Lecture 25 - Harmonic approximation of functions

Fourier series

Polynomials → Orthogonal polynomials → Harmonic functions → Diffusion equation on a circular plate

↳ Laplace equation on the sphere

↓

Solve well-known ODE & PDEs

Lecture 12

Synthesis $f(x) = \sum_{i=0}^{\infty} a_i P_i(x)$

P_i → Orthogonal Polynomial

a_i ↳ coefficients

Analysis $a_i = \dfrac{\int_c^d f(x) P_i(x)\, dx}{\int_c^d P_i(x) P_i(x)\, dx}$

$\int_c^d P_i(x) P_i(x)\, dx$ → Orthogonality integral

$$\int_c^d P_i(x) P_j(x)\, dx = \text{value}\ \delta_{ij}$$

δ_{ij} ↑ Kronecker delta

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Fourier series

$m=0$	1	0			
$m=1$	$\cos\theta$	$\sin\theta$	$\Rightarrow$	$\cos m\theta$	$\sin m\theta$
$m=2$	$\cos^2\theta - \sin^2\theta$	$2\cos\theta\sin\theta$			
$m=3$					

$$e^{im\theta} = \cos m\theta + i\sin m\theta$$

↳ natural solution of the Harmonic equations

Simple harmonic oscillator

$$\frac{d^2y}{dx^2} + y = 0$$

$\theta \in [0, 2\pi)$ Domain in 2π-periodic

$f(x) = f(x+N)$ → Periodic function

N ↓ Period

$$a_m = \int_0^{2\pi} f(x)\, e^{-imx}\, dx$$

$f(x)$ → real-valued function; ↓ $x \in [0, 2\pi]$

e^{-imx} → complex

$$\boxed{f(x)} = \sum_{m=-\infty}^{\infty} a_m e^{imx}$$

a_m → occur in conjugate pairs for real-valued $f(x)$

↳ $a_m^* = a_{-m}$

Harmonic functions

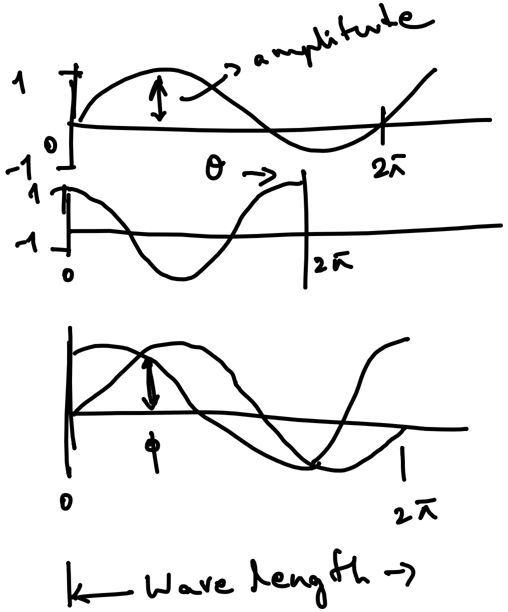
Periodic function

Conjugate pairs

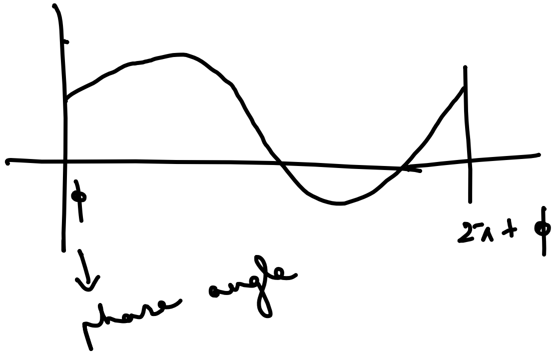
$\cos\theta$

$\sin\theta$

$e^{im\theta}$



One wave of the sinusoid



$m \rightarrow$ How frequently is the wave reappearing?

$\boxed{\text{Frequency}}$

$$a_1 = \int_0^{2\pi} f(x)\, e^{-ix}\, dx$$

coefficient of the polynomial

$$f(x) = \begin{array}{|c|} a_0 \\ +\ a_1 \\ +\ a_2 \end{array} \begin{array}{l} \underline{e^{i0x}} \\ \underline{e^{ix}} \\ \underline{e^{i2x}} \end{array}$$

periodic orthogonal functions — mean

1 wave between $[0, 2\pi)$

2 waves

$$\Rightarrow \quad a_0 = \int_0^{2\pi} f(x)\, e^{-i\overset{0}{\underline{m}}x}\, dx$$

$$a_0 = \int_0^{2\pi} f(x) \cdot 1\, dx$$

$$a_0 = \int_0^{2\pi} f(x)\, dx$$

Orthogonal harmonic function only apply to certain types of functions

Dirichlet conditions
- $f(x)$ must be smooth [finite maxima & minima]
- $f(x)$ must be bounded $\quad |f(x)| \leq M$
- $\int_a^b |f(x)|\, dx =$ finite — integrable function

$\int_a^b f^2(x) =$ finite — Hilbert space — Square integrable function

- $f(x)$ must be piecewise continuous

$$a_0 = \frac{\int_0^{2\pi} f(x)\, e^{-i0x}\, dx}{\int_0^{2\pi} e^{-i0x}\, e^{i0x}\, dx} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx = \bar{f}(x)$$

$$f(x) = \bar{f}(x) + \sum_{|m|=1}^{\infty} a_m e^{imx}$$

We are keeping the mean value as the base and adding the anomalies on top of them.

$$f(x) - \bar{f}(x) = \sum_{|m|=1}^{\infty} a_m e^{imx}$$

Zero-mean processes $\leftarrow$ $\boxed{\delta f(x)} = \sum_{|m|=1}^{\infty} a_m e^{imx}$

Anomaly of the function $f(x)$

$$\frac{1}{2\pi}\int_0^{2\pi} \left(f(x) - \bar{f}(x)\right) dx =$$

$\bar{f}(x)$ → constant value

(1)

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx - \frac{\bar{f}(x)}{2\pi}\int_0^{2\pi} dx = 0,$$

$$\frac{1}{2\pi}\int_0^{2\pi} \delta f(x) = 0$$

$$(1)\ \frac{1}{2\pi}\int_0^{2\pi}\left(f(x) - \int_0^{2\pi} f(x')\, dx'\right) dx$$

$$\underline{a_m} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-imx}\, dx$$

↓

denotes the number of waves in the sinusoid

↙

frequencies

$\underline{\text{Global integral}}$

↳ Harmonic functions

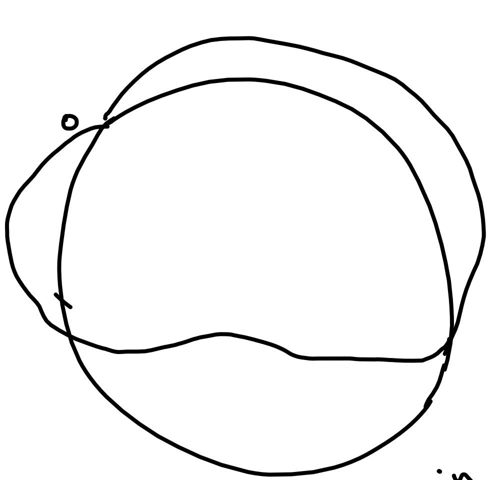
$$f(x) = \sum_{i=-\infty}^{\infty} a_{\underline{m}}\, e^{imx}\, dx$$

$m = 1$

$m = 1.\underline{2345678}$

$\phi_1 \quad \phi_2$

$f(x)$ must be provided over the entire domain



Circular domain

Confined domain

Curvilinear

Spectrum

$f(x)$ → Obeys Dirichlets → approximate it with harmonics functions → $\boxed{\text{Coefficients of the harmonic functions gives us a new way of looking at the data}}$

$f(x)$ → split into frequencies → a_m indicate the amplitude of each frequency

$a_m \to$ complex $\Rightarrow$ $a_m = \underbrace{c_m}_{\text{cosine part}} + i \underbrace{d_m}_{\text{sine part}}$

cosine $\to$ even function $f(-x) = f(x)$

sine $\to$ odd function $f(-x) = -f(x)$

$f(x) \to$ Fourier series approximation $\to$ a_m $\nearrow$ complex, $\searrow$ real

$f(x)$ $\to$ even function

$\hookrightarrow$ cosine series

cosine transform

frequency $\to$ m

Amplitude $\to$ a_m

Spectrum $\to$ a_m, $m \in (-\infty, \infty)$