

# Pingala Series

Omkar Raijade

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**Abstract**—This manual provides a simple introduction to Transforms

### 1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \quad (1.1)$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \quad (1.2)$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^n a_k = a_{n+2} - 1, \quad n \geq 1 \quad (1.3)$$

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \quad (1.4)$$

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \quad (1.5)$$

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \quad (1.6)$$

All four problems are verified in the following code:

```
wget https://github.com/omkar30122001/
pingala/blob/main/1.py
```

## 2 PINGALA SERIES

2.1 The *one sided* Z-transform of  $x(n)$  is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \quad (2.1)$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \quad (2.2)$$

Generate a stem plot for  $x(n)$ .

**Solution:** The following code plots  $x(n)$ :

```
wget https://github.com/omkar30122001/
pingala/blob/main/2.2.py
```

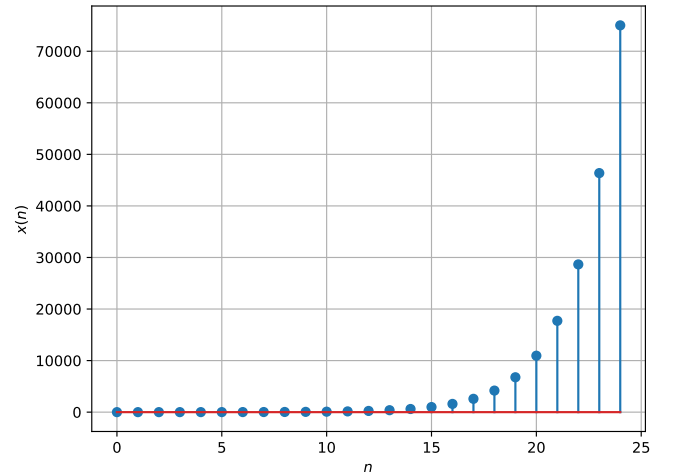


Fig. 2.2: Plot of  $x(n)$

2.3 Find  $X^+(z)$ .

**Solution:** Taking one sided Z-Transform on both sides of eq. 2.2

$$\mathcal{Z}^+[x(n+2)] = \mathcal{Z}^+[x(n+1)] + \mathcal{Z}^+[x(n)] \quad (2.3)$$

$$z^2 X^+(z) - z^2 x(0) - z x(1) = z X^+(z) - z x(0) + X^+(z) \quad (2.4)$$

$$(z^2 - z - 1) X^+(z) = z^2 \quad (2.5)$$

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \quad (2.6)$$

$$X^+(z) = \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})}, \quad |z| > \alpha \quad (2.7)$$

2.4 Find  $x(n)$ .

**Solution:** Expanding  $X^+(z)$  using partial fractions

$$X^+(z) = \frac{1}{(\alpha - \beta)z^{-1}} \left[ \frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}} \right] \quad (2.8)$$

$$X^+(z) = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} (\alpha^n - \beta^n) z^{-n+1} \quad (2.9)$$

$$X^+(z) = \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} z^{-n+1} \quad (2.10)$$

$$X^+(z) = \sum_{k=0}^{\infty} \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} z^{-k} \quad (2.11)$$

Thus

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) \quad (2.12)$$

$$x(n) = a_{n+1} u(n) \quad (2.13)$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \quad (2.14)$$

**Solution:** Following code plots  $y(n)$

```
wget https://github.com/omkar30122001/
pingala/blob/main/2.5.py
```

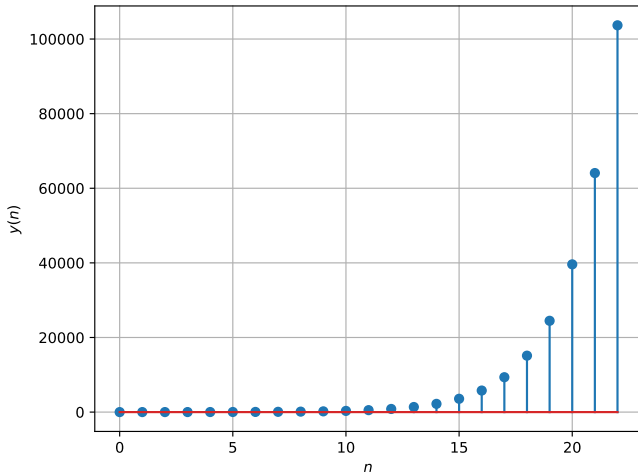


Fig. 2.5: Plot of  $y(n)$

2.6 Find  $Y^+(z)$ .

**Solution:** Take one sided Z-Transform on both sides of eq. 2.14

$$\mathcal{Z}^+[y(n)] = \mathcal{Z}^+[x(n+1)] + \mathcal{Z}^+[x(n-1)] \quad (2.15)$$

$$Y^+(z) = zX^+(z) - zx(0) + z^{-1}X^+(z) + zx(-1) \quad (2.16)$$

$$Y^+(z) = \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \quad (2.17)$$

$$Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > \alpha \quad (2.18)$$

2.7 Find  $y(n)$ .

**Solution:** From eq. 2.14

$$y(n) = [x(n-1) + x(n+1)] u(n) \quad (2.19)$$

Using eq. 2.13

$$y(n) = [a_n + a_{n+2}] u(n) \quad (2.20)$$

Using eq. 1.2

$$y(n) = b_{n+1} u(n) \quad (2.21)$$

Using eq. 1.5 (As it is verified in the code)

$$y(n) = [\alpha^{n+1} + \beta^{n+1}] u(n) \quad (2.22)$$

### 3 POWER OF THE Z TRANSFORM

3.1 Show that

$$\sum_{k=1}^n a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \quad (3.1)$$

**Solution:**

$$\sum_{k=1}^n a_k = \sum_{k=0}^{n-1} x(k) \quad (3.2)$$

$$\sum_{k=1}^n a_k = \sum_{k=-\infty}^{n-1} x(k) \quad (3.3)$$

$$\sum_{k=1}^n a_k = \sum_{k=-\infty}^{\infty} x(k) u(n-1-k) \quad (3.4)$$

$$\sum_{k=1}^n a_k = x(n) * u(n-1) \quad (3.5)$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \quad (3.6)$$

can be expressed as

$$[x(n+1) - 1]u(n) \quad (3.7)$$

**Solution:** From eq. 2.13

$$a_{n+2} - 1 = [x(n+1) - 1], \quad n \geq 0 \quad (3.8)$$

Using the definition of  $u(n)$

$$a_{n+2} - 1 = [x(n+1) - 1]u(n) \quad (3.9)$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \quad (3.10)$$

**Solution:**

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \quad (3.11)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \quad (3.12)$$

Using the definition of one sided Z-Transform

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} X^+(10) \quad (3.13)$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \quad (3.14)$$

can be expressed as

$$w(n) = (\alpha^{n+1} + \beta^{n+1})u(n) \quad (3.15)$$

and find  $W(z)$ .

**Solution:** In eq. 3.14, put  $n = k + 1$  and from the definition of  $u(n)$ , we get

$$\alpha^n + \beta^n = (\alpha^{k+1} + \beta^{k+1})u(k) \quad (3.16)$$

Hence 3.14 can be expressed as

$$w(n) = (\alpha^{n+1} + \beta^{n+1})u(n) \quad (3.17)$$

And  $w(n)$  is same as  $y(n)$ . Hence

$$W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad (3.18)$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \quad (3.19)$$

**Solution:**

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \quad (3.20)$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \quad (3.21)$$

Using the definition of one sided Z-Transform

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} Y^+(10) \quad (3.22)$$

3.6 Solve the JEE 2019 problem.

**Solution:** We know that

$$\sum_{k=1}^n a_k = x(n) * u(n-1) \quad (3.23)$$

But

$$x(n) * u(n-1) \stackrel{Z}{\rightleftharpoons} X(z)z^{-1}U(z) \quad (3.24)$$

$$= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \quad (3.25)$$

$$= z \left[ \frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right] \quad (3.26)$$

$$\stackrel{Z}{\rightleftharpoons} z \sum_{n=0}^{\infty} [x(n) - 1] z^{-n} \quad (3.27)$$

$$= \sum_{n=0}^{\infty} [x(n) - 1] z^{-n+1} \quad (3.28)$$

$$= \sum_{n=0}^{\infty} [x(n+1) - 1] z^{-n} \quad (3.29)$$

From eq. 2.13, we get

$$\sum_{k=1}^n a_k = a_{n+2} - 1 \quad (3.30)$$

We have already established the remaining options in order in the problems 3.3, 2.7 and 3.5

Therefore, options 1, 2 and 3 are correct and option 4 is incorrect.