Lecture 17 - Gaussian Elimination

Solving of linear systems of equations

→ model/design

mathematical model ← $A x = b$ → observation/data

x ?

→ inverse

A is square → uniquely determined system

rectangular ✗

Find x without inverting A

→ Cramer's rule → $n \leq 3$

Elimination → Gaussian elimination → Forward elimination → Factorization

→ Backward substitution → Estimation

↓

→ Factorization

→ $A = LU$ → Upper triangular

↓

Lower triangular with ones in the diagonal

→ $A = L D L^T$

↓

Symmetric linear system

$$\begin{array}{c} \\ \text{pivot} \\ \\ \end{array}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{array}{l} -\ 1 \\ -\ 2 \\ -\ 3 \end{array} \Rightarrow \begin{array}{l} 2' = 2 - 1\,\dfrac{a_{21}}{a_{11}} \Rightarrow \\ 3' = 3 - 1\,\dfrac{a_{31}}{a_{11}} \Rightarrow \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & \underline{a'_{32}} & a'_{33} \end{bmatrix}\begin{array}{l} -\ 1' \\ -\ 2' \\ -\ 3' \end{array} \Rightarrow \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \end{bmatrix}$$

$$3'' = 3' - 2'\,\frac{a'_{32}}{a'_{22}} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \end{bmatrix}$$

Forward elimination

Naive Gauss Elimination

$$x_3 = \frac{b''_3}{a''_{33}} \Rightarrow$$

$$a'_{22}\,x_2 + a'_{23}\,x_3 = b'_2$$

$$a'_{22}\,x_2 + a'_{23}\,\frac{b''_3}{a''_{33}} = b'_2$$

$$x_2 = \frac{b'_2 - a'_{23}\,\dfrac{b''_3}{a''_{33}}}{a'_{22}}$$

Back substitution

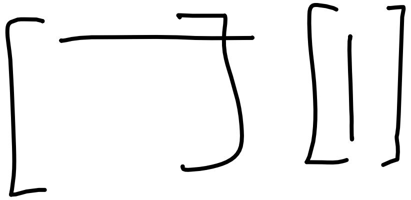
Back substitution

recursive

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}} \Rightarrow x_3 = \frac{b_3''}{a_{33}''}$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}}$$

$\longrightarrow$ linear sum of the upper triangle terms in the row



$$[\ \ ]x_1 + [\ \ ]x_2 + [\ \ ]x_3 + \dots$$

Partial pivoting

helps in avoiding scenarios where the pivot is very close to zero

a

a'

a''

$a^{(n)}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$D = (-1)^p \rightarrow$ pivoting steps used

Gauss elimination (inputs, options

↳ naive

↳ partial

↳ complete

naive → Forward

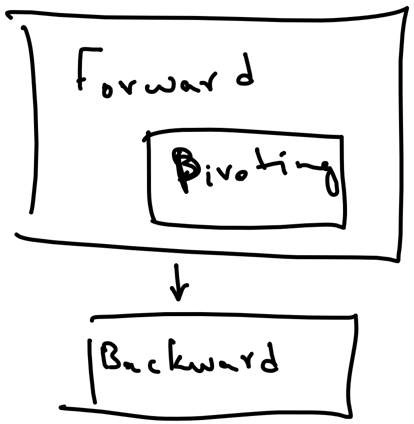
naive → Backward

partial → Forward

→ Backward

→ Pivoting → requires keeping log of row interchanges

Gauss Elimination



Gauss elimination $\longrightarrow$ U $\rightarrow$ upper row echelon form

upper triangular form

$$\begin{bmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \longrightarrow$$

$$D = a_{11}\, a'_{22}\, a''_{33} \implies$$

$$D = \prod \operatorname{diag}(U)$$

$$D = a_{11}\,(\overset{\text{minor}}{a'_{22}\, a''_{33} - a'_{23}\, 0})$$

$$- a_{12}\,(\overset{\text{minor}}{a''_{33}\, 0 - a'_{23}\, 0})$$

$$+ a_{13}\,(0 - a'_{32}\, 0)$$

$$= a_{11}\, a'_{22}\, a''_{33}$$

Singular system $\longrightarrow$ $D = 0$

$D \approx 0$

ill-conditioned system

commonly found system in nature

$$\left.\begin{aligned} x_1 + 2x_2 &= 10 \\ 1.1x_1 + 2x_2 &= 10.4 \end{aligned}\right]$$

ill-conditioned system $\longrightarrow$ perturbation analysis

↳ anomalous part

Gauss Elimination

Forward $\searrow$

Pivot

Backward $\swarrow$

ill-conditioned systems $\rightarrow$ can have determinants close to zero but not zero

$$x_1 + 2x_2 = 10$$
$$1.1x_1 + 2x_2 = 10.4$$

$$\begin{bmatrix} 1 & 2 \\ 1.1 & 2 \end{bmatrix} \Rightarrow D = 2 - 2.2 = -0.2$$

$\Downarrow$

10

$$D = \begin{bmatrix} 10 & 20 \\ 11 & 20 \end{bmatrix} = 200 - 220 = -20$$

Gauss Elimination with normalization

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{21} & a'_{22} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

$$\Downarrow$$

normalized

$$\begin{bmatrix} 1 & \bar{a}_{12} & \bar{a}_{13} \\ 0 & 1 & \bar{a}'_{22} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

Gauss-Jordan

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$