Lecture 22 – Orthogonal factorizations of matrices

$$Ax = b$$

$$QRx = b$$

$$Q^T Q Rx = Q^T b$$

$$Rx = Q^T b$$

QR factorization

Q → orthogonal, R → upper triangular

$$\underset{m\times n}{A}\ \underset{n\times 1}{x} = b$$

$A \to A^{-1}$

$LU \to A \Rightarrow \underline{A^{-1}}$

A → backward substitution

Eigenvalue decomposition

$$E \Lambda E^T = A$$

E → orthogonal, Λ → diagonal

Symmetric matrices

Square $n \times n$

Singular value decomposition

SVD

$$USV^T = A$$

S → diagonal, U, V → orthogonal

$m \times n$

Orthogonalization of matrices

Gram-Schmidt orthogonalization

$$A = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k]$$

↓ ↓

orthogonalize vectors

$$A^0 = [\vec{u}_1, \vec{u}_2, \dots \vec{u}_k]$$

$$\vec{v}_1 x_1 + \vec{v}_2 x_2 + \vec{v}_3 x_3 \dots + \vec{v}_k x_k = \underline{b}$$

$$A^{0^T} A^0 = I$$

$$\vec{u}_1 \cdot \vec{u}_2 = 0$$

$$\vec{u}_1 \cdot \vec{u}_1 = \|\vec{u}_1\|^2$$

$$\vec{u}_1 \cdot \vec{u}_2 = \vec{v}_1 \cdot \vec{v}_2 - \langle \vec{v}_2, \vec{v}_1 \rangle \cancel{\frac{\vec{v}_1 \cdot \vec{v}_1}{\langle \vec{v}_1, \vec{v}_1 \rangle}}$$

$$= \vec{v}_1 \cdot \vec{v}_2 - \vec{v}_2 \cdot \vec{v}_1 = 0$$

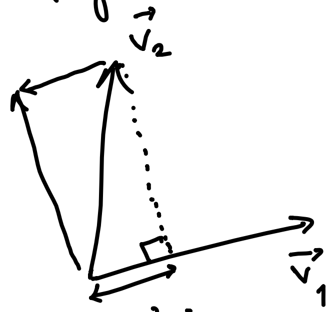
$$\vec{u}_1 = \vec{v}_1$$

$$\vec{u}_2 = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \frac{\vec{u}_1}{\langle \vec{u}_1, \vec{u}_1 \rangle}$$

$$\vec{u}_3 = \vec{v}_3 - \langle \vec{v}_3, \vec{u}_1 \rangle \frac{\vec{u}_1}{\langle \vec{u}_1, \vec{u}_1 \rangle} - \langle \vec{v}_3, \vec{u}_2 \rangle \frac{\vec{u}_2}{\langle \vec{u}_2, \vec{u}_2 \rangle}$$

$$\boxed{\vec{u}_k = \vec{v}_k - \sum_{j=1}^{k-1} \langle \vec{v}_k, \vec{u}_j \rangle \frac{\vec{u}_j}{\langle \vec{u}_j, \vec{u}_j \rangle}}$$

Algorithm



$$\langle \vec{v}_1, \vec{v}_2 \rangle = \vec{v}_1 \cdot \vec{v}_2$$

$$\vec{u}_1 = \vec{v}_1$$

$$\vec{u}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \vec{u}_1$$

$$\vec{u}_2 = \vec{v}_2 - \langle \vec{v}_1, \vec{u}_1 \rangle \frac{\vec{e}_{u_1}}{\|\vec{u}_1\|}$$

vector

$\langle \cdot, \cdot \rangle$ inner product

$$\vec{u}_k = \vec{v}_k - \sum_{j=1}^{k-1} \underbrace{\langle \vec{v}_k, \vec{u}_j \rangle \frac{\vec{u}_j}{\langle \vec{u}_j, \vec{u}_j \rangle}}_{proj_{\vec{u}_j}(\vec{v}_k)}$$

$$\vec{u}_k = \vec{v}_k - \sum_{j=1}^{k-1} proj_{\vec{u}_j}(\vec{v}_k)$$

Gram-Schmidt $\rightarrow$ not very accurate

$\downarrow$

$\vec{u}_1, \vec{u}_2 \neq$ orthogonal

$Ax = b$

$QRx = b \Rightarrow \underline{R}x = Q^T b$

$\hookrightarrow$ solve x by backward substitution

QR factorization using Gram-Schmidt

$$A = [\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_k]$$

$$\underset{m \times n}{A} = \underset{m \times m}{Q} \underset{m \times n}{R} = \underbrace{[\vec{e}_{u_1}, \vec{e}_{u_2}, \vec{e}_{u_3}, \dots, \vec{e}_{u_k}]}_{Q} \overbrace{\begin{bmatrix} \vec{v}_1 \cdot \vec{e}_{u_1} & \vec{v}_2 \cdot \vec{e}_{u_1} & \vec{v}_3 \cdot \vec{e}_{u_1} & \dots & \vec{v}_k \cdot \vec{e}_{u_1} \\ 0 & \vec{v}_2 \cdot \vec{e}_{u_2} & \vec{v}_3 \cdot \vec{e}_{u_2} & \dots & \vec{v}_k \cdot \vec{e}_{u_2} \\ 0 & 0 & \vec{v}_3 \cdot \vec{e}_{u_3} & \dots & \vec{v}_k \cdot \vec{e}_{u_3} \\ 0 & & & & \vdots \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}}^{R}$$

Householder reflections

$P \rightarrow$ projection matrix

orthogonal

symmetric

square.

$P = P^T$

$PP^T = \underline{I}$

$PP = \underline{I}$

$Px = x$

$P = \underline{I} - 2uu^T \rightarrow$ one inner product

Identity matrix

how to find u? $\rightarrow$ vector

$$A = \underbrace{P_k P_{k-1} \cdots P_2 P_3}_{P} A$$

$u \rightarrow \vec{v}_1 \Rightarrow u = \|\vec{v}_1\| \vec{e}_{v_1}$

$$u = \begin{bmatrix} \|\vec{v}_1\| \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underset{5\times 4}{A} = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \qquad \underset{5\times 4}{A_1} = \underset{5\times 5}{P_1} \underset{5\times 4}{A} = \overset{R'}{\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix}}$$

Goal - upper triangular matrix & an orthogonal matrix

length of the column vectors

$$QR \rightarrow \begin{cases} QR \\ QR \text{ with pivoting} \end{cases}$$

$$A_2 = \overset{P_2}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & P_2' & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix}} \begin{bmatrix} A_1 \end{bmatrix} = \underbrace{\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}}$$

$P_2' \rightarrow 4\times 4$

$P_4 P_3 P_2 P_1$

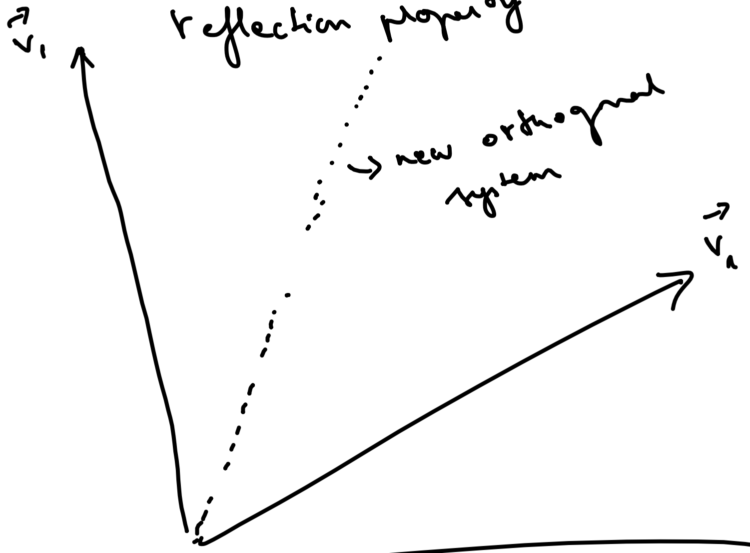
$$R = \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Q = \prod_{j=1}^{k-1} P_j \qquad P_j = I - 2u_j u_j^T \qquad u_k = \frac{\vec{v}_k + \alpha \vec{e}_j}{\| \vec{v}_k + \alpha \vec{e}_j \|}$$

where $\alpha = -\operatorname{sign}(\vec{v}_k(1)) \|\vec{v}_k\|$

Householder reflections - Geometry

$$P \to \underbrace{Px = x}_{\text{reflection property}}$$



Matrix Computations - Golub & van Loan

Givens Rotation