Lecture 28 – FFT, Convolution, Sampling, Power spectrum

$$f[\underline{x}] = \sum_{k=-N/2}^{N/2-1} a_k \, e^{i\frac{2\pi}{\underline{p}} k x}$$

$$f[\underline{n\Delta x}] = \sum_{k=-N/2}^{N/2-1} a_k \, e^{i 2\pi/p \, k \underline{n\Delta x}}$$

$$f[n] = \sum_{k=-N/2}^{N/2-1} a_k \, e^{i 2\pi/\cancel{p} \, k n \cancel{p}/N}$$

$$f[n] = \sum_{k=-N/2}^{N/2-1} a_k \underbrace{e^{i 2\pi/N}}_{W}{}^{kn}$$

$$f[n] = \sum_{k=-N/2}^{N/2-1} a_k W^{kn}$$

$$\Rightarrow \quad a[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] \, W^{*kn}$$

Transform pair

$P \to$ Period

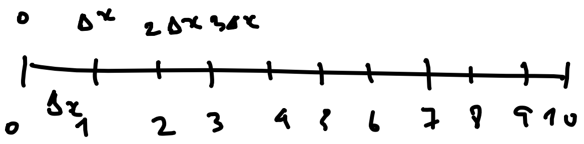
$N \to$ number of frequencies

$N \to$ number of data points

$x \to$ sampled on an equispaced grid

$\underline{P = N\Delta x}$ → space between two adjacent points

$\Delta x = P/N$



$$F = \begin{array}{c|c} & k \longrightarrow \\ \hline n \downarrow & \begin{bmatrix} W^{nk} & \cdots & \cdots \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & \ddots \end{bmatrix} \end{array}$$

$n = 0 \text{ to } 3 \qquad k = -2 \text{ to } 1$

$$f_0 = a_{-2} W^{-2\cdot 0} + a_{-1} W^{-1\cdot 0} + a_0 W^{0\cdot 0} + a_1 W^{1\cdot 0}$$

$$f_1 = a_{-2} W^{-2\cdot 1} + a_{-1} W^{-1\cdot 1} + a_0 W^{0\cdot 1} + a_1 W^{1\cdot 1}$$

$$f_2 = a_{2} W^{-2\cdot 2} + a_{-1} W^{-1\cdot 2} + a_0 W^{0\cdot 2} + a_1 W^{1\cdot 2}$$

$$f_3 = a_{-2} W^{-2\cdot 3} + a_{-2} W^{-1\cdot 3} + a_0 W^{0\cdot 3} + a_1 W^{1\cdot 3}$$

$$W = e^{i 2\pi/4}$$

$$W = e^{i\pi/2} = i$$

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = \overset{F}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ i^{-2} & i^{-1} & 1 & i \\ i^{-4} & i^{-2} & 1 & i^{2} \\ i^{-6} & i^{-3} & 1 & i^{3} \end{bmatrix}} \begin{bmatrix} a_{-2} \\ a_{-1} \\ a_0 \\ a_1 \end{bmatrix}$$

transpose of F & multiplication with $\underline{f}$ gives $\underline{a}$

$\downarrow$

This costs N^2 operations.

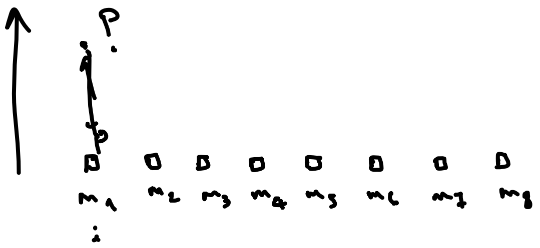
$f_0 \quad 2k \qquad 0 \qquad\qquad 2\underline{k} \quad 0 \text{ to } N/2$

$f_2 \qquad\qquad 2$

$f_1 \quad 2\cancel{k}\boxed{+1} \quad k=0 \quad 1$

$f_3 \qquad\qquad k=1 \quad 3$

Convolution



$$\Rightarrow \quad \vec{a}_p = -G \sum_{i=1}^{8} \frac{m_i}{l_{ip}^3} \vec{l}_{ip}$$

$$m_i \rightarrow dm$$

$$|\vec{a}_p| = -G \int_0^L \frac{dm_Q}{l_{PQ}^3} \vec{l}_{PQ}$$

$$a_p = -G \int_0^L \frac{dm_Q}{l_{PQ}^2}$$

$$a_p = -G \int_0^L K(P,Q)\, dm_Q$$

$$a_p = -G \int_0^L \underbrace{K(P,Q)}_{\text{Kernel}} \rho_Q\, dV$$

One - dimension

$$a_p = -G \int_0^L \underline{K(P-Q)\, \rho_Q}\, dV$$

Convolution

$f(x)$

$g(x)$

$$h(x) = f(x) \boxed{*}\, g(x) = (f * g)(x)$$

$\boxed{*} \rightarrow$ convolution

$$h(x) = \int_{x'} f(x')\, g(x'-x)\, dx'$$

$$f(x') = \int_{f} a(f) e^{i f x'}\, df$$

$$g(x') = \int_{f'} b(f') e^{i f' x'}\, df'$$

$$g(x'-x) = \int_{f'} b(f') e^{i f'(x'-x)}\, df' \quad ?$$

$$= \int_{f'} b(f') e^{i f' x'} e^{-i f' x}\, df'$$

$$h(x) = \int_{x'} \int_{f} a(f) e^{i f x'} \int_{f'} b(f') e^{i f' x'} e^{-i f' x}\, dx'$$

$f = -f'$

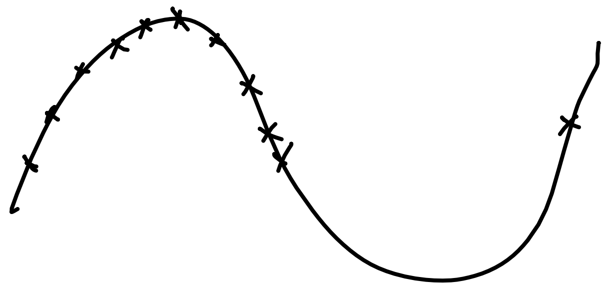
$$\underline{\underline{h(x) = \int_{f} \int_{f'} a(f)\, b(f')\, e^{-i f' x}}} \underbrace{\int_{x'} e^{i f x'} e^{i f' x'}\, dx'}_{\text{orthogonality integral}}$$

$$h(x) = \int_f a(f)\, b(-f)\, e^{-i(-f)x}\, df$$

$$h(x) = \int_f \underline{a(f)\, b(-f)}\, e^{ifx}\, df$$

$$= \int_f a(f)\, b^*(f)\, e^{ifx}\, df$$

Fourier series   Fourier Integral   DFT ← FFT



Aliasing

Shannon-Nyquist Information theorem

$$f_{\text{sampling}} \geq 2f$$

Power spectrum

$$\int_x \underline{f^2(x)}\, dx = \int_f \underline{a^2(f)}\, df \quad \text{Parseval's theorem}$$

$a^2(f) \longrightarrow \dfrac{a^2(f)}{f}$ density

↳ Power spectrum