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Tutorial-3 - Fourier Series

1. Obtain fourier series for

$$f(x) = \left(\frac{\pi - x}{2}\right)^2 \text{ in } 0 \le x \le 2\pi \text{ and } f(x + 2\pi) = f(x)$$

S.T. $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\pi - x}{2}\right]^2 dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{(\pi - x)^2}{4}\,dx = \frac{1}{8\pi}\int_0^{2\pi}(\pi - x)^2\,dx$$

$$= \frac{1}{8\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = \frac{1}{8\pi}\cdot\frac{1}{(-3)}\left[(\pi - 2\pi)^3 - (\pi - 0)^3\right]$$

$$= \frac{-1}{24\pi}\left[-\pi^3 - \pi^3\right] = \frac{-1}{\underset{12}{\cancel{24}}\pi}\left(-\cancel{2}\pi^{\cancel{3}}\right)$$

$$= \frac{\pi^2}{12}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \cos nx\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi - x)^2\cos nx\,dx$$

$$\int (\pi - x)^2 \cdot \cos nx \Big|_0^{2\pi}$$
$$+ (\pi - x)^2 \quad \sin nx/n$$
$$+ 2(\pi - x) \quad -\cos nx/n^2$$
$$-2(-1) = 2 \quad -\sin nx/n^3$$

$$= \frac{1}{4\pi}\left[-2(\pi - x)\frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-2(\pi - 2\pi)\frac{\cos 2n\pi}{n^2} - \left(-2\pi\frac{\cos 0n}{n^2}\right)\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi\cos 2n\pi}{n^2} + \frac{2\pi}{n^2}\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{1}{4\pi}\left[\frac{4\pi}{n^2}\right] = \frac{1}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi-x}{2}\right)^2 \sin nx \, dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi-x)^2 \sin nx \, dx$$

$$\int (\pi-x)^2 \sin nx$$
$$+(\pi-x)^2 \quad -\cos nx/n$$
$$+2(\pi-x) \quad -\sin nx/n^2$$
$$2 \quad \cos nx/n^3$$

$$= \frac{1}{4\pi}\left[-\frac{(\pi-x)^2 \cos nx}{n} + \frac{2\cos nx}{n^3} \right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-\frac{(\pi-2\pi)^2 \cos 2n\pi}{n} + \frac{2\cos 2n\pi}{n^3} - \left(-\frac{(\pi-0)^2 \cos n0}{n} + \frac{2\cos n0}{n^3} \right) \right]$$

$$= \frac{1}{4\pi}\left[-\frac{(-\pi)^2 \cos 2n\pi}{n} + \frac{2\cos 2n\pi}{n^3} - \left(-\frac{\pi^2 \cos n0}{n} + \frac{2\cos n0}{n^3} \right) \right]$$

$$= \frac{1}{4\pi}\left[\cancel{-\frac{\pi^2}{n}} + \cancel{\frac{2}{n^3}} + \cancel{\frac{\pi^2}{n}} - \cancel{\frac{2}{n^3}} \right]$$

$$= 0.$$

$$\therefore \quad a_0 = \frac{\pi^2}{12}; \quad a_n = \frac{1}{n^2} \text{ and } b_n = 0.$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\therefore \frac{1}{2\pi}\int_0^{2\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum (a_n^2 + b_n^2)$$

Now; $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} [f(x)]^2 dx = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\pi-x}{2}\right)^4 dx$

$$= \frac{1}{32\pi}\int_0^{2\pi} \left[\pi^4 - 4\pi^3 x + 6\pi^2 x^2 - 4\pi x^3 + x^4\right] dx$$

$$= \frac{1}{32\pi}\left[\pi^4 x - 2\pi^3 x^2 + 2\pi^2 x^3 - \pi x^4 + \frac{x^5}{5} \right]_0^{2\pi}$$

$$= \frac{1}{32\pi}\left[2\pi^5 - 8\pi^5 + \cancel{16\pi^5} - \cancel{16\pi^5} + \frac{32\pi^5}{5} \right]$$

$$= \frac{1}{32\pi} \cdot \frac{2\pi^5}{5} = \frac{\pi^4}{80}.$$

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$$\therefore \frac{\pi^4}{80} = \left(\frac{\pi^2}{12}\right)^2 + \frac{1}{2} \Sigma \left(\frac{1}{n^2}\right)^2$$

$$\frac{\pi^4}{80} = \frac{\pi^4}{144} + \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right)$$

$$\Rightarrow \pi^4\left(\frac{1}{80} - \frac{1}{144}\right) = \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right)$$

$$\pi^4\left(\frac{9-5}{720}\right) = \frac{\pi^4}{180} = \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right)$$

$$\therefore \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$$

Hence Proved

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Q.5. Find Fourier series for $f(x) = x + x^2$ in $(-\pi, \pi)$ and $f(x+2\pi) = f(x)$.

$f(x) = x + x^2$ is the sum of odd function 'x' and even function 'x^2'.

$\therefore f_1(x) = x$ and $f_2(x) = x^2$.

Let $f_1(x) = x = \sum_{n=1}^{\infty} b_n \sin nx \quad (\because a_0 = a_n = 0)$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2} + 0 + \frac{\sin n0}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\pi(-1) \cdot \frac{(-1)^n}{n} - 0\right]$$

$$= \frac{2(-1)^{n+1}}{n}.$$

$$\therefore f_1(x) = x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

Now since $f_2(x) = x^2$ is an even function, $(\because b_n = 0)$

$$f_2(x) = x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x) \, dx = \frac{1}{\pi}\int_0^{\pi} x^2 \, dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{\pi^2}{3}.$$

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$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2x\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2}\right] = \frac{4(-1)^n}{n^2}.$$

$$\therefore f_2(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$\therefore x + x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

3. Obtain Fourier series for $f(x) = 9 - x^2$ in $(-3, 3)$.

$f(x) = 9 - x^2$. interval $(-3, 3)$ $\therefore l = 3$.

We have $f(-x) = 9 - (-x^2) = 9 - x^2 = f(x)$

Hence $f(x)$ is even.

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{3}\int_0^3 (9 - x^2)\,dx$$

$$= \frac{1}{3}\left[9x - \frac{x^3}{3}\right]_0^3$$

$$= \frac{1}{3}\left[9\times 3 - \frac{27}{3}\right]$$

$$= \frac{1}{3}\left[27 - 9\right] = \frac{18}{3} = 6.$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{3}\int_0^3 (9-x^2)\cos\frac{n\pi x}{3}\,dx.$$

$$= \frac{2}{3}\left[(9-x^2)\sin\frac{n\pi x}{3}\left(\frac{3}{n\pi}\right) - 2x\cdot\cos\frac{n\pi x}{3}\left(\frac{3}{n\pi}\right)^2 + 2\sin\frac{n\pi x}{3}\left(\frac{3}{n\pi}\right)^3\right]_0^3$$

$\int (9-x^2)\cos\frac{n\pi x}{3}$

$+(9-x^2)$ $\quad \frac{\sin n\pi x}{3}\left(\frac{3}{n\pi}\right)$

$-(-2x)$ $\quad -\frac{\cos n\pi x}{3}\left(\frac{3}{n\pi}\right)^2$

$+(-2)$ $\quad -\frac{\sin n\pi x}{3}\left(\frac{3}{n\pi}\right)^3$

$$= \frac{2}{3}\left[0 - 6\cos n\pi\left(\frac{3}{n\pi}\right)^2 + 2\sin n\pi\left(\frac{3}{n\pi}\right)^3 - \left(9\sin n0\left(\frac{3}{n\pi}\right) - 0 + 2\sin n0\left(\frac{3}{n\pi}\right)^3\right]\right.$$

$$= \frac{2}{3}\left[-6\left(\frac{9}{n^2\pi^2}\cos n\pi\right)\right]$$

$$= \frac{-36}{n^2\pi^2}(-1)^n.$$

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~~f(x) = 6~~

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}.$$

But $b_n = 0$.

$$\therefore f(x) = 6 + \sum_{n=1}^{\infty} \frac{-36}{n^2\pi^2}(-1)^n \cos\frac{n\pi x}{l} + 0.$$

$$\therefore f(x) = 6 - \frac{36}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\frac{n\pi x}{l}$$

~~= 6 − 6 − 36~~

$$f(x) = 6 - \frac{36}{\pi^2}\left[\frac{-1}{1^2}\cos\frac{n\pi x}{3} + \frac{1}{2^2}\cos\frac{2\pi x}{3} - \frac{1}{3^2}\cos\frac{3\pi x}{3} + \cdots\right]$$

$$f(x) = 6 + \frac{36}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{3} - \frac{1}{2^2}\cos\frac{2\pi x}{3} + \frac{1}{3^2}\cos\frac{3\pi x}{3} - \cdots\right]$$

This is the Fourier Series.

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Q4. Obtain half range Sine Series for $f(x) = lx - x^2$ in $0 < x < l$
Hence ~~to~~ prove that

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots$$

Let $f(x) = \sum b_n \sin\frac{n\pi x}{l}$.

$$b_n = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx-x^2)\left(-\cos\frac{n\pi x}{l}\left(\frac{l}{n\pi}\right)\right) - (l-2x)\left(-\sin\frac{n\pi x}{l}\left(\frac{l^2}{(n\pi)^2}\right)\right) + (-2)\left(\cos\frac{n\pi x}{l}\left(\frac{l^3}{(n\pi)^3}\right)\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left(0 - 0 - \frac{2l^3}{n^3\pi^3}\cos n\pi\right) - \left(0 - 0 - \frac{2l^3}{n^3\pi^3}\right)\right]$$

$$= \frac{2}{l}\left[-\frac{2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{2}{\cancel{l}}\cdot\frac{2l^{\cancel{3}2}}{n^3\pi^3}(1 - \cos n\pi)$$

$$= \begin{cases} \dfrac{8l^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

$$\therefore f(x) = lx - x^2 = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \dots\right]$$

Now put $x = \frac{l}{2}$.

$$\therefore l\left(\frac{l}{2}\right) - \left(\frac{l}{2}\right)^2 = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots\right]$$

$$\therefore \frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots$$

Hence Proved.

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5. Obtain half range cosine series for $f(x) = e^x$ in $0 < x < 1$

$f(x) = e^x$ interval $0 < x < 1$. $\therefore l = 1$.

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{1}\int_0^1 e^x\,dx$$

$$= \left[e^x\right]_0^1 = \left[e^1 - e^0\right]$$

$$= e - 1.$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{1}\int_0^1 e^x \cos n\pi x\,dx$$

$$= 2\left[\frac{e^x}{1+n^2\pi^2}\left(\cos n\pi x + n\pi \sin n\pi x\right)\right]_0^1$$

$$= 2\left[\frac{e}{1+n^2\pi^2}(-1)^n - \frac{1}{1+n^2\pi^2}\right]$$

$$\therefore e^x = (e-1) + 2\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2}\left[e(-1)^n - 1\right]\cos n\pi x.$$

$$= (e-1) + 2\left[-\frac{1}{1+1^2\pi^2}(e+1)\cos\pi x + \frac{1}{1+2^2\pi^2}(e-1)\cos 2\pi x - \frac{1}{1+3^2\pi^2}(e+1)\cos 3\pi x + \cdots\right]$$

This is the fourier series

6. Obtain Fourier Series for $f(x) = |x|$ in $(-\pi, \pi)$.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$f(x) = |x| = x$

$f(-x) = |-x| = x$

$\therefore f(x) = f(-x) = x$

$\therefore f(x)$ is an even function.

$b_n = 0$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right]$$

$$= \pi.$$

$\therefore a_0 = \pi$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x.\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

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$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$\therefore a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right].$$

Now, $a_n = \begin{cases} 0 & \text{, if } n \text{ is even} \\ -4/\pi n^2 & \text{, if } n \text{ is odd} \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$$