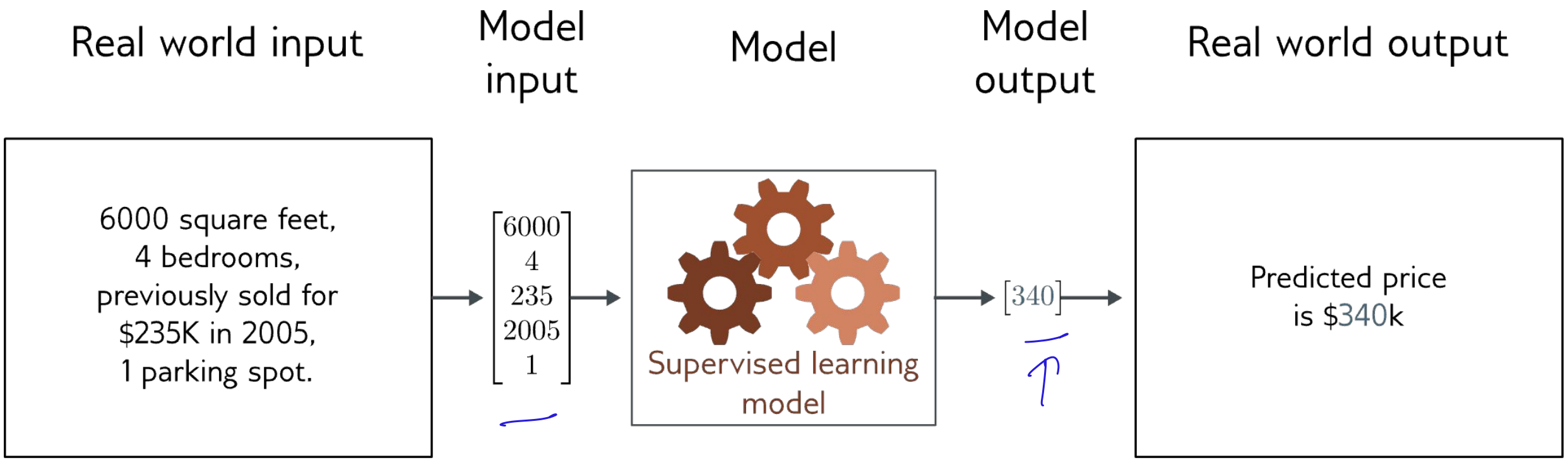
Shallow Neural Networks

January 8-9th, 2025

Deep Learning (CS60010)

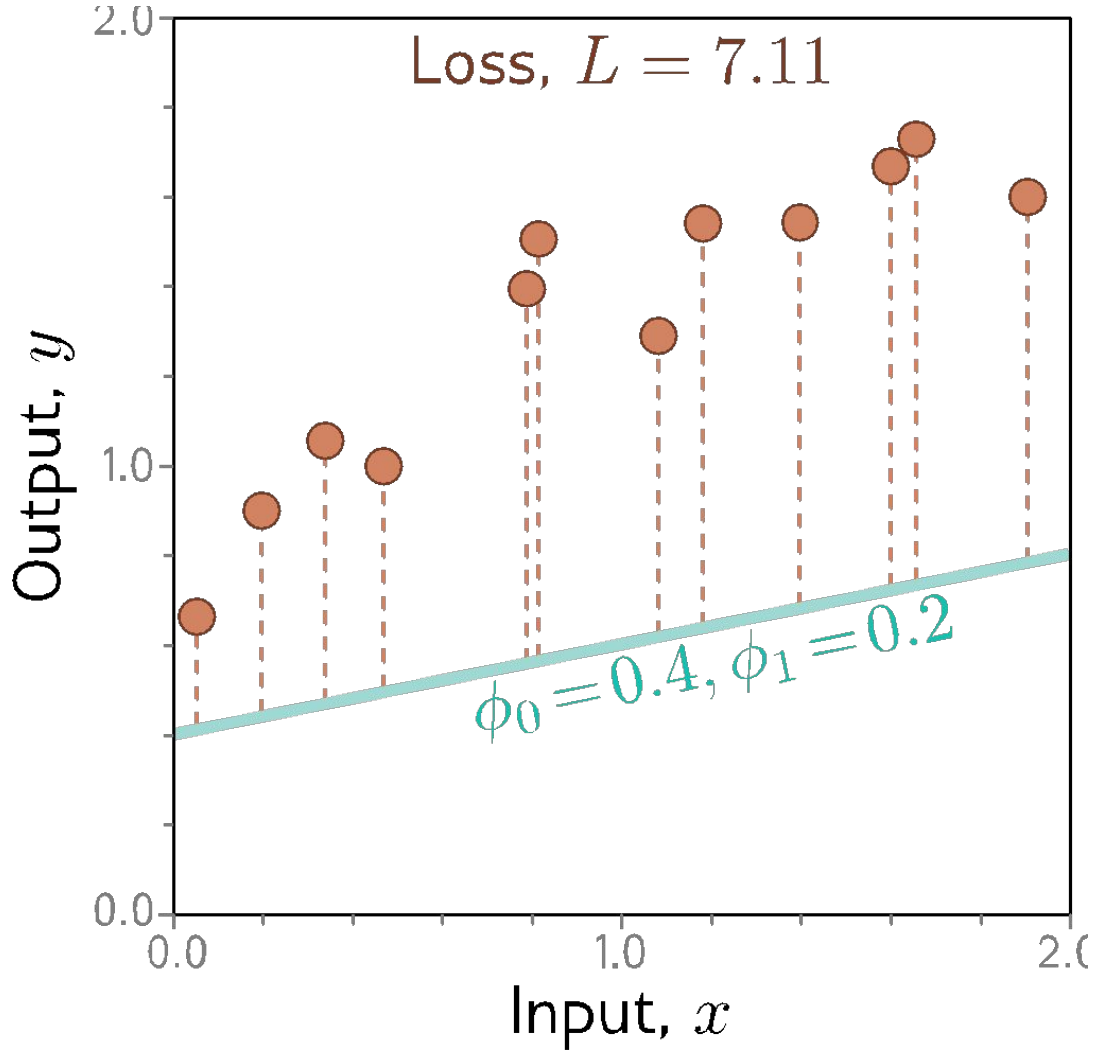
**Slides adapted from http://udlbook.com*

Regression



- Univariate regression problem (one output, real value)
- Fully connected network

Example: 1D Linear regression loss function



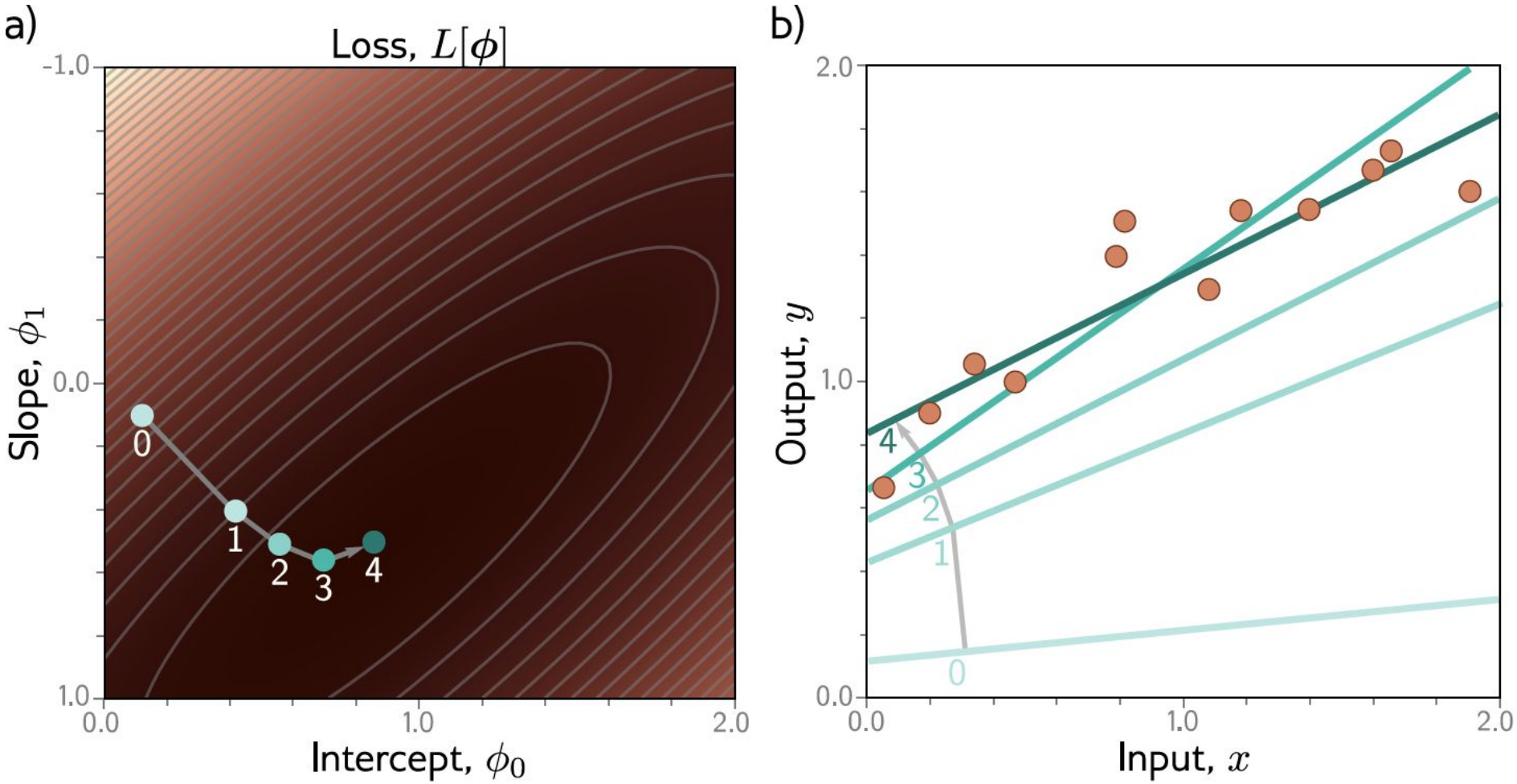
Loss function:

$$L[\boldsymbol{\phi}] = \sum_{i=1}^{I} (\mathrm{f}[x_i, \boldsymbol{\phi}] - y_i)^2$$

$$= \sum_{i=1}^{I} (\phi_0 + \phi_1 x_i - y_i)^2$$

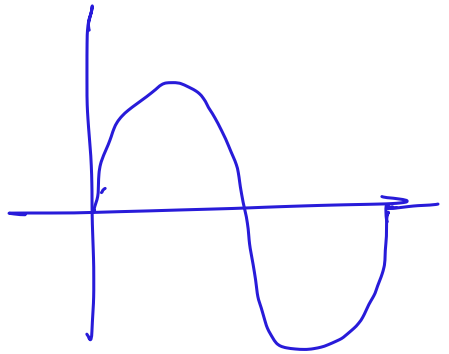
"Least squares loss function"

Example: 1D Linear regression training



This technique is known as gradient descent

Shallow neural networks



- 1D regression model is obviously limited
  - Want to be able to describe input/output that are not lines
  - Want multiple inputs
  - Want multiple outputs

- Shallow neural networks
  - Flexible enough to describe arbitrarily complex input/output mappings
  - Can have as many inputs as we want
  - Can have as many outputs as we want

Shallow neural networks

- Example network, 1 input, 1 output
- Universal approximation theorem
- More than one output
- More than one input
- General case
- Number of regions
- Terminology

1D Linear Regression

$$y = \mathrm{f}[x, \boldsymbol{\phi}]$$
$$= \phi_0 + \phi_1 x$$

Example shallow network

$$y = \mathrm{f}[x, \boldsymbol{\phi}]$$
$$= \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11} x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21} x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31} x]$$

Number of hidden layers?

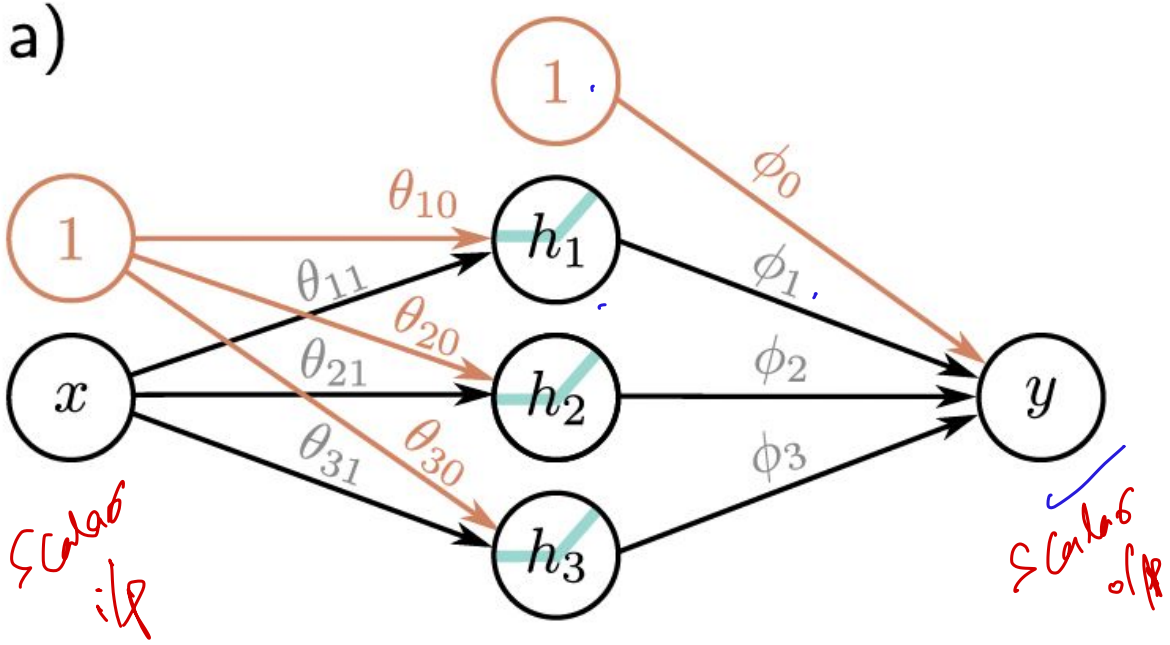
Example shallow network

$$h_1 = a[\theta_{10} + \theta_{11} x]$$

$$\phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$

$$y = f[x, \boldsymbol{\phi}]$$
$$= \phi_0 + \phi_1 a[\theta_{10} + \theta_{11} x] + \phi_2 a[\theta_{20} + \theta_{21} x] + \phi_3 a[\theta_{30} + \theta_{31} x]$$

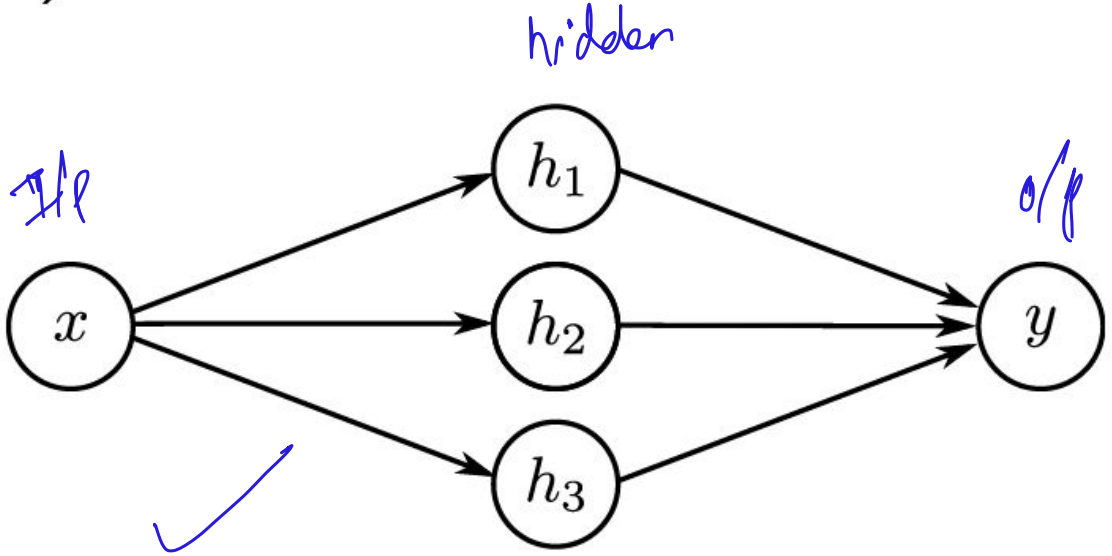
a)



Scalar i/p

Scalar o/p

b)



Example shallow network

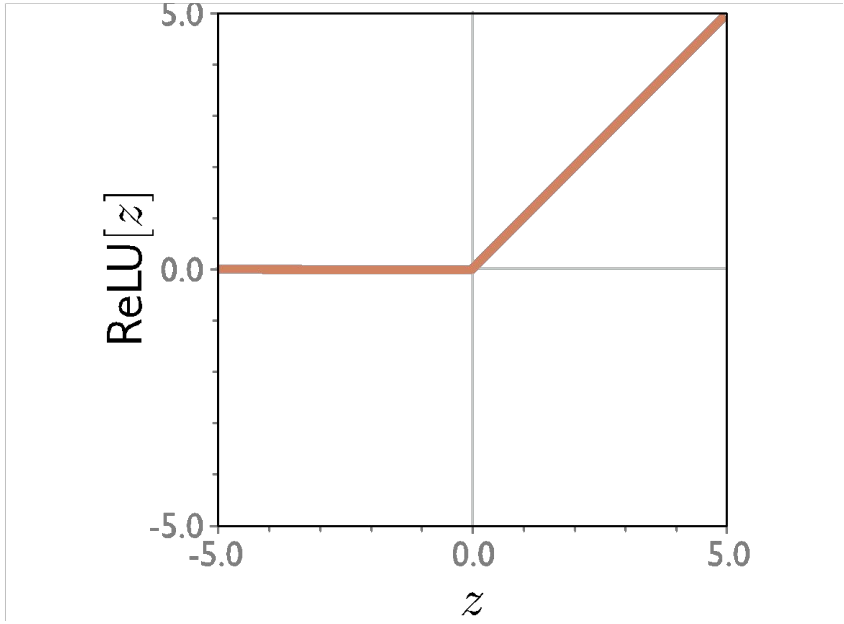
Activation function

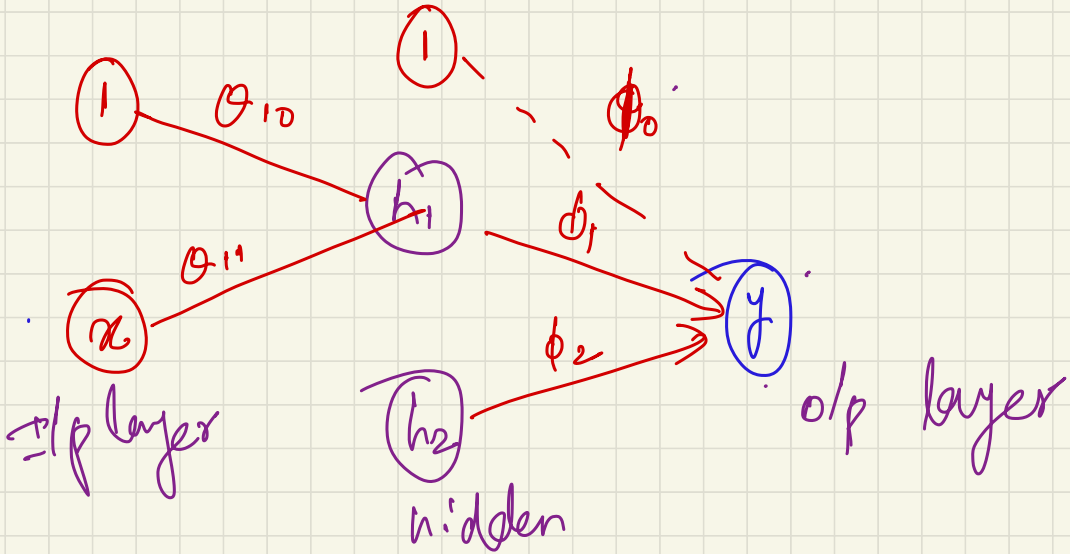
$$y = \mathrm{f}[x, \boldsymbol{\phi}]$$
$$= \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11}x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21}x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31}x]$$

$$\mathrm{a}[z] = \mathrm{ReLU}[z] = \begin{cases} 0 & z < 0 \\ z & z \geq 0 \end{cases}.$$

Rectified Linear Unit

(particular kind of activation function)





$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2$$

$$= \phi_0 + \phi_1 \left(a[\theta_{10} + \theta_{11} x] \right) + \phi_2 \left(a[\theta_{20} + \theta_{21} x] \right)$$

Example shallow network

$$y = \mathrm{f}[x, \boldsymbol{\phi}]$$
$$= \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11} x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21} x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31} x]$$

This model has 10 parameters:

$$\boldsymbol{\phi} = \{\phi_0, \phi_1, \phi_2, \phi_3, \theta_{10}, \theta_{11}, \theta_{20}, \theta_{21}, \theta_{30}, \theta_{31}\}$$

- Represents a family of functions
- Parameters determine particular function
- Given parameters can perform inference (run equation)
- Given training dataset $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^{I}$
- Define loss function $L[\boldsymbol{\phi}]$ (least squares)
- Change parameters to minimize loss function

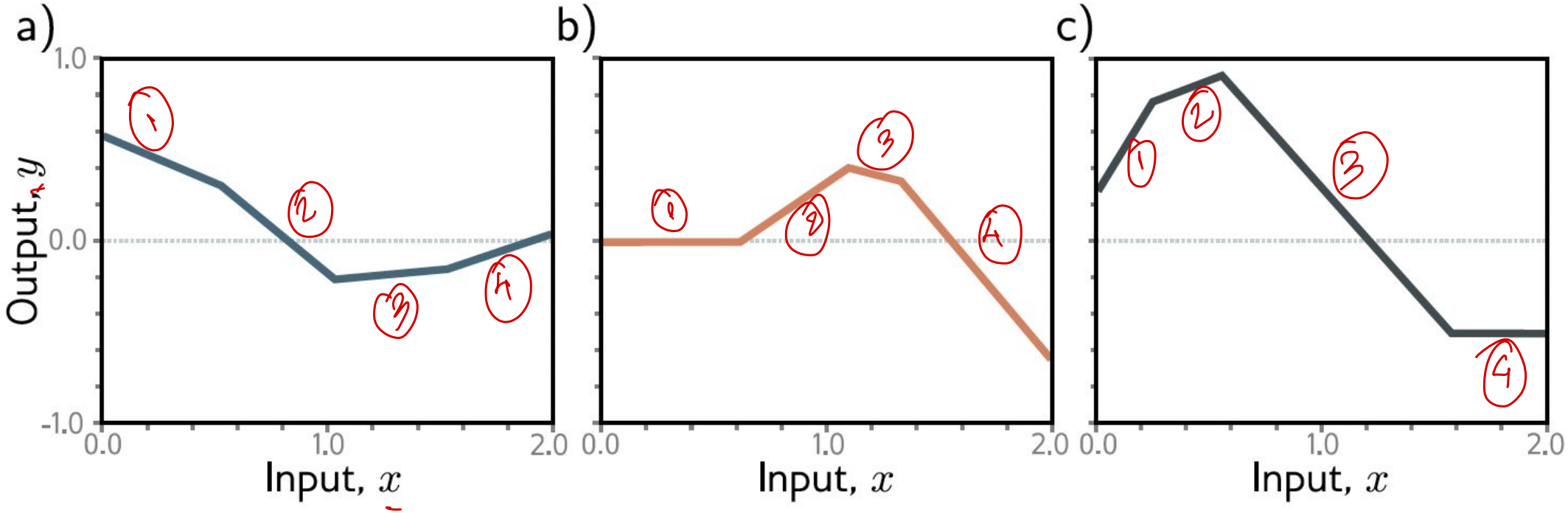
Example shallow network

$$y = \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11}x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21}x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31}x].$$

Example shallow network

$y = f(x)$

$$y = \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11}x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21}x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31}x].$$



Family of functions defined by the equation for three choices of parameters

Hidden units

$$y = \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11}x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21}x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31}x].$$

h_1 h_2 h_3

Break down into two parts:

$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$

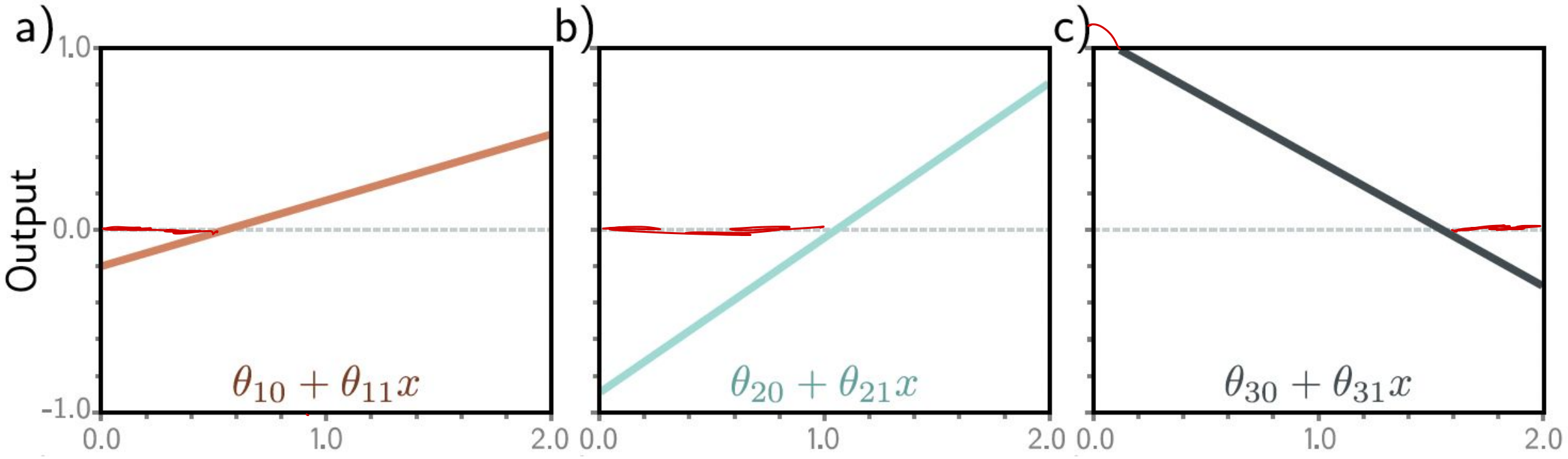
where:

Hidden units

$$\begin{aligned} h_1 &= \mathrm{a}[\theta_{10} + \theta_{11}x] \\ h_2 &= \mathrm{a}[\theta_{20} + \theta_{21}x] \\ h_3 &= \mathrm{a}[\theta_{30} + \theta_{31}x] \end{aligned}$$

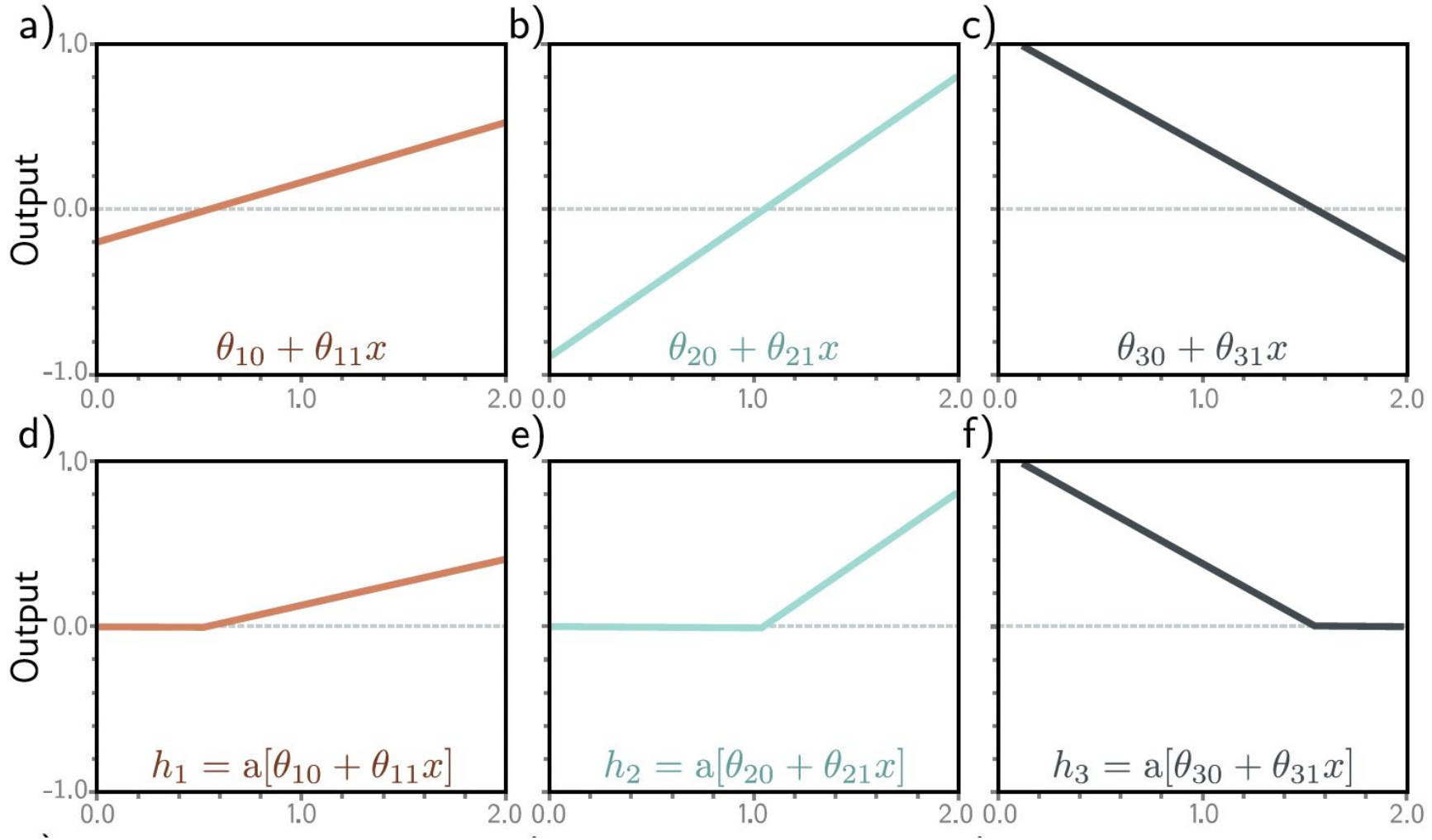
1. compute three linear functions

pre-activation

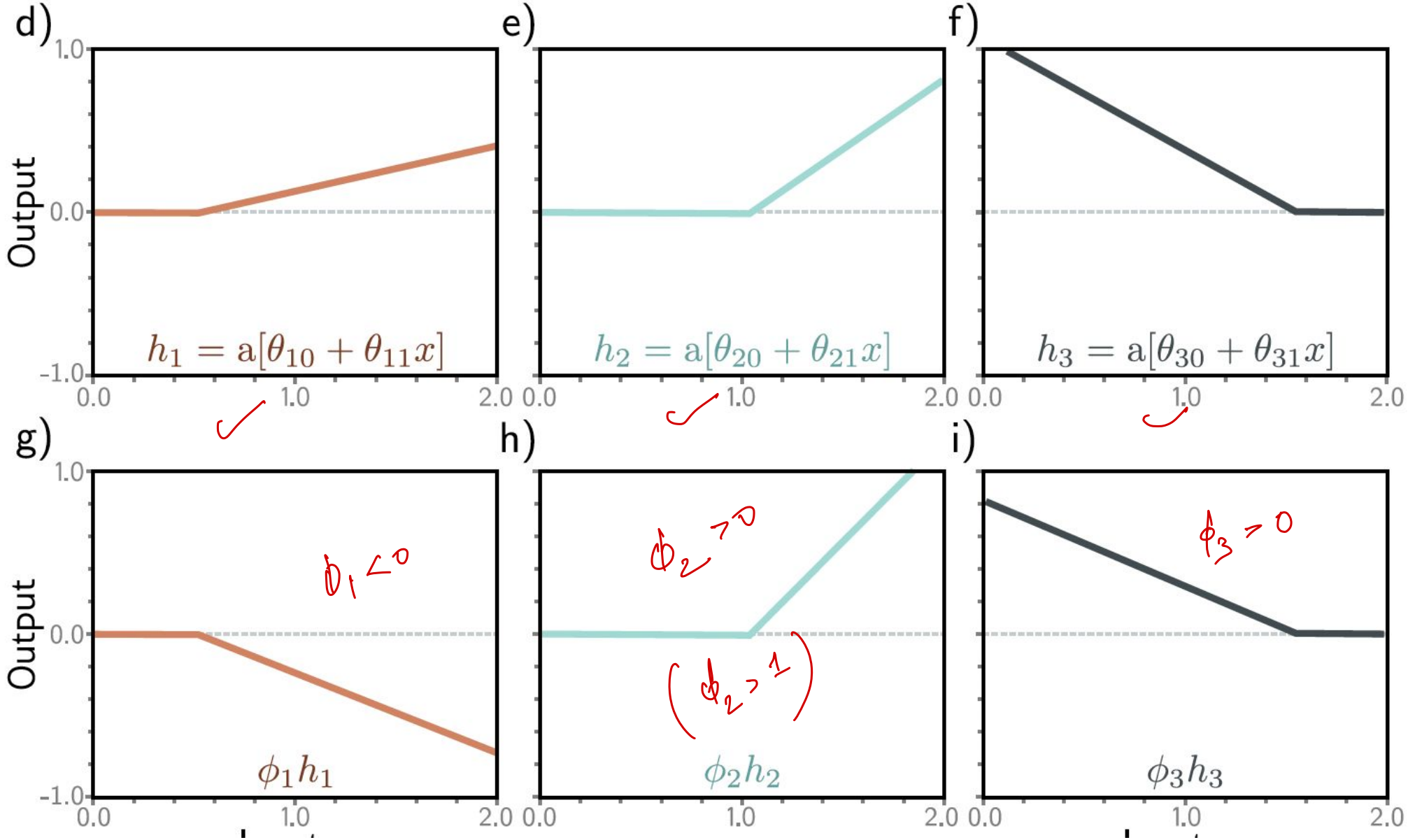


2. Pass through ReLU functions
(creates hidden units)

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x],$$



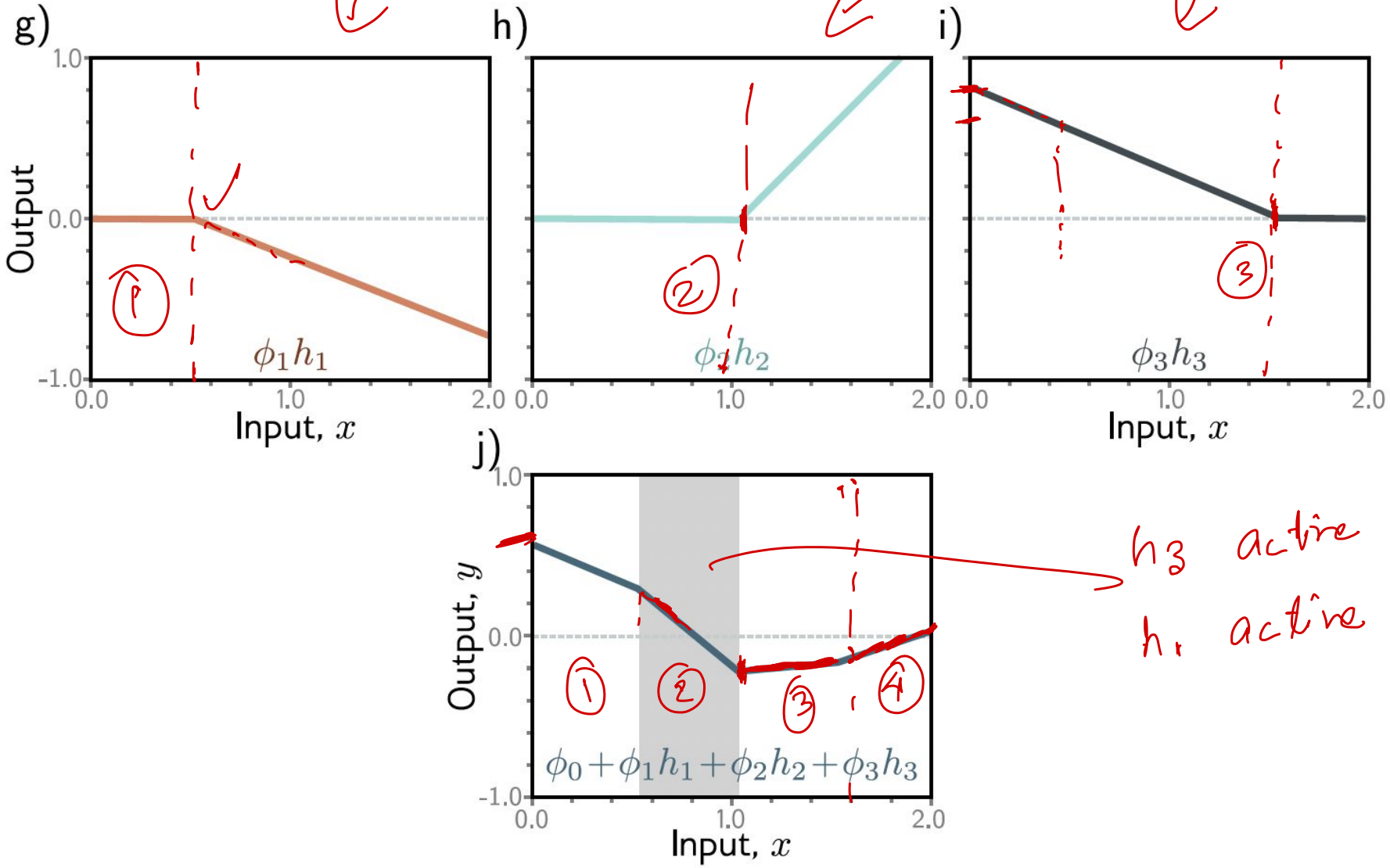
2. Weight the hidden units



4. Sum the weighted hidden units to create output

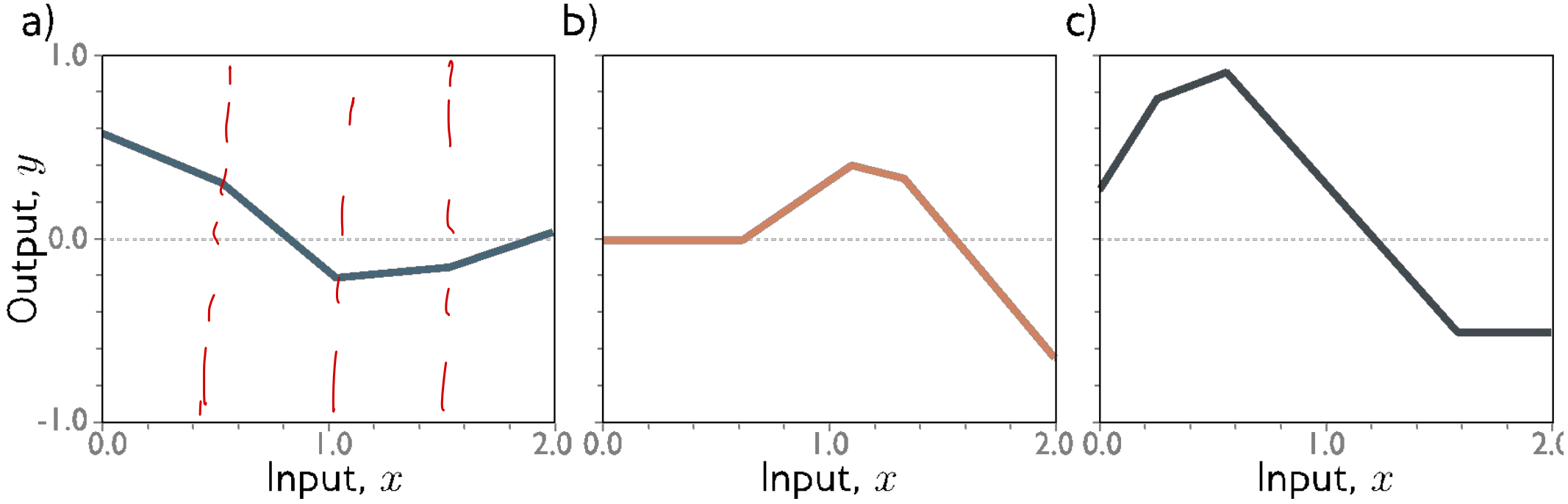
$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$

3 joints



Example shallow network

$$y = \phi_0 + \phi_1 \mathrm{a}[\theta_{10} + \theta_{11}x] + \phi_2 \mathrm{a}[\theta_{20} + \theta_{21}x] + \phi_3 \mathrm{a}[\theta_{30} + \theta_{31}x].$$

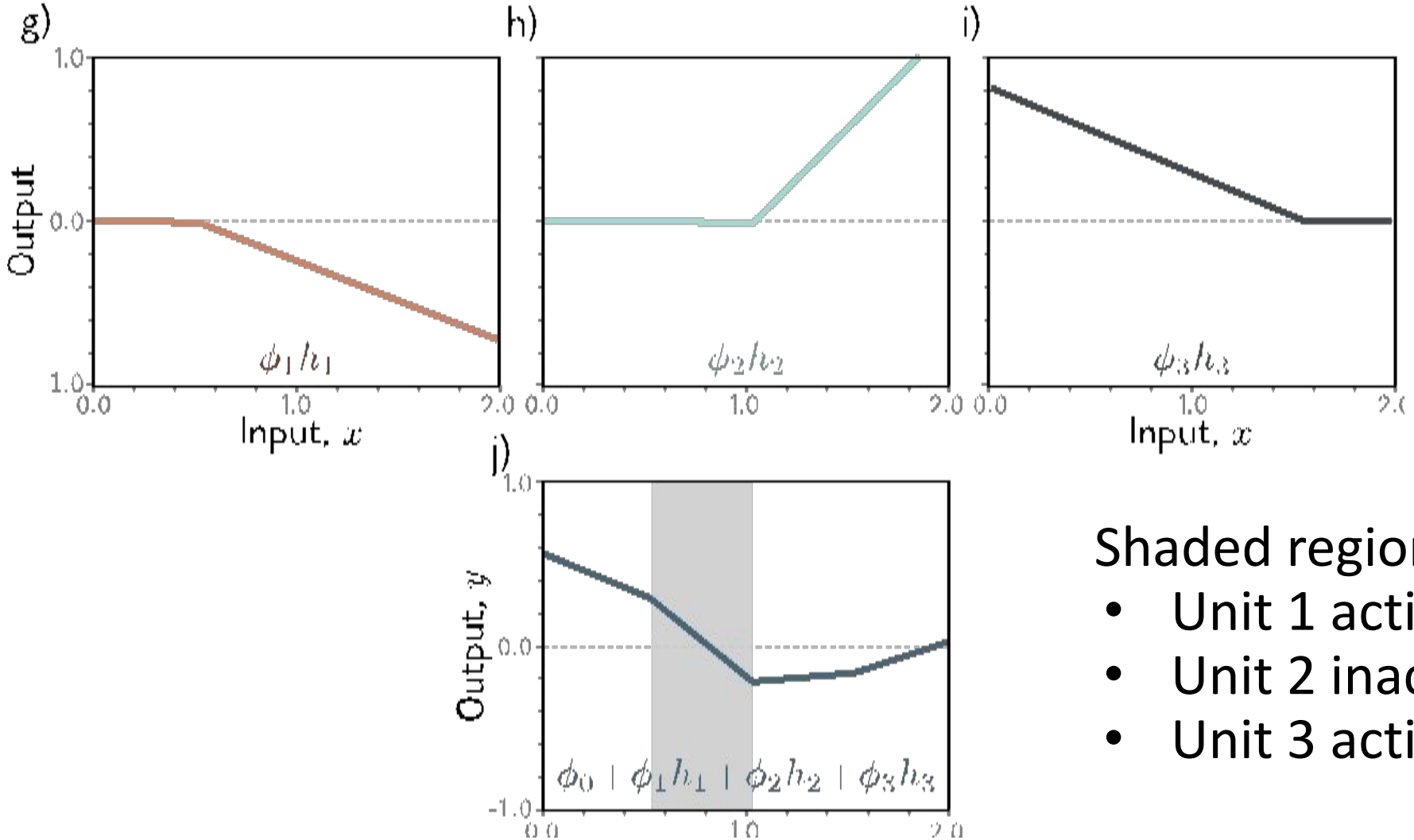


Example shallow network = piecewise linear functions

1 "joint" per ReLU function

3 joints

Activation pattern = which hidden units are activated



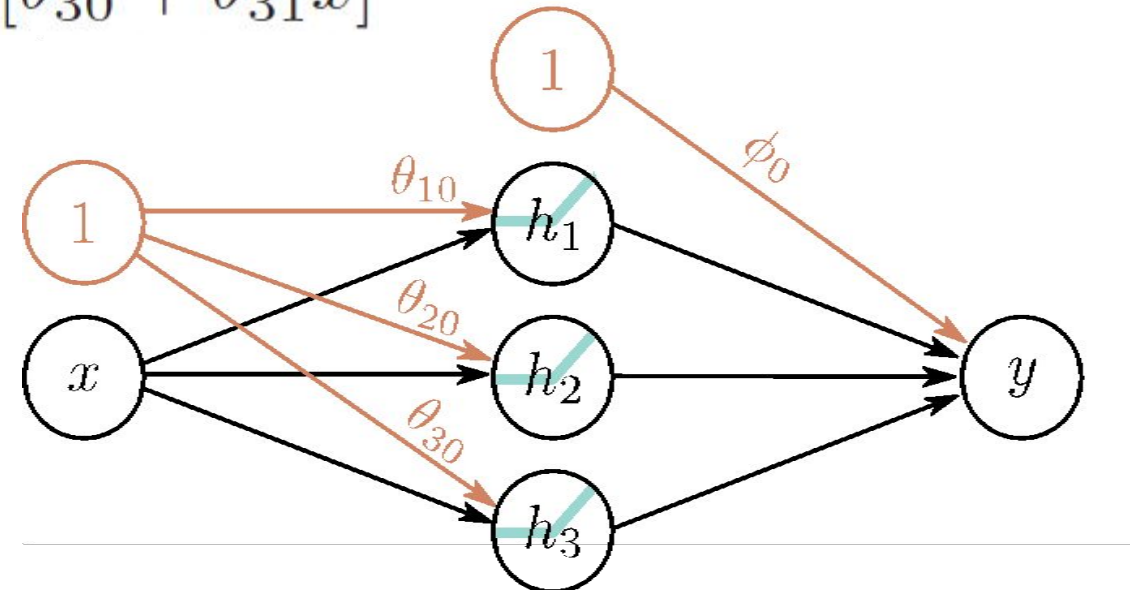
Shaded region:

- Unit 1 active
- Unit 2 inactive
- Unit 3 active

Depicting neural networks

$$
\begin{aligned}
h_1 &= \mathrm{a}[\theta_{10} + \theta_{11}x] \\
h_2 &= \mathrm{a}[\theta_{20} + \theta_{21}x] \\
h_3 &= \mathrm{a}[\theta_{30} + \theta_{31}x]
\end{aligned}
$$

$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$

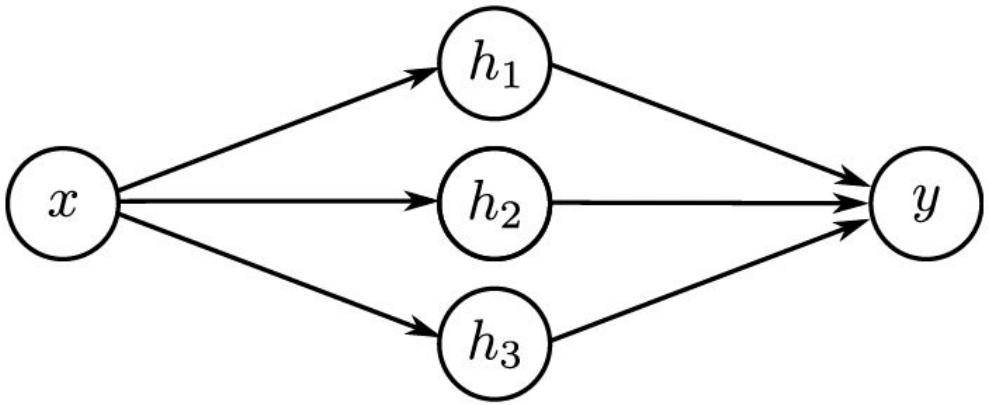


Each parameter multiplies its source and adds to its target

Depicting neural networks

$$
\begin{aligned}
h_1 &= \mathrm{a}[\theta_{10} + \theta_{11}x] \\
h_2 &= \mathrm{a}[\theta_{20} + \theta_{21}x] \\
h_3 &= \mathrm{a}[\theta_{30} + \theta_{31}x]
\end{aligned}
$$

$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$



Shallow neural networks

- Example network, 1 input, 1 output
- Universal approximation theorem
- More than one output
- More than one input
- General case
- Number of regions
- Terminology

With 3 hidden units:

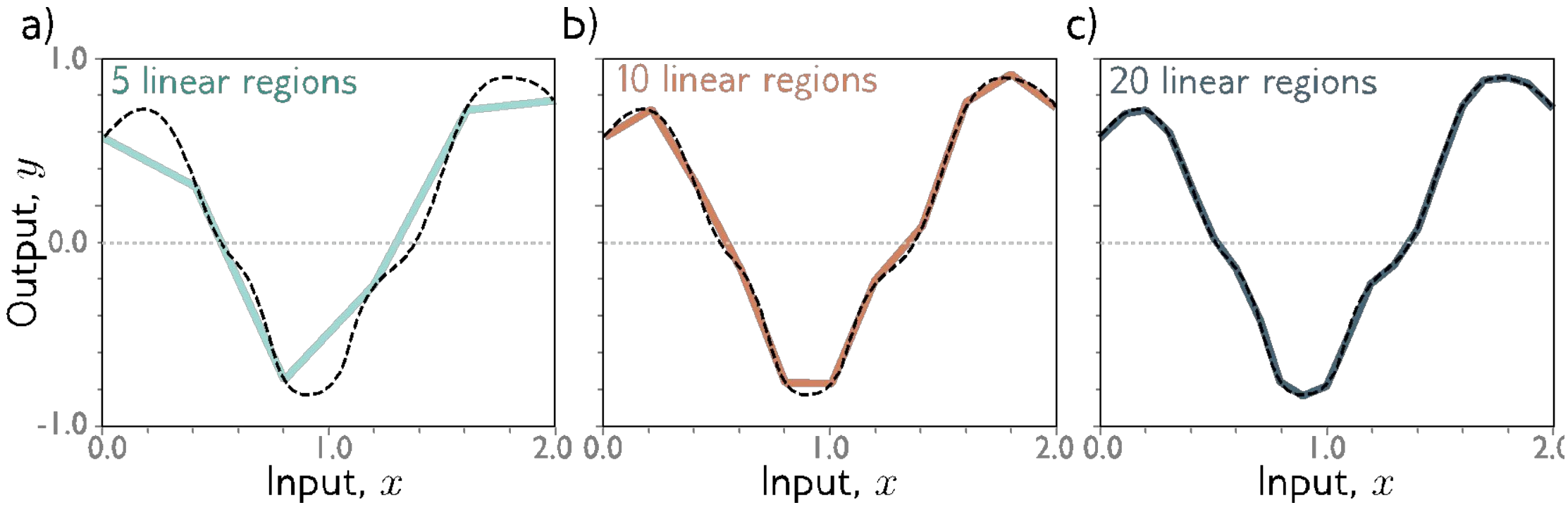
$$
\begin{aligned}
h_1 &= \mathrm{a}[\theta_{10} + \theta_{11}x] \\
h_2 &= \mathrm{a}[\theta_{20} + \theta_{21}x] \\
h_3 &= \mathrm{a}[\theta_{30} + \theta_{31}x]
\end{aligned}
\qquad\qquad
y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3
$$

With D hidden units:

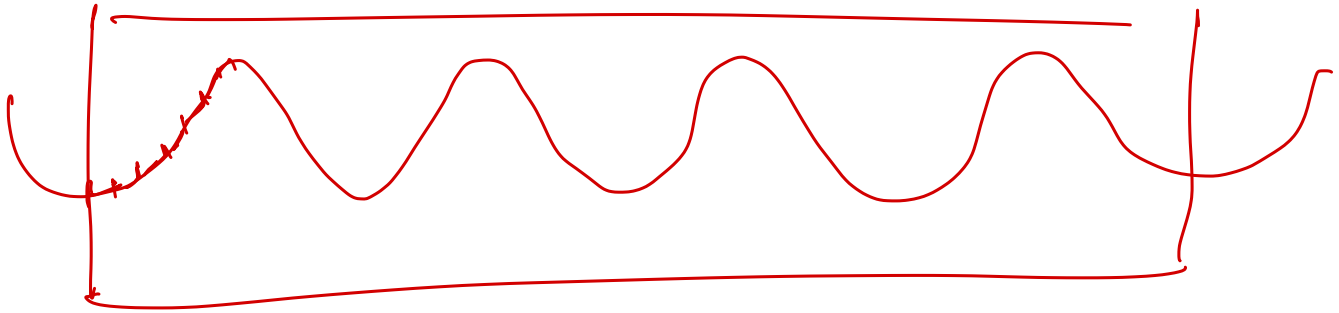
$$
h_d = \mathrm{a}[\theta_{d0} + \theta_{d1}x
\qquad\qquad
y = \phi_0 + \sum_{d=1}^{D} \phi_d h_d
$$

With enough hidden units…

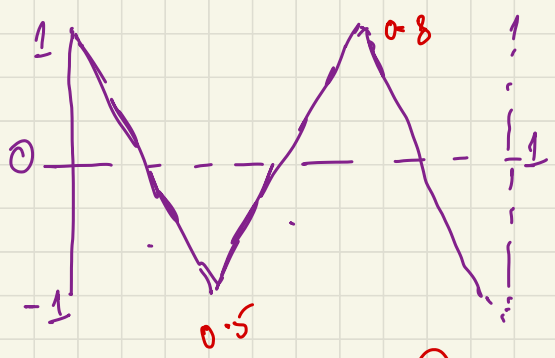
… we can describe any 1D function to arbitrary accuracy



Universal approximation theorem



"a formal proof that, with enough hidden units, a shallow neural network can describe any continuous function on a compact subset of $\mathbb{R}^D$ to arbitrary precision"



Can you model this using a shallow NN with 2 neurons? If yes, give the construction. If no, provide the proof.

[HW]

$\phi_0 + \phi_1 h_1 + \phi_2 h_2$

Can you model this using 3 neurons? If yes, provide construction. If no, prove.

Shallow neural networks

- Example network, 1 input, 1 output
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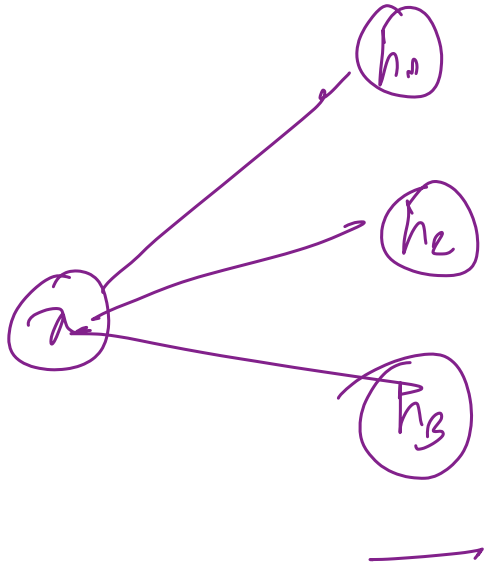
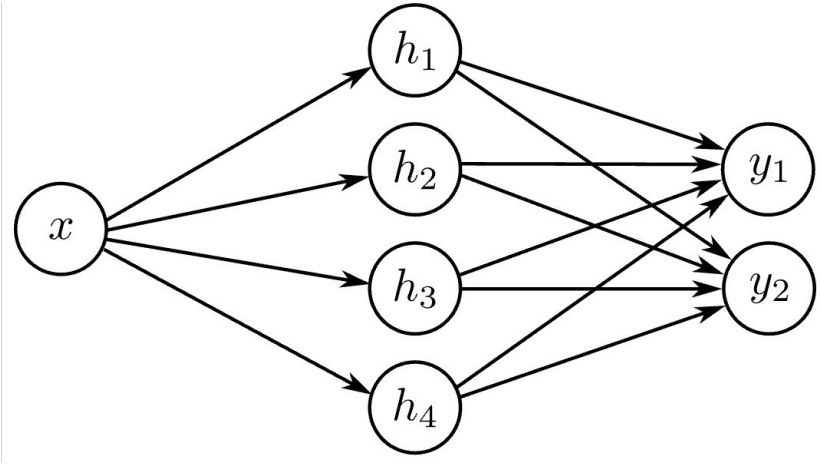


Two outputs

- 1 input, 4 hidden units, 2 outputs

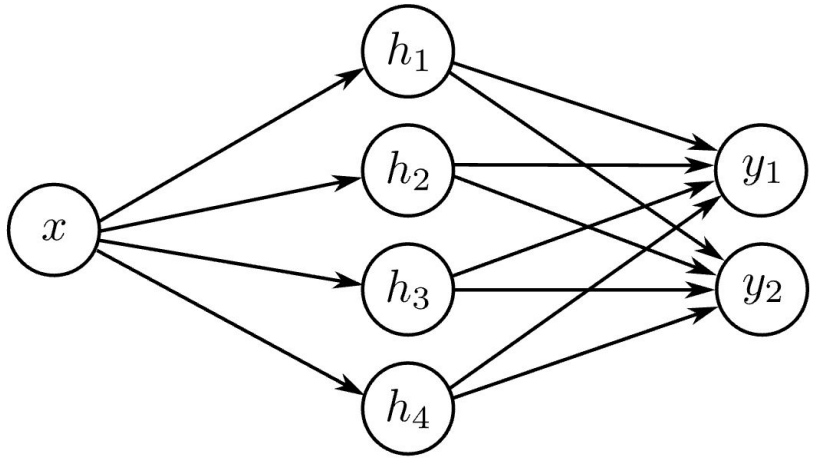


Two outputs

- 1 input, 4 hidden units, 2 outputs

$$
\begin{aligned}
h_1 &= \mathrm{a}[\theta_{10} + \theta_{11}x] \\
h_2 &= \mathrm{a}[\theta_{20} + \theta_{21}x] \\
h_3 &= \mathrm{a}[\theta_{30} + \theta_{31}x] \\
h_4 &= \mathrm{a}[\theta_{40} + \theta_{41}x]
\end{aligned}
$$

$$
\begin{aligned}
y_1 &= \phi_{10} + \phi_{11}h_1 + \phi_{12}h_2 + \phi_{13}h_3 + \phi_{14}h_4 \\
y_2 &= \phi_{20} + \phi_{21}h_1 + \phi_{22}h_2 + \phi_{23}h_3 + \phi_{24}h_4
\end{aligned}
$$



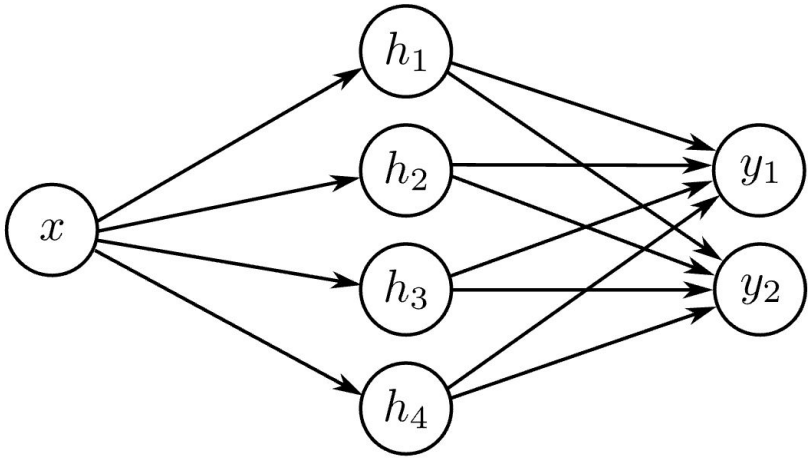
How would the outputs look like?

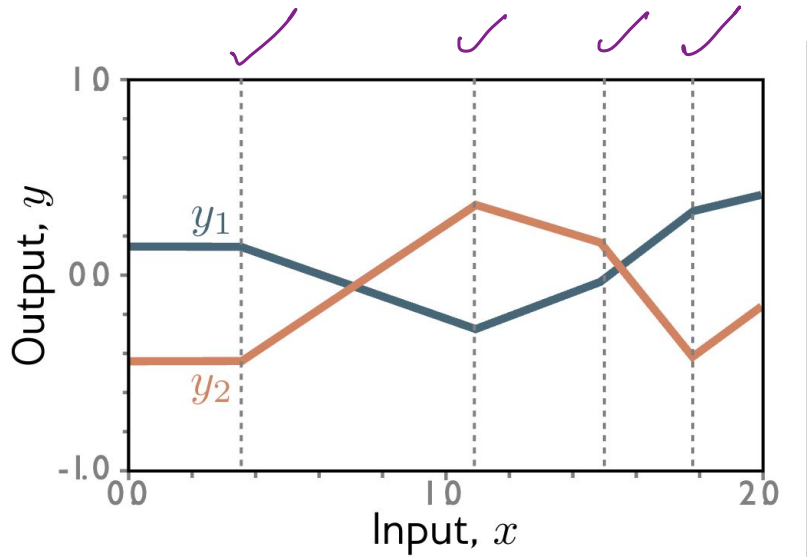
Two outputs

- 1 input, 4 hidden units, 2 outputs

$$
\begin{aligned}
h_1 &= a[\theta_{10} + \theta_{11}x] \\
h_2 &= a[\theta_{20} + \theta_{21}x] \\
h_3 &= a[\theta_{30} + \theta_{31}x] \\
h_4 &= a[\theta_{40} + \theta_{41}x]
\end{aligned}
$$

$$
\begin{aligned}
y_1 &= \phi_{10} + \phi_{11}h_1 + \phi_{12}h_2 + \phi_{13}h_3 + \phi_{14}h_4 \\
y_2 &= \phi_{20} + \phi_{21}h_1 + \phi_{22}h_2 + \phi_{23}h_3 + \phi_{24}h_4
\end{aligned}
$$



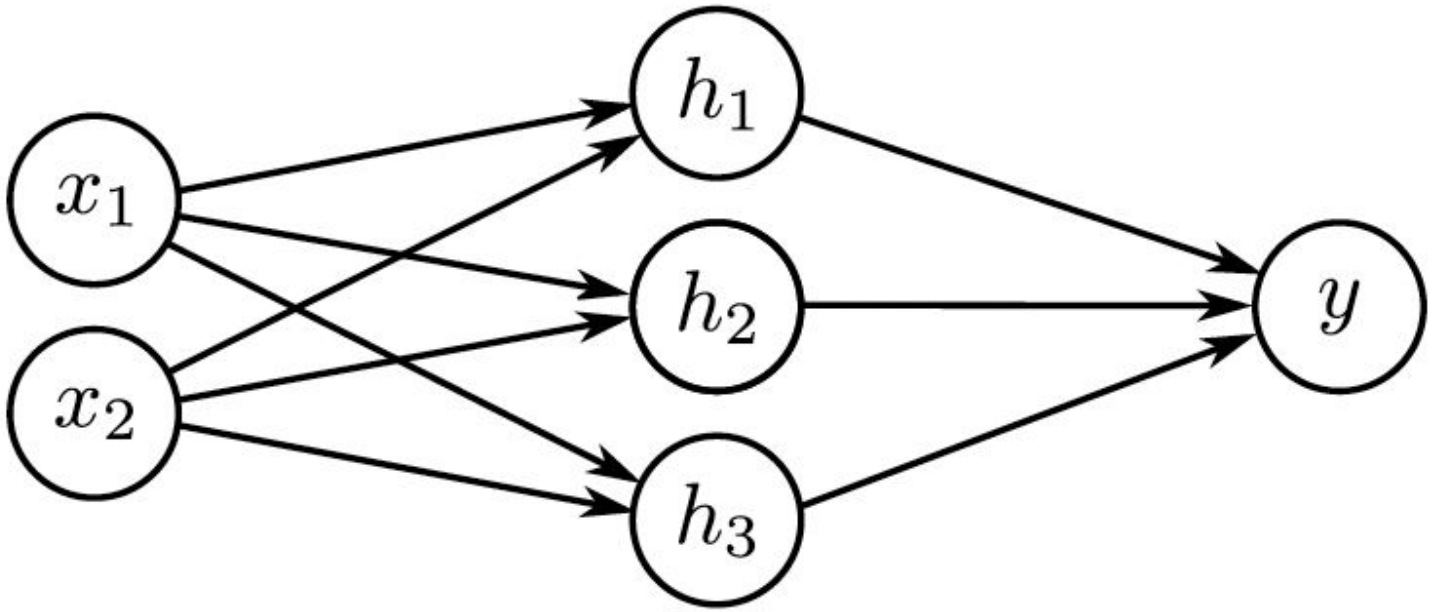


Shallow neural networks

- Example network, 1 input, 1 ouput
- Universal approximation theorem
- More than one output
- More than one input
- General case
- Number of regions
- Terminology

Two inputs

- 2 inputs, 3 hidden units, 1 output

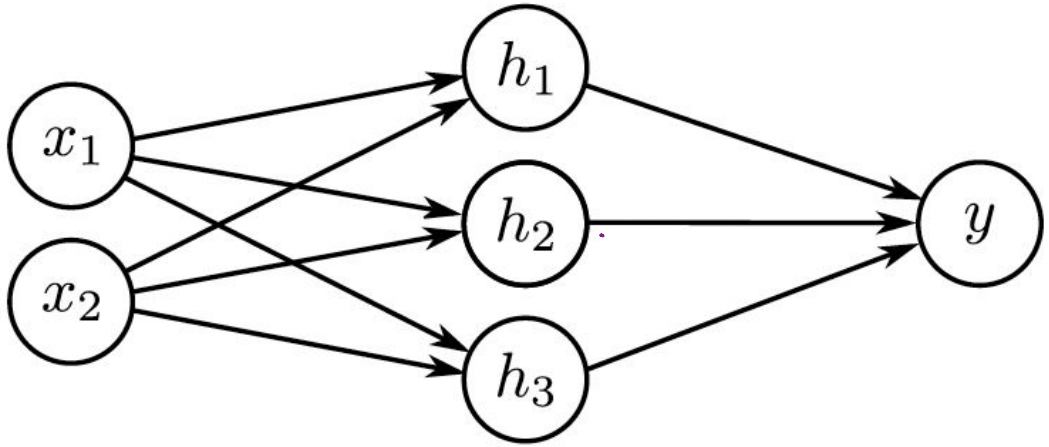


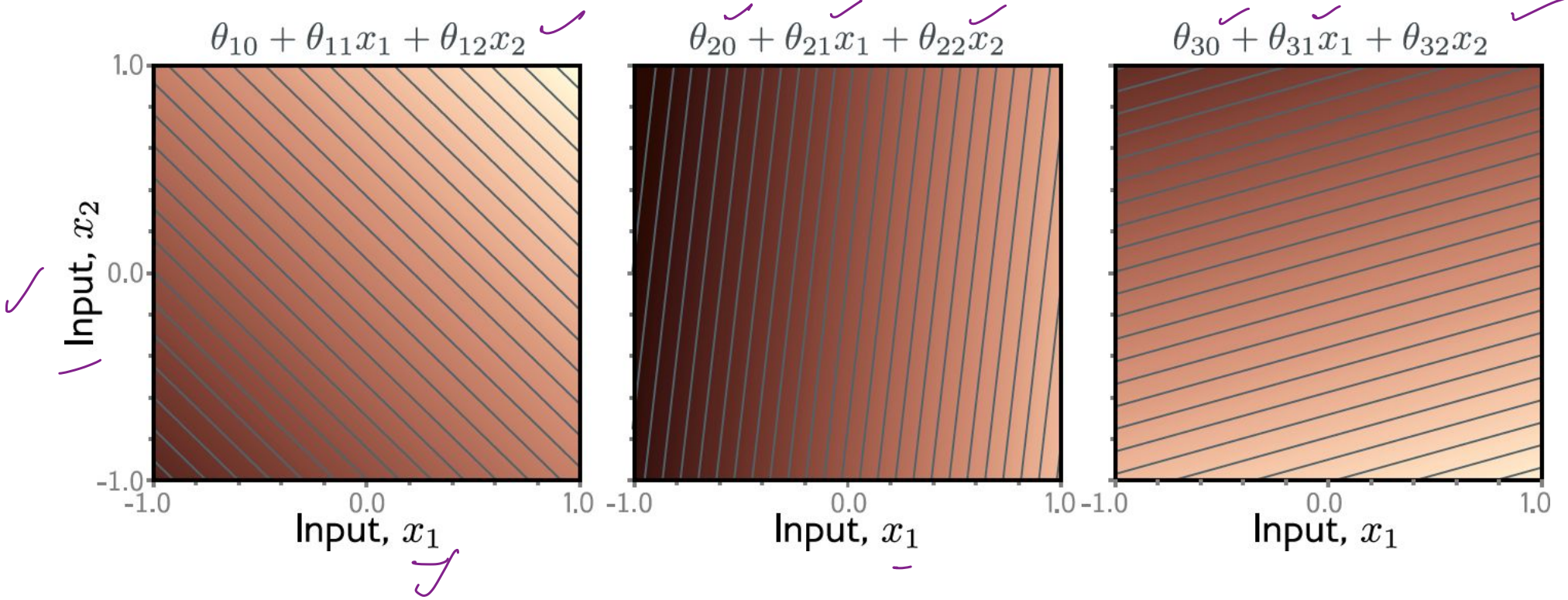
Two inputs

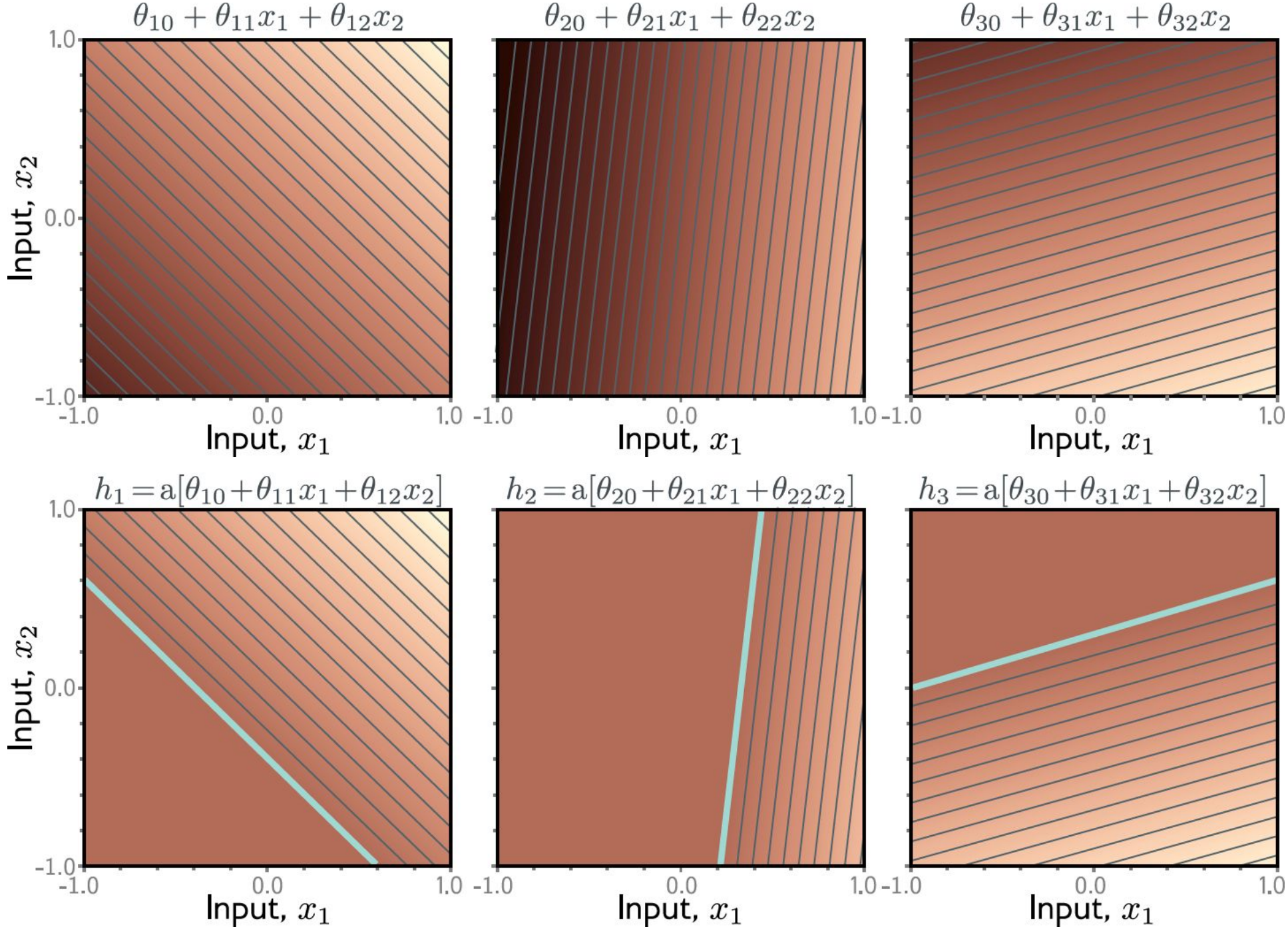
- 2 inputs, 3 hidden units, 1 output

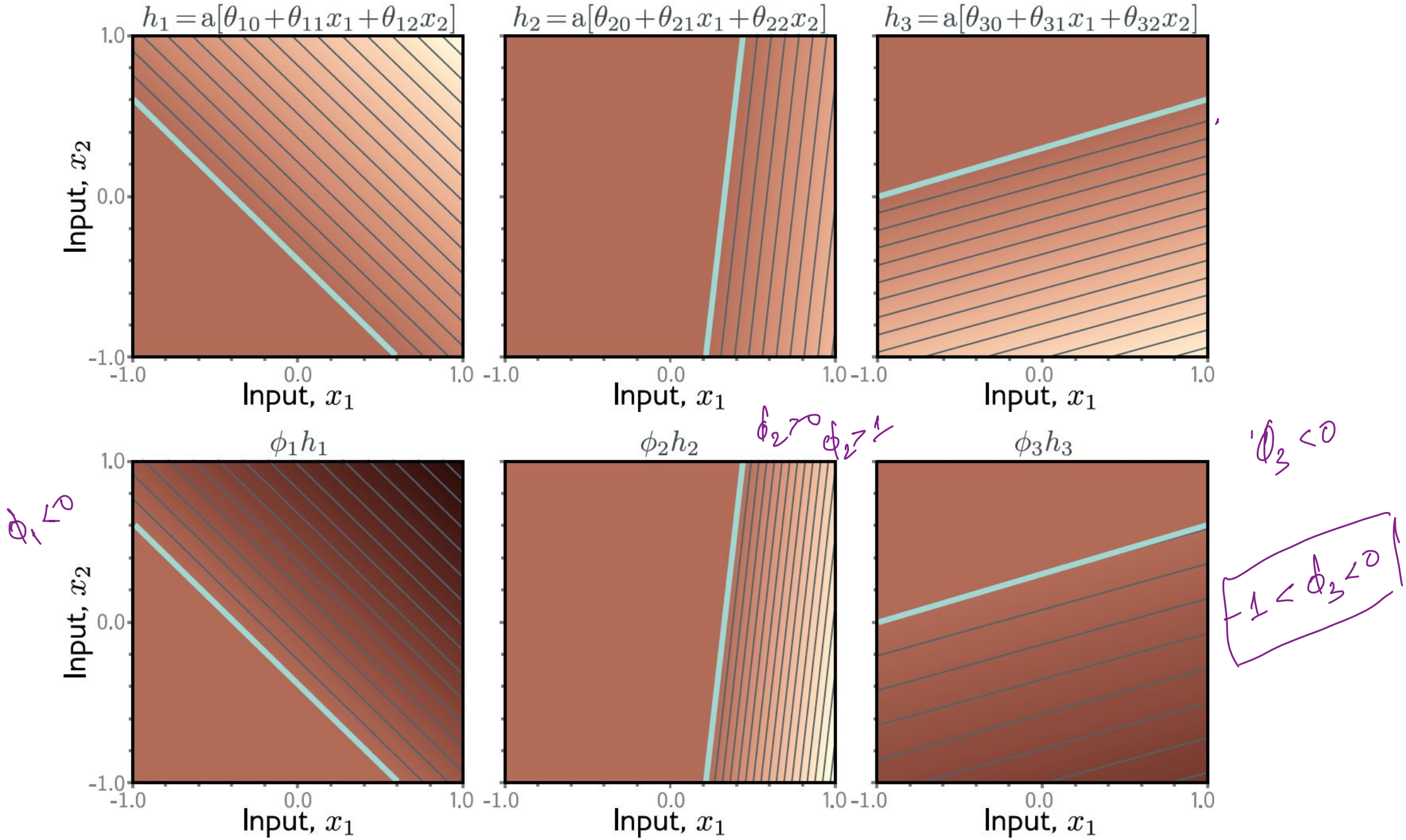
$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x_1 + \theta_{12}x_2]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x_1 + \theta_{22}x_2]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x_1 + \theta_{32}x_2]$$

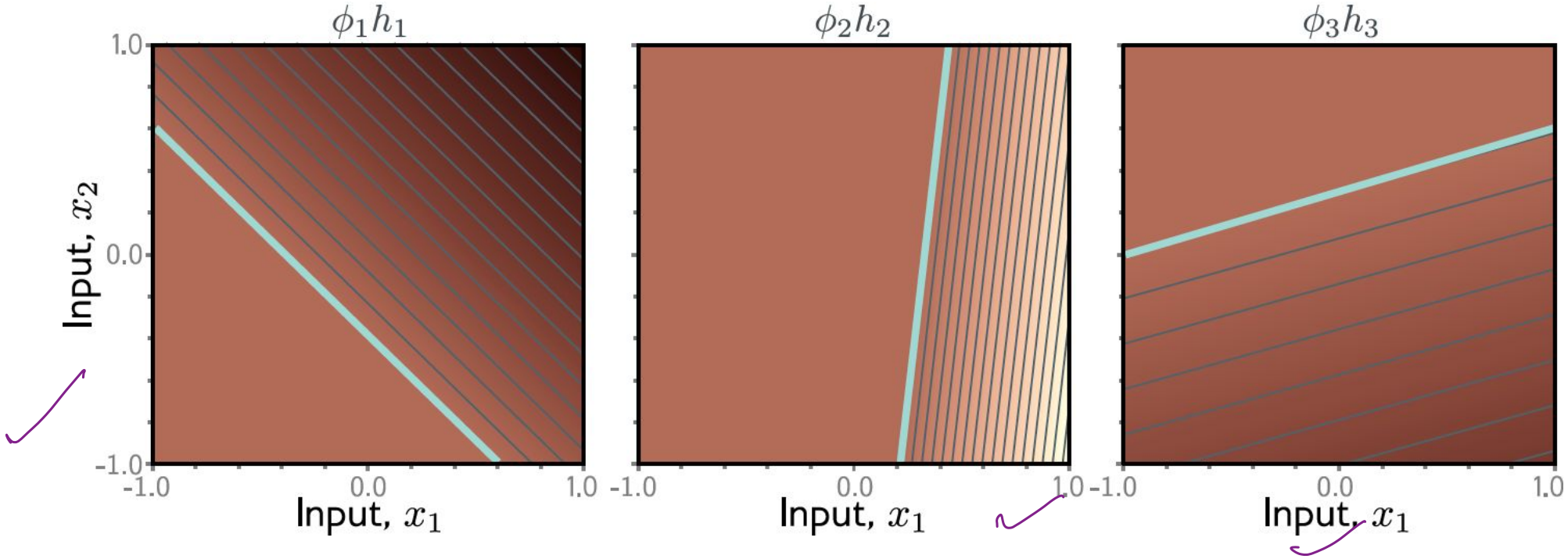
$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$



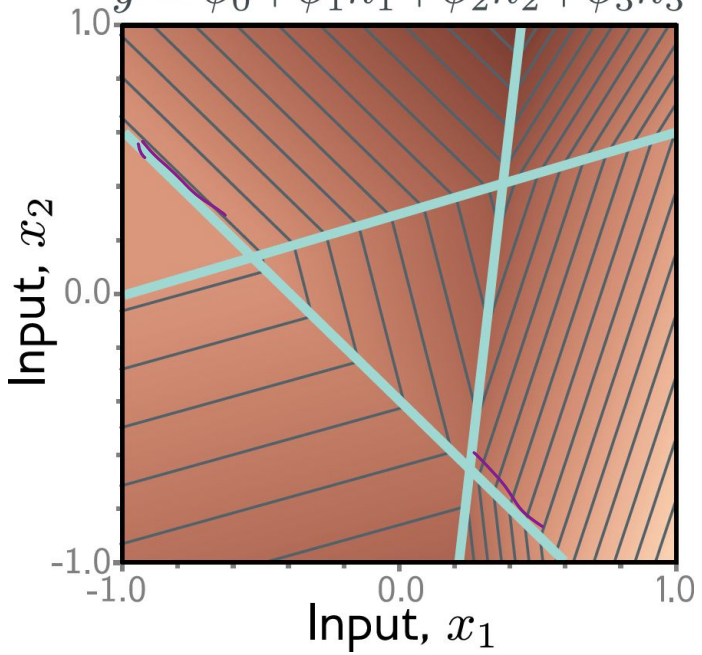

$\theta_{10} + \theta_{11}x_1 + \theta_{12}x_2$
$\theta_{20} + \theta_{21}x_1 + \theta_{22}x_2$
$\theta_{30} + \theta_{31}x_1 + \theta_{32}x_2$
Input, x_2
1.0
0.0
-1.0
-1.0
0.0
1.0
Input, x_1
Input, x_1
Input, x_1

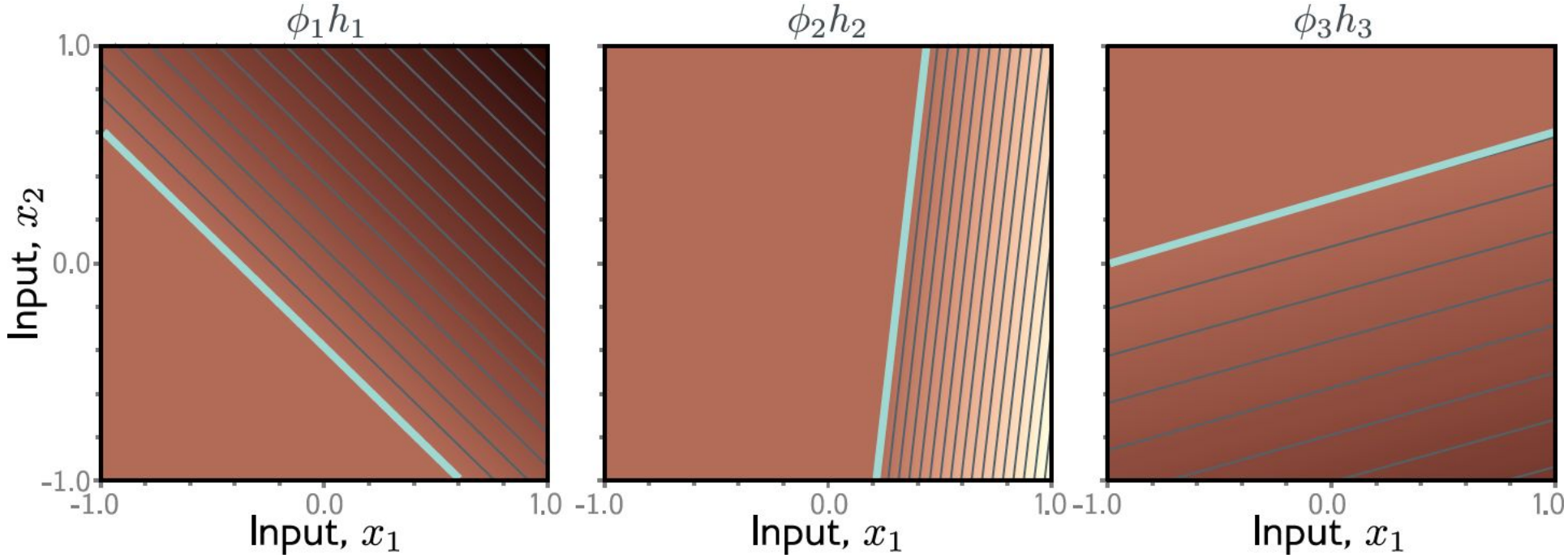

$\theta_{10} + \theta_{11}x_1 + \theta_{12}x_2$
$\theta_{20} + \theta_{21}x_1 + \theta_{22}x_2$
$\theta_{30} + \theta_{31}x_1 + \theta_{32}x_2$
$h_1 = \mathrm{a}[\theta_{10} + \theta_{11}x_1 + \theta_{12}x_2]$
$h_2 = \mathrm{a}[\theta_{20} + \theta_{21}x_1 + \theta_{22}x_2]$
$h_3 = \mathrm{a}[\theta_{30} + \theta_{31}x_1 + \theta_{32}x_2]$
Input, x_1
Input, x_2
-1.0
0.0
1.0

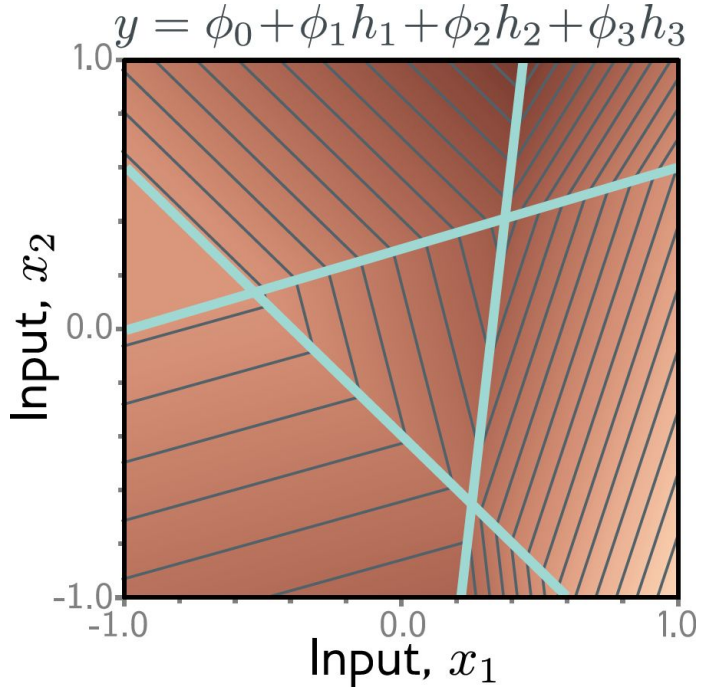

$h_1 = a[\theta_{10} + \theta_{11}x_1 + \theta_{12}x_2]$
$h_2 = a[\theta_{20} + \theta_{21}x_1 + \theta_{22}x_2]$
$h_3 = a[\theta_{30} + \theta_{31}x_1 + \theta_{32}x_2]$
$\phi_1 h_1$
$\phi_2 h_2$
$\phi_3 h_3$
Input, x_1
Input, x_2
1.0
0.0
-1.0
$\phi_2 > 0$
$\phi_2 > 1$
$\phi_3 < 0$
$\phi_1 < 0$
$-1 < \phi_3 < 0$



$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$

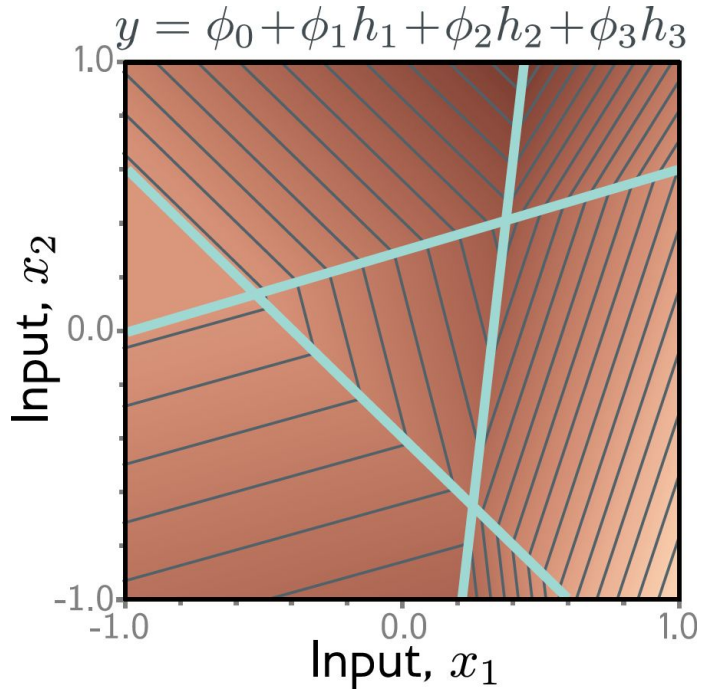


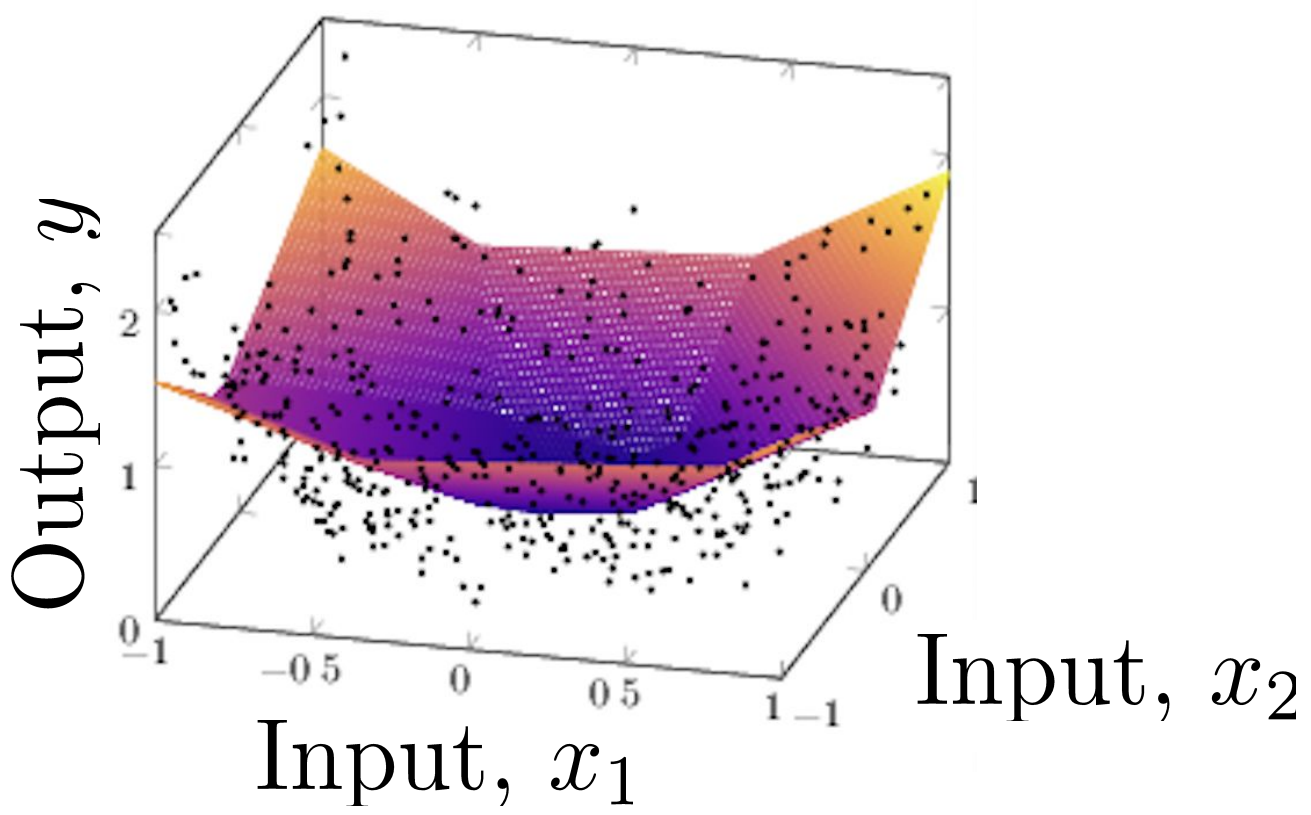




Convex polygons

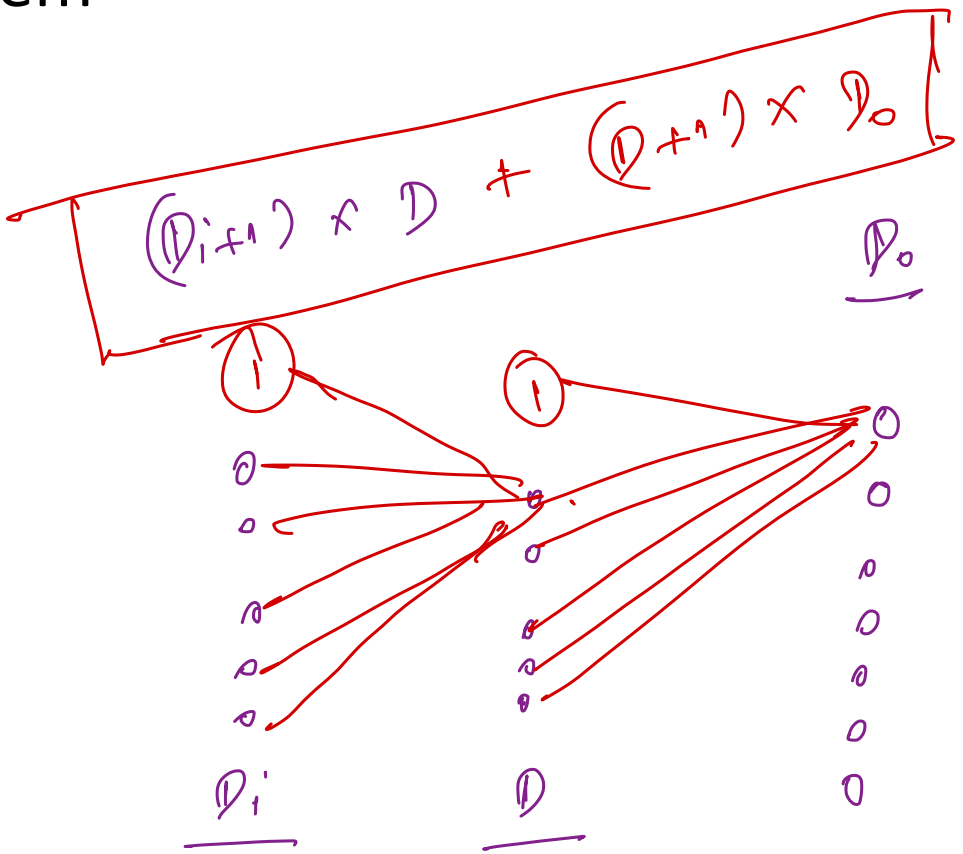
Fitting


$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$
1.0
0.0
-1.0
Input, x_2
-1.0
0.0
1.0
Input, x_1


Output, y
2
1
0
−1
−0.5
0
0.5
1
Input, x_1
1
0
−1
Input, x_2

Shallow neural networks

- Example network, 1 input, 1 output
- Universal approximation theorem
- More than one output
- More than one input
- General case
- Number of regions
- Terminology

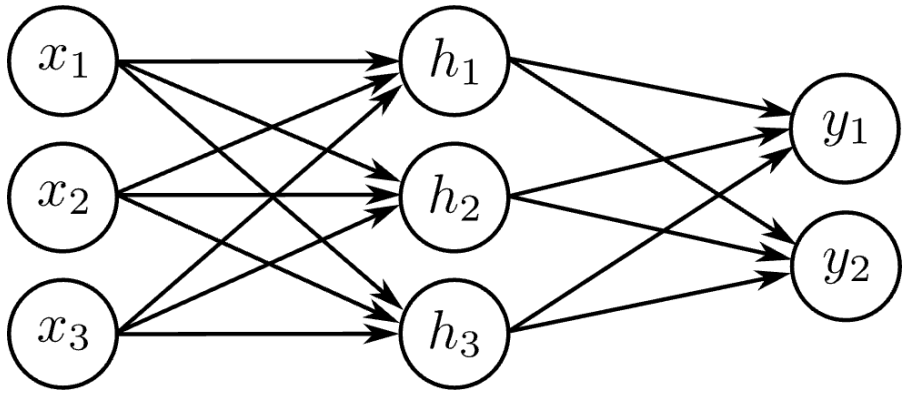


Arbitrary inputs, hidden units, outputs

- D_o Outputs, D hidden units, and D_i inputs

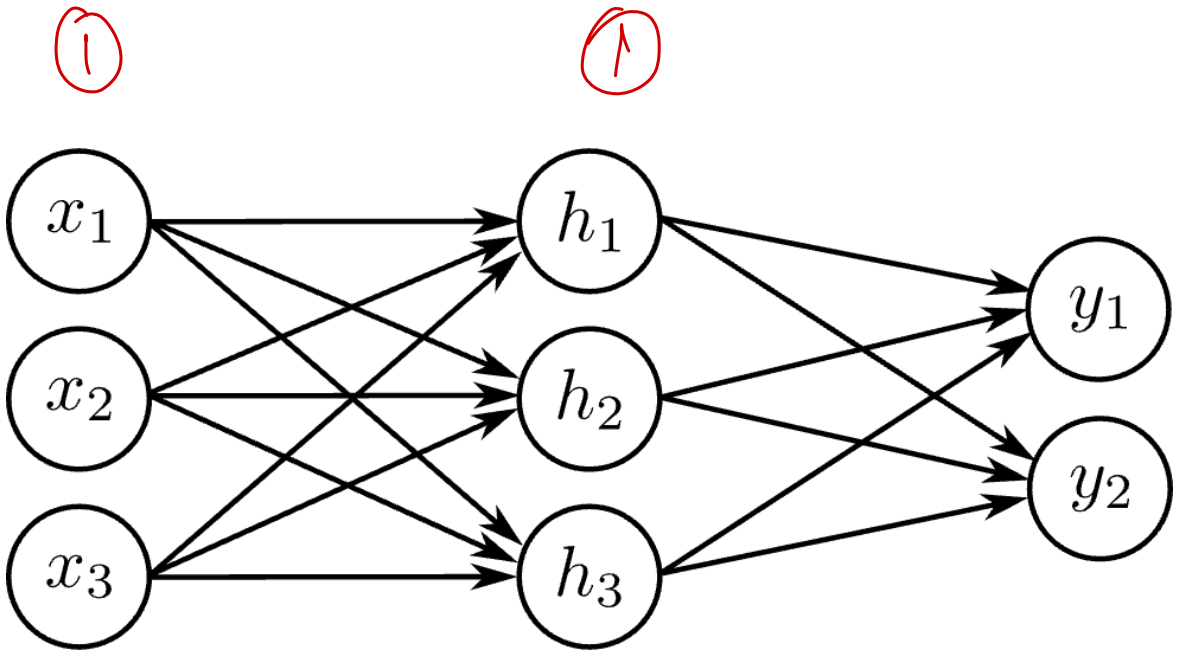
$$h_d = \mathrm{a}\left[\theta_{d0} + \sum_{i=1}^{D_i} \theta_{di} x_i\right] \qquad y_j = \phi_{j0} + \sum_{d=1}^{D} \phi_{jd} h_d$$

- e.g., Three inputs, three hidden units, two outputs



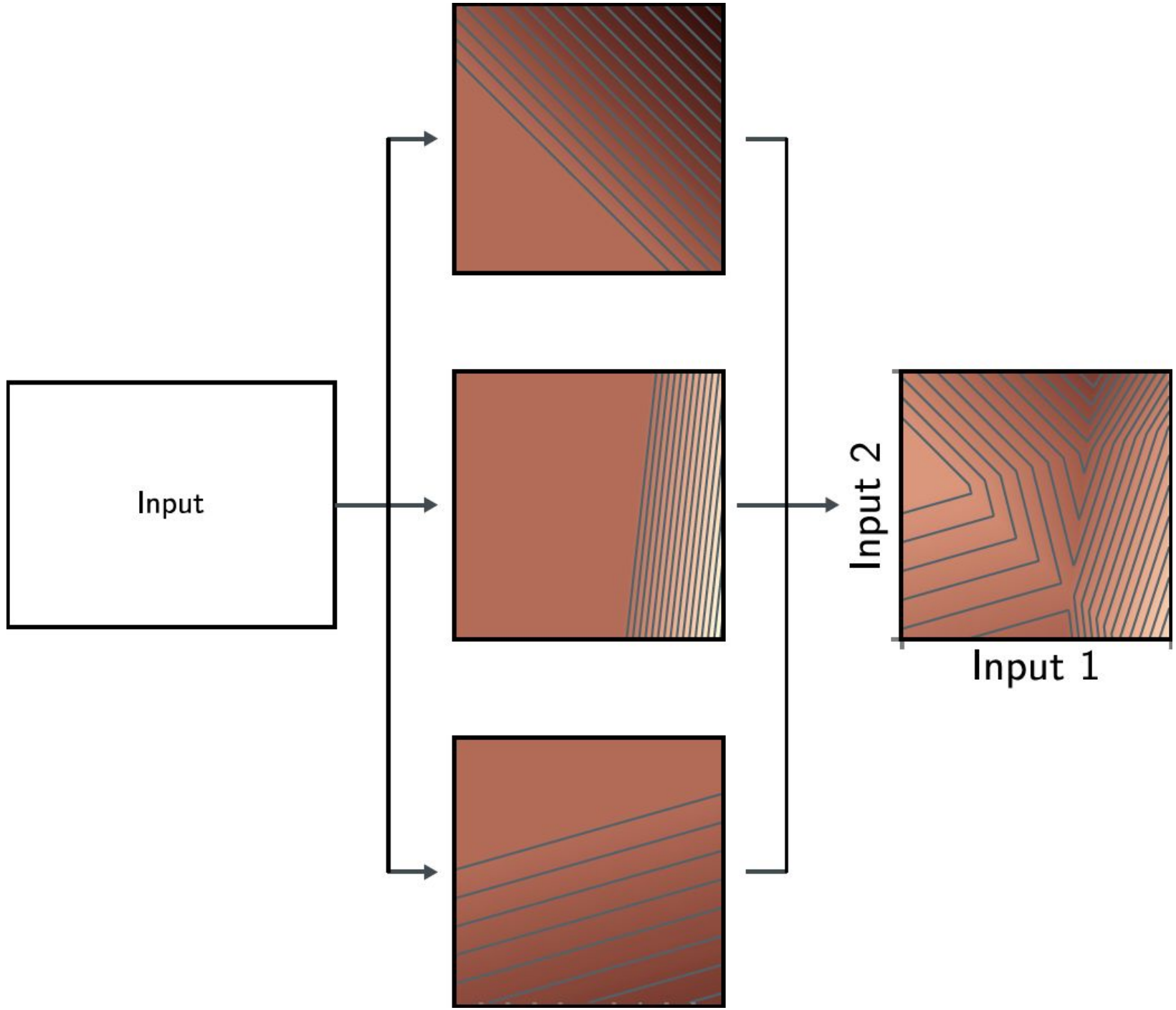
Question:

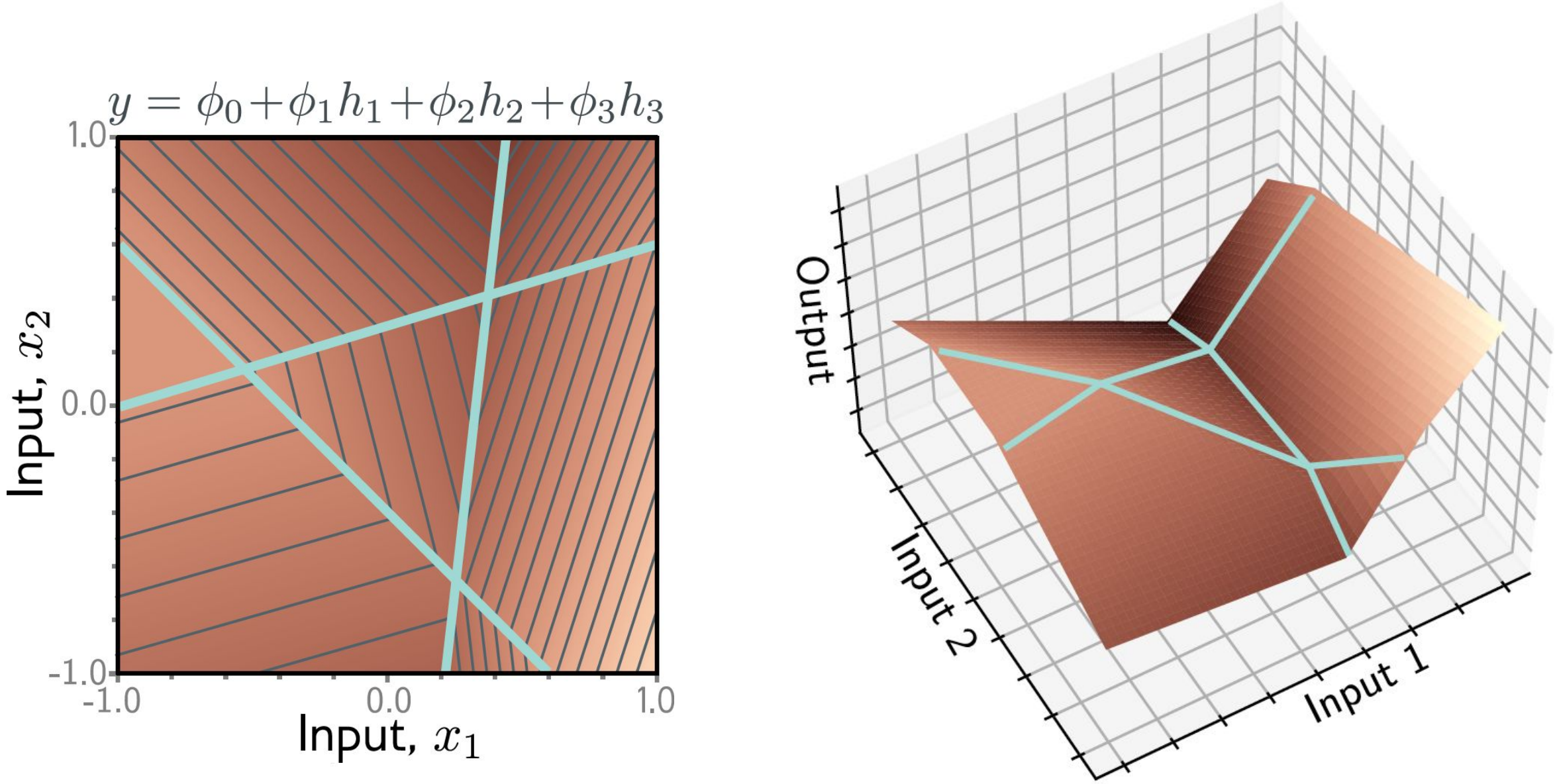
- How many parameters does this model have?

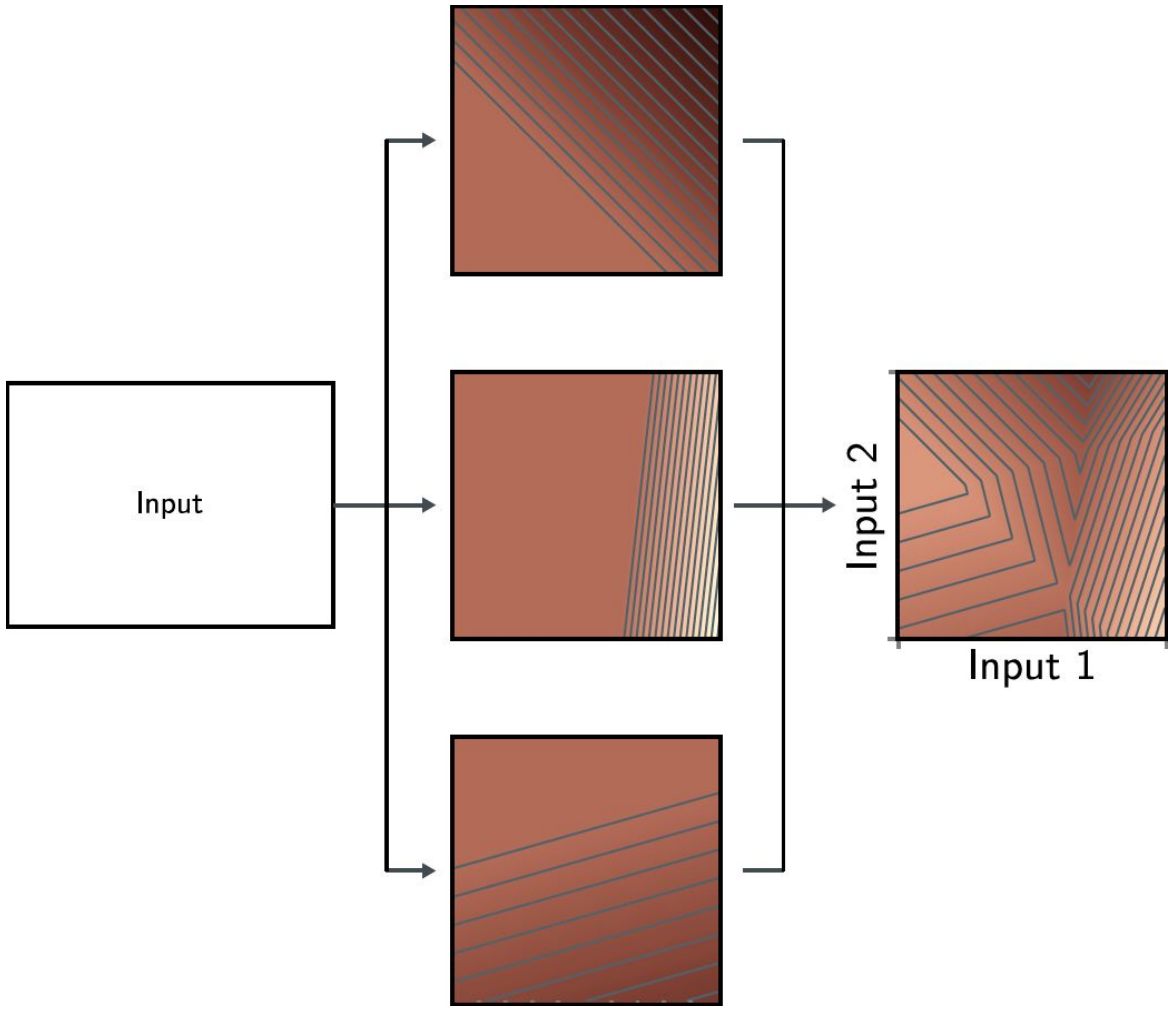


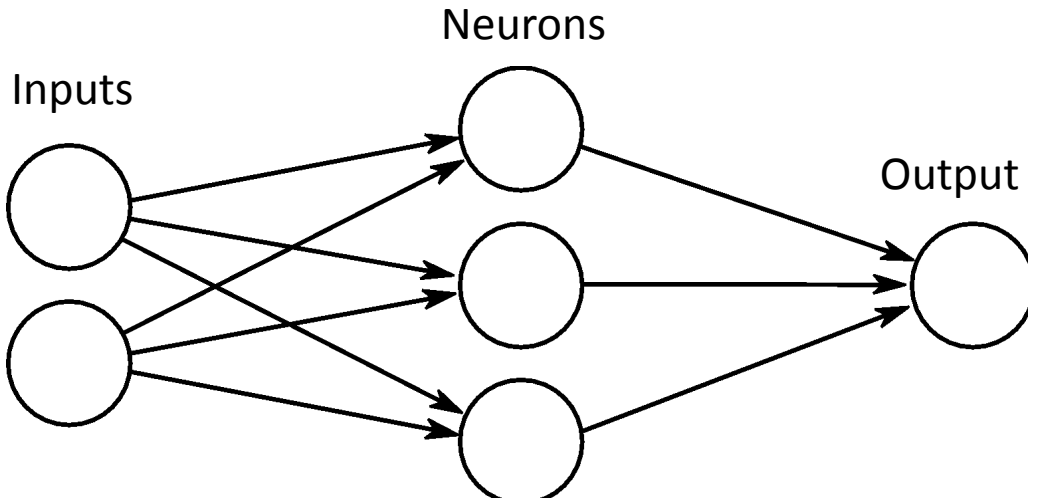
4x3+4x2

= 20 parameters


Input
Input 2
Input 1


$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$
1.0
0.0
-1.0
Input, x_2
-1.0
0.0
1.0
Input, x_1
Output
Input 2
Input 1





"neural network"

$$y = \phi_0 + \phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3$$

$$h_1 = \mathrm{a}[\theta_{10} + \theta_{11} x_1 + \theta_{12} x_2]$$
$$h_2 = \mathrm{a}[\theta_{20} + \theta_{21} x_1 + \theta_{22} x_2]$$
$$h_3 = \mathrm{a}[\theta_{30} + \theta_{31} x_1 + \theta_{32} x_2]$$

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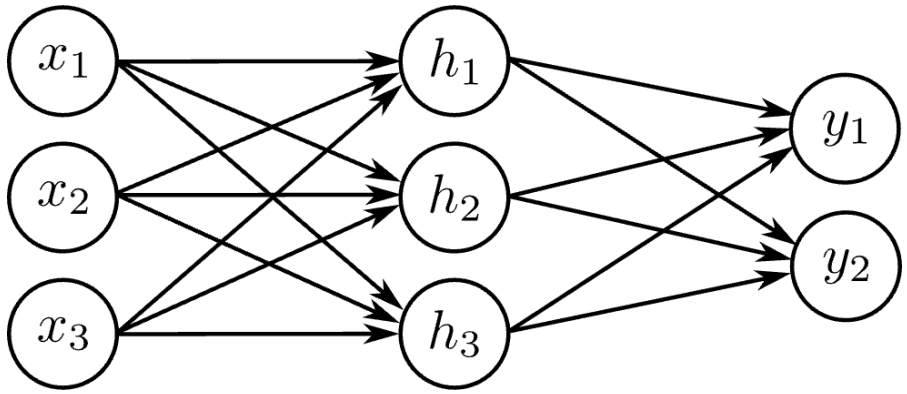
- e.g., Three inputs, three hidden units, two outputs



Shallow neural networks

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- More than one input
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- Number of regions
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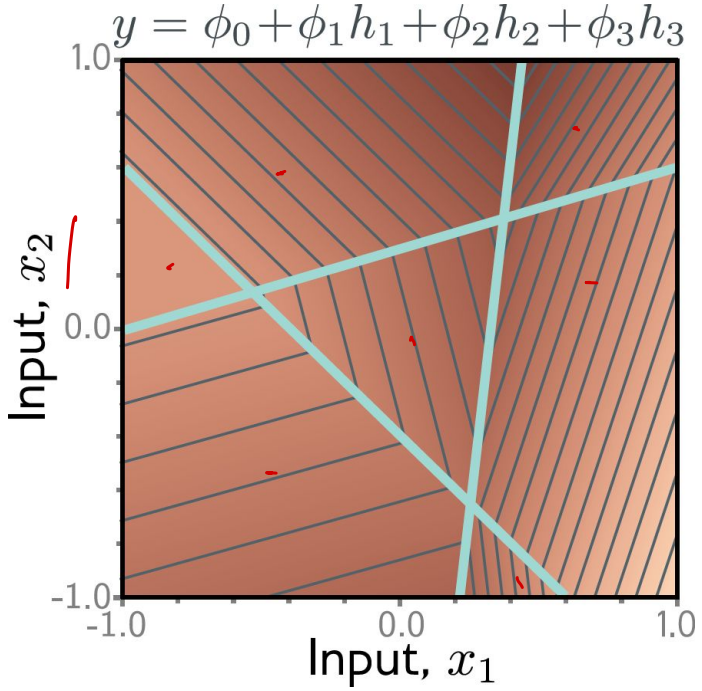
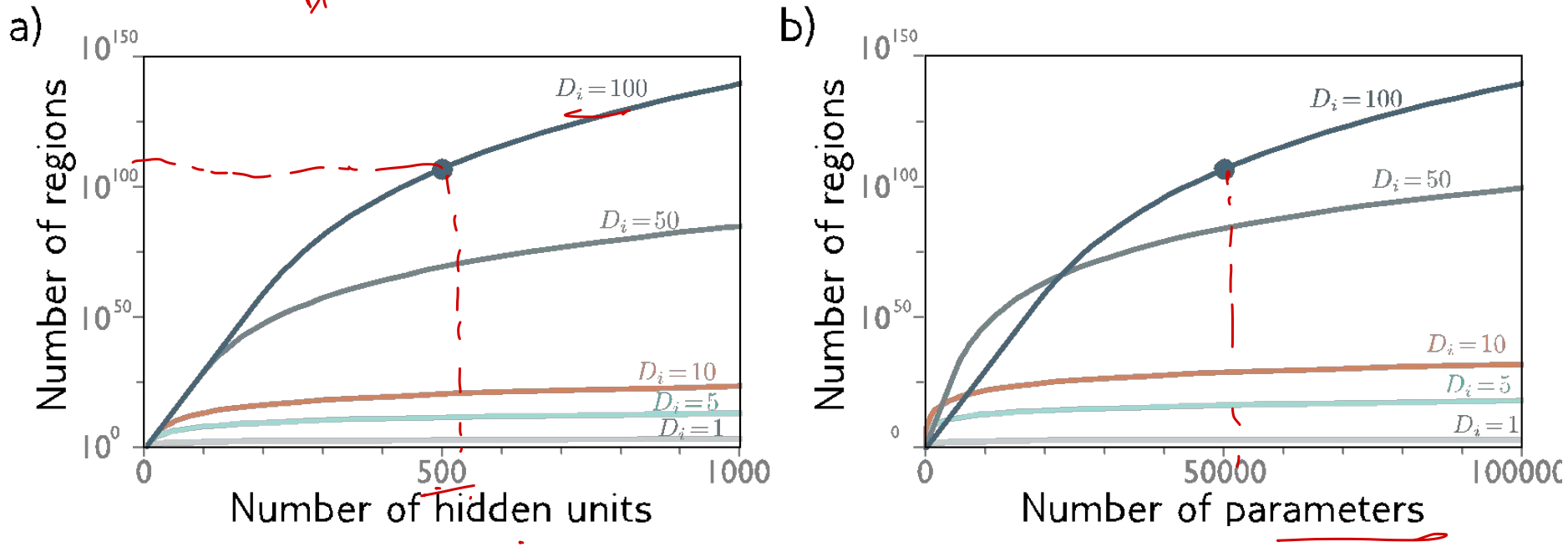
Number of output regions

- In general, each output consists of D dimensional convex polytopes
- With two inputs, and three outputs, we saw there were seven polygons:



Number of output regions

- In general, each output consists of D dimensional convex polytopes
- How many?



Highlighted point = 500 hidden units or 51,001 parameters