Gauss Divergence theorem

The Surface integral of the normal Component of a vector over a closed surface S is equal to the volume integral of the divergence of F throughout the volume bounded by S.

$$\iint_S \bar{N}\cdot\bar{F}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

where ~~N~~ $\bar{N}$ is the unit outward normal.

$dv = dx\,dy\,dz$

<u>Ex (1)</u> Use Gauss Divergence theorem to evaluate $\iint_S \bar{N}\cdot\bar{F}\,ds$ where $\bar{F} = x^2 i + zj + yzk$ and S is the surface of the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$

Soln Gauss Divergence theorem

$$\iint_S \bar{N}\cdot\bar{F}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz)$$

$$= 2x + 0 + y$$

$$= 2x + y$$

$$\therefore \iiint_V \nabla.\bar{F}\, dv = \int_0^1\int_0^1\int_0^1 (2x+y)\, dz\, dy\, dx$$

$$= \int_0^1\int_0^1 \left((2x+y)\, z\right)_0^1 dy\, dx$$

$$= \int_0^1\int_0^1 (2x+y)(1-0)\, dy\, dx$$

$$= \int_0^1 \left[2xy + \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 (2x(1) + \tfrac{1}{2})\, dx$$

$$= \left(2\frac{x^2}{2} + \frac{1}{2}x\right)_0^1$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

Ex ② Use Gauss Divergence thm to evaluate $\iint_S \bar{N}.\bar{F}\, ds$ where $\bar{F} = 4x\,i + 3y\,j - 2z\,k$ and S is the surface bounded by $x=0, y=0, z=0$ and $2x+2y+z=4$

Soln Divergence thm

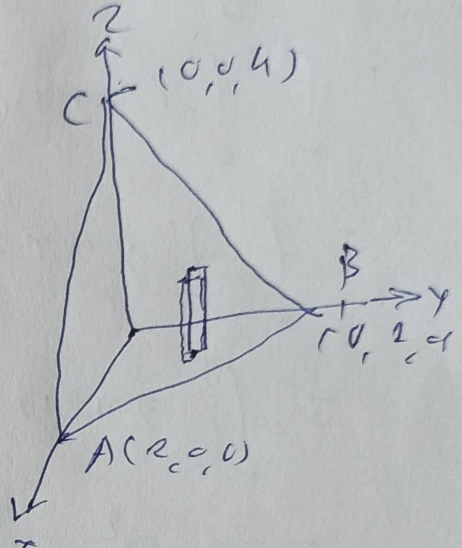
$$\iint_S \bar{N}.\bar{F}\, ds = \iiint_V \nabla.\bar{F}\, dv$$

(8)

$\bar{F} = 4x\,i + 3y\,j - 2z\,k$

$\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(-2z)$

$= 4 + 3 - 2 = 5$



$x=0, y=0, z=0$

$2x+2y+z=4$

$y=0, z=0, 2x=4, \; x=2$

$(2,0,0)$

$x=0, z=0, \; 2y=4 \; y=2$

$(0,0,2)$

$z=4 \; (0,0,4)$

$z=0, \; z=4-2x-2y$

$\iint_S \hat{n}\cdot \bar{F}\, ds = \iiint \nabla\cdot\bar{F}\, dv$ $\quad 2y=4-x$

$y=2-x$

$z=0$

$= \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} 5 \, dz\, dy\, dx$ $\quad x=0, x=2$

$= 5\int_0^2 \int_0^{2-x} (z)_0^{4-2x-2y} \, dy\, dx$

$= 5\int_0^2 \int_0^{2-x} (4-2x-2y)\, dy\, dx$

$= 5\int_0^2 \left(4y - 2xy - \frac{2y^2}{2}\right)_0^{2-x} dx$

$= 5\int_0^2 4(2-x) - 2x(2-x) - (2-x)^2 \, dx$

~~$= 5\left[4\left(2x-\frac{x^2}{2}\right) - 2\left(\frac{2x^2}{2} - \frac{x^3}{3}\right) - \frac{(2-x)^3}{-3}\right]_0^2$~~

$= 5[4(4-2) - 2(4-\frac{8}{3}) -$

$$= 5 \int_0^2 4(2-x) - 2x(2-x) - (2-x)^2 \, dx$$

$$= 5 \int_0^2 (8 - 4x - 4x + 2x^2 - 4 + 4x - x^2 \, dx$$

$$= 5 \int_0^2 4 - 4x + x^2 \, dx$$

$$= 5\left[4x - 4\frac{x^2}{2} + \frac{x^3}{3}\right]_0^2$$

$$= 5(8 - 2(4) + \frac{8}{3} - 0)$$

$$= 5\left(\frac{8}{3}\right)$$

$$= \frac{40}{3}$$

Ex (3) Use Gauss's divergence thm to evaluate $\iint_S \bar{N} \cdot \bar{F} \, ds$ where $\bar{F} = 4xi - 2y^2j + z^2k$ and S is the region bounded by $x^2 + y^2 = 4$

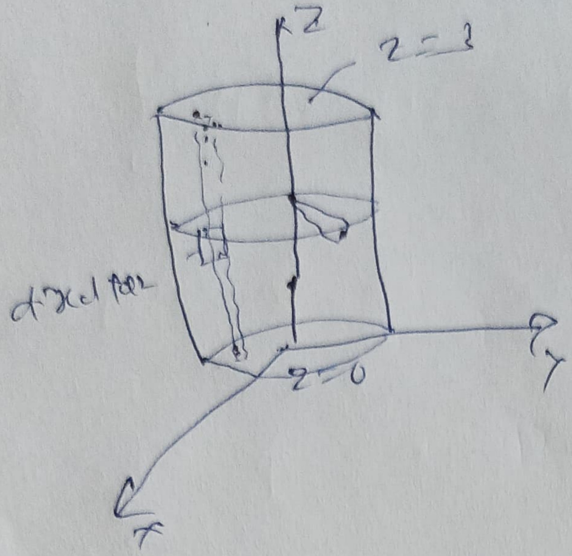
$z = 0, z = 3$ (15) 15

Sol.n Gauss Divergence thm

$$\iint_S \bar{N} \cdot \bar{F} \, ds = \iiint_V \nabla \cdot \bar{F} \, dv$$

$\bar{F} = 4xi - 2y^2j + z^2k$ $\therefore \nabla \cdot \bar{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$

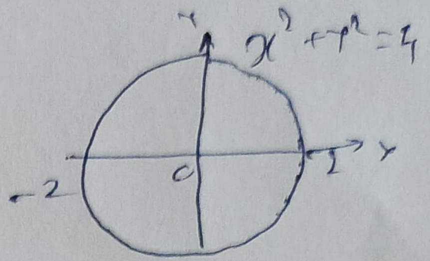
$\nabla \cdot \bar{F} = 4 - 4y + 2z$



$z=0 \quad z=3$

$y^2 = 4-x^2$

$x =$



$$\iiint_V \nabla\cdot F \, dV = \iiint_V (4-4y+2z)\,dx\,dy\,dz$$

$z=0,\ z=3,\ y=-\sqrt{4-x^2}\ \ y=\sqrt{4-x^2},\ x=\pm 2$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4-4y+2z)\,dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4z-4yz+\frac{2z^2}{2}\right)_0^3 dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12-4y(3)+9)\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21-12y)\,dy\,dx$$

$$= \int_{-2}^{2}\left[21y-\frac{12y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2}\left[21(\sqrt{4-x^2}+\sqrt{4-x^2}) - 6(\sqrt{4-x^2}-\sqrt{4-x^2})\right] dx$$

$$= 42\cdot 2\int_0^2 \sqrt{2^2-x^2}\,dx = 84\left(\frac{x}{2}\sqrt{4-x^2}+\frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right)_0^2$$

$$= 84\left(0+2\sin^{-1}\left(\frac{2}{2}\right)\right) - 84\cdot \ldots$$