Lecture 29 - Numerical calculus

Input Outputs

$$P - E - R = \frac{dS}{dt}$$

Runoff → storage change

Water budget equation

Precipitation Evaporation residual

Flux quantity

$\frac{dS}{dt}$ → Satellite gravimetry

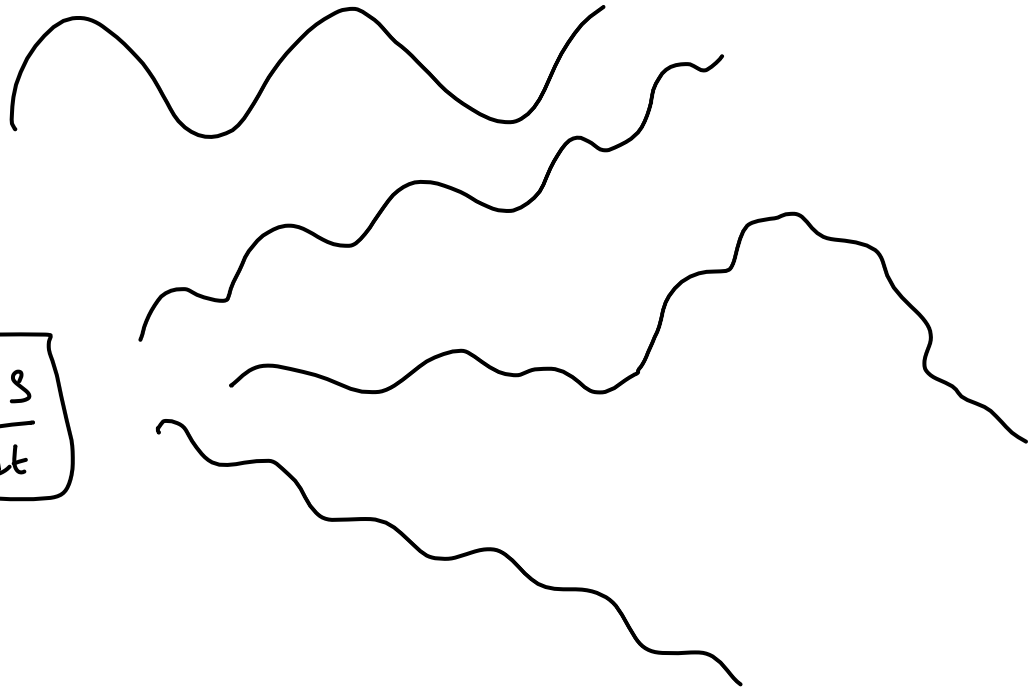
ΔS → Storage anomaly → $\boxed{\frac{dS}{dt}}$

$$\underline{S(t)} - \boxed{\bar{S}} = \Delta S$$

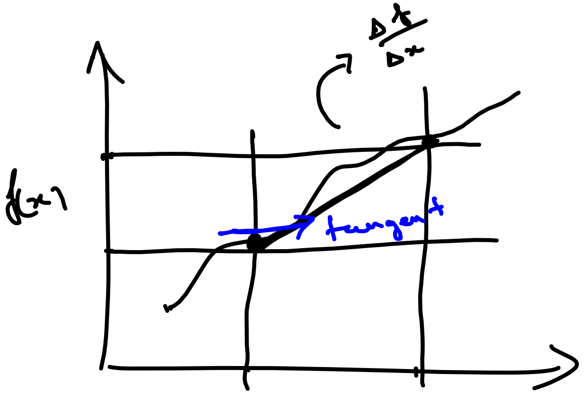
mean storage

mean

Hydrol
Cryo + Solid Earth

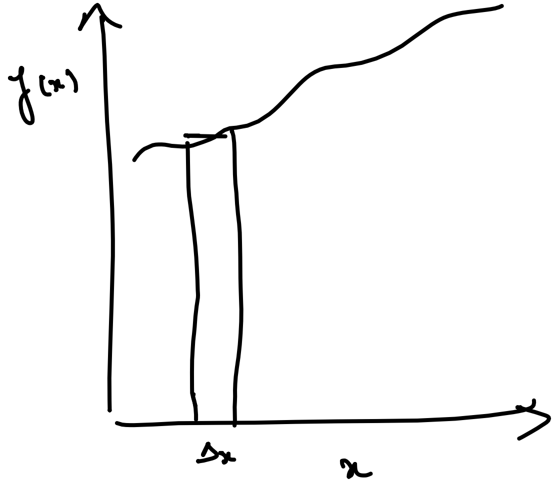


$$f(x) \to \frac{df}{dx} = \lim_{\Delta x \to 0} \boxed{\frac{\Delta f}{\Delta x}}$$



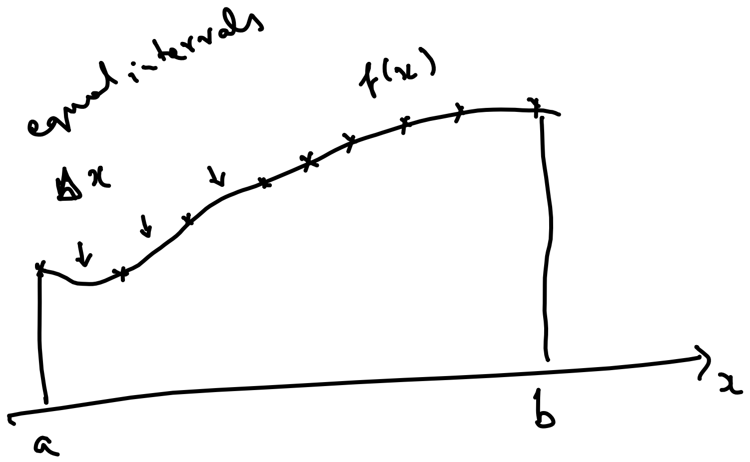
$$\frac{d}{dx}\left(\frac{df}{dx}\right) \to \text{curvature}$$

$$\int_a^b f(x)\, dx \to \lim_{\substack{N \to \infty \\ \Delta x \to 0}} \sum_{i=1}^{N} \underbrace{\boxed{f(x_i)\ \Delta x}}_{\text{Area}}$$



Numerical integration

Newton - Cotes integration methods



closed form integral → $f(x)$ → a
→ b given

open form integral → $f(x)$ → a
→ b not given

Newton - Cotes → fit a polynomial
between a and b
and integrate

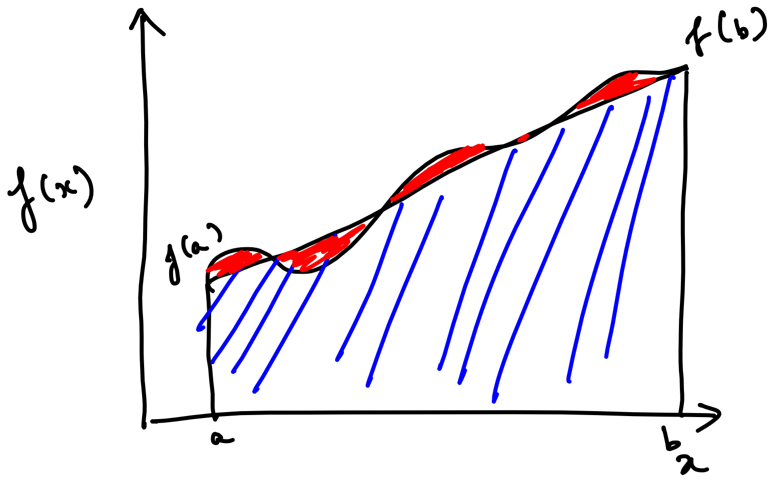
Types of functions

1) Polynomials, trigonometric
functions, exponentials
↳ Analytical integration

2) Complex non-linear
functions, improbable
integrals

3) Tabulated values

Polynomial of degree 1 $\to$ line



Trapezoid

$\hookrightarrow$ find the area of the Trapezoid $(a, b, f(a), f(b))$

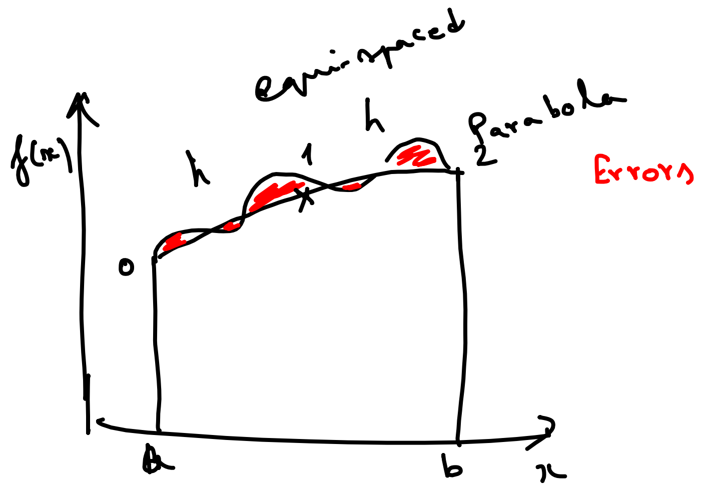
$$= (b-a)\,\frac{f(a)+f(b)}{2}$$

Errors at the level of third order

$$E_t = -\frac{1}{12} f''(\xi)\,(b-a)^3$$

Trapezoid Rule

Error



- Lagrange polynomial fitting points
  x_0, x_1, x_2
- Integrate the polynomial

This leads to a formulation called as the Simpson's 1/3 rule

Simpson's 1/3 formula

$$I = (b-a)\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}\right]$$

$x_0 = a$

$x_2 = b$

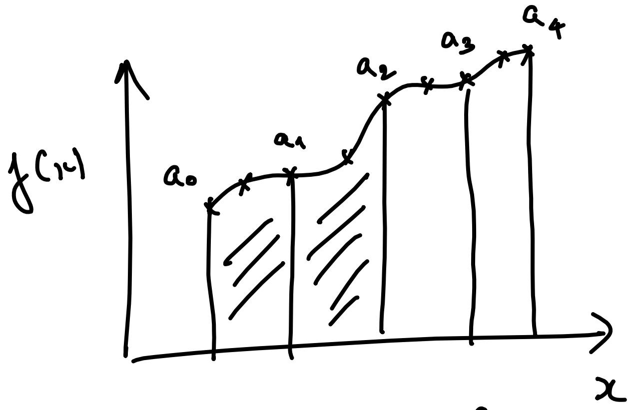
$(b-a) = 2h$

$$I = \frac{\cancel{2}h}{\cancel{6}_3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$E_s = -\frac{(b-a)^5}{2880} f''''(\xi)$$

Practical scenario – Multiple application of integral formulas



$$I = \int_{a_0}^{a_4} f(x)\,dx = \int_{a_0}^{a_1} f(x)\,dx + \int_{a_1}^{a_2} f(x)\,dx + \int_{a_2}^{a_3} f(x)\,dx + \int_{a_3}^{a_4} f(x)\,dx$$