Lecture 18 – Gaussian Elimination

$$\underset{n \times n \text{ square}}{A}\ \ ?x = b$$

Forward Elimination → $LU = A$ $L =$ (ones) $U =$ (real numbers)

Backward substitution

3×3

$$\underbrace{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A}$$

$$LU = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{13}+l_{32} & l_{33}+l_{32}u_{23} \\ & & +l_{31}u_{13} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$l_{11} = a_{11}, \quad l_{11}u_{12} = a_{12}, \quad l_{11}u_{13} = a_{13}$$

$$u_{12} = \frac{a_{12}}{l_{11}} \qquad \Rightarrow u_{13} = a_{13}/l_{11}$$

$$l_{21} = a_{21}, \quad l_{21}u_{12} + l_{22} = a_{22}, \quad l_{21}u_{13} + l_{22}u_{23} = a_{23}$$

$$l_{22} = a_{22} - l_{21}u_{12} \qquad u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}}$$

$$l_{31} = a_{31}, \quad l_{31}u_{13} + l_{32} = a_{32}, \quad l_{33} + l_{32}u_{23} + l_{31}u_{13} = a_{33}$$

$$l_{32} = a_{32} - l_{31}u_{13} \qquad l_{33} = a_{33} - l_{32}u_{23} + l_{31}u_{13}$$

$$l_{i1} = a_{i1}, \quad l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj}, \quad u_{jk} = \left[a_{jk} - \sum_{i=1}^{j-1} l_{ji}u_{ik}\right]/l_{jj}, \quad l_{nn} = a_{nn} - \sum_{i=1}^{n-1} l_{ni}u_{in}$$

$LU = A$

$Ax = b$

$LUx = b$

$Ux = c$

$Lc = b$

Matrix inverse

$L\overset{\text{one } U}{U}x = Ax = b$ ↓ several b

$Lc = b$

x

↓ Back substitution.

$U\underline{a}'_1 = y$

$Ly = \underline{a}_1$

$LUA^{-1} = AA^{-1} = [\, I \,]$

$[\;]$

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\underline{a}'_1 \; \underline{a}'_2 \quad \underline{a}_1 \; \underline{a}_2 \; \underline{a}_3 \; \dots$

$LUA^{-1} = \begin{bmatrix} 1 & & \cdot \\ 0 & 1 & \\ 0 & 0 & 1 \\ 0 & 0 & \end{bmatrix}$

$[\;][\;][\;]$

$Q_{\hat{x}\hat{x}} = (A^T Q_{bb}^{-1} A)^{-1}$

Stochastic information

Random

Dispersion → Variance - Covariance matrices

$\underbrace{\boxed{(A^T A)^{-1}}}_{\text{Normal system}} \boxed{A^T b} = \underline{\hat{x}}$

$(A^T A)^{-1} \qquad Q_{\hat{x}\hat{x}}$

Error propagation.

$D\{b\} = Q_{bb}$ Weight $= Q_{bb}^{-1}$

↳ weight my observations b

$\underbrace{\boxed{(A^T Q_{bb}^{-1} A)^{-1} \; A^T Q_{bb}^{-1}}}_{\text{linear operator}} b = \hat{x}$

$Q_{\hat{x}\hat{x}} = \cancel{(A^T Q_{bb}^{-1} A)^{-1} A^T Q_{bb}^{-1} Q_{bb}}$

$\cancel{Q_{bb}^{-1} A (A^T Q_{bb}^{-1} A)^{-1}}$

Complexity of the algorithm

FLOPS – Floating point operations

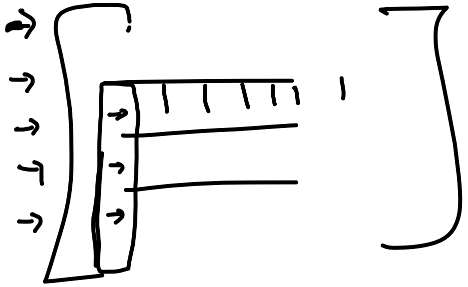
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Gaussian Elimination

For loop along rows

For loop find the factor

For loop element-wise



$i = 1$

	Multiplication	subtractions	b
$\frac{a_{21}}{a_{11}}$	2	2	1,1
	2+2	2+1	

Outer	Middle	Addition/Subtraction	Multiplication/Division
1	2, n	$(n-1)(n)$	$(n-1)(n+1)$
2		$\vdots$	
$\vdots$			
k		$(n-1)(n+1-k)$	$(n-1)(n+2-k)$
$\vdots$			

$$\sum_{k=1}^{n-1} (n-1)(n+1-k)$$

$$+$$

$$\sum_{k=1}^{n-1} (n-1)(n+2-k)$$

fraction of n^3 $\boxed{2/3}$