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Tutorial - 1. - Laplace Transform.

1. Q. Find L.T of $\sinh^5 t$.

$$(a+b)^n = \sum_{r=0}^{r=n} {}^nC_r\, a^{n-r} b^r$$

$\therefore a = e^t \quad b = -e^{-t}$

$n = 5$.

$$L[(\sinh t)^5]$$

$$= L\left[\left(\frac{e^t - e^{-t}}{2}\right)^5\right]$$

$$= L\left[\frac{1}{2^5}(e^t - e^{-t})^5\right]$$

$$= \frac{1}{32} L[(e^t - e^{-t})^5]$$

$$= \frac{1}{32} L\left[{}^5C_0 (e^t)^5 (-e^{-t})^0 + {}^5C_1 (e^t)^4 (-e^{-t})^1 + {}^5C_2 (e^t)^3 (-e^{-t})^2 + {}^5C_3 (e^t)^2 (-e^{-t})^3 + {}^5C_4 (e^t)^1 (-e^{-t})^4 + {}^5C_5 (e^t)^0 (-e^{-t})^5\right]$$

$$= \frac{1}{32} L\left[e^{5t} + 5e^{4t}(-e^{-t}) + 10e^{3t}(e^{-2t}) + 10e^{2t}(-e^{-3t}) + 5e^{t}(e^{-4t}) + (-e^{-5t})\right]$$

$$= \frac{1}{32} L\left[e^{5t} - 5e^{4t}e^{-t} + 10e^{3t}e^{-2t} - 10e^{2t}e^{-3t} + 5e^{t}e^{-4t} - e^{-5t}\right]$$

$$= \frac{1}{32} L\left[e^{5t} - e^{-5t} - 5(e^{3t} - e^{-3t}) + 10(e^t - e^{-t})\right]$$

$$= \frac{2}{32} L\left[\frac{e^{5t} - e^{-5t}}{2} - \frac{5(e^{3t} - e^{-3t})}{2} + \frac{10(e^t - e^{-t})}{2}\right]$$

$$= \frac{1}{16} L\left[\sinh 5t - 5\sinh 3t + 10\sinh t\right]$$

$$= \frac{1}{16}\left[\frac{5}{s^2-25} - \frac{15}{s^2-9} + \frac{10}{s^2-1}\right]$$

$$= \frac{1}{16}\left[\frac{5(s^2-9)(s^2-1) - 15(s^2-25)(s^2-1) + 10(s^2-25)(s^2-9)}{(s^2-25)(s^2-9)(s^2-1)}\right]$$

$$= \frac{1}{16}\left[\frac{5(s^4 - s^2 - 9s^2 + 9) - 15(s^4 - s^2 - 25s^2 + 25) + 10(s^4 - 25s^2 - 9s^2 + 225)}{(s^2-25)(s^2-9)(s^2-1)}\right]$$

$$= \frac{1}{16}\left[\frac{5s^4 - 50s^2 + 45 - 15s^4 + 390s^2 - 375 + 10s^4 - 340s^2 + 2250}{(s^2-25)(s^2-9)(s^2-1)}\right]$$

$$= \frac{1}{16}\left[\frac{1920}{(s^2-25)(s^2-9)(s^2-1)}\right] = \frac{120}{(s^2-25)(s^2-9)(s^2-1)}$$

2. Evaluate $\int_0^\infty e^{-2t} t^5 \cosh t \, dt$.

considering $L[t^5 \cos ht]$

$$= L\left[t^5\left(\frac{e^t + e^{-t}}{2}\right)\right]$$

$$= \frac{1}{2}\left[L(e^t t^5) + L(e^{-t} t^5)\right]$$

$$= \frac{1}{2}\left[\frac{5!}{(s-1)^6} + \frac{5!}{(s+1)^6}\right]$$

$$= \frac{5!}{2}\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$$

$$\therefore \int_0^\infty e^{-2t} t^5 \cosh t \, dt = \frac{5!}{2}\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$$

Now put $s = 2$.

$$\therefore \int_0^\infty e^{-2t} t^5 \cosh t \, dt = \frac{5!}{2}\left[\frac{1}{(2-1)^6} + \frac{1}{(2+1)^6}\right]$$

$$= \frac{5\times4\times3\times2}{2}\left[1 + \frac{1}{3^6}\right]$$

$$= 60\left[1 + \frac{1}{3^6}\right]$$

3. Find LT of $e^t \sin 2t \sin 3t$.

$$L[e^t \sin 2t \sin 3t] = \frac{1}{2} L[e^t(2\sin 3t \sin 2t)]$$

$$= \frac{-1}{2} L[e^t(\cos(3t+2t) - \cos(3t-2t))]$$

$$= \frac{-1}{2} L[e^t(\cos 5t - \cos t)] = \frac{-1}{2} L[e^t \cos 5t - e^t \cos t]$$

$$= -\frac{1}{2}\left[\frac{s-1}{(s-1)^2 + 5^2} - \frac{s-1}{(s-1)^2 + 1^2}\right]$$

$$= \frac{-1}{2}\left[\frac{s-1}{s^2-2s+26} - \frac{s-1}{s^2-2s+2}\right]$$

$$= \frac{-1}{2}\left[\frac{(s^2-2s+2)(s-1) - (s-1)(s^2-2s+26)}{(s^2-2s+2)(s^2-2s+26)}\right]$$

$$= \frac{-1}{2}\left[\frac{\cancel{s^3}-\cancel{2s^2}+\cancel{2s}-\cancel{s^2}+2s-2-\cancel{s^3}+\cancel{2s^2}-26+s^2-2s+26}{(s^2-2s+2)(s^2-2s+26)}\right]$$

$$= \frac{-1}{\cancel{2}}\left[\frac{\cancel{2}(s-1)}{(s^2-2s+2)(s^2-2s+26)}\right]$$

$$= \frac{-(s-1)}{(s^2-2s+2)(s^2-2s+26)}$$

4. Find L.T of $te^{-3t}\cos 2t \cos 3t$

$$e^{-3t}\cos 2t\cos 3t = \frac{e^{-3t}}{2}\left[2\cos 3t\cos 2t\right]$$

$$= \frac{e^{-3t}}{2}\left[\cos 5t + \cos t\right]$$

$$\therefore L\left[e^{-3t}\cos 2t\cos 3t\right] = \frac{1}{2}\left\{L\left[e^{-3t}\cos 5t + e^{-3t}\cos t\right]\right.$$

$$= \frac{1}{2}\left[\frac{s+3}{(s+3)^2+5^2} + \frac{s+3}{(s+3)^2+1^2}\right]$$

$$= \frac{1}{2}\left[\frac{s+3}{s^2+6s+34} + \frac{s+3}{s^2+6s+10}\right]$$

$$\therefore L\left[te^{-3t}\cos 2t\cos 3t\right] = \frac{-d}{ds}\left[\frac{1}{2}\left(\frac{s+3}{(s^2+6s+34)} + \frac{s+3}{(s^2+6s+10)}\right)\right]$$

$$= \frac{1}{2}\left[\frac{s^2+6s-16}{(s^2+6s+34)^2} + \frac{s^2+6s+8}{(s^2+6s+10)^2}\right]$$

5. P.T. $\int_0^\infty \frac{\sin 2t + \sin 3t}{te^t}\, dt = \frac{3\pi}{4}$

$$L[\sin 2t] = \frac{2}{s^2+4} \quad ; \quad L[\sin 3t] = \frac{3}{s^2+9}$$

$$\therefore L\left[\frac{\sin 2t + \sin 3t}{t}\right] = \int_s^\infty \left[\frac{2}{s^2+4} + \frac{3}{s^2+9}\right] ds$$

$$= \left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty + \left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

$$= \left(\frac{\pi}{2} - \tan^{-1}\frac{s}{2}\right) + \left(\frac{\pi}{2} - \tan^{-1}\frac{s}{3}\right)$$

$$= \pi - \left(\tan^{-1}\frac{s}{2} + \tan^{-1}\frac{s}{3}\right).$$

also,

$$\tan^{-1}\frac{s}{2} = \alpha \text{ and } \tan^{-1}\frac{s}{3} = \beta.$$

$$\therefore \frac{s}{2} = \tan\alpha \text{ and } \frac{s}{3} = \tan\beta.$$

$$\therefore \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{(s/2) + (s/3)}{1 - \left(\frac{s}{2}\right)\left(\frac{s}{3}\right)} = \frac{5s/6}{\frac{6-s^2}{6}}$$

$$= \frac{5s}{6-s^2}$$

$$\therefore \alpha + \beta = \tan^{-1}\left(\frac{5s}{6-s^2}\right)$$

$$\therefore L\left[\frac{\sin 2t + \sin 3t}{t}\right] = \pi - \tan^{-1}\left(\frac{5s}{6-s^2}\right)$$

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$$\therefore \int_s^\infty e^{-t}\left(\frac{\sin 2t+\sin 3t}{te^t}\right)dt = \pi - \tan^{-1}\left(\frac{5s}{6-s^2}\right)$$

$$\therefore \int_s^\infty e^{-t}\left(\frac{\sin 2t+\sin 3t}{t}\right)dt = \pi - \tan^{-1}\left(\frac{5s}{6-s^2}\right)$$

put $s=1$.

$$\therefore \int_s^\infty e^{-t}\left(\frac{\sin 2t+\sin 3t}{t}\right)dt = \pi - \tan^{-1}\left(\frac{5}{6-1}\right).$$

$$= \pi - \tan^{-1}\left(\frac{5}{5}\right)$$

$$= \pi - \tan^{-1}(1)$$

$$= \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

$$\therefore \int_s^\infty e^{-t}\left(\frac{\sin 2t+\sin 3t}{2t}\right)dt = \frac{3\pi}{4}.$$

Hence proved

6. Find $L\left[\cosh t \int_0^t e^u \cosh u \, du\right]$

$\Rightarrow$

We have $L[\cosh u] = \dfrac{s}{s^2-1}$

$$\therefore L[e^u \cosh u] = \frac{s-1}{(s-1)^2-1}.$$

$$= \frac{s-1}{s^2-2s+\cancel{1}-\cancel{1}} = \frac{s-1}{s(s-2)}$$

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$$\therefore L\left[\int_0^t e^u \cosh u\, du\right] = \frac{1}{s}\left[\frac{s-1}{s(s-2)}\right]$$

$$= \frac{s-1}{s^2(s-2)}$$

$$\therefore L\left[\cosh t \int_0^t e^u \cosh u\, du\right] = L\left[\frac{e^t+e^{-t}}{2}\int_0^t e^u \cosh u\, du\right]$$

$$= \frac{1}{2}\left\{L\left[e^t\left(\int_0^t e^u \cosh u\, du\right)\right] + L\left[e^{-t}\left(\int_0^t e^u \cosh u\, du\right)\right]\right\}$$

$$= \frac{1}{2}\left[\frac{(s-1)-1}{(s-1)^2(s-1-2)} + \frac{(s+1)-1}{(s+1)^2(s+1-2)}\right]$$

$$= \frac{1}{2}\left[\frac{s-2}{(s-1)^2(s-3)} + \frac{s}{(s+1)^2(s-1)}\right]$$