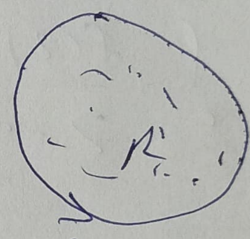
① Cauchy's theorem ①

Cauchy's integral theorem

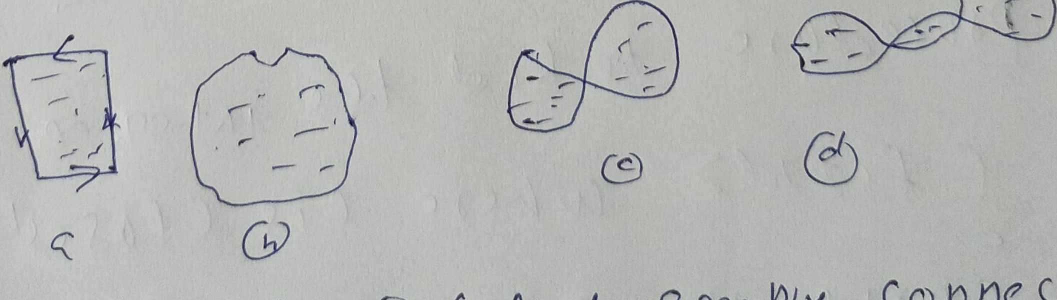
If $f(z)$ is an analytic function and if its derivative $f'(z)$ is continuous at each point within and on a simple closed curve C then the integral of $f(z)$ along the closed curve C is zero i.e.

$$\oint f(z)\,dz = 0$$



If a closed curve does not intersect itself it is called a simple closed curve

If a closed curve intersects itself it is called a multiple curve



a (b) (c) (d)

A region R is called simply connected region

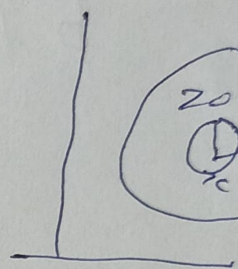
Cauchy's integral formula

(Fundamental formula)

If $f(z)$ is analytic inside and on a closed curve C of a simply connected region R and if z_0 is any point within C then

$$\boxed{\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)}$$

i.e. $\displaystyle\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$



(1) Cauchy's integral formula for derivatives

$$\boxed{\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!}\, f^{n-1}(z_0)}$$

Procedure to find the integral $\int_C f(z)\,dz$ where C is closed curve in a given region R.

(i) when $\int_C F(z)dz = \int_C \frac{\phi(z)}{z-z_0}dz$ or $\int_C \frac{\phi(z)}{(z-z_0)^n}dz$

if z_0 is a point outside of C, $F(z)$ is analytic in & on C we put $F(z) = f(z)$

by Cauchy's theorem

$$\int_C f(z)dz = 0$$

If z_0 is a point inside C, $f(z)$ is not analytic in C we put $\phi(z) = f(z)$ which may be analytic in C, Cauchy's formula

$$\int_C F(z)dz = \int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$

(ii) when $\int_C F(z)dz = \int_C \frac{\phi(z)}{(z-z_1)(z-z_2)}dz$

if z_1, z_2 are both inside of C

$F(z)$ is not analytic in C

we put $\frac{1}{(z-z_1)(z-z_2)} = \frac{K_1}{z-z_1} + \frac{K_2}{z-z_2}$

then $\phi(z) = f(z)$ may be analytic in C

$$\int_C F(z)\,dz = k_1 \int_C \frac{f(z)}{z-z_1}\,dz + k_2 \int_C \frac{f(z)}{z-z_2}\,dz$$

$$= k_1\, 2\pi i\, f(z_1) + k_2\, 2\pi i\, f(z_2)$$

(ii) If z_1, z_2 both outside of C, $F(z) = f(z)$ then Cauchy's theorem.

$$\int_C f(z)\,dz = \int_C f(z)\,dz = 0$$

(iv) If z_1 lies outside of C, z_2 lies inside of C. $F(z)$ is not analytic in C.

$f(z) = \frac{\phi(z)}{z-z_1}$, $f(z)$ may be analytic

by Cauchy's formula.

$$\int_C f(z)\,dz = \int_C \frac{f(z)}{z-z_0}\,dz =$$

$$= \int_C \frac{\frac{\phi(z)}{z-z_1}}{z-z_2}\,dz = 2\pi i\, f(z_2)$$

Ex (1) Evaluate $\int_C \frac{1}{z}\cos z\,dz$

where C is ellipse $9x^2 + 4y^2 = 1$

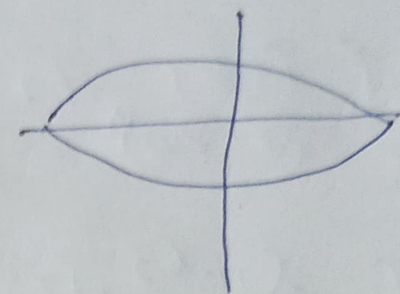
Soln

Q

Sol^n Point $z = 0$

$$9x^2 + 4y^2 = 1$$

$$\frac{x^2}{\frac{1}{9}} + \frac{y^2}{1/4} = 1$$



$z = 0$ lies inside of ellipse $f(z) = \cos z$ may be analytic

Cauchy's integral formula

$$\int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

$$\int_C \frac{\cos z}{z} dz = 2\pi i f(0)$$

$$= 2\pi i \cos 0$$

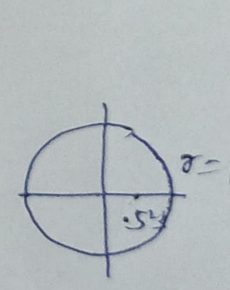
$$= 2\pi i (1)$$

$$= 2\pi i$$

Ex (2) Evaluate $\int_C \frac{\sin^6 z}{(z - \frac{\pi}{6})^3} dz$

where C is $|z| = 1$

Sol^n $|z| = 1$ is a circle with centre at origin $r = 1$



$z = \frac{\pi}{6}$ $\therefore |z| = |\frac{\pi}{6}| = |\frac{3.142}{6}| = \frac{3.142}{6}$

$= 0.523 < 1$ point lies inside of C

$f(z) = \sin^6 z$ is analytic inside C 6

Cauchy's integral formula for derivative

$$\int_C \frac{f(z)}{(z-z_0)^n}\, dz = 2\pi i f^{n-1}(z_0)$$

$n = 3$ $\qquad z_0 = \frac{\pi}{6}$

$$\int_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3}\, dz = \frac{2\pi i}{(n-1)!} f^{2}\left(\frac{\pi}{6}\right)$$

$f(z) = \sin^6 z$

$f'(z) = 6 \sin^5 z \cos z$

$f''(z) = 6[\sin^5 z(-\sin z) + \cos z \cdot 5 \sin^4 z \cos z]$

$= 6[-\sin^6 z + 5 \sin^4 z \cos^2 z]$

$f''(\frac{\pi}{6}) = 6[-(\sin\frac{\pi}{6})^6 + 5(\sin(\frac{\pi}{6}))^4 (\cos\frac{\pi}{6})^2]$

$= 6[-(\frac{1}{2})^6 + 5(\frac{1}{2})^4 (\frac{\sqrt{3}}{2})^2]$

$= 6[-\frac{1}{64} + 5 \cdot \frac{1}{16} \cdot \frac{3}{4}]$

$= 6(-\frac{1}{64} + \frac{15}{64}) = 6(\frac{14}{64})$

$= \frac{3(7)}{16} = \frac{21}{16}$

$$\therefore \int_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3}\, dz = \frac{2\pi i}{(3-1)!} \cdot \frac{21}{16}$$

$$= \frac{2\pi i}{2} \cdot \frac{21}{16} = \frac{21\pi i}{16}$$

Ex ③ Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz$ ⑦

where C is circle $|z| = 4$

Soln: $|z| = 4$ is a circle with centre at origin & radius $r = 4$

$(z-2)(z-3) = 0 \quad z = 2, z = 3$

if $z = 2 \quad |z| = |2| = 2 < 4$

if $z = 3 \quad |z| = |3| = 3 < 4$

$z = 2, 3$ lies inside of C

f(z) is not analytic

$f(z) = \sin \pi z^2 + \cos \pi z^2$ may be analytic

$\therefore \frac{1}{(z-2)(z-3)} = \frac{K_1}{z-2} + \frac{K_2}{z-3} = \frac{-1}{z-2} + \frac{1}{z-3}$

$1 = K_1(z-3) + K_2(z-2)$

$z = 2 \qquad z = 3$

$1 = K_1(-1) \qquad 1 = K_2(1)$

$K_1 = -1 \qquad K_2 = 1$

$\therefore$ Cauchy's integral formula

$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz =$

(8)

$$= -\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\, dz + \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-3}\, dz$$

$$= -2\pi i f(2) + 2\pi i f(3)$$

$$= -2\pi i [\sin 4\pi + \cos 4\pi)$$

$$+ 2\pi i (\sin 9\pi + \cos 9\pi)$$

$$= -2\pi i (0 + 1) + 2\pi i (0 + (-1))]$$

$$= -2\pi i - 2\pi i$$

$$= -4\pi i$$

<u>Ex (4)</u> Evaluate $\int_C \frac{z+2}{(z-3)(z-4)}\, dz$

when C is circle $|z| = 1$

Soln $|z| = 1$ Bacircle $(0,0)$ $r = 1$

$z = 3, 4$, $|z| = |3| = 3 > 1$
$|z| = |4| = 4 > 1$

$z = 3, 4$ lies out side of C

$f(z) = \frac{z+2}{(z-3)(z-4)}$ is analytic

Cauchy's therm $\int_C f(z)\,dz = 0$, $\int_C \frac{z+2}{(z-3)(z-4)}\, dz = 0$