(9)

$L\{e^{-at} f(t)\} = F(s+a)$

$s \to s+a$

$L\{e^{-3t} f(2t)\} = \frac{s+3}{(s+3)^2 + 2(s+3) + 16}$

$= \frac{s+3}{s^2+6s+9+2s+6+16}$

$L\{e^{-3t} f(2t)\} = \frac{s+3}{s^2+8s+31}$

Ex (2) Find $L\{\cosh 2t \cdot \cos 2t\}$ 13

Soln $L\{\cosh 2t \cdot \cos 2t\}$

$= L\left[\left(\frac{e^{2t}+e^{-2t}}{2}\right)\cos 2t\right]$

$= \frac{1}{2}\left[L(e^{2t}\cos 2t) + L(e^{-2t}\cos 2t)\right]$

$s \to s-2$ $\quad s \to s+2$

$L(\cos 2t) = \frac{s}{s^2+4}$

$= \frac{1}{2}\left[\frac{s-2}{(s-2)^2+4} + \frac{s+2}{(s+2)^2+4}\right]$

$= \frac{1}{2}\left[\frac{s-2}{s^2-4s+4+4} + \frac{s+2}{s^2+4s+4+4}\right]$

$= \frac{1}{2}\left[\frac{s^3+4s^2+8s-2s^2-8s-16 + s^3-4s^2+8s+2s^2-8s+16}{(s^2+8-4s)(s^2+8+4s)}\right]$

$= \frac{1}{2}\left[\frac{2s^3}{(s^2+8)^2-(4s)^2}\right] = \frac{s^3}{s^4+16s^2+64-16s^2}$

$= \frac{s^3}{s^4+64}$

(10) Ex(1) Find L.T. of $\sinh at \sin at$

Soln $L[\sinh at \sin at]$ $\qquad \sinh at = \dfrac{e^{at} - e^{-at}}{2}$

$$= L\left[\left(\frac{e^{at} - e^{-at}}{2}\right) \sin at\right]$$

$$= \frac{1}{2}\left[L(e^{at} \sin at) - L(e^{-at} \sin at)\right]$$

$s \to s-a \qquad s \to s+a$

$$\therefore L(\sin at) = \frac{a}{s^2+a^2}$$

$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{2}a\left[\frac{1}{s^2-2as+a^2+a^2} - \frac{1}{s^2+2as+a^2+a^2}\right]$$

$$= \frac{a}{2}\left[\frac{\cancel{s^2}+2as+\cancel{2a^2}-\cancel{s^2}+2as-\cancel{2a^2}}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right]$$

$$= \frac{a}{2}\left[\frac{4as}{(s^2+2a^2)^2-(2as)^2}\right]$$

$$= \frac{2a^2s}{s^4+\cancel{4a^2s^2}+4a^4-\cancel{4a^2s^2}}$$

$$= \frac{2a^2s}{s^4+4a^4}$$

(11) Ex (2) Find L.T. of $\frac{\cos 2t \sin t}{e^t}$

Soln $L[e^{-t} \cos 2t \sin t]$

$= \frac{1}{2} L[e^{-t}(\sin(2t+t) - \sin(2t-t))]$

$= \frac{1}{2} L[e^{-t}(\sin 3t - \sin t)]$

$= \frac{1}{2}[L[e^{-t}\sin 3t]_{s \to s+1} - L[e^{-t}\sin t]_{s \to s+1}]$

$L[\sin 3t] = \frac{3}{s^2+9}$ $\quad L[\sin t] = \frac{1}{s^2+1}$

$= \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]$

$= \frac{1}{2}\left[\frac{3}{s^2+2s+1+9} - \frac{1}{s^2+2s+1+1}\right]$

$= \frac{1}{2}\left[\frac{3s^2+6s+6-s^2-2s-10}{(s^2+2s+10)(s^2+2s+2)}\right]$

$= \frac{1}{2}\left[\frac{2s^2+4s-4}{—''—}\right]$

$= \frac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}$

(12) Ex(3): Find $L[\sin 2t \cos t \cdot \cosh 2t]$ 11, 14

Soln $L[\sin 2t \cos t \cosh 2t]$

$= L\left[\left(\frac{e^{2t}+e^{-2t}}{2}\right) \cdot \frac{1}{2}\left(2\sin 2t \cos t\right)\right]$

$= \frac{1}{4} L\left[\left(e^{2t}+e^{-2t}\right)\left(\sin(3t)+\sin t\right)\right]$

$= \frac{1}{4}\Big[L(e^{2t}\sin 3t) + L(e^{-2t}\sin 3t)$
$\quad\quad\quad s\to s-2 \quad\quad\quad s\to s+2$

$+ L(e^{2t}\sin t) + L(e^{-2t}\sin t)\Big]$
$\quad\quad s\to s-2 \quad\quad\quad s\to s+2$

$L\{\sin 3t\} = \frac{3}{s^2+9}$ $\quad L[\sin t] = \frac{1}{s^2+1}$

$= \frac{1}{4}\left[\frac{3}{(s-2)^2+9} + \frac{3}{(s+2)^2+9} + \frac{1}{(s-2)^2+1} + \frac{1}{(s+2)^2+1}\right]$

$= \frac{1}{4}\left[\frac{3}{s^2-4s+13} + \frac{3}{s^2+4s+13} + \frac{1}{s^2-4s+5} + \frac{1}{s^2+4s+5}\right]$

Ex(5): Evaluate $\int_0^\infty e^{-t} \sin\frac{t}{2} \sinh\frac{\sqrt{3}}{2}t\, dt$ 12

Soln $\int_0^\infty e^{-st} \sinh\frac{\sqrt{3}}{2}t \sin\frac{t}{2}\, dt$

$= L\left[\sinh\frac{\sqrt{3}}{2}t \sin\frac{t}{2}\right]$

$= L\left[\left(\frac{e^{\frac{\sqrt{3}}{2}t} - e^{-\frac{\sqrt{3}}{2}t}}{2}\right)\sin\frac{t}{2}\right]$

$= \frac{1}{2}\left[L\left(e^{\frac{\sqrt{3}}{2}t}\sin\frac{t}{2}\right) - L\left(e^{-\frac{\sqrt{3}}{2}t}\sin\frac{t}{2}\right)\right]$

(13)

$$L[\sin\tfrac{t}{2}] = \frac{\frac{1}{2}}{s^2+\frac{1}{4}}$$

$$= \frac{1}{2}\left[\frac{\frac{1}{2}}{(s-\frac{\sqrt{3}}{2})^2+\frac{1}{4}} - \frac{\frac{1}{2}}{(s+\frac{\sqrt{3}}{2})^2+\frac{1}{4}}\right]$$

$$= \frac{1}{4}\left[\frac{1}{s^2-\sqrt{3}s+\frac{3}{4}+\frac{1}{4}} - \frac{1}{s^2+\sqrt{3}s+\frac{3}{4}+\frac{1}{4}}\right]$$

put $s=1$

$$\int_0^\infty e^{-t} \sin\frac{t}{2} \sinh\frac{\sqrt{3}}{2}t\,dt$$

$$= \frac{1}{4}\left[\frac{1}{1-\sqrt{3}+1} - \frac{1}{1+\sqrt{3}+1}\right]$$

$$= \frac{1}{4}\left[\frac{1}{2-\sqrt{3}} - \frac{1}{2+\sqrt{3}}\right]$$

$$= \frac{1}{4}\left[\frac{\cancel{2}+\sqrt{3}-\cancel{2}+\sqrt{3}}{4-3}\right)$$

$$= \frac{1}{4}\left(\frac{2\sqrt{3}}{1}\right) = \frac{\sqrt{3}}{2}$$

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(3) Effect of multiplication of t

If $L[f(t)] = F(s)$

then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

$n=1$ $L\{t\,f(t)\} = -1\frac{d}{ds}F(s)$

$L\{t^2 f(t)\} = \frac{d^2}{ds^2}F(s)$

(14) $L[t^3 f(t)] = -\frac{d^3}{ds^3} F(s)$

Ex ① Find LT of $t e^{-t} \cosh 2t$ — 15

Soln $L[t e^{-t} \cosh 2t]$

$= L\left[t e^{-t}\left(\frac{e^{2t}+e^{-2t}}{2}\right)\right]$

$= \frac{1}{2} L[t(e^{t}+e^{-3t})]$

$= \frac{1}{2}[L[te^{t}] + L(te^{-3t})]$

$L\{e^{t}\} = \frac{1}{s-1}, \quad L\{e^{-3t}\} = \frac{1}{s+3}$

$= \frac{1}{2}\left[(-1)\frac{d}{ds}\left(\frac{1}{s-1}\right) + (-1)\frac{d}{ds}\left(\frac{1}{s+3}\right)\right]$

$= \frac{1}{2}\left[-\left(\frac{-1}{(s-1)^2}\right)(0) - 1\left(-\frac{1}{(s+3)^2}\right)\right]$

$= \frac{1}{2}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+3)^2}\right]$

Ex ② Find $L[t e^{3t} \sin 4t]$ — 16

Soln $L[\sin 4t] = \frac{4}{s^2+16}$

$L[e^{3t}\sin 4t] = \frac{4}{(s-3)^2+16} = \frac{4}{s^2-6s+25}$

$L\{t e^{3t}\sin 4t\} = (-1)\frac{d}{ds}F(s) = -\frac{d}{ds}\left(\frac{4}{s^2-6s+25}\right)$

$= -4\left[-\frac{1}{(s^2-6s+25)^2}\cdot(2s-6)\right] = \frac{8(s-3)}{(s^2-6s+25)^2}$

(15) HW. $L[t \sin^3 t]$ 15.

$L[t \cos^2 t]$ 12.

E (1) Find $L[t\sqrt{1+\sin t}]$ 13, 15

Evaluate $\int_0^\infty e^{-t}(t\sqrt{1+\sin t})\,dt$ 16

Soln $\int_0^\infty e^{-st}\, t\sqrt{1+\sin t}\,dt$

$= L[t\sqrt{1+\sin t}]$

$\therefore L[\sqrt{1+\sin t}] = L[\sqrt{\sin^2\frac{t}{2}+\cos^2\frac{t}{2}+2\sin\frac{t}{2}\cos\frac{t}{2}}]$

$= L[(\sin\frac{t}{2}+\cos\frac{t}{2})^2]^{\frac{1}{2}}$

$= L[\sin\frac{t}{2}] + L[\cos\frac{t}{2}]$

$= \dfrac{\frac{1}{2}}{s^2+\frac{1}{4}} + \dfrac{s}{s^2+\frac{1}{4}} = \dfrac{\frac{1}{2}4^2}{4s^2+1} + \dfrac{4s}{4s^2+1}$

$= \dfrac{2+4s}{4s^2+1}$

$L[t\sqrt{1+\sin t}] = -1\dfrac{d}{ds}\left(\dfrac{2+4s}{4s^2+1}\right)$

$= -\left[\dfrac{(4s^2+1)(4)-(2+4s)(8s+0)}{(4s^2+1)^2}\right]$

$= -\left[\dfrac{16s^2+4-16s-32s^2}{(4s^2+1)^2}\right] = -\dfrac{(-16s^2-16s+4)}{(4s^2+1)^2}$

(1)

$$= \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

Put $s=1$

$$\int_0^\infty e^{-t}\, t \sqrt{1+\sin t}\, dt = \frac{4(4+4-1)}{(4+1)^2}$$

$$= \frac{28}{25}$$

Ex (4) $\displaystyle\int_0^\infty \frac{t^2 \sin 3t}{e^{2t}}\, dt$ 09

Soln $\displaystyle\int_0^\infty e^{-2t}\, t^2 \sin 3t\, dt$

$$\int_0^\infty e^{-st}\, t^2 \sin 3t\, dt = L[t^2 \sin 3t]$$

$$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)$$

$$= 3\frac{d}{ds}\left[\frac{d}{ds}\left(\frac{1}{s^2+9}\right)\right]$$

$$= 3\frac{d}{ds}\left(\frac{-1}{(s^2+9)^2}(2s)\right) = \frac{-d\,6s}{ds\,(s^2+9)^2}$$

$$= -6\frac{d}{ds}\left(\frac{s}{(s^2+9)^2}\right) = -6\left[\frac{(s^2+9)^2(1) - s \cdot 2(s^2+9)(2s)}{(s^2+9)^4}\right]$$

$$= (s^2+9)(-6)\left[\frac{s^2+9-4s^2}{(s^2+9)^{4\,3}}\right) = \frac{-6(9-3s^2)}{(s^2+9)^3}$$

(17) put $s = 2$

$$\int_0^\infty e^{-2t} t^2 \sin 3t \, dt$$

$$= -\frac{6(9-3(4))}{(4+9)^3} = -\frac{6(-3)}{(13)^2}$$

$$= \frac{18}{2197}$$

<u>HW.</u> Find $\int_0^\infty e^{-3t} t \cos t \, dt$

15

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(4) <u>division of t</u>

If $L[f(t)] = F(s)$ then

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s) \, ds$$

(1) $\int \frac{1}{x} dx = \log x$

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\int \frac{2x}{x^2+a^2} dx = \log(x^2+a^2)$$