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Tutorial-2 - Inverse Laplace Transform

1. Find $L^{-1}\left[\dfrac{5s^2-15s-11}{(s+1)(s-2)^2}\right]$

Let $\left[\dfrac{5s^2-15s-11}{(s+1)(s-2)^2}\right] = \dfrac{A}{s+1} + \dfrac{B}{s-2} + \dfrac{C}{s-2}$

$\therefore \dfrac{5s^2-15s-11}{\cancel{(s+1)(s-2)^2}} = \dfrac{A(s-2)^2 + B(s+1)(s-2) + C(s+1)}{\cancel{(s+1)(s-2)^2}}$

$\therefore 5s^2-15s-11 = A(s-2)^2 + B(s+1)(s-2) + C(s+1)$

$\therefore 5s^2-15s-11 = As^2 - 4sA + 4A + Bs^2 - Bs - 2B + Cs + C$.

$\therefore A = \dfrac{5(-1)^2-15(-1)-11}{(-1-2)^2} = \dfrac{9}{9} = 1$. $\therefore \boxed{A=1}$

$C = \dfrac{5(2)-15(2)-11}{2+1} = \dfrac{20-30-11}{3} = -7$ $\boxed{C=-7}$

$\therefore 5 = A + B$

$5 = 1 + B$

$\boxed{B = 4}$

$\therefore 5s^2-15s-11 = \dfrac{1}{s+1} + \dfrac{4}{s-2} - \dfrac{7}{(s-2)}$

$\therefore \quad = L^{-1}\left[\dfrac{1}{s+1} + \dfrac{4}{s-2} - \dfrac{7}{(s-2)^2}\right]$

$= e^{-t} + 4e^{2t} - 7te^{2t}$

2. Use convolution theorem to find $L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right]$

$$L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right] = L^{-1}\left[\frac{1}{(s^2+4s+4+9)^2}\right]$$

$$= L^{-1}\left[\frac{1}{(s+2)^2+1^2}\right]^2$$

$$= e^{-2t} \cdot L^{-1}\left[\frac{1}{s^2+3^2}\right]^2$$

$$f(s) = G(s) = \frac{1}{s^2+3^2}$$

$$\therefore f(t) = g(t) = \frac{\sin 3t}{3},$$

$$\int_0^t \frac{\sin 3u}{3} \cdot \frac{\sin 3(t-u)}{3}\, du$$

$$= \frac{1}{9}\int_0^t \sin 3u \cdot \sin(3t-3u)\, du$$

$$= \frac{1}{18}\int_0^t 2 \cdot \sin 3u \cdot \sin(3t-3u)\, du$$

$$= \frac{-1}{18}\int_0^t \left[\cos 3t - \cos(6u-3t)\right] du$$

$$= \frac{-1}{18}\left[u\cos 3t - \frac{\sin(6u-3t)}{6}\right]_0^t$$

$$= \frac{-1}{18}\left[t\cos 3t - \frac{\sin 3t}{6} - \frac{\sin 3t}{6}\right]$$

$$= \frac{1}{18}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right] = \frac{e^{-2t}}{18}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$$

3 Use convolution theorem to find $L^{-1}\left[\frac{(s-1)^2}{(s^2-2s+5)^2}\right]$

$L^{-1}\left[\frac{(s-1)^2}{(s^2-2s+5)^2}\right]$

$= L^{-1}\left[\frac{(s-1)^2}{(s^2-2s+1+5)^2}\right] = L^{-1}\left[\frac{(s-1)^2}{((s-1)^2+4)^2}\right]$

$= e^t \{ L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right]$

$= e^t L^{-1}\left[\frac{s}{s^2+2^2} \cdot \frac{s}{s^2+2^2}\right]$

$F(s) = G(s) = \frac{s}{s^2+2^2}$

$\therefore f(t) = g(t) = \cos 2t$

$\therefore \int_0^t \cos 2u \cdot \cos(2t-2u)\, du$

$= \frac{1}{2}\int_0^t 2\cos 2u \cdot \cos(2t-2u)\, du$

$= \frac{1}{2}\int_0^t \cos 2t + \cos(4u-2t)\, du$

$= \frac{1}{2}\left[u\cos 2t + \frac{\sin(4u-2t)}{4}\right]_0^t$

$= \frac{1}{2}\left[t\cos 2t + \frac{\sin(4t-2t)}{4} - 0 - \frac{\sin(-2t)}{4}\right]$

$= \frac{1}{2}\left[t\cos 2t + \frac{\sin 2t}{4} + \frac{\sin 2t}{4}\right]$

$= \frac{1}{2}\left[\frac{\sin 2t}{2} + t\cos 2t\right]$

$= \frac{e^t}{2}\left[\frac{\sin 2t}{2} + t\cos 2t\right]$

4. Find $L^{-1}[2\tanh^{-1}s]$

$L^{-1}[2\tanh^{-1}s]$

$= L^{-1}\left[2 \cdot \frac{1}{2}\log\left(\frac{1+s}{1-s}\right)\right]$

$= L^{-1}\left[\log\left(\frac{1+s}{1-s}\right)\right]$

$= \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\log\left(\frac{1+s}{1-s}\right)\right]$

$= \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\log(1+s) - \log(1-s)\right]$

$= \frac{-1}{t} L^{-1}\left[\frac{1}{1+s} - \frac{1}{1-s}(-1)\right]$

$= \frac{-1}{t} L^{-1}\left[\frac{1}{1+s} + \frac{1}{1-s}\right]$

$= \frac{-1}{t} L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$

$= \frac{-1}{t}\left[e^{-t} - e^{t}\right]$

$= \frac{2}{t}\left[\frac{e^{t} - e^{-t}}{2}\right]$

$= \frac{2}{t}\sinh t.$

5. Find $L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$.

$L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$

$= L^{-1}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]$

$$= \frac{-1}{t} L^{-1}\left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right]$$

$$= \frac{-1}{t}\left[2\cos at - 2\cos bt\right]$$

$$= \frac{2}{t}(\cos bt - \cos at)$$

6. Find $L^{-1}\left[\frac{1}{s(s^2+4)}\right]$

$$= \int_0^t L^{-1}\left[\frac{1}{s^2+4}\right] du$$

$$= \int_0^t \frac{\sin 2u}{2} du$$

$$= \frac{-1}{2}\left[\frac{\cos 2u}{2}\right]_0^t$$

$$= \frac{-1}{4}\left[\cos 2t - 1\right]$$

$$= \frac{1}{4}\left[1 - \cos 2t\right]$$