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Tutorial 5 - Matrices

1. Find the eigen values and eigen vectors of matrix

$$\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

characteristic Equation:

$$\lambda^3 - (SOD)\lambda^2 + (SOCD)\lambda - |A| = 0.$$

where SOD = sum of diagonal

SOCD = sum of cofactor diagonal.

$|A| = 36$

$SOD = 11$

$SOCD = 36$

$$\therefore \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0.$$

$$\therefore \lambda = 2, 3, 6. \longrightarrow \text{Eigen values}.$$

~~i) λ = 2~~. Now,

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}.$$

And $[A - \lambda I]X = 0.$

$\therefore$ i) $\lambda = 2.$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \frac{x_1}{\begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix}} = \frac{x_2}{-\begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 3 \\ 1 & -1 \end{vmatrix}}$$

$\therefore$

$$\frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{-2}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = X = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

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Hence corresponding to $\lambda = 2$, the eigenvector is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

ii) $\lambda = 3$.

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\therefore \frac{x_1}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = \frac{x_2}{-\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, corresponding to $\lambda = 3$, the eigenvector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

iii) $\lambda = 6$.

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 + R_1$

$R_3 + 3R_1$

$$\begin{bmatrix} -3 & -1 & 1 \\ -4 & -2 & 0 \\ -8 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - \frac{1}{2} R_2$

$\frac{1}{2} R_2$

$$\begin{bmatrix} -3 & -1 & 1 \\ -2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -3x_1 - x_2 + x_3 = 0$

and $-2x_1 - x_2 = 0$.

Put $x_2 = -2t$

$\therefore 2x_1 = -x_2 = 2t$

$\therefore x_1 = t$.

$\therefore x_3 = 3x_1 + x_2$

$x_3 = 3t \cancel{+} -2t$

$x_3 = t$.

$$\therefore x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Hence, ~~the~~ corresponding to $\lambda = 6$, eigenvector is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

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2. Find the eigenvalues of adj A and of A^2-2A+I where

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$\lambda^3 - (SOD)\lambda^2 + (SOCD)\lambda - |A| = 0.$

$\lambda^3 - 9\lambda^2 + 26\lambda - 24 = 0.$

$\lambda = 2, 4, 3.$

If $A \rightarrow \lambda$

$\text{Adj } A \rightarrow \frac{|A|}{\lambda}$

$= \frac{24}{\lambda}$

$\therefore \lambda = 2 \quad \therefore \text{Adj } A = 12$

$\lambda = 4 \quad \text{Adj } A = 6$

$\lambda = 3 \quad \text{Adj } A = 8.$

$\text{Adj } A = 12, 6, 8.$

$A^2 - 2A + I = \lambda^2 - 2\lambda + 1$

$\lambda = 2$

$\therefore 4 - 4 + 1 = 1.$

$\lambda = 4$

$\therefore 16 - 8 + 1 = 9$

$\lambda = 3$

$\therefore 9 - 6 + 1 = 4.$

$\therefore A^2 - 2A + I = 1, 9, 4.$

3. Verify Cayley-Hamilton theorem and hence find A^{-1} and A^4

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

$\lambda^3 - (SOD)\lambda^2 + (SOCD)\lambda - |A| = 0.$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$

By CHT (Cayley-Hamilton Theorem),

$\therefore A^3 - 6A^2 + 11A - 6I = 0.$

LHS $\Rightarrow A^3 - 6A^2 + 11A - 6I$

$$= \begin{bmatrix} 14 & 0 & -13 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix} + \begin{bmatrix} 22 & 0 & -11 \\ 0 & 22 & 0 \\ -11 & 0 & 22 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 0 & -13 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{bmatrix} - \begin{bmatrix} 30 & 0 & -24 \\ 0 & 24 & 0 \\ -24 & 0 & 30 \end{bmatrix} + \begin{bmatrix} 16 & 0 & -11 \\ 0 & 16 & 0 \\ -11 & 0 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Hence CHT (Cayley-Hamilton Theorem) is verified.

Now to find A^{-1}

By CHT,

$A^3 - 6A^2 + 11A - 6I = 0$

multiply by A^{-1}

$\therefore A^2 - 6A + 11I - 6A^{-1} = 0.$

$\therefore 6A^{-1} = A^2 - 6A + 11I$

$A^{-1} = \frac{1}{6}(A^2 - 6A + 11I)$

$$\cancel{A^{-1} = \frac{1}{6}\begin{bmatrix} 7 & 0 & -5 \\ 0 & 6 & 0 \\ -5 & 0 & 7 \end{bmatrix}}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} -7 & 0 & 2 \\ 0 & -8 & 0 \\ 2 & 0 & -7 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

Now, $A^3 - 6A^2 + 11A - 6I = 0$

Multiply by A

$A^4 - 6A^3 + 11A^2 - 6A = 0$

$A^4 = 6A^3 - 11A^2 + 6A.$

$$\therefore A^4 = \begin{bmatrix} 41 & 0 & -40 \\ 0 & 16 & 0 \\ -40 & 0 & 41 \end{bmatrix}.$$

4. If $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$, find $A^7 + 31A^2 + I$.

Characteristic Equation.

$\lambda^2 - 2\lambda - 3 = 0.$

$\therefore A^2 - 2A - 3I = 0.$

Now, $$A^2 = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 2 & 5 \end{bmatrix}.$$

$$A^4 = [A^2]^2 = \begin{bmatrix} 5 & 8 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 8 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 41 & 80 \\ 20 & 41 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 5 & 8 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 28 \\ 7 & 13 \end{bmatrix}$$

$\because A^4 \cdot A^3$

$$\therefore A^7 = \begin{bmatrix} 41 & 80 \\ 20 & 41 \end{bmatrix} \begin{bmatrix} 13 & 28 \\ 7 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 1093 & 2188 \\ 547 & 1093 \end{bmatrix}$$

$$\therefore A^7 + 31A^2 + I = \begin{bmatrix} 1093 & 2188 \\ 547 & 1093 \end{bmatrix} + 31 \begin{bmatrix} 5 & 8 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1093 & 2188 \\ 547 & 1093 \end{bmatrix} + \begin{bmatrix} 155 & 248 \\ 62 & 155 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1249 & 2436 \\ 609 & 1249 \end{bmatrix}$$

$$= \begin{bmatrix} 609 & 2436 \\ 609 & 609 \end{bmatrix} + 640 I$$

$$= 609 \cdot A + 640 I.$$

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5 Show that $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalisable. Find the diagonal form and transforming matrix.

characteristic equation.

$\lambda^3 - (SOD)\lambda^2 + (SOCD)\lambda - |A| = 0.$

~~$\lambda^3 + \lambda^2 + 21 - 45 = 0$~~

$\lambda^3 + \lambda^2 - 21 - 45 = 0.$

$\therefore \quad \lambda = 5, -3, -3.$

Now, $\lambda_1 = 5.$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$R_2 = R_2 + \frac{2}{7} R_1$

$$\therefore \begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ -1 & -2 & -5 \end{bmatrix}$$

$R_3 = R_3 - \frac{1}{7} R_1$

$$\therefore \begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ 0 & -16/7 & -32/7 \end{bmatrix}$$

$R_3 = R_3 - \left(\frac{2}{3}\right) R_2$

$$\therefore \begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore R_2 = R_2\left(1/12\right)$ and $R_1 = (-1)R_1$

$$\begin{bmatrix} 7 & -2 & 3 \\ 0 & -2/7 & -4/7 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 = R_1/7 - R_2$

$$\therefore \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2/7 & -4/7 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 = R_2\left(-7/2\right)$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $x_3 = t$ $\therefore x_1 = -t$ and $x_2 = -2t$

$$\therefore X = \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Now, $\lambda_2 = -3$ $\therefore \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 = R_2 - 2R_1$

$$\therefore \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ -1 & -2 & 3 \end{bmatrix}$$

$R_3 = R_3 + R_1$

$$\therefore \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $x_2 = t$; $x_3 = s$ then $x_1 = 3s - 2t$

$$\therefore x = \begin{bmatrix} 3s-2t \\ t \\ s \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} s$$

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∴ The diagonalising matrix,

$$M = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

∴ Diagonal matrix is

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

6. If $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$, Prove that $3\tan A = A\tan 3$

$\lambda^2 - (SOD)\lambda + |A| = 0$.

$\lambda^2 - 9 = 0$.

$\lambda^2 = 9$

$\lambda = \pm 3$.

$\therefore \lambda = 3, -3$

$f(\lambda) = 3\tan\lambda$

$f(\lambda) = (\lambda^2 - 9)\phi(\lambda) + (a\lambda + b)$

where $\lambda^2 - 9$ = divisor,

$\phi(\lambda)$ = quotient

$\cancel{a(\lambda+b)}$ $(a\lambda + b)$ = remainder.

$f(\lambda) = a\lambda + b$

$\because \lambda^2 - 9 = 0$

$\therefore$ $\cancel{f(\lambda)}$

$3\tan\lambda = a\lambda + b$

i) $\lambda = 3$

$\therefore 3\tan 3 = 3a + b$ — ①

ii) $\lambda = -3$.

$3\tan(-3) = -3a + b$

$-3\tan 3 = -3a + b$ — ②.

Adding ① + ②.

$\therefore 0 = 2b$

$\boxed{b = 0}$

$\therefore \boxed{a = \tan 3}$

$\therefore 3\tan\lambda = a\lambda + b$

$3\tan\lambda = \lambda\tan 3 + 0$

$3\tan\lambda = \lambda\tan 3$.

$3\tan A = A\tan 3$. Hence Proved.