Numerical Differentiation

As engineers, we are going to continously deal with systems and processes that change. Therefore we need to really understand two mathematical concepts :-

1. Differentiation
2. Integration.

We have already discussed about Integration concepts in detail in the last chapter. This chapter is all about Numerical Differentiation

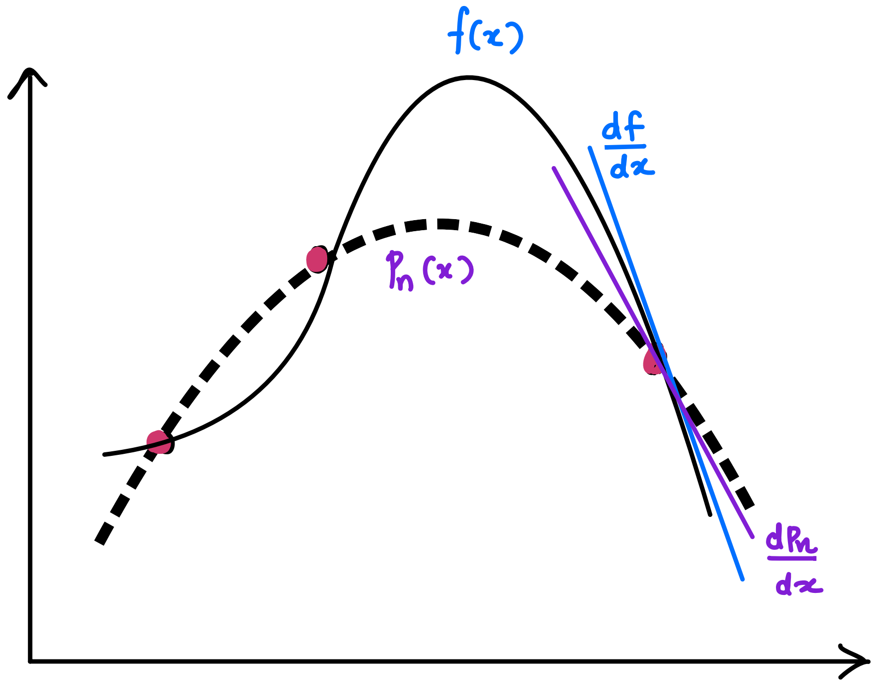
Numerical differentiation

What is differentiation?

Differentiate means to percieve the difference in or between two things. Mathematically to find differentiation we go for derivatives. Hence, derivatives serve as the fundamental vehicle for differentiation and they represent the rate of change of a dependent variable w.r.t the independent variable.

Given few data points an approximating polynomial can be fitted to pass through all the data points. $[P_n(x)]$

$$\frac{d}{dx} f(x) \approx \frac{d}{dx} P_n(x)$$



Numerical Differentiation

- Unequally Spaced data
- Equally Spaced data
- Taylor Series Approach
- Error Estimation

Unequally Spaced Data

1. Direct fit Polynomials
2. Lagrange Polynomials
3. Divided difference Polynomials

Direct fit Polynomials

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$P_n(x)$ – Exact n^{th} degree polynomial

If $N > n+1$ points ; n^{th} degree polynomial that best fits the data points – least square method

Lagrange Polynomials

* Can be used for both equally and unequally spaced data

Given $[a, f(a)]$ & $[b, f(b)]$

$$P_1(x) = \frac{(x-b)}{(a-b)} f(a) + \frac{(x-a)}{(b-a)} f(b)$$

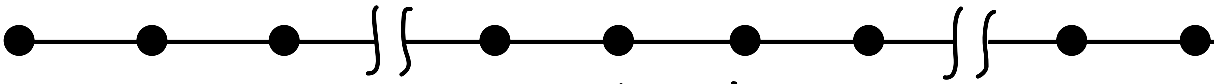
$$P_n(x) = \frac{(x-b)(x-c)\cdots(x-a_n)}{(a-b)(a-c)\cdots(a-a_n)} f(a) + \frac{(x-a)(x-c)\cdots(x-a_n)}{(b-a)(b-c)\cdots(b-a_n)} f(b)$$
$$+ \cdots + \frac{(x-a)(x-b)\cdots(x-a_{n-1})}{(a_n-a)(a_n-b)\cdots(a_n-a_{n-1})} f(a_n)$$

$P_2(x)$ be 2^{nd} degree Lagrange Polynomial

$$P_2(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)} f(a) + \frac{(x-a)(x-c)}{(b-a)(b-c)} f(b) + \frac{(x-a)(x-b)}{(c-a)(c-b)} f(c)$$

$$P_2'(x) = \frac{2x-(b+c)}{(a-b)(a-c)} f(a) + \frac{2x-(a+c)}{(b-a)(b-c)} f(b) + \frac{2x-(a+b)}{(c-a)(c-b)} f(c)$$

$$P_2''(x) = \frac{2f(a)}{(a-b)(a-c)} + \frac{2f(b)}{(b-a)(b-c)} + \frac{2f(c)}{(c-a)(c-b)}$$



Node numbers	0	1	2	$i-1$	i	$i+1$	$n-1$	n
x-values	x_0	x_1	x_2	x_{i-1}	x_i	x_{i+1}	x_{n-1}	x_n
y-values	y_0	y_1	y_2	y_{i-1}	y_i	y_{i+1}	y_{n-1}	y_n

Let us compute dy/dx or df/dx at node i.

Denote the difference operators.

$$\Delta x = x_{i+1} - x_i$$

$$\nabla x = x_i - x_{i-1}$$

$$\delta x = x_{i+1/2} - x_{i-1/2}$$

Forward Difference.

Approximate the function between (x_i, x_{i+1}) as :-

$$f(x) = \frac{x - x_i}{x_{i+1} - x_i} f(x_{i+1}) + \frac{x - x_{i+1}}{x_i - x_{i+1}} f(x_i) = \frac{f(x_{i+1})}{\Delta x}(x - x_i) - \frac{f(x_i)}{\Delta x}(x - x_{i+1})$$

$$\frac{df}{dx} = \frac{f(x_{i+1})}{\Delta x} - \frac{f(x_i)}{\Delta x} = \frac{\Delta f}{\Delta x}$$

Backward Difference

Approximate function between (x_{i-1}, x_i) as :-

$$f(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i) + \frac{x - x_i}{x_{i-1} - x_i} f(x_{i-1})$$

$$= \frac{f(x_i)}{\nabla x}(x - x_{i-1}) - \frac{f(x_{i-1})}{\nabla x}(x - x_i)$$

$$\frac{df}{dx} = \frac{f(x_i)}{\nabla x} - \frac{f(x_{i-1})}{\nabla x} = \frac{\nabla f}{\nabla x}$$

Central Difference

Approximate function between (x_{i-1}, x_i, x_{i+1}) as:

$$f(x) = \frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} f(x_{i-1}) + \frac{(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})} f(x_i) + \frac{(x-x_{i-1})(x-x_i)}{(x_{i+1}-x_i)(x_{i+1}-x_{i-1})} f(x_{i+1})$$

$$f(x) = \frac{(x-x_i)(x-x_{i+1})}{2h^2} f(x_{i-1}) + \frac{(x-x_{i-1})(x-x_{i+1})}{-h^2} f(x_i) + \frac{(x-x_{i-1})(x-x_i)}{2h^2} f(x_{i+1})$$

Now evaluate central difference approximations of $\frac{df}{dx}$ and $\frac{d^2f}{dx^2}$ at $x = x_i$:

$$\frac{df}{dx} = \frac{f(x_{i-1})}{2h^2}\left[2x - x_i - x_{i+1}\right] - \frac{f(x_i)}{h^2}\left[2x - x_{i-1} - x_{i+1}\right] + \frac{f(x_{i+1})}{2h^2}\left[2x - x_{i-1} - x_i\right]$$

at $x = x_i$

$$\left.\frac{df}{dx}\right|_{x=x_i} = \frac{f(x_{i-1})}{2h^2}\left[x_i - x_{i+1}\right] - \frac{f(x_i)}{h^2}\left[2x_i - x_{i-1} - x_{i+1}\right] + \frac{f(x_{i+1})}{2h^2}\left[x_i - x_{i-1}\right]$$

$$= \frac{f(x_{i-1})\left[-\cancel{h}\right]}{2h^{\cancel{2}}} - \cancelto{0}{\frac{f(x_i)}{h^2}\left[h - h\right]} + \frac{f(x_{i+1})\left[\cancel{h}\right]}{2h^{\cancel{2}}}$$

$$= -\frac{f(x_{i-1})}{2h} + \frac{f(x_{i+1})}{2h} = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

$$\boxed{\left.\frac{df}{dx}\right|_{x=x_i} = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}}$$

$$\left.\frac{d^2f}{dx^2}\right|_{x_i} = \frac{f(x_{i-1})}{\cancel{2}h^2}\left[\cancel{2}\right] - \frac{f(x_i)}{h^2}\left[2\right] + \frac{f(x_{i+1})}{\cancel{2}h^2}\left[\cancel{2}\right]$$

$$\boxed{\left.\frac{d^2f}{dx^2}\right|_{x_i} = \frac{f(x_{i-1}) - 2f(x_i) + f(x_{i+1})}{h^2}}$$

Similarly one can approximate the function b/w 3 points x_i, x_{i+1}, x_{i+2} and obtain the forward difference expressions of the first and second derivative as follows:-

$$f(x) = \frac{(x-x_{i+1})(x-x_{i+2})}{(x_i-x_{i+1})(x_i-x_{i+2})} f(x_i) + \frac{(x-x_i)(x-x_{i+2})}{(x_{i+1}-x_i)(x_{i+1}-x_{i+2})} f(x_{i+1}) + \frac{(x-x_i)(x-x_{i+1})}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})} f(x_{i+2})$$

$$= \frac{(x-x_{i+1})(x-x_{i+2})}{2h^2} f(x_i) - \frac{(x-x_i)(x-x_{i+2})}{h^2} f(x_{i+1}) + \frac{(x-x_i)(x-x_{i+1})}{2h^2} f(x_{i+2})$$

$$\frac{df}{dx} = \frac{f(x_i)}{2h^2}\left[2x - x_{i+1} - x_{i+2}\right] - \frac{f(x_{i+1})}{h^2}\left[2x - x_i - x_{i+2}\right] + \frac{f(x_{i+2})}{2h^2}\left[2x - x_i - x_{i+1}\right]$$

at $x = x_i$

$$\left.\frac{df}{dx}\right|_{x_i} = \frac{f(x_i)}{2h^2}\overset{-3h}{\left[2x_i - x_{i+1} - x_{i+2}\right]} - \frac{f(x_{i+1})}{h^2}\overset{-2h}{\left[2x_i - x_i - x_{i+2}\right]} + \frac{f(x_{i+2})}{2h^2}\overset{-h}{\left[2x_i - x_i - x_{i+1}\right]}$$

$$= \frac{f(x_i)}{2h^{\cancel{2}}}\left[-3\cancel{h}\right] + \frac{f(x_{i+1})}{h^{\cancel{2}}}(2\cancel{h}) + \frac{f(x_{i+2})}{2h^{\cancel{2}}}\left[-\cancel{h}\right]$$

$$\left.\frac{df}{dx}\right|_{x_i} = \frac{-3f(x_i) + 4f(x_{i+1}) - f(x_{i+2})}{2h}$$

$$\frac{d^2f}{dx^2} = \frac{2f(x_i)}{2h^2} - \frac{2f(x_{i+1})}{h^2} + \frac{2f(x_{i+2})}{2h^2}$$

$$\left.\frac{d^2f}{dx^2}\right|_{x_i} = \frac{f(x_i) - 2f(x_{i+1}) + f(x_{i+2})}{h^2}$$

Therefore :

$$\left.\frac{df}{dx}\right|_{x_i} = \frac{-3f(x_i) + 4f(x_{i+1}) - f(x_{i+2})}{2h}$$

$$\left.\frac{d^2f}{dx^2}\right|_{x_i} = \frac{f(x_i) - 2f(x_{i+1}) + f(x_{i+2})}{h^2}$$

Similarly one can approximate the function between three points $\{x_{i-2}, x_{i-1}, x_i\}$ and obtain the backward difference

$$f(x) = \frac{(x-x_i)(x-x_{i-1})}{(x_{i-2}-x_i)(x_{i-2}-x_{i-1})} f(x_{i-2}) + \frac{(x-x_i)(x-x_{i-2})}{(x_{i-1}-x_{i-2})(x_{i-1}-x_i)} f(x_{i-1}) + \frac{(x-x_{i-1})(x-x_{i-2})}{(x_i-x_{i-1})(x_i-x_{i-2})} f(x_i)$$

$$= \frac{(x-x_i)(x-x_{i-1})}{2h^2} f(x_{i-2}) + \frac{(x-x_i)(x-x_{i-2})}{-h^2} f(x_{i-1}) + \frac{(x-x_{i-1})(x-x_{i-2})}{2h^2} f(x_i)$$

$$\frac{df}{dx} = \frac{f(x_{i-2})}{2h^2}(2x - x_i - x_{i-1}) - \frac{f(x_{i-1})}{h^2}(2x - x_i - x_{i-2}) + \frac{f(x_i)}{2h^2}(2x - x_{i-1} - x_{i-2})$$

at $x = x_i$

$$\left.\frac{df}{dx}\right|_{x_i} = \frac{f(x_{i-2})}{2h^2}\overset{h}{(x_i - x_{i-1})} - \frac{f(x_{i-1})}{h^2}\overset{2h}{(x_i - x_{i-2})} + \frac{f(x_i)}{2h^2}\overset{h+2h}{(2x_i - x_{i-1} - x_{i-2})}$$

$$= \frac{f(x_{i-2})}{2h} - \frac{2f(x_{i-1})}{h} + \frac{3f(x_i)}{2h}$$

$$\left.\frac{df}{dx}\right|_{x_i} = \frac{f(x_{i-2}) - 4f(x_{i-1}) + 3f(x_i)}{2h}$$

$$\left.\frac{d^2f}{dx^2}\right|_{x_i} = \frac{f(x_{i-2})}{h^2} - \frac{2f(x_{i-1})}{h^2} + \frac{f(x_i)}{h^2}$$

$$\left.\frac{d^2f}{dx^2}\right|_{x_i} = \frac{f(x_{i-2}) - 2f(x_{i-1}) + f(x_i)}{h^2}$$

$$\boxed{\begin{aligned} \left.\frac{df}{dx}\right|_{x_i} &= \frac{f(x_{i-2}) - 4f(x_{i-1}) + 3f(x_i)}{2h} \\ \left.\frac{d^2f}{dx^2}\right|_{x_i} &= \frac{f(x_{i-2}) - 2f(x_{i-1}) + f(x_i)}{h^2} \end{aligned}}$$

Advantage of Lagrange Polynomial is that the numerical differentiation can be found even if the points are unevenly located. [ie. grids are uneven].

Divided Difference Polynomials

$$P_n(x) = f_i^0 + (x-x_0) f_i^1 + (x-x_0)(x-x_1) f_i^2 + (x-x_0)(x-x_1)(x-x_2) f_i^3 + \cdots$$

$$f'(x) = P_n'(x) = f_i^{(1)} + 2x - (x_0+x_1) f_i^2 +$$

$$\left[3x^2 - 2(x_0+x_1+x_2)x + (x_0x_1 + x_0x_2 + x_1x_2)\right] f_i^{(3)} + \cdots$$

Equally Spaced Data

Newton's Divided Difference formula

$$P_n(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\triangle^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \ldots + Eror$$

$$s = \frac{x - x_0}{h}$$

$$x = x_0 + sh$$

$$f'(x) \approx \frac{d}{dx} P_n(x) \approx P_n'(x) = \frac{d}{ds} P_n(s) \frac{ds}{dx}$$

$$\frac{ds}{dx} = \frac{1}{h}$$

$$f'(x) \cong \frac{1}{h} \frac{d}{ds} P_n(s)$$

$$P_n'(x) = \frac{1}{h}\left(\Delta f_0 + \frac{2s-1}{2!} \Delta^2 f_0 + \frac{3s^2 - 6s + 2}{6} \Delta^3 f_0 + \cdots \right)$$

Example

x	$f(x)$	∇f / Δf	$\nabla^2 f$ / $\Delta^2 f$	$\nabla^3 f$ / $\Delta^3 f$	$\nabla^4 f$ / $\Delta^4 f$
0	1.0000				
		−0.0025			
0.1	0.9975		−0.005		
		−0.0075		0.0001	
0.2	0.9900		−0.0049		−0.1
		−0.0124		−0.0999	
0.3	0.9776		−0.1048		
		−0.1172			
0.4	0.8604				

forward diff.

Backward diff

$$\left(\frac{dy}{dx}\right)_{x=0} = 0.25033$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = -9.67667$$

Taylor Series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f''''(x) + \cdots$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f''''(x) + \cdots$$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2!} f''(x) + \frac{(2h)^3}{3!} f'''(x) + \frac{(2h)^4}{4!} f''''(x) + \cdots$$

$$f(x-2h) = f(x) - 2hf'(x) + \frac{(2h)^2}{2!} f''(x) - \frac{(2h)^3}{3!} f'''(x) + \frac{(2h)^4}{4!} f''''(x) + \cdots$$

First Central Difference Approximations

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f''''(x) + \cdots \quad -(1)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f''''(x) + \cdots \quad -(2)$$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + \frac{(2h)^4}{4!}f''''(x) + \cdots \quad -(3)$$

$$f(x-2h) = f(x) - 2hf'(x) + \frac{(2h)^2}{2!}f''(x) - \frac{(2h)^3}{3!}f'''(x) + \frac{(2h)^4}{4!}f''''(x) + \cdots \quad -(4)$$

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + \frac{h^4}{12}f''''(x) + \cdots \quad -(5)$$

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{3}f'''(x) + \cdots \quad -(6)$$

$$f(x+2h) + f(x-2h) = 2f(x) + 4h^2 f''(x) + \frac{4h^4}{3}f^{4}(x) + \cdots \quad -(7)$$

$$f(x+2h) - f(x-2h) = 4hf'(x) + \frac{8h^3}{3}f'''(x) + \cdots \quad -(8)$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f'''(x)$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

$$f(x+2h) - f(x-2h) = 4h f'(x) + \frac{8h^3}{3} f'''(x) + \cdots \quad -8$$

$$2f(x+h) - 2f(x-h) = 4h f'(x) + \frac{2h^3}{3} f'''(x) + \cdots \quad -6$$

$$f(x+2h) - 2f(x+h) + 2f(x+h) - f(x-2h)$$

$$= 2\,\frac{\cancel{6}h^3}{\cancel{3}} f'''(x) + O(h^2)$$

$$f'''(x) = \frac{f(x+2h) - 2f(x+h) + 2f(x+h) - f(x-2h)}{2h^3} + O(h^2)$$

First Non-Central Difference Approximations

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f''''(x) + \cdots \quad — ①$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f''''(x) + \cdots \quad — ②$$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2!} f''(x) + \frac{(2h)^3}{3!} f'''(x) + \frac{(2h)^4}{4!} f''''(x) + \cdots \quad — ③$$

$$f(x-2h) = f(x) - 2hf'(x) + \frac{(2h)^2}{2!} f''(x) - \frac{(2h)^3}{3!} f'''(x) + \frac{(2h)^4}{4!} f''''(x) + \cdots \quad — ④$$

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + \frac{h^4}{12} f''''(x) + \cdots \quad — ⑤$$

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{3} f'''(x) + \cdots \quad — ⑥$$

$$f(x+2h) + f(x-2h) = 2f(x) + 4h^2 f''(x) + \frac{4h^4}{3} f^{4}(x) + \cdots \quad — ⑦$$

$$f(x+2h) - f(x-2h) = 4hf'(x) + \frac{8h^3}{3} f'''(x) + \cdots \quad — ⑧$$

$$f'(x) = \frac{f(x+h) - f(x)}{h} + o(h)$$

$$f'(x) = \frac{f(x) - f(x-h)}{h} + o(h)$$

$$f(x+2h) = f(x) + 2h f'(x) + \frac{(2h)^2}{2!} f''(x) + \frac{(2h)^3}{3!} f'''(x) + \frac{(2h)^4}{4!} f''''(x) + \cdots \quad — \ 3$$

$$2f(x+h) = 2f(x) + 2h f'(x) + \frac{2h^2}{2!} f''(x) + \frac{2h^3}{3!} f'''(x) + \frac{2h^4}{4!} f''''(x) + \cdots \quad — \ 1$$

$$f(x+2h) - 2f(x+h) = -f(x) + \frac{2h^2}{2} f''(x) + \frac{6h^3}{6} f'''(x) + \cdots$$

$$f''(x) = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} + o(h)$$

Second Non-Central Difference Approximations

$$4\,f(x+h) = 4f(x) + 4h f'(x) + \frac{4h^2}{2!} f''(x) + \frac{4h^3}{3!} f'''(x) + \frac{4h^4}{4!} f''''(x) + \cdots \quad \text{— } ①$$

$$f(x+2h) = f(x) + 2h f'(x) + \frac{(2h)^2}{2!} f''(x) + \frac{(2h)^3}{3!} f'''(x) + \frac{(2h)^4}{4!} f''''(x) + \cdots \quad \text{— } ③$$

$$f(x+2h) - 4f(x+h) = -3f(x) - 2h f'(x) - \frac{\cancel{4}^{\,2} h^3}{\cancel{3!}_{\,3}} f'''(x)$$

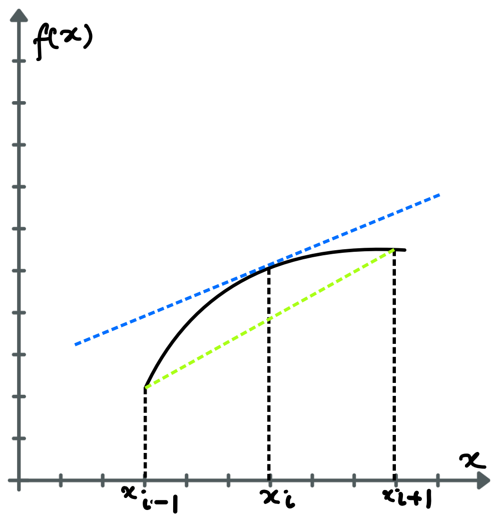
$$f'(x) = \frac{4f(x+h) - 3f(x) - f(x+2h)}{2h} + O(h^2)$$

First forward Difference

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{(x_{i+1} - x_i)} + O(x_{i+1} - x_i)$$

$$\boxed{f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x)}$$

This is called the first forward difference.



First backward difference

$$f(x_{i-1}) = f(x_i) - \Delta x\, f'(x_i) + \frac{\Delta x^2}{2!} f''(x_i) - \frac{\Delta x^3}{3!} f'''(x_i) + \cdots \quad (2)$$

$$\boxed{f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} + O(\Delta x)}$$

It is called as the first backward difference.

Central difference.

Eqn (1) − Eqn (2)

$$f(x_{i+1}) - f(x_{i-1}) = 2\Delta x\, f'(x_i) + \frac{2\Delta x^3}{3!} f'''(x_i) + \cdots$$

$$\boxed{f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O(\Delta x^2)}$$

Richardson's extrapolation

- Proposed in 1927 for the numerical whether prediction.
- This method can be applied whenever it is known that an approximation technique has an error which has a predictable form and that error depends on a parameter which is usually the interval of size 'h'.

- True error in multiple segment Trapezoidal rule

$$E_t \simeq \propto \frac{1}{n^2} \qquad n : \text{no. of segment}$$

- We take advantage of this property and use it to come up with a better estimate of Integrals.

$$E \simeq \frac{c}{n^2}$$

$$E_t = \text{True value} - \text{Aprox.value}$$

$$T.V = \text{Aprox. value} + E_t$$

$$T.V = (\text{Aprox Value})_n + (E_t)_n$$

$$\boxed{TV \approx (\text{Approx Value})_n + \frac{c}{n^2}} \qquad \text{—— } ①$$

So now if we double the no. of segments

$$\boxed{TV \approx (\text{Approx Value})_{2n} + \frac{c}{4n^2}} \qquad \text{—— } ②$$

Now let's say you are computing $f'(x)$ depends on step size h

$$f'(x) \cong f(h) + E(h)$$

$$E(h) \cong ch^y$$

$$f'(x) = f(h_1) + ch_1^y$$

$$f'(x) = f(h_2) + ch_2^y$$

$$f(h_1) - f(h_2) + c(h_1^y - h_2^y) = 0$$

$$\frac{f(h_1) - f(h_2)}{h_2^y - h_1^y} = c$$

$$f'(x) = f(h_1) + \frac{f(h_1) - f(h_2)}{h_2^y - h_1^y} h_1^y$$

$$= \frac{f(h_1)\{h_2^y - \cancel{h_1^y}\} + \cancel{f(h_1) h_1^y} - f(h_2) h_1^y}{h_2^y - h_1^y}$$

$$= \frac{f(h_1) h_2^y - f(h_2) h_1^y}{h_2^y - h_1^y}$$

$$= \frac{f(h_1) - \left(\frac{h_1}{h_2}\right)^y f(h_2)}{\left(\frac{h_2}{h_2}\right)^y - \left(\frac{h_1}{h_2}\right)^y}$$

$$= \frac{f(h_1) - \left(\frac{h_1}{h_2}\right)^y f(h_2)}{1 - (h_1/h_2)^y}$$

$$= \frac{\left(\frac{h_1}{h_2}\right)^y f(h_2) - f(h_1)}{(h_1/h_2)^y - 1}$$

$$f'(x) = \frac{\left(\frac{h_1}{h_2}\right)^y f(h_2) - f(h_1)}{(h_1/h_2)^y - 1}$$

Example

x	0	0.1	0.2	0.3	0.4
$f(x)$	0	0.0819	0.1341	0.1646	0.1797

Calculate $f'(0)$ Using Richardson's Extrapolation

$$f'(x) = \frac{4f(x+h) - 3f(x) - f(x+2h)}{2h} + O(h^2)$$

$$f'(0) = \frac{4f(0.2) - 3f(0) - f(0+0.4)}{2 \times 0.2}$$

$$f'(0) = \frac{4f(0.1) - 3f(0) - f(0+0.2)}{2 \times 0.1}$$

$$f'(0_{0.2}) = \frac{-3f(0)+4f(0.2)-f(0.4)}{2(0.2)} = 0.8918$$

$$f'(0_{0.1}) = \frac{-3f(0)+4f(0.1)-f(0.2)}{2(0.1)} = 0.9675$$

$$f'(x) = \frac{\left(\frac{h_1}{h_2}\right)^y f(h_2) - f(h_1)}{(h_1/h_2)^y - 1}$$

$y = 2$

$$f'(0) = \frac{\left(\frac{0.2}{0.1}\right)^2 0.9675 - 0.8918}{(0.2/0.1)^2 - 1}$$

$$= \underline{\underline{0.9927}}$$