(5) Evaluation of $\int_0^\infty e^{-at} f(t)\,dt$

$$\int_0^\infty e^{-st} f(t)\,dt = L\{f(t)\} = F(s)$$

put $s = a$

Ex ① Evaluate $\int_0^\infty e^{-2t} \sin^3 t\,dt$. 09

Soln

$$\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$$

$$\int_0^\infty e^{-st} \sin^3 t\,dt = L[\sin^3 t]$$

$$= L\left[\frac{3\sin t - \sin 3t}{4}\right]$$

$$= \frac{1}{4}\left[3L[\sin t] - L[\sin 3t]\right]$$

$$= \frac{1}{4}\left[3 \cdot \frac{1}{s^2+1} - \frac{3}{s^2+9}\right]$$

$$= \frac{3}{4}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right) = F(s)$$

put $s = 2$

$$\therefore \int_0^\infty e^{-2t} \sin^3 t\,dt = \frac{3}{4}\left(\frac{1}{5} - \frac{1}{13}\right)$$

$$= \frac{3}{4}\left(\frac{13-5}{65}\right) = \frac{3}{4}\left(\frac{8}{65}\right)$$

$$= \frac{6}{65}$$

Ex ② If $\int_0^\infty e^{-st} \sin(t+\alpha) \cdot \cos(t-\alpha)\,dt$

$= \frac{3}{8}$ Find α 09, 14, 17

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(6)

Sol^n $\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$

$\therefore \int_0^\infty e^{-st} \sin(t+\alpha)\cos(t-\alpha)\,dt$

$= L[\sin(t+\alpha).\cos(t-\alpha)]$

$= \frac{1}{2} L[\sin(t+\alpha+t-\alpha) + \sin(t+\alpha-t+\alpha)]$

$= \frac{1}{2}[L\{\sin 2t\} + \sin 2\alpha\, L\{1\}]$

$= \frac{1}{2}\left[\frac{2}{s^2+4} + \sin 2\alpha \cdot \frac{1}{s}\right]$

put $s = 2$

$\int_0^\infty e^{-2t} \sin(t+\alpha).\cos(t-\alpha)\,dt$
$= \frac{1}{2}\left[\frac{2}{8} + \frac{\sin 2\alpha}{2}\right]$

$\therefore \frac{3}{8} = \frac{1}{8} + \frac{1}{4}\sin 2\alpha$

$\frac{3-1}{8} = \frac{1}{4}\sin 2\alpha \quad \therefore \frac{1}{4} = \frac{1}{4}\sin 2\alpha$

$\sin 2\alpha = 1 \quad \therefore 2\alpha = \frac{\pi}{2}$

$\alpha = \frac{\pi}{4}$

H.W If $\int_0^\infty e^{-2t} \sin(t+\alpha)\cos(t-\alpha)\,dt$ (7)

$= \frac{1}{4}$ Find α.

(7) (i) Change of Scale Property

Let $L\{f(t)\} = F(s)$

then $L\{f(at)\} = \frac{1}{a} F(s/a)$

Ex (1) If $L\{f(t)\} = \frac{20-4s}{s^2-4s+20}$ find $L\{f(3t)\}$

soln: $L\{f(t)\} = \frac{20-4s}{s^2-4s+20} = F(s)$

$L\{f(at)\} = \frac{1}{a} F(s/a)$

$a = 3$

$L\{f(3t)\} = \frac{1}{3} F(s/3)$

$= \frac{1}{3} \cdot \frac{(20 - 4(\frac{s}{3}))}{\frac{s^2}{9} - 4\frac{s}{3} + 20}$

$= \frac{\frac{60-4s}{9}}{\frac{s^2-12s+180}{9}}$

$= \frac{60-4s}{s^2-12s+180}$

Ex If $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s\sqrt{s}} e^{-\frac{1}{4s}}$ Find $L\{\sin 2\sqrt{t}\}$

soln $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s\cdot\sqrt{s}} e^{-\frac{1}{4s}} = F(s)$

$L\{\sin 2\sqrt{t}\} = L\{\sin\sqrt{4t}\} = \frac{1}{4} F(s/4)$

$= \frac{1}{4} \cdot \frac{\sqrt{\pi}}{2\frac{s}{4}\cdot\frac{\sqrt{s}}{2}} \cdot e^{-\frac{1}{4 s/4}} = \frac{\sqrt{\pi}}{s\sqrt{s}} e^{-\frac{1}{s}}$

(8) First shifting theorem

If $L\{f(t)\} = F(s)$ then

$L\{e^{at} f(t)\} = F(s-a)$

$L\{e^{-at} f(t)\} = F(s+a)$

If $f(t)$ by e^{at} $\rightarrow s \rightarrow s-a$

(8) $L\{t^n\} = \dfrac{n!}{s^{n+1}} = \dfrac{\overline{|n+1}}{s^{n+1}}$

$L\{e^{at} t^n\} = \dfrac{n!}{(s-a)^{n+1}} = \dfrac{\overline{|n+1}}{(s-a)^{n+1}}$

(9) $L\{\sin kt\} = \dfrac{k}{s^2+k^2}$

$L\{e^{at} \sin kt\} = \dfrac{k}{(s-a)^2+k^2}$

(10) $L\{\cos kt\} = \dfrac{s}{s^2+k^2}$

$L\{e^{at} \cos kt\} = \dfrac{s-a}{(s-a)^2+k^2}$

(11) $L\{e^{at} \sinh kt\} = \dfrac{k}{(s-a)^2-k^2}$

(12) $L\{e^{at} \cosh kt\} = \dfrac{s-a}{(s-a)^2-k^2}$

Ex (1) If $L\{f(t)\} = \dfrac{s}{s^2+s+4}$ Find $L\{e^{-3t} f(2t)\}$

Soln: $L[f(t)] = \dfrac{s}{s^2+s+4} = F(s)$

$L[f(2t)] = \dfrac{1}{2} \cdot \dfrac{\frac{s}{2}}{\frac{s^2}{4}+\frac{s}{2}+4} = \dfrac{s/4}{\frac{s^2+2s+16}{4}}$

$= \dfrac{s}{s^2+2s+16}$

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$$L\{e^{-at} f(t)\} = F(s+a)$$
$$s \to s+a$$

$$L\{e^{-3t} f(t)\} = \frac{s+3}{(s+3)^2 + 2(s+3) + 16}$$

$$= \frac{s+3}{s^2+6s+9+2s+6+16}$$

$$L\{e^{-3t} f(t)\} = \frac{s+3}{s^2+8s+31}$$