③ ① Fourier series

Fourier Coefficients (Euler's formulae)

Let $f(x)$ be a periodic function of period 2π which can be represented in the interval $(c, c+2\pi)$ by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad — ①$$

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx \quad — ②$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx \quad — ③$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx \quad — ④$$

a_0, a_n, b_n are fourier coefficients

Case ① Put $c = 0$ in $(c, c+2\pi)$ then $(0, 2\pi)$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

② Case II Put $c = -\pi$, in $(c, c+2\pi)$

then $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

① $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

② $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

③ $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

④ $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

① $\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$

② $\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$, $\cos^2 \frac{x}{2} = \frac{1+\cos x}{2}$

③ $\sin^3 x = \frac{3\sin x - \sin 3x}{4}$,

④ $\cos^3 x = \frac{3\cos x + \cos 3x}{4}$

① $\sin 0 = 0$, $\sin n\pi = 0$, $n = 1, 2, 3 \ldots$, $\sin 2n\pi = 0$

② $\cos 0 = 1$, $\cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = 0$, $\cos n\pi = (-1)^n$,

$\cos 2n\pi = 1$, $\sin\frac{\pi}{2} = 1$, $\sin\frac{3\pi}{2} = -1$

(3) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(1) $\sin(n \pm 1)\pi = 0$, $\sin(n \pm 1)2\pi = 0$

(2) $\cos(n \pm 1)\pi = \cos(n\pi + \pi) = \cos n\pi \cos\pi \mp \underbrace{\sin n\pi \cdot \sin\pi}_{0}$

$= -\cos n\pi = -(-1)^n$

$\cos(n \pm 1)2\pi = \cos(2n\pi + 2\pi) = \cos 2n\pi = 1$

(3) $\sin(n \cdot 2\pi \pm \theta) = \pm\sin\theta$

(4) $\cos(n \cdot 2\pi \pm \theta) = \cos\theta$

LIATE

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4$

$\int e^{ax}dx = e^{ax} \cdot \frac{1}{a}$, $\int \cos nx\,dx = \frac{\sin nx}{n}$

$\int \frac{1}{x}dx = \log x$ $\quad \int \frac{1}{\sqrt{x}}dx = 2\sqrt{x}$

$\int \sin nx\,dx = -\frac{\cos nx}{n}$ $\quad k\int 1\,dx = kx$

$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$\int_0^{2\pi} \sin nx\,dx = -\left(\frac{\cos nx}{n}\right)_0^{2\pi} = -\frac{1}{n}(\underset{1}{\cos 2n\pi} - \underset{1}{\cos 0})$

$= -\frac{1}{n}(1-1) = 0$

$\int_0^{2\pi} \cos nx\,dx = \left(\frac{\sin nx}{n}\right)_0^{2\pi} = \frac{1}{n}(\underset{0}{\sin 2n\pi} - \underset{0}{\sin 0})$

$= \frac{1}{n}(0) = 0$

(4) Ex(1): Find a fourier series to represent $f(x) = x^2$ in $(0, 2\pi)$ and hence deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - -$

10, 12, 17

Soln: interval is $(0, 2\pi)$

Fourier series in $(0, 2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad -(1)$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{2\pi} x^2\, dx$$

$$= \frac{1}{\pi} \left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{3\pi} (8\pi^{3} - 0)$$

$$\boxed{a_0 = \frac{8\pi^2}{3}}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx\, dx$$

$$= \frac{1}{\pi} \left[x^2 \cdot \left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{2}{n^2} (x \cos nx)_0^{2\pi} \right]$$

$$= \frac{2}{\pi n^2} [2\pi \cos 2n\pi - 0]$$

$$= \frac{2}{\pi n^2} (2\pi(1)) = \frac{4}{n^2}$$

$$\boxed{a_n = \frac{4}{n^2}}$$

⑤ $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\,dx$

$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$

$b_n = \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$

$= \frac{1}{\pi}\left[-\frac{1}{n}\left(x^2 \cos nx\right)\Big|_0^{2\pi} + \frac{2}{n^3}\left(\cos nx\right)\Big|_0^{2\pi}\right.$

$= \frac{1}{\pi}\left[-\frac{1}{n}\left(4\pi^2 \underbrace{\cos 2n\pi}_{(1)} - 0\right) + 0\right.$

$\boxed{b_n = -\frac{4\pi}{n}}$

Put a_0, a_n, b_n in eqn ①

$f(x) = \frac{1}{2}\cdot\frac{8\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum\frac{1}{n}\sin nx$

$x^2 = \frac{4\pi^2}{3} + 4\left(\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - -\right)$

$- 4\pi\left(\frac{1}{1}\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - - -\right) - ②$

Put $x = \pi$ in ②

$\pi^2 = \frac{4\pi^2}{3} + 4\left(\frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi\,.\right)$

$- 4\pi\left(\frac{1}{1}\sin\pi + \frac{1}{2}\sin 2\pi + \frac{1}{3}\sin 3\pi\ \ -\right]$

$\pi^2 - \frac{4\pi^2}{3} = 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} - - \right) - 0$

$-\frac{\pi^2}{3} = -4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - - \right.$

$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - -$

① Q Find the fourier expansion for

$f(x) = \sqrt{1-\cos x}$ in $(0, 2\pi)$ Hence

deduce that $\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

<u>Soln</u> Fourier series for $f(x)$ in $(0, 2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \sqrt{1-\cos x} = \sqrt{2\sin^2\frac{x}{2}} = \sqrt{2}\sin\frac{x}{2}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\,dx$$

$$= \frac{\sqrt{2}}{\pi}\left(-\cos\frac{x}{2}\,2\right)_0^{2\pi} = -\frac{2\sqrt{2}}{\pi}\left(\cos\frac{2\pi}{2} - \cos 0\right)$$

$$= -\frac{2\sqrt{2}}{\pi}(-1-1) = \frac{4\sqrt{2}}{\pi}$$

$$\boxed{a_0 = \frac{4\sqrt{2}}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\cos nx\,dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\cos nx\sin\frac{x}{2}\,dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} \sin\left(n+\tfrac{1}{2}\right)x - \sin\left(n-\tfrac{1}{2}\right)x\,dx$$

$$= \frac{\sqrt{2}}{2\pi}\left[-\frac{\cos\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}} - \left(-\frac{\cos\left(n-\frac{1}{2}\right)x}{n-\frac{1}{2}}\right)\right]_0^{2\pi}$$

(7) $a_n = \frac{\sqrt{2}}{2\pi}\left[-\frac{2}{2n+1}\left(\cos\left(\frac{2n+1}{2}\right)2\pi - \cos 0\right)\right.$

$\left.+\frac{2}{2n-1}\left(\cos\left(\frac{2n-1}{2}\right)2\pi - \cos 0\right)\right]$

$\cos(2n+1)\pi = \cos(2n\pi+\pi)$
$= \cos\pi = -1$

$= \frac{\sqrt{2}}{2\pi}\left[-\frac{2}{2n+1}(-1-1)+\frac{2}{2n-1}(-1-1)\right]$

$= \frac{\sqrt{2}}{2\pi}\left(\frac{4}{2n+1}-\frac{4}{2n-1}\right) = \frac{4\sqrt{2}}{2\pi}\left(\frac{2n-1-2n-1}{4n^2-1}\right)$

$= \frac{2\sqrt{2}}{\pi}\left(\frac{-2}{4n^2-1}\right) = \frac{-4\sqrt{2}}{\pi(4n^2-1)}$

$\boxed{a_n = \frac{-4\sqrt{2}}{\pi(4n^2-1)}}$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\sin nx\,dx$

$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\sin nx\sin\frac{x}{2}\,dx$

$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\cos\left(n-\frac{1}{2}\right)x - \cos\left(n+\frac{1}{2}\right)x\,dx$

$= \frac{\sqrt{2}}{2\pi}\left[\frac{\sin\left(n-\frac{1}{2}\right)x}{n-\frac{1}{2}} - \frac{\sin\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}}\right]_0^{2\pi}$

(8)

$$b_n = \frac{\sqrt{2}}{2\pi}\left[\frac{2}{2n-1}\left(\sin\frac{(2n-1)}{2}2\pi - \sin 0\right)\right.$$
$$\left. - \frac{2}{2n+1}\left(\sin\frac{(2n+1)}{2}2\pi - \sin 0\right)\right]$$

$$\sin(2n\pm 1)\pi = 0$$

$$b_n = 0$$

put in (1)

$$f(x) = \frac{1}{2}\cdot\frac{4\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}\frac{1}{4n^2-1}\cos nx + 0$$

$$\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}\frac{1}{4n^2-1}\cos nx$$

put $x=0$

$$\sqrt{1-\cos 0} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}\frac{1}{4n^2-1}\cos 0$$

$$0 - \frac{2\sqrt{2}}{\pi} = -\frac{4\sqrt{2}}{\pi}\sum\frac{1}{4n^2-1}$$

$$\frac{1}{2} = \sum_{n=1}\frac{1}{4n^2-1}$$