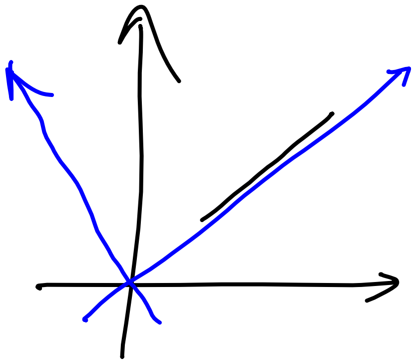
Lecture 24 - Singular value decomposition

Eigenvalue decomposition

Similarity transformation of matrices

Similar matrices

Matrix similarity



Scale

translation

rotation

Gauss elimination

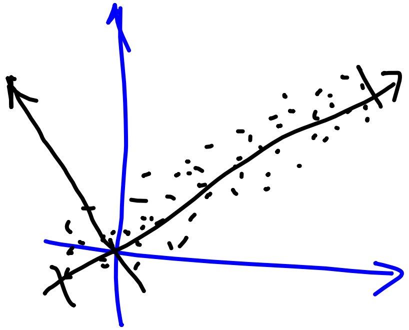
$$\left.\begin{array}{l} LU \\ LL^T \\ LDL^T \end{array}\right\} \text{Non-orthogonal decompositions}$$

QR - orthogonal decomposition

diagonal

USV^T

orthogonal



$$\underset{n\times n}{\boxed{M^{-1}}}\ \underset{n\times n}{A}\ \underset{n\times n}{\boxed{M}} = \underset{n\times n}{B}$$

A is similar to B

$M^{-1} A M = B$ has to be diagonal

Eigenvalue decomposition

$M^{-1} = M^T \Rightarrow M^T A M = B$ ↳ diagonal

M ↳ orthogonal

$$Ax = b$$

$$\begin{bmatrix} \\ \vec{a}_1 \\ \\ \end{bmatrix} x_1 + \begin{bmatrix} \\ \vec{a}_2 \\ \\ \end{bmatrix} x_2 + \dots + \begin{bmatrix} \\ \vec{a}_n \\ \\ \end{bmatrix} x_n = b$$

$$Ax = 0$$

↳ homogeneous linear system

Eigenvalue decomposition

- square matrices only
- Eigenvectors need not be orthogonal

Linear system $\underset{m\times n}{A}$

underdetermined $m<n$ lack of information

uniquely determined $m=n$ sufficient information.

overdetermined $m>n$ redundant information

$A = U S V^T$

S → diagonal

U, V → orthogonal

$A\vec{v}_k = \vec{u}_k \sigma_k$

σ_k → singular values

$\vec{v}_k$ → orthogonal

$\vec{u}_k$ → orthogonal

$AV = US$

V → $[v_k]$

U → $[u_k]$

S → $\begin{bmatrix} \sigma_k & \\ & \ddots \end{bmatrix}$

$\underset{m\times n}{A} \Rightarrow A^T A \, , \, A A^T$

→ symmetric → Eigenvalue decomposition

$m > n \rightarrow \text{rank} \leq n$
$\qquad\qquad\qquad r$

same rank as A $\left\{ \begin{array}{l} \underset{n \times n}{A^T A} \\ \\ \underset{m \times m}{A A^T} \end{array} \right.$

$$\left.\begin{array}{ll} A^T A \Rightarrow M_I^T A^T A M_I = \underset{n \times n}{\Sigma_I} M_I^T & \begin{array}{l} r \text{ eigenvalues} \\ n-r \text{ zeros} \end{array} \\ A A^T \Rightarrow M_0^T A A^T M_0 = \underset{m \times m}{\Sigma_0} M_0^T & \begin{array}{l} r \text{ eigenvalues} \\ m-r \text{ zeros} \end{array} \end{array}\right\} \longrightarrow 1$$

$$A = USV^T$$

$$\left.\begin{array}{rl} A^T A & = V S \underbrace{U^T U}_{I} S V^T \\ & = V S^2 V^T \\ \\ A A^T & = U S V^T V S U^T \\ & = U S^2 U^T \end{array}\right\} \quad 2$$

$S^2 = \Sigma$ eigenvalues

$U = M_0$

$V = M_I$

$$\underset{m\times n}{A} = \underset{m\times m}{U}\;\underset{m\times n}{S}\;\underset{n\times n}{V^T}$$

$$S = \begin{array}{c} \begin{bmatrix} \ddots & & 0 \\ 0 & \ddots & \\ & & \ddots \\ \hline 0 & 0 \; 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \uparrow n \downarrow \\ \updownarrow m-n \end{array}$$

Orthogonalize A

$$QR = A$$

Gram-Schmidt   Householder   Givens rotation

$P_4 \; P_3 \; P_2 \; P_1$

SVD

Householder(A)

Householder(A^T)

5×4

$$\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \rightarrow \text{Bidiagonal system}$$

Orthogonalization

$$\begin{bmatrix} x & & \\ 0 & & \\ 0 & 0 & \\ 0 & 0 & \\ 0 & 0 & \\ 0 & 0 & \end{bmatrix}$$

- Making particular values of the system 0
- Finding orthogonal vectors for each column vector via _Proj
  - Reflection
  - Rotation

$$B = P_n \ldots P_2 P_1 A P_1' P_2' P_3' \ldots P_{n-1}'$$

$$B = U'^T A V'$$

$LU \to \det(A) \to \mathrm{Prod}(u_{ii})$

$QR \to \det(A) \to \mathrm{Prod}(R_{ii})$

$USV^T \to \det(A) \to (\mathrm{Prod}(s))^2$

Matrix Computations

- Chapter 2
- Chapter 5 - 5.1, 5.2

Algorithm for SVD

1) Bidiagonalization - Golub-Kahan algorithm

$$\left.\begin{array}{l} \text{Householder}(A) \\ \text{Householder}(A^T) \end{array}\right\} B$$

2) Givens Rotation to make the upper diagonal zero

$\therefore U'^T A V' = B$

$\Rightarrow A = USV^T$

$\therefore B G_1 G_2 \dots = S$

$A = U' B G_1 G_2 \cdots G V'^T$

$A = USV^T$