Lecture 13 - Interpolation

Curve fitting → regression $\quad y_i = \underline{f}(x_i)$

Curve fitting → interpolation $\quad y_i \Rightarrow y_i^* \rightarrow$ unknown "amongst" the given data

$y_i \longrightarrow$ fit some function

find values "amongst" the data

i) $\quad \underbrace{f(x_{i-1}) \quad f^*(x_i) \quad f(x_{i+1})}$

$$f^*(x_i) = \frac{f(x_i+1) + f(x_i-1)}{2}$$

ii) $\quad f(x_{i-1}) \xrightarrow{\text{slope}} f(x_{i+1})$, $f(x_i)$

iii) nearest neighbour → contiguity $\quad f(x_{i-1}) \underset{d_{i-1}}{\longleftrightarrow} f(x_i) \underset{d_{i+1}}{\longleftrightarrow} f(x_{i+1})$

Orthogonal polynomials

$$\int_a^b f(x)\, P_i(x)\, dx = a_i \int_a^b P_i(x)\, P_j(x)\, d$$

$$\Rightarrow a_i = \int_a^b f(x)\, P_i(x)\, dx \Big/ \boxed{\int_a^b P_j(x)\, P_j(x)\, dx}$$

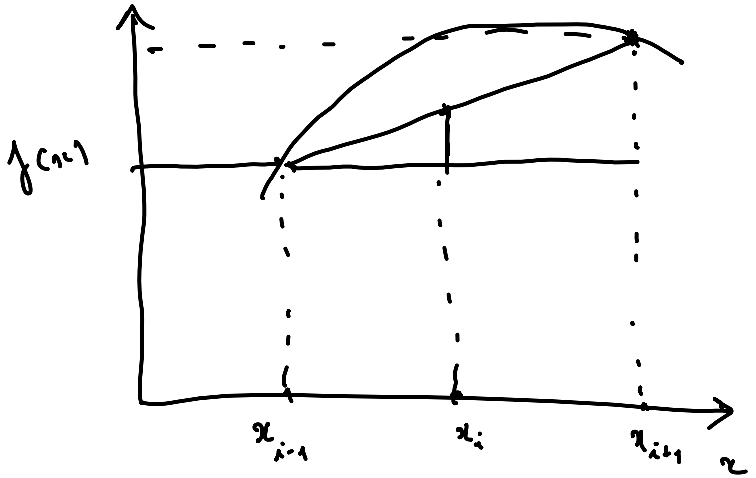
Normalization factor

$i = j$

$$\bar{P}_i(x) = \frac{P_i(x)}{\sqrt{\int_a^b P_i(x)\, P_i(x)\, dx}} \Rightarrow \int_a^b \bar{P}_i\, \bar{P}_i\, dx = 1$$

orthonormal polynomial

Linear interpolation



$$\frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}} = \frac{f(x) - f(x_{i-1})}{x - x_{i-1}}$$

$$f(x) = f(x_{i-1}) + \boxed{\frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}}}(x - x_{i-1})$$

derivative

Finite divided difference

$$f(x) = f(x_{i-1}) + \frac{df}{dx}(x - x_{i-1}) + R_1$$

$$f(x) = f(x_{i-1}) + \frac{f(x_{i+1}) - f(x_{i-1})}{(x_{i+1} - x_{i-1})} \underbrace{(x - x_{i-1})}_{\Delta x \to \text{small}}$$

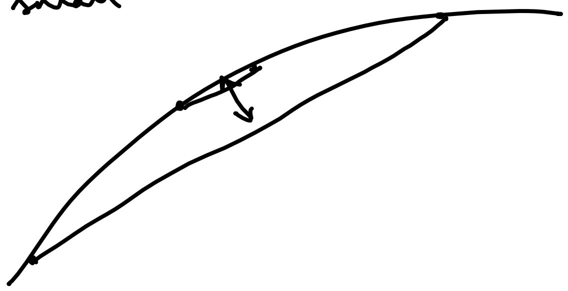
↓

including errors

↳ noise associated with the data

↓

signal / observation method



$6x \to \frac{n(n+1)}{2}$

$3+$

2 1

$f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$

$= ((a_3 x + a_2)x + a_1)x + a_0$

$3x$ n

$3+$ n

Divided differences → polynomials

Newton's form → factorized

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

x_0

$x = x_0$ $f(x_0) = b_0$

$x = x_1$ $f(x_1) = b_0 + b_1(x_1 - x_0)$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

↓

first derivative

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

3 → equation $\begin{matrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{matrix}$ $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$ $\begin{vmatrix} b_0 \\ b_1 \\ b_2 \end{vmatrix}$

3 →

n+1

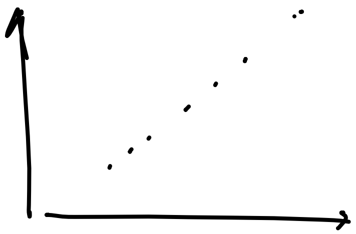
→ polynomial → 2

$$\begin{matrix} x_0 & f(x_0) & \\ & & f[x_0, x_1] \\ x_1 & f(x_1) & & f[x_0, x_1, x_2] \\ & & f[x_1, x_2] \\ x_2 & f(x_2) & \end{matrix}$$

Recursive

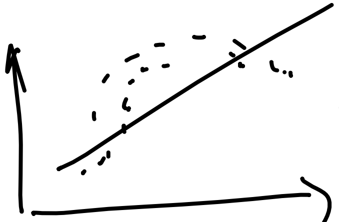
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix}$$

$-1 \quad 1 \qquad f(x_3)$



→ Over fitting

Goodness of fit



under fitting

n+1 data points

n degree

finite divided differences

$$f(x) = b_0 + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$+ \ldots + f[x_0, x_1, x_2, \ldots, x_{n+1}](x - x_0)(x - x_1)\ldots(x - x_n)$$

error estimate $\rightarrow R_n$

Lagrange interpolation

$$f_n(x) = \sum_{i=0}^{n} L_i(x)\, f(x_i)$$

Product

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$f(x_1) \rightarrow f(x_1)$$