Lecture 14 - Lagrange interpolating polynomials & Splines

$$y \rightarrow y^*$$

$f(x) \nearrow$

$\uparrow$

polynomial $\rightarrow$ linear interpolation

$\downarrow$

instability for higher degrees

- mean between data points [average]
- nearest neighbour
- linear interpolation

$a_0 + a_1 x + a_2 x^2$ ↳ Newton's form

$$f(x) = f(x_0) + \underbrace{f[x_0, x_1]}_{\text{Finite divided differences}}(x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots$$

Lagrange interpolating polynomial

$$f_n(x) = \sum_{i=0}^{n} \underbrace{L_i(x)}_{\text{Kernel}} \underbrace{f(x_i)}_{\text{value}} \Rightarrow L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$\Rightarrow f_n(x) = \sum_{i=0}^{n} \overbrace{\prod_{j=0}^{n} \left(\frac{x - x_j}{x_i - x_j} \right)}^{\text{Polynomial}} f(x_i)$$

$$f_2(x) = \sum_{i=0}^{2} \left[\prod_{\substack{j=0 \\ j \neq i}}^{2} \frac{(x - x_j)}{(x_i - x_j)} \right] f(x_i)$$

Evaluated values

$i = 0 \Rightarrow \prod_{\substack{j=0 \\ j \neq i}}^{2} \frac{(x - x_j)}{(x_i - x_j)} f(x_i) \Rightarrow$

quadratic functions

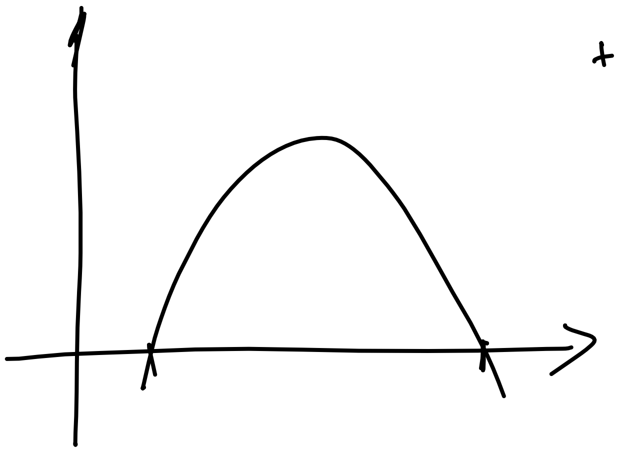
$$\underline{\left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)} f(x_0)$$

$i = 1$

$$+ \underline{\left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)} f(x_1)$$

$i = 2$

$$+ \underline{\left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)} f(x_2)$$



Lagrange $\longleftrightarrow$ Newton

$$f(x) = f(x_0) + \boxed{\frac{f(x_1) - f(x_0)}{x_1 - x_0}} (x - x_0) \quad — \text{①}$$

$\longrightarrow$ Newton's 1st degree polynomial

$\downarrow$

$$f[x_0, x_1] = \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1} \quad — \text{②}$$

② in ①

$$f(x) = f(x_0) + \left(\frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1} \right) (x - x_0)$$

$$= \frac{(x_0 - x_1) f(x_0) + (x - x_0) f(x_0)}{x_0 - x_1} + \frac{f(x_1)(x - x_0)}{x_1 - x_0}$$

$$f(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1)$$

$$\boxed{\frac{x - x_j}{x_i - x_j} f(x_i)} \longrightarrow \text{Lagrange form}$$

Comment $\rightarrow$ 1) Approximation is the same as Taylor Series approximation error

2) All these interpolation techniques degenerate to linear interpolation

Spline interpolation

1 Linear
2 Quadratic → for every interval fit a quadratic polynomial → conditions apply
3 Cubic

natural splines ←

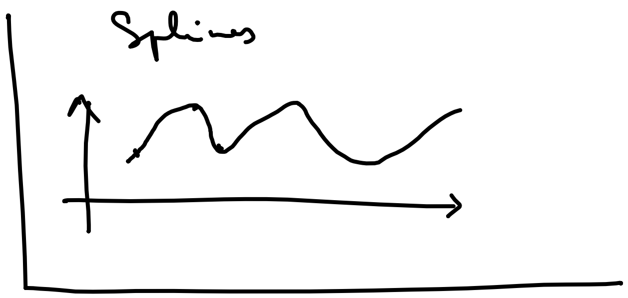
n intervals

a_2, b_2, c_2

Data points

$a_1x^2 + b_1x + c_1$ $\quad a_3, b_3, c_3$

n+1 $\boxed{\text{knots}}$ → internal knots
→ Boundary knot



$i-1, \quad i$

$a_{i-1}x_{i-1}^2 + b_{i-1}x_{i-1} + c_{i-1}$

$= a_i x_{i-1}^2 + b_i x_{i-1} + c_i$

Conditions

1. Function values $f(x)$ of adjacent polynomials must be equal at the interior knots
2. The first & last functions must pass through the end points
3. The first derivatives at the internal knots must be equal
4. Assume that the second derivative is zero

Quadratic

→ $a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$

$a_n x_n^2 + b_n x_n + c_n = f(x_n)$