Lecture 35 - Runge-Kutta methods

ODE ← $f(x, y)$ → function we are interested in

↓

independent

ODE

↓

first derivative / first order ODEs

↓

Solve → integrate them

ODE

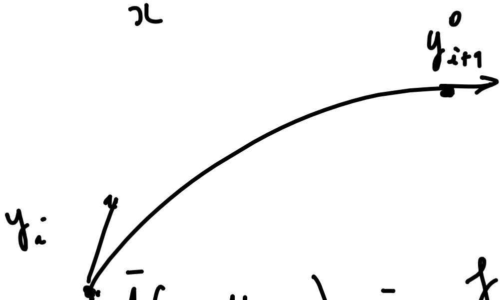
$$y_{i+1} = y_i + f(x_i, y_i)\, h$$

First-order Taylor series approximation.

y_i ↓ initial value

↑ Euler method

h → Step size in x



Heun's method

predictor

$$\bar{f}(x, y_{i+1}) = \frac{f(x_{i+1}, y^0_{i+1}) + f(x_i, y_i)}{2}$$

↓

corrector

Mid-point polygon method

$$y_{i+1} = y_i + \phi(x_i, y_i, h)\, h$$

$$\downarrow$$

unknowns

$$a_j \,,\; p_{j-1} \,,\; q_{j-1,m}$$

$j \in [1,n]$

$m \in [1,j]$

n - order of the method

x, y, h

weights

$$\phi(x_i, y_i, h) = a_1 k_1 + a_2 k_2 + a_3 k_3 + \dots + a_n k_n$$

evaluation of slope within h

$$k_1 = f(x_i, y_i) \quad \rightarrow \text{ODE / slope}$$

$$k_2 = f(x_i + p_1 h \,,\; y_i + q_{11} k_1 h)$$

$$k_3 = f(x_i + p_2 h \,,\; y_i + q_{21} k_1 h + q_{22} k_2 h)$$

$$\vdots$$

$$k_n = f(x_i + p_{n-1} h \,,\; y_i + q_{n-1,1} k_1 h + \dots + q_{n-1,n-1} k_{n-1} h)$$

k_j are determined via a recursive relationship

Runge-Kutta Method of order 2

$$y_{i+1} = y_i + (\overbrace{a_1 k_1 + a_2 k_2}^{\phi})\, \underbrace{h}_{\text{step-size}}$$

$$\underline{k_1} = \boxed{f(x_i, y_i)}$$

$$\underline{k_2} = f(x_i + p_1 h,\ y_i + q_{11} k_1 h)$$ Use Taylor series

$$= \boxed{f(x_i, y_i)} + p_1 h \frac{\partial f}{\partial x} + q_{11} k_1 h \frac{\partial f}{\partial y}$$

$$y_{i+1} = y_i + \Big[a_1 f(x_i, y_i) + a_2 f(x_i, y_i) + a_2 p_1 h \frac{\partial f}{\partial x} + a_2 q_{11} f(x_i, y_i) h \frac{\partial f}{\partial y} \Big] h$$

$$y_{i+1} = y_i + \Big[(a_1 + a_2) f(x_i, y_i) + a_2 p_1 h \frac{\partial f}{\partial x} + a_2 q_{11} f(x_i, y_i) h \frac{\partial f}{\partial y} \Big] h$$

$$y_{i+1} = y_i + f(x_i, y_i) h + \frac{f'(x_i, y_i)}{2!} h^2$$

$$f'(x, y) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x, y)$$

$$y_{i+1} = y_i + f(x_i, y_i) h + \Big[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x, y) \Big] \frac{h^2}{2}$$

Also

$$g(x + r, y + s) = g(x, y) + r \frac{\partial g}{\partial x} + s \frac{\partial g}{\partial y}$$

$$y_{i+1} = y_i + \left[(a_1+a_2)\, f(x_i, y_i) \right] h$$

$$+ \left[a_2 p_1 \frac{\partial f}{\partial x} + a_2 q_{11} f(x_i, y_i) \frac{\partial f}{\partial y} \right] h^2$$

$\hookrightarrow$ numerically compute

local truncation error

global truncation error

round off error

$\to$ $$y_{i+1} = y_i + f(x_i, y_i)\, h + f'(x_i, y_i) \frac{h^2}{2}$$

Single-step algorithms

$a_1 + a_2 = 1 \to$ condition equation

$a_2 \underline{p_1} = 1/2$

$a_2 \underline{q_{11}} = 1/2$

RKn tables

p_1	q_{11}
$\vdots$	q_{21}
p_{n-1}	

Scenarios

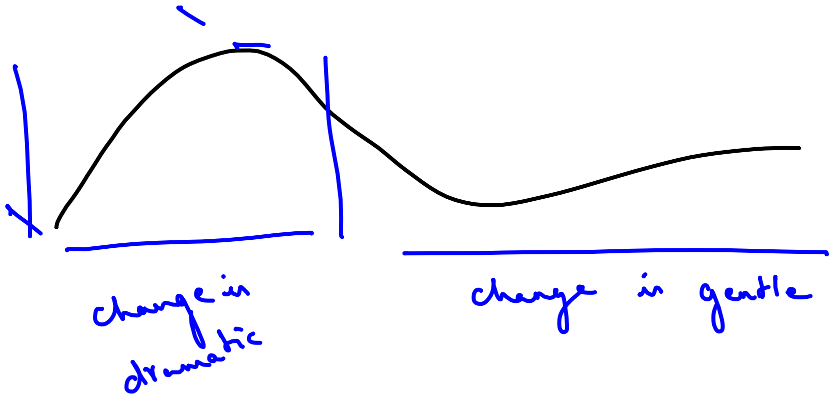
$a_1 = 1,\ a_2 = 0 \Rightarrow \underbrace{p_1 = 0,\ q_{11} = 0}$

$\therefore$ condition cannot be satisfied

$a_1 = 0,\ a_2 = 1 \Rightarrow p_1 = 1/2,\ q_{11} = 1/2 \to$ Heun's method

$a_1 = a_2 = 1/2 \Rightarrow p_1 = 1,\ q_{11} = 1 \to$ Classical RK2 method

$a_2 = 2/3,\ a_1 = 1/3 \Rightarrow p_1 = 3/4,\ q_{11} = 3/4 \to$ Ralston's RK2 method



use step-size according to the change in the function

Adaptive step-size methods