Lecture 31 – Gauss Quadrature

Equally spaced data points

Numerical integration

Tabulated value $f[x_k]$ => Polynomial => Integrate polynomial => Error Tolerance not fixable

To determine the truncation error

Trapezoidal
Simpson's

Newton - Cotes

function $f(x)$ => Polynomial => Integrate polynomial → Fix error Tolerance

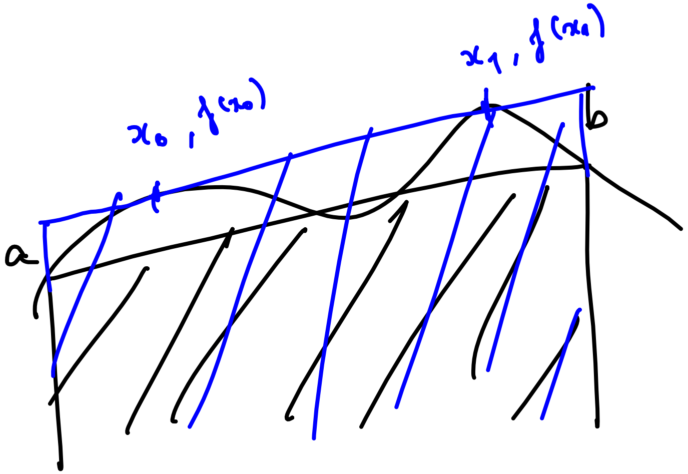
↓ approximation error

↓ number of segments

↓ equal segments → Romberg integration

Richardson's extrapolation

Gauss Quadrature $\rightarrow$ Gauss-Legendre Quadrature



Trapezoidal $\rightarrow$ c_0, c_1

Gauss $\rightarrow$ c_0, c_1, x_0, x_1

Method of undetermined coefficients

$$I = c_0 f(a) + c_1 f(b)$$

$$c_0 f(a) + c_1 f(b) = \int_a^b 1 \, dx$$

$$c_0 f(a) + c_1 f(b) = \int_a^b x \, dx$$

$$I = c_0 f(a) + c_1 f(b)$$

$$I = c_0' f(x_0) + c_1' f(x_1)$$

$$\left.\begin{aligned} c_0 + c_1 &= b - a \\ c_0 a + c_1 \times b &= \frac{b^2 - a^2}{2} \end{aligned}\right| \quad c_0 = c_1 = \frac{b-a}{2}$$

Gauss Quadrature $\rightarrow$ $f(x)$ known

c_0
c_1
x_0
x_1

$$c_0 f(x_0) + c_1 f(x_1) = \int_a^b 1 \, dx$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_a^b x \, dx$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_a^b x^2 \, dx$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_a^b x^3 \, dx$$

$$c_0 + c_1 = b - a$$

$$c_0 x_0 + c_1 x_1 = \frac{b^2 - a^2}{2}$$

$$c_0 x_0^2 + c_2 x_1^2 = \frac{b^3 - a^3}{3}$$

$$c_0 x_0^3 + c_1 x_1^3 = \frac{b^4 - a^4}{4}$$

Non-linear system of simultaneous equations

c_0, c_1, x_0, x_1

Two-point Gauss Quadrature

$b - a \rightarrow 0 \quad 1$

$\boxed{-1 \quad 1}$

Romberg

- Tolerances

- Iterative

- Pros
  + Specify tolerance
  + Problem-specific

- Cons
  + Problem-specific
  + Approximation error is what is used for tolerance

Gauss

- Accuracy & Coefficients } Pre-determined

- One stroke

- Pros
  + For the same order better than Newton-Cotes
  + General
  + repeated evaluations

- Cons
  + General
  + Cannot specify tolerance

Spectral analysis Gauss-Legendre

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \begin{cases} 0, & n \neq m \\ v, & n = m \end{cases}$$

$$\sum_{i=1}^{N} P_n(x_i)\, P_m(x_i)\, \Delta x = \begin{cases} v_a, & n \neq m \\ v_a', & n = m \end{cases}$$

Quadrature check-points

- Bounds
- Weights
- Locations of the function evaluation

Gauss Quadrature

$$E_t = \frac{2^{2n+3}\left[(n+1)!\right]^4}{(2n+3)\left[(2n+2)!\right]^3} \boxed{f^{(2n+2)}(\xi)}$$

$n \rightarrow$ number of points used in the quadrature $- 1$

$\int^2 f(x)\,dx =$ finite

$$E_t = \frac{2^{2+3}\left[2!\right]^4}{(2+3)\left[4!\right]^3} f^4(\xi)$$

Gauss Quadrature weights from orthogonal polynomials

Chebyshev

Legendre

$$\int_{-1}^{1} W(x)\, P_n(x)\, P_m(x)\, dx = \begin{cases} 0, & n \neq m \\ \text{value}, & n = m \end{cases}$$

$$\sum_j w_j\, P_n(x_j)\, P_m(x_j) = \begin{cases} 0, & n \neq m \\ \text{value}, & n = m \end{cases}$$

$\downarrow$

$x_j \rightarrow$ zeros of the polynomial

Improper Integrals

- Multiple times an integral scheme