Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1
    python3−scipy python3−numpy
    python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.
    githubusercontent.com/
    gadepall/
EE1310/master/filter/codes/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('
    Sound_Noise.wav')

#sampling frequency of Input signal
sampl_freq=fs
```

```
#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and
    denominator polynomials
    respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with
    butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)

#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav
    file
sf.write('
    Sound_With_ReducedNoise.
    wav', output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/
    gadepall/EE1310/raw/
    master/filter/codes/xnyn.
    py
```

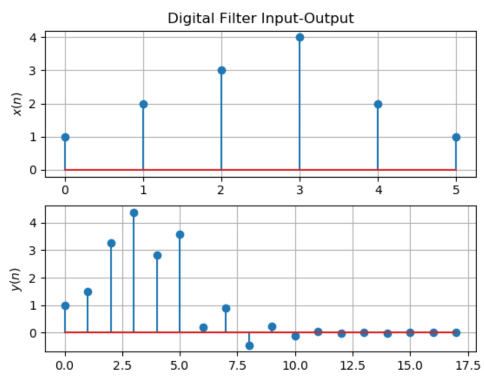

Fig. 3.2

3.3 Repeat the above exercise using a C code.

Solution: C code for creating .dat files:

```
#include <stdio.h>
#include <stdlib.h>
```

```
int main(void){

float x[6] = {1,2,3,4,2,1};
float y[18];

y[0] = 1;
y[1] = 1.5;

for(unsigned i=2;i<=5;i++)
{
        y[i] = x[i] + x[i-2] - 0.5*y[i
            -1];
}
for(unsigned i = 6;i<=7;i++)
{
        y[i] = x[i-2] - 0.5*y[i-1];
}
for(unsigned i=8;i<=17;i++)
{
        y[i] = -0.5*y[i-1];
}

FILE *fpx = fopen("sketchfnx.dat","w+")
    ;

for (unsigned i=0;i<=5;i++)
{
        fprintf(fpx,"%f\n",x[i]);
}

fclose(fpx);

FILE *fpy = fopen("sketchfny.dat","w+")
    ;

for (unsigned i=0;i<=17;i++)
{
        fprintf(fpy,"%f\n",y[i]);
}

fclose(fpy);
        return 0;
}
```

Python code for plotting from the .dat files:

```
import numpy as np
import matplotlib.pyplot as plt

xn = np.loadtxt("sketchfnx.dat")

plt.subplot(2, 1, 1)
plt.stem(range(0,6),xn)
plt.title('Digital Filter Input-Output')
plt.ylabel('$x(n)$')
plt.grid()# minor

yn = np.loadtxt("sketchfny.dat")

plt.subplot(2, 1, 2)
plt.stem(range(0,18),yn)
plt.xlabel('$n$')
plt.ylabel('$y(n)$')
plt.grid()# minor

plt.show()
```

4 $\mathcal{Z}$-transform

4.1 The $\mathcal{Z}$-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.1)$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \quad (4.2)$$

and find

$$\mathcal{Z}\{x(n-k)\} \quad (4.3)$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \quad (4.4)$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.5)$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \quad (4.6)$$

$$\text{let } y(n) = x(n-k) \quad (4.7)$$

$$Y(z) = \mathcal{Z}\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n)z^{-n} \quad (4.8)$$

$$\Longrightarrow Y(z) = \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \quad (4.9)$$

$$\text{put } n-k = p \quad (4.10)$$

$$Y(z) = \sum_{p=-\infty}^{\infty} x(p)z^{-(p+k)} \quad (4.11)$$

$$Y(z) = \mathcal{Z}\{y(n)\} = z^{-k}\sum_{p=-\infty}^{\infty} x(p)z^{-p} \quad (4.12)$$

$$\mathcal{Z}\{x(n-k)\} = z^{-k}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.13)$$

$$\boxed{\mathcal{Z}\{x(n-k)\} = z^{-k}X(z)} \quad (4.14)$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1. **Solution:**

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.15)$$

From (3.1)

$$X(z) = \sum_{n=0}^{5} x(n)z^{-n} \quad (4.16)$$

$$X(z) = x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2} + \quad (4.17)$$

$$x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5} \quad (4.18)$$

$$\boxed{X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5}} \quad (4.19)$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.20)$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (4.21)$$

$$\Longrightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \quad (4.22)$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.23)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.24)$$

is

$$U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1 \quad (4.25)$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.26}$$

and from (4.30),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.27}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.28}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.29}$$

Solution:

$$a^n u(n) = \begin{cases} a^n & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.30}$$

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.31}$$

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.32}$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.33}$$

Plot $|H(e^{\jmath\omega})|$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.

Solution: The following code plots Fig. 4.6.

```
wget https://raw.
    githubusercontent.com/
    gadepall/EE1310/master/
    filter/codes/dtft.py
```

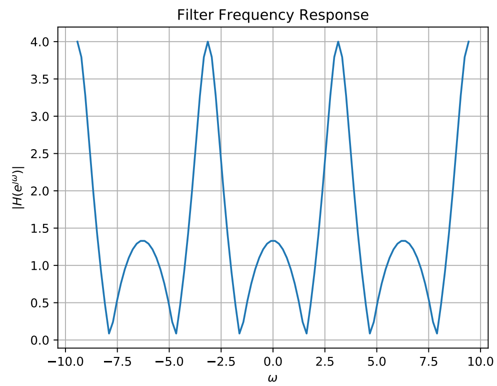

Fig. 4.6: $|H(e^{\jmath\omega})|$

Solution:

$$H(e^{\jmath\omega}) = \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}} \tag{4.34}$$

$$H(e^{\jmath\omega}) = \frac{1 + \cos 2\omega - \imath \sin 2\omega}{1 + \frac{1}{2}(\cos\omega - \imath\sin\omega)} \tag{4.35}$$

1. $(1 + \cos 2\omega - \imath \sin 2\omega)$ is periodic with period π

2. $\left(1 + \frac{1}{2}(\cos\omega - \imath\sin\omega)\right)$ is periodic with period 2π

Hence, $H(e^{\jmath\omega})$ is periodic with period LCM(π,2π) which is 2π

4.7 Express $h(n)$ in terms of $H(e^{\jmath\omega})$.

Solution: Proof of inverse dtft:

$$H(e^{\jmath\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-\jmath\omega n} \tag{4.36}$$

Multiplying both sides with $e^{\jmath\omega k}$ and integrating from $-\pi$ to π with respect to ω we get:

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega k} d\omega = \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{-\jmath\omega n} e^{\jmath\omega k} d\omega \tag{4.37}$$

Case 1: $n = k$

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega k}d\omega = h(k)\int_{-\pi}^{\pi} d\omega \quad (4.38)$$

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega k}d\omega = 2\pi h(k) \quad (4.39)$$

$$\implies h(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega k}d\omega \quad (4.40)$$

Case 2: $n \neq k$

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega k}d\omega = \sum_{n=-\infty}^{\infty} h(n)\int_{-\pi}^{\pi} e^{\jmath\omega(k-n)}d\omega \quad (4.41)$$

$$= \sum_{n=-\infty}^{\infty} h(n)\frac{e^{\jmath\omega(k-n)}}{\jmath(k-n)} \quad \text{from } -\pi \text{ to } \pi \quad (4.42)$$

$$= \sum_{n=-\infty}^{\infty} \frac{h(n)}{\jmath(k-n)}\left[e^{\jmath\pi(k-n)} - e^{-\jmath\pi(k-n)}\right] \quad (4.43)$$

$$= \sum_{n=-\infty}^{\infty} \frac{h(n)}{\jmath(k-n)}\left[\cos\pi(k-n) - \cos\pi(k-n)\right] \quad (4.44)$$

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega k}d\omega = 0 \quad (4.45)$$

$$\boxed{h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega} \quad (4.46)$$

Solving for $h(n)$:

$$H(e^{\jmath\omega}) = \frac{1 + e^{-\jmath 2\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}} \quad (4.47)$$

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (4.48)$$

$$H(z) = (1 + z^{-2})\left(1 + \frac{1}{2}z^{-1}\right)^{-1} \quad (4.49)$$

$$H(z) = \left(1 + z^{-2}\right)\left(1 + \frac{1}{2z}\right)^{-1} \quad (4.50)$$

$$H(z) = \left(1 + z^{-2}\right)\left(1 - \frac{1}{2z} + \frac{1}{(2z)^2} - \frac{1}{(2z)^3} + ...\right) \quad (4.51)$$

$$H(z) = \left(1 + z^{-2}\right)\left(1 - \frac{z^{-1}}{2} + \frac{z^{-2}}{4} - \frac{z^{-3}}{8} + ...\right) \quad (4.52)$$

$$H(z) = \left(1 - \frac{z^{-1}}{2} + \frac{z^{-2}}{4} - \frac{z^{-3}}{8} + ...\right) + \left(z^{-2} - \frac{z^{-3}}{2} + \frac{z^{-4}}{4} - ...\right)$$

$$H(z) = 1 - \frac{z^{-1}}{2} + \frac{5}{4}z^{-2} - \frac{5}{8}z^{-3} + \frac{5}{16}z^{-4} - ... \quad (4.53)$$

$$H(e^{\jmath\omega}) = 1 - \frac{e^{-\jmath\omega}}{2} + \frac{5}{4}e^{-\jmath 2\omega} - \frac{5}{8}e^{-\jmath 3\omega} + \frac{5}{16}e^{-\jmath 4\omega} - ... \quad (4.54)$$

We have proved earlier that:

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega \quad (4.55)$$

Therefore

$$\boxed{h(n) = \left\{1, \frac{-1}{2}, \frac{5}{4}, \frac{-5}{8}, \frac{5}{16}, \frac{-5}{32}, ...\right\}} \quad (4.56)$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \quad (5.1)$$

for H(z) in (4.22).

Solution:

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (5.2)$$

The ROC is $|z| > \frac{1}{2}$

$$
\begin{array}{r l}
 & 1 - \frac{1}{2}z^{-1} \\
1 + \frac{1}{2}z^{-1} \big) & 1 + \quad z^{-2} \\
 & 1 + \frac{1}{2}z^{-1} \\
\hline
 & -\frac{1}{2}z^{-1} + z^{-2} \\
 & -\frac{1}{2}z^{-1} - \frac{1}{4}z^{-2} \\
\hline
 & \frac{5}{4}z^{-2}
\end{array}
$$

$$H(z) = \left(1 - \frac{1}{2}z^{-1}\right) + \frac{\frac{5}{4}z^{-2}}{\left(1 + \frac{1}{2}z^{-1}\right)} \tag{5.3}$$

$$= \left(1 - \frac{1}{2}z^{-1}\right) + \frac{5}{4}z^{-2}\left(1 + \frac{1}{2}z^{-1}\right)^{-1} \tag{5.4}$$

$$= \left(1 - \frac{1}{2}z^{-1}\right) + \frac{5}{4}z^{-2}\left(1 - \frac{z^{-1}}{2} + \frac{z^{-2}}{4} - \frac{z^{-3}}{8}...\right)$$

$$= 1 - \frac{1}{2}z^{-1} + \frac{5}{4}z^{-2} - \frac{5}{8}z^{-3} + \frac{5}{16}z^{-4} \tag{5.5}$$

For $n < 5$

$$h(n) = \left\{1, -\frac{1}{2}, \frac{5}{4}, -\frac{5}{8}, \frac{5}{16}\right\} \tag{5.6}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.7}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: From (4.22),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.8}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.9}$$

using (4.29) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.

Solution: The following code plots Fig. 5.3.

```
wget https://raw.
    githubusercontent.com/
    gadepall/EE1310/master/
    filter/codes/hn.py
```

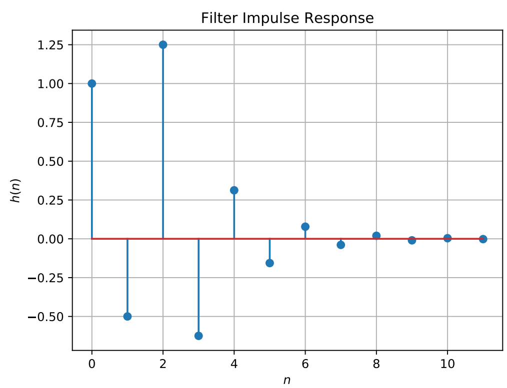

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

Yes, it is bounded. The supremum is $\frac{5}{4}$ and the infrimum is $-\frac{5}{8}$ as all other points of the sequence lie in between these two which can be easily observed.

5.4 Convergent? Justify using the ratio test.
Solution: $h(n)$ is:

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.10}$$

Ratio test:

$$L = \lim_{n\to\infty} \left|\frac{h(n+1)}{h(n)}\right| = \left|\frac{\left(-\frac{1}{2}\right)^{n+1} + \left(-\frac{1}{2}\right)^{n-1}}{\left(-\frac{1}{2}\right)^{n} + \left(-\frac{1}{2}\right)^{n-2}}\right| \tag{5.11}$$

$$L = \left|\frac{-\frac{1}{2}\left[\left(-\frac{1}{2}\right)^{n} + \left(-\frac{1}{2}\right)^{n-2}\right]}{\left(-\frac{1}{2}\right)^{n} + \left(-\frac{1}{2}\right)^{n-2}}\right| \tag{5.12}$$

$$L = \left|-\frac{1}{2}\right| \tag{5.13}$$

$$L = \frac{1}{2} \tag{5.14}$$

As $L < 1$, the series is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.15}$$

Is the system defined by (3.2) stable for the impulse response in (5.7)?
Solution: $h(n)$ is zero for $n < 0$, hence:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty} h(n) \tag{5.16}$$

$$= 1 - \frac{1}{2} + \frac{5}{4} - \frac{5}{8} + \frac{5}{16} - \frac{5}{32} + \dots \tag{5.17}$$

$$= \frac{1}{2} + \frac{5}{4}\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots\right) \tag{5.18}$$

$$= \frac{1}{2} + \frac{5}{4}\left(\frac{1}{1 + \frac{1}{2}}\right) \tag{5.19}$$

$$= \frac{4}{3} \tag{5.20}$$

$$\implies \sum_{n=-\infty}^{\infty} h(n) = \frac{4}{3} < \infty \tag{5.21}$$

Hence the system is stable.

5.6 Verify the above result using a python code.
Solution: The following Python code verifies above result:

```
wget https://github.com/
    omkar30122001/
    Assign_1/blob/main/
    sumhn.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.22}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://raw.
    githubusercontent.com/
    gadepall/EE1310/master/
    filter/codes/hndef.py
```

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{5.23}$$

Comment. The operation in (5.23) is known as *convolution*.
Solution: The following code plots

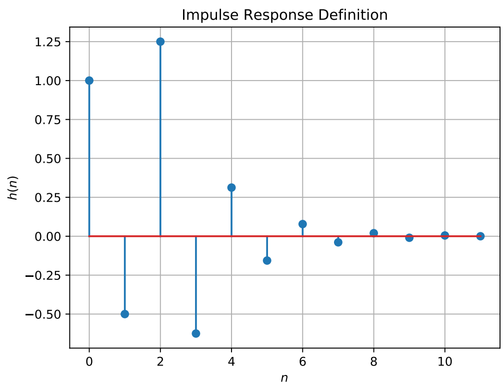

Fig. 5.7: $h(n)$ from the definition

Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/gadepall/EE1310/master/filter/codes/ynconv.py
```

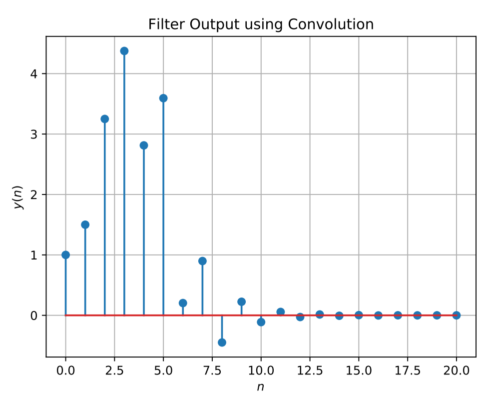

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a Teoplitz matrix.

Solution:

$$y(n) = \begin{pmatrix} h(0) & 0 & 0 & 0 & 0 & 0 \\ h(1) & h(0) & 0 & 0 & 0 & 0 \\ h(2) & h(1) & h(0) & 0 & 0 & 0 \\ h(3) & h(2) & h(1) & h(0) & 0 & 0 \\ h(4) & h(3) & h(2) & h(1) & h(0) & 0 \\ h(5) & h(4) & h(3) & h(2) & h(1) & h(0) \\ h(6) & h(5) & h(4) & h(3) & h(2) & h(1) \\ h(7) & h(6) & h(5) & h(4) & h(3) & h(2) \\ h(8) & h(7) & h(6) & h(5) & h(4) & h(3) \\ h(9) & h(8) & h(7) & h(6) & h(5) & h(4) \\ h(10) & h(9) & h(8) & h(7) & h(6) & h(5) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{5.24}$$

$$y(n) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 & 0 & 0 \\ 5/4 & -1/2 & 1 & 0 & 0 & 0 \\ -5/8 & 5/4 & -1/2 & 1 & 0 & 0 \\ 5/16 & -5/8 & 5/4 & -1/2 & 1 & 0 \\ -5/32 & 5/16 & -5/8 & 5/4 & -1/2 & 1 \\ 5/64 & -5/32 & 5/16 & -5/8 & 5/4 & -1/2 \\ -5/128 & 5/64 & -5/32 & 5/16 & -5/8 & 5/4 \\ 5/256 & -5/128 & 5/64 & -5/32 & 5/16 & -5/8 \\ -5/512 & 5/256 & -5/128 & 5/64 & -5/32 & 5/16 \\ 5/1024 & -5/512 & 5/256 & -5/128 & 5/64 & -5/32 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{5.25}$$

$$y(n) = \begin{pmatrix} 1 \\ 3/2 \\ 13/4 \\ 35/8 \\ 35/16 \\ 115/32 \\ 13/64 \\ 115/128 \\ -115/256 \\ 115/512 \\ -115/1024 \end{pmatrix} \tag{5.26}$$

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.27}$$

Solution: From (5.23) we have

$$y(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{5.28}$$

Replace n by $m+k$

$$y(m+k) = \sum_{m+k=-\infty}^{\infty} x(k)h(m) \tag{5.29}$$

Now replace k by $n-m$

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-m)h(m) \quad (5.30)$$

$$\implies \boxed{y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k)} \quad (5.31)$$

6 DFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0,1,\dots,N \quad (6.1)$$

and $H(k)$ using $h(n)$.

6.2 Compute

$$Y(k) = X(k)H(k) \quad (6.2)$$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\cdot e^{\jmath 2\pi kn/N}, \quad n = 0,1,\dots \quad (6.3)$$

Solution: The following code plots Fig. (6.3) and computes $X(k)$ and $Y(k)$. Note that this is the same as $y(n)$ in Fig. (3.2). Download the code using

```
wget https://github.com/
    omkar30122001/
    Assign_1/blob/main
    /6.3.py
```

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: Download the code from

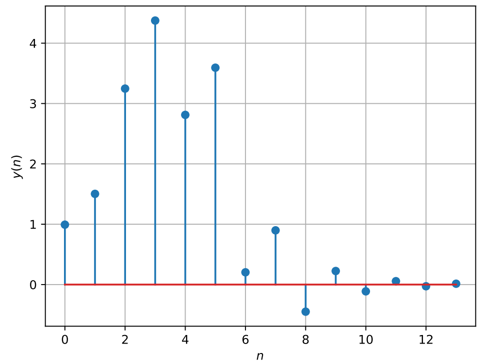

Fig. 6.3: $y(n)$ from the DFT

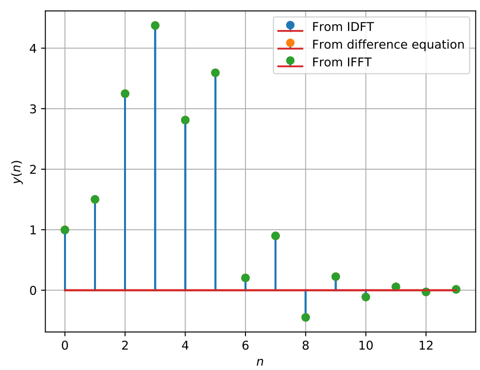

Fig. 6.4: $y(n)$ using FFT and IFFT

```
wget https://github.com/
    omkar30122001/
    Assign_1/blob/main
    /6.4.py
```

The values of $y(n)$ using all the three methods have been plotted on one stem plot for convenience. Note that there is very little difference in the values of $y(n)$.

7 FFT

1. The DFT of $x(n)$ is given by
$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$
2. Let
$$W_N = e^{-j2\pi/N} \tag{7.2}$$
Then the N-point *DFT matrix* is defined as
$$\vec{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.3}$$
where W_N^{mn} are the elements of $\vec{F}_N$.
3. Let
$$\vec{I}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^2 & \vec{e}_4^3 & \vec{e}_4^4 \end{pmatrix} \tag{7.4}$$
be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as
$$\vec{P}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix} \tag{7.5}$$
4. The 4 point *DFT diagonal matrix* is defined as
$$\vec{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$
5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$
Solution:
$$W_N = e^{-\jmath 2\pi/N} \tag{7.8}$$
$$\implies W_{N/2} = e^{-\jmath 4\pi/N} \tag{7.9}$$
And
$$W_N^2 = e^{-\jmath 4\pi/N} \tag{7.10}$$
$$\boxed{W_N^2 = W_{N/2}} \tag{7.11}$$
6. Show that
$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.12}$$
Solution:
$$\vec{F}_2 = \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.13}$$
$$\vec{D}_{4/2} = \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \tag{7.14}$$
$$\text{R.H.S} = \begin{bmatrix} \vec{F}_2 & \vec{D}_{4/2}\vec{F}_2 \\ \vec{F}_2 & -\vec{D}_{4/2}\vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.15}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -i & i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i & -i \end{bmatrix} \vec{P}_4 \tag{7.16}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \tag{7.17}$$
$$\text{LHS} = \vec{F}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \tag{7.18}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \tag{7.19}$$
Hence, LHS = RHS

7. Show that

$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N \tag{7.20}$$

Solution:

$$\vec{F}_N = \begin{bmatrix} W_N^{00} & \cdots & W_N^{0(N-1)} \\ \vdots & & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \tag{7.21}$$

On multiplying by $\vec{P}_N$

$$\vec{F}_N \vec{P}_N = \begin{bmatrix} W_N^{00} & W_N^{02} & \cdots & \cdots & W_N^{01} & W_N^{03} & \cdots \\ \vdots & & & & \vdots & & \\ W_N^{(N-1)0} & W_N^{(N-1)2} & \cdots & \cdots & W_N^{(N-1)1} & W_N^{(N-1)3} & \cdots \end{bmatrix} \tag{7.22}$$

$$= \begin{bmatrix} (W_N^0)^0 & (W_N^0)^2 & \cdots & W_N^1 (W_N^0)^0 & W_N^1 (W_N^0)^2 & \cdots \\ \vdots & & & \vdots & & \\ (W_N^{N-1})^0 & (W_N^{N-1})^2 & \cdots & W_N^{N-1} (W_N^{N-1})^0 & W_{N/2}^{N-1} (W_N^{N-1})^2 & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \left(W_{N/2}^0\right)^0 & \left(W_{N/2}^0\right)^1 & \cdots & W_N^1 \left(W_{N/2}^0\right)^0 & W_{N/2+1}^1 \left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & & & \vdots & & \\ (W_N^{N-1})^0 & (W_N^{N-1})^2 & \cdots & W_N^{N-1} \left(W_{N/2}^{N-1}\right)^0 & W_N^{N-1} \left(W_{N/2}^{N-1}\right)^1 & \cdots \end{bmatrix} \tag{7.23}$$

$$\begin{bmatrix} \left(W_{N/2}^0\right)^0 & \left(W_{N/2}^0\right)^1 & \cdots & W_{N/2}^0 \left(W_{N/2}^0\right)^0 & W_{N/2}^0 \left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & & & \vdots & & \\ \left(W_{N/2}^{N/2-1}\right)^0 & \left(W_{N/2}^{N/2-1}\right)^1 & \cdots & W_N^{N/2-1} \left(W_{N/2}^{N/2+1}\right)^0 & W_N^{N/2-1} \left(W_{N/2}^{N/2-1}\right)^1 & \cdots \\ \left(W_{N/2}^{N/2+0}\right)^0 & \left(W_{N/2}^{N/2+0}\right)^1 & \cdots & W_N^{N/2+0} \left(W_{N/2}^{N/2+0}\right)^0 & W_N^{N/2} \left(W_{N/2}^{N/2+0}\right)^1 & \cdots \\ \vdots & & & \vdots & & \\ (W_N^{N-1})^0 & (W_N^{N-1})^2 & \cdots & W_N^{N/2+(N/2-1)} \left(W_{N/2}^{N-1}\right)^0 & W_{N/2+(N/2-1)}^{N-1} \left(W_{N/2}^{N-1}\right)^1 & \cdots \end{bmatrix} \tag{7.24}$$

$$W_N^{N/2+k} = -W_N^k \tag{7.25}$$

$$W_{N/2}^{N/2+k} = W_{N/2}^k \tag{7.26}$$

$$\begin{bmatrix} \left(W_{N/2}^0\right)^0 & \left(W_{N/2}^0\right)^1 & \cdots & W_{N/2}^0 \left(W_{N/2}^0\right)^0 & W_{N/2}^0 \left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & & & \vdots & & \\ \left(W_{N/2}^{N/2-1}\right)^0 & \left(W_{N/2}^{N/2-1}\right)^1 & \cdots & W_N^{N/2-1} \left(W_{N/2}^{N/2+1}\right)^0 & W_N^{N/2-1} \left(W_{N/2}^{N/2-1}\right)^1 & \cdots \\ \left(W_{N/2}^0\right)^0 & \left(W_{N/2}^0\right)^1 & \cdots & -W_N^0 \left(W_{N/2}^0\right)^0 & -W_N^0 \left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & & & \vdots & & \\ (W_N^{N-1})^0 & (W_N^{N-1})^2 & \cdots & -W_N^{N/2-1} \left(W_{N/2}^{N/2-1}\right)^0 & -W_{(N/2-1)}^{N-1} \left(W_{N/2}^{N/2-1}\right)^1 & \cdots \end{bmatrix} \tag{7.27}$$

$$\begin{bmatrix} \begin{bmatrix} \left(W_{N/2}^0\right)^0 & \left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & \vdots & \\ \left(W_{N/2}^{N/2-1}\right)^0 & \left(W_{N/2}^{N/2-1}\right)^1 & \cdots \end{bmatrix} & \begin{bmatrix} W_{N/2}^0 \left(W_{N/2}^0\right)^0 & W_{N/2}^0 \left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & \vdots & \\ W_N^{N/2-1} \left(W_{N/2}^{N/2-1}\right)^0 & W_N^{N/2-1} \left(W_{N/2}^{N/2-1}\right)^1 & \cdots \end{bmatrix} \\ \begin{bmatrix} \left(W_{N/2}^0\right)^0 & \left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & \vdots & \\ \left(W_{N/2}^{N/2-1}\right)^0 & \left(W_{N/2}^{N/2-1}\right)^1 & \cdots \end{bmatrix} & -\begin{bmatrix} W_{N/2}^0 \left(W_{N/2}^0\right)^0 & W_{N/2}^0 \left(W_{N/2}^0\right)^1 & \cdots \\ \vdots & \vdots & \\ W_N^{N/2-1} \left(W_{N/2}^{N/2-1}\right)^0 & W_N^{N/2-1} \left(W_{N/2}^{N/2-1}\right)^1 & \cdots \end{bmatrix} \end{bmatrix} \tag{7.28}$$

$$= \begin{bmatrix} \vec{F}_{N/2} & \vec{D}_{N/2} \vec{F}_{N/2} \\ \vec{F}_{N/2} & -\vec{D}_{N/2} \vec{F}_{N/2} \end{bmatrix} \tag{7.29}$$

Where

$$\vec{D}_{N/2} = diag\left(W_N^0 \quad \cdots \quad W_N^{N/2-1}\right) \tag{7.30}$$

$$\vec{F}_{N/2} = \begin{bmatrix} \left(W_{N/2}^0\right)^0 & \left(W_{N/2}^0\right)^1 & \dots \\ \vdots & \vdots & \\ \left(W_{N/2}^{N/2-1}\right)^0 & \left(W_{N/2}^{N/2-1}\right)^1 & \dots \end{bmatrix} \tag{7.31}$$

$$\implies \vec{F}_N\vec{P}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \tag{7.32}$$

Multiplying both side by $\vec{P}_N$ as $\left(\vec{P}_N\right)^2 = \vec{I}$

$$\boxed{\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N} \tag{7.33}$$

8. Find

$$\vec{P}_4\vec{x} \tag{7.34}$$

Solution:

$$\vec{P}_4 = \left(\vec{e}_4^{\,1} \quad \vec{e}_4^{\,3} \quad \vec{e}_4^{\,2} \quad \vec{e}_4^{\,4}\right) \tag{7.35}$$

From (3.1)

$$\vec{x} = \{1, 2, 3, 4\} \tag{7.36}$$

$$\vec{P}_4\vec{x} = (1, 3, 2, 4) \tag{7.37}$$

9. Show that

$$\vec{X} = \vec{F}_N\vec{x} \tag{7.38}$$

where $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution:

$$\left(\vec{F}_N\vec{x}\right)_k = \sum_{m=0}^{N-1} W_N^{mk} x(m) \tag{7.39}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi km/N} = X(k) = \vec{X}_k \tag{7.40}$$

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.41}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.42}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.43}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.44}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.45}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.46}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.47}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.48}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.49}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.50}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.51}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.52}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.53}$$

Solution: We write out the values of performing an 8-point FFT on $\vec{x}$ as follows.

$$X(k) = \sum_{n=0}^{7} x(n)e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.54}$$

$$= \sum_{n=0}^{3} \left(x(2n)e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x(2n+1)e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.55}$$

$$= X_1(k) + e^{-\frac{\jmath 2k\pi}{4}} X_2(k) \tag{7.56}$$

where $\vec{X}_1$ is the 4-point FFT of the even-numbered terms and $\vec{X}_2$ is the 4-point FFT of the odd numbered terms. Noticing that for $k \geq 4$,

$$X_1(k) = X_1(k-4) \tag{7.57}$$

$$e^{-\frac{\jmath 2k\pi}{8}} = -e^{-\frac{\jmath 2(k-4)\pi}{8}} \tag{7.58}$$

we can now write out $X(k)$ in matrix form as in (**??**) and (**??**). We also need to solve the two 4-point FFT terms so formed.

$$X_1(k) = \sum_{n=0}^{3} x_1(n)e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.59}$$

$$= \sum_{n=0}^{1} \left(x_1(2n)e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x_2(2n+1)e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.60}$$

$$= X_3(k) + e^{-\frac{\jmath 2k\pi}{4}} X_4(k) \tag{7.61}$$

using $x_1(n) = x(2n)$ and $x_2(n) = x(2n+1)$. Thus we can write the 2-point FFTs

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.62}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.63}$$

Using a similar idea for the terms

X_2,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.64}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.65}$$

But observe that from (**??**),

$$\vec{P}_8 \vec{x} = \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \end{pmatrix} \tag{7.66}$$

$$\vec{P}_4 \vec{x}_1 = \begin{pmatrix} \vec{x}_3 \\ \vec{x}_4 \end{pmatrix} \tag{7.67}$$

$$\vec{P}_4 \vec{x}_2 = \begin{pmatrix} \vec{x}_5 \\ \vec{x}_6 \end{pmatrix} \tag{7.68}$$

where we define $x_3(k) = x(4k)$, $x_4(k) = x(4k+2)$, $x_5(k) = x(4k+1)$, and $x_6(k) = x(4k+3)$ for $k = 0, 1$.

11. For

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.69}$$

compte the DFT using (7.38)
Solution:

$$\vec{X} = \vec{F}_6 \vec{x} \tag{7.70}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right) & \left(e^{\frac{-j2\pi}{6}}\right)^2 & \left(e^{\frac{-j2\pi}{6}}\right)^3 & \left(e^{\frac{-j2\pi}{6}}\right)^4 & \left(e^{\frac{-j2\pi}{6}}\right)^5 \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^2 & \left(e^{\frac{-j2\pi}{6}}\right)^4 & \left(e^{\frac{-j2\pi}{6}}\right)^6 & \left(e^{\frac{-j2\pi}{6}}\right)^8 & \left(e^{\frac{-j2\pi}{6}}\right)^{10} \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^3 & \left(e^{\frac{-j2\pi}{6}}\right)^6 & \left(e^{\frac{-j2\pi}{6}}\right)^9 & \left(e^{\frac{-j2\pi}{6}}\right)^{12} & \left(e^{\frac{-j2\pi}{6}}\right)^{15} \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^4 & \left(e^{\frac{-j2\pi}{6}}\right)^8 & \left(e^{\frac{-j2\pi}{6}}\right)^{12} & \left(e^{\frac{-j2\pi}{6}}\right)^{16} & \left(e^{\frac{-j2\pi}{6}}\right)^{20} \\ 1 & \left(e^{\frac{-j2\pi}{6}}\right)^5 & \left(e^{\frac{-j2\pi}{6}}\right)^{10} & \left(e^{\frac{-j2\pi}{6}}\right)^{15} & \left(e^{\frac{-j2\pi}{6}}\right)^{20} & \left(e^{\frac{-j2\pi}{6}}\right)^{25} \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.71}$$

$$= \begin{pmatrix} 13 \\ -4 - \sqrt{3}j \\ 1 \\ -1 \\ 1 \\ -4 + \sqrt{3}j \end{pmatrix} \tag{7.72}$$

12. Repeat the above exercise using the FFT after zero padding $\vec{x}$.
Solution:
The code:

```
wget https://github.com/
    omkar30122001/
    Assign_1/blob/main
    /7.12.py
```

The result:

$$\begin{bmatrix} 13 \\ -3.1213 - 6.5355j \\ j \\ 1.1213 - 0.5355j \\ -1 \\ 1.1213 + 0.5355j \\ -j \\ -3.1213 + 6.5355j \end{bmatrix} \tag{7.73}$$

13. Write a C program to compute the 8-point FFT.
Solution:
The code:

```
wget https://github.com/
    omkar30122001/
    Assign_1/blob/main
    /7.13.c
```

The output:

$$\begin{bmatrix} 13 \\ -3.1327 - j6.5545 \\ j \\ 1.1327 - j0.5545 \\ -1 \\ 1.1327 + j0.5545 \\ -j \\ -3.1327 + j6.5545 \end{bmatrix} \tag{7.74}$$

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.
    lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify. **Solution:**

```
wget https://github.com/
    omkar30122001/
    Assign_1/blob/main
    /8.1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b. **Solution:** For the given values, the difference equation is

$$\begin{aligned} & y(n) - (4.44)\,y(n-1) + (8.78)\,y(n-2) \\ & - (9.93)\,y(n-3) + (6.90)\,y(n-4) \\ & - (2.93)\,y(n-5) + (0.70)\,y(n-6) \\ & - (0.07)\,y(n-7) = \left(5.02\times10^{-5}\right)x(n) \\ & + \left(3.52\times10^{-4}\right)x(n-1) + \left(1.05\times10^{-3}\right)x(n-2) \\ & + \left(1.76\times10^{-3}\right)x(n-3) + \left(1.76\times10^{-3}\right)x(n-4) \\ & + \left(1.05\times10^{-3}\right)x(n-5) + \left(3.52\times10^{-4}\right)x(n-6) \\ & + \left(5.02\times10^{-5}\right)x(n-7) \end{aligned} \tag{8.2}$$

From (8.1), we see that the transfer function can be written as follows

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{\sum_{k=0}^{M} a(k) z^{-k}} \tag{8.3}$$

$$= \sum_i \frac{r(i)}{1 - p(i) z^{-1}} + \sum_j k(j) z^{-j} \tag{8.4}$$

where $r(i)$, $p(i)$, are called residues and poles respectively of the partial fraction expansion of $H(z)$. $k(i)$ are the coefficients of the direct polynomial terms that might be left over. We can now take the inverse z-transform of (8.4) and get using (4.29),

$$h(n) = \sum_i r(i)[p(i)]^n u(n) + \sum_j k(j)\delta(n-j) \tag{8.5}$$

Substituting the values,

$$h(n) = [(2.76)(0.55)^n + (-1.05 - 1.84\jmath)(0.57 + 0.16\jmath)^n + (-1.05 + 1.84\jmath)(0.57 - 0.16\jmath)^n + (-0.53 + 0.08\jmath)(0.63 + 0.32\jmath)^n + (-0.53 - 0.08\jmath)(0.63 - 0.32\jmath)^n + (0.20 + 0.004\jmath)(0.75 + 0.47\jmath)^n + (0.20 - 0.004\jmath)(0.75 - 0.47\jmath)^n]u(n) + \left(-6.81 \times 10^{-4}\right)\delta(n) \tag{8.6}$$

The values $r(i)$, $p(i)$, $k(i)$ and thus the impulse response function are computed and plotted at

```
wget https://github.com/
    omkar30122001/
    Assign_1/blob/main
    /8.2.1.py
```

The filter frequency response is plotted at

```
wget https://github.com/
    omkar30122001/
    Assign_1/blob/main
    /8.2.2.py
```

Observe that for a series $t_n = r^n$, $\frac{t_{n+1}}{t_n} = r$. By the ratio test, t_n converges if $|r| < 1$. We observe that for all i, $|p(i)| < 1$ and so, as $h(n)$ is the sum of many convergent series, we see that $h(n)$ converges and is bounded.

$$\sum_{n=0}^{\infty} h(n) = H(1) = \frac{\sum_{k=0}^{N} b(k)}{\sum_{k=0}^{M} a(k)} = 1 < \infty \tag{8.7}$$

Therefore, the system is stable. From Fig. (8.4), $h(n)$ is negligible after $n \geq 64$, and we can apply a 64-bit FFT to get y(n). The following code uses the DFT matrix to generate $y(n)$ in Fig. (8.4).

```
wget https://github.com/
    omkar30122001/
    Assign_1/blob/main
    /8.2.3.py
```

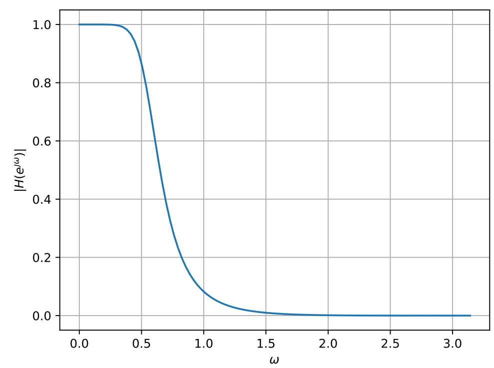

Fig. 8.4: Plot of $H(e^{\jmath\omega})$

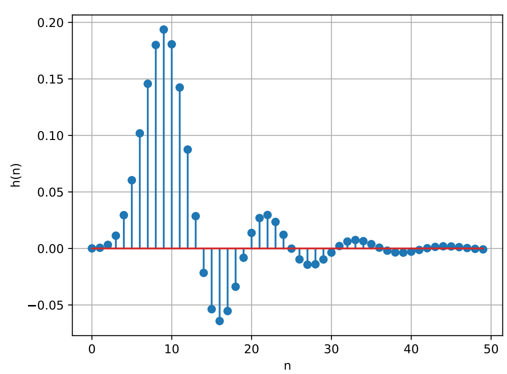

Fig. 8.4: Filter frequency response

8.3 What is the sampling frequency of the input signal?

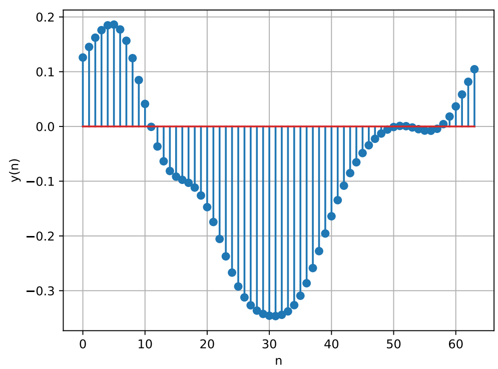

Fig. 8.4: Plot of $y(n)$

Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.
8.5 Modifying the code with different input parameters and to get the best possible output.
Solution: We make the order of the filter = 7
The code:

```
wget https://github.com/
    omkar30122001/
    Assign_1/blob/main
    /8.5.py
```