

Laplace transform - I

Let $f(t)$ be a function of $t > 0$ is defined by definite integral $\int_0^{\infty} e^{-st} f(t) dt$ if it exists is called Laplace transform is denoted by $L\{f(t)\}$

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = F(s) \text{ or } \hat{f}(s) = \phi(s)$$

$$t \rightarrow s$$

L.T. of standard functions

$$\textcircled{1} L\{e^{at}\} = \frac{1}{s-a}$$

$$L\{e^{-at}\} = \frac{1}{s+a}$$

$$\textcircled{2} L[1] = \frac{1}{s}$$

$$\textcircled{3} L\{t^n\} = \frac{n!}{s^{n+1}} \text{ if } n \text{ is +ve integer}$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \text{ if } n \text{ is fraction.}$$

$$\textcircled{1} \Gamma(n+1) = n! \text{ if } n \text{ is +ve integer}$$

$$= n\Gamma n \text{ if } n \text{ is fraction}$$

eg. $\Gamma 4 = \Gamma 3+1 = 3! = 1 \cdot 2 \cdot 3 = 6$

$$\Gamma \frac{3}{2} = \Gamma \frac{1}{2} + 1 = \frac{1}{2} \Gamma \frac{1}{2} = \frac{\sqrt{\pi}}{2}, \therefore \Gamma \frac{1}{2} = \sqrt{\pi}$$

eg $\Gamma \frac{9}{2} = \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma \frac{1}{2} = \frac{105}{16} \sqrt{\pi}$

$$(4) \quad L[\sin kt] = \frac{k}{s^2 + k^2}$$

$$(5) \quad L[\cos kt] = \frac{s}{s^2 + k^2}$$

$$(6) \quad L[\sinh kt] = \frac{k}{s^2 - k^2}$$

$$(7) \quad L[\cosh kt] = \frac{s}{s^2 - k^2}$$

$$(8) \quad L[af_1(t) + bf_2(t)] = aL[f_1(t)] + bL[f_2(t)]$$

$$(9) \quad \text{Find L.T of } (\sin 2t - \cos 2t)^2$$

$$(1) \quad 2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$(2) \quad 2 \cos A \sin B = \sin(A+B) - \sin(A-B)$$

$$(3) \quad 2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$(4) \quad 2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$2 \sin A \sin B$$

$$\text{Soln } L[\sin]$$

$$\sin 2x = 2 \sin x \cos x, \quad \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$\sin^2 x + \cos^2 x = 1$$

$$(1) \quad \sin^2 x = \frac{1 - \cos 2x}{2}, \quad \sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$$

$$(2) \quad \cos^2 x = \frac{1 + \cos 2x}{2}, \quad \cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$$

$$\sin^3 x = \frac{3 \sin x - \sin 3x}{4}$$

$$\cos^3 x = \frac{3 \cos x + \cos 3x}{4}$$

$$\cosh^2 x = \frac{1 + \cosh 2x}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\sqrt{1 + \sin x} = \sqrt{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}$$

$$= \sqrt{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2} = \sin \frac{x}{2} + \cos \frac{x}{2}$$

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Ex (1) Find L.T. of $(\sin 2t - \cos 2t)^2$

Soln: $L[(\sin 2t - \cos 2t)^2]$

$$= L[\sin^2 2t - 2\sin 2t \cos 2t + \cos^2 2t]$$

$$= L[1 - \sin 2(2t)]$$

$$= L[1] - L[\sin 4t]$$

$$= \frac{1}{s} - \frac{s}{s^2 + 16}$$

Ex: (2) Find L.T. of $\sqrt{1 + \sin 4t}$

Soln: $L[\sqrt{1 + \sin 4t}]$

$$= L[\sqrt{\frac{\sin^2 2t}{2} + \frac{\cos^2 2t}{2} + 2\sin 2t \cos 2t}]$$

$$= L[(\sin 2t + \cos 2t)^2]^{\frac{1}{2}}$$

$$= L[\sin 2t] + L[\cos 2t]$$

$$= \frac{\frac{1}{2}}{s^2 + \frac{1}{4}} + \frac{s}{s^2 + \frac{1}{4}} = \frac{\frac{1}{2} - s}{4s^2 + 1} + \frac{4s}{4s^2 + 1}$$

$$= \frac{2 + 4s}{4s^2 + 1}$$

Ex (3) Find L.T. of $e^{2t} + 4t^3 - \sin 2t \cos 3t$

Soln: $L[e^{2t} + 4t^3 - \sin 2t \cos 3t]$

$$= L[e^{2t}] + 4L[t^3] - \frac{1}{2} L[2\sin 2t \cos 3t]$$

$$= \frac{1}{s-2} + 4 \cdot \frac{3!}{s^3+1} - \frac{1}{2} L[\sin 5t + \sin(-t)]$$

$$= \frac{1}{s-2} + \frac{24}{s^3+1} - \frac{1}{2} \left[\frac{s}{s^2+25} - \frac{1}{s^2+1} \right]$$

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$$(1) \quad L[\cos t, \cos 3t, \cos 5t]$$

$$(1) \quad \text{Find } L[\sin^5 t] \quad L[\sinh^5 t]$$

$$\text{Soln: } \sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$(\sin t)^5 = \left(\frac{e^{it} - e^{-it}}{2i} \right)^5$$

$$\begin{aligned} \sin^5 t = & \frac{1}{32i} \left[5(e^{it})^5 (e^{-it})^0 - 5(e^{it})^4 (e^{-it})^1 \right. \\ & + 10(e^{it})^3 (e^{-it})^2 - 10(e^{it})^2 (e^{-it})^3 \\ & + 5(e^{it})^1 (e^{-it})^4 - 5(e^{it})^0 (e^{-it})^5 \end{aligned}$$

$$\sin^5 t = \frac{1}{32i} \left[\frac{e^{5it}}{1} - 5 \frac{e^{4it-iit}}{1} + 10 \frac{e^{3it-2it}}{1} - 10 \frac{e^{2it-3it}}{1} + 5 \frac{e^{it-4it}}{1} - 5 \frac{e^{-5it}}{1} \right]$$

$$= \frac{1}{32i} \left[\frac{e^{5it}}{1} - 5 \frac{e^{3it}}{1} + 10 \frac{e^{it}}{1} - 10 \frac{e^{-it}}{1} + 5 \frac{e^{-3it}}{1} - \frac{e^{-5it}}{1} \right]$$

$$= \frac{1}{16} \left[\left(\frac{e^{5it} - e^{-5it}}{2i} \right) - 5 \left(\frac{e^{3it} - e^{-3it}}{2i} \right) + 10 \left(\frac{e^{it} - e^{-it}}{2i} \right) \right]$$

$$\sin^5 t = \frac{1}{16} [\sin 5t - 5 \sin 3t + 10 \sin t]$$

$$L[\sin^5 t] = \frac{1}{16} [L[\sin 5t] - 5 L[\sin 3t] + 10 L[\sin t]]$$

$$= \frac{1}{16} \left[\frac{5}{s^2 + 25} - 5 \cdot \frac{3}{s^2 + 9} + 10 \cdot \frac{1}{s^2 + 1} \right]$$

$$= \frac{5}{16} \left[\frac{1}{s^2 + 25} - \frac{3}{s^2 + 9} + \frac{2}{s^2 + 1} \right]$$