

## Residue thm (2)

(1)

Ex(1) Evaluate  $\int_C \frac{z^2}{(z-1)^2(z+1)} dz$  where

$C$  is (1)  $|z| = \frac{1}{2}$  (2)  $|z| = 2$

Soln  $(z-1)^2(z+1) = 0$

$$z = 1, -1$$

$\therefore z = -1$  is a simple pole

$z = 1$  is a pole of order 2

(1) If  $C$  is  $|z| = \frac{1}{2}$

if  $z = 1$ ,  $|z| = 1/1 = 1 > \frac{1}{2}$

$z = -1$ ,  $|z| = 1 - 1/1 = 1 > \frac{1}{2}$

$z = 1 - i$  possibly outside of  $C$

by Cauchy's thm

$$\int f(z) dz = \int_C \frac{z^2}{(z-1)^2(z+1)} dz = 0$$

(2)  $|z| = 2$

if  $z = 1$ ,  $|z| = 1/1 = 1 < 2$

$z = -1$ ,  $|z| = 1 - 1/1 = 1 < 2$

both poles lies inside of  $C$

$$\text{Residue} = \lim_{z \rightarrow -1} (z+1) \frac{z^2}{(z-1)^2(z+1)}$$

$$= \frac{1}{(-1-1)^2} = \frac{1}{4}$$

$$\text{Residue at } z=1 = \lim_{z \rightarrow 2} \frac{1}{(m-1)!} \lim_{z \rightarrow 2} \frac{d^{m-1}}{dz^{m-1}} (z-2)^m f(z) \quad (1)$$

$$= \frac{1}{(2-1)!} \lim_{z \rightarrow 1} \frac{d}{dz} (z-1)^2 \frac{z^2}{z+1} \Big|_{z=1}$$

$$= \frac{1}{1!} \lim_{z \rightarrow 1} \frac{d}{dz} \left( \frac{z^2}{z+1} \right)$$

$$= \lim_{z \rightarrow 1} \frac{(z+1) \cdot 2z - z^2 \cdot 1}{(z+1)^2}$$

$$= \frac{(1+1)(2(1))-1^2}{(1+1)^2}$$

$$= \frac{2(2)-1}{4} = \frac{3}{4}$$

use Cauchy's residue theorem

$\oint_C f(z) dz = 2\pi i (\text{residue at } z=-1, z=1)$

$$= 2\pi i \left( \frac{1}{4} + \frac{3}{4} \right) =$$

$$= 2\pi i (1)$$

$$= 2\pi$$

Ex ② Evaluate using Cauchy's

$$\text{Residue theorem } \oint_C \frac{1-2z}{z(z-1)(z-2)} dz$$

where C is  $|z|=1.5$  1.5

$$\sin f(z) = \frac{1-z^2}{z(z-1)(z-2)}$$

(3)

$$|z|=1.5$$

$$z(z-1)(z-2)=0$$

$z=0, 1, 2$  are simple poles

$$|f(z)|=1.5$$

$$\begin{array}{ll} \text{if } z=0 & |z|=|0|=0<1.5 \\ z=1 & |z|=|1|=1<1.5 \\ z=2 & |z|=|2|=2>1.5 \end{array} \quad \begin{array}{l} \text{little circle} \\ \text{little circle} \\ \text{large circle} \end{array}$$

$$\text{Residue at } z=0 = \lim_{z \rightarrow 0} (z-0) \frac{1-z^2}{z(z-1)(z-2)}$$

$$= \frac{1-0}{(0-1)(0-2)} = \frac{1}{2}$$

$$\text{Residue at } z=1 = \lim_{z \rightarrow 1} (z-1) \frac{(1-z^2)}{z(z-1)(z-2)}$$

$$= \frac{1-2^1}{1(1-2)} = \frac{1-2}{1-2} = \frac{-1}{-1} = 1$$

$$\int \frac{1-z^2}{z(z-1)(z-2)} dz = 2\pi i \left( \text{Sum of residues at } z=0, z=1 \right)$$

$$= 2\pi i \left( \frac{1}{2} + 1 \right)$$

$$= 2\pi i \left( \frac{3}{2} \right)$$

$$= 3\pi i$$

Ex(3) Using Residue theorem evaluate (4)

$$\oint_C \frac{e^{2z}}{(z-\pi i)^3} dz \text{ where } C: |z-2i|=2$$

$$|z-2i|=2$$

$$\text{Solve } z-\pi i=0$$

$$z=\pi i, \pi i, \pi i$$

$z=\pi i$  is a pole of order 3

$$\text{if } z=\pi i \quad |z-2i|=|\pi i-2i|=|\pi i-2i|=1.157 < 2$$

$z=\pi i$  lies inside of C

$$\begin{aligned} \text{Residue of } f(z) &= \lim_{(z-\pi i)^3 \rightarrow 0} \frac{d^2}{dz^2} (z-\pi i)^3 \cdot \frac{e^{2z}}{z-\pi i} \\ z=\pi i, m=3 &= 8! \lim_{z \rightarrow \pi i} \frac{d^2}{dz^2} e^{2z} \end{aligned}$$

$$= \frac{1}{2} \lim_{z \rightarrow \pi i} \frac{d^2}{dz^2} e^{2z}$$

$$= \frac{1}{2} \lim_{z \rightarrow \pi i} 2 e^{2z-2}$$

$$= \frac{1}{2} \lim_{z \rightarrow \pi i} e^{2z}$$

$$= 20 e^{2i\pi}$$

$$= 2 (\cos 2\pi + i \sin 2\pi)$$

$$= 2$$

$$\int_C f(z) dz = 2\pi i (n = 4\pi i)$$

(5)

Ex(4) Using residue theorem evaluate

$$\oint_C \frac{z-1}{z^2+2z+5} dz \text{ where } C \text{ is circle}$$

$$\textcircled{1} |z|=1 \quad \textcircled{2} |z+1+i|=2, \quad \textcircled{3} |z+1-i|=2$$

$$\text{Solve } z^2 + 2z + 5 = 0$$

$$z = -2 \pm \sqrt{4-20} = -2 \pm \sqrt{-16}$$

$$= -2 \pm 4i = -1 \pm 2i \quad \text{are simple poles}$$

$$\text{if } z = -1+2i \quad |z| = \sqrt{(-1)^2 + 2^2} = \sqrt{5} > 1$$

$$z = -1-2i \quad |z| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5} > 1$$

both points lies outside or C cancel it from

$$\oint_C f(z) dz = 0$$

$$\textcircled{1} \quad |z+1+i|^2 = |z+1+i| = |-1+2i+1+i| \\ \text{if } z = -1+2i \quad |z+1+i| = \sqrt{(-1+2i)^2 + i^2} = \sqrt{3i} \not\leq 3 > 2$$

$$z = -1-2i \quad |z+1+i| = \sqrt{(-1-2i)^2 + i^2} = \sqrt{-5} = \sqrt{2}$$

$z = -1+2i$  lying outside or C

$z = -1-2i$  lying inside or C

$$\text{Residue at } z = \lim_{z \rightarrow -1-2i} (z+1+i) \frac{(z-1)}{(z+1+i)(z+1-i)}$$

$$= \frac{-1-2i-1}{-1-2i+1-2i} = \frac{-2-2i}{-4i} = \frac{1+i}{2i}$$

(6)

$$\int \frac{z-1}{z^2+4z} dz = 2\pi i \left( \frac{1}{2i} \right) = \pi (1+i)$$

Ex.  $\int_C \frac{1}{z^3(z+4)} dz$  when  $|z|=2$

(1)  $\int_C \frac{e^z}{z^2+1} dz$   $|z|=4$

(2)  $\int_C \frac{\sin^6 z}{(z-\frac{\pi}{8})^3} dz$   $|z|=1$