Green's theorem

If P and Q are two functions of x, y and their partial derivatives $\frac{\partial P}{\partial y}$ $\frac{\partial Q}{\partial x}$ are continuous single valued functions over the closed region bounded by a curve then

$$\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Coro: vector form of Green's thm

If we put $\bar{F} = Pi + Qj$ and $\bar{r} = xi + yj$ ~~+ zk~~ then Green's thm

$$\int_C \bar{F} \cdot d\bar{r} = \iint_R \bar{N} \cdot (\nabla \times \bar{F})\, ds$$

Where $\bar{N}$ is the unit vector ~~to~~ along the z axis

Ex ① Evaluate by Green's theorem

$\int_C e^{-x} \sin y\, dx + e^{-x} \cos y\, dy$ where C is the rectangle whose vertices are $(0,0)$ $(\pi, 0)$ $(\pi, \frac{\pi}{2})$ $(0, \frac{\pi}{2})$

Soln: Green's theorem

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

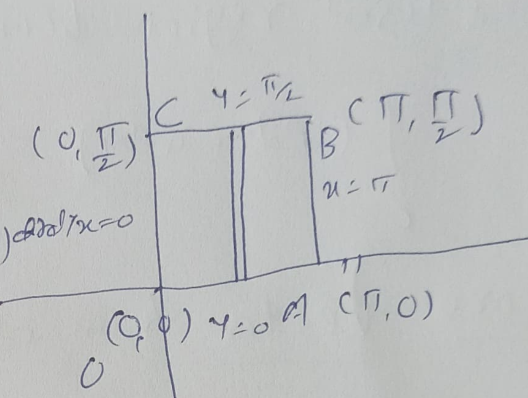
$\because\; P = e^{-x}\sin y, \quad Q = e^{-x}\cos y$

$$\int_C e^{-x}\sin y\,dx + e^{-x}\cos y\,dy = \int_C P\,dx + Q\,dy$$

$$\frac{\partial Q}{\partial x} = -e^{-x}\cos y \qquad \frac{\partial P}{\partial y} = e^{-x}\cos y$$

$y \Rightarrow y = 0$ to $y = \frac{\pi}{2}$

$x \Rightarrow x = 0$ to $x = \pi$



$$\int_C P\,dx + Q\,dy = \iint_R \left(-e^{-x}\cos y - e^{-x}\cos y\right)dx\,dy$$

$$= \int_0^{\pi}\int_0^{\pi/2} -2e^{-x}\cos y\,dy\,dx$$

$$= -2\int_0^{\pi} e^{-x}\left(\sin y\right)_0^{\pi/2} dx$$

$$= -2\int_0^{\pi} e^{-x}\big(\underbrace{\sin \tfrac{\pi}{2}}_{1} - \underbrace{\sin 0}_{0}\big)\,dx$$

$$= -2\int_0^{\pi} e^{-x}(1-0)\,dx$$

$$= -2\left(\frac{e^{-x}}{-1}\right)_0^{\pi}$$

$$= 2\left(e^{-\pi} - e^{0}\right)$$

$$= 2\left(e^{-\pi} - 1\right)$$

Ex ② Verify Green's thm for $\bar{F} = x^2 i - xy j$ and C is the triangle having vertices A(0, 2) B(2, 0) C(4, 2)

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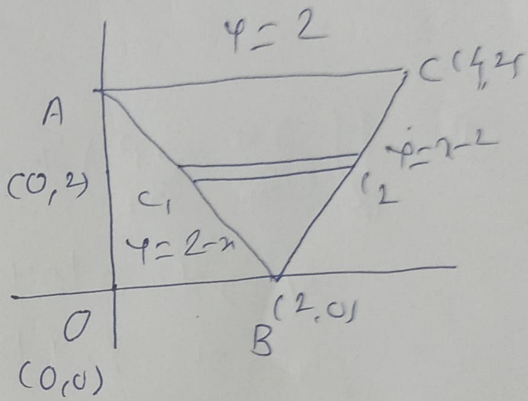
Soln. Green's theorem $\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$\bar{F} = x^2 i - xy j$ $\quad P = x^2,\ Q = -xy$

$\frac{\partial Q}{\partial x} = -y,\ \frac{\partial P}{\partial y} = 0$



① along AB
eqn of AB is

$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)$

$y - 2 = \frac{2 - 0}{0 - 2}(x - 0)$

$y - 2 = -x$ $\quad \therefore y = 2 - x$ $\quad x \Rightarrow x = 0$ to $x = 2$

$x + y = 2$ $\quad dy = -dx$

$\oint_{C_1} P\,dx + Q\,dy = \int_{C_1} x^2 dx - xy\,dy \quad \int_0^2 x^2 dx - x(2-x)(-dx)$

$= \int_0^2 x^2 + 2x - x^2\,dx = 2\left(\frac{x^2}{2}\right)_0^2 = 4 - 0 = 4$

along BC eqn of BC $y - 0 = \frac{0 - 2}{2 - 4}(x - 2)$

$y = x - 2$, $dy = dx$, $x \Rightarrow x = 2$ to $x = 4$

$\int_{C_2} P\,dx + Q\,dy = \int_2^4 x^2 dx - x(x-2)\,dx = \int_2^4 x^2 - x^2 + 2x\,dx$

$= 2\left(\frac{x^2}{2}\right)_2^4 = 16 - 4 = 12$

earlier CAB

along CA $\underline{y=2}$, $dy=0$

$x \Rightarrow x=4$ to $x=0$

$$\int_{C_3} P\,dx + Q\,dy = \int_4^0 x^2\,dx - xy\,dy = \int_4^0 x^2\,dx - 0$$

$$= \left(\frac{x^3}{3}\right)_4^0 = \frac{1}{3}(0-64) = -\frac{64}{3}$$

$$\therefore \int_C P\,dx + Q\,dy = 4 + 12 - \frac{64}{3} = 16 - \frac{64}{3}$$

$$= \frac{-16}{3} \quad \text{—} \ (1)$$

(ii) $$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_0^2 \int_{2-y}^{2+y} -y\,dx\,dy$$

$y \Rightarrow y=0$ to $y=2$

$x \Rightarrow x=2-y$ to $x=2+y$

$$= \int_0^2 -y\,(x)_{2-y}^{2+y}\,dy = -\int_0^2 y\,(2+y-2+y)\,dy$$

$$= -\int_0^2 y\,(2y)\,dy = -2\int_0^2 y^2\,dy = -2\left(\frac{y^3}{3}\right)_0^2$$

$$= -\frac{2}{3}(8-0) = \frac{-16}{3} \quad \text{—} \ (2)$$

verified Green's thm.

this question ask only evaluation

Ex (3) verify Green's thm in the plane for $\int_C (xy+y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$

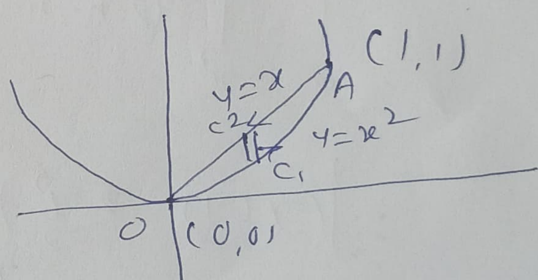
Soln. Green's thm $\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

For $\int_C (xy+y^2)\,dx + x^2\,dy$ $\quad P = xy+y^2,\; Q = x^2$

$\frac{\partial Q}{\partial x} = 2x \quad \frac{\partial P}{\partial y} = x+2y$

put $y=x$ in $y=x^2$: $x = x^2$ $\quad x^2 - x = 0 \quad x(x-1)=0$

$x=0,\, x=1$ $\quad$ if $x=0$ $y=0$; $x=1$ $y=1$ $\quad y=0$ $(0,0)$; $y=1$ $(1,1)$



(1) along OA

$y = x^2,\; dy = 2x\,dx$

$x \Rightarrow x=0$ to $x=1$

$$\int_C P\,dx + Q\,dy = \int_C (xy+y^2)\,dx + x^2\,dy$$

$$= \int_0^1 (x^3+x^4)\,dx + x^2\, 2x\,dx = \int_0^1 (x^3+x^4+2x^3)\,dx$$

$$= \int_0^1 3x^3 + x^4\,dx = 3\left(\frac{x^4}{4}\right) + \frac{x^5}{5}\Big|_0^1$$

$$= \frac{3}{4}(1-0) + \frac{1}{5}(1-0) = \frac{3}{4} + \frac{1}{5} = \frac{15+4}{20}$$

$$= \frac{19}{20}$$

along AO_{C_2} $y=x$, $dy=dx$

$$\int_{C_2} P\,dx+Q\,dy=\int_{C_2}(xy+y^2)dx+x^2\,dy$$

$$=\int_1^0 x(x)+x^2)\,dx+x^2\,dx=\int_1^0 (x^2+x^2+x^2)dx$$

$$=\int_1^0 3x^2\,dx=3\left(\frac{x^3}{3}\right)_1^0=0-1=-1$$

$$\int_C P\,dx+Q\,dy=\frac{19}{20}-1=-\frac{1}{20}$$

(ii) $$\iint_R \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dx\,dy=\iint_R 2x-(x+2y)\,dy\,dx$$

$y \Rightarrow y=x^2$ to $y=x$, $x \Rightarrow x=0$ to $x=1$

$$=\int_0^1\int_{x^2}^{x}(2x-x-2y)\,dy\,dx=\int_0^1\int_{x^2}^{x}(x-2y)\,dy\,dx$$

$$=\int_0^1\left(x(y)-\frac{2y^2}{2}\right)_{x^2}^{x}dx=\int_0^1 x(x-x^2)-(x^2-x^4)\,dx$$

$$=\int_0^1(x^2-x^3-x^2+x^4)\,dx=-\left(\frac{x^4}{4}\right)_0^1+\left(\frac{x^5}{5}\right)_0^1$$

$$=-\frac{1}{4}(1-0)+\frac{1}{5}(1-0)=-\frac{1}{4}+\frac{1}{5}=\frac{-5+4}{20}=-\frac{1}{20}$$

verify Green's thm

Ex (4) Use Green's theorem to evaluate

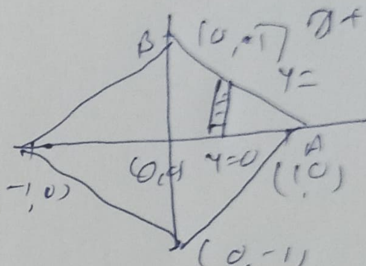
$\oint_C (x^2+xy)\,dx + (x^2+y^2)\,dy$ where C is the square formed by the line $\pm 1, \pm 1$ (17)

Soln $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$\oint_C (x^2+xy)\,dx + (x^2+y^2)\,dy$ $\quad P = x^2+xy$

$Q = x^2+y^2$

$\frac{\partial Q}{\partial x} = 2x, \quad \frac{\partial P}{\partial y} = x$



eqn of AB $y-0 = \frac{0-1}{1-0}(x-1)$

$y = -(x-1) = -x+1$

$x+y=1$ $\quad y \Rightarrow 0$ to $y = 1-x$

$x \Rightarrow x=0$ to $x=1$

$= \int_0^1 \int_0^{1-x} (2x - x)\,dy\,dx = \int_0^1 \int_0^{1-x} x\,dy\,dx$

$= \int_0^1 x(y)_0^{1-x} dx = \int_0^1 x(1-x)\,dx = \int_0^1 x - x^2\,dx$

$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$

Total $= 4 \iint_{\Delta OAB} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = 4\left(\frac{1}{6}\right) = \frac{2}{3}$

HW (1) evaluate using Green's thm $\oint_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$ where C is the boundary of surface enclosed by the lines $x=0, y=0, x=2, y=3$

Ans $= 12$