Ex (5) Evaluate $\int_C \frac{z+3}{z^2+2z+5}\,dz$ where

C is circle (i) $|z|=1$ (ii) $|z+1-2i|=2$

Soln: $z^2+2z+5=0$

$z=\frac{-2\pm\sqrt{4-20}}{2} = \frac{-2\pm\sqrt{-16}}{2} = \frac{-2\pm 4i}{2}$

$z=-1\pm 2i$

$z=-1+2i \quad -1-2i$

(i) if $z=-1+2i$ $\quad |z|=|-1+2i|=\sqrt{1+4}=\sqrt{5}>1$

$z=-1-2i$ $\quad |z|=|-1-2i|=\sqrt{1+4}=\sqrt{5}>1$

$z=-1+2i$, $-1-2i$ lies outside of C $\therefore f(z)=\frac{z+3}{z^2+2z+5}$

by Cauchy's thm

$\int_C f(z)dz=0$

$\therefore \int_C \frac{z+3}{z^2+2z+5}dz=0$

(ii) $|z+1-2i|=2$

if $z=-1+2i$, $|z+1-2i|=|-1+2i+1-2i|$

$=0<2$

$z=-1-2i$ $\quad |z+1-2i|=|-1-2i+1-2i|=|-4i|$

$=4>2$

$z=-1+2i$ lies inside of C

$z=-1-2i$ lies outside of C

$\therefore f(z) = \frac{z+3}{z+1+2i}$ may be analytic in C (2)

by Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-z_0)} dz = 2\pi i f(z_0)$$

$$\therefore \int_C \frac{z+3}{z^2+2z+5} dz = \int_C \frac{\frac{z+3}{z+1+2i}}{z+1-2i} dz$$

$$= 2\pi i f(-1+2i)$$

$$= 2\pi i \left[\frac{z+3}{z+1+2i}\right]$$

$$= 2\pi i \left[\frac{-1+2i+3}{-1+2i+1+2i}\right]$$

$$= 2\pi i \left(\frac{2+2i}{4i}\right)$$

$$= \frac{\pi}{2}(2+2i)$$

$$= \pi(1+i)$$

Ex (6) Evaluate $\int_C \frac{z+6}{z^2-4} dz$ where

C is the circle (i) $|z|=1$ (ii) $|z-2|=1$

(iii) $|z+2|=1$

Soln: $z^2-4=0$, $(z-2)(z+2)=0$

$z=2,-2$

(i) $|z|=1$ if $z=2$ $|z|=|2|=2>1$

$z=-2$, $|z|=|-2|=2>1$

$z = 2, -2$ lies outside of C $\quad f(z) = \frac{z+6}{z^2-4}$ (3)

by Cauchy's thm

$$\oint_C f(z)\,dz = 0$$

$$\therefore \int_C \frac{z+6}{z^2-4}\,dz = 0$$

(ii) $|z-2| = 1$

if $z = 2$, $|z-2| = |2-2| = 0 < 1$

$z = -2$, $|z-2| = |-2-2| = |-4| = 4 > 1$

$z = 2$ lies inside of C

$z = -2$ lies outside of C

$f(z) = \frac{z+6}{z+2}$ may be analytic

$\therefore$ by Cauchy's integral formula

$$\int_C \frac{z+6}{z^2-4}\,dz = \int_C \frac{\frac{z+6}{z+2}}{z-2}\,dz = 2\pi i f(2)$$

$$= 2\pi i \left[\frac{z+6}{z+2}\right]_{z=2} = 2\pi i \left[\frac{2+6}{2+2}\right]$$

$$= 2\pi i \left(\frac{8}{4}\right) = 4\pi i$$

(iii) $|z+2| = 2$

if $z = -2$, $|z+2| = |-2+2| = 0 < 2$

$z = 2$ $\quad |z+2| = |2+2| = 4 > 2$

$z = -2$ lies inside of C

$z = 2$ lies outside of C

$f(z) = \frac{z+6}{z-2}$ may be analytic in C

$z = 2, -2$ lies outside of C $f(z) = \frac{z+6}{z^2-4}$ (3)

by Cauchy's thm

$\int_C f(z) dz = 0$

$\therefore \int_C \frac{z+6}{z^2-4} dz = 0$

(ii) $|z-2| = 1$

if $z = 2$, $|z-2| = |2-2| = 0 < 1$

$z = -2$, $|z-2| = |-2-2| = |-4| = 4 > 1$

$z = 2$ lies inside of C

$z = -2$ lies outside of C

$f(z) = \frac{z+6}{z+2}$ maybe analytic

$\therefore$ by Cauchy's integral formula

$\int_C \frac{z+6}{z^2-4} dz = \int_C \frac{\frac{z+6}{z+2}}{z-2} dz = 2\pi i f(2)$

$= 2\pi i \left[\frac{z+6}{z+2}\right]_{z=2} = 2\pi i \left[\frac{2+6}{2+2}\right]$

$= 2\pi i \left(\frac{8}{4}\right) = 4\pi i$

(iii) $|z+2| = 2$

if $z = -2$, $|z+2| = |-2+2| = 0 < 2$

$z = 2$ $|z+2| = |2+2| = 4 > 2$

$z = -2$ lies inside of C

$z = 2$ lies outside of C

$f(z) = \frac{z+6}{z-2}$ maybe analytic in C

∴ Cauchy's formula

$$\int_C \frac{z+6}{z^2-4}\,dz = \int_C \frac{\frac{z+6}{z-2}}{z+2}\,dz$$

$$= 2\pi i\, f(-2)$$

$$= 2\pi i \left[\frac{z+6}{z-2}\right]_{z=-2}$$

$$= 2\pi i \left[\frac{-2+6}{-2-2}\right]$$

$$= 2\pi i \left[\frac{4}{-4}\right]$$

$$= -2\pi i$$

Ex (7) Evaluate $\int_C \frac{dz}{z^3(z+4)}$ where C is the circle $|z|=2$

Soln: $|z|=2$, $z=0$, $z+4=0$ $z=-4$

if $z=-4$ $|z| = |-4| = 4 > 2$

$z=0$ $|z| = |0| = 0 < 2$

$z=-4$ lies outside of C

$z=0$ lies inside of C

by Cauchy's integral formula for derivative

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i\, f^{n-1}(z_0)}{(n-1)!} \qquad z_0=0 \quad n=3$$

$$\int_C \frac{1}{z^3(z+4)}\,dz = \int_C \frac{\frac{1}{z+4}}{z^3}\,dz =$$

$$= \frac{2\pi i}{(3-1)!} f''(0)$$

$f(z) = \frac{1}{z+4}$

$f'(z) = -\frac{1}{(z+4)^2} = -(z+4)^{-2}$

$f''(z) = -(-2)(z+4)^{-3} = \frac{2}{(z+4)^3}$

$= \frac{2\pi i}{2!} f''(0)$

$= \frac{2\pi i}{2} \left(\frac{2}{(z+4)^3}\right)_{z=0}$

$= \pi i \left(\frac{2}{(4)^3}\right)$

$= \frac{2\pi i}{64} = \frac{\pi i}{32}$

Ex 8 Evaluate $\int_C \frac{1}{(z^3-1)^2} dz$ where

C is $|z-1| = 1$

Soln: $z^3 - 1 = 0 \quad (z-1)(z^2+z+1) = 0$

$z - 1 = 0, \quad z = 1$

$z^2 + z + 1 = 0 \quad z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$

$z = \frac{-1 \pm \sqrt{3} i}{2} \quad \therefore z = -\frac{1}{2} + \frac{\sqrt{3}}{2} i, \quad z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

if $z = 1 \quad |z-1| = |1-1| = 0 < 1$

$z = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \quad |z-1| = \left|-\frac{1}{2} + \frac{\sqrt{3}}{2} i - 1\right|$

$= \left|-\frac{3}{2} + i\frac{\sqrt{3}}{2}\right| = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3} > 1$

$z = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \quad |z-1| = \left|-\frac{1}{2} - i\frac{\sqrt{3}}{2} - 1\right| = \left|-\frac{3}{2} - i\frac{\sqrt{3}}{2}\right|$

$= \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3} > 1$

$z = 1$ lies inside of C

$z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ lies outside of C

$f(z) = \frac{1}{(z^2+z+1)^2}$ analytic and the

by Cauchy's formula for derivative

$$\int_C \frac{1}{(z^3-1)^2}\,dz = \int_C \frac{\frac{1}{(z^2+z+1)^2}}{(z-1)^2}\,dz$$

$$= \frac{2\pi i}{(n-1)!} f^{n-1}(1) \qquad n = 2,\ z_0 = 1$$

$$= \frac{2\pi i}{(2-1)!} f'(1)$$

$$= \frac{2\pi i}{1} f'(1)$$

$$f(z) = \frac{1}{(z^2+z+1)^2} = (z^2+z+1)^{-2}$$

$$f'(z) = -2(z^2+z+1)^{-3}(2z+1)$$

$$= -\frac{2(2z+1)}{(z^2+z+1)^3}$$

$$f'(1) = -\frac{2(2+1)}{(1+1+1)^3} = -\frac{2(3)}{27} = -\frac{2}{9}$$

$$= \frac{2\pi i}{1}\left(-\frac{2}{9}\right)$$

$$= -\frac{4\pi i}{9}$$

Ex (8) To find the value of the function

$$f(\xi) = \int_c \frac{\phi(z)}{z-\xi} \quad \text{at } \xi = \xi_0$$

If $f(\xi) = \int_c \frac{3z^2+2z+1}{z-\xi}\,dz$ where

c is the circle $x^2+y^2=4$ find the value

of (i) $f(3)$ (ii) $f'(1-i)$ (iii) $f''(1-i)$

Soln $x^2+y^2=(2)^2$, is a circle $r=2$

$|z|=2$

$z=3$ ~~to~~ $|z|=|3|=3>2$

point lies out side of circle

by Cauchy's thm

$$f(3) = \int_c \frac{3z^2+2z+1}{z-3}\,dz = 0$$

(ii) $\xi = 1-i$ $\quad |z| = |1-i| = \sqrt{1+1} = \sqrt{2} < 2$

$\xi = 1-i$ lies inside of C

~~$f'(\xi)$~~ $\phi(z) = 3z^2+2z+1$

$\phi(\xi) = 3\xi^2+2\xi+1$

$\phi'(\xi) = 6\xi+2$

$\phi''(\xi) = 6$

$$f(\xi) = \int_c \frac{3z^2+2z+1}{z-\xi}\,dz = 2\pi i\,(3\xi^2+2\xi+1)$$

$f'(\xi) = 2\pi i\,(6\xi+2)$

$f''(z) = 2\pi i(6)$

$f'(1-i) = 2\pi i(6(1-i)+2)$

$= 2\pi i[6-6i+2$

$= 2\pi i[8-6i]$

$=$

$f''(1-i) = 2\pi i(6)$

$= 12\pi i$