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Date
Page

Maths Tutorial - 6 Numerical Methods for P.D.E.

1. Solve the partial differential equation $\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial y} = z$ given $z(x,0) = 3x^{-5x} + 2e^{-3x}$ by the method of separation of variables.

Let us assume the trial solution of the given equation in the form

$$z = X.Y.$$

where X is function of x and Y is function of y.

$$\therefore \frac{\partial z}{\partial x} = X'Y \quad \text{and} \quad \frac{\partial z}{\partial y} = XY'.$$

$\therefore$ Putting the above values in the equation, we get

$$X'Y - 2XY' = XY$$

$$\therefore X'Y - XY = 2XY'$$

$$\therefore (X'-X)Y = 2XY'$$

$$\therefore \frac{X'-X}{X} = \frac{2Y'}{Y} = a \quad (say).$$

$$\therefore \frac{X'-X}{X} = a$$

$$\therefore X'-X = aX$$

$$\therefore X' = aX + X$$

$$\therefore X' = X(a+1) \quad \text{--- (1)}.$$

and $\frac{2Y'}{Y} = a$

$$\therefore Y' = \frac{a}{2}Y \quad \text{--- (2)}.$$

Now equation (1) can be written as

$$\frac{dX}{dx} = X(a+1)$$

$$\therefore \frac{dX}{X} = (a+1)dx$$

Since now the variables are separated, by integration, we get

$$\log X = (a+1)x + \log c_1$$

$$\therefore \log \frac{X}{c_1} = (a+1)x$$

$$\therefore X = c_1 e^{(a+1)x} \quad \text{--- (3)}.$$

Equation (2) can be written as

$$\frac{dY}{dy} = \left(\frac{a}{2}\right) Y.$$

$$\therefore \frac{dY}{Y} = \frac{a}{2} dy.$$

Now variables are separated. Hence by integration,

$$\log Y = \left(\frac{a}{2}\right) y + \log c_2$$

$$\therefore \log \frac{Y}{c_2} = \frac{a}{2} \cdot y$$

$$\therefore Y = c_2 e^{ay/2} \quad \text{--- (4)}.$$

Now put the values of (3) and (4) in our equation

$$z = X \cdot Y.$$

$$\therefore z = c_1 e^{(a+1)x} \cdot c_2 e^{(ay/2)}$$

$$z = c e^{(a+1)x} \cdot e^{(ay/2)} \quad \text{--- (5)}.$$

But by data, $y = 0$, $z = 3e^{-5x} + 2e^{-3x}$ --- (6)

$\therefore$ Put $y = 0$.

$$\therefore z = c e^{(a+1)x} \quad \text{--- (7)}.$$

Comparing (6) and (7), (first term)

$\therefore c = 3$ and $(a+1) = -5$ $\therefore a = -6$.

Comparing (6) and (7) (second term)

$\therefore c = 2$ and $(a+1) = -3$ $\therefore a = -4$.

Using the two solution obtained, we get

$$z = 3e^{-5x} e^{-3y} + 2e^{-3x} \cdot e^{-2y}$$

$$\therefore z = 3e^{-(5x+3y)} + 2e^{-(3x+2y)}$$

This is the required solution of the given differential equation

2. A tightly stretched string with fixed end points $x=0$ and $x=l$ in the shape defined by $y = kx(l-x)$ where k is a constant is released from this position of rest. Find $y(x,t)$ the vertical displacement if $\frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Since we have $x=0$ and $x=l$,

we get $m = \frac{n\pi}{l}$.

$$\therefore y = c_5 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad \text{--- (1)}$$

Putting $n = 1, 2, 3$.

$$y = \Sigma b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad \text{--- (2)}$$

Now, $t=0$, $y = kx(l-x)$

Putting $t=0$ in (2), we get,

$$y = \Sigma b_n \sin \frac{n\pi x}{l} \quad \text{--- (3)}$$

where $y = kx(l-x)$.

But eq^n (3) is a fourier half range sine series for the function $f(x) = kx(l-x)$.

The coefficients b_n can be determined from

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_0^l kx(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\left[x(l-x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[\frac{-2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{4kl^2}{n^3\pi^3}(1-\cos n\pi)$$

Hence putting the value of b_n in ②, the solution is

$$y = \frac{4kl^2}{\pi^3}\sum\left(\frac{1-\cos n\pi}{n^3}\right)\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}, \quad n=1,2,3\ldots$$

i.e. $y(x,t) = \frac{8kl^2}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l} + \cdots\right]$

or

$$y(x,t) = \frac{8kl^2}{\pi^3}\sum\frac{1}{(2r-1)^3}\sin\frac{(2r-1)\pi x}{l}\cos\frac{(2r-1)\pi ct}{l}.$$

3 A rod of length l has its ends A and B kept at 0°C and 100°C respectively until steady states conditions prevail. If the temperature at B is reduced suddenly to 0°C and kept so while that of A is maintained. Find the temperature $u(x,t)$ at a distance x from A and at time t.

The differential equation of one-dimensional heat flow is of the form

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— (1)}$$

The solution of this equation is of the form

$$u = (c_1 \cos mx + c_2 \sin mx) e^{-m^2 c^2 t} \quad \text{——— (2)}$$

Given initial boundary conditions are

i) $u=0$ at $x=0$ for all values of t.

ii) $u=0$ at $x=l$ for all values of t.

iii) steady state at $t=0$.

Applying these conditions:-

i) put $x=0, u=0$ in (2), we get

$$0 = c_1 e^{-m^2 c^2 t}$$

$$\therefore c_1 = 0.$$

$\therefore$ (2) becomes $u = c_2 \sin mx \, e^{-m^2 c^2 t}$ ——— (3).

ii) Putting $x=l, u=0$ in (3), we get,

$$0 = c_2 \sin ml \, e^{-m^2c^2t}$$

$$\therefore ml = n\pi$$

$$\therefore m = \frac{n\pi}{l} \text{ where } n = 1,2,3\ldots$$

$\therefore$ (3) becomes $u = c_2 \sin \frac{n\pi x}{l} e^{-n^2\pi^2c^2t/l^2}$ where $n = 1,2,3\ldots$

Hence general solution is

$$u = \sum b_n \sin \frac{n\pi x}{l} e^{-n^2\pi^2c^2t/l^2} \quad \text{—— (4)}.$$

(ii) $t = 0$.

$$\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{i.e.} \quad \frac{d^2 u}{dx^2} = 0$$

$$\therefore \frac{du}{dx} = a \quad \text{and} \quad u = ax + b.$$

But when $x = 0, u = 0 \quad \therefore b = 0$

$$\therefore u = ax$$

and when $x = l, u = 100 \quad \therefore 100 = al$

$\therefore$ ~~$u = 0$~~

$$\therefore a = \frac{100}{l}$$

$$\therefore u = \frac{100x}{l} \quad \text{—— (5)}.$$

Using this condition (5) at $t = 0$ in (4), we get,

$$\frac{100x}{l} = \sum b_n \sin \frac{n\pi x}{l} \quad \text{—— (6)}.$$

But this is a Fourier half range sine series for the function $f(x) = \frac{100x}{l}$.

where $b_n = \frac{2}{l}\int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l}\, dx$

$$= \frac{200}{l^2}\int_0^l x \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\left[x\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right) - (1)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{200}{l^2}\left[\frac{-l^2}{n\pi}\cos n\pi\right]$$

$$= \frac{200}{\pi}\left(-\frac{\cos n\pi}{n}\right)$$

Hence from (4), we get the general solution as

$$u = \sum \frac{200}{\pi}\left(-\frac{\cos n\pi}{n}\right) \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2c^2t/l^2}$$

$$\therefore\ u = \frac{200}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l}\, e^{-\pi^2c^2t/l^2} - \frac{1}{2}\sin\frac{2\pi x}{l}\, e^{-4\pi^2c^2t/l^2} + \dots\right.$$

i.e. $u = \frac{200}{\pi}\sum \frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}\, e^{-n^2\pi^2c^2t/l^2}$

4\. Solve $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$, $u(0,t)=0$, $u(5,t)=0$, $u(x,0)=x^2(25-x^2)$

$h=1$, ~~upto 3~~ upto 3 seconds using Bender-Schmidt relation.

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \qquad \therefore\ a=1.\quad h=1$$

$$k = \frac{a h^2}{2}$$

$$k = \frac{1}{2}(1)^2 = \frac{1}{2}.$$

Since $h=1$ and range of x is 0 to 5, we divide x into 6 intervals. ~~from~~

We divide time interval by taking $k=1/2$ and since $t=3$

$\therefore\ t_0 = 0;\ t_1 = 1/2;\ t_2 = 1;\ t_3 = 3/2;\ t_4 = 2;\ t_5 = 5/2;\ t_6 = 3.$

By data, ~~u(x,~~ $u(0,t) = 0$.

Hence for all $t = 0, 1/2, 1, 3/2, 2, 5/2, 3,$

$u(0,0) = 0 \qquad u(0,3/2) = 0$

$u(0,1/2) = 0 \qquad u(0,2) = 0$

$u(0,1) = 0 \qquad u(0,5/2) = 0$

$u(0,3) = 0.$

By data, $u(5,t) = 0$

Hence for all $t = 0, 1/2, 1, 3/2, 2, 5/2, 3$

$u(5,0) = 0$ $\qquad$ $u(5,2) = 0$

$u(5,1/2) = 0$ $\qquad$ $u(5,5/2) = 0$

$u(5,1) = 0$ $\qquad$ $u(5,3) = 0$.

$u(5,3/2) = 0$

By data, $u(x,0) = x^2(25 - x^2)$

when $x = 0, 1, 2, 3, 4, 5,$

$u(0,0) = 0$.

$u(1,0) = 1(25-1) = 24$

$u(2,0) = 4(25-4) = 84$

$u(3,0) = 9(25-9) = 144$

$u(4,0) = 16(25-16) = 144$

$u(5,0) = 25(25-25) = 0$.

Thus we get the following table

$x \rightarrow h = 1$

$t \downarrow$ $\quad k = 1/2$

t \ x	0	1	2	3	4	5
0	0	24	84	144	144	0
1/2	0	42	84	114	72	0
1	0	42	78	78	57	0
3/2	0	39	60	67.5	39	0
2	0	30	53.25	49.5	33.75	0
5/2	0	26.625	39.75	43.5	24.75	0
3	0	19.875	35.0625	32.25	21.75	0

5. Using Crank-Nicholson method, solve $\frac{\partial^2 u}{\partial x^2} - 16\frac{\partial u}{\partial t} = 0$.

$0 < x < 1$, $t > 0$, $u(x,0) = 0$, $u(0,t) = 0$, $u(1,t) = 200t$.

Compute u for one step in t division taking $h = \frac{1}{4}$

$\frac{\partial^2 u}{\partial x^2} - 16\frac{\partial u}{\partial t} = 0$ $\therefore a = 16$ one step $\therefore t = 1$.

and $h = \frac{1}{4} = 0.25$

$K = ah^2$

$K = 16 \times \left(\frac{1}{4}\right)^2 = 16 \times \frac{1}{16}$

$K = 1$.

The interval of x is from 0 to 1.

subinterval is of size $h = \frac{1}{4}$.

$x_0 = 0$; $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$.

By data, $u(x,0) = 0$.

$\therefore u(0,0) = 0$

$u(0.25, 0) = 0$ . . . . $u(1,0) = 0$.

By data, $u(0,t) = 0$.

$\therefore u(0,0) = 0$ and $u(0,1) = 0$.

By data; $u(1,t) = 200t$.

$\therefore u(1,0) = \cancel{2000}\,0$ and $u(1,1) = 200$.

Thus we get the following table:-

$x \to h = 0.25$

t ↓	t \ x	0	0.25	0.5	0.75	1
	0	0	0	0	0	0
K=1	1	0	A	B	C	200

classmate

Date

Page

Now, by Crank-Nicholson formula, we calculate the remaining values of u i.e. A, B, C.

We use $e = \frac{1}{4}(a+b+c+d)$

To find $\underline{A}$:

$$A = \frac{1}{4}(0+0+0+B)$$

$$A = \frac{B}{4} \quad \text{—— (1)}$$

$\underline{B}$:

$$B = \frac{1}{4}(A+0+0+C)$$

$$B = \frac{A+C}{4} \quad \text{—— (2)}$$

C:

$$C = \frac{1}{4}(B+0+0+200)$$

$$C = \frac{B+200}{4} \quad \text{—— (3)}.$$

Putting values of A and C in (2)

$$\therefore B = \frac{1}{4}\left[\frac{B}{4} + \frac{B+200}{4}\right]$$

$$B = \frac{1}{4}\left[\frac{2B+200}{4}\right] = \frac{2B+200}{16}$$

$$B = \frac{B}{8} + \frac{200}{16}.$$

$$B - \frac{B}{8} = \frac{200}{16}$$

$$\therefore \frac{7}{8}B = \frac{50}{4}$$

$$\therefore \boxed{B = \frac{100}{7}}$$

Now by ①;

$$A = \frac{B}{4} = \frac{100}{4\times 7} = \frac{25}{7}$$

$$\therefore \boxed{A = \frac{25}{7}}$$

By ④;

$$C = \frac{1}{4}(B+200)$$

$$C = \frac{1}{4}\left(\frac{100}{7} + 200\right)$$

$$C = \frac{1500}{4\times 7} = \frac{375}{7}$$

$$\boxed{C = \frac{375}{7}}$$

$\therefore A = \frac{25}{7} = 3.5714 \qquad B = \frac{100}{7} = 14.2857 \qquad C = \frac{375}{7} = 53.5714$

Thus, final table.

$x \rightarrow h = 0.25$

t \ x	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	3.5714	14.2857	53.5714	200

t ↓ k = 1

6. Solve by Crank ~~Nico~~ Nicholson method $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$

$0 < x < 1$, $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = 100x(1-x)$ taking $h = 0.25$ for one-time step.

$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$. $\therefore a = 1$. $t = 1$.

$h = 0.25 = \frac{1}{4}$

$k = ah^2 = 1 \times \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

$\therefore t_1 = 0$ and $t_2 = 1/16$.

since $t = 1$ and is divided into $1/16$ size.

Now $h = 0.25$

and x interval is from 0 to 1

$\therefore x_0 = 0$; $x_1 = 0.25$; $x_2 = 0.5$; $x_3 = 0.75$; $x_4 = 1$.

Now by data,

$u(0,t) = 0$ and $t = 1, 1/16$

$\therefore u(0,1) = 0$ and $u(0, 1/16) = 0$

by data,

$u(1,t) = 0$.

$u(1,1) = 0$ and $u(1, 1/16) = 0$.

by data,

$u(x,0) = 100x(1-x)$

$u(0,0) = \cancel{100}\ 0$

$u(0.25,0) = 100 \times 0.25(1-0.25) = 18.75$

$u(0.5,0) = 100 \times 0.5(1-0.5) = 25$

$u(0.75,0) = 100 \times 0.75(1-0.75) = 18.75$

$u(1,0) = 100 \times 1(1-1) = 0$,

∴ Thus we get the following table,

$x \to h = 0.25$

$t \downarrow$	$t \backslash x$	0	0.25	0.5	0.75	1
	0	0	18.75	25	18.75	0
$k = 1/16$	1/16	0	A	B	C	0

Now, by Crank-Nicholson formula, we calculate the remaining values of U i.e. A, B, C.

We use $e = \frac{1}{4}(a+b+c+d)$.

To find A: $A = \frac{1}{4}(0+0+25+B)$

$$A = \frac{25+B}{4} \quad \text{—— (1)}$$

B: $B = \frac{1}{4}(A+18.75+18.75+C)$

$$B = \frac{37.5+A+C}{4} \quad \text{—— (2)}$$

C: $C = \frac{1}{4}(B+25+0+0)$

$$C = \frac{25+B}{4} \quad \text{—— (3)}.$$

Putting values of A and C in (2).

$$B = \frac{1}{4}\left[37.5 + \frac{25+B}{4} + \frac{25+B}{4}\right]$$

$$B = \frac{1}{4}\left[37.5 + \frac{50+2B}{4}\right]$$

$$B = \frac{1}{4}\left[\frac{150+50+2B}{4}\right] = \frac{200+2B}{16}$$

$$B = \frac{25}{2} + \frac{2B}{16}$$

Date
Page

$$B = \frac{25}{2} + \frac{B}{8}.$$

$$B - \frac{B}{8} = \frac{25}{2}.$$

$$\frac{7B}{8} = \frac{25}{2}$$

$$\boxed{B = \frac{100}{7}}$$

Now, $A = C = \frac{25+B}{4}$

$$= \frac{25 + 100/7}{4}$$

$$= \frac{175+100}{28}$$

$$= 9.821$$

$\therefore A = 9.821$; $B = \frac{100}{7} = 14.28$; $C = 9.821$

$\therefore$ The final table is

$x \rightarrow h = 0.25$

	t \ x	0	0.25	0.5	0.75	1
t	0	0	18.75	25	18.75	0
↓	1/16	0	9.821	14.28	9.821	0
k=1/16						