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Maths Tutorial 4 - Complex Variable

1. ~~Analy~~ Is $f(z) = \frac{z}{\bar{z}}$ analytic?

$$f(z) = \frac{z}{\bar{z}} = \frac{x+iy}{x-iy} = \frac{(x+iy)(x+iy)}{(x-iy)(x+iy)}$$

$$= \frac{(x+iy)^2}{x^2+y^2}$$

$$= \frac{x^2+2ixy+y^2}{x^2+y^2}$$

$$= \frac{x^2+y^2}{x^2+y^2} + i\frac{2xy}{x^2+y^2}$$

$$= 1 + i\frac{2xy}{x^2+y^2}$$

$\therefore u = 1$ and $v = \frac{2xy}{x^2+y^2}$

Now $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$.

But $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are not zero.

$\therefore f(z)$ is not analytic.

2. S.T. $f(z) = \frac{1}{r^2}(\cos 2\theta - i\sin 2\theta)$ is analytic

Let $W = f(z) = u + iv = \frac{1}{r^2}(\cos 2\theta - i\sin 2\theta)$

$$= \frac{\cos 2\theta}{r^2} - i\frac{\sin 2\theta}{r^2}$$

$\therefore u = \frac{\cos 2\theta}{r^2}$ and $v = \frac{-\sin 2\theta}{r^2}$

~~$\frac{\partial u}{\partial r} = \frac{r^2\cos 2\theta - \cos 2\theta(2r)}{r^4}$~~

$$\frac{\partial u}{\partial r} = \cos 2\theta \left(\frac{-2}{r^3}\right) = -\frac{2\cos 2\theta}{r^3} \quad \text{——} \ (1)$$

$$\frac{\partial u}{\partial \theta} = \frac{1}{r^2}(-\sin 2\theta).2. = \frac{-2\sin 2\theta}{r^2} \quad \text{——} \ (2)$$

$$\frac{\partial v}{\partial r} = \frac{2\sin 2\theta}{r^3} \quad \text{——} \ (3)$$

$$\frac{\partial v}{\partial \theta} = \frac{-1}{r^2}\cos 2\theta . 2$$

$$= -\frac{2\cos 2\theta}{r^2} \quad \text{——} \ (4)$$

Now equating (1) and (4),

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

And,

equating (2) and (3).

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

Both the above equations are CR equations.

Hence this satisfies the CR equations for polar co-ordinates.

Hence it is proved that $f(z)$ is analytic.

3. Find the imaginary part of the analytic function whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$. Also verify that v is harmonic.

Let $u = e^{2x}(x\cos 2y - y\sin 2y)$.

$\frac{\partial u}{\partial x} = e^{2x} \cdot 2(x\cos 2y - y\sin 2y) + e^{2x}(\cos 2y)$

$= e^{2x}(2x\cos 2y - 2y\sin 2y + \cos 2y)$

$\frac{\partial u}{\partial y} = e^{2x}(-2x\sin 2y) + e^{2x}(-\sin 2y - 2y\cos 2y)$

$= e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y)$

$\therefore f(z) = u + iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}$

$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$f'(z) = e^{2x}(2x\cos 2y - 2y\sin 2y + \cos 2y)$
$- i\left(e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y)\right)$

$y = 0,\ x = z$

$\therefore f'(z) = e^{2z}(2z+1) - i(0)$

$f'(z) = e^{2z}(2z+1)$

$f(z) = \frac{(1+2z)e^{2z}}{2} - \frac{e^{2z}}{2}$

$\int (2z+1) e^{2z}$
$+(2z+1) \quad e^{2z}/2$
$-2 \quad e^{2z}/4$

$f(z) = \frac{e^{2z}}{\cancel{2}}(\cancel{1} + 2z - \cancel{1})$

$f(z) = ze^{2z} + c$

Now finding v.

$f(z) = ze^{2z}$

$f(z) = (x+iy)e^{2(x+iy)}$

$= (x+iy)e^{2x} \cdot e^{2iy}$

$= (x+iy)e^{2x}(\cos 2y + i\sin 2y)$

$= e^{2x}(x+iy)(\cos 2y + i\sin 2y)$

$= e^{2x}(x\cos 2y - y\sin 2y) + i(x\sin 2y + y\cos 2y) + c$

$\therefore v = (x \sin 2y + y \cos 2y) e^{2x}$

Now, $\frac{\partial v}{\partial x} = 2e^{2x}(y\cos 2y + x\sin 2y) + e^{2x}(\sin 2y)$

$\frac{\partial^2 v}{\partial x^2} = 4e^{2x}(y\cos 2y + x\sin 2y) + 2e^{2x}\sin 2y + 2e^{2x}\sin 2y$

$\frac{\partial^2 v}{\partial x^2} = 4e^{2x}(y\cos 2y + x\sin 2y) + 4e^{2x}\sin 2y$

$= 4e^{2x}(y\cos 2y + x\sin 2y + \sin 2y)$

Now, $\frac{\partial v}{\partial y} = e^{2x}(\cos 2y - 2y\sin 2y + 2x\cos 2y)$

$\frac{\partial^2 v}{\partial y^2} = e^{2x}(-2\sin 2y - 2\sin 2y - 4y\cos 2y - 4x\sin 2y)$

$= e^{2x}(-4\sin 2y - 4y\cos 2y - 4x\sin 2y)$

$= 4e^{2x}(-\sin 2y - y\cos 2y - x\sin 2y)$

$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\therefore$ v is harmonic

4. Find the analytic function $f(z) = u + iv$ such that $u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$ when $f\left(\frac{\pi}{2}\right) = 0$.

We have $f(z) = u + iv$

$\therefore if(z) = iu - v$

$(1+i)f(z) = (u-v) + i(u+v)$

$= U + iV$

$\therefore \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(u-v)$

$$= \frac{(2\cos x - e^{y} - e^{-y})(-\sin x + \cos x) - (\cos x + \sin x - e^{-y})(-2\sin x)}{(2\cos x - e^{y} - e^{-y})^2}$$

$$\frac{\partial U}{\partial x} = \frac{2 + e^{y}\sin x - e^{-y}\sin x - e^{-y}\cos x - e^{-y}\cos x}{(2\cos x - e^{y} - e^{-y})^2} \quad \text{--- (1)}$$

$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}(u-v) = \frac{(2\cos x - e^{y} - e^{-y})(e^{-y}) - (\cos x + \sin x - e^{-y})(-e^{-y} + e^{-y})}{(2\cos x - e^{y} - e^{-y})^2}$$

$$= \frac{\cos x \cdot e^{-y} + \cos x\, e^{y} + \sin x\, e^{y} - \sin x\, e^{y} - 2}{(2\cos x - e^{y} - e^{-y})^2} \quad \text{---} (ii)$$

Since $U-V$ is given,

$$\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

$$\therefore (1+i) f(z) = U + iV$$

$$(1+i) f'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$$

$$(1+i) f'(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} \qquad y=0,\ x=z.$$

$$= \frac{2 + \sin z - \sin z - \cos z - \cos z}{(2\cos z - 2)^2} - i \cdot \frac{\cos z + \cos z + \sin z - \sin z - 2}{(2\cos z - 2)^2}$$

$$= \frac{2(1-\cos z)}{4(\cos z - 1)^2} - i\,\frac{2(\cos z - 1)}{4(\cos z - 1)^2}$$

$$= -\frac{1}{2(\cos z - 1)} - i \cdot \frac{1}{2(\cos z - 1)}$$

$$= (1+i)\frac{1}{2(1-\cos z)}$$

$$= (1+i)\frac{1}{4\sin^2(z/2)}$$

$$= \frac{(1+i)}{4}\operatorname{cosec}^2\left(\frac{z}{2}\right)$$

$$\therefore (1+i) f'(z) = \frac{(1+i)}{4}\operatorname{cosec}^2\left(\frac{z}{2}\right)$$

$$f'(z) = \frac{1}{4}\operatorname{cosec}^2\left(\frac{z}{2}\right)$$

$$f(z) = \frac{1}{4}\int \operatorname{cosec}^2 \frac{z}{2}\, dz$$

$$= \frac{1}{4_2} - \cot\left(\frac{z}{2}\right) \cdot 2. = \frac{-1}{2}\cot\frac{z}{2} + c.$$

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But when $z = \frac{\pi}{2}$, $f(z) = 0$.

$$\therefore 0 = \frac{-1}{2} + c$$

$$\therefore c = \frac{1}{2}.$$

$$\therefore f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right)$$

5. Show that $u = \cos x \cosh y$ is a harmonic function. Find its harmonic conjugate and corresponding analytic function.

$u = \cos x \cosh y$.

$\frac{\partial u}{\partial x} = -\sin x \cosh y$ $\quad \frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y$.

$\frac{\partial u}{\partial y} = \cos x \sinh y$ $\quad \frac{\partial^2 u}{\partial y} = \cos x \cosh y$.

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\cos x \cosh y + \cos x \cosh y = 0.$$

$\therefore u$ is a harmonic function.

Now,

$\frac{\partial u}{\partial x} = -\sin x \cosh y$ $\quad \left(\frac{\partial u}{\partial x}\right)_{x=z,\, y=0} = -\sin z$

$\frac{\partial u}{\partial y} = \cos x \sinh y$ $\quad \left(\frac{\partial u}{\partial y}\right)_{x=z,\, y=0} = 0$

$$\therefore f'(z) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -\sin z$$

$$f(z) = \int -\sin z \, dz = \cos z + c$$

This is the required analytic function.

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$$f(z) = \cos z$$
$$= \cos(x+iy)$$
$$= \cos x \cos iy - \sin x \sin iy$$
$$\therefore u+iv = \cos x \cosh y - i \sin x \sinh y$$
$$\therefore v = -\sin x \sinh y$$

This is the required harmonic conjugate.

6. Find the orthogonal trajectories of the family of curves $e^{-x}\cos y + xy = \alpha$ where α is a real constant in XY-Plane

Take $u = e^{-x}\cos y + xy$

$$\therefore \frac{\partial u}{\partial x} = -e^{-x}\cos y + y \qquad \frac{\partial u}{\partial y} = -e^{-x}\sin y + x.$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$f'(z) = (-e^{-x}\cos y + y) - i(-e^{-x}\sin y + x)$$
$$y = 0,\ x = z$$
$$f'(z) = -e^{-z} - iz$$
$$\therefore f(z) = \int -e^{-z} - iz$$

$$= e^{-z} - \frac{iz^2}{2} + c$$

$$f(z) = e^{-(x+iy)} - \frac{i(x+iy)^2}{2} + c$$
$$= e^{-x}(\cos y - i\sin y) - \frac{i}{2}(x^2 + 2ixy - y^2) + c$$

$\therefore$ Imaginary part, $v = -e^{-x}\sin y - \frac{1}{2}(x^2 - y^2)$

Hence, the required orthogonal trajectories are
$$e^{-x}\sin y + \frac{1}{2}(x^2 - y^2) = c^2$$