

① Vector calculus

1.1 Solenoidal and irrotational vector fields

Vector operator Del ∇

$$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

Gradient : If ϕ is scalar point function then vector function $\nabla\phi$ is called the gradient of ϕ

$$\nabla\phi = i \frac{\partial\phi}{\partial x} + j \frac{\partial\phi}{\partial y} + k \frac{\partial\phi}{\partial z}$$

Divergence :

Defⁿ: Let $\vec{F} = f_1 i + f_2 j + f_3 k$ then

$$\nabla \cdot \vec{F} = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (f_1 i + f_2 j + f_3 k)$$

$$= \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3$$

is called the divergence of \vec{F} then

$$\boxed{\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

If \vec{F} is vector point function such that

$\nabla \cdot \vec{F} = 0$ then \vec{F} is called solenoidal

(2)

curl : If $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$ then

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

called curl of \vec{F}

$$\therefore \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

If \vec{F} is a vector point function such that $\text{curl } \vec{F} = \nabla \times \vec{F} = 0$ then \vec{F} is called irrotational or conservative field.

Ex 1 If $\vec{F} = xy e^{2z} \hat{i} + xy^2 \cos z \hat{j} + x^2 \cos y \hat{k}$

Find $\text{div } \vec{F}$ and $\text{curl } \vec{F}$

Soln: $\vec{F} = xy e^{2z} \hat{i} + xy^2 \cos z \hat{j} + x^2 \cos y \hat{k}$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3$$

$$= \frac{\partial}{\partial x} (xy e^{2z}) + \frac{\partial}{\partial y} (xy^2 \cos z) + \frac{\partial}{\partial z} (x^2 \cos y)$$

$$= y e^{2z} (1) + x (2y \cos z) + 0$$

$$\text{div } \vec{F} = y e^{2z} + 2xy \cos z$$

③

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy e^{2z} & x^2 (\cos y) & x^2 (\sin y) \end{vmatrix}$$

$$= \hat{i} \left[\frac{\partial}{\partial y} (x^2 (\sin y)) - \frac{\partial}{\partial z} (xy^2 (\cos y)) \right]$$

$$- \hat{j} \left[\frac{\partial}{\partial x} (x^2 (\sin y)) - \frac{\partial}{\partial z} (xy e^{2z}) \right]$$

$$+ \hat{k} \left[\frac{\partial}{\partial x} (xy^2 (\cos y)) - \frac{\partial}{\partial y} (xy e^{2z}) \right]$$

$$= \hat{i} \left[x^2 (-\sin y) \cdot x + (\cos y - (xy^2 (-\sin y))) \right]$$

$$- \hat{j} \left[x^2 (-\sin y) \cdot y + (\cos y \cdot 2x - (xy e^{2z} \cdot 2)) \right]$$

$$+ \hat{k} \left[y^2 (\cos y) - x e^{2z} \right]$$

$$= \hat{i} \left[-x^3 \sin y + xy^2 \sin y \right] - \hat{j} \left[-x^2 y \sin y \right]$$

$$+ 2x (\cos y - 2xy e^{2z})$$

$$+ \hat{k} \left[y^2 (\cos y - x e^{2z}) \right]$$

(4)

A vector \vec{F} is said to be solenoidal
if $\nabla \cdot \vec{F} = 0$

A vector \vec{F} is said to be irrotational
if $\nabla \times \vec{F} = 0$

Ex (2) If $\vec{F} = (x+3y)\mathbf{i} + (y-2z)\mathbf{j} + (az+x)\mathbf{k}$ is solenoidal find a

Soln: If \vec{F} is solenoidal then $\nabla \cdot \vec{F} = 0$

$$\begin{aligned}\therefore \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(F_1) + \frac{\partial}{\partial y}(F_2) + \frac{\partial}{\partial z} F_3 \\ &= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(az+x) = 0\end{aligned}$$

$$\therefore 1 + 1 + a = 0$$

$$a + 2 = 0 \therefore a = -2$$

Ex (3) Prove that $\vec{F} = (x+2y+az)\mathbf{i} + (bx-3y-z)\mathbf{j} + (4x+(y+2z))\mathbf{k}$ is solenoidal and find a, b, c if \vec{F} is irrotational.

Soln: If \vec{F} is solenoidal then $\nabla \cdot \vec{F} = 0$

$$\begin{aligned}\therefore \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x+2y+az) + \frac{\partial}{\partial y}(bx-3y-z) \\ &\quad + \frac{\partial}{\partial z}(4x+(y+2z))\end{aligned}$$

$$= 1 - 3 + 2 = 0$$

$\nabla \cdot \vec{F} = 0$ then \vec{F} is solenoidal

(5)

\vec{F} is irrotational then $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & 4x+y+2z & bx-3y-z \end{vmatrix} = 0$$

$$\begin{aligned} \therefore & \hat{i} \left[\frac{\partial}{\partial y} (4x+y+2z) - \frac{\partial}{\partial z} (bx-3y-z) \right] \\ & - \hat{j} \left[\frac{\partial}{\partial x} (4x+y+2z) - \frac{\partial}{\partial z} (x+2y+az) \right] \\ & + \hat{k} \left[\frac{\partial}{\partial x} (bx-3y-z) - \frac{\partial}{\partial y} (x+2y+az) \right] = 0 \end{aligned}$$

$$\therefore \hat{i} [c - (-1)] - \hat{j} [4 - a] + \hat{k} [b - 2] = 0$$

$$(c+1)\hat{i} - \hat{j}(4-a) + \hat{k}(b-2) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$c+1=0 \quad 4-a=0 \quad b-2=0$$

$$c=-1 \quad a=4 \quad b=2$$

$$a=4 \quad b=2 \quad c=-1$$

Ex (4)

A vector field is given by

6

$$\vec{F} = (x^2 + 2xy^2)\vec{i} + (y^2 + x^2y)\vec{j}. \text{ Show}$$

that \vec{F} is irrotational and find its scalar potential.

17

Soln. we s.t. $\nabla \times \vec{F} = 0$

$$\therefore \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + 2xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$

$$= \vec{i} \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y^2 + x^2y) \right) - \vec{j} \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2 + 2xy^2) \right)$$

$$+ \vec{k} \left(\frac{\partial}{\partial x}(y^2 + x^2y) - \frac{\partial}{\partial y}(x^2 + 2xy^2) \right)$$

$$= \vec{i} [0 - 0] - \vec{j} [0 - 0] + \vec{k} [0 + y \cdot 2x - (0 + 2xy)]$$

$$= 0\vec{i} - 0\vec{j} + \vec{k} (2xy - 2xy)$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k}$$

$$= 0$$

$$\nabla \times \vec{F} = 0$$

$\therefore \vec{F}$ is irrotational.

If ϕ is scalar potential then

(7)

$$\vec{F} = \nabla \phi$$

$$(x^2 + xy^2)\mathbf{i} + (y^2 + x^2y)\mathbf{j} + 0\mathbf{k} = \mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z}$$

$$\therefore \frac{\partial \phi}{\partial x} = x^2 + xy^2 \quad - (1)$$

$$\frac{\partial \phi}{\partial y} = y^2 + x^2y \quad - (2)$$

$$\frac{\partial \phi}{\partial z} = 0 \quad - (3)$$

$$(1) \Rightarrow \int \frac{\partial \phi}{\partial x} = \int x^2 + xy^2 \, dx + \psi_1(y, z)$$

$$\phi = \frac{x^3}{3} + \frac{x^2}{2}y^2 + \psi_1(y, z) \quad - (4)$$

$$(2) \Rightarrow \int \frac{\partial \phi}{\partial y} = \int y^2 + x^2y \, dy + \psi_2(x, z)$$

$$\phi = \frac{y^3}{3} + x^2 \frac{y^2}{2} + \psi_2(x, z) \quad - (5)$$

$$(3) \Rightarrow \int \frac{\partial \phi}{\partial z} = \int 0 \, dz + \psi_3(x, y)$$

$$\phi = 0 + \psi_3(x, y) \quad - (6)$$

from (4) (5) & (6)

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + C$$

EX 5 P.T. $\vec{F} = (z^2 + 2x + 3y)\vec{i} + (3x + 2y + 2z)\vec{j}$ (5)
 $+ (y + 2z^2)\vec{k}$ is irrotational and

find scalar potential function of such that

$$\vec{F} = \nabla \phi \text{ and } \phi(1,1,1) = 9$$