(8)

Convolution thm corr.

If $L^{-1}\{\frac{1}{s}F(s)\} = \int_0^t f(u)\,du$

(i) $L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{\frac{d}{ds}F(s)\}$

Ex (1) Find $L^{-1}\{\frac{1}{s}\log(1+\frac{1}{s^2})\}$ 16

(ii) $L^{-1}\{\log(1+\frac{a^2}{s^2})\}$, (iii) $L^{-1}\{\log(\frac{s+a}{s+b})\}$, 13

Soln $F(s) = \log(1+\frac{1}{s^2})$

use $L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{\frac{d}{ds}F(s)\}$

$= -\frac{1}{t}L^{-1}\{\frac{d}{ds}\log(1+\frac{1}{s^2})\}]$

$= -\frac{1}{t}L^{-1}\{\frac{d}{ds}\log(\frac{s^2+1}{s^2})\}]$

$= -\frac{1}{t}L^{-1}\{\frac{d}{ds}(\log(s^2+1) - \log s^2)\}]$

$= -\frac{1}{t}L^{-1}\{\frac{d}{ds}\log(s^2+1) - 2\log s\}$

$= -\frac{1}{t}L^{-1}\{\frac{1}{s^2+1}\cdot 2s - \frac{2}{s}\}$

$= -\frac{2}{t}[L^{-1}\{\frac{s}{s^2+1} - \frac{1}{s}\}$

$L^{-1}\{\log(1+\frac{1}{s^2})\} = -\frac{2}{t}[\cos t - 1] = \frac{2}{t}(1-\cos t)$

(19) $L^{-1}\{\frac{1}{s}F(s)\} = \int_0^t f(u)\,du$

$\therefore L^{-1}\left\{\frac{1}{s}\log(1+\frac{1}{s^2})\right\} = \int_0^t \frac{2}{u}(1-\cos u)\,du$

<u>Ex 2</u>: Find $L^{-1}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$ 13.

(ii) $L^{-1}\left\{\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right\}$ 14

<u>Soln</u> (ii) $F(s) = \log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)$

use $L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{\frac{d}{ds}F(s)\}$

$L^{-1}\left\{\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right\} = -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right\}$

$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left(\log(s^2+a^2) - \log(\sqrt{s+b})\right)\right]$

$= -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}\log(s^2+a^2) - \frac{1}{2}\log(s+b)\right.$

$= \cancel{-\frac{1}{t}} -\frac{1}{t}L^{-1}\left\{\frac{1}{s^2+a^2}2s - \frac{1}{2}\frac{1}{s+b}\right\}$

$= -\frac{1}{t}\left[2L^{-1}\left\{\frac{s}{s^2+a^2}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+b}\right\}\right]$

$= -\frac{1}{t}\left[2\cos at - \frac{1}{2}e^{-bt}\right)$

$= \frac{1}{t}\left(\frac{1}{2}e^{-bt} - 2\cos at\right)$

(20) Ex (3) $L^{-1}\{\tan^{-1}\frac{a}{s}\}$ 12, 13

$L^{-1}\{\cot^{-1}(s+1)\}$ 46

$L^{-1}\{\tan^{-1}(\frac{s+a}{b})\}$ 15

Soln $F(s) = \cot^{-1}(s+1)$

Use $L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{\frac{d}{ds}F(s)\}$

$L^{-1}\{\cot^{-1}(s+1)\} = -\frac{1}{t}L^{-1}\{\frac{d}{ds}\cot^{-1}(s+1)\}$

$= -\frac{1}{t}L^{-1}\{-\frac{1}{1+(s+1)^2}\cdot 1\}$

$= \frac{1}{t}L^{-1}\{\frac{1}{(s+1)^2+1}\}$

$= \frac{1}{t}e^{-t}L^{-1}\{\frac{1}{s^2+1}\}$

$= \frac{e^{-t}}{t}\sin t$

HW $L^{-1}\{\cot^{-1}(\frac{s+3}{2})\}$

Ex (5) Find $L^{-1}\{\tan^{-1}\frac{2}{s^2}\}$

Soln $L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{\frac{d}{ds}F(s)\}$

$L^{-1}\{\tan^{-1}\frac{2}{s^2}\} = -\frac{1}{t}L^{-1}\{\frac{d}{ds}\tan^{-1}\frac{2}{s^2}\}$

$= -\frac{1}{t}\left[\frac{1}{1+\frac{4}{s^4}}\cdot 2(-2s^{-3})\right]$

(21) $L^{-1}\{\tan^{-1}\frac{2}{s^2}\} = -\frac{1}{t} L^{-1}\{ \frac{1}{\frac{s^4+4}{s^4}} (-\frac{4}{s^3})\}$

$$= \frac{4}{t} L^{-1}\{ \frac{s}{s^4+4} \} \quad -(1)$$

Let $\frac{s}{s^4+4} = \frac{s}{(s^4+4s^2+4)-4s^2} = \frac{s}{(s^2+2)^2-(2s)^2}$

$$= \frac{s}{(s^2+2-2s)(s^2+2+2s)} = \frac{s}{(s^2-2s+2)(s^2+2s+2)}$$

Let $\frac{1}{S \cdot B} = \frac{1}{B-S}\left(\frac{1}{S}-\frac{1}{B}\right)$

$$\frac{1}{(s^2-2s+2)(s^2+2s+2)} = \frac{1}{4s}\left(\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right)$$

$$\frac{1 \cdot s}{(s^2-2s+2)(s^2+2s+2)} = \frac{1s}{4s}\left(\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right)$$

from (1)

$$L^{-1}\{\frac{s}{s^4+4}\} = \frac{1}{4} L^{-1}\{ \frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2} \}$$

$$= \frac{1}{4}\left[L^{-1}\{ \frac{1}{s^2-2s+1+1} - \frac{1}{s^2+2s+1+1} \}\right.$$

$$= \frac{1}{4}\left[L^{-1}\{ \frac{1}{(s-1)^2+1} \} - L^{-1}\{\frac{1}{(s+1)^2+1}\}\right.$$

$$= \frac{1}{4}\left[e^{t}\sin t - e^{-t}\sin t \right) =$$

$$= \frac{1}{2}\sin t\left(\frac{e^t - e^{-t}}{2}\right) = \frac{1}{2}\sin t \sinh t$$

from (1)

$$= \frac{4}{t} \cdot \frac{1}{2}\sin t \sinh t = \frac{2}{t}\sin t \sinh t$$

(12) use convolution thm PT.

$$L^{-1}\left\{\frac{1}{s}\log\left(\frac{s+a}{s+b}\right)\right\} = \int_0^t \frac{e^{-bu}-e^{-au}}{u}\,du$$

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Soln (i) $L^{-1}\left\{\frac{1}{s}\tan^{-1}\left(\frac{s+a}{b}\right)\right\} = \int_0^t -\frac{1}{u}e^{-au}\sin bu\,du$

Soln (ii) $F(s) = \tan^{-1}\left(\frac{s+a}{b}\right)$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}F(s)\right\}$$

$$L^{-1}\left\{\tan^{-1}\left(\frac{s+a}{b}\right)\right\} = -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}\tan^{-1}\left(\frac{s+a}{b}\right)\right\}$$

$$= -\frac{1}{t}L^{-1}\left\{\frac{1}{1+\frac{(s+a)^2}{b^2}}\cdot\frac{1}{b}\cdot 1\right\}$$

$$= -\frac{1}{t}L^{-1}\left\{\frac{b^2}{(s+a)^2+b^2}\cdot\frac{1}{b}\right\}$$

$$= -\frac{b}{t}L^{-1}\left\{\frac{1}{(s+a)^2+b^2}\right\}$$

$$= -\frac{b}{t}e^{-at}L^{-1}\left\{\frac{1}{s^2+b^2}\right\}$$

$$= -\frac{b}{t}\cdot\frac{1}{b}e^{-at}\sin bt$$

$$= -\frac{1}{t}e^{-at}\sin bt = f(u)$$

$$L^{-1}\left\{\frac{1}{s}\tan^{-1}\left(\frac{s+a}{b}\right)\right\} = \int_0^t f(u)\,du$$

$$= -\int_0^t \frac{1}{u}e^{-au}\sin bu\,du$$

$$= -\int_0^t \frac{1}{u}e^{-au}\sin bu\,du$$