Lecture 32 - Numerical Differentiation

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \to \text{ratio}$$

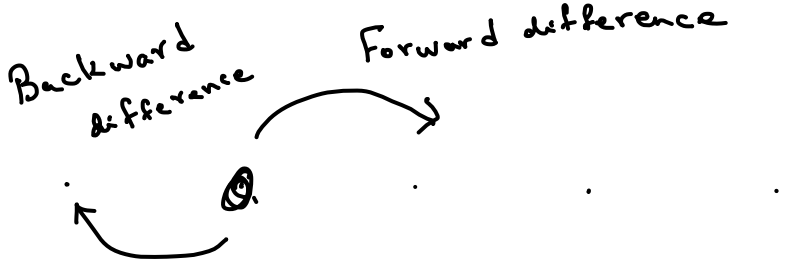
Taylor series

$$f(x_{i+1}) = f(x_i) + f'(x)\frac{(x_{i+1} - x_i)}{1!} + f''(x)\frac{(x_{i+1} - x_i)^2}{2!} + \dots$$

$$f'(x) = \frac{f(x_{i+1}) - f(x_i)}{(x_{i+1} - x_i)} - \frac{f''(x)\,(x_{i+1} - x_i) - \dots}{\text{Omit}}$$

$$+\ O(\Delta x)$$

Truncation Error $\rightarrow$ Backward = forward



$$f(x_i) = f(x_{i-1}) + f'(x)\,\frac{(x_i - x_{i-1})}{1!} + f''(x)\,\frac{(x_i - x_{i-1})^2}{2!}$$ Backward difference

$$f(x_{i-1}) = f(x_i) - f'(x)\,\frac{(x_i - x_{i-1})}{1!} - f''(x)\,\frac{(x_i - x_{i-1})^2}{2!}$$

Central difference $O(\Delta x^2)$

Backward $f'(x_i) = \dfrac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$

Forward $f'(x_i) = \dfrac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$

$x_i - x_{i-1} = x_{i+1} - x_i = \Delta x$

$$2 f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{\Delta x}$$

$$\Rightarrow f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x}$$

Error $O(\Delta x^2)\, f'''(\xi)$

Backward $\nabla^{(n)}$

Forward $\Delta^{(n)}$

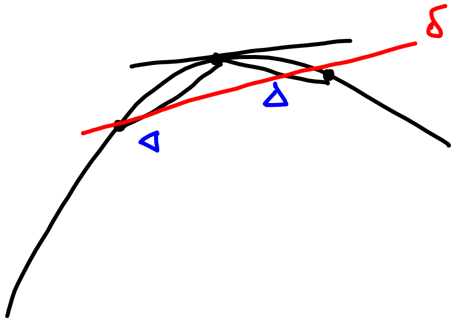
Central δ ?

Equispaced datasets

Difference table

	∇	∇^2	
x_0			
x_1	$f(x_1)-f(x_0)$		
x_2	$f(x_2)-f(x_1)$	$\nabla f(x_2)-\nabla f(x_1)$	
x_3	$\vdots$	$\vdots$	$\nabla^2 f(x_3)-\nabla^2 f(x_2)$
x_4	$\vdots$	$\vdots$	
x_5			

Edge effect



Difference table δ

	δ	δ^2	δ^3
x_0	$\boxed{f(x_1)-f(x_0)}$		
x_1	$\frac{f(x_2)-f(x_0)}{2\Delta x}$	x	
x_2	$\frac{f(x_1)-f(x_3)}{2\Delta x}$	$\frac{\delta f(x_3)-\delta f(x_1)}{2\Delta x}$	✓
x_3	$\frac{f(x_4)-f(x_2)}{2\Delta x}$	✓	
x_4	$\frac{f(x_5)-f(x_3)}{2\Delta x}$	✗	
x_5	$\boxed{f(x_5)-f(x_4)}$		

Unequally spaced data

- fit a Lagrange polynomial

- Differentiate Lagrange polynomial

Lagrange polynomial of degree n

$$f(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)} f(x_i)$$

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) \Rightarrow$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

$$f'(x) = \frac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} f(x_0)$$

$$+ \frac{2x - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$

$$+ \frac{2x - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)}$$

Richardson's Extrapolation

- Do numerical derivatives at two different intervals
- Get an estimate of the error of the smaller interval
- Add it to the derivative from the smaller interval

$h \rightarrow h/2 \rightarrow h/4$

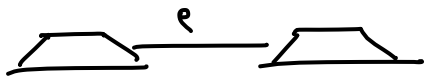
$$D \approx \boxed{4/3} D(h/2) - \boxed{1/3} D(h)$$

$$4/3 - 1/3 = 1$$

If we have noise, then?

$$y_i = \tilde{y}_i + \varepsilon_i$$

GRACE ρ



μm $\approx$

$\dot{\rho}$

$$\rho_1 = \tilde{\rho}_1 + \varepsilon$$

$$\rho_2 = \tilde{\rho}_2 + \varepsilon \Rightarrow \Delta\rho_{12} = \tilde{\rho}_2 - \tilde{\rho}_1$$