Lecture 12 - Curve fitting with orthogonal polynomials

Curve fitting $\rightarrow$ regression

$\rightarrow$ interpolation

regression $\rightarrow$ $y_i \rightarrow \underline{f}(x_i)$

Sum of the Squared residuals $\sum e_i^2 = \sum (y - f(x_i))^2$

Linear regression

$\tilde{f}(x) = a_0 + a_1 x$

$\tilde{f}(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

(1) $\leftarrow$ $= a e^{bx}$ $\rightarrow$ logarithmic

$= \dfrac{x}{b + x}$ $\rightarrow$ $\dfrac{1}{x}, \dfrac{1}{y}$

(3) $\leftarrow$ $= a x^b$

(1) $\ln(\tilde{f}(x)) = \ln a + bx$

(3) $\ln(\tilde{f}(x)) = \ln a + b \ln x$

Polynomials

$$P_n(x) = \boxed{a_0 + a_1 x} + a_2 x^2 + \dots + a_n x^n$$

$\nearrow$ $a_0, a_1, a_2 \dots a_n$ are related / correlated

$\rightarrow$ $n \rightarrow$ increases $\rightarrow$ $\underline{A}$ becomes unstable / ill-conditioned

$$y_i = a_0 + a_1 x_i$$

$$\underset{n\times 1}{\begin{bmatrix} y_i \end{bmatrix}} = \underset{n\times 2}{\begin{bmatrix} 1 & x_i \\ & \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}}$$

$\underline{y}$ $\quad$ $\underline{A}$ $\quad$ $\underline{x}$

$$S = \|\underline{e}\|^2 = \underbrace{(\underline{y} - \underline{A}\,\underline{x})^T (\underline{y} - \underline{A}\,\underline{x})}_{\text{inner product / vector norm}}$$

$\underline{y} \rightarrow$ known; $\underline{A} \rightarrow$ mapping; $\underline{x} \rightarrow$ unknown

$$\frac{\partial S}{\partial \underline{x}^T} = -\underline{A}^T(\underline{y} - \underline{A}\,\underline{x}) = 0$$

$$\Rightarrow -\underline{A}^T\underline{y} + \underline{A}^T\underline{A}\,\underline{x} = 0$$

$$\Rightarrow \underline{A}^T\underline{A}\,\underline{x} = \underline{A}^T\underline{y}$$

$$\hat{\underline{x}} = (\underline{A}^T\underline{A})^{-1}\underline{A}^T\underline{y}$$

$$\underline{A}^T\underline{A} = \begin{bmatrix} n & \boxed{\Sigma x_i} \\ \boxed{\Sigma x_i} & \Sigma x_i^2 \end{bmatrix} \rightarrow (\underline{A}^T\underline{A})^{-1} = \frac{\begin{array}{c|cc} & a_0 & a_1 \\ \hline a_0 & \Sigma x_i^2 & -\Sigma x_i \\ a_1 & -\Sigma x_i & n \end{array}}{\det(\underline{A}^T\underline{A})}$$

$$\underline{A}^T\underline{y} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix}$$

Orthogonal polynomial

$\left.\begin{array}{l} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{array}\right\}$ linked $\rightarrow$ matrix algebra $A^T A = \underline{I}$ $\hookrightarrow$ orthogonal

$$f(x) = a_0 + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x) + \dots + a_n P_n(x)$$

$$= \sum_{i=0}^{n} a_i P_i(x)$$

$$\int_a^b P_i(x) P_j(x) = \begin{cases} 0, & i \neq j \\ \text{value}, & i = j \end{cases}$$ $\longrightarrow$ orthogonality relation

$$S \to e^2 = \int_a^b \left(f(x) - \sum a_i P_i(x) \right)^2 dx$$

$$\frac{\partial S}{\partial P_j(x)} = -2 \int_a^b \left(f(x) - \sum a_i P_i(x) \right) P_j(x) \, dx$$

$$-2\left[\int_a^b \left(f(x)\, P_j(x)\right) dx - \int_a^b P_j(x) \sum a_i\, P_i(x)\, dx\right] = 0$$

$\delta_{ij} \rightarrow$ Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\int_a^b f(x)\, P_j(x)\, dx - a_i \int_a^b \underbrace{P_i(x)\, P_j(x)}_{\delta_{ij}}\, dx = 0$$

$$\Leftarrow \int_a^b f(x)\, P_j(x)\, dx - a_i\, \delta_{ij} = 0$$

$$\boxed{a_i = \int_a^b f(x)\, P_i(x)\, dx = 0}$$

$\because\ \delta_{ij} = 1$ when $i = j$

Sequential Polynomials

Orthogonal

Legendre Polynomials

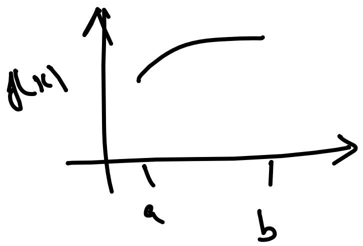
$[-1, 1]$

$[a, b] \longrightarrow$ linear transformation to

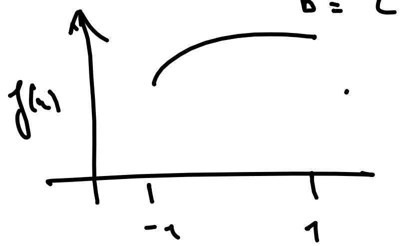
$[-1, 1]$

$a = cu_{M} + d$

$b = cu_{u} + d$



$\longrightarrow$



$$a = C u_l + d \quad \Rightarrow \quad u_l = -1 \;,\; u_u = 1$$

$$b = C u_u + d$$

$$a = -C + d$$

$$b = C + d$$

$$b - a = 2C \quad \Rightarrow \quad C = \frac{b-a}{2}$$

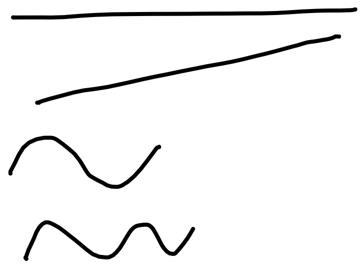
$$a + b = 2d \quad \Rightarrow \quad d = \frac{a+b}{2}$$

$\boxed{P_0}$ =

P_1 =

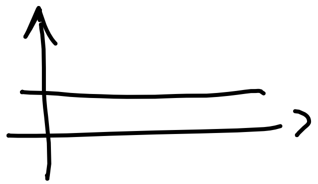
P_2 =

P_3 =



n-roots

$f(x)$



Fourier Transformation

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$$

roots → sampling a function at its roots allows us to use quadrature rules

$$\int_a^b P_i(x)\, P_j(x) = \begin{cases} \text{value}, & i = j \\ 0, & i \neq j \end{cases} \quad \text{– orthogonality}$$

$$a_i = \int_a^b f(x)\, P_i(x)\, dx \quad \text{– analysis}$$

$$\hat{f}(x) = \sum a_i P_i(x) \quad \text{– synthesis}$$

Legendre polynomials

$P_l(x) \longrightarrow x \in [-1, 1]$ $\qquad$ $P_l(\cos\theta) \rightarrow \theta \in [0, \pi]$

$P_n(x)$

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l (x^2-1)^l}{dx^l} \qquad \text{(Rodrigues formule)}$$

$P_0(x) = 1$

$P_1(x) = \frac{1}{2 \cdot 1!} \frac{d(x^2-1)}{dx} = \frac{2x}{2} = x \rightarrow$ straight lin

$P_2(x) = \frac{1}{2^2 2!} \frac{d^2 (x^2-1)^2}{dx^2} = \frac{d}{dx}\left[4x(x^2-1)\right] = 4(x^2-1) + 8x^2$

$\rightarrow$ quadratic