Lecture 26 – Fourier series properties & Fourier Integral

$$\underset{\substack{\text{data/}\\ \text{signal}}}{f(x)} = \sum_{m=-\infty}^{\infty} \underset{\text{coefficient}}{a_m} \underbrace{e^{imx}}_{\text{mathematical function / orthogonal}} \quad x \in [0, 2\pi)$$

base function

complex

domain

Synthesis

$$a_m = \frac{1}{2\pi} \underbrace{\int_0^{2\pi} f(x)\, e^{-imx}\, dx}_{\text{global integral}}$$

conjugate

complex

$a_{\pm m}$

Analysis

$$\int_0^{2\pi} e^{imx}\, e^{-ikx}\, dx = 2\pi\, \delta_{mk}$$

signal

spectrum

base function

Synthesis

Analysis

Orthogonality

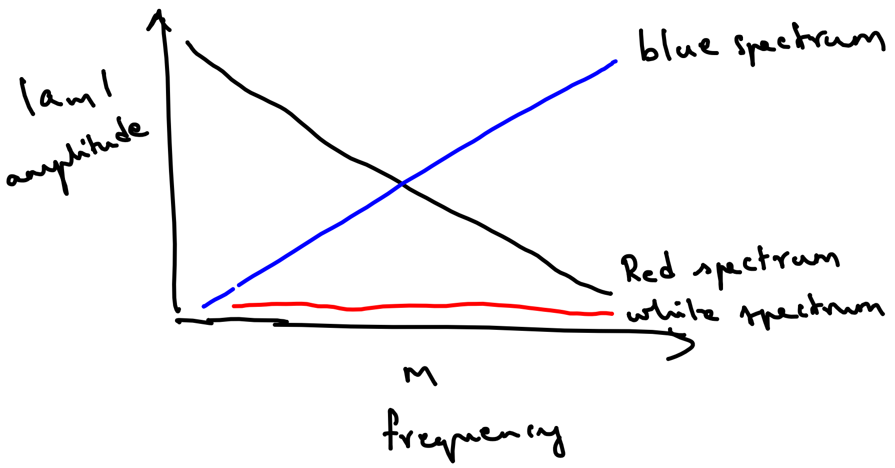
Integrable functions

$$\int_0^{2\pi} |f(x)|\, dx = \text{finite value} \quad | \quad \text{sufficient}$$

$$\int_0^{2\pi} f^2(x)\, dx = \text{finite value} \quad | \quad \text{extra handle on the signal}$$

Power spectrum analysis → Hilbert space

inner product space



Parneval's theorem

$$\int_0^{2\pi} f^2(x)\, dx = 2\pi \sum_{m=-\infty}^{\infty} |a_m|^2$$

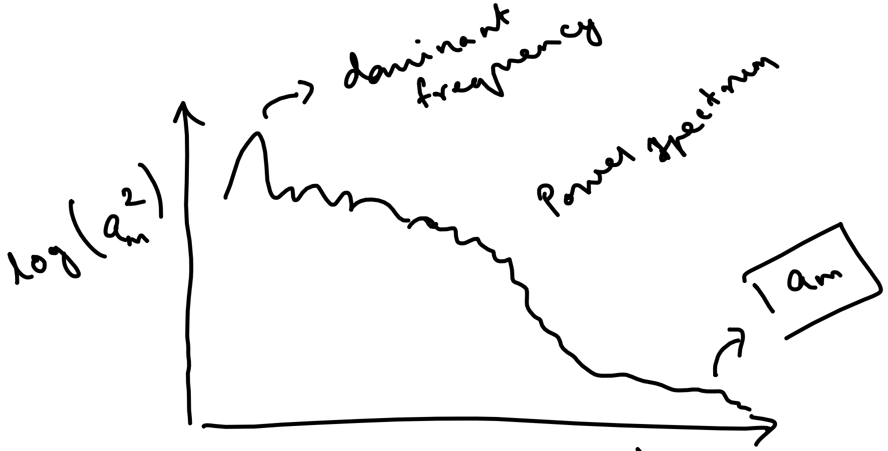
Energy integral

$$\int_0^{2\pi} f^2(x)\, dx = \int_0^{2\pi} \sum_{m=-\infty}^{\infty} a_m e^{imx} \sum_{k=-\infty}^{\infty} a_k e^{-ikx}\, dx$$

$$= \sum_{m=-\infty}^{\infty} a_m \sum_{k=-\infty}^{\infty} a_k \int_0^{2\pi} e^{imx} e^{-ikx}\, dx$$

$$= \sum_{m=-\infty}^{\infty} a_m \sum_{k=-\infty}^{\infty} a_k\, \delta_{mk}\, 2\pi = 2\pi \sum_{m=-\infty}^{\infty} |a_m|^2$$

$$\sum_{m=-\infty}^{\infty} a_m^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2 + \dots +$$



$m >> 1$

Basic tool for analysing signals

Superposition principle

$\underbrace{f(x)}_{\text{specific frequencies}} + \underbrace{g(x)}_{\text{rest of the frequencies}} \longleftrightarrow a_m + b_m$

$f(x) - g(x) \longleftrightarrow a_m - b_m$

Precursor to filtering

1. transform pair $\quad f(x) \longleftrightarrow a_m$

2. Proportionality $\quad c f(x) \longleftrightarrow c a_m$

3. Superposition $\quad f(x) + g(x) \longleftrightarrow a_m + b_m$

$x \in [0, 2\pi)$

4. Symmetry $\quad f(-x) \longleftrightarrow a_{-m}$

5. Translation

- signal $\quad f(x + x_0) \longleftrightarrow a_m e^{imx_0}$

- spectrum $\quad f(x) \underline{e^{-ikx}} \longleftrightarrow a_{m+k}$

conjugate

6. Differentiation $\quad \frac{d^p}{dx^p} f(x) \longleftrightarrow (im)^p a_m \Rightarrow$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \sum a_m \underline{\underline{e^{imx}}}$$

$$= \sum \underline{im\, a_m}\, e^{imx}$$

7. Integration $\quad \int_0^{2\pi} f(x) \longleftrightarrow a_0$

Proof by applying the orthogonality rule after substituting $f(x) = \sum a_m e^{imx}$

Fourier series $\rightarrow$ circle

$-\infty$ ——— ∞

line

$$\frac{1}{2\pi}\int_0^{2\pi} \longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} \rightarrow 1/2\pi\int_{-P/2}^{P/2} \rightarrow \lim_{P\to\infty} 1/2\pi\int_{-P/2}^{P/2} f(x)\, e^{-im\frac{2\pi}{P}p}\, dp \Rightarrow \frac{2\pi}{P}m = f$$

$P = 2\pi$

$x = \frac{2\pi}{P}p$

$$\downarrow$$

$$\lim_{P\to\infty} 1/2\pi\int_{-P/2}^{P/2} f(x)\, e^{-ifp}\, \underline{dp} \Leftrightarrow \int_{-\infty}^{\infty} a(f)\, e^{ifp}\, \underline{df}$$

domain of the line

domain of the frequency

$$\boxed{\begin{aligned} 1/2\pi\int_{-\infty}^{\infty} f(x)\, e^{-ifp}\, dp &= a(f) \\ \int_{-\infty}^{\infty} a(f)\, e^{ifp}\, df &= f(x) \end{aligned}}$$

transform pair of the Fourier Integral