Ex (4) obtain Taylor's and Laurent's expansion of $f(z) = \frac{z-1}{z^2-2z-3}$ indicating region of convergence.

Soln $f(z) = \frac{z-1}{z^2-2z-3} = \frac{z-1}{(z-3)(z+1)} = \frac{a}{z+1} + \frac{b}{z-3}$

$z-1 = a(z-3) + b(z+1)$ — ①

put $z = -1$

$-1-1 = a(-1-3)$

$-2 = -4a$

$a = \frac{1}{2}$

$z = 3$

$3-1 = b(3+1)$

$2 = 4b$

$b = \frac{1}{2}$

$f(z) = \frac{\frac{1}{2}}{z+1} + \frac{\frac{1}{2}}{z-3}$

$f(z)$ is not analytic at $z=-1$, $z=3$

$\therefore f(z)$ is analytic in ① $|z|<1$, ⓘⓘ $1<|z|<3$

(iii) $|z|>3$

case ① $|z|<1 \therefore |z|<3$

$\left|\frac{z}{3}\right| < 1$

$f(z) = \frac{1}{2(1+z)} + \frac{1}{2(-3+z)}$

$= \frac{1}{2(1+z)} + \frac{1}{-6(1-\frac{z}{3})}$

$= \frac{1}{2}(1+z)^{-1} - \frac{1}{6}(1-\frac{z}{3})^{-1}$

$= \frac{1}{2}(1-z+z^2-z^3 -) - \frac{1}{6}(1+\frac{z}{3}+\frac{z^2}{3^2} -)$

(ii) $1 < |z| < 3$

$1 < |z| \quad |z| < 3$

$\frac{1}{|z|} < 1 \quad |\frac{z}{3}| < 1$

$$f(z) = \frac{1}{2(z+1)} + \frac{1}{2(z-3)} = \frac{1}{2z(1+\frac{1}{z})} + \frac{1}{2(-3)(1-\frac{z}{3})}$$

$$= \frac{1}{2z}(1+\tfrac{1}{z})^{-1} - \frac{1}{6}(1-\tfrac{z}{3})^{-1}$$

$$= \frac{1}{2z}(1 - \tfrac{1}{z} + \tfrac{1}{z^2} - \tfrac{1}{z^3} \cdots) - \frac{1}{6}(1 + \tfrac{z}{3} + \tfrac{z^2}{3^2} \cdots)$$

$$= \frac{1}{2}(\tfrac{1}{z} - \tfrac{1}{z^2} + \tfrac{1}{z^3} \cdots) - \frac{1}{6}(1 + \tfrac{z}{3} + \tfrac{z^2}{3^2} \cdots)$$

B Laurent series

(iii) $|z| > 3 \quad |z| > 1$

$3 < |z| \quad 1 < |z|$

$\frac{3}{|z|} < 1 \quad \frac{1}{|z|} < 1$

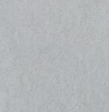
$$f(z) = \frac{1}{2(z+1)} + \frac{1}{2(z-3)}$$

$$= \frac{1}{2z(1+\frac{1}{z})} + \frac{1}{2z(1-\frac{3}{z})}$$

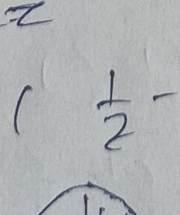
$$= \frac{1}{2z}(1+\tfrac{1}{z})^{-1} + \frac{1}{2z}(1-\tfrac{3}{z})^{-1}$$

$$= \frac{1}{2z}(1 - \tfrac{1}{z} + \tfrac{1}{z^2} \cdots) + \frac{1}{2z}(1 + \tfrac{3}{z} + \tfrac{3^2}{z^2} \cdots)$$

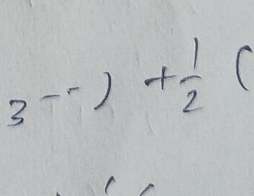
$$= \frac{1}{2}(\tfrac{1}{z} - \tfrac{1}{z^2} + \tfrac{1}{z^3} \cdots) + \frac{1}{2}(\tfrac{1}{z} + \tfrac{3}{z^2} + \tfrac{3^2}{z^3} \cdots)$$



$|z| < 1$



$1 < |z| < 3$



$|z| > 3$

Ex (5) Expand $f(z) = \dfrac{z^2-1}{z^2+5z+6}$ around $z=1$

Soln $f(z) = \dfrac{z^2-1}{z^2+5z+6} = \dfrac{z^2-1}{(z+2)(z+3)}$

$\dfrac{N}{D} = Q + \dfrac{R}{D}$ $\therefore N = Q \cdot D + R$

$$z^2+5z+6 \overline{\big)\, z^2-1 \,}\big|\, 1$$
$$\underline{z^2+5z+6}$$
$$-5z-7$$

$\dfrac{z^2-1}{z^2+5z+6} = 1 + \dfrac{-5z-7}{(z+2)(z+3)}$

$-\dfrac{(5z+7)}{(z+2)(z+3)} = \dfrac{a}{z+2} + \dfrac{b}{z+3}$

$-(5z+7) = a(z+3) + b(z+2)$

$z=-2$

$-10+7 = a(-2+3)$

$-3 = a$

$a = -3$

$z=-3$

$-15+7 = b(-3+2)$

$-8 = -b$

$b = 8$

$\dfrac{z^2-1}{z^2+5z+6} = 1 - \left(\dfrac{-3}{z+2} + \dfrac{8}{z+3}\right)$

$= 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$

around $z=1$ we obtain Laurent's series in powers of $(z-1)$

$$\frac{z^2-1}{z^2+5z+6} = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3}$$

① $|z-1|<3$, $3<|z-1|<4$, $|z-1|>4$

Case ① $|z-1|<3$ $|z-1|<4$

$\left|\frac{z-1}{3}\right|<1$, $\left|\frac{z-1}{4}\right|<1$

$$f(z) = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3}$$

$$= 1 - \frac{8}{4\left(1+\frac{z-1}{4}\right)} + \frac{3}{3\left(1+\frac{z-1}{3}\right)}$$

$$= 1 - 2\left(1+\frac{z-1}{4}\right)^{-1} + \left(1+\frac{z-1}{3}\right)^{-1}$$

$$= 1 - 2\left(1-\left(\frac{z-1}{4}\right)+\left(\frac{z-1}{4}\right)^2-\left(\frac{z-1}{4}\right)^3 - -\right)$$

$$+\left(1-\left(\frac{z-1}{3}\right)+\left(\frac{z-1}{3}\right)^2-\left(\frac{z-1}{3}\right)^3 - -\right]$$

Case ⑪ $3<|z-1|<4$

$3<|z-1|$ $|z-1|<4$

$\frac{3}{|z-1|}<1$ $\frac{|z-1|}{4}<1$

$$f(z) = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3}$$

$$= 1 - \frac{8}{4\left(1+\left(\frac{z-1}{4}\right)\right)} + \frac{3}{(z-1)\left(1+\frac{3}{z-1}\right)}$$

$$= 1 - 2\left[1+\left(\frac{z-1}{4}\right)\right]^{-1} + \frac{3}{z-1}\left[1+\frac{3}{z-1}\right]^{-1}$$

$$= 1 - 2\left[1-\left(\frac{z-1}{4}\right)+\frac{(z-1)^2}{4^2}-\frac{(z-1)^3}{4^3} - \right]$$

$$+\frac{3}{z-1}\left[1-\frac{3}{(z-1)}+\frac{3^2}{(z-1)^2}-\frac{3^3}{(z-1)^3}\right]-$$

$$=1-2\left[1-\frac{(z-1)}{4}+\frac{(z-1)^2}{16}-\frac{(z-1)^3}{64}--\right]$$

$$+3\left[\frac{1}{z-1}-\frac{3}{(z-1)^2}+\frac{9}{(z-1)^3}--\right]$$

case (iii) $\quad |z-1|>4 \quad |z-1|>3$

$4<|z-1| \quad 3<|z-1|$

$\frac{4}{|z-1|}<1 \quad \frac{3}{|z-1|}<1$

$$H(z)=1-\frac{8}{(z-1)+4}+\frac{3}{(z-1)+3}$$

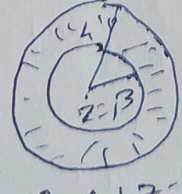
$$=1-\frac{8}{(z-1)\left(1+\frac{4}{z-1}\right)}+\frac{3}{(z-1)\left(1+\frac{3}{z-1}\right)}$$

$$=1-\frac{8}{z-1}\left[1+\frac{4}{z-1}\right]^{-1}+\frac{3}{z-1}\left[1+\frac{3}{z-1}\right]^{-1}$$

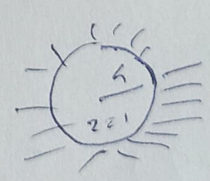
$$=1-\frac{8}{z-1}\left[1-\frac{4}{z-1}+\frac{4^2}{(z-1)^2}-\frac{4^3}{(z-1)^3}--\right]$$

$$+\frac{3}{(z-1)}\left(1-\frac{3}{z-1}+\frac{3^2}{(z-1)^2}-\frac{3^3}{(z-1)^3}-\right)$$

$|z-1|<3$



$3<|z-1|<4$



$|z-1|>4$

Ex 6 Find all possible Laurent's expansion of the function $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ about $z=-1$ (6)

Soln: $f(z) = \frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$

$7z-2 = A(z-2)(z+1) + B(z)(z+1) + C\,z(z-2)$ — (1)

$z=0$ | $z=2$ | $z=-1$

$-2 = A(-2)(1)$ | $12 = B(2)(3)$ | $-9 = C(-1)(-3)$

$A=1$ | $B=2$ | $-9 = 3C$, $C=-3$

$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$ about $z=-1$, $z+1=$

$= -\frac{3}{z+1} + \frac{2}{(z+1)-3} + \frac{1}{(z+1)-1}$

$= -\frac{3}{z+1} + \frac{1}{(z+1)-1} + \frac{2}{(z+1)-3}$

$|z+1|<1$, $1<|z+1|<3$, $|z+1|>3$

case 1 $|z+1|<1$, $|z+1|<3$

case (1) $|z+1|<1$, $\left|\frac{z+1}{3}\right|<1$

$f(z) = -\frac{3}{z+1} + \frac{1}{-1+(z+1)} + \frac{2}{-3+(z+1)}$

$= -\frac{3}{z+1} - \frac{1}{1-(z+1)} - \frac{2}{3\left(1-\left(\frac{z+1}{3}\right)\right)}$

$= -\frac{3}{z+1} - (1-(z+1))^{-1} - \frac{2}{3}\left(1-\left(\frac{z+1}{3}\right)\right)^{-1}$

$= -\frac{3}{z+1} - [1+(z+1)+(z+1)^2+(z+1)^3 - -]$

$- \frac{2}{3}[1+\left(\frac{z+1}{3}\right)+\frac{(z+1)^2}{3^2} - -]$

Case (II) $1 < |z+1| < 3$

$1 < |z+1|, \quad |z+1| < 3$

$\frac{1}{|z+1|} < 1, \quad \frac{|z+1|}{3} < 1$

$$f(z) = -\frac{3}{z+1} + \frac{1}{-1+(z+1)} + \frac{2}{(z+1)-3}$$

$$= -\frac{3}{z+1} + \frac{1}{z+1\left(1-\frac{1}{(z+1)}\right)} + \frac{2}{-3\left(1-\frac{(z+1)}{3}\right)}$$

$$= -\frac{3}{z+1} + \frac{1}{z+1}\left[1-\frac{1}{z+1}\right]^{-1} - \frac{2}{3}\left[1-\left(\frac{z+1}{3}\right)\right]^{-1}$$

$$= -\frac{3}{z+1} + \frac{1}{z+1}\left[1+\frac{1}{z+1}+\frac{1}{(z+1)^2} - -\right]$$

$$-\frac{2}{3}\left[1+\left(\frac{z+1}{3}\right)+\frac{(z+1)^2}{3^2} - -\right]$$

Case (III) $|z+1| > 4 \quad |z+1| > 3$

$4 < |z+1| \quad 3 < |z+1|$

$\frac{4}{|z+1|} < 1, \quad \frac{3}{|z+1|} < 1$

$$f(z) = -\frac{3}{z+1} + \frac{1}{(z+1)-1} + \frac{2}{(z+1)-3}$$

$$= -\frac{3}{z+1} + \frac{1}{z+1\left(1-\frac{1}{z+1}\right)} + \frac{2}{z+1\left(1-\frac{3}{z+1}\right)}$$

$$= -\frac{3}{z+1} + \frac{1}{z+1}\left[1-\frac{1}{z+1}\right]^{-1} + \frac{2}{z+1}\left[1-\frac{3}{z+1}\right]^{-1}$$

$$= -\frac{3}{z+1} + \frac{1}{z+1}\left(1+\frac{1}{z+1}+\frac{1}{(z+1)^2} - -\right)$$

$$+\frac{2}{z+1}\left(1+\frac{3}{z+1}+\frac{3^2}{(z+1)^2} - -\right)$$