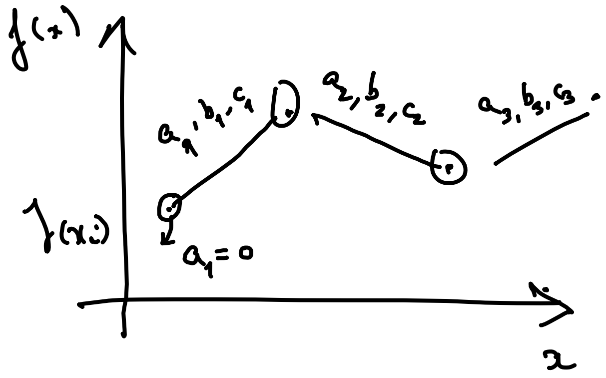
Lecture 15 - Spline interpolation

Fundamental basis → fit piecewise polynomials
→ ensure continuity by maintaining continuous derivatives
} objective

1 - linear → linear interpolation
2 - quadratic
3 - cubic



4 data points
3 intervals
4 knots ↗ internal ↘ boundary

Quadratic splines

1) Function values at adjacent polynomials must be equal at the internal knots

$$a_{i-1} x_{i-1}^2 + b_{i-1} x_{i-1} + c_{i-1} = f(x_{i-1})$$
$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$
} $2n-2$ conditions

$n+1$

n

$n-1$ internal knots

2) The first and the last functions must pass through the end points

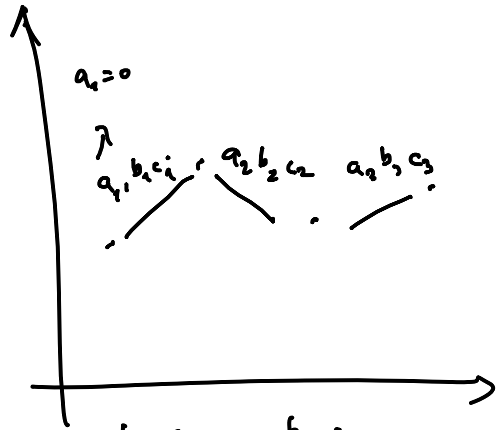
$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$
$$a_n x_n^2 + b_n x_n + c_n = f(x_n)$$
} 2 conditions

$3n$ unknowns

3. The first derivatives at the interior knots must be equal

$$f'(x) = 2ax + b$$

$$\Rightarrow \quad 2a_{i-1}x_{i-1} + b_{i-1} = 2a_i x_{i-1} + b_i \quad \rightarrow \quad n-2$$



Constraint equations

$3n$ unknowns

$$\left\{\begin{array}{ll} 1) & 2n - \cancel{2} \\ 2) & \cancel{2} \\ 3) & n-1 \end{array}\right\} \text{ equations}$$

$3n-1 \rightarrow$ equation

$$\boxed{a_1 = 0}$$

$3n-1$ equations, $3n-1$ unknowns
uniquely determined
Square matrix

$$a_1 x_1^2 + b_1 x_1 + c_1 = f(x_1)$$
$$a_2 x_1^2 + b_2 x_1 + c_2 = f(x_1)$$
$$a_2 x_2^2 + b_2 x_2 + c_2 = f(x_2)$$
$$a_3 x_2^2 + b_3 x_2 + c_3 = f(x_2)$$
$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$
$$a_3 x_3^2 + b_3 x_3 + c_3 = f(x_3)$$
$$2a_1 x_1 + b_1 - 2a_2 x_1 - b_2 = 0$$
$$2a_2 x_2 + b_2 - 2a_3 x_2 - b_3 = 0$$

$$\begin{array}{cccccccc} b_1 & c_1 & a_2 & b_2 & c_2 & a_3 & b_3 & c_3 \\ x_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1^2 & x_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & x_2^2 & x_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2^2 & x_2 & 1 \\ x_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3^2 & x_3 & 1 \\ 1 & 0 & -2x_1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2x_2 & 1 & 0 & -2x_3 & -1 & 0 \end{array}$$

Cubic Splines

interior 1) $a_{i-1} x_{i-1}^3 + b_{i-1} x_{i-1}^2 + c_{i-1} x_{i-1} + d_{i-1} = f(x_{i-1})$ $\quad \cancel{2}(n-1)$ $\quad$ | $4n$

$a_i x_{i-1}^3 + b_i \ldots\ldots = f(x_{i-1})$ $\quad 2n-2$

2) $a_1 x_0^3 + b_1 x_0^2 + c_1 x_0 + d_1 = f(x_0)$ boundary knots $\quad 2$

$a_n x_n^3 + b_n x_n^2 + c_n x_n + d_n = f(x_n)$

3) first derivative $\quad$ interior knots $\quad n-1$

4) second derivative $\quad n-1$

$4n-2$ $\quad$ 2 degrees of freedom

$a_1 = 0$

$a_n = 0$

Box 18.3 → 516 Sixth edition Chapra & Canale

$$f''(x) = f''_i(x_{i-1})\left(\frac{x - x_i}{x_{i-1} - x_i}\right) + f''_i(x_i)\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)$$

1) Double integrate

$$\iint f''(x)\, dx^2 = \int (f'(x) + c)\, dx = f(x) + \underline{c}x + \underline{d}$$

2) Find constants

$x_{i-1} \Rightarrow f(x_{i-1})$

$x_i \quad f(x_i)$

$$f(x) = f''_i(x_{i-1}) \frac{(x_i - x)^3}{6(x_i - x_{i-1})} + f''_i(x_i) \frac{(x - x_{i-1})^3}{6(x_i - x_{i-1})}$$

$$+ \left[\frac{f(x_{i-1})}{x_i - x_{i-1}} - \frac{f''(x_{i-1})(x_i - x_{i-1})}{6}\right](x_i - x)$$

$$+ \left[\frac{f(x_i)}{x_i - x_{i-1}} - \frac{f''(x_i)(x_i - x_{i-1})}{6}\right](x - x_{i-1})$$

3) Differentiate

$$f'_i(x_i) = f'_{i+1}(x_i)$$

$$(x_i - x_{i-1}) f''(x_{i-1}) + 2(x_{i+1} - x_{i-1}) f''(x_i) + (x_{i+1} - x_i) f''(x_{i+1}) = \text{term}\left[f(x)_{i-1, i, i+1}\right]$$