Lecture 16 – Solving linear systems of equations

Linear systems of equations

Modelling → output = Model input

↙ outcome = $\boxed{\text{Model}}$ parameter →

outcome = Model parameters ? ⇒ inverse problem

↓

outcome = Model parameters → forward modelling

Parametric model

Constraints → Cubic spline → f'

f''

Constraint model

Known ⟷ Mapping ⟷ unknown

Cubic spline

parameters f''

$Ax = b$ → uniquely determined system

↓

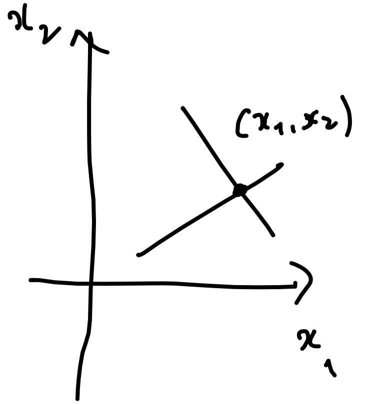
n equations & n unknowns

matrix with constraints

data point

enough information

$$\boxed{\begin{aligned} a_1 x_1 + a_2 x_2 &= c \\ b_1 x_1 + b_2 x_2 &= d \end{aligned}}$$



not enough information

$$\left|\begin{aligned} a_1 x_1 + a_2 x_2 &= c \\ 2a_1 x_1 + 2a_2 x_2 &= 2c \end{aligned}\right.$$

enough information

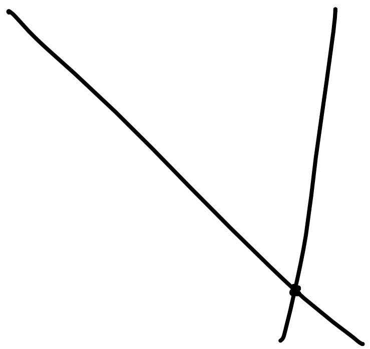
- unique equations
  - ↳ they are not linearly dependent another equations

- not enough information
  - ↳ the equations are linearly dependent

Model / Design

row space vectors

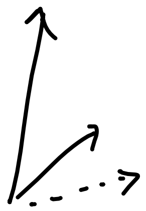
$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$



$$a_1 x_1 + a_2 x_2 = c \to \text{line}$$

column space

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} x_1 + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} x_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$



vectors

Matrix analysis of out model matrix

Cramer's rule

$$a_1 x_1 + a_2 x_2 = c \quad - \quad ①$$
$$b_1 x_1 + b_2 x_2 = d \quad - \quad ②$$

elimination

↓

Factorizes our matrix

$$- \begin{cases} ① \times b_1 & \cancel{a_1 b_1 x_1} + a_2 b_1 x_2 = b_1 c \\ ② \times a_1 & \cancel{a_1 b_1 x_1} + a_1 b_2 x_2 = a_1 d \end{cases}$$

$$(a_2 b_1 - a_1 b_2) x_2 = b_1 c - a_1 d$$

$$\Rightarrow x_2 = \frac{b_1 c - a_1 d}{\underline{a_2 b_1 - a_1 b_2}}$$

$\Rightarrow x_1$

↳ determinant of the model matrix

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \rightarrow D = a_1 b_2 - a_2 b_1$$

$\Rightarrow$ x_1

x_2

x_3

$\vdots$

Matrix factorizations $\rightarrow$

decomposition

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array} \overset{A}{\begin{bmatrix} 2 & 4 \\ \boxed{4} & 11 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{b}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

$n \times n$

L>

L>

Eliminate the elements to derive one equation with one variable

Back substitute

$$\begin{array}{c} a_1 \\ a_2 - a_1 \times 2 \end{array} \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$3x_2 = -3 \Rightarrow x_2 = -1$

Matrix factorization

- evade matrix inversion
- forming factors that are simpler in form than the model matrix

Gaussian Elimination

$$\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \overset{b}{\begin{bmatrix} 2 \\ 1 \\ 4 \\ 8 \end{bmatrix}} \Rightarrow \begin{array}{l} Ax = b \\ Ux = c \end{array}$$

$$c = \begin{bmatrix} 2 \\ 0 \\ 4 \\ 5 \end{bmatrix}$$

→ Pivots

$a_2 - a_1 \times 1/2$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$a_3 - a_2 \times 2/3$

$$\rightarrow \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$a_4 - a_3 \times 3/4$

$$\rightarrow \underset{U}{\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}}$$

row echelon

Steps / levels

$$\begin{array}{cccc} & c & & b \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix} & \begin{bmatrix} 2 \\ 0 \\ 4 \\ 5 \end{bmatrix} & = & \begin{bmatrix} 2 \\ 1 \\ 4 \\ 8 \end{bmatrix} \\ L & & & \end{array}$$

Forward elimination

$$b = Lc$$

$$\Rightarrow \quad LUx = b$$

$$Ax = b$$

LU-factorization

$$U = DL^T$$

$$U = \begin{bmatrix} 2 & & & \\ & 3/2 & & \\ & & 4/3 & \\ & & & 5/4 \end{bmatrix}$$

$$A = LDL^T$$