Pingala Series

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1
$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

1.2
$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

1.3
$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

1.4
$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

All four problems are verified in the following code:

```
/home/ubuntu/Desktop/Pingala
Assignment/1.py
```

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2}$$

Generate a stem plot for $x(n)$.
Solution: The following code plots $x(n)$:

```
/home/ubuntu/Desktop/Pingala
Assignment/2.2.py
```

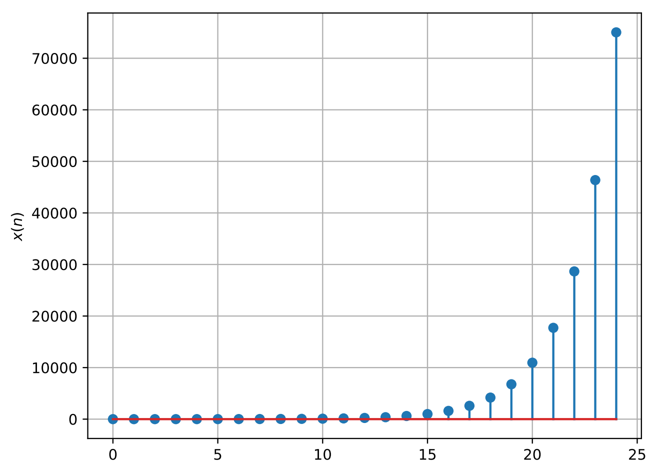

Fig. 2.2: Plot of $x(n)$

2.3 Find $X^+(z)$.
Solution: Taking one sided Z-Transform on both sides of eq. 2.2

$$\mathcal{Z}^+ [x(n+2)] = \mathcal{Z}^+ [x(n+1)] + \mathcal{Z}^+ [x(n)] \tag{2.3}$$

$$z^2 X^+(z) - z^2 x(0) - z x(1) = z X^+(z) - z x(0) + X^+(z) \tag{2.4}$$

$$\left(z^2 - z - 1\right) X^+(z) = z^2 \tag{2.5}$$

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.6}$$

$$\boxed{X^+(z) = \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})}, \quad |z| > \alpha} \tag{2.7}$$

2.4 Find $x(n)$.
Solution: Expanding $X^+(z)$ using partial fractions

$$X^+(z) = \frac{1}{(\alpha - \beta) z^{-1}} \left[\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}} \right] \tag{2.8}$$

$$X^+(z) = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} (\alpha^n - \beta^n) z^{-n+1} \tag{2.9}$$

$$X^+(z) = \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} z^{-n+1} \tag{2.10}$$

$$X^+(z) = \sum_{k=0}^{\infty} \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} z^{-k} \tag{2.11}$$

Thus

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) \tag{2.12}$$

$$\boxed{x(n) = a_{n+1} u(n)} \tag{2.13}$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \tag{2.14}$$

Solution: Following code plots $y(n)$

/home/ubuntu/Desktop/Pingala Assignment/2.5.py

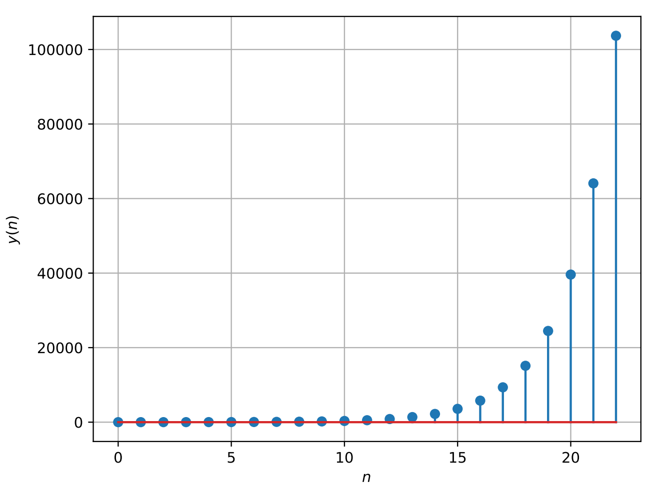

Fig. 2.5: Plot of $y(n)$

2.6 Find $Y^+(z)$.
Solution: Take one sided Z-Transform on both sides of eq. 2.14

$$\mathcal{Z}^+ [y(n)] = \mathcal{Z}^+ [x(n+1)] + \mathcal{Z}^+ [x(n-1)] \tag{2.15}$$

$$Y^+(z) = zX^+(z) - zx(0) + z^{-1}X^+(z) + zx(-1) \tag{2.16}$$

$$Y^+(z) = \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \tag{2.17}$$

$$\boxed{Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > \alpha} \tag{2.18}$$

2.7 Find $y(n)$.
Solution: From eq. 2.14

$$y(n) = [x(n-1) + x(n+1)] u(n) \tag{2.19}$$

Using eq. 2.13

$$y(n) = [a_n + a_{n+2}] u(n) \tag{2.20}$$

Using eq. 1.2

$$y(n) = b_{n+1} u(n) \tag{2.21}$$

Using eq. 1.5 (As it is verified in the code)

$$\boxed{y(n) = \left[\alpha^{n+1} + \beta^{n+1}\right] u(n)} \tag{2.22}$$

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \tag{3.1}$$

Solution:

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \tag{3.2}$$

$$\sum_{k=1}^{n} a_k = \sum_{k=-\infty}^{n-1} x(k) \tag{3.3}$$

$$\sum_{k=1}^{n} a_k = \sum_{k=-\infty}^{\infty} x(k) u(n-1-k) \tag{3.4}$$

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \tag{3.5}$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.6}$$

can be expressed as

$$[x(n+1)-1]\,u(n) \tag{3.7}$$

Solution: From eq. 2.13

$$a_{n+2}-1 = [x(n+1)-1], \quad n \geq 0 \tag{3.8}$$

Using the definition of $u(n)$

$$\boxed{a_{n+2}-1 = [x(n+1)-1]\,u(n)} \tag{3.9}$$

3.3 Show that

$$\sum_{k=1}^{\infty}\frac{a_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty}\frac{x(k)}{10^k} = \frac{1}{10}X^{+}(10) \tag{3.10}$$

Solution:

$$\sum_{k=1}^{\infty}\frac{a_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty}\frac{a_{k+1}}{10^k} \tag{3.11}$$

$$= \frac{1}{10}\sum_{k=0}^{\infty}\frac{x(k)}{10^k} \tag{3.12}$$

Using the definition of one sided Z-Transform

$$\boxed{\sum_{k=1}^{\infty}\frac{a_k}{10^k} = \frac{1}{10}X^{+}(10)} \tag{3.13}$$

3.4 Show that

$$\alpha^n+\beta^n, \quad n \geq 1 \tag{3.14}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1}+\beta^{n+1}\right)u(n) \tag{3.15}$$

and find $W(z)$.
Solution: In eq. 3.14, put $n = k+1$ and from the definition of $u(n)$, we get

$$\alpha^n+\beta^n = \left(\alpha^{k+1}+\beta^{k+1}\right)u(k) \tag{3.16}$$

Hence 3.14 can be expressed as

$$w(n) = \left(\alpha^{n+1}+\beta^{n+1}\right)u(n) \tag{3.17}$$

And $w(n)$ is same as $y(n)$. Hence

$$\boxed{W(z) = Y(z) = \frac{1+2z^{-1}}{1-z^{-1}-z^{-2}}} \tag{3.18}$$

3.5 Show that

$$\sum_{k=1}^{\infty}\frac{b_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty}\frac{y(k)}{10^k} = \frac{1}{10}Y^{+}(10) \tag{3.19}$$

Solution:

$$\sum_{k=1}^{\infty}\frac{b_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty}\frac{b_{k+1}}{10^k} \tag{3.20}$$

$$= \frac{1}{10}\sum_{k=0}^{\infty}\frac{y(k)}{10^k} \tag{3.21}$$

Using the definition of one sided Z-Transform

$$\boxed{\sum_{k=1}^{\infty}\frac{b_k}{10^k} = \frac{1}{10}Y^{+}(10)} \tag{3.22}$$

3.6 Solve the JEE 2019 problem.
Solution: We know that

$$\sum_{k=1}^{n}a_k = x(n)*u(n-1) \tag{3.23}$$

But

$$x(n)*u(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} X(z)z^{-1}U(z) \tag{3.24}$$

$$= \frac{z^{-1}}{(1-z^{-1}-z^{-2})(1-z^{-1})} \tag{3.25}$$

$$= z\left[\frac{1}{1-z^{-1}-z^{-2}} - \frac{1}{1-z^{-1}}\right] \tag{3.26}$$

$$\overset{\mathcal{Z}}{\rightleftharpoons} z\sum_{n=0}^{\infty}[x(n)-1]\,z^{-n} \tag{3.27}$$

$$= \sum_{n=0}^{\infty}[x(n)-1]\,z^{-n+1} \tag{3.28}$$

$$= \sum_{n=0}^{\infty}[x(n+1)-1]\,z^{-n} \tag{3.29}$$

From eq. 2.13, we get

$$\sum_{k=1}^{n}a_k = a_{n+2}-1 \tag{3.30}$$

We have already established the remaining options in order in the problems 3.3, 2.7 and 3.5
Therefore, options 1, 2 and 3 are correct and option 4 is incorrect.