt{3}}$

$$I = C_0 f(x_0) + C_1 f(x_1)$$

$$= f(-1/\sqrt{3}) + f(1/\sqrt{3})$$

$$= e^{1-1/\sqrt{3}} + e^{1+1/\sqrt{3}}$$

$$= \underline{\underline{6.3681}}$$

$$e_{true} = \left| \frac{6.3891 - 6.3681}{6.3891} \right| \times 100 = \underline{\underline{0.313\ \%}}$$

Remark : Because Gauss-Quadrature requires function evaluation at non-uniformly spaced points within the integration interval, it is **NOT useful** for cases where the function is **Unknown**. So it is not suited for engineering problems, because in engineering problems we have a tabulated data.

IMPROPER INTEGRALS

So far we have talked about integrals that have finite limits and bounded integrands. These are the integrals that are commonly found in engineering.

But in certain situations we may encounter improper integrals and we have to evaluate them.

What are Improper Integrals ?

- If either a or b or both are infinity
- If the function to be integrated is undefined / discontinous at any point in the integral.

This is meaningful only if the Integral converges.

Case I : Either a or b is infinity.

(i) $ab > 0$

$$\int_a^b f(x)\,dx = ? \qquad \text{if} \quad a = -\infty \quad \text{or} \quad b = \infty$$

Evaluated by making a change of variable. i.e transform the infinite range to a finite range.

Identity used for transformation :-

$$\boxed{\int_a^b f(x)\,dx = \int_{1/b}^{1/a} \frac{1}{t^2} f\left(1/t\right) dt}$$

This identity will work for any function that decreases towards zero atleast as fast as $\frac{1}{x^2}$ as $x \to \infty$.

(ii) $ab < 0$

a is $-\infty$ and b is positive

(or) a is negative and b is ∞.

In this case divide the integral into two parts :-

$$\int_{-\infty}^{b} f(x)\,dx = \underbrace{\int_{-\infty}^{-A} f(x)\,dx}_{\mathrm{I}} + \underbrace{\int_{-A}^{b} f(x)\,dx}_{\mathrm{II}}$$

$-A$: 'A' is chosen as a sufficiently large negative value, so that $f(x)$ has begun to approach zero asymptotically atleast as fast as $\frac{1}{x^2}$.

Part I can be solved by the previously discussed Identity.

Part II can be solved by any of the Integration Techniques such as Simpson's $1/3^{rd}$ Rule (or) Trapezoidal Rule.

Problem Encountered :- When we are trying to transform this Integral using the Identity we have talked about earlier; it is possible that at certain points the transformed function f will be singular; in one of these limits.

In those cases you can go for Open Integration formulas, that do not require evaluation of Integral at the end points of the Integration Interval.

Summary.

- Romberg Integration
- Method of undetermined coefficients.
- Improper integrals

PROBLEMS.

Qn 1) Using Trapezoidal Rule Integrate:

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5 \text{ from}$$

$a = 0$ to $b = 0.8$. True Solution $= 1.640533$.

Soln:- One Interval

$$f(0) = 1/5$$

$$f(0.8) = 0.232$$

$$I = h\,\frac{f(a) + f(b)}{2} \approx 0.8\left[\frac{1/5 + 0.232}{2}\right]$$

$$I \approx \underline{0.1728}.$$

$$E_r = \left|\frac{1.640533 - 0.1728}{1.640533}\right| \times 100 = \underline{89.47}\ \%.$$

$$E_r = \frac{-1}{12} f''(\xi)\, h^3$$

$$\bar{f''}(x) = \frac{\int_0^{0.8} f''(x)}{0.8 - 0} = \frac{\int_0^{0.8} 8000x^3 - 10800x^2 + 4050x - 400}{0.8}$$

$$= \frac{-48}{0.8} = -60$$

$$E_a = \frac{-1}{12}(-60)(0.8)^3 = \underline{\underline{2.56}}.$$

Two Interval

$h = 0.4.$

$$I = \int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx \approx h\left[\frac{f_0}{2} + \frac{f_n}{2} + \sum_{i=1}^{n-1} f_i\right]$$

$$I = 0.4\left[\frac{1}{10} + \frac{0.232}{2} + 2.456\right]$$

$$= \underline{\underline{1.0688}}$$

$$E_r = \left|\frac{1.64053 - 1.0688}{1.64053}\right| \times 100 = \underline{\underline{34.85\%}}$$

$$E_a = -\frac{(b-a)^3}{12n^2} f''(\xi)$$

$$= -\frac{0.8^3}{12 \times 2^2} \times -60 = \underline{\underline{0.64}}.$$

Qn) Use Simpson's $1/3^{rd}$ Rule to find $\int_0^{0.8} f(x)\,dx$. where

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$.

$$I = \frac{h}{3}\left[f_0 + f_n + 4 * \sum_{i=odd} f_i + 2 * \sum_{i=even} f(i)\right]$$

$$I = \frac{0.4}{3}\left[f(x_0) + 4f(x_1) + f(n)\right]$$

$$I = \frac{0.4}{3}\left[0.2 + 4 \times [2.456] + 0.232\right] = \underline{\underline{1.367}}.$$

$$E_r = -\frac{(b-a)^5}{2880} f''''(\xi)$$