A technical detail of AftFore

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A method for forecasting the number of earthquakes with $M > M_t$ in the testing period [S, T] based on the data of earthquakes $\boldsymbol{D} = \{t_i, M_i\}_{i=1}^{N}$ in the testing period [0, T] is described below. Note that we make use of all the observed data including earthquakes below the completeness magnitude.

1. Model description

A joint rate intensity rate of aftershocks at time t after the main shock with magnitude M is modelled by the Omori-Utsu and Gutenberg-Richter laws, given as

$$\lambda(t, M|K, p, c, \beta) = \frac{K}{(t+c)^p}\beta e^{-\beta(M-M_0)}, \tag{1}$$

where K, p, c, and β are parameters and M_0 represents the main shock magnitude. We also consider the detection rate of aftershocks that depends on time and magnitude to consider missing of early aftershocks, given as

$$\Phi(M|\mu(t), \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{M} dx \exp\left[-\frac{\left(x-\mu(t)\right)^2}{2\sigma^2}\right], \tag{2}$$

where $\mu(t)$ is a time-varying parameter that represents the magnitude with 50% detection rate and σ is a parameter representing the magnitude range of partially detected events. To make the following estimation plausible, we decompose the time-varying parameter $\mu(t)$ to the time-varying part $\mu_0(\mathrm{t})$ and the constant term μ_1, $\mu(t) = \mu_0(\mathrm{t}) + \mu_1$, and fix the $\mu_0(\mathrm{t})$ to the one estimated by the Bayesian smoothing method proposed in our previous studies (Omi *et al.,* 2013: also see Appendix for the detail). In this way, the time-varying parameter $\mu(t)$ is now reduced to a single parameter μ_1. Finally our model is characterized by a parameter set $\theta = \{K, p, c, \beta, \sigma, \mu_1\}$

2. Bayesian Estimation

We here estimate the parameter set θ given the observed aftershock data $\boldsymbol{D}$. In the context of Bayesian statistics, the plausibility of the parameter values given the data is quantified by the posterior probability distribution given by Bayes' theorem as

$$posterior(\theta|\boldsymbol{D}) \propto L(\theta|\boldsymbol{D})prior(\theta), \quad (3)$$

where $L(\theta|\boldsymbol{D})$ and $prior(\theta)$ are the likelihood function and prior probability distribution respectively. If we assume that the observed earthquakes follow the inhomogeneous Poisson process with the intensity rate $\nu_\theta(t,M) = \lambda(t,M|K,p,c,\beta)\Phi(M|\mu(t),\sigma)$, the log-likelihood function can be obtained as

$$\ln L(\theta|\boldsymbol{D}) = \sum_{0<t_i<T} \nu_\theta(t_i,M_i) - \int_{-\infty}^{\infty} dM \int_0^T dt \nu_\theta(t,M). \quad (4)$$

We use independent priors for the p, c, β, and σ parameters, $prior(\theta) = prior(p) \cdot prior(c) \cdot prior(\beta) \cdot prior(\sigma)$. Here the respective prior is given by $N(1.05, 0.13^2)$, $LN(-4.02, 1.42^2)$, $N(1.96, 0.34^2)$, and $LN(-1.61, 1.0^2)$, where N denotes the normal distribution and LN denotes the log-normal distribution based on *Omi et al*., (2016).

To appropriately account for the estimation uncertainty, we combine the forecasts from many probable parameter sets (Bayesian forecasting). For this purpose, we sample many parameter sets $\{\theta_i\}_{i=1}^m$ from the posterior probability distribution with the Markov chain Monte Carlo method. For our method, we use 1000 parameter sets.

3. Bayesian Forecasting

Given a parameter set θ, the predictive distribution $\mathrm{P}(n|\theta,M_t)$ of the number n of earthquakes with with $M > M_t$ in the testing period $[S, T]$ is the Poisson distribution with mean given by

$$\bar{n} = \int_{M_t}^{\infty} dM \int_0^T dt\, \lambda(t,M|K,p,c,\beta). \quad (5)$$

For the Bayesian forecasting, the predictive distribution $\mathrm{P}(n|\{\theta_i\}_{i=1}^m, M_t)$ is given by

$$\mathrm{P}(n|\{\theta_i\}_{i=1}^m, M_t) = \frac{1}{m}\sum_{i=1}^m \mathrm{P}(n|\theta_i, M_t). \quad (6)$$

Appendix . Bayesian smoothing method for the time-varying detection rate

A time-varying detection rate is estimated based on the Bayesian smoothing method. We first discretize the time-varying parameter $\mu(t)$ as $\mu(t) = \mu_i \quad (t_{i-1} < t \le t_i)$, where t_i is the occurrence time of i-th aftershock and we set $t_0 = 0$. Thus the time-varying parameter $\mu(t)$ is now represented by a N-dimensional vector $\boldsymbol{\mu} = \{\mu_i\}_{i=1}^{N}$, where N is the number of observed aftershocks in the learning period.

The likelihood function of $\boldsymbol{\mu}$ given the observed magnitude sequence $\boldsymbol{M} = \{M_i\}_{i=1}^{N}$ is given by

$$P_{\beta,\sigma}(\boldsymbol{M}|\boldsymbol{\mu}) = \prod_{i=1}^{N} \beta e^{-\beta(M_i-\mu_i)-\frac{(\beta\sigma)^2}{2}} \Phi(M_i|\mu_i,\sigma), \tag{7}$$

(see *Omi et al.,* 2013). To estimate $\boldsymbol{\mu}$**,** which has the same length as the data, we introduce smoothness constraint that penalizes the time-variation of $\boldsymbol{\mu}$, given as

$$\mathrm{P}_{\mathrm{V}}(\boldsymbol{\mu}) = \prod_{i=3}^{N} \frac{1}{\sqrt{2\pi V}} e^{-\frac{(\mu_i-2\mu_{i-1}+\mu_{i-2})^2}{2V}}, \tag{8}$$

where V is a hyper-parameter that controls the smoothness of $\boldsymbol{\mu}$. From the Bayes' theorem, the posterior probability distribution of $\boldsymbol{\mu}$ given the data $\boldsymbol{M}$ under the hyper parameters $\{\beta,\sigma,V\}$ is given by

$$\mathrm{P}_{\beta,\sigma,\mathrm{V}}(\boldsymbol{\mu}|\boldsymbol{M}) \propto \mathrm{P}_{\beta,\sigma}(\boldsymbol{M}|\boldsymbol{\mu})\mathrm{P}_{\mathrm{V}}(\boldsymbol{\mu}). \tag{3}$$

The MAP estimate $\boldsymbol{\mu}^*$ given the hyper-parameters $\{\beta,\sigma,V\}$, $\boldsymbol{\mu}^* = \arg\max_{\boldsymbol{\mu}} P_{\beta,\sigma,\mathrm{V}}(\boldsymbol{\mu}|\boldsymbol{M})$,

can be readily found by using the Newton method.

The Bayesian smoothing method aims to find the MAP estimate $\boldsymbol{\mu}^*$ under the optimal estimates of the hyper-parameters $\{\beta,\sigma,V\}$. The hyper-parameters are optimized by maximizing the posterior probability distribution of the hyper-parameters given as

$$P(\beta,\sigma,V|\boldsymbol{M}) \propto P(\boldsymbol{M}|\beta,\sigma,V)P(\beta,\sigma,V). \tag{3}$$

Here $\mathrm{P}(\boldsymbol{M}|\beta,\sigma,\mathrm{V})$ is the marginal likelihood function,

$$P(\boldsymbol{M}|\beta,\sigma,V) = \int \boldsymbol{d\mu} P_{\beta,\sigma}(\boldsymbol{M}|\boldsymbol{\mu})P_V(\boldsymbol{\mu}), \tag{3}$$

and we approximate it using the Laplace approximation as

$$P(\boldsymbol{M}|\beta,\sigma,V) \approx (2\pi)^{\frac{N}{2}}|-H|^{-\frac{1}{2}} P_{\beta,\sigma}(\boldsymbol{M}|\boldsymbol{\mu}^*)P_V(\boldsymbol{\mu}^*), \tag{3}$$

where $\boldsymbol{\mu}^*$ is the MAP estimate, and H is the Hessian of $\ln P_{\beta,\sigma,\mathrm{V}}(\boldsymbol{\mu}|\boldsymbol{M})$ at $\boldsymbol{\mu} = \boldsymbol{\mu}^*$. $\mathrm{P}(\boldsymbol{M}|\beta, \sigma, \mathrm{V})$ is the prior probability distribution of the hyper-parameters. We employ the priors for the β and σ, and set them to the same one as are employed in Section 2. The hyper-parameters are optimized using the Quasi Newton method, where the gradient is numerically obtained.

References:

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