

① Vector calculus

1.1 Solenoidal and irrotational vector fields

vector operator Del ∇

$$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

Gradient: If ϕ is scalar point function then vector function $\nabla \phi$ is called the gradient of ϕ

$$\nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

Divergence:

Defn: Let $\bar{F} = f_1 i + f_2 j + f_3 k$ then

$$\nabla \cdot \bar{F} = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (f_1 i + f_2 j + f_3 k)$$

$$= \frac{\partial}{\partial x} f_1 + \frac{\partial}{\partial y} f_2 + \frac{\partial}{\partial z} f_3$$

is called the divergence of \bar{F} fun

$$\boxed{\text{div } \bar{F} = \nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

If \bar{F} is vector point function such that $\nabla \cdot \bar{F} = 0$ then \bar{F} is called solenoidal

②

CURL : If $\vec{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$ then

$$\nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \text{ is}$$

Called free curl of \vec{F}

$$\therefore \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

If \vec{F} is a vector point function such that
 $\text{curl } \vec{F} = \nabla \times \vec{F} = 0$ then \vec{F} is called
 irrotational or conservative field.

Ex ① If $\vec{F} = xy e^{xz} \mathbf{i} + x^2 \cos z \mathbf{j} + x^2 \cos y \mathbf{k}$

Find $\text{div } \vec{F}$ and $\text{curl } \vec{F}$

Soln: $\vec{F} = xy e^{xz} \mathbf{i} + x^2 \cos z \mathbf{j} + x^2 \cos y \mathbf{k}$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3$$

$$= \frac{\partial}{\partial x} (xy e^{xz}) + \frac{\partial}{\partial y} (x^2 \cos z) + \frac{\partial}{\partial z} (x^2 \cos y)$$

$$= ye^{xz} (1) + x(\cos z, 2y) + 0$$

$$\text{div } \vec{F} = ye^{xz} + 2xy \cos z$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy e^{2z} & x^2 \cos y & x^2 \cos y \end{vmatrix}$$

$$= i \left[\frac{\partial}{\partial y} (x^2 \cos y) - \frac{\partial}{\partial z} (xy e^{2z}) \right]$$

$$-j \left[\frac{\partial}{\partial x} (x^2 \cos y) - \frac{\partial}{\partial z} (xy e^{2z}) \right]$$

$$+k \left[\frac{\partial}{\partial x} (xy e^{2z}) - \frac{\partial}{\partial y} (x^2 \cos y) \right]$$

$$= i \left[x^2 (-\sin y) \cdot x + \cancel{\cos} - (x^2 (-\sin y)) \right]$$

$$-j \left[x^2 (-\sin y) \cdot y + (\cos y) \cdot 2x - (x^2 e^{2z} \cdot 2) \right]$$

$$+k \left[y^2 \cos y - x e^{2z} \right]$$

$$= i \left[-x^3 \sin y + x^2 y \sin y \right] - j \left[-x^2 y \sin y \right]$$

$$+ 2x \cos y - 2x y e^{2z}$$

$$+ k \left[y^2 \cos y - x e^{2z} \right]$$

(4)

A vector \vec{F} is said to be solenoidal if $\nabla \cdot \vec{F} = 0$

A vector \vec{F} is said to be irrotational if $\nabla \times \vec{F} = 0$

Ex(2) If $\vec{F} = (x+3y)j + (y-2z)j + (az+x)k$ is solenoidal find a

Soln: If \vec{F} is solenoidal then $\nabla \cdot \vec{F} = 0$

$$\begin{aligned}\therefore \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(F_1) + \frac{\partial}{\partial y}(F_2) + \frac{\partial}{\partial z}(F_3) \\ &= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(az+x) = 0\end{aligned}$$

$$\therefore 1 + 1 + a = 0$$

$$a + 2 = 0 \therefore a = -2$$

Ex(3) Prove that $\vec{F} = (x+2y+az)i + (bx-3y-z)j + (4x+y+2z)k$ is

$$+ (bx-3y-z)j + (4x+y+2z)k$$

solenoidal and find a, b, c if \vec{F} is irrotational.

Soln: If \vec{F} is solenoidal then $\nabla \cdot \vec{F} = 0$

$$\therefore \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x+2y+az) + \frac{\partial}{\partial y}(bx-3y-z)$$

$$+ \frac{\partial}{\partial z}(4x+y+2z)$$

$$= 1 - 3 + 2 = 0$$

$\nabla \cdot \vec{F} = 0$ then \vec{F} is solenoidal

(5)

\vec{F} is irrotational then $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+z^2 & 4x+y+2z & bx-3y-z \end{vmatrix} = 0$$

$$\therefore i \left[\frac{\partial}{\partial y} (4x+y+2z) - \frac{\partial}{\partial z} (bx-3y-z) \right] - j \left[\frac{\partial}{\partial x} (4x+y+2z) - \frac{\partial}{\partial z} (x+2y+z^2) \right] + k \left[\frac{\partial}{\partial x} (bx-3y-z) - \frac{\partial}{\partial y} (x+2y+z^2) \right] = 0$$

$$\therefore i [c - (-1)] - j [4 - a] + k [b - 2] = 0$$
$$(c+i)i - j(4-a) + k(b-2) = 0i + 0j + 0k$$

$$c+1=0, \quad 4-a=0, \quad b-2=0$$

$$c=-1, \quad a=4, \quad b=2$$

$$a=4, \quad b=2, \quad c=-1$$

E*4)

A vector field is given by

(6)

$$\bar{F} = (x^2 + 2xy^2) \mathbf{i} + (y^2 + x^2y) \mathbf{j}. \text{ Show}$$

that \bar{F} is irrotational and find its scalar potential.

17

Soln. we s.t. $\nabla \times \bar{F} = 0$

$$\therefore \nabla \times \bar{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + 2xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$

$$= \mathbf{i} \left(\frac{\partial}{\partial y} (0) - \frac{\partial}{\partial z} (y^2 + x^2y) \right) - \mathbf{j} \left(\frac{\partial}{\partial x} (0) - \frac{\partial}{\partial z} (x^2 + 2xy^2) \right) + \mathbf{k} \left(\frac{\partial}{\partial x} (y^2 + x^2y) - \frac{\partial}{\partial y} (x^2 + 2xy^2) \right)$$

$$= \mathbf{i} [0 - 0] - \mathbf{j} [0 - 0] + \mathbf{k} [0 + 2y^2 - (0 + 2x^2)]$$

$$= 0\mathbf{i} - 0\mathbf{j} + \mathbf{k}(2y^2 - 2x^2)$$

$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k}$$

$$= 0$$

$$\nabla \times \bar{F} = 0$$

Hence \bar{F} is irrotational.

FF ϕ is scalar potential fun (7)

$$\vec{F} = \nabla \phi$$

$$(x^2 + 2xy^2)\hat{i} + (y^2 + x^2y)\hat{j} + 0\hat{k} = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

$$\therefore \frac{\partial \phi}{\partial x} = x^2 + 2xy^2 \quad - (1)$$

$$\frac{\partial \phi}{\partial y} = y^2 + x^2y \quad - (2)$$

$$\frac{\partial \phi}{\partial z} = 0 \quad - (3)$$

$$(1) \Rightarrow \int 1 \partial \phi = \int x^2 + 2xy^2 \partial x + \Psi_1(y, z)$$

$$\phi = \frac{x^3}{3} + \frac{x^2}{2}y^2 + \Psi_1(y, z) \quad - (4)$$

$$(2) \Rightarrow \int 1 \partial \phi = \int y^2 + x^2y \partial y + \Psi_2(x, z)$$

$$\phi = \frac{y^3}{3} + \frac{x^2y^2}{2} + \Psi_2(x, z) \quad - (5)$$

$$(3) \Rightarrow \int 1 \partial \phi = \int 0 \partial z + \Psi_3(x, y)$$

$$\phi = 0 + \Psi_3(x, y) \quad - (6)$$

from (4) (5) & (6)

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + C$$

Ex 5 P.T. $\vec{F} = (2^i + 2x + 3t) \vec{i} + (3x + 2y + 2) \vec{j}$ (5)
+ ($y + 2z^2$) \vec{k} is irrotational & quest

Find scalar potential function of such that

$$\vec{F} = \nabla \phi \text{ and } \phi(1, 1, 0) = 9$$