Vector integration

line integral

Let $\bar{F}$ be a vector function defined through out some region of space and let c be any curve in that region.

$\bar{r} = xi + yj + zk$

$d\bar{r} = dxi + dyj + dzk$

$\bar{F} = F_1 i + F_2 j + F_3 k$

$$\int_C \bar{F} \cdot d\bar{r} = \int_C (F_1 i + F_2 j + F_3 k)(dxi + dyj + dzk)$$

$$= \int_C F_1 dx + F_2 dy + F_3 dz$$

line integral in Parametric form

curve C can be expressed in the Parametric form

$x = f_1(t),\ y = f_2(t),\ z = f_3(t)$

line integral will be reduced to $\int \phi(t) dt$.

integral betn the limits t_1 and t_2

t is parameter

condition for independence of the path in the line integral

If $\bar{F}$ is the gradient of some scalar point function ϕ i.e. if $\bar{F} = \nabla\phi$ then the line integral is independent of the path form A to B

$$\Rightarrow \quad \bar{F} = \nabla\phi = \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right)$$

$$\int_C \bar{F}.d\bar{r} = \int_A^B \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right)(dx\,i + dy\,j + dz\,k)$$

$$= \int_A^B \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \int_A^B d\phi = (\phi)_A^B$$

line integral doesnot depend on path but depends upon the end points A and B.

① If $\bar{F} = \nabla\phi$ such a field is called conservative.

② If the curve is closed and the field is conservative then $\phi_A = \phi_B$

$$\therefore \int_C \bar{F}\,d\bar{r} = \phi_A - \phi_A = 0$$

If the circulation of $\bar{F}$ along every closed curve in the region is zero then $\bar{F}$ is called irrotational

(3)

Ex(1) Evaluate $\int \bar{F} \times d\bar{r}$ where

$\bar{F} = (2xy + z^2)i + x^2 j + 3xz^2 k$ along the curve

$x = t,\ y = t^2,\ z = t^3$ from $(0,0,0)$ to $(1,1,1)$

Soln:

$$\bar{F} \times d\bar{r} = \begin{vmatrix} i & j & k \\ 2xy + z^2 & x^2 & 3xz^2 \\ dx & dy & dz \end{vmatrix}$$

$x = t,\ y = t^2,\ z = t^3 \quad dx = dt,\ dy = 2t\,dt$

$dz = 3t^2 dt$

$$\bar{F} \times d\bar{r} = \begin{vmatrix} i & j & k \\ 2t^3 + t^6 & t^2 & 3t^{7} \\ dt & 2t\,dt & 3t^2 dt \end{vmatrix}$$

$= i[3t^4 - 6t^8]dt - j(6t^5 + 3t^8 - 3t^7)$

$dt + k(\cancel{6t^5 + 3t^8}\ 4t^4 + 2t^7 - t^2)dt$

$x = t \quad x = 0,\ t = 0 \quad x = 1,\ t = 1$

$$\int_C \bar{F} \times d\bar{r} = \int_0^1 (3t^4 - 6t^8)dt\, i - j(6t^5 + 3t^8 - 3t^7)$$

$dt + (4t^4 + 2t^7 - t^2)dt\, k$

$= \left(\frac{3t^5}{5} - \frac{6t^9}{9}\right)_0^1 i - j\left(\frac{6t^6}{6} + \frac{3t^9}{9} - \frac{3t^8}{8}\right)_0^1$

$+ \left(\frac{4t^5}{5} + \frac{2t^8}{8} - \frac{t^3}{3}\right)_0^1 k$

$= \left(\frac{3}{5} - \frac{4}{3}\right) i - j\left(1 + \frac{1}{3} - \frac{3}{8}\right) + \left(\frac{4}{5} + \frac{1}{4} - \frac{1}{3}\right) k$

$= -\frac{11}{15} i + \frac{23}{24} j + \frac{43}{60} k$

(4)

Ex (2) Find the total work done in moving particle in the force field

$\bar{F} = 3xy\,i - 5z\,j + 10x\,k$ along $x = t^2+1$

$y = 2t^2$, $z = t^3$ from $t=1$ and $t=2$

Soln $\bar{F}\cdot d\bar{r} = (3xy\,i - 5z\,j + 10x\,k)\cdot(dx\,i + dy\,j + dz\,k)$

$= 3xy\,dx - 5z\,dy + 10x\,dz$

$= 3(t^2+1)(2t^2)\,2t\,dt - 5(t^3)(4t\,dt)$

$+ 10(t^2+1)(3t^2\,dt)$

$= 12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2\,dt$

$= (12t^5 + 12t^3 + 10t^4 + 30t^2)\,dt$

~~work done~~ work done $= \int_C \bar{F}\cdot d\bar{r}$

$= \int_1^2 (12t^5 + 12t^3 + 10t^4 + 30t^4)\,dt$

$= \left(\frac{12t^6}{6} + \frac{12t^4}{4} + \frac{10t^5}{5} + \frac{30t^3}{3}\right)_1^2$

$= (2t^6 + 3t^4 + 2t^5 + 10t^3)_1^2$

$= 303$

Ex (3): P.T. $\bar{F} = (y^2\cos x + z^3)\,i + (2y\sin x - 4)\,j + (3xz^2 + 4)\,k$ is a conservative field

Find (i) Scalar potential for $\bar{F}$ (ii)

(ii) the work done in ~~the~~ moving an object in this field from $(0,1,-1)$ to $(\frac{\pi}{2}, -1, 2)$

Soln The field is conservative then curl $\vec{F} = 0$

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(3xz^2 + 2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right)$$
$$- j\left(\frac{\partial}{\partial x}(3xz^2 + 2) - \frac{\partial}{\partial z}(y^2\cos x + z^3)\right)$$
$$+ k\left(\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x + z^3)\right)$$

$$= i[0 - 0) - j(3z^2 - 3z^2) + k(2y\cos x - 2y\cos x)$$

$$= 0i - 0j + 0k$$

curl $\vec{F} = 0$

$\vec{F}$ is conservative field.

(i) Since $\vec{F}$ is conservative then there exists scalar potential ϕ such that

$$\vec{F} = \nabla\phi$$

$(y^2\cos x + z^3)i + (2y\sin x - 4)j$ (6)

$+ (3xz^2 + 2)k = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$

$\therefore \frac{\partial \phi}{\partial x} = y^2\cos x + z^3$ — (1)

$\frac{\partial \phi}{\partial y} = 2y\sin x - 4$ — (2)

$\frac{\partial \phi}{\partial z} = 3xz^2 + 2$ — (3)

(1) $\Rightarrow$ $\int 1\,\partial\phi = \int y^2\cos x + z^3\,\partial x + \psi_1(y, z)$

$\phi = \underline{y^2\sin x} + \underline{z^3 x} + \psi_1(y, z)$ — (4)

(2) $\Rightarrow$ $\int 1\,\partial\phi = \int 2y\sin x - 4\,\partial y + \psi_2(x, z)$

$\phi = \frac{2y^2}{2}\sin x - 4y + \psi_2(x, z)$

$\phi = \underline{y^2\sin x} - \underline{4y} + \psi_2(x, z)$ — (5)

(3) $\Rightarrow$ $\int 1\,\partial\phi = \int 3xz^2 + 2\,\partial z + \psi_3(x, y)$

$\phi = \frac{3xz^3}{3} + 2z + \psi_3(x, y)$

$\phi = \underline{xz^3} + \underline{2z} + \psi_3(x, y)$ — (6)

(4) (5) + (6)

$\phi = y^2\sin x + xz^3 - 4y + 2z$

workdone $= \int_c \bar{F} \cdot d\bar{r}$

$= \int_c [(y^2 \cos x + z^3) i + (2y \sin x - 4) j + (3xz^2 + 2) k] [dx i + dy j + dz k]$

$= \int_c (y^2 \cos x + z^3) dx + (2y \sin x - 4) dy + (3xz^2 + 2) dz$

$= \int_c d(y^2 \sin x + xz^3 - 4y + 2z)$

$= [y^2 \sin x + xz^3 - 4y + 2z]_{(0,1,-1)}^{(\frac{\pi}{2}, -1, 2)}$

$= (\sin \frac{\pi}{2} + \frac{\pi}{2}(8) - 4(-1) + 2(2)) - (0 + 0 - 4(1) + 2(-1))$

$= (1 + 4\pi + 4 + 4) + 6$

$= 4\pi + 15$

Ex (5) If the vector field $\bar{F}$ is irrotational Find the constants a, b, c where $\bar{F}$ is given by $\bar{F} = (x + 2y + az) i + (bx - 3y - z) j + (4x + cy + 2z) k$. S.T. $\bar{F}$ can be expressed as the gradient of a scalar function. then find the workdone in moving a particle in this field from $(1, 2, -4)$ to $(3, 3, 2)$ along the straight line joining these points.

Soln. The field $\bar{F}$ is irrotational

then curl $\bar{F} = 0$

$$\text{curl } \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right]$$
$$-j\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$$
$$+k\left[\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right]$$

$$= i[c-(-1)] - j[4-a] + k[b-2] = 0$$

$$\therefore (c+1)i - j(4-a) + (b-2)k = 0i + 0j + 0k$$

$$c+1=0 \quad 4-a=0 \quad b-2=0$$
$$c=-1 \quad a=4 \quad b=2$$

$$\bar{F} = (x+2y+4z)i + (2x-3y-z)j + (4x-y+2z)k$$

Since $\bar{F}$ is irrotational then there exists scalar potential function ϕ such that

$$\bar{F} = \nabla\phi$$

$$\therefore (x+2y+4z)i + (2x-3y-z)j + (4x-y+2z)k = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

$$\therefore \frac{\partial \phi}{\partial x} = x + 2y + 4z \quad -(1)$$

$$\frac{\partial \phi}{\partial y} = 2x - 3y - z \quad -(2)$$

$$\frac{\partial \phi}{\partial z} = 4x - y + 2z \quad -(3)$$

$$(1) \Rightarrow \int \partial \phi = \int x + 2y + 4z \, \partial x + \psi_1(y, z)$$

$$\phi = \frac{x^2}{2} + 2xy + 4zx + \psi_1(y, z) \quad -(4)$$

$$(2) \Rightarrow \int \partial \phi = \int 2x - 3y - z \, \partial y + \psi_2(x, z)$$

$$\phi = 2xy - \frac{3y^2}{2} - zy + \psi_2(x, z) \quad -(5)$$

$$(3) \Rightarrow \int \partial \phi = \int 4x - y + 2z \, \partial z + \psi_3(x, y)$$

$$\phi = 4xz - yz + 2\frac{z^2}{2} + \psi_3(x, y)$$

$$\phi = 4xz - yz + z^2 + \psi_3(x, y) \quad -(6)$$

(4) (5) & (6)

$$\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3}{2}y^2 - yz + z^2$$

work done $= \int_C F \cdot d\bar{r} = \int_C (x + 2y + 4z)\,dx$

$$+ (2x - 3y - z)\,dy + (4x - y + 2z)\,dz$$

$$= \int_C d\left(\frac{x^2}{2} - \frac{3y^2}{2} - yz + 2xy + 4xz + z^2\right)$$

$$= \left(\frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz\right)_{(1, 2, -4)}^{(3, 3, 2)}$$

$$= \frac{9}{2} - \frac{27}{2} + 4 + 18 + 24 - 6 - \left(\frac{1}{2} - 6 + 16 + 4 - 16 + 8\right)$$

$$= \frac{49}{2}$$