Lecture 36 – Runge-Kutta methods

- Adaptive techniques
- Embedded methods

$$y_{i+1} = y_i + \phi(x_i, y_i, h)\boxed{h}$$

$$\phi = \overset{\text{weights}}{a_1} \underset{\text{slopes}}{k_1} + a_2 k_2 + a_3 k_3 + \dots + a_n k_n$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, \; y_i + q_{11} k_1 h)$$

$$\vdots$$

$$k_n = f(x_i + p_{n-1} h, \; y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \dots q_{n-1,n-1} k_{n-1} h)$$

$$\begin{array}{c|cccc}
p_1 & q_{11} & & & \\
p_2 & q_{21} & q_{22} & & \\
 & \vdots & & & \\
p_{n-1} & q_{n-1,1} & \dots & & q_{n-1,n-1} \\
\hline
 & & & &
\end{array}$$

Find $\boxed{a_j, \; p_{j-1}, \; q_{j-1,1}}$ $j \in [1, n]$

- Assume certain values of a_j
- Solve condition equations

Recursive

Step size → key ingredient

↓

change step size

↓

decrease the step size

Order of the method

↳ increase in computational complexity with increased order

↓

increase order

Embedded RK methods

Get a hold of the truncation error

Software package

RK78

RK45

RK34

- Use methods of two different orders, but successive orders
- Use the lower order as the main driver
- Use the higher order to estimate the truncation error

Runge-Kutta Fehlberg

RK45 method

k_1

k_2

k_3

k_4

k_5

k_6

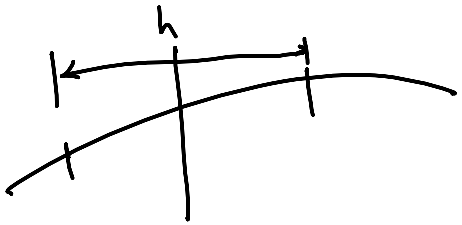
order 4 $\rightarrow$ k_1, k_3, k_4, k_6

order 5 $\rightarrow$ k_1, k_3, k_4, k_5, k_6

Adaptive methods

$\downarrow$ order-based

$\downarrow$ step-size-based



Tolerance → Absolute

→ Relative

↓

used for determining the adaptive step size

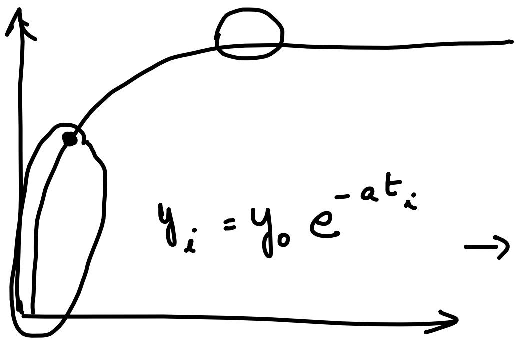
Implicit Euler Method

$$y_{i+1} = y_i + f(x_i, y_i)\, h$$

$$= y_i + \boxed{\left.\frac{dy}{dx}\right|_{x_i}} h$$

↳ x_{i+1}

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_{i+1}} h$$



$y_i = y_0 e^{-at_i}$ → eqn.

$$\frac{dy}{dt} = -ay$$

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i} h$$

$$y_{i+1} = y_i - a y_i h$$

$$= y_i (1 - ah)$$

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_{i+1}} h$$

$$y_{i+1} = y_i - a y_{i+1} h$$

$$y_{i+1} + a y_{i+1} h = y_i$$

$$y_{i+1} = \frac{y_i}{1 + ah}$$

Bulirsch - Stoer Extrapolation

→ It uses central differences

$$\frac{dy}{dx} = \frac{y_{x+h} - y_{x-h}}{2h}$$

$$y_{x+h} = y_{x-h} + \frac{dy}{dx} 2h$$

→ Use Richardson's extrapolation to estimate the truncation error & correct the value