Residues

If a function which is analytic in a region R is equal to zero at a point $z = z_0$ in R then z_0 is called zero of $f(z)$ in R

If $f(z_0) = 0$ but $f'(z_0) \neq 0$ then z_0 is called a simple zero or zero of first order

If $f(z_0) = 0$ and $f'(z_0) = f''(z_0) = f^{n-1}(z_0) = 0$ but $f^n(z_0) \neq 0$ then z_0 is called zero of order n

Taylor's series

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) \cdots$$

(i) If $\cancel{f(z_0)=0}$ $z = z_0$ is a simple pole then

$f(z_0) = 0$ & $f'(z_0) \neq 0$

(ii) if $z = z_0$ is a zero order n then $f(z_0) = f'(z_0) = f''(z_0) = 0$

$$f(z) = \frac{(z-z_0)^n}{n!} f^n(z_0)$$

Ex (1) Find the zeros of $f(z) = \sin z$

Soln $\sin z = 0$ when $z = 0, \pm\pi, \pm 2\pi, \dots$

$f'(z) = \cos z$ is not equal to zero

for these values

$f(z) = \sin z$ has simple zeros

at $z = 0$, $z = \pm\pi, \pm 2\pi, \dots$

Singular Points

If a function $f(z)$ is analytic at every point in the neighbourhood of a point z_0 itself then $z = z_0$ is called a singular point or singularity of $f(z)$

eg $f(z) = \frac{z^2}{z-2}$ $z = 2$ is a ~~singularity~~ singularity of $f(z)$.

Isolated Singularity

If the singular point z_0 of $f(z)$ is such that there is no other singular point in the neighbourhood of z_0 then such a singularity is called an isolated singularity

Pole If $z = z_0$ is an isolated singularity of $f(z)$ then we can find a region $0 < |z - z_0| < \delta$ in which $f(z)$ is analytic. Laurent's Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n} \quad -(1)$$

$$= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} \cdots + \frac{b_n}{(z - z_0)^n} \quad -(2)$$

(1) if $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$

$$+ \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} - - + \frac{b_n}{(z - z_0)^n}$$

$$b_{n+1} = b_{n+2} = 0$$

then $z = z_0$ is called pole of order n

(ii) A pole of order 1 is called a simple pole.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0}$$

Ex (1) S.T. $f(z) = \dfrac{1}{(z-1)^2 (z-2)^3}$

has a pole of order 2 at $z = 1$ order 3 at $z = 2$

Soln:

$z-1=0 \quad z-2=0$

$z=1 \quad z=2$

We can expand $f(z)$ around

$z=1, \; z=2$

$$f(z)=\frac{1}{(z-1)^2(z-2)^3} = \frac{1}{(z-1)^2[(z-1)-1]^3}$$

$$=-\frac{1}{(z-1)^2}\frac{1}{[1-(z-1)]^3}$$

$$=-\frac{1}{(z-1)^2}[1-(z-1)]^{-3}$$

$$(1-z)^{-3}=1+3z+6z^2+10z^3 \ldots$$

$$=-\frac{1}{(z-1)^2}[1+3(z-1)+6(z-1)^2+10(z-1)^3 \ldots]$$

$$=-\frac{1}{(z-1)^2}-\frac{3}{z-1}-6-10(z-1) \ldots$$

$$=-6-10(z-1)-\frac{3}{z-1}-\frac{1}{(z-1)^2}$$

$b_1 \neq 0, \; b_2 \neq 0, \; b_3=b_4=0$

$\therefore z=1$ is a pole order 2

$$f(z)=\frac{1}{(z-2)^3(z-1)^2}$$

$$=\frac{1}{(z-2)^3}\frac{1}{(z-2+1)^2}$$

$$=\frac{1}{(z-2)^3}[1+(z-2)]^{-2}$$

$$=\frac{1}{(z-2)^3}[1-2(z-2)+3(z-2)^2-4(z-2)^3 +5(z-2)^4 \ldots]$$

$$= \frac{1}{(z-2)^3} - \frac{2}{(z-2)^2} + \frac{3}{z-2} - 4 + 5(z-2)$$

$$= -4 + 5(z-2) + \frac{3}{z-2} - \frac{2}{(z-2)^2} + \frac{1}{(z-2)^3} - -$$

$b_1 \neq 0, b_2 \neq 0, b_3 \neq 0$ $b_n = 0$

$z = 2$ is a Pole of order $n = 3$

Removable Singularity

If $z = z_0$ is a singularity of $f(z)$ such that $\lim_{z \to z_0} f(z)$ exists then $z = z_0$ is called removable singularity

Residues

If $z = z_0$ is an isolated singularity then the constants b_1 ie coefficient of $\frac{1}{z-z_0}$ in the Laurent's expansion of $f(z)$ at $z = z_0$ is called the residue of $f(z)$ at $z = z_0$

$\therefore$ Residue of $f(z)$ at $z = z_0 = b_1$

$= \text{coefficient of } \frac{1}{z-z_0}$

$$= \frac{1}{2\pi i} \oint_C f(z)\,dz$$

or $\oint_C f(z)\,dz = 2\pi i$ (residue at $z = z_0$)

Calculation of residue at pole

(i) If $z = z_0$ is a simple pole of $f(z)$

then residue of $f(z)$ at $z = z_0$ $= \lim_{z \to z_0} (z - z_0) f(z)$

(ii) If $z = z_0$ is a simple pole of $f(z) = \dfrac{P(z)}{Q(z)}$

then residue of $f(z)$ at $z = z_0$ $= \lim_{z \to z_0} \dfrac{P(z)}{Q'(z)}$

(iii) If $z = z_0$ is a pole of order m

then

residue of $f(z)$ at $z = z_0$ $= \dfrac{1}{(m-1)!} \lim_{z \to z_0} \dfrac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right]$

Ex (1) Find the pole of the function

$$f(z) = \frac{z^2}{(z-1)^2 (z+2)}$$ and also find the residue at each pole

Soln. $(z-1)^2 (z+2) = 0$

$z = 1, 1, -2$

$\therefore$ $z = -2$ is a simple pole

$z = 1$ is a pole of order 2

$\therefore$ Residue of $f(z)$ at $z = z_0$ $= \lim\limits_{z \to z_0} (z - z_0) f(z)$

$\therefore$ Residue of $f(z)$ at $z = -2$ $= \lim\limits_{z \to -2} (z+2) \dfrac{z^2}{(z-1)^2 (z+2)}$

$$= \frac{(-2)^2}{(-2-1)^2} = \frac{4}{9}$$

Residue at $z = z_0$ order m $= \dfrac{1}{(m-1)!} \lim\limits_{z \to z_0} \dfrac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)]$

$\therefore$ Residue of $f(z)$ at $z = 1$ of order $m = 2$ $= \dfrac{1}{(2-1)!} \lim\limits_{z \to 1} \dfrac{d^{2-1}}{dz^{2-1}} \left[(z-1)^2 \dfrac{z^2}{(z-1)^2 (z+2)}\right]$

$$= \frac{1}{1} \lim_{z \to 1} \frac{d}{dz} \left[\frac{z^2}{z+2}\right]$$

$$= \lim_{z \to 1} \left[\frac{(z+2)\,2z - z^2(1)}{(z+2)^2}\right]$$

$$= \frac{(1+2)(2(1)) - (1)^2}{(1+2)^2}$$

$$= \frac{3(2) - 1}{3^2} = \frac{6-1}{9} = \frac{5}{9}$$

Ex (2) Find the residues

of $f(z) = \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)^2}$ at its poles

Soln $(z-1)(z-2)^2 = 0$

$z = 1, 2, 2$

$z = 1$ simple pole

$z = 2$ is pole of order $m = 2$

Residue at $z=1$ $= \lim_{z\to 1} (z-1) f(z)$

$= \lim_{z\to 1} \cancel{(z-1)} \left[\frac{\sin\pi z^2 + \cos\pi z^2}{\cancel{(z-1)}(z-2)^2} \right]$

$= \frac{\sin\pi + \cos\pi}{(1-2)^2} = \frac{0+(-1)}{(-1)^2}$

$= -1$

Residue at $z=2$, $m=2$ $= \frac{1}{(2-1)!} \lim_{z\to 2} \frac{d}{dz} [(z-2)^2 f(z)]$

$= \frac{1}{1} \lim_{z\to 2} \frac{d}{dz} \cancel{(z-2)^2} \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)\cancel{(z-2)^2}}$

$= \lim_{z\to 2} \frac{(z-1)(\cos\pi z^2 . 2\pi z - \sin\pi z^2 2\pi z) - (\sin\pi z^2 + \cos\pi z^2) 1}{(z-1)^2}$

$= \frac{1(4\pi\cos 4\pi - 4\pi\sin 4\pi) - (\sin 4\pi + \cos 4\pi)}{(2-1)^2}$

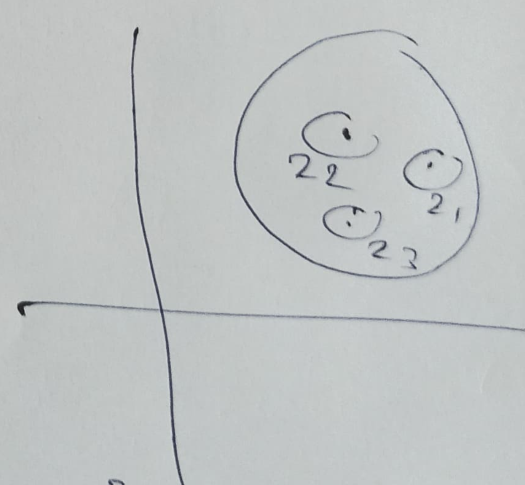
$$= \frac{4\pi(1) - 0 - (0+1)}{1}$$

$$= 4\pi - 1$$

Cauchy's Residue thm

If $f(z)$ is analytic inside and on a simple closed curve C, ~~expect~~ except at a finite number of isolated singular points $z_1, z_2 \ldots z_n$ inside C then

$$\oint_C f(z)\,dz = 2\pi i\,(\text{Sum of residues at } z_1, z_2, \ldots z_n)$$



Ex(1) Evaluate $\int_C \frac{z^2}{(z-1)^2(z+1)}\,dz$ where

C is (i) $|z| = \frac{1}{2}$ (ii) $|z| = 2$