Taylor's and Laurent's Series

A series of the type $(a_1 + ib_1) + (a_2 + ib_2)$ $\ldots + (a_n + ib_n) \ldots$ where $a_1, a_2, \ldots a_n$, $b_1, b_2 \ldots b_n$ are real nos. is called series of complex numbers denoted by

$$\sum_{n=1}^{\infty} (a_n + ib_n)$$

A series of the type $c_0 + c_1(z-a) + c_2(z-a)^2 \ldots + c_n(z-a)^n \ldots$ is called a power series in powers of $(z-a)$

$$\sum_{n=1}^{\infty} c_n (z-a)^n$$

z is complex no. The constants $c_0, c_1, c_2 \ldots c_n$ are called coefficients, the constant a is called the centre of the series

Power series $\sum c_n(z-a)^n$ is convergent for $|z-a| < R$, is divergent for $|z-a| > R$

$|z-a| = R$ circle of convergence.

Taylor's Series

If $f(z)$ is analytic inside a circle C with centre at z_0 then for all z inside C $f(z)$ can be expanded as

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) \cdots$$

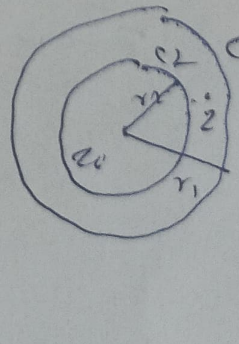
The series is convergent at every point inside C

Laurent's Series

If C_1 and C_2 are two concentric circles of radii r_1 and r_2 with centre at ~~z_0~~ z_0 and if $f(z)$ is analytic on C_1 and C_2 and in the annular region R between two circles then for any z in R

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-z_0)^{n+1}} \, dw$$



$$b_n = \frac{1}{2\pi i} \int_{C} \frac{f(w)}{(w-z_0)^{-n+1}} \, dw$$

the part $\sum_{n=0}^{\infty} a_n(z-a)^n$ consisting of positive integral power of $(z-a)$ is called the analytic part or regular part and the part $\sum_{n=1}^{\infty} b_n(z-a)^{-n}$ consisting of negative integral power of $(z-a)$ is called principal part of Laurent's series

<u>Maclaurin's series</u>

1. $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} - -$
2. $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - -$
3. $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - -$
4. $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} - -$
5. $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} - -$
6. $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} - -$
7. $(1+z)^{-1} = 1 - z + z^2 - z^3 - -$

$(1-z)^{-1} = 1 + z + z^2 + z^3 - -$

$(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 - -$

$(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 - -$

Ex (1) Expand cos z of Taylor's series at $z = \frac{\pi}{2}$

Soln $f(z) = \cos z$

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a)$$

$\cdots$ — $\cdots$

$f(z) = \cos z$, $f(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

$f'(z) = -\sin z$ $\quad f'(\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1$

$f''(z) = -\cos z$, $f''(\frac{\pi}{2}) = -\cos\frac{\pi}{2} = 0$

$f'''(z) = \sin z$ $\quad f'''(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

$$f(z) = 0 + (z - \tfrac{\pi}{2})(-1) + 0 + \frac{(z-\frac{\pi}{2})^3}{3!}(1)$$

$$+ \frac{(z - \frac{\pi}{2})^5}{5!}(-1) - -$$

Ex (2) Find the Laurents series for

$f(z) = \dfrac{4z+3}{z(z-3)(z+2)}$ valid for $2 < |z| < 3$

Soln $f(z) = \dfrac{4z+3}{z(z-3)(z+2)} = \dfrac{A}{z} + \dfrac{B}{z-3} + \dfrac{C}{z+2}$

$$4z+3 = A(z-3)(z+2) + B(z)(z+2) + C(z)(z-3)$$

Ex ① Expand $\cos z$ of Taylor's series at $z = \frac{\pi}{2}$

Soln $f(z) = \cos z$

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a)$$

$$\cdots$$

$f(z) = \cos z \quad f(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

$f'(z) = -\sin z \quad f'(\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1$

$f''(z) = -\cos z \quad f''(\frac{\pi}{2}) = -\cos\frac{\pi}{2} = 0$

$f'''(z) = \sin z \quad f'''(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

$$f(z) = 0 + (z - \tfrac{\pi}{2})(-1) + 0 + \frac{(z-\frac{\pi}{2})^3}{3!}(1)$$

$$+ \frac{(z-\frac{\pi}{2})^5}{5!}(-1) \cdots$$

Ex ② Find the Laurent's series for

$f(z) = \dfrac{4z+3}{z(z-3)(z+2)}$ valid for $2 < |z| < 3$

Soln $f(z) = \dfrac{4z+3}{z(z-3)(z+2)} = \dfrac{A}{z} + \dfrac{B}{z-3} + \dfrac{C}{z+2}$

$$4z+3 = A(z-3)(z+2) + B(z)(z+2) + C(z)(z-3)$$

$z = 0$

$3 = A(-3)(2) + 0$

$3 = -6A$

$A = -\frac{1}{2}$

$z = 3$

$12 + 3 = 0 + B(3)(5) + 0$

$15 = 15B$

$B = 1$

$z = -2$

$-8 + 3 = 0 + C(-2)(-2-3)$

$-5 = 10C$

$C = -\frac{1}{2}$

$\therefore f(z) = \frac{-\frac{1}{2}}{z} + \frac{1}{z-3} + \frac{-\frac{1}{2}}{z+2}$

$2 < |z| < 3$

$2 < |z| \qquad |z| < 3$

$\frac{2}{|z|} < 1 \qquad \frac{|z|}{3} < 1$ series is convergent

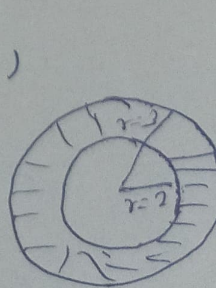
$f(z) = -\frac{1}{2z} + \frac{1}{-3+z} + \frac{-\frac{1}{2}}{z+2}$

$= -\frac{1}{2z} + \frac{1}{-3(1-\frac{z}{3})} \frac{-\frac{1}{2}}{z(1+\frac{2}{z})}$

$= -\frac{1}{2z} - \frac{1}{3}(1-\frac{z}{3})^{-1} - \frac{1}{2z}(1+\frac{2}{z})^{-1}$

$= -\frac{1}{2z} - \frac{1}{3}(1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} - -)$

$-\frac{1}{2z}(1 - \frac{2}{z} + \frac{2^2}{z^2} - -)$



ROC is

$2 < |z| < 3$

(6)

Ex (2) Find all possible Laurent's expansion of the function

$$f(z) = \frac{2-z^2}{z(1-z)(2-z)} \quad \text{about } z=0$$

indicating region of convergence in each case

Soln
$$f(z) = \frac{2-z^2}{z(1-z)(2-z)} = \frac{A}{z} + \frac{B}{1-z} + \frac{C}{2-z}$$

$$2-z^2 = A(1-z)(2-z) + B\,z(2-z) + C\,z(1-z) \quad -(1)$$

put $z=0$	$z=1$	$z=2$
$2 = A(1)(2)$	$2-1 = B(1)(2-1)$	$2-4 = C\,2(1-2)$
	$1 = B$	$-2 = -2C$
$A = 1$		$C = 1$

$$f(z) = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

(i) $0 < |z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$

$|z| < 1$ $|z| < 2$ $\therefore \left|\frac{z}{2}\right| < 1$

$|z| < a$
$\frac{1}{a-z} = \frac{1}{a(1-\frac{z}{a})}$
$a < |z|$
$\frac{1}{-z(1-\frac{a}{z})}$

$$f(z) = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$
$$= \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2(1-\frac{z}{2})}$$
$$= \frac{1}{z} + (1-z)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$
$$= \frac{1}{z} + 1 + z + z^2 + z^3 - - + \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} - -\right)$$

case (ii) $1<|z|<2$

$1<|z|$ $|z|<2$

$\frac{1}{|z|}<1$ $\left|\frac{z}{2}\right|<1$

$$f(z)=\frac{1}{z}+\frac{1}{1-z}+\frac{1}{2-z}$$

$$=\frac{1}{z}+\frac{1}{-z+1}+\frac{1}{2(1-\frac{z}{2})}$$

$$=\frac{1}{z}+\frac{1}{-z(1-\frac{1}{z})}+\frac{1}{2(1-\frac{z}{2})}$$

$$=\frac{1}{z}-\frac{1}{z}(1-\frac{1}{z})^{-1}+\frac{1}{2}(1-\frac{z}{2})^{-1}$$

$$=\frac{1}{z}-\frac{1}{z}(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}\cdots)$$

$$+\frac{1}{2}(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}\cdots)$$

(iii) $|z|>2$, $|z|>1$

$1<|z|$, $2<|z|$

$\frac{1}{|z|}<1$ $\frac{2}{|z|}<1$

$$f(z)=\frac{1}{z}+\frac{1}{-z+1}+\frac{1}{-z+2}$$

$$=\frac{1}{z}-\frac{1}{z(1-\frac{1}{z})}-\frac{1}{z(1-\frac{2}{z})}$$

$$=\frac{1}{z}-\frac{1}{z}(1-\frac{1}{z})^{-1}-\frac{1}{z}(1-\frac{2}{z})^{-1}$$

$$=\frac{1}{z}-\frac{1}{z}(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}\cdots)$$

$$-\frac{1}{z}(1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}\cdots)$$

ROC's

$0<|z|<1$ $\quad$ $1<|z|<2$ $\quad$ $2<|z|<\infty$

Ex (3). Find Laurent's series

$$f(z) = \frac{2}{(z-1)(z-2)} \quad 0,\ |z|<1$$

$1<|z|<2 \qquad |z|>2$

Ex (5) obtain Taylor's & Laurent's expansion of

$$f(z) = \frac{z-1}{z^2-2z-3}$$

indicating region of convergence.