Stoke's theorem

The integral of the normal component of the curl of a vector F over a surfaces is equal to the line integral of the tangential component of $\bar{F}$ around the curve bounding s

$$\iint_S \bar{N}\cdot(\nabla\times\bar{F})\,ds = \int_C \bar{F}\cdot d\bar{r}$$

where $\bar{N}$ is the unit outward normal vector to the element ds.

$ds = dx\,dy \qquad ds = r\,dr\,d\theta$

Ex ① Use stoke's theorem to evaluate $\int_C F\cdot d\bar{r}$ where $F = 4xyi - y^2 j + yz k$ and c is the boundary of the circle $x=0, y=0$ $\underline{16}$.

~~z=0~~, $x^2+y^2=1$ and $z=0$

Soln stokes theorem

$$\int_C F\cdot d\bar{r} = \iint_S \bar{N}\cdot(\nabla\times\bar{F})\,ds$$

$$\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy & -y^2 & yz \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(-y^2)\right] - j\left[\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(4xy)\right] + k\left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(4xy)\right]$$

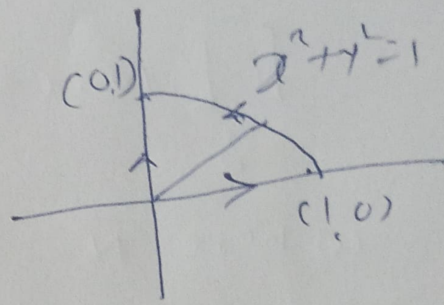
$\nabla \times \bar{F} = (z-0)i - j(0-0) + k(0-4x)$

$= zi - 0j - 4xk$

in the xy plane $\bar{N} = k$, $ds = dx\,dy$

$\iint_S \bar{N}\cdot(\nabla\times\bar{F})\,ds = \iint k\cdot(zi - 4xk)\,dx\,dy$

$= \iint -4x\,dx\,dy$



$x^2+y^2=1$

$x=0$, $y=\pm 1$ $\quad y=0$ $x=\pm 1$

$= -4\iint x\,dx\,dy$ $\quad x = r\cos\theta$ $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$ $\quad r=1$

$x^2+y^2 = r^2 = 1$

$r \Rightarrow r=0$ to $r=1$

$\theta \Rightarrow \theta = 0$ to $\theta = \frac{\pi}{2}$

$= -4\int_0^{\frac{\pi}{2}}\int_0^1 r\cos\theta\, r\,dr\,d\theta$

$= -4\int_0^{\pi/2} \cos\theta \left(\int_0^1 r^2\,dr\right)d\theta = -4\int_0^{\pi/2}\cos\theta\left(\frac{r^3}{3}\right)_0^1 d\theta$

$= -\frac{4}{3}\int_0^{\pi/2}\cos\theta\,d\theta$

$= -\frac{4}{3}(\sin\theta)_0^{\pi/2} = -\frac{4}{3}\left(\sin\frac{\pi}{2} - \sin 0\right)$

$= -\frac{4}{3}$

$= -\frac{4}{3}$

Ex 2 Use Stokes theorem to evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = x^2 i + xy j$ and C is the boundary of the rectangle $x=0, y=0, x=a, x=b$

Soln Stokes theorem

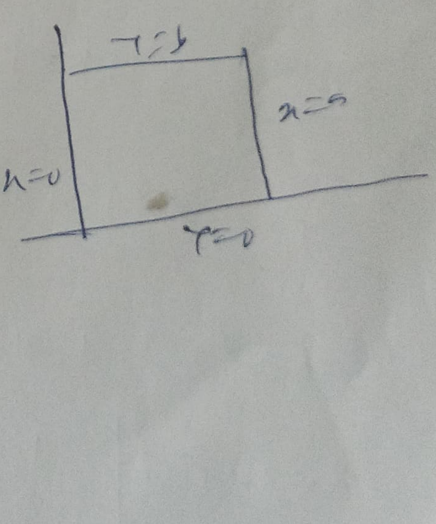
$$\int_C \bar{F}\cdot d\bar{r} = \iint_S \bar{N}\cdot(\nabla\times\bar{F})\, ds$$

$$\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(xy)\right] - j\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2)\right] + k\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= i(0-0) - j(0-0) + k(y-0)$$

$$= yk$$



in the xy plane

$ds = dx\,dy \quad \bar{N} = k$

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S \bar{n}\cdot(\nabla\times\bar{F})\,ds$$

$$= \iint_S k\cdot(yk)\,dx\,dy =$$

$$= \int_0^a\int_0^b y\,dy\,dx = \int_0^a \left(\frac{y^2}{2}\right)_0^b dx$$

$$= \frac{1}{2}\int_0^a b^2\,dx = \frac{b^2}{2}(x)_0^a$$

$$= \frac{ab^2}{2}$$

__Ex(3)__ Evaluate by Stokes thm $\int_C (xy\,dx + xy^2\,dy)$ where C is the square in the xy plane with vertices $(1,0)$, $(0,1)$, $(-1,0)$ & $(0,-1)$

__Soln__ Stokes thm $\int_C \bar{F}\cdot d\bar{r} = \iint_S \bar{n}\cdot(\nabla\times\bar{F})\,ds$

$\bar{F} = xy\,i + xy^2\,j$ $\qquad \bar{r} = xi + yj$

$\qquad d\bar{r} = dx\,i + dy\,j$

~~F.dr~~ $\bar{F}\cdot d\bar{r} = xy\,dx + xy^2\,dy \cdot (xy\,i + xy^2 j)(dx\,i + dy\,j)$

$$\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}$$

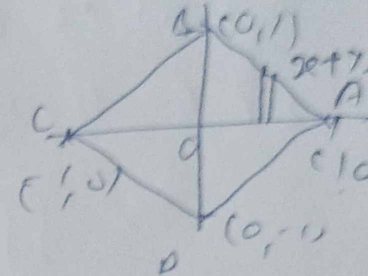
5

$$\nabla\times\vec{F} = i\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(xy^2)\right) - j\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(xy)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(xy)\right]$$

$$= i(0-0) - j(0-0) + k(y^2 - x)$$

$$= k(y^2 - x)$$



$\hat{n} = k \qquad ds = dx\,dy$

$$\iint_S \hat{n}\cdot(\nabla\times\vec{F})\,ds = \iint_S k(y^2 - x)k\,dx\,dy$$

$$= \iint_S (y^2 - x)\,dx\,dy$$

$y \Rightarrow y = 0$ to $y = 1-x$
$x \Rightarrow x = 0$ to $x = 1$

$$= 4\int_0^1 \int_0^{1-x} (y^2 - x)\,dy\,dx$$

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$$= 4\int_0^1 \left[\frac{y^3}{3} - x(y)\right]_0^{1-x} dx$$

$$= 4\int_0^1 \frac{1}{3}(1-x)^3 - x(1-x)\,dx$$

$$= 4\left[\frac{1}{3}\frac{(1-x)^4}{-4} - \frac{x^2}{2} + \frac{x^3}{3}\right]_0^1$$

$$= 4\left(-\frac{1}{12}(0-1) - \frac{1}{2} + \frac{1}{3}\right)$$

$$= 4\left(\frac{1}{12} - \frac{1}{2} + \frac{1}{3}\right) = 4\left(\frac{1-6+4}{12}\right) = 4\left(\frac{-1}{12}\right)$$

$$= -\frac{1}{3}$$

Gauss Divergence theorem

The surface integral of the normal component of a vector over a closed surface S is equal to the volume integral of the divergence of F throughout the volume bounded by S.

$$\iint_S \bar{N}\cdot\bar{F}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

where ~~$\bar{N}$~~ $\bar{N}$ is the unit outward normal.

(1) $dv = dx\,dy\,dz$

Ex (1) Use Gauss Divergence theorem to evaluate $\iint_S \bar{N}\cdot\bar{F}\,ds$ where $\bar{F} = x^2 i + zj + yzk$ and S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$

Soln Gauss Divergence theorem

$$\iint_S \bar{N}\cdot\bar{F}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv$$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz)$$

$$= 2x + 0 + y$$

$$= 2x + y$$

$$\therefore \iiint_V \nabla\cdot \bar{F}\, dv = \int_0^1\int_0^1\int_0^1 (2x+y)\, dz\, dy\, dx$$

$$= \int_0^1\int_0^1 \left((2x+y)z\right)_0^1 dy\, dx$$

$$= \int_0^1\int_0^1 (2x+y)(1-0)\, dy\, dx$$

$$= \int_0^1 \left[2xy + \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 2x(1) + \frac{1}{2}\, dx$$

$$= \left(2\frac{x^2}{2} + \frac{1}{2}x\right)_0^1$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

Ex ② Use Gauss Divergence thm to evaluate $\iint_S \bar{N}\cdot\bar{F}\, ds$ where $\bar{F} = 4xi + 3yj - 2zk$ and S is the surface bounded by $x=0, y=0, z=0$ and $2x+2y+z=4$

Soln Divergence thm