Lecture 19 – Gauss-Jordan elimination & Matrix condition

L U

$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} u_{11} & \\ & u_{22} \end{bmatrix}$

Symmetric $(A^T A)$

Gauss elimination $\longrightarrow$ LU decomposition $\longrightarrow$ LDL^T $\longrightarrow$ LL^T

↳ Croat's method $\begin{bmatrix} l_{11} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ (L U)

Banded

↳ Thomas method -> Banded matrices

$\boxed{\text{Gauss - Jordan}}$

- We keep track of what we are doing

- We are able to calculate inverse

SQUARE MATRIX

Matrix Condition

Three types of linear systems

Overdetermined	Full rank
Uniquely determined	Rank deficient / Singular
underdetermined	Ill-posed

$\underbrace{\text{Overdetermined, Uniquely determined, underdetermined}}_{\text{Information}}$

$\underbrace{\text{Full rank, Rank deficient / Singular, Ill-posed}}_{\text{Quality of the information}}$

$\frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{4} = 1$

$b = [\;1 \quad 2 \quad 7 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \quad 6 \quad 3\;]$

$\boxed{\frac{1}{2} + 1 + \frac{7}{4}}$

$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2 = 1\frac{1}{2}$

$\frac{1}{4} \cdot 1 + \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 7 =$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & & & & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & & \\ & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & \\ 0 & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \\ & & & & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} \bar{b} \\ \\ \\ \end{bmatrix}$$

Banded

Gauss-Jordan

$$\begin{array}{c} (1) - \\ (2) - \\ (3) - \end{array}\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Gauss Elimination

$$(2) - \frac{a_{21}}{a_{11}} \cdot (1)$$

$$\begin{array}{ccc} & 0 & 0 \\ \left[\begin{array}{c} 1 \\ 0 \\ a_{31} \end{array}\right. & \begin{array}{c} \boxed{\frac{a_{12}}{a_{11}}} \\ a_{22} - a'_{12}\, a_{21} \\ a_{32} \end{array} & \left.\begin{array}{c} \boxed{a_{13}/a_{11}} \\ a_{23} - a'_{13}\, a_{21}\ 0 \\ a_{33} \end{array}\right] \\ 0 & 0 & \end{array}$$

$$\left[\begin{array}{ccc} 1 & a_{12}/a_{11} & a_{13}/a_{11} \\ 0 & \boxed{a_{22} - a'_{12}\, a_{21}} & a_{23} - a'_{13}\, a_{31} \\ 0 & a_{32} - a'_{12}\, a_{31} & a_{33} - a'_{13}\, a_{31} \end{array}\right] - (2')$$

a'_{22} base eliminate a'_{12} and a'_{32}

Gauss-Jordan

$$a'_{11} = (1)/a_{11}$$

$$(2) \cdot \frac{a_{21}}{a'_{11}}\, (1)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b''_1 \\ b''_2 \\ b''_3 \end{bmatrix}$$

Gauss Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22}' & a_{23}' \\ & & a_{33}'' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3'' \end{bmatrix}$$

$$\frac{2n^3}{3}$$

LU

$$\begin{bmatrix} 1 & & 0 \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ & u_{22} & u_{23} \\ 0 & & \boxed{u_{33}} \end{bmatrix}$$

$$\frac{2n^3}{3}$$ → number of rows/columns

↓

Atleast $2n^3/3$

Floating Point Operations

Gauss-Jordan

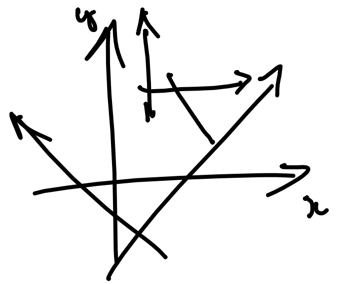
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1'' \\ b_2'' \\ b_3'' \end{bmatrix}$$

n^3

$Ax = b$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

↓



$$\leftarrow \left| \begin{array}{l} x + 2y = 4 \\ x + 2.3y = 4.1 \end{array} \right|$$ ill-posedness

Full rank → well-posed systems → trouble free

Rank deficient → trouble free

Ill-posed problem → Full rank / rank deficient] Matrix condition

$$\begin{bmatrix} 1000p & & \\ & 0.003 & \\ & & 1 \end{bmatrix}$$

Norms → Finiteness of the system

Vectors

L_2-norm

Euclidean norm

$\underline{x} \Rightarrow L_2\text{-norm} = \sqrt{\underline{x}^T x}$

$\|\underline{x}\|$ → How big is the norm? Length of the vector

L_∞-norm $\|\underline{x}\|_\infty = \max |x_i|$

L_1-norm $\|\underline{x}\|_1 = \sum |x_i|$

L_p-norm $\|\underline{x}\|_p = \sqrt[p]{\sum |x_i|^p}$

Matrices

$\|A\|_e = \sqrt{\sum_i \sum_j a_{ij}^2}$

L_1-norm

$\|A_1\| = \max_j \sum_i |a_{ij}|$

$\|A\|_p$

$\|A\|_\infty = \max_i \sum_j |a_{ij}|$

row

1 2 3 4 5 6 2 1 → $\sqrt{\dfrac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 2^2 + 1^2}{8}}$

$$\text{RMS} = \sqrt{\frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 2^2 + 1^2}{8}}$$