

## Residue thm (2)

(1)

EX ① Evaluate  $\int_C \frac{z^2}{(z-1)^2(z+1)} dz$  where

$C$  is ①  $|z| = \frac{1}{2}$  ②  $|z| = 2$

Soln  $(z-1)^2(z+1) = 0$

$$z = 1, 1, -1$$

$\therefore z = -1$  is a simple pole

$z = 1$  is a pole of order 2

①  $\nexists$   $C$  is  $|z| = \frac{1}{2}$

if  $z = 1$   $|z| = |1| = 1 > \frac{1}{2}$

$z = -1$   $|z| = |-1| = 1 > \frac{1}{2}$

$z = 1, -1$  poles lying outside of  $C$   
by Cauchy's thm

$$\int_C f(z) dz = \int_C \frac{z^2}{(z-1)^2(z+1)} dz = 0$$

②  $|z| = 2$

if  $z = 1$   $|z| = |1| = 1 < 2$

$z = -1$   $|z| = |-1| = 1 < 2$

both poles lie inside of  $C$

if  $z = -1$  Residue =  $\lim_{z \rightarrow -1} (z+1) \frac{z^2}{(z-1)^2(z+1)}$   
at  $z = -1$

$$= \frac{(-1)^2}{(-1-1)^2} = \frac{1}{4}$$

Residue at  $z=1$  =  $\lim_{z \rightarrow 1} \frac{1}{(m-1)!} \lim_{z \rightarrow 1} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$  ①  
 or order  $m=2$

$$= \frac{1}{(2-1)!} \lim_{z \rightarrow 1} \frac{d}{dz} (z-1)^2 \frac{z^2}{(z+1)^2 (z+1)}$$

$$= \frac{1}{1} \lim_{z \rightarrow 1} \frac{d}{dz} \left( \frac{z^2}{(z+1)^3} \right)$$

$$= \lim_{z \rightarrow 1} \frac{(z+1)^3 \cdot 2z - z^2 \cdot 3(z+1)^2}{(z+1)^6}$$

$$= \frac{(1+1)^3 \cdot 2(1) - 1^2 \cdot 3(1+1)^2}{(1+1)^6}$$

$$= \frac{2(2) - 1 \cdot 3}{4} = \frac{3}{4}$$

Use Cauchy's residue thm

$$\oint_C f(z) dz = 2\pi i (\text{residue at } z=-1, z=1)$$

$$= 2\pi i \left( \frac{1}{4} + \frac{3}{4} \right) =$$

$$= 2\pi i (1)$$

$$= 2\pi i$$

Ex ② Evaluate using Cauchy's

Residue thm  $\oint_C \frac{1-zz}{z(z-1)(z-2)} dz$

Where  $C: |z|=1.5$  1.5

$$\text{Given } f(z) = \frac{1-z^2}{z(z-1)(z-2)}$$

(3)

$$z(z-1)(z-2)$$

$$|z| = 1.5$$

$$z(z-1)(z-2) = 0$$

$z=0, 1, 2$  are simple poles

$$\text{if } |z| = 1.5$$

$$\text{if } z=0 \quad |z|=|0|=0 < 1.5 \quad \left. \begin{array}{l} z=1 \quad |z|=|1|=1 < 1.5 \\ z=2 \quad |z|=|2|=2 > 1.5 \end{array} \right\} \text{ lies inside or on}$$

$$\text{lies outside}$$

$$\text{Residue at } z=0 = \lim_{z \rightarrow 0} (z-0) \frac{1-z^2}{z(z-1)(z-2)}$$

$$= \frac{1-0}{(0-1)(0-2)} = \frac{1}{2}$$

$$\text{Residue at } z=1 = \lim_{z \rightarrow 1} (z-1) \frac{(1-z^2)}{z(z-1)(z-2)}$$

$$= \frac{1-2(1)}{1(1-2)} = \frac{1-2}{1-2} = \frac{-1}{-1} = 1$$

$$\int \frac{1-z^2}{z(z-1)(z-2)} dz = 2\pi i \left( \text{Sum of residues at } z=0, z=1 \right)$$

$$= 2\pi i \left( \frac{1}{2} + 1 \right)$$

$$= 2\pi i \left( \frac{3}{2} \right)$$

$$= \underline{3\pi i}$$

Ex 3) Using residue theorem evaluate

(4)

$$\oint_C \frac{e^{2z}}{(z-\pi i)^3} dz \text{ where } C: |z-2i|=2$$

$$|z-2i|=2$$

Soln.  $z-\pi i=0$

$$z=\pi i, \pi i, \pi i$$

$z=\pi i$  is a pole of order 3

if  $z=\pi i$   $|z-2i|=|\pi i-2i|=|3ki-2i|$

$$=|1.1ki|=1.04 < 2$$

$z=\pi i$  lies inside of  $C$

Residue of  $f(z)$  at  $z=\pi i$   $m=3$   $\frac{1}{(3-1)!} \lim_{z \rightarrow \pi i} \frac{d^2}{dz^2} \frac{(z-\pi i)^3 \cdot e^{2z}}{(z-\pi i)^3}$

$$= \frac{1}{2!} \lim_{z \rightarrow \pi i} \frac{d^2}{dz^2} e^{2z}$$

$$= \frac{1}{2} \lim_{z \rightarrow \pi i} \frac{d}{dz} 2e^{2z}$$

$$= \frac{1}{2} \lim_{z \rightarrow \pi i} 2 \cdot 2e^{2z}$$

$$= \frac{4}{2} \lim_{z \rightarrow \pi i} e^{2z}$$

$$= 2 \cdot e^{2i\pi}$$

$$= 2 \left( \cos 2\pi + i \sin 2\pi \right)$$

$$= 2$$

$$\int_C f(z) dz = 2\pi i \left( \sum \text{residues} \right) = 4\pi i$$

EX (4) Using residue theorem evaluate

$$\oint_C \frac{z-1}{z^2+z+5} dz \text{ where } C \text{ is circle}$$

①  $|z|=1$  ②  $|z+1+i|=2$  ③  $|z+1-i|=2$

Soln  $z^2+z+5=0$

$$z = \frac{-1 \pm \sqrt{1-20}}{2} = \frac{-1 \pm \sqrt{-19}}{2}$$

$$= \frac{-1 \pm 4i}{2} = -1 \pm 2i \text{ are simple poles}$$

if  $z = -1+2i$   $|z| = |-1+2i| = \sqrt{5} > 1$

$z = -1-2i$   $|z| = |-1-2i| = \sqrt{5} > 1$

both points lie outside of C (Cauchy's theorem)

$$\int_C f(z) dz = 0$$

①  $|z+1+i|=2$

if  $z = -1+2i$   $|z+1+i| = |-1+2i+1+i| = |3i| = 3 > 2$

$z = -1-2i$   $|z+1+i| = |-1-2i+1+i| = |-1-i| = \sqrt{2} < 2$

$z = -1+2i$  lies outside of C

$z = -1-2i$  lies inside of C

Residue at  $z = -1-2i$   $\lim_{z \rightarrow -1-2i} (z+1+i) \frac{(z-1)}{(z+1+i)(z+1-i)} = \frac{-1-2i-1}{-1-2i+1-i} = \frac{-2-2i}{-3i} = \frac{2+i}{3}$

$$= \frac{-1-2i-1}{-1-2i+1-i} = \frac{-2-2i}{-3i} = \frac{2+i}{3}$$

$$\int \frac{z-1}{z^2+2+45} dz = 2\pi i \left( \frac{1+i}{2i} \right) \\ = \pi (1+i)$$

Ex.  $\int_C \frac{1}{z^3(z+4)} dz$  when  $|z|=2$

(i)  $\int_C \frac{e^z}{z^2+11z+2} dz$   $|z|=4$

(ii)  $\int_C \frac{\sin 6z}{(z-\frac{\pi}{8})^3} dz$   $|z|=1$